Weekly Puzzle
Vectors

Thomas Winrow-Campbell

28/01/2025 - 03/02/2025

What you need to know:

Trigonometric identities
Triangle area formula
Determinant of a matrix
Differentiation

Questions:

Vector Operations:

For this question, $\mathbf{a}$ and $\mathbf{b}$ will be vectors in 2 dimensions, where $\mathbf{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_x \\ b_y \end{pmatrix}$. $\mathbf{u}$, $\mathbf{w}$ and $\mathbf{v}$ will be vectors in 3 dimensions, where $\mathbf{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix}$.

1. (a) We shall define the dot product, $\cdot$, as $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between the two vectors. Prove that $\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y$.

   The dot product of $\mathbf{u}$ and $\mathbf{v}$ is $u_x v_x + u_y b_y + u_z v_z$.

   (b) We shall define the cross product, $\times$, as $\mathbf{u} \times \mathbf{v} = |\mathbf{u}||\mathbf{v}|\sin\theta\,\mathbf{n}$, where θ is the angle between the two vectors, and $\mathbf{n}$ is the unit vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.[1] Prove that the square of the magnitude of the cross product of $\mathbf{u}$ and $\mathbf{v}$ is $(u_y v_z - u_z v_y)^2 + (u_z v_x - u_x v_z)^2 + (u_x v_y - u_y v_x)^2$. (Hint: Use the trigonometric identity $\sin^2\theta + \cos^2\theta \equiv 1$, and express $\cos\theta$ by rearranging the dot product.

   (c) Let the triangle, T have vertices O, A and B. Prove that the area of T is equal to half the magnitude of the cross product of $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

   The cross product of two vectors, $\mathbf{u}$ and $\mathbf{v}$ in three dimensions can be thought of as the determinant of the matrix, $\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{j} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{pmatrix}$.

[1] The direction of the cross product of $\mathbf{u}$ and $\mathbf{v}$ can be found by using the right-hand rule, where $\mathbf{u}$ is represented by the index finger, $\mathbf{v}$ is represented by the middle finger, and the thumb represents the direction of $\mathbf{u} \times \mathbf{v}$, when positioned $90°$ from each of the two fingers.

(d) The scalar triple product of three vectors, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is given by $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$. Prove that $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \det\begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}$.

(e) The vector triple product of three vectors, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is given by $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$. Prove the triple product expansion, which states that $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$.

An Introduction to Vector Calculus:

Partial differentiation is similar to differentiation, except it is used for multivariable functions. When one differentiates with respect to one of the variables, it treats the other variables as constants. For example, let the function f be defined as $f(x,y) = \sin(x^2+y^2)$. Then, $\frac{\partial f}{\partial x} = 2x\cos(x^2+y^2)$.

2. (a) The gradient of a scalar function, f, with 3 variable inputs, x, y and z is given by $\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$. For example, if $f(x,y,z) = 4x + 7y^3 + \cos(z)$, then $\nabla f = 4\mathbf{i} + 21y^2\mathbf{j} - \sin(z)\mathbf{k}$. What is ∇g where $g(x,y,z) = \cos^2(xyz)$?

   (b) The divergence of a vector field, $\mathbf{F}$, where $\mathbf{F} = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ is given by $\nabla\cdot\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$. What is the divergence of the vector field, $\mathbf{G}$, where $\mathbf{G} = x\mathbf{i} + xyz\mathbf{j} + y\sin(x^2+z)\mathbf{k}$?

   You may have noticed that this ∇ symbol is very unusual. It is called the nabla, and it is not a vector, nor is it a scalar. It is actually a vector differential operator. It can be thought of as $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$. You can use this to confirm the definitions above.

   (c) The curl of a vector field, $\mathbf{F}$ is given by $\nabla\times\mathbf{F}$. By using the definition of the cross product, $\times$, what is the curl of $\mathbf{F}$?