Weekly Puzzle - Solutions

Geometry

Thomas Winrow-Campbell

10/11/2024 - 16/12/2024

Questions:

Cyclic quadrilaterals:

1. (a) A quadrilateral whose vertices all lie on one circle.

   (b) They sum to π radians ($180°$).

   (c) Let A, B and C be distinct points on the circumference of a circle, such that AB is the diameter of that circle. Reflect C along the diameter AB to get the point C', which will lie on the circumference. As the angles ACB and $AC'B$ are equal, due to the reflection, and due to the property of the sum of opposite angles, the angle ACB is $90°$.

   (d) Let the cyclic quadrilateral be $ABCD$.
   From the law of sines, $\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c} = 2R$, where R is the triangle's circumradius, then let angles ADB, BDC be x, y respectively.
   Then, $AB = 2R\sin(x)$ and $BC = 2R\sin(y)$, but also $AC = 2R\sin(x+y)$.
   From the circle theorem that says that angles at the circumference subtended by the same arc are equal, angles ADB and ACB are equivalent. Let angles ACD, CBD be w, z respectively.
   The angles in a triangle sum to $180°$, so $x+y+z+w = 180°$, so we shall replace w with $180° - (x+y+z)$.
   So $CD = 2R\sin(z)$ and $AD = 2R\sin(x+y+z)$.
   The original equation can therefore be written as $4R^2\sin(x+y)\sin(y+z) = 4R^2\sin(x)\sin(z) + 4R^2\sin(y)\sin(x+y+z)$, which reduces to proving $\sin(x+y)\sin(y+z) = \sin(x)\sin(z)+\sin(y)\sin(x+y+z)$.

   $LHS = (\sin(x)\cos(y)+\cos(x)\sin(y))(\sin(y)\cos(z)+\cos(y)\sin(z))$
   $= \sin(x)\sin(y)\cos(y)\cos(z)+\sin(x)\cos(y)^2\sin(z)+\cos(x)\sin(y)^2\cos(z)+\cos(x)\sin(y)\cos(y)\sin(z)$.

   $RHS = \sin(x)\sin(z)+\sin(y)(\sin(x+y)\cos(z)+\cos(x+y)\sin(z)) = \sin(x)\sin(z)+\sin(y)((\sin(x)\cos(y)+\cos(x)\sin(y))\cos(z) + (\cos(x)\cos(y)-\sin(x)\sin(y))\sin(z))$
   $= \sin(x)\sin(z)+\sin(x)\sin(y)\cos(y)\cos(z)+\cos(x)\sin(y)^2\cos(z)+\cos(x)\sin(y)\cos(y)\sin(z)-\sin(x)\sin(y)^2\sin(z)$

   One can combine $\sin(x)\sin(z)-\sin(x)\sin(y)^2\sin(z)$ as $\sin(x)\cos(y)^2\sin(z)$,
   giving $RHS = \sin(x)\sin(y)\cos(y)\cos(z)+\cos(x)\sin(y)^2\cos(z)+\cos(x)\sin(y)\cos(y)\sin(z)+\sin(x)\cos(y)^2\sin(z)$.
   so $LHS = RHS$.

Area and Perimeter:

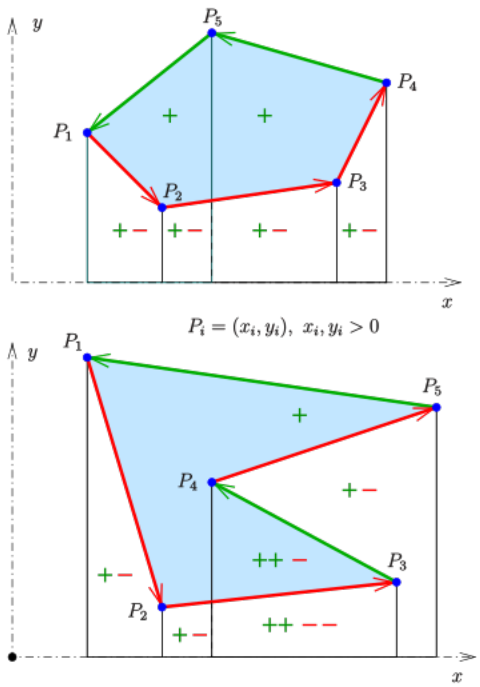
2. (a) By constructing n congruent isosceles triangles with common vertex of the centre, and the other two vertices the points on the end of the side lengths. The vertical height of the isosceles triangles, h, can be found by TOA. $\tan(\frac{\pi}{n}) = \frac{\frac{l}{2}}{h} = \frac{l}{2h}$, so $h = \frac{l}{2}\cot(\frac{\pi}{n})$ So $A = n\frac{1}{2}lh = \frac{nl^2}{4}\cot(\frac{\pi}{n})$.

