Chapter 1

14. The Paired $t-$Test

1.1 Two Sample Inference Procedures

- Previously we looked at inference procedures (Confidence Intervals and Hypothesis Testing) for single samples.
- Yesterday we looked at ***paired*** samples, with two sets of paired measurements. With paired measurements, we are specifically interested in the ***case-wise*** differences.
- Although there are two sets of data, we consider the single data set of case-wise differences.
- Now we look at the case of two independent sample procedures.
- Independent samples are distinct from paired samples, in that data in one set are not paired with data in another set.

1.1.1 Mean Difference Between Matched Data Pairs

The approach described in this lesson is valid whenever the following conditions are met:

- The data set is a simple random sample of observations from the population of interest.
- Each element of the population includes measurements on two paired variables (e.g., x and y) such that the paired difference between x and y is: d = x - y.
- The sampling distribution of the mean difference between data pairs (d) is approximately normally distributed.

The observed data are from the same subject or from a matched subject and are drawn from a population with a normal distribution does not assume that the variance of both populations are equal.

1.2 Paired T test

A paired t-test is used to compare two population means where you have two samples in which observations in one sample can be paired with observations in the other sample. Examples of where this might occur are:

(i) Before-and-after observations on the same subjects (e.g. students diagnostic test results before and after a particular module or course).

(ii) A comparison of two different methods of measurement or two different treatments where the measurements/treatments are applied to the **same** subjects (e.g. simultaneous blood pressure measurements using a stethoscope and a dynamap on each patient in a study).

The difference between two paired measurements is known as a ***case-wise*** difference.

1.3 Paired T test

The mean and standard deviation of the sample d values are obtained by use of the basic formulas in Chapters 3 and 4, except that d is substituted for X.

The mean difference for a set of differences between paired observations is $\bar{d} = \frac{\sum d_i}{n}$.

The deviations formula and the computational formula for the standard deviation of the differences between paired observations are, respectively,

$$S_d = \sqrt{\frac{\sum (d_i - \bar{d})^2}{n-1}} = \sqrt{\frac{\sum (d^2) - n(\bar{d}^2)}{n-1}} \tag{1.1}$$

(1.2)

The standard error of the mean difference between paired observations is obtained for the standard error of the mean.

Hypotheses

$$H_0 : \mu_d = 0$$
$$H_1 : \mu_d \neq 0$$

1.3.1 How a paired t test works

- The paired t test compares two paired groups.
- It calculates the difference between each set of pairs, and analyzes that list of differences based on the assumption that the differences in the entire population follow a Gaussian distribution.

- First we calculate the difference between each set of pairs, keeping track of sign.
- If the value in column B is larger, then the difference is positive. If the value in column A is larger, then the difference is negative.
- The t ratio for a paired t test is the mean of these differences divided by the standard error of the differences. If the t ratio is large (or is a large negative number), the P value will be small. The number of degrees of freedom equals the number of pairs minus 1.

1.4 Important

- Firstly we have to compute each of the case-wise differences.
- Then we have to compute the mean value of these differences.
- Lastly we also have to compute the standard deviation of the differences.

1.5 Procedure

To test the null hypothesis that the true mean difference is zero, the procedure is as follows:

1. Calculate the difference ($di = y_i x_i$) between the two observations on each pair, making sure you distinguish between positive and negative differences.
2. Calculate the mean difference, $\bar{d}$.
3. Calculate the standard deviation of the differences, s_d, and use this to calculate the standard error of the mean difference, $SE(\bar{d}) = \frac{s_d}{\sqrt{n}}$
4. Calculate the t-statistic, which is given by $T = \frac{\bar{d}}{SE(\bar{d})}$.
5. Under the null hypothesis, this statistic follows a t-distribution with $n1$ degrees of freedom.

1.6 The Paired t-test

Using the sample to make inferences about the general population of case-wise differences.

- Often we are making conclusions for the population of differences. (Is a training regime effective? - based on a paired data sample.)
- Let μ_d be mean value for the population of case-wise differences.
- The null hypothesis is that that $\mu_d = 0$ (i.e. no difference)
- Given $\bar{d}$ mean value for the sample of differences, and s_d standard deviation of the differences for the paired sample data, we can perform inference procedures as we have done previously.

