# Weekly Puzzle

Mechanics - Solutions

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## Solutions:

### Newtonian Mechanics:

1. (a) Let the two particles be $A$, and $B$, with masses $m_A$ and $m_B$ respectively, and initial velocities $u_A$ and $u_B$ respectively, with final velocities $v_A$ and $v_B$ respectively.

   However, we can change the reference frame so that $B$ is initially stationary. Therefore, we can relabel the velocities as $u_{A1} = u_A - u_B$, $u_{B1} = u_B - u_B = 0$, $v_{A1} = v_A - u_B$ and $v_{B1} = v_B - u_B$.

   From the conservation of momentum, $m_A u_{A1} + m_B u_{B1} = m_A v_{A1} + m_B v_{B1}$, so $m_A u_{A1} = m_A v_{A1} + m_B v_{B1}$.

   From the conservation of energy, $\frac{1}{2}m_A {u_{A1}}^2 + \frac{1}{2}m_B {u_{B1}}^2 = \frac{1}{2}m_A {v_{A1}}^2 + \frac{1}{2}m_B {v_{B1}}^2$, so $m_A {u_{A1}}^2 = m_A {v_{A1}}^2 + m_B {v_{B1}}^2$.

   We can additionally divide both of these by $m_A$, and $m = \frac{m_B}{m_A}$.

   By squaring the conservation of momentum formula and dividing we have ${u_{A1}}^2 = {v_{A1}}^2 + m^2 {v_{B1}}^2 + 2m v_{A1} v_{B1}$.

   So ${v_{A1}}^2 + k^2 {v_{B1}}^2 + 2k v_{A1} v_{B1} = {v_{A1}}^2 + k {v_{B1}}^2$ so $k^2 {v_{B1}}^2 + 2k v_{A1} v_{B1} = k {v_{B1}}^2$ and so $k {v_{B1}}^2 + 2 v_{A1} v_{B1} = {v_{B1}}^2$

   Therefore $(k-1){v_{B1}}^2 + 2 v_{A1} v_{B1} = 0$. So, either $v_{B1} = 0$ or $(k-1)v_{B1} + 2v_{A1} = 0$. If $v_{B1} = 0$, then $u_{A1} = v_{B1}$, but this is not possible, as $A$ must have been moving towards $B$ for a collision, as $B$ was stationary, so then it must have passed through $B$, which isn't possible. Therefore, $(k-1)v_{B1} + 2v_{A1} = 0$, so $v_{A1} = \frac{1-k}{2} v_{B1}$.

   We can substitute this into $u_{A1} = v_{A1} + k v_{B1}$ to get $u_{A1} = \frac{1-k}{2} v_{B1} + k v_{B1} = \frac{1+k}{2} v_{B1}$ so $v_{B1} = \frac{2}{1+k} u_{A1}$ and so $v_{A1} = \frac{1-k}{2} \frac{2}{1+k} u_{A1} = \frac{1-k}{1+k} u_{A1}$.

   Therefore, we can revert the refrence frame change to get $v_A - u_B = \frac{1-k}{1+k} u_A - \frac{1-k}{1+k} u_B$. $v_A = \frac{1-k}{1+k} u_A + \frac{2k}{1+k} u_B$.

   And $v_B - u_B = \frac{2}{1+k} u_A - \frac{2}{1+k} u_B$ so $v_B = \frac{2}{1+k} u_A + \frac{k-1}{1+k} u_B$.

   So, $v_A = \frac{m_A - m_B}{m_A + m_B} u_A + \frac{2m_B}{m_A + m_B} u_B$ and $v_B = \frac{2m_A}{m_A + m_B} u_A + \frac{m_B - m_A}{m_A + m_B} u_B$

   (b) Note 1: this can be done without the use of the two unit vectors in polar coordinates, though it is not as concise. It will be at the end of this document.

   Note 2: It is important to recognise that the magnitude of the derivative of $\mathbf{f}$ is not the same as the derivative of the magnitude of $\mathbf{f}$. For example, if $\mathbf{f}$ is the velocity of an object moving in circular motion with constant rate of change of angle, the derivative of the magnitude of $\mathbf{f}$ is 0, whereas the magnitude of the derivative of $\mathbf{f}$ is non-zero.

   The angle, $\theta$, is a function of time, so $\theta = \theta(t)$. $r$ is constant.

   Let $\boldsymbol{r} = (r, \theta)$. (Note: $\boldsymbol{r}$ is the position, and $r$ is the radius.)

$\boldsymbol{r} = r\hat{\boldsymbol{r}}$, where $\hat{\boldsymbol{r}}$ is the radial unit vector. The unit vectors in polar coordinates are $\hat{\boldsymbol{r}} = \cos(\theta)\boldsymbol{i} + \sin(\theta)\boldsymbol{j}$, and $\hat{\boldsymbol{\theta}} = -\sin(\theta)\boldsymbol{i} + \cos(\theta)\boldsymbol{j}$.

Using these, we have $\boldsymbol{v} = r\dot{\boldsymbol{r}} = r(-\sin(\theta)\dot{\theta}\boldsymbol{i} + \cos(\theta)\dot{\theta}\boldsymbol{j}) = r\dot{\theta}\hat{\boldsymbol{\theta}}$.

So as $\boldsymbol{a} = \dot{\boldsymbol{v}}$, then $\boldsymbol{a} = r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\dot{\hat{\boldsymbol{\theta}}}$.

$\dot{\hat{\boldsymbol{\theta}}} = -\sin(\theta)\boldsymbol{i} \dot{+} \cos(\theta)\boldsymbol{j} = -\cos(\theta)\dot{\theta}\boldsymbol{i} - \sin(\theta)\dot{\theta}\boldsymbol{j} = -\dot{\theta}\hat{\boldsymbol{r}}$.

