Weekly Puzzle - Solutions

TW-C

25/11/2024 - 1/12/2024

Solutions:

1. (a) What is $\cosh(i)$ in terms of trigonometric functions?

   $\cosh(\theta) = \frac{e^\theta + e^{-\theta}}{2}$, so by substituting, we have $\cosh(i) = \frac{e^i + e^{-i}}{2}$.
   Using Euler's formula, we have $e^i = \cos(1) + i\sin(1)$ and $e^{-i} = \cos(-1) + i\sin(-1) = \cos(1) - i\sin(1)$.
   So $\cosh(i) = \frac{\cos(1)+i\sin(1)+\cos(1)-i\sin(1)}{2} = \frac{2\cos(1)}{2} = \cos(1)$.

   (b) What is $\cosh(ix)$ in terms of trigonometric functions?

   $\cosh(\theta) = \frac{e^\theta + e^{-\theta}}{2}$, so by substituting, we have $\cosh(ix) = \frac{e^{ix} + e^{-ix}}{2}$.
   Using Euler's formula, we have $e^{ix} = \cos(x) + i\sin(x)$ and $e^{-ix} = \cos(-x) + i\sin(-x) = \cos(x) - i\sin(x)$.
   So $\cosh(ix) = \frac{\cos(x)+i\sin(x)+\cos(x)-i\sin(x)}{2} = \frac{2\cos(x)}{2} = \cos(x)$.

   (c) Hence, by making a suitable substitution, what is $\cos(i)$?

   As $\cosh(ix) = \cos(x)$, we can substitute to obtain $\cos(i) = \cosh(i \times i) = \cosh(-1) = \cosh(1)$.

   (d) What is $\sin(i)$?

   $\sinh(\theta) = \frac{e^\theta - e^{-\theta}}{2}$, so by substituting, we have $\sinh(ix) = \frac{e^{ix} - e^{-ix}}{2}$.
   Using Euler's formula, we have $e^{ix} = \cos(x) + i\sin(x)$ and $e^{-ix} = \cos(-x) + i\sin(-x) = \cos(x) - i\sin(x)$.
   So $\sinh(ix) = \frac{\cos(x)+i\sin(x)-(\cos(x)-i\sin(x))}{2} = \frac{2i\sin(x)}{2} = i\sin(x)$.
   Therefore, by dividing both sides by i, we have $\sin(x) = \frac{\sinh(ix)}{i}$, so by substituting, we have $\sin(i) = \frac{\sinh(i \times i)}{i} = \frac{\sinh(-1)}{i} = \frac{-\sinh(1)}{i} = i\sinh(1)$.

   (e) Express $\cos(z)$, where z is a complex number in the form $a + bi$, in terms of trigonometric and hyperbolic functions of real numbers.

   $\cos(z) = \cos(a + bi) = \cos(a)\cos(bi) - \sin(a)\sin(bi)$
   As $\cos(x) = \cosh(ix)$, then $\cos(bi) = \cosh(bi \times i) = \cosh(-b) = \cosh(b)$, and as $\sin(x) = \frac{\sinh(ix)}{i}$, then $\sin(bi) = \frac{\sinh(bi \times i}{i} = \frac{\sinh(-b)}{i} = \frac{-\sinh(b)}{i} = i\sinh(b)$.
   Therefore, $\cos(z) = \cos(a)\cosh(b) - i\sin(a)\sinh(b)$.

2. (a) By using the definition of the natural logarithm of real numbers in relation to exponents, come up with a definition for the logarithm of complex numbers.

   We can define $ln(a) = b$ to mean $e^b = a$, so we can define $ln(z) = w$ to mean $e^w = z$.

(b) Why is this not a function? (Hint: why do we restrict the domains of trigonometric functions when defining their inverses)

As $e^{2\pi ni} = 1$, where $n \in \mathbb{Z}$, then $e^{w+2\pi ni} = e^w \times e^{2\pi ni} = z \times 1 = z$.
So w is not unique, as one can add any multiple of $2\pi i$ to it, so our definition is one-to-many, and so it is not a function.

We shall use the notation $Log(z)$ to represent the principal branch of the complex logarithm, meaning the argument of z will be restricted to lie in the interval $(-\pi, \pi]$.

(c) By using the exponential form of complex numbers, what is $Log(z)$?

Let $\theta \in (-\pi, \pi]$.
$z = re^{i\theta}$, so $Log(z) = Log(re^{i\theta}) = ln(r) + Log(e^{i\theta}) = ln(r) + i\theta$.