1.7 Calculations for Case-wise Differences

- We are usually required to compute the case-wise difference for each data pairing.
- We will often be required to compute the case-wise differences, the average of those differences and the standard deviation of those difference.
- The mean difference for a set of differences between paired observations is

$$\frac{\bar{d} = \sum d_i}{n}$$

- The computational formula for the standard deviation of the differences between paired observations is

$$s_d = \sqrt{\frac{\sum d_i^2 - n\bar{d}^2}{n-1}}$$

- It is nearly always a small sample test.
- Importantly, although we start out with two samples of data, we can look at the data as a single sample of ***case-wise differences***.

$$d_i = x_i - y_i$$

- We can use the same methodologies that we have encountered previously for making decisions based on paired data.
- (Remark: For most paired data studies, the sample sizes are very small.)

Remark: Make sure to keep your case-wise diffference calculated consistently, i.e. always "***After-Before***".
Before-After would also fine, as long as each calculation is done the same way.

$$(\bar{X} - \bar{Y}) \pm \left[\text{Quantile } \times S.E(\bar{X} - \bar{Y})\right]$$

- If the combined sample size of X and Y is greater than 30, even if the individual sample sizes are less than 30, then we consider it to be a large sample.

- The quantile is calculated according to the procedure we met in the previous class.

1.8 The Paired $t-$Test

The standard error of the mean difference between paired observations is obtained for the standard error of the mean.

1.9 Hypotheses

$$H_0 : \mu_d = 0$$
$$H_1 : \mu_d \neq 0$$

1.10 Paired T test

- A paired sample t-test is used to determine whether there is a significant difference between the average values of the same measurement made under two different conditions.

- Both measurements are made on each unit in a sample, and the test is based on the paired differences between these two values.

- The usual null hypothesis is that the difference in the mean values is zero. For example, the yield of two strains of barley is measured in successive years in twenty different plots of agricultural land (the units) to investigate whether one crop gives a significantly greater yield than the other, on average.

Confidence Interval:

- Recall our sample mean $\bar{x}$, the standard error $S.E(\bar{x})$ and the quantile from the $t-$ distribution.

- We can use these values to compute a 95% confidence interval.

- The 95% confidence interval can be computed as $0.17 \pm (2.262 \times 0.17 = (-0.21, 0.55)$

- Notice that 0 is within that range of values. This supports our conclusion to reject the Null Hypothesis.

Paired t test The paired t test was devloped by Guinness Brewery employee, Gossett, in1908.

- First we have to compute the case-wise differences.
- Then we compute the variance of those differences.

$$(\bar{X} - \bar{Y}) \pm \left[\text{Quantile } \times S.E(\bar{X} - \bar{Y})\right]$$

- If the combined sample size of X and Y is greater than 30, even if the individual sample sizes are less than 30, then we consider it to be a large sample.
- The quantile is calculated according to the procedure we met in the previous class.
- Assume that the mean (μ) and the variance (σ) of the distribution of people taking the drug are 50 and 25 respectively and that the mean (μ) and the variance (σ) of the distribution of people not taking the drug are 40 and 24 respectively.

1.11 Paired t test

1.11.1 Inference : Paired values

- Know how to compute case-wise differences.
- Know how to compute the mean of the case-wise differences (see formulae).
- Know how to compute the standard deviation of the casewise differences (see formulae).
- Consider two populations X and Y that are indepedently distributed from each other.
- That is to say, the true value of correlation is zero.

$$\rho_{XY} = 0$$

- In the context of a linear regression model, in the form $Y = \beta_0 + \beta_1 X$, a true correlation value of zero is equivalent to a true slope value of Zero.

$$\text{“}\rho_{XY} = 0\text{”} \longleftrightarrow \text{“}\beta_1 = 0\text{”}$$

1.11.2 Paired t-test

$$(\bar{X} - \bar{Y}) \pm \left[\text{Quantile } \times S.E(\bar{X} - \bar{Y})\right]$$

- If the combined sample size of X and Y is greater than 30, even if the individual sample sizes are less than 30, then we consider it to be a large sample.
- The quantile is calculated according to the procedure we met in the previous class.

