0.1 Hypothesis Test for the Mean of a Single Sample

This procedure is used to assess whether the population mean has a specified value, based on the sample mean. The hypotheses are conventionally written in a form similar to below (here the hypothesized population mean is zero).

There are two hypothesis test for the mean of a single sample.

1. The sample is of a normally-distributed variable for which the population standard deviation (σ) is known.
2. The sample is of a normally-distributed variable where is estimated by the sample standard deviation (s).

In practice, the population standard deviation is rarely known. For this reason, we will consider the second case only in this course.

In most statistical packages, this analysis is performed in the summary statistics functions.

0.2 Independent one-sample t-test

In testing the null hypothesis that the population mean is equal to a specified value μ_0, one uses the statistic

$$t = \frac{\overline{x} - \mu_0}{s/\sqrt{n}} \tag{1}$$

where s is the sample standard deviation and n is the sample size. The degrees of freedom used in this test is $n - 1$.

Hypothesis Testing

The four key elements of this are

1. Formally stating the Null and Alternative hypotheses
2. Determine the test statistic value
   This is based on the data given in the question , and the formulae at the back of the exam paper.
3. Determining the critical value for the test This depends on the sample size 'n' (and hence the degree of freedom), the number of tails 'k' , and the required significance level .
4. Employing the Decision Rule.
   This compares the absolute value of the test statistic and the critical value.

The procedure is more practicable for written exams, so we will use this one more. The first two steps are the same.

- This procedure is similar to large sample hypothesis tests for the mean.
- The crucial differnce is that we use as our critcal value a $t-$ quantile from the Student $t-$ distribution.
- This value depends on sample size n (and hence degrees of freedom $n-1$), the significance level α, and whether the procedure is a onetailed test or a two tailed test)

Part 1: One Sample t-test

Question here

The null and alternative hypotheses

$$H_0 : \mu = 40kg$$
$$H_1 : \mu \neq 40kg$$

0.2.1 2 sided test

A two-sided test is used when we are concerned about a possible deviation in either direction from the hypothesized value of the mean. The formula used to establish the critical values of the sample mean is similar to the formula for determining confidence limits for estimating the population mean, except that the hypothesized value of the population mean m0 is the reference point rather than the sample mean.

Hypothesis Tests for single samples

- We could have inference procedures for single sample studies. We would base an argument on the either the sample mean or sample proportion as appropriate.
- A hypothesis test can be used to determine how "confident" we can be with our data in making that statements.
- The lower the significance level (The margin for Type I error) the stronger our data must be.
- Large samples lead to more confident conclusion.
- We could have either hypothesis test for the sample mean or the sample proportion, to test a statement about the population as a whole (i.e something about the population mean)
- We make our argument in the form of the null and alternative hypotheses.
- The Hypothesis testing procedure determines the strength of evidence in making our arguments.
- We simply follow the four step procedure.
- All of the components are the same used in confidence intervals.
- The critical value is simply a quantile from the Z or $t-$distribution.

- The standard errors are also as before. Although when performing a hypothesis test for proportions, we use the expected value under the null hypothesis, rather than point estimate. (reason beyond scope of course.)

Example 1 (a) Small Sample Hypothesis Test

- The manufacturer claims that average tube life for a particular brand of ultraviolet tube is 9,000 hr.
- Test this claim at the 5 percent level of significance against the alternative hypothesis that the mean life is not 9,000 hr
- We are given the following information: a sample of $n = 10$ tubes the mean operating life was $\bar{x} = 8,800$ hr. The sample standard deviation is be $s = 500$ hr.

- $H_0 : \ \mu = 9000$ (Average life span is 9000 hours.)
- $H_1 : \ \mu \neq 9000$ (Average life span is not 9000 hours.)

- The observed difference is -200 hours. (i.e. 8,800 - 9,000 hours)
- The standard error is determined from formulae.

$$S.E.(\bar{x}) = \frac{s}{\sqrt{n}} = \frac{500}{\sqrt{10}} = 158.1139$$

Test Statistic and Critical Value

$$TS = \frac{8800 - 9000}{158.11}$$

- The test statistic $TS = -1.265$
- The CV is determined with $\alpha = 0.05$ and $k = 2$ (column $= \alpha/k = 0.025$).
- The sample is small n = 10 $df = n - 1 = 9$ (i.e. row =9).
- Therefore $CV = 2.262$
- (Remark: If the sample was large, we could use $CV = 1.96$).

