Weekly Puzzle

Differential equations

Thomas Winrow-Campbell

11/02/2025 - 17/02/2025

What you need to know:

Solving first order differential equations
Partial fractions
Complex numbers and Euler's formula

Questions:

First Order:

1. (a) Solve $\frac{dy}{dx} + y = xe^x$.

   (b) Solve $\frac{dy}{dx} = y^2 + 4y + 3$. (Hint: Partial fractions.)

   (c) A Bernoulli differential equation is a differential equation of the form $\frac{dy}{dx} + P(x)y = Q(x)y^n$, where n is a real number. If $n \neq 0$ and $n \neq 1$, one may use the substitution $u = y^{1-n}$ to reduce it to a linear differential equation. Solve the differential equation, $x\frac{dy}{dx} + 3y = e^x y^2$, given that $y = \frac{1}{1+e}$ when $x = 1$.

Second Order:

2. (a) Solve the differential equation $y'' + 7y' + 6 = 0$, by using a substitution.

Second order differential equations of the form $y'' + py' + qy = 0$ may be solved by replacing y'' with λ^2, y' with λ, and y with 1, giving a quadratic equation, called the characteristic equation. The roots of this quadratic, λ_1 and λ_2, may then be found. The solution to the differential equation is then $Ae^{\lambda_1 x} + Be^{\lambda_2 x}$ if $\lambda_1 \neq \lambda_2$, where A and B are constants (even complex). If $\lambda_1 = \lambda_2$, then the solution is $Ae^{\lambda_1 x} + Bxe^{\lambda_1 x}$ (Note that there is no condition that λ_1 and λ_2 must be real!)

   (b) Solve $y'' + 7y' + 6y = 0$.

   (c) Solve $y'' + 2y' + 2y = 0$. (Note that this involves complex numbers.)

   (d) Given that the solutions to the characteristic equation are $a + bi$ and $a - bi$, find the solution to the second-differential equation $y'' + 2ay' + (a^2 + b^2)y = 0$ using trigonometric functions.

A non-homogeneous second-differential equation of the form $y'' + py' + qy = f(x)$ may be solved similarly to one of the form $y'' + py' + qy = 0$, except one must add a particular function to the solution. You will be given the form of the particular solution.

(e) Solve $y'' + 4y' + 4y = e^x$. The form of the particular solution when $f(x) = e^x$ is Ae^x.

Other differential equations:

3. (a) Solve for x and y in terms of t, where they satisfy $\frac{dy}{dt} = x + 2y$ and $\frac{dx}{dt} = 2x - y$. (Hint: Eliminate one of the two variables.)

   (b) The logistic equation is the solution to $\frac{dy}{dx} = ky(1 - \frac{y}{K})$. What is the logistic equation?