Statistics for Computing

MA4413 Lecture 8B

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Two Sample Inference Procedures

- Previously we looked at inference procedures (Confidence Intervals and Hypothesis Testing) for single samples.
- Yesterday we looked at ***paired*** samples, with two sets of paired measurements. With paired measurements, we are specifically interested in the ***case-wise*** differences.
- Although there are two sets of data, we consider the single data set of case-wise differences.
- Now we look at the case of two independent sample procedures.
- Independent samples are distinct from paired samples, in that data in one set are not paired with data in another set.

Two Sample Inference Procedures

- Firstly, we will look at the difference in the means of two independent populations.
- Let us assume that the both populations are normally distributed $N(\mu_X, \sigma_X^2)$ and $N(\mu_Y, \sigma_Y^2)$
- The difference in means for two independent populations X and Y is denoted $\mu_X - \mu_Y$.
- Almost always, this value is unknown, and is instead estimated by the difference in sample means: $\bar{X} - \bar{Y}$.
- The sample sizes do not need to be equal necessarily. We denote the respective sample sizes n_X and n_Y.
- For the moment, we will assume that both n_X and n_Y are large samples (≥ 30).

Sampling

- The sampling distribution of the difference in means is normally distributed, when both samples sizes are greater than 30.
- The expected value of this distribution is $\mu_X - \mu_Y$.
- Importantly, the standard error of this distribution is

$$S.E(\bar{X} - \bar{Y}) = \sqrt{\frac{\sigma_X^2}{n_X} + \frac{\sigma_Y^2}{n_Y}}$$

- The standard deviations for populations X and Y are σ_X and σ_Y respectively.
- Usually these population standard deviations are estimated by the sample standard deviations s_X and s_Y respectively.

95% Confidence Intervals

The 95% confidence interval $\mu_X - \mu_Y$ is computed as

$$(\bar{X} - \bar{Y}) \pm 1.96\sqrt{\frac{s_X^2}{n_X} + \frac{s_Y^2}{n_Y}}$$

We will use this in an example shortly.

Hypothesis Testing

- Hypothesis testing works in much the same way as material we have covered already, in that we will use a four step process.
- The final two steps (critical value step and decision rule step) are precisely the same as previously.
- We will now discuss the first two steps.

Hypothesis Testing: Null and Alternative Hypothesis

We are often interested in whether or not two populations have equal mean values. Accordingly, we would construct the hypotheses accordingly.

H_0 $\mu_X = \mu_Y$

H_1 $\mu_X \neq \mu_Y$

Equivalently we may view in the context of the difference in the populations means, where a difference of zero indicates equality of means.

H_0 $\mu_X - \mu_Y = 0$

H_1 $\mu_X - \mu_Y \neq 0$

This second approach is more intuitive in the context of constructing the test statistic.

Hypothesis Testing: Test Statistic

- The standard error for difference in means has been introduced previously

$$S.E(\bar{X} - \bar{Y}) = \sqrt{\frac{\sigma_X^2}{n_X} + \frac{\sigma_Y^2}{n_Y}}$$

- **Null value**: The expected value of the difference under the null hypothesis $\mu_X - \mu_Y$ is always 0, when the equality of population means is in question.
- **Observed Value**: The observed difference between sample means is $\bar{X} - \bar{Y}$.

Example 1: Difference in Means (a)

Two sets of patients are given courses of treatment under two different drugs. The benefits derived from each drug can be stated numerically in terms of the recovery times. There are 40 patients in treatment group 1 (i.e. Drug 1), and 45 patients in treatment group 2 (i.e. Drug 2). The mean recovery time and standard deviations are as follows:

- Drug 1: $n_1 = 40$, $\bar{x}_1 = 3.3$ days and $s_1 = 1.524$
- Drug 2: $n_2 = 45$, $\bar{x}_2 = 4.3$ days and $s_2 = 1.951$

Example 1: Difference in Means (b)

- The first step in hypothesis testing is to specify the null hypothesis and an alternative hypothesis.
- When testing differences between mean recovery times, the null hypothesis is that the two population means are equal.
- That is, the null hypothesis is:
  $H_0 : \mu_1 = \mu_2$ (The population means are equal)
  $H_1 : \mu_1 \neq \mu_2$ (The population means are different)

(Remark: Two Tailed Test, therefore $k = 2$, and $\alpha = 0.05$)