1.12 Confidence interval for the true mean difference

The in above example the estimated average improvement is just over 2 points. Note that although this is statistically significant, it is actually quite a small increase. It would be useful to calculate a confidence interval for the mean difference to tell us within what limits the true difference is likely to lie.

Using our example: We have a mean difference of 2.05. The 2.5% point of the t-distribution with 19 degrees of freedom is 2.093. The 95% confidence interval for the true mean difference is therefore: $2.05 \pm (2.093 \times 0.634) = 2.05 \pm 1.33 = (0.72, 3.38)$.

This confirms that, although the difference in scores is statistically significant, it is actually relatively small. We can be 95% sure that the true mean increase lies somewhere between just under one point and just over 3 points.

1.13 Difference in Two means

For this calculation, we will assume that the variances in each of the two populations are equal. This assumption is called the assumption of homogeneity of variance.

The first step is to compute the estimate of the standard error of the difference between means ().

$$S.E.(\bar{X} - \bar{Y}) = \sqrt{\frac{s_x^2}{n_x} + \frac{s_y^2}{n_y}}$$

- s_x^2 and s_x^2 is the variance of both samples.
- n_x and n_y is the sample size of both samples.

The degrees of freedom is $n_x + n_y - 2$.

1.14 Worked Examples

Two procedures for sintering copper are compared by testing each procedure on six different types of powder. The measurement of interest is the porosity of each test specimen. The results of the test are as follows:

$Powder$	$Procedure1$	$Procedure2$
1	21	23
2	27	26
3	18	21
4	22	24
5	26	25
6	19	16

ls there a difference between the true average porosity measurements for the two procedures, at a significance level of 5%.

1.15 Worked Examples : Paired T test

- The weight of 6 individuals (in kgs) was observed before and after a diet regime (diet given below)
- Compute the mean difference and standard deviation of the differences.
- Remark : Sample size n= 6

$Person$	$WeightBefore$	$WeightAfter$	$Difference$
A	89	87	2
B	79	76	3
C	106	105	1
D	92	92	0
E	88	84	4
F	98	96	2

Before we start, we need to compute the average difference and the standard deviations of the differences.

What is the mean difference d

$$\bar{d} = \sum \frac{d_i}{n} = \frac{2+3+1+0+4+2}{6} = 2$$

Standard Deviation: Sx=(xi-x)2n-1

$Person$	di	$di-d$	$(di-d)2$
A	2	0	0
B	3	1	1
C	1	-1	1
D	0	-2	4
E	4	2	4
F	2	0	0

Sx=(di-d)26-1=(0 + 1 +1 +4 +4 +0)6-1
Sd=105=2
Now we are ready to perform our hypothesis test.

1.16 Step1 : Formally state the null and alternative hypotheses

- d : true difference in weight before and after the diet regime
- Null Hypothesis Ho:d = 0 True difference is zero
- Alternative Hypothesis Ha:d 0 True difference is not zero
- Remark: This is a two tailed test

1.17 Step 2 : Compute the test statistic.

Remember the general structure of a test statistic
TS =Observed Value-Null ValueStd. Error
From the formulae
We have to compute the standard error for a sample mean.
(From formulae at back of exam paper)
S.E.(d) =Sdn=26= 0.5773

1.18 Step 3 : Determine the Critical Value

- Small sample (group is less than 30). (Population variance is unknown.)
- Use t distribution with n-1 degrees of Freedom.
- We use Murdoch Barnes Table 7 (Student T distribution)
- Significance levels is 1%. This is a two tailed test.

[Blackboard]
Critical Value is 2.571
Step 4 : Decision

[Blackboard]

Is the test statistic in the acceptance region or the rejection region?

It is in the rejection region. We reject the null hypothesis. The diet does work.

Worked Example paired T test