Therefore, $\boldsymbol{a} = r\ddot{\theta}\hat{\boldsymbol{\theta}} - r\dot{\theta}^2\hat{\boldsymbol{r}}$.

Therefore, $a = |\boldsymbol{a}| = r\sqrt{\ddot{\theta}^2 + \dot{\theta}^4}$ so $|\boldsymbol{a}|^2 = r^2(\ddot{\theta}^2 + \dot{\theta}^4)$.

Also, $v = |\boldsymbol{v}| = r\dot{\theta}$ so $a^2 = \dot{v}^2 + \frac{v^4}{r^2}$ .

Note: in the case of no tangential acceleration, $r\dot{v}$, then this becomes $a = \frac{v^2}{r}$, which is a result familiar to physics students.

(c) Weight is given by $W = mg$. As $\rho = \frac{m}{V}$, where $\rho$ is the density and $V = \frac{4}{3}\pi r^3$, then $W = \frac{4}{3}\pi r^3 g\rho_s$, where $\rho_s$ is the density of the sphere. Stokes' drag force is $6\pi\eta rv$. The volume of the displaced fluid is the same as the volume of the sphere, which is $\frac{4}{3}\pi r^3$. Therefore, as density, $\rho$, is $\frac{m}{V}$, then the mass of the fluid is $\frac{4}{3}\pi r^3\rho_f$, where $\rho_f$ is the density of the fluid. Hence the upthrust is $\frac{4}{3}\pi r^3 g\rho_f$. At the maximum speed, the acceleration is 0, so the resultant force is 0, so the weight and the combined upthrust and Stokes' drag force equal each other.

Therefore, $\frac{4}{3}\pi r^3 g\rho_s = \frac{4}{3}\pi r^3 g\rho_f + 6\pi\eta rv$. Upon rearranging, we get $v = \frac{2r^2 g(\rho_s - \rho_f)}{9\eta}$.

(d) The weight is given by $W = mg$. The force along the slope is $mg\sin\theta$ and perpendicular to the slope is $mg\cos\theta$. The normal force is equal in magnitude to the weight perpendicular to the slope, so $F_f = \mu mg\cos\theta$.

Therefore, the resultant force is $mg\sin\theta - \mu mg\cos\theta$. As $F = ma$, then $a = g(\sin\theta - \mu\cos\theta)$

(e) The tangential component of the weight is given by $-mg\sin\theta$. Tangential acceleration is related to angular acceleration by $a = l\ddot{\theta}$ (this is also found in part (b)). Therefore, $-mg\sin\theta = ml\ddot{\theta}$.

So $g\sin\theta = -l\ddot{\theta}$, so $\ddot{\theta} + \frac{g}{l}\sin\theta = 0$.

Alternative method to 1b:

Let $\mathbf{r} = (r, \theta)$.
So $\mathbf{r} = r\cos(\theta)\mathbf{i} + r\sin(\theta)\mathbf{j}$.
So $\dot{\mathbf{r}} = -r\sin(\theta)\dot{\theta}\mathbf{i} + r\cos(\theta)\dot{\theta}\mathbf{j}$.
So $\ddot{\mathbf{r}} = -r(\cos(\theta)\dot{\theta}\dot{\theta} + \sin(\theta)\ddot{\theta})\mathbf{i} + r(-\sin(\theta)\dot{\theta}\dot{\theta} + \cos(\theta)th\ddot{e}ta)\mathbf{j} = -r(\cos(\theta)\dot{\theta}^2 + \sin(\theta)\ddot{\theta})\mathbf{i} + r(-\sin(\theta)\dot{\theta}^2 + \cos(\theta)\ddot{\theta})\mathbf{j}$.

We only want the magnitude of the acceleration, so
$|\ddot{\mathbf{r}}| = \sqrt{(-r(\cos(\theta)\dot{\theta}^2 + \sin(\theta)\ddot{\theta}))^2 + (r(-\sin(\theta)\dot{\theta}^2 + \cos(\theta)\ddot{\theta}))^2}$.
So $|\ddot{\mathbf{r}}| = r\sqrt{(\cos(\theta)\dot{\theta}^2 + \sin(\theta)\ddot{\theta})^2 + (-\sin(\theta)\dot{\theta}^2 + \cos(\theta)\ddot{\theta})^2}$.
So $|\ddot{\mathbf{r}}| = r\sqrt{(\cos^2(\theta)\dot{\theta}^4 + \sin^2(\theta)\ddot{\theta}^2 + 2\sin(\theta)\cos(\theta)\ddot{\theta}\dot{\theta}^2) + (\sin^2(\theta)\dot{\theta}^4 + \cos^2(\theta)\ddot{\theta}^2 - 2\sin(\theta)\cos(\theta)\ddot{\theta}\dot{\theta}^2)}$.

Luckily, this simplifies nicely to give $a = |\ddot{\mathbf{r}}| = r\sqrt{\dot{\theta}^4 + \ddot{\theta}^2}$.
However, we must express this in terms of $v = |\dot{\mathbf{r}}|$.
$v = \sqrt{(-r\sin(\theta)\dot{\theta})^2 + (r\cos(\theta)\dot{\theta})^2}$.
So $v = r\sqrt{(\sin(\theta)\dot{\theta})^2 + (\cos(\theta)\dot{\theta})^2} = r\sqrt{\dot{\theta}^2} = r\dot{\theta}$ so $a^2 = \dot{v}^2 + \frac{v^4}{r^2}$.