Weekly Puzzle

Calculus

TW-C

02/11/2024 - 08/12/2024

What you need to know:

Triangle inequality
Simple limit laws

Questions:

We shall define $\lim_{x \to a} f(x) = L$ to mean that for every positive real number ϵ, there exists a positive real number δ such that, for all x satisfying $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$. This is called the epsilon-delta definition.

Worked Example:

We shall prove that $\lim_{x \to 4} x = 4$, as an example.

We wish to show for each positive real ϵ, there exists a positive real δ such that for all x satisfying $0 < |x - 4| < \delta$, then $|f(x) - 4| < \epsilon$, but as $f(x) = x$, then we need to show $|x - 4| < \epsilon$. But we also have $|x - 4| < \delta$, so we can choose $\delta = \epsilon$, so our proof is complete. (Note, we do not have to choose this specifically, we could have chosen $\delta = \frac{\epsilon}{2}$, and this would be fine.)

Limits:

1. (a) Prove $\lim_{x \to 6} 3x + 7 = 25$, using the epsilon-delta definition.

   (b) Prove $\lim_{x \to 2} x^2 = 4$, using the epsilon-delta definition.

   The sum law for limits says that if $\lim_{x \to a} f(x) = m$ and $\lim_{x \to a} g(x) = n$, then $\lim_{x \to a} (f(x) + g(x)) = m + n$.

   (c) Prove the sum law for limits. (The triangle equality is useful here.)

   The constant multiple law for limits says that if $\lim_{x \to a} f(x) = L$, then $\lim_{x \to a} (cf(x)) = cL$.

   (d) Prove the constant multiple law for limits.

From this point on, you may use the sum law, difference law, constant multiple law, product law, quotient law, power law and root law for limits without proof.

Differentiation:

2. The definition of the derivative of a function f is that $\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$.

   (a) Derive the derivative of the squaring function, f, where $f(x) = x^2$.

   We shall define continuous at a to mean if the function f satisfies $\lim_{x \to a} f(x) = f(a)$.

   (b) Provide an example of a function which is discontinuous at some real a.

   (c) Prove that if a function is differentiable at a, then it is therefore continuous at a.

   (d) Prove the product rule, which says $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$.

   You may assume, without proof, that $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$, when the derivatives are defined, for the following part.

   (e) Derive the derivative of the function $arcsin$.

Integration:

3. An indefinite integral of a continuous function f is a differentiable function F, which satisfies $\frac{dF}{dx} = f(x)$.

   (a) What are the indefinite integrals of the square function?

   We shall use the Riemann integral definition for following definite integrals. We will take a partition of the interval $[a, b]$, meaning the following sequence of real numbers which follow $a = x_0 < x_1 < \ldots < x_n = b$ where the difference between consecutive x values is constant and is Δx. Then the definition of $\int_a^b f(x)dx$ is $\lim_{n \to \infty} \sum_{i=0}^{n-1} f(x_i)\Delta x$.

   (b) Prove the fundamental theorem of calculus, which says $\int_a^b f(x)dx = F(b) - F(a)$, where F is a function which satisfies $\frac{dF}{dx} = f(x)$.

   (c) Evaluate $\int_2^4 (x^2 + 3)dx$.

   (d) What is $\int_1^5 \frac{1}{x}dx$?

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