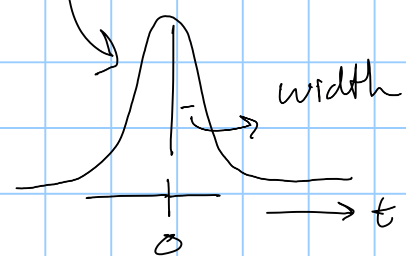
Gaussian Conv.

$e^{-x \cdot \frac{\alpha}{2} t^2}$



$$g(t) = g_0 + g_1 (t-0) + \frac{g_2}{2!}(t-0)^2 + \frac{g_3}{3!}(t-0)^3 + \cdots$$

$\int_{-1}^{+1} e^{-x\frac{\alpha}{2}t^2} \cdot g(t)\,dt$ only depends on even coefficients g_0, g_2, g_4, etc...

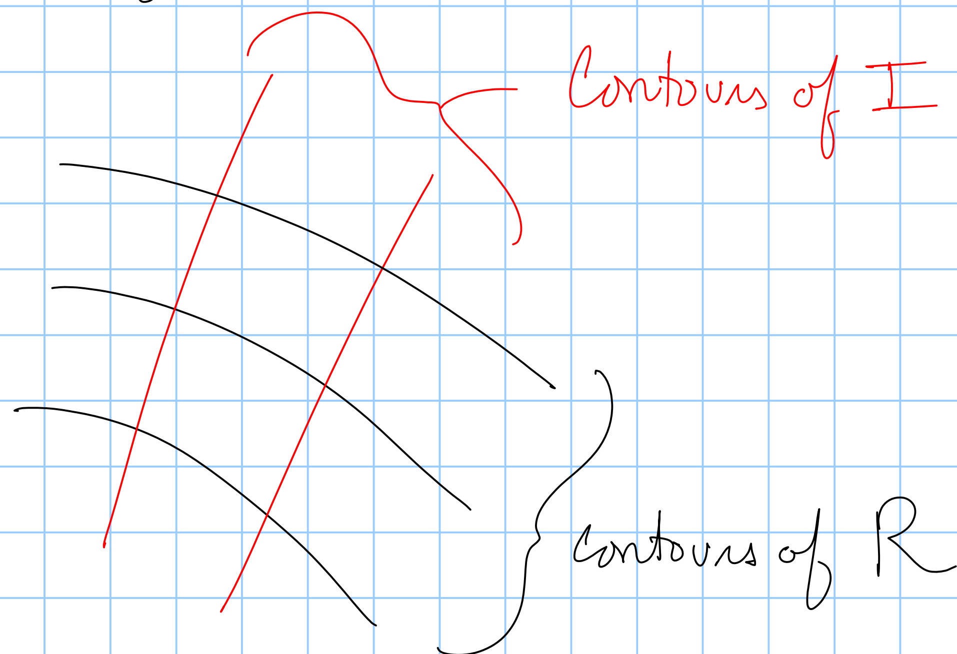
X on the concavity α of the exponent $\frac{\alpha}{2}t^2$.

doesn't depend on endpoints ± 1, as long as they are $\sim \frac{1}{\sqrt{x}}$ away from $t = 0$.

Laplace's Method: Now if we consider $\int_{-1}^{+1} e^{\alpha h(t)} \cdot g(t)\, dt$, we need only look at the behavior of $g(t)$ near the maximum $t_c = \operatorname{argmax}_{t \in [-1,+1]} h(t)$. If $h'(t_c) = 0$ (e.g. h smooth & t_c in the interior), then we only need to pay attention to the concavity $h''(t_c)$ and the even derivatives of g @ $t = t_c$.

If, on the other hand, $h'(t_c) \neq 0$ (e.g. h has a cusp or t_c occurs on the boundary of the interval), then we need to pay attention to the odd-derivatives of g @ t_c.

Cauchy-Reimann Conditions: $h(z) = R(z) + i\,I(z)$



Cauchy-Reimann conditions imply the R-contours are $\perp$ to the I-contours.

Laplacian: $\Delta u = \nabla \cdot \nabla u$.