Example 1: Difference in Means (c)

- The observed difference in means is 1 day.
- The relevant formula for the standard error is

$$S.E(\bar{x}_1 - \bar{x}_2) = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

- $$S.E(\bar{x}_1 - \bar{x}_2) = \sqrt{\frac{(1.524)^2}{40} + \frac{(1.951)^2}{45}}$$

- $$S.E(\bar{x}_1 - \bar{x}_2) = 0.377 \text{ days}$$

Example 1: Difference in Means (d)

- The Test statistic is therefore

$$TS = \frac{\text{observed} - \text{null}}{\text{Std. Error}} = \frac{1-0}{0.377} = 2.65$$

- Two Tailed Test, therefore $k=2$, and $\alpha = 0.05$. Also two large samples. The CV is 1.96.

Example 1: Difference in Means (e)

Decision Rule

$$|TS| > CV?$$

- If Yes: Reject the null Hypothesis
- If No : Fail to reject the Null Hypothesis

$|2.65|$ is greater than 1.96. We reject the null hypothesis. There is a significant difference in the effectiveness of the two drug treatments.

Two Small Samples Case

- Previously we have looked at large samples, now we will consider small samples.
- (For the sake of clarity, I will not use small samples that have a combined sample size of greater than 30.
- Additionally we require the assumption that both samples have equal variance. This assumption **must** be tested with another formal hypothesis test. We will revisit this later, and in the mean time, assume that the assumption of equal variance holds.

Two Small Samples Case

- The key differences between the large sample case and the small sample cases arise in the following steps.
  - The standard error is computed in a different way (see next slide).
  - The degrees of freedom used to compute the critical value is $(n_X - 1) + (n_Y - 1))$ or equivalently $(n_X + n_Y - 2)$.
  - Also - a formal test of equality of variances is required beforehand (End of Year Exam)

Two Small Samples Case: Standard Error

Computing the standard error requires a two step calculation. From the formulae, we have the two equations below. The first term s_p^2 is called the ***pooled variance*** of the combined samples.

$$
\begin{aligned}
s_p^2 &= \frac{s_X^2(n_X-1)+s_Y^2(n_Y-1)}{n_X+n_Y-2}. \\
S.E.(\bar{X}-\bar{Y}) &= \sqrt{s_p^2\left(\frac{1}{n_X}+\frac{1}{n_Y}\right)}.
\end{aligned}
$$

Example 2: Difference in Means (a)

- For a random sample of 10 light bulbs for a particular brand, the mean bulb life is 4,000 hr with a sample standard deviation of 200 hours.
- For another brand of bulbs, a random sample of 8 has a sample mean lifetime of 4,300 hours and a sample standard deviation of 250 hours.
- Test the hypothesis that there is no difference between the mean operating life of the two brands of bulbs, using the 5 percent level of significance

Example 2: Difference in Means (b)

- $n_1 = 10$ and $n_2 = 8$.
- $\bar{x}_1 = 4000$, $\bar{x}_2 = 4,300$, therefore $\bar{x}_1 - \bar{x}_2 = -300$ hours
- $s_1 = 200$, $s_2 = 250$ hours.
- Small sample - Degrees of freedom $n_1 + n_2 - 2 = 10 + 8 - 2 = 16$

Example 2: Difference in Means (c)

Pooled variance estimate

$$s_p^2 = \frac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2} = \frac{(9\times 200^2)+(7\times 250^2)}{16}$$

$$s_p^2 = 49843.75$$

Example 2: Difference in Means (d)

Computing the Standard Error

$$S.E(x_1 - x_2) = \sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

$$S.E(x_1 - x_2) = \sqrt{49843.75\left(\frac{1}{10} + \frac{1}{8}\right)}$$

$$S.E(x_1 - x_2) = \sqrt{11214.84} = 105.9$$

Example 2: Difference in Means (e)

Test Statistic and Critical Value

- The Test Statistic is

$$TS = \frac{(-300) - 0}{105.9} = -2.83$$

- The Critical Value is determined with $\alpha = 0.05$, $k = 2$, $df = 16$
- $CV = 2.120$
- We can now apply the decision rule : Is the absolute value of the Test Statistic greater than the Critical Value?
- Is $2.83 > 2.12$? Yes We reject H_0. There is evidence of a difference in means.