Weekly Puzzle

Differential equations

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11/02/2025 - 17/02/2025

Questions:

First Order:

1. (a) The integrating factor for a differential equation of the form $\frac{dy}{dx} + P(x)y = Q(x)$ is $e^{\int P(x)\,dx}$, so as $P(x) = 1$, then our integrating factor is e^x, so $e^x \frac{dy}{dx} + e^x y = xe^{2x}$. Therefore, $\frac{d(ye^x)}{dx} = xe^{2x}$.
   We can use integration by parts to integrate xe^{2x}.
   Let $u = x$ and $v' = e^{2x}$. Then, $u' = 1$ and $v = \frac{1}{2}e^{2x}$.
   So, $\int xe^{2x}\,dx = x\frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x}\,dx = \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C = \frac{(2x-1)e^{2x}}{4} + C$.
   Therefore, $ye^x = \frac{(2x-1)e^{2x}}{4} + C$, so $y = \frac{(2x-1)e^x}{4} + Ce^{-x}$.

   (b) If $y^2 + 4y + 3 \neq 0$, then $y \neq -1$ and $y \neq -3$, and so $\frac{1}{y^2+4y+3}\frac{dy}{dx} = 1$.

   We can use partial fractions to solve this.
   $\frac{1}{y^2+4y+3} = \frac{1}{(y+1)(y+3)} = \frac{A}{y+1} + \frac{B}{y+3} = \frac{A(y+3)}{(y+1)(y+3)} + \frac{B(y+1)}{(y+1)(y+3)}$.
   So $1 = A(y+3) + B(y+1) = y(A+B) + 3A + B$. So $A + B = 0$ and $3A + B = 1$, so $2A = 1$ so $A = \frac{1}{2}$, so $B = -\frac{1}{2}$.
   So $\frac{1}{y^2+4y+3} = \frac{1}{2(y+1)} - \frac{1}{2(y+3)}$.

   Therefore, $(\frac{1}{2(y+1)} - \frac{1}{2(y+3)})\frac{dy}{dx} = 1$.
   So by integrating, we have $\int(\frac{1}{2(y+1)} - \frac{1}{2(y+3)})\,dy = \int 1\,dx = x + C$.
   Hence $\frac{1}{2}\ln|y+1| - \frac{1}{2}\ln|y+3| = x + C$, which can be written as $\ln|\frac{y+1}{y+3}| = 2x + 2C$.
   Therefore $|\frac{y+1}{y+3}| = e^{2x+2C} = A_1 e^{2x}$, where A_1 is a positive constant.
   So $\frac{y+1}{y+3} = Ae^{2x}$, where A is a non-zero constant and upon rewriting, we get $y = \frac{2}{1-Ae^{2x}} - 3 = \frac{3Ae^{2x}-1}{1-Ae^{2x}}$.

   We must also consider the cases when $y = -1$ and $y = -3$.
   If $y = -1$ or $y = -3$, then $\frac{dy}{dx} = y^2 + 4y + 3 = 0$, so $y = -1$ and $y = -3$ are valid constant solutions.
   We may condense this, as $y = -1$ is the same as $y = \frac{3Ae^{2x}-1}{1-Ae^{2x}}$ when $A = 0$.
   So our solutions are $y = \frac{3Ae^{2x}-1}{1-Ae^{2x}}$ and $y = -3$, where A is a constant.

   (c) If $y \neq 0$, $u = y^{-1}$, so $y = u^{-1}$, so $\frac{dy}{du} = -u^{-2}$, so $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = u^{-2}\frac{du}{dx}$.
   So $-xu^{-2}\frac{du}{dx} + 3u^{-1} = e^x u^{-2}$.
   Upon multiplying by $-\frac{u^2}{x}$, we have $\frac{du}{dx} - \frac{3}{x}u = -\frac{e^x}{x}$ if $x \neq 0$.
   Hence, our integration factor is $e^{\int -\frac{3}{x}\,dx} = e^{-3(\ln(x))} = x^{-3}$.

