

Condorcet's Nightmare

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The field of social choice theory deals with how to take the preferences of a group of people and combine them into a single decision for everyone. This includes voting systems, where each voter ranks a set of candidates first to last and a single candidate is chosen as the winner. A simple example is plurality: take the candidate who is ranked first by the most people. However, plurality is terrible for a number of reasons. For example, if there are more extreme candidates A and C who divide the voter base almost evenly for first choice, but also a moderate candidate B who is most people's second choice, only A and C have a real chance to win the election despite B arguably being the "best" decision.

In place of plurality, a number of other systems have been proposed, all with various "desirable" properties. At the same time, it has been proven that certain combinations of these properties are impossible to achieve at the same time, the most famous example of which is Arrow's theorem. One property is known as Condorcet consistency, which states that if a candidate beats every other candidate in a "head-to-head" election, i.e. an election with all but the two candidates removed, then they should also win the overall election. In this case, the candidate is called the Condorcet winner. The mathematician for whom this is named, Marquis de Condorcet, also showed that a Condorcet winner does not always exist by providing the following counterexample with 3 candidates A, B, and C and 3 voters:

Ranks	#1	#2	#3
Voter 1	A	B	C
Voter 2	B	C	A
Voter 3	C	A	B

In head-to-head elections, A beats B, B beats C, and C beats A like a game of rock-paper-scissors, which comes from the symmetric nature of the preferences: Voters 2 and 3 have preferences which are "rotations" of Voter 1's, so

But can similar things be done for other non-transitive games? For example, there is a variant of RPS which adds two new options: lizard and Spock. Each option beats two others and loses to two others. We can construct an analogous example with 5 candidates and 5 voters:

Ranks	#1	#2	#3	#4	#5
Voter 1	A	B	C	D	E
Voter 2	B	C	D	E	A
Voter 3	C	D	E	A	B
Voter 4	D	E	A	B	C
Voter 5	E	A	B	C	D

However, it's also possible to construct the same game (up to relabeling) with only 3 voters:

Ranks	#1	#2	#3	#4	#5
Voter 1	A	B	C	D	E
Voter 2	C	D	E	A	B
Voter 3	E	B	D	A	C

Ranks	#1	#2	#3	#4	#5	#6	#7
Voter 1	A	B	C	D	E	F	G
Voter 2	F	C	G	D	A	E	B
Voter 3	E	G	B	D	F	A	C