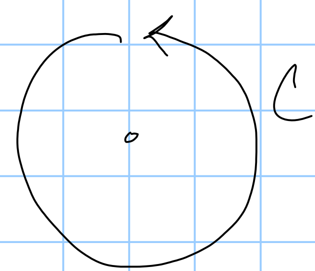
Given a ball B_R of radius R, we can average u over B_R:

$$\bar{u}(R) = \frac{1}{|\partial B_R|} \int_{B_R} u(\vec{x})\, dS = \text{average of } u \text{ over surface } B_R.$$

The derivative $\frac{d}{dR} \bar{u}(R)$ is proportional to $\int_{B_R} \Delta u(\vec{x})\, d\vec{x}$, which is the volume integral of Δu in the interior of B_R.

$\Rightarrow$ mean-value theorem & maximum principle for harmonic functions (i.e., when $\Delta u = 0$).

Residue theorem:



$$\oint_C \frac{f(z)}{z^{k+1}}\,dz = 2\pi i\,\frac{f^{(k)}(0)}{k!}$$

Method of Steepest descent: Consider $F(M) = \int_C \exp^{M h(z)} \cdot g(z)\,dz$; where C is a contour in the complex-plane. (& $M \in \mathbb{R}^+$ is large).

if the phase $\mathrm{Im}(h(z))$ is constant along C, then we can just pull out the imaginary part of the exponent (ie, $\exp^{M\,\mathrm{Im}(h)\cdot i}$) and use Laplace's method on the remaining (Real) exponent.

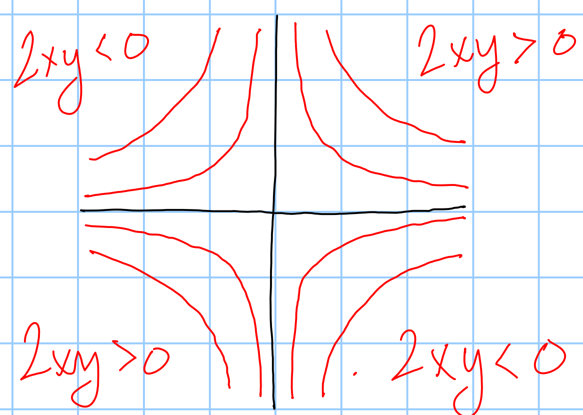
If $\mathrm{Im}(h(z))$ is not constant along C, then we cannot immediately use Laplace's method, since the exponent will oscillate wildly as $M \to \infty$ in the integrand.

If $\text{Im}(h) \neq$ constant along C, then we may be able to deform C so that $\text{Im}(h)$ <u>is</u> constant along the new C'.

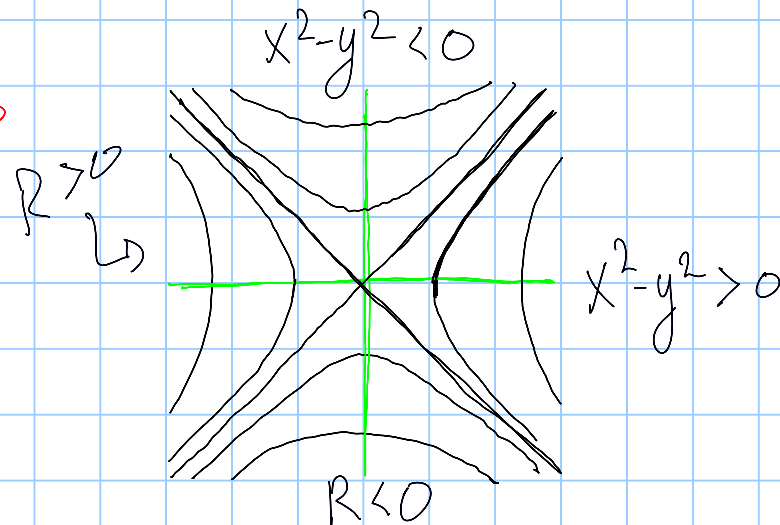
because of Cauchy-Reimann conditions, we know that constant-I-curves (ie, contours of $\text{Im}(h)$) are in fact paths of steepest-ascent (or steepest-descent) for $R = \text{Re}(h(z))$.