So, $x^{-3}\frac{du}{dx} - x^{-4} \times 3u = \frac{d(x^{-3}u)}{dx} = -x^{-4}e^x$.
This gives $x^{-3}u = \int -x^{-4}e^x\,dx$, so let $I(x) = \int x^{-4}e^x\,dx$.
Then, $x^{-3}u = -I(x) + C$, so by multiplying by x^3, we have $u = (-I(x) + C)x^3$.
Therefore $y = \frac{1}{x^3(-I(x)+C)}$.
So, if $x = 1$, $\frac{1}{1+e} = \frac{1}{-I(1)+C}$ so $C = 1 + e + I(1)$.
Therefore, $y = \frac{1}{x^3(I(1)-I(x)+e+1)}$.
We must also consider $x = 0$, which gives $3y = y^2$, so $y = 0$ or $y = 3$. $y = 3$ is continuous with $y = \frac{1}{x^3(I(1)-I(x)+e+1)}$, otherwise we couldn't differentiate y at $x = 0$, but the proof is very long and difficult.
Additionally, $y = 0$ must be considered, which gives a valid constant solution.
So our solutions are $y = 0$, and $y = \frac{1}{x^3(I(1)-I(x)+e+1)}$ if $x \neq 0$ and $y = 3$ if $x = 0$.

Second Order:

2. (a) Let $u = y'$, then $u' + 7u + 6 = 0$ so $u' + 7u = -6$.
   Our integration factor is $e^{\int 7\,dx} = e^{7x}$.
   $e^{7x}u' + 7e^{7x}u = \frac{d(e^{7x}u)}{dx} = -6e^{7x}$.
   $\int \frac{d(e^{7x}u)}{dx}\,dx = \int -6e^{7x}\,dx$.
   $e^{7x}u = -\frac{6}{7}e^{7x} + C$. After dividing by e^{7x}, it becomes $u = Ce^{-7x} - \frac{6}{7}$.
   As $\frac{dy}{dx} = u$, then $\frac{dy}{dx} = Ce^{-7x} - \frac{6}{7}$.
   So $\int \frac{dy}{dx}\,dx = \int (Ce^{-7x} - \frac{6}{7})\,dx$.
   $y = \frac{C}{-7}e^{-7x} - \frac{6}{7}x + B$.
   $y = Ae^{-7x} - \frac{6}{7}x + B$.

   (b) $\lambda^2 + 7\lambda + 6 = (\lambda + 6)(\lambda + 1) = 0$.
   Therefore, the solutions to the characteristic equation are $\lambda = -1$ and $\lambda = -6$. and so the solution to the differential equation is $Ae^{-x} + Be^{-6x}$.

   (c) $\lambda^2 + 2\lambda + 2 = 0$, so $\lambda = \frac{-2\pm\sqrt{2^2-4\times1\times2}}{2\times1} = -1 \pm i$.
   Therefore, the solution to the differential equation is $Ae^{(-1+i)x} + Be^{(-1-i)x}$.
   However, we do not need complex numbers for our solution, using Euler's formula, we have $e^{i\theta} = \cos\theta + i\sin\theta$.
   Hence, $Ae^{(-1+i)x} + Be^{(-1-i)x} = e^{-x}(Ae^{ix} + Be^{-ix}) = e^{-x}(A(\cos x + i\sin x) + B(\cos -x + i\sin -x))$.
   We know that cos is even, meaning $\cos -x = \cos x$. sin is odd, meaning $\sin -x = -\sin x$.
   Thefore, $e^{-x}(A(\cos x + i\sin x) + B(\cos -x + i\sin -x)) = e^{-x}(A(\cos x + i\sin x) + B(\cos x - i\sin x)) = e^{-x}((A + B)\cos x + (A - B)i\sin x)$.
   We can choose constants, C and D, as $C = A + B$, $D = (A - B)i$, which allows all possible A and B, so our solution is $e^{-x}(C\cos x + D\sin x)$.

   (d) If $b \neq 0$, then the solution is $Ae^{(a+bi)x} + Be^{(a-bi)x} = e^{ax}(Ae^{ibx} + Be^{-ibx})$.
   $e^{ax}(A(\cos(bx) + i\sin(bx)) + B(\cos(-bx) + i\sin(-bx))) = e^{ax}(A(\cos(bx) + i\sin(bx)) + B(\cos(bx) - i\sin(bx)))$.
   $= e^{ax}((A + B)\cos(bx) + (A - B)i\sin(bx))$. We can choose constants, C and D, as $C = A + B$, $D = (A - B)i$, which allows all possible A and B, so our solution is $e^{ax}(C\cos(bx) + D\sin(bx))$.
   If $b = 0$, then the roots are both a, so our solution is $Ae^{ax} + Bxe^{ax}$.

