Weekly Puzzle
Matrices - Solutions

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Questions:

Matrices:

1. (a) First, subtract $\frac{4}{7}$ of row 1 from row 2, to get $\begin{pmatrix} 7 & 4 & 3 \\ 0 & -\frac{79}{7} & -\frac{5}{7} \\ 5 & 0 & 2 \end{pmatrix}$.

   Next, subtract $\frac{5}{7}$ of row 1 from row 3, to get $\begin{pmatrix} 7 & 4 & 3 \\ 0 & -\frac{79}{7} & -\frac{5}{7} \\ 0 & -\frac{20}{7} & -\frac{1}{7} \end{pmatrix}$

   Next, subtract $\frac{20}{79}$ of row 2 from row 3, to get $\begin{pmatrix} 7 & 4 & 3 \\ 0 & -\frac{79}{7} & -\frac{5}{7} \\ 0 & 0 & \frac{3}{79} \end{pmatrix}$

   So the determinant is $7 \times -\frac{79}{7} \times \frac{3}{79} = -3$.

   (b) Let $\mathbf{M} = \begin{pmatrix} 1 & 2 & -6 \\ 3 & 1 & -4 \\ 5 & 3 & 2 \end{pmatrix}$, then we can left-multiply by $\mathbf{M}^{-1}$, giving $\mathbf{M}^{-1}\mathbf{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} -34 \\ -14 \\ 40 \end{pmatrix}$.

   $LHS = \mathbf{I}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, so $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} -34 \\ -14 \\ 40 \end{pmatrix}$

   $\mathbf{M}^{-1} = \frac{1}{62}\begin{pmatrix} -14 & 22 & 2 \\ 26 & -32 & 14 \\ -4 & -7 & 5 \end{pmatrix}$,

   so $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} -34 \\ -14 \\ 40 \end{pmatrix} = \frac{1}{62}\begin{pmatrix} -14 & 22 & 2 \\ 26 & -32 & 14 \\ -4 & -7 & 5 \end{pmatrix}\begin{pmatrix} -34 \\ -14 \\ 40 \end{pmatrix} = \frac{1}{62}\begin{pmatrix} -14 \times -34 + 22 \times -14 + 2 \times 40 \\ 26 \times -34 + -32 \times -14 + 14 \times 40 \\ -4 \times -34 + -7 \times -14 + 5 \times 40 \end{pmatrix} =$
   $\frac{1}{62}\begin{pmatrix} 248 \\ 124 \\ 434 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix}$.

   (c) Let $\mathbf{M} = \begin{pmatrix} 3 & 2 \\ 5 & -6 \end{pmatrix}$, and let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$.

   If $\mathbf{Mv} = \lambda\mathbf{v}$, then $(\mathbf{M} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$.

   $\begin{pmatrix} 3-\lambda & 2 \\ 5 & -6-\lambda \end{pmatrix}\mathbf{v} = \mathbf{0}$.

   So $\begin{pmatrix} 3-\lambda & 2 \\ 5 & -6-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$.

The determinant of $\begin{pmatrix} 3-\lambda & 2 \\ 5 & -6-\lambda \end{pmatrix}$ is 0, so $(3-\lambda)(-6-\lambda) - 10 = 0$, so $\lambda^2 + 3\lambda - 28 = 0$ so $(\lambda+7)(\lambda-4) = 0$, so $\lambda = -7$ or $\lambda = 4$.

Therefore, $(3-\lambda)x + 2y = 0$, and $5x + (-6-\lambda)y = 0$.

If $\lambda = -7$, then $10x + 2y = 0$, so $5x + y = 0$, so $\begin{pmatrix} 1 \\ -5 \end{pmatrix}$ is an eigenvector.

If $\lambda = 4$, then $-x + 2y = 0$, so $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector.

(d) From part (c), what is the link between the square matrix and the product of the eigenvalues? Prove this link for any square matrix $\mathbf{M}$ of size $n \times n$ with eigenvalues $\lambda_1, \dots, \lambda_n$.

The eigenvalues $\lambda_1, \dots, \lambda_n$ are solutions to $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.

$\det(\mathbf{A} - \lambda\mathbf{I}) = k(\lambda - \lambda_1)(\lambda - \lambda_2)\dots(\lambda - \lambda_n)$, where k is a scalar.

However, the leading term, λ^n would have coefficient $(-1)^n$, because the highest-degree term in $\det(\mathbf{A} - \lambda\mathbf{I})$ comes from the product of the diagonal elements.

Therefore, $k = (-1)^n$, so $\det(\mathbf{A} - \lambda\mathbf{I}) = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\dots(\lambda - \lambda_n) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\dots(\lambda_n - \lambda)$.

Therefore, by setting $\lambda = 0$, then $\det(\mathbf{A}) = \lambda_1\lambda_2\dots\lambda_n$.

(e) The sum of the eigenvalues is equal to the trace.