Ex: $h(z) = z^2$, $R(x,y) = x^2 - y^2$. $I(x,y) = 2xy$.

Given an arbitrary contour C, we can deform $C \to C'$ so that $I(x,y)$ along C' is constant (ie, so $2xy = I_0$).



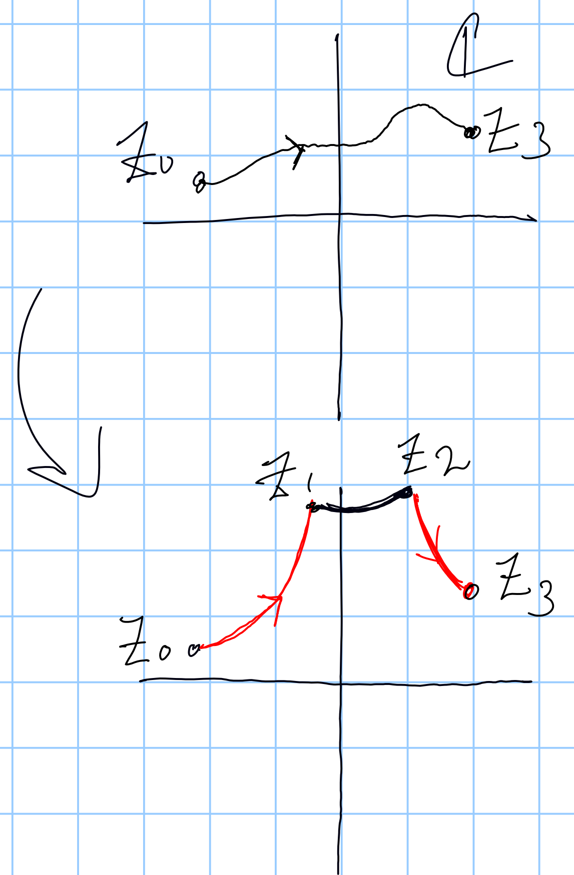
I-contours



R-contours

Note: contours intersect $\perp$

Given a C:



we can deform to

$C_{01} + C_{12} + C_{13}$

I constant,
R greatest @ z_3

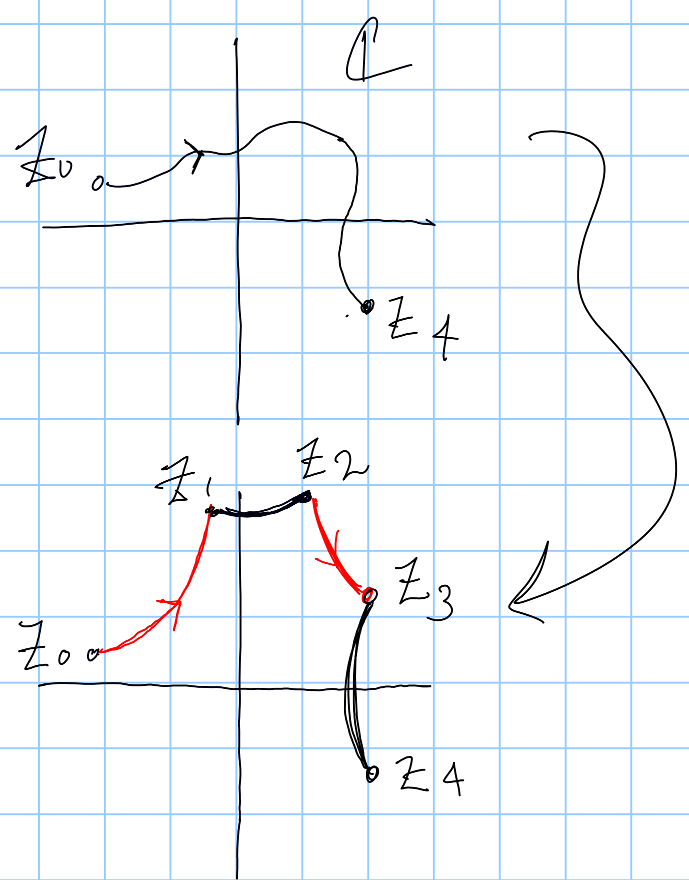
R constant, but very negative

Ignore!