   (b) $A = \frac{1}{2}ab\sin(c)$.

   From the cosine rule $\cos(c) = \frac{a^2+b^2-c^2}{2ab}$. As $\cos(c)^2 + \sin(c)^2 = 1$, and as $\sin(c) > 0$, (as $0 < c < \pi$), then $\sin(c) = \sqrt{1 - (\frac{a^2+b^2-c^2}{2ab})^2}$.

   so $A = \frac{1}{2}\frac{ab}{2ab}\sqrt{4a^2b^2 - (a^2+b^2-c^2)^2} = \frac{1}{4}\sqrt{4a^2b^2 - (a^2+b^2-c^2)^2}$
   $= \frac{1}{4}\sqrt{(2ab - (a^2+b^2-c^2))(2ab + (a^2+b^2-c^2))} = \frac{1}{4}\sqrt{(c^2 - (a-b)^2)((a+b)^2 - c^2)}$
   $= \frac{1}{4}\sqrt{(c-(a-b))(c+(a-b))(a+b-c)(a+b+c)} = \frac{1}{4}\sqrt{(c+b-a)(c+a-b)(a+b-c)(a+b+c)}$.
   $= \sqrt{\frac{(c+b-a)}{2}\frac{(c+a-b)}{2}\frac{(a+b-c)}{2}\frac{(a+b+c)}{2}} = \sqrt{s(s-a)(s-b)(s-c)}$.

   (c) Let us initially assume all the points lie in the first quadrant (we will consider if they do not later). Please see the following diagram by Ag2gaeh (Source: https://commons.wikimedia.org/wiki/File:Trapez-formel-prinz.svg) (License: https://creativecommons.org/licenses/by-sa/4.0/).



   One can see if $x_{i+1} - x_i > 0$, then one must subtract the area of the trapeziums shown, and if $x_{i+1} - x_i < 0$, then one must add this area. Area of trapezium $= \frac{1}{2}(a+b)h = |\frac{1}{2}(x_i - x_{i+1})(y_{i+1} + y_i)|$. However, due to the subtraction and addition, one can simply add $\frac{1}{2}(x_i - x_{i+1})(y_{i+1} + y_i)$, regardless of the sign of $x_i - x_{i+1}$. Therefore, summing these for all the trapeziums will give the total area, giving $\frac{1}{2}\sum_{i=1}^{n}(y_i + y_{i+1})(x_i - x_{i+1})$.

   We shall now discuss if the points do not necessarily lie in the first quadrant. There exists some translation that we can do to move them to the first quadrant, so therefore there is a translation we can do to move the points from the first quadrant back. Let that be translation T, which can be split into $x \mapsto x + a$ and $y \mapsto y + b$.

So our previous equation will become $\frac{1}{2}\sum_{i=1}^{n}(y_i + b + y_{i+1} + b)(x_i + a - x_{i+1} - a)$, which simplifies to $\frac{1}{2}\sum_{i=1}^{n}(y_i + y_{i+1} + 2b)(x_i - x_{i+1}) = \frac{1}{2}\sum_{i=1}^{n}(y_i + y_{i+1})(x_i - x_{i+1}) + b\sum_{i=1}^{n}(x_i - x_{i+1})$ The part on the right hand side of the $+$ is the same as $b \times (x_1 - x_{n+1})$. However, as P_{n+1} and P_1 are the same point, then $x_{n+1} = x_1$, so this part is 0, so the equation is still $\frac{1}{2}\sum_{i=1}^{n}(y_i + y_{i+1})(x_i - x_{i+1})$.

(d) One can expand the previous part to give $\frac{1}{2}\sum_{i=1}^{n}(y_i x_i - y_i x_{i+1} + y_{i+1} x_i - y_{i+1} x_{i+1}) = \frac{1}{2}\sum_{i=1}^{n}(x_i y_{i+1} - x_{i+1} y_i) + \frac{1}{2}\sum_{i=1}^{n}(y_i x_i - y_{i+1} x_{i+1})$.
The part on the right hand side of the $+$ is $\frac{1}{2}\sum_{i=1}^{n}(y_i x_i) - \frac{1}{2}\sum_{i=1}^{n}(y_{i+1} x_{i+1}) = \frac{1}{2}\sum_{i=1}^{n}(y_i x_i) - \frac{1}{2}\sum_{i=2}^{n+1}(y_i x_i) = \frac{1}{2}(y_1 x_1 - y_{n+1} x_{n+1})$. Like before, P_{n+1} and P_1 are the same point, so this simplifies to 0, so we are left with $\frac{1}{2}\sum_{i=1}^{n}(x_i y_{i+1} - x_{i+1} y_i)$.