(e) The homogeneous equation is $y'' + 4y' + 4y = 0$, which has characteristic equation $\lambda^2 + 4\lambda + 4 = 0$.
$(\lambda + 2)(\lambda + 2) = 0$, so $\lambda = -2$.
So the solution is $e^{-2x}(A + Bx)$.
We are told that we must add on the particular function, which is of the form Ae^x, but we already used A, so we may change it to Ce^x, giving us $e^{-2x}(A + Bx) + Ce^x$.
As this will work for all A and B, we may choose A and B as both 0, so Ce^x should be a solution.
$y = Ce^x$, so $y' = Ce^x$ and $y'' = Ce^x$.
So, $Ce^x + 4Ce^x + 4Ce^x = e^x$, so $9C = 1$, so $C = \frac{1}{9}$.
Therefore, our solution is $e^{-2x}(A + Bx) + \frac{1}{9}e^x$.

Other differential equations:

3. (a) We may differentiate the first equation with respect to t, giving $\frac{d^2y}{dt^2} = \frac{dx}{dt} + 2\frac{dy}{dt}$.
$\frac{d^2y}{dt^2} = 2x - y + 2(x + 2y) = 4x + 3y$. $x = \frac{dy}{dt} - 2y$, so $\frac{d^2y}{dt^2} = 4\frac{dy}{dt} - 8y + 3y = 4\frac{dy}{dt} - 5y$.
So $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 5y = 0$. So our characteristic equation is $\lambda^2 - 4\lambda + 5 = 0$, so $\lambda = \frac{4 \pm \sqrt{4^2 - 4 \times 1 \times 5}}{2 \times 1} = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm i$.
So, from 2d we have our solution is $y = e^{2t}(C\cos t + D\sin t)$.
So as $x = \frac{dy}{dt} - 2y$, we have $x = \frac{e^{2t}(C\cos t + D\sin t)}{dt} - 2e^{2t}(C\cos t + D\sin t)$.
$x = \frac{d(e^{2t})}{dt}(C\cos t + D\sin t) + e^{2t}\frac{d(C\cos t + D\sin t)}{dt} - 2e^{2t}(C\cos t + D\sin t)$.
$x = 2e^{2t}(C\cos t + D\sin t) + e^{2t}(-C\sin t + D\cos t) - 2e^{2t}(C\cos t + D\sin t) = e^{2t}(D\cos t - C\sin t)$.

(b) If $y \neq 0$ and $y \neq K$, then $\frac{1}{y(K-y)}\frac{dy}{dx} = \frac{k}{K}$.
$\frac{1}{y(K-y)} = \frac{A}{y} + \frac{B}{K-y} = \frac{A(K-y)}{y(K-y)} + \frac{By}{y(K-y)}$.
$1 = A(K - y) + By = (B - A)y + AK$. So $B - A = 0$ so $A = B$ and $AK = 1$ so $A = \frac{1}{K}$.
So $(\frac{1}{Ky} + \frac{1}{K(K-y)})\frac{dy}{dx} = \frac{k}{K}$. Upon multiplying by K, we have $(\frac{1}{y} + \frac{1}{(K-y)})\frac{dy}{dx} = k$.
By integrating, we have $\int(\frac{1}{y} + \frac{1}{K-y})\,dy = \int k\,dx$.
Thefore $\ln|y| - \ln|K - y| = \ln|\frac{y}{K-y}| = kx + C$.
So $|\frac{y}{K-y}| = e^{kx+C} = A_1e^{kx}$ where A_1 is a positive constant.
Hence $\frac{y}{K-y} = Ae^{kx}$ where A is a non-zero constant.
Upon rewriting, we have $y = K - \frac{K}{Ae^{kx}+1}$, which can then be written as $y = \frac{K}{1+\frac{1}{A}e^{-kx}}$.
Considering $y = 0$ and $y = K$, both of these give $\frac{dy}{dx} = 0$, so both are valid constant solutions.
So the solutions are $y = \frac{K}{1+\frac{1}{A}e^{-kx}}$, $y = 0$ and $y = K$, where A is a non-zero constant.