

Weekly Puzzle - Solutions

Vectors

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Questions:

Vector Operations:

1. (a) $|\mathbf{a}| = \sqrt{a_x^2 + a_y^2}$ and $|\mathbf{b}| = \sqrt{b_x^2 + b_y^2}$
 $|\mathbf{c}| = |\mathbf{b} - \mathbf{a}| = \sqrt{(b_x - a_x)^2 + (b_y - a_y)^2}$
 Using the cosine rule, $\cos \theta = \frac{|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{c}|^2}{2|\mathbf{a}||\mathbf{b}|}$
 so $\cos \theta = \frac{a_x^2 + a_y^2 + b_x^2 + b_y^2 - ((b_x - a_x)^2 + (b_y - a_y)^2)}{2|\mathbf{a}||\mathbf{b}|} = \frac{a_x b_x + a_y b_y}{|\mathbf{a}||\mathbf{b}|}$.
 So $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \times \frac{a_x b_x + a_y b_y}{|\mathbf{a}||\mathbf{b}|} = a_x b_x + a_y b_y$.
 (b) $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta$ so $|\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2 |\mathbf{v}|^2 \sin^2 \theta$
 $\sin^2 \theta = 1 - \cos^2 \theta = 1 - \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)^2 = \frac{|\mathbf{u}|^2 |\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2}{|\mathbf{u}|^2 |\mathbf{v}|^2}$.
 so $|\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2 |\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2 = (u_x^2 + u_y^2 + u_z^2)(v_x^2 + v_y^2 + v_z^2) - (u_x v_x + u_y v_y + u_z v_z)^2 =$
 $(u_x^2 v_x^2 + u_x^2 v_y^2 + u_x^2 v_z^2 + u_y^2 v_x^2 + u_y^2 v_y^2 + u_y^2 v_z^2 + u_z^2 v_x^2 + u_z^2 v_y^2 + u_z^2 v_z^2) - (u_x^2 v_x^2 + u_y^2 v_y^2 + u_z^2 v_z^2 + 2u_x u_y v_x v_y +$
 $2u_x u_z v_x v_z + 2u_y u_z v_y v_z)$.
 This can be simplified to $(u_y v_z - u_z v_y)^2 + (u_z v_x - u_x v_z)^2 + (u_x v_y - u_y v_x)^2$.
 (c) From the area of triangle formula, the area is equal to $\frac{1}{2}ab \sin C$. As $a = |\vec{OA}|$ and $b = |\vec{OB}|$, and
 as C is the angle between \vec{OA} and \vec{OB} , therefore, the area is equal to $\frac{1}{2}|\vec{OA} \times \vec{OB}|$.
 (d) From the given matrix determinant, $\mathbf{v} \times \mathbf{w} = \mathbf{i} \det \begin{pmatrix} v_y & v_z \\ w_y & w_z \end{pmatrix} - \mathbf{j} \det \begin{pmatrix} v_x & v_z \\ w_x & w_z \end{pmatrix} + \mathbf{k} \det \begin{pmatrix} v_x & v_y \\ w_x & w_y \end{pmatrix}$.
 Therefore, $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = u_x \det \begin{pmatrix} v_y & v_z \\ w_y & w_z \end{pmatrix} - u_y \det \begin{pmatrix} v_x & v_z \\ w_x & w_z \end{pmatrix} + u_z \det \begin{pmatrix} v_x & v_y \\ w_x & w_y \end{pmatrix} = \det \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}$.
 (e) $(\mathbf{u} \times (\mathbf{v} \times \mathbf{w}))_x = \det \begin{pmatrix} u_y & u_z \\ (\mathbf{v} \times \mathbf{w})_y & (\mathbf{v} \times \mathbf{w})_z \end{pmatrix} = u_y(v_x w_y - v_y w_x) - u_z(v_z w_x - v_x w_z) = v_x(u_y w_y +$
 $u_z w_z) - w_x(u_y v_y + u_z v_z) = v_x(u_y w_y + u_z w_z) - w_x(u_y v_y + u_z v_z) + u_x v_x w_x - u_x v_x w_x = v_x(u_x w_x +$
 $u_y w_y + u_z w_z) - w_x(u_x u_x + u_y v_y + u_z v_z) = v_x(\mathbf{u} \cdot \mathbf{w}) - w_x(\mathbf{u} \cdot \mathbf{v})$.
 A similar technique can be used for $(\mathbf{u} \times (\mathbf{v} \times \mathbf{w}))_y$ and $(\mathbf{u} \times (\mathbf{v} \times \mathbf{w}))_z$.

An Introduction to Vector Calculus:

2. (a) $g(x, y, z) = \cos^2(xyz)$, so $\frac{\partial g}{\partial x} = 2 \cos(xyz) \times -\sin(xyz) \times yz = -yz \sin(2xyz)$. As g is a symmetric
 function, then $\frac{\partial g}{\partial y} = -xz \sin(2xyz)$ and $\frac{\partial g}{\partial z} = -xy \sin(2xyz)$.

Therefore, $\nabla g = -yz \sin(2xyz)\mathbf{i} - xz \sin(2xyz)\mathbf{j} - xy \sin(2xyz)\mathbf{k}$.

- (b) $\mathbf{G} = x\mathbf{i} + xyz\mathbf{j} + y \sin(x^2 + z)\mathbf{k}$, so $P(x, y, z) = x$, $Q(x, y, z) = xyz$ and $R(x, y, z) = y \sin(x^2 + z)$.
Therefore, $\frac{\partial P}{\partial x} = 1$, $\frac{\partial Q}{\partial y} = xz$ and $\frac{\partial R}{\partial z} = y \cos(x^2 + z)$. So, $\nabla \cdot \mathbf{G} = 1 + xz + y \cos(x^2 + z)$.

(c) $\nabla \times \mathbf{F} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{pmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \mathbf{k}.$