Weekly Puzzle - Solutions
Vectors

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Questions:

Vector Operations:

1. (a) $|\mathbf{a}| = \sqrt{a_x^2 + a_y^2}$ and $|\mathbf{b}| = \sqrt{b_x^2 + b_y^2}$
$|\mathbf{c}| = |\mathbf{b} - \mathbf{a}| = \sqrt{(b_x - a_x)^2 + (b_y - a_y)^2}$
Using the cosine rule, $\cos\theta = \frac{|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{c}|^2}{2|\mathbf{a}||\mathbf{b}|}$
so $\cos\theta = \frac{a_x^2 + a_y^2 + b_x^2 + b_y^2 - ((b_x^2 + a_x^2 - 2b_x a_x) + (b_y^2 + a_y^2 - 2b_y a_y))}{2|\mathbf{a}||\mathbf{b}|} = \frac{a_x b_x + a_y b_y}{|\mathbf{a}||\mathbf{b}|}$.
So $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \times \frac{a_x b_x + a_y b_y}{|\mathbf{a}||\mathbf{b}|} = a_x b_x + a_y b_y$.

(b) $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$ so $|\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2|\mathbf{v}|^2 \sin^2\theta$
$\sin^2\theta = 1 - \cos^2\theta = 1 - (\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|})^2 = \frac{|\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u}\cdot\mathbf{v})^2}{|\mathbf{u}|^2|\mathbf{v}|^2}$.
so $|\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2 = (u_x^2 + u_y^2 + u_z^2)(v_x^2 + v_y^2 + v_z^2) - (u_x v_x + u_y v_y + u_z v_z)^2 = (u_x^2 v_x^2 + u_x^2 v_y^2 + u_x^2 v_z^2 + u_y^2 v_x^2 + u_y^2 v_y^2 + u_y^2 v_z^2 + u_z^2 v_x^2 + u_z^2 v_y^2 + u_z^2 v_z^2) - (u_x^2 v_x^2 + u_y^2 v_y^2 + u_z^2 v_z^2 + 2u_x u_y v_x v_y + 2u_x u_z v_x v_z + 2u_y u_z v_y v_z)$.
This can be simplified to $(u_y v_z - u_z v_y)^2 + (u_z v_x - u_x v_z)^2 + (u_x v_y - u_y v_x)^2$.

(c) From the area of triangle formula, the area is equal to $\frac{1}{2}ab\sin C$. As $a = |\overrightarrow{OA}|$ and $b = |\overrightarrow{OB}|$, and as C is the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$, therefore, the area is equal to $\frac{1}{2}|\overrightarrow{OA} \times \overrightarrow{OB}|$.

(d) From the given matrix determinant, $\mathbf{v} \times \mathbf{w} = \mathbf{i}\det\begin{pmatrix} v_y & v_z \\ w_y & w_z \end{pmatrix} - \mathbf{j}\det\begin{pmatrix} v_x & v_z \\ w_x & w_z \end{pmatrix} + \mathbf{k}\det\begin{pmatrix} v_x & v_y \\ w_x & w_y \end{pmatrix}$.
Therefore, $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = u_x\det\begin{pmatrix} v_y & v_z \\ w_y & w_z \end{pmatrix} - u_y\det\begin{pmatrix} v_x & v_z \\ w_x & w_z \end{pmatrix} + u_z\det\begin{pmatrix} v_x & v_y \\ w_x & w_y \end{pmatrix} = \det\begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}$.

(e) $(\mathbf{u} \times (\mathbf{v} \times \mathbf{w}))_x = \det\begin{pmatrix} u_y & u_z \\ (\mathbf{v} \times \mathbf{w})_y & (\mathbf{v} \times \mathbf{w})_z \end{pmatrix} = u_y(v_x w_y - v_y w_x) - u_z(v_z w_x - v_x w_z) = v_x(u_y w_y + u_z w_z) - w_x(u_y v_y + u_z v_z) = v_x(u_y w_y + u_z w_z) - w_x(u_y v_y + u_z v_z) + u_x v_x w_x - u_x v_x w_x = v_x(u_x w_x + u_y w_y + u_z w_z) - w_x(u_x u_x + u_y v_y + u_z v_z) = v_x(\mathbf{u} \cdot \mathbf{w}) - w_x(\mathbf{u} \cdot \mathbf{v})$.
A similar technique can be used for $(\mathbf{u} \times (\mathbf{v} \times \mathbf{w}))_y$ and $(\mathbf{u} \times (\mathbf{v} \times \mathbf{w}))_z$.

An Introduction to Vector Calculus:

2. (a) $g(x, y, z) = \cos^2(xyz)$, so $\frac{\partial g}{\partial x} = 2\cos(xyz) \times -\sin(xyz) \times yz = -yz\sin(2xyz)$. As g is a symmetric function, then $\frac{\partial g}{\partial y} = -xz\sin(2xyz)$ and $\frac{\partial g}{\partial z} = -xy\sin(2xyz)$.

Therefore, $\nabla g = -yz\sin(2xyz)\mathbf{i} - xz\sin(2xyz)\mathbf{j} - xy\sin(2xyz)\mathbf{k}$.

(b) $\mathbf{G} = x\mathbf{i} + xyz\mathbf{j} + y\sin(x^2+z)\mathbf{k}$, so $P(x,y,z) = x$, $Q(x,y,z) = xyz$ and $R(x,y,z) = y\sin(x^2+z)$. Therefore, $\frac{\partial P}{\partial x} = 1$, $\frac{\partial Q}{\partial y} = xz$ and $\frac{\partial R}{\partial z} = y\cos(x^2+z)$. So, $\nabla \cdot \mathbf{G} = 1 + xz + y\cos(x^2+z)$.

(c) $\nabla \times \mathbf{F} = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{pmatrix} = (\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z})\mathbf{i} + (\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x})\mathbf{j} + (\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y})\mathbf{k}$.