I constant,
R greatest @ z_0

Compare R @ z_0 with R @ z_3.
Typically one will dominate.

But what if we can't do this? Consider:

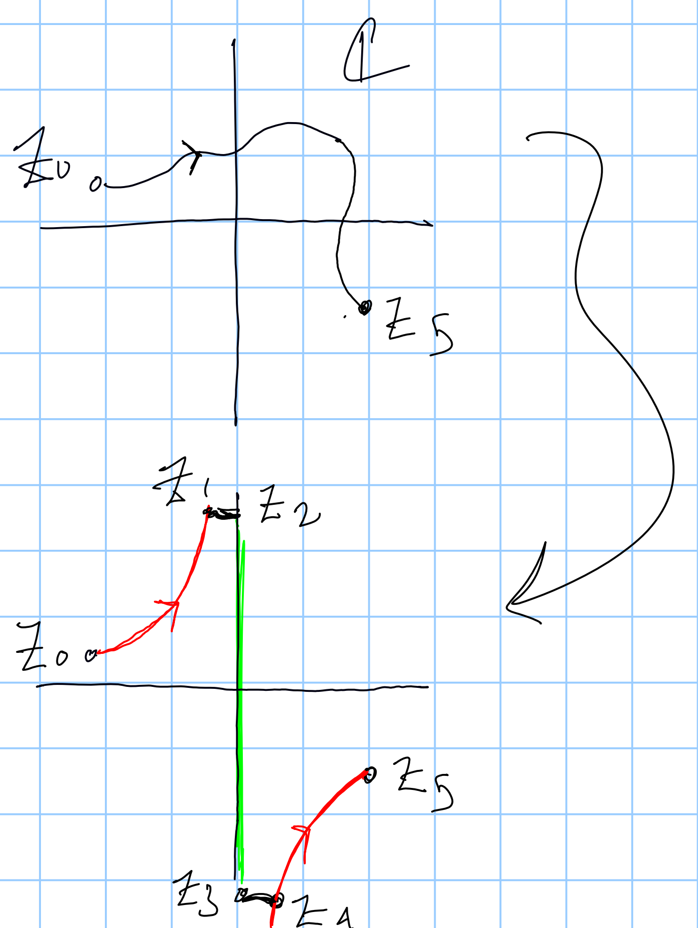


This deformation doesn't work!

The last component C_{34} has constant R and fluctuating I.

The R @ z_4 competes with the R @ z_0, then we can't continue (but see "stationary phase" later on).

Instead, try the deformation:



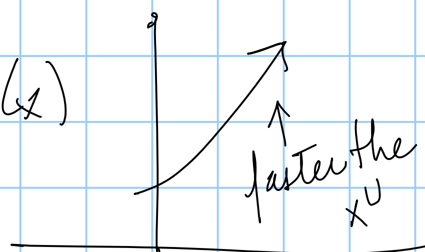
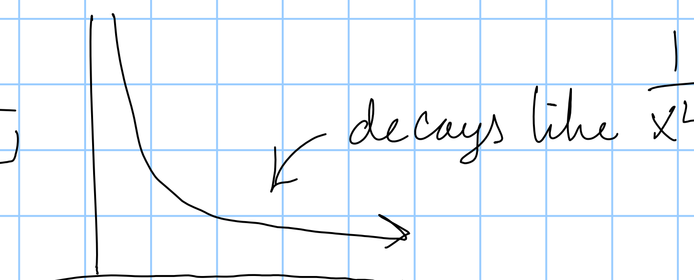
This time, I is constant on C_{01} & C_{45}, and the R-values along C_{12} & C_{34} are very negative (thus ignorable). The one extra consideration is the behaviour along C_{23}, which depends on the value (& concavity) of C_{23} at the saddle-point $z_c = 0$.

in the previous example, the saddle-point could not compete with $R(z_0)$ or $R(z_s)$, so we can ignore it. However, if the contour is closed or extends to ∞, the saddle-point can become dominant.

