Weekly Puzzle

Matrices

Thomas Winrow-Campbell

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What you need to know:

Matrix addition, multiplication with scalars and inverse
The product of matrices and vectors

Questions:

Matrices:

1. (a) Gaussian elimination and Gauss-Jordan elimination are process for calculating the determinant and inverse of a matrix respectively. For Gaussian elimination, there are 3 operations that one can do. One can swap two rows, which multiplies the determinant by -1. One can multiply a row by a non-zero scalar, which multiplies the determinant by the same scalar. One can also add a scalar multiple of one row to another, and this has no effect on the determinant.

   An example is provided:
   Let $\mathbf{M} = \begin{pmatrix} 1 & 3 & 8 \\ 5 & 4 & 3 \\ 6 & 2 & 3 \end{pmatrix}$.

   We are first going to add -5 of the first row to the second row, resulting in $\begin{pmatrix} 1 & 3 & 8 \\ 0 & -11 & -37 \\ 6 & 2 & 3 \end{pmatrix}$.

   We are now going to add -6 of the first row to the third row, resulting in $\begin{pmatrix} 1 & 3 & 8 \\ 0 & -11 & -37 \\ 0 & -16 & -45 \end{pmatrix}$.

   Finally, we are going to add $-\frac{16}{11}$ of the second row to the third row, resulting in $\begin{pmatrix} 1 & 3 & 8 \\ 0 & -11 & -37 \\ 0 & 0 & \frac{97}{11} \end{pmatrix}$.

   This is called the upper triangular form, which is when the elements below the leading diagonal are all 0. The determinant of this can be calculated by multiplying the elements in the leading diagonal, giving us -97.

   By using Gaussian elimination, what is the determinant of $\begin{pmatrix} 7 & 4 & 3 \\ 4 & -9 & 1 \\ 5 & 0 & 2 \end{pmatrix}$?

   (b) Solving simultaneous equations can be simplified greatly with the use of matrices. For instance, if we wish to solve
   $x + 2y - 6z = -34$, $3x + y - 4z = -14$, $5x + 3y + 2z = 40$,

then we can write $\begin{pmatrix}1 & 4 & 3\\ 4 & -9 & 1\\ 5 & 0 & 2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}-34\\ -14\\ 40\end{pmatrix}$. One can then left-multiply both sides by the inverse of the matrix, giving us the solution. Using this method, what are the solutions to the simultaneous equations?

(c) The eigenvectors, $\mathbf{v}$ and the eigenvalues, λ, of a square matrix, $\mathbf{M}$ satisfy the equation, $\mathbf{Mv} = \lambda\mathbf{v}$, where $\mathbf{v} \neq \mathbf{0}$.

What are the eigenvectors and eigenvalues of the matrix $\begin{pmatrix}3 & 2\\ 5 & -6\end{pmatrix}$?

(Hint: Rearrange $\mathbf{Mv} = \lambda\mathbf{v}$ to give $(\mathbf{M} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$.)
(Hint 2: As $\mathbf{v} \neq \mathbf{0}$, then the determinant of $\mathbf{M} - \lambda\mathbf{I}$ is 0.)

(d) From part (c), what is the link between the square matrix and the product of the eigenvalues? Prove this link for any square matrix $\mathbf{M}$ of size $n \times n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$.

(e) The trace of a square matrix is equal to the sum of the elements on the leading diagonal. What is the link between the trace of a square matrix and its eigenvalues? (You need not prove it, though you may if you wish.)