Weekly Puzzle
Calculus

TW-C

02/11/2024 - 08/12/2024

Limits:

1. (a) We wish to show for each positive real ϵ, there exists is a positive real δ such that for all x satisfying $0 < |x - 6| < \delta$, then $|f(x) - 25| < \epsilon$, but as $f(x) = 3x + 7$, then we need to show $|3x + 7 - 25| < \epsilon$ which is the same as $|3x - 18| < \epsilon$. But we also have $|x - 6| < \delta$, so we can choose $\delta = \frac{\epsilon}{3}$, so our proof is complete.

   (b) We wish to show for each positive real ϵ, there exists is a positive real δ such that for all x satisfying $0 < |x - 2| < \delta$, then $|f(x) - 4| < \epsilon$, but as $f(x) = x^2$, then we need to show $|x^2 - 4| < \epsilon$ which is the same as $|x + 2| \times |x - 2| < \epsilon$.

   We can now do some work to derive an answer, and then prove it to ensure it is correct. Suppose $\delta < 1$, then $|x - 2| < 1$, so $1 < x < 3$, so $3 < x + 2 < 5$. Therefore, as $|x + 2| < 5$ and $|x - 2| < \delta$. So $|x^2 - 4| < 5\delta = \epsilon$. So as we have both $\delta < 1$ and $\delta = \frac{\epsilon}{5}$. So, let $\delta = min\{1, \frac{\epsilon}{5}\}$.

   Proof:

   We shall have 2 cases, $\delta < 1$ or $\delta < \frac{\epsilon}{5}$. (If $\epsilon = 5$, then use the case $\delta < 1$.)
   1: If $\delta < 1$, then as $0 < |x - 2| < \delta$, so $0 < |x^2 - 4| < |x + 2|\delta$. As $|x + 2| < 5$, and $\delta < 1$, then $|x^2 - 4| < 5$. But, as $\delta < 1$, then we know that $\frac{\epsilon}{5} \geq 1$ from the min function, so $\epsilon \geq 5$, so $|x^2 - 4| < \epsilon$.
   2: If $\delta < \frac{\epsilon}{5}$, then $\frac{\epsilon}{5} < 1$, from the min function. So, then $\delta < 1$, so follow the first case.

   (c) We know that for every $\epsilon_1 > 0$, there exists a $\delta_1 > 0$ such that for all x in $0 < |x - a| < \delta_1$, then $|f(x) - m| < \epsilon_1$. We also know that for every $\epsilon_2 > 0$, there exists a $\delta_2 > 0$ such that for all x in $0 < |x - a| < \delta_2$, then $|g(x) - n| < \epsilon_2$.

   If we take both ϵ_1 and ϵ_2 as $\frac{\epsilon}{2}$.
   Let $\delta = min\{\delta_1, \delta_2\}$. So if $0 < |x - a| < \delta$, then $|f(x) - m| < \frac{\epsilon}{2}$ and $|g(x) - n| < \frac{\epsilon}{2}$.
   So $|f(x) - m| + |g(x) - n| < \epsilon$. Using the triangle inequality, which says $|A + B| \leq |A| + |B|$, we have $|(f(x) + g(x)) - (m + n)| < \epsilon$.

   So if $0 < |x-a| < \delta$, we have $|(f(x)+g(x))-(m+n)| < \epsilon$. This is the definition of $\lim_{x \to a}(f(x)+g(x)) = m + n$, so we are done.

   (d) We have that for every $\epsilon_1 > 0$, there exists a $\delta > 0$ such that for all x in $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon_1$. Let $\epsilon_1 = \frac{\epsilon}{c}$.

   So $|f(x) - L| < \frac{\epsilon}{c}$, so $|cf(x) - cL| < \epsilon$, so $\lim_{x \to a}(cf(x)) = cL$.

Differentiation:

2. (a) $\frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2-x^2}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0} 2x+h = 2x.$

   (b) Any correct answer, such as x^b, where b is any negative real.

   (c) Let the function be f.
   Then $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$. So $\lim_{h\to 0}(h \times f'(a)) = \lim_{h\to 0}(f(a+h) - f(a)) = \lim_{x\to a}(f(x) - f(a))$
   So $\lim_{x\to a}(f(x) - f(a)) = f'(a) \times \lim_{h\to 0} h = 0$, so $\lim_{x\to a} f(x) = f(a)$, so f is continuous at a. As this can be applied to all a in the domain of f, then if f is differentiable, it is therefore continuous.

   (d) $\frac{du}{dx} = \lim_{h\to 0} \frac{u(x+h)-u(x)}{h}$ and $\frac{dv}{dx} = \lim_{h\to 0} \frac{v(x+h)-v(x)}{h}$.
   $\frac{d(uv)}{dx} = \lim_{h\to 0} \frac{u(x+h)\times v(x+h)-u(x)\times v(x)}{h} = \lim_{h\to 0} \frac{u(x+h)\times v(x+h)-u(x)\times v(x+h)+u(x)\times v(x+h)-u(x)\times v(x)}{h}$
   $= \lim_{h\to 0} \frac{u(x+h)\times v(x+h)-u(x)\times v(x+h)}{h} + \lim_{h\to 0} \frac{u(x)\times v(x+h)-u(x)\times v(x)}{h}$
   $= (\lim_{h\to 0} v(x+h)) \times (\lim_{h\to 0} \frac{u(x+h)-u(x)}{h}) + u(x) \times \lim_{h\to 0} \frac{v(x+h)-v(x)}{h}$.

   From the previous part, we have shown that as v is differentiable, it is continuous.

   So $\frac{d(uv)}{dx} = v(x)\frac{du}{dx} + u(x)\frac{dv}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$.

   (e) $y = arcsin(x)$, so $x = sin(y)$, so as $\frac{dx}{dy} = cos(y)$, so $\frac{dy}{dx} = \frac{1}{cos(y)}$. As $-\frac{\pi}{2} < y \leq \frac{\pi}{2}$ (from the range of the $arcsin$ function), then $cos(y) \geq 0$, so $cos(y) = \sqrt{1-x^2}$. Therefore, $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$.

Integration:

3. (a) $\frac{x^3}{3} + C$.

   (b) Let $h = \frac{b-a}{n}$. We shall also take $\Delta x = h$ and so $x_i = a + ih$.
   So $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\Delta x = \lim_{h\to 0} \sum_{i=0}^{\frac{b-a}{h}-1} f(a+ih)h$.
   As $f(x) = \frac{dF}{dx} = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h}$, then $\int_a^b f(x)dx = \lim_{h\to 0} \sum_{i=0}^{\frac{b-a}{h}-1} (\frac{F(a+ih+h)-F(a+ih)}{h} \times h)$
   $= \lim_{h\to 0} \sum_{i=0}^{\frac{b-a}{h}-1}(F(a+ih+h) - F(a+ih)) = F(a + (\frac{b-a}{h} - 1)h + h) - F(a) = F(b) - F(a)$.

   (c) As $\frac{x^3}{3} + 3x$ differentiates to give $x^2 + 3$, then the answer is $\frac{4^3}{3} + 3 \times 4 - \frac{2^3}{3} - 3 \times 2 = \frac{74}{3}$.

   (d) As $ln(x)$ differentiates to give $\frac{1}{x}$, then the answer is $ln(5) - ln(1) = ln(5)$.