Weekly Puzzle
Combinatorics

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04/03/2025 - 10/03/2025

What you need to know:

Proof by induction
Meaning of a connected planar graph

Questions:

Derangement:

1. (a) The derangement, $!n$, of a set of n elements is the number of ways to order n elements such that no element appears in its original position. For example, the derangements of ABC are BCA and CAB. Prove that $!n = (n-1)(!(n-1)+!(n-2))$ for $n \geq 2$. Note that $!0 = 1$ and $!1 = 0$.

   (b) Prove that $!n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$ for all non-negative integers, n. (Hint: Proof by induction.)

   (c) As $n \to \infty$, what is the probability of a random permutation being a derangement? (Hint: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.)

Catalan numbers:

2. (a) Consider a mountain range, which consists of n upstrokes and n downstrokes, which each increase the elevation by 1 and -1 respectively, and the elevation at any point must not be negative. It must start at elevation 0. How many mountain ranges are there, C_n? (Hint: Consider flipping the directions of the strokes if it goes below evelation 0.)

   (b) By using probabilities, evaluate $\sum_{k=0}^{\infty} \frac{C_k}{4^k}$. (Hint: Consider a random walk starting at 0 trying to get to -1. The total probability of a recurrent walk is 1.)

Combinatorial Geometry - Pick's theorem:

3. Euler's formula for a connected planar graph states $V - E + F = 2$, where V is the number of vertices, E is the number of edges and F is the number of faces, including the exterior face. Consider a simple polygon (one that does not intersect itself) and its vertices lie on integer coordinates. Using Euler's formula, prove that the area of the polygon is $A = i + \frac{b}{2} - 1$, where i is the number of integer points inside the polygon and b is the number of integer points on the boundary.