Example 1 (d): Decision Rule

- **Decision:**Is $|TS| > CV$? Is $1.265 > 2.262$?
- No. We fail to reject the null hypothesis.
- There is not enough evidence to say that the mean lifespan is not 9000 hours.

Confidence Interval for a mean (1) Finally, an example to finish the class:

- For a given week, a random sample of 100 hourly employees selected from a very large number of employees in a manufacturing firm has a sample mean wage of $\bar{x} = 280$ dollars, with a sample standard deviation of $s = 40$ dollars.
- Estimate the mean wage for all hourly employees in the firm with an interval estimate such that we can be 95 percent confident that the interval includes the value of the population mean.
- The point estimate in this case is the sample mean $\bar{x} = 280$ dollars.
- We have a large sample (n=100) and the confidence level is 95%. Therefore the quantile is 1.96.
- The standard error is computed as follows:

$$S.E(\bar{x}) = \frac{s}{\sqrt{n}} = \frac{40}{\sqrt{100}} = 4$$

- **Confidence Interval for mean**

$$280 \pm (1.96 \times 4) = (280 \pm 7.84) = (\ 272.16\ ,\ 287.84\ )$$

Two Small Samples Case

- Previously we have looked at large samples, now we will consider small samples.
- (For the sake of clarity, I will not use small samples that have a combined sample size of greater than 30.
- Additionally we require the assumption that both samples have equal variance. This assumption **must** be tested with another formal hypothesis test. We will revisit this later, and in the mean time, assume that the assumption of equal variance holds.

Two Small Samples Case

- The key differences between the large sample case and the small sample cases arise in the following steps.
  - The standard error is computed in a different way (see next slide).
  - The degrees of freedom used to compute the critical value is $(n_X - 1) + (n_Y - 1))$ or equivalently $(n_X + n_Y - 2)$.
  - Also - a formal test of equality of variances is required beforehand (End of Year Exam)

Two Small Samples Case: Standard Error

Computing the standard error requires a two step calculation. From the formulae, we have the two equations below. The first term s_p^2 is called the ***pooled variance*** of the combined samples.

$$\begin{aligned} s_p^2 &= \frac{s_X^2(n_X-1) + s_Y^2(n_Y-1)}{n_X+n_Y-2}. \\ S.E.(\bar{X}-\bar{Y}) &= \sqrt{s_p^2\left(\frac{1}{n_X}+\frac{1}{n_Y}\right)}. \end{aligned}$$

Example 2: Difference in Means (a)

- For a random sample of 10 light bulbs for a particular brand, the mean bulb life is 4,000 hr with a sample standard deviation of 200 hours.
- For another brand of bulbs, a random sample of 8 has a sample mean lifetime of 4,300 hours and a sample standard deviation of 250 hours.
- Test the hypothesis that there is no difference between the mean operating life of the two brands of bulbs, using the 5 percent level of significance

Example 2: Difference in Means (b)

- $n_1 = 10$ and $n_2 = 8$.
- $\bar{x}_1 = 4000$, $\bar{x}_2 = 4,300$, therefore $\bar{x}_1 - \bar{x}_2 = -300$ hours
- $s_1 = 200$, $s_2 = 250$ hours.
- Small sample - Degrees of freedom $n_1 + n_2 - 2 = 10 + 8 - 2 = 16$

Example 2: Difference in Means (c)

Pooled variance estimate

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2} = \frac{(9 \times 200^2) + (7 \times 250^2)}{16}$$

$$s_p^2 = 49843.75$$

Example 2: Difference in Means (d)

Computing the Standard Error

$$S.E(x_1 - x_2) = \sqrt{s_p^2 \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

$$S.E(x_1 - x_2) = \sqrt{49843.75 \left(\frac{1}{10} + \frac{1}{8}\right)}$$

$$S.E(x_1 - x_2) = \sqrt{11214.84} = 105.9$$

Example 2: Difference in Means (e)

Test Statistic and Critical Value

- The Test Statistic is
$$TS = \frac{(-300) - 0}{105.9} = -2.83$$
- The Critical Value is determined with $\alpha = 0.05$, $k = 2$, $df = 16$
- $CV = 2.120$
- We can now apply the decision rule : Is the absolute value of the Test Statistic greater than the Critical Value?
- Is $2.83 > 2.12$? Yes We reject H_0. There is evidence of a difference in means.