Generically speaking, most saddle-points are "quadratic" and look like a scaled & rotated version of z^2.
(ie $\alpha z^2 \cdot \exp(i\theta)$ for some real α and θ).
Sometimes, one might encounter higher order saddles, e.g. z^4.

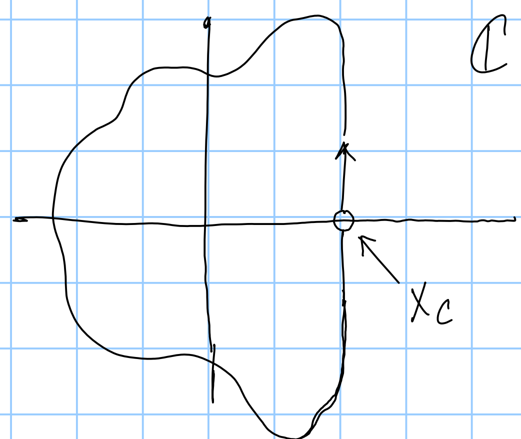
Darwin-Fowler derivation: $\Gamma_{M,N} = \frac{1}{2\pi i} \oint_C \frac{f(z)^M}{z^{MU+1}}\, dz$

Contour-C is closed, so if we use method of steepest-descent (+ Laplace's method), then only saddle-points matter!

Note: $f(x)$  while $\frac{1}{x^U}$ 

So we can expect a minimum at some $x_c \in \mathbb{R}^+$.

When calculating $\Gamma_{M,\mathcal{U}} = \frac{1}{2\pi i} \oint_C \frac{f(z)^M}{z^{M\upsilon+1}} dz$, choose contour



We expect behavior of $h(z)$ near x_c to be like $(z-x_c)^2$, except scaled and rotated so that one $I=0$ contour lies along the positive real axis $\mathbb{R}^+$, and the other $I=0$ contour lies along the line $Re(z) = x_c$.

Moreover, because x_c is a saddle point and x_c is a minimum along $\mathbb{R}^+$, it must be a maximum on $Re(z) = x_c$.

Note: if we choose C to be a circle with radius x_c, then we can see that $h(z)$ must have the largest real-part at $z = x_c \exp i\theta$ with $\theta = 0$.

Specifically, when $\theta = 0$ $f(x_c) = x_c^{E_0} + x_c^{E_1} + \dots + x_c^{E_{N-1}}$.

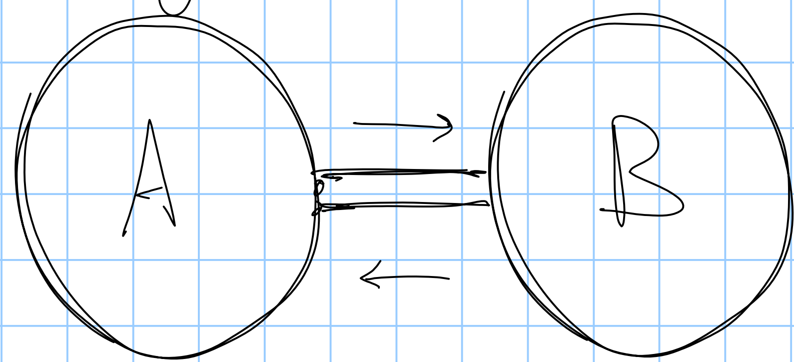
For each other θ we get interference:

$$f(x_c \exp i\theta) = x_c^{E_0} \exp iE_0\theta + x_c^{E_1} \exp iE_1\theta + \dots + x_c^{E_{N-1}} \exp iE_{N-1}\theta.$$

WLOG choose $E_0 = 0$ & $E_1 = 1$.

Now vectors cannot align perfectly unless $\theta = 0$.

Relating β to temperature:



allow energy to be exchanged (slowly) between A & B.

Assume that this exchange is not "too specific" so that A & B remain essentially independent. Also assume exchange is very slow compared to equilibration time.

(ie, systems quickly adopt critical distribution for each U).