Infinite lists

An ordinal is either 0 or an infinite list of ordinals written right to left, which is all 0 after some point
$\gamma = (\ldots, 0, 0, \gamma_n, \ldots, \gamma_0)$, so $\gamma_k = 0$ for all $k > n$
For all i (including n and larger), $\gamma_i < (\ldots, \gamma_{i+3}, \gamma_{i+2}, \gamma_{i+1} + 1, 0, \ldots, 0)$

Finite lists

Leading 0s may be removed from infinite list and length of remaining list put in front: $(n : \gamma_{n-1}, \ldots, \gamma_0)$
Requires construction of naturals first, unlike infinite lists
Still lexicographic since lengths are compared first
No need for base case since list may be empty: could set $(0 :) = 0$, $(1 : (0 :)) = (1 : 0) = 1$, $(1 : 1) = 2$, etc.
This would offset naturals by 1 from infinite list representation, but otherwise identical

Ordinal-indexed lists

"Redundant" finite lists: place index of each element before it: $(n : \gamma_n, \ldots, \gamma_0) \to (n : \gamma_n, \ldots, 0 : \gamma_0)$
"Sparse" finite lists: remove all 0 entries from list (along with their indices),
turn all entries $\alpha \to -1 + \alpha$ to fill gap: $(2 : 1, 1 : 0, 0 : \omega) \to (2 : 0, 0 : \omega)$
Ordinal-indexed lists: allow indices to be ordinals themselves: $(\omega : 0) = \sup(\{(0 : 0), (1 : 0), (2 : 0), \ldots\})$
No longer requires naturals to be constructed first

Formal definition

Ordinal is a finite list of pairs $(\beta_n : \gamma_n, \beta_{n-1} : \gamma_{n-1}, \ldots, \beta_0 : \gamma_0)$, ordered lexicographically
Indices are strictly decreasing: $\beta_n > \beta_{n-1} > \ldots > \beta_0$
For all i, if $i < n$ and $\beta_{i+1} = \beta_i + 1$, then $\gamma_i < (\beta_n : \gamma_n, \ldots, \beta_{i+1} : \gamma_{i+1} + 1)$
Otherwise, $\gamma_i < (\beta_n : \gamma_n, \ldots, \beta_{i+1} : \gamma_{i+1}, \beta_i + 1 : 0)$

Examples

$$
\begin{aligned}
() &= 0 \\
(0 : 0) &= 1 \\
(0 : 1) &= 2 \\
(0 : n) &= 1 + n, && n < (1 : 0) \\
(1 : 0) &= \omega \\
(1 : 0, 0 : \gamma) &= \omega + (1 + \gamma), && \gamma < (1 : 1) \\
(1 : 1) &= \omega^2 \\
(1 : \gamma) &= \omega^{1+\gamma}, && \gamma < (2 : 0) \\
(1 : \gamma_1, 0 : \gamma_0) &= \omega^{1+\gamma_1} + (1 + \gamma_0), && \gamma_1 < (2 : 0), \gamma_0 < (1 : \gamma_1 + 1) = \omega^{1+\gamma_1+1} \\
(2 : 0) &= \varepsilon_0 \\
(2 : 0, 1 : \gamma_1) &= \varepsilon_0 \omega^{1+\gamma_1} && \gamma_1 < (2 : 1) \\
(2 : 0, 1 : \gamma_1, 0 : \gamma_0) &= \varepsilon_0 \omega^{1+\gamma_1} + (1 + \gamma_0) && \gamma_1 < (2 : 1), \gamma_0 < (2 : 0, 1 : \gamma_1 + 1) = \varepsilon_0 \omega^{1+\gamma_1+1} \\
(3 : 0) &= \zeta_0 \\
(\beta : \gamma) &= \varphi_{-1+\beta}(\gamma) && \beta \geq 2
\end{aligned}
$$

Relation to Veblen functions

The notation goes up to $\Gamma_0 = \varphi(1, 0, 0)$, since each lower-$\beta$ term essentially applies another two-variable Veblen function, except for the last, which is added to the result. It is closely related to Veblen normal form, but restricted to two arguments and preserving lexicographical order.

$$
\begin{aligned}
[0, \varepsilon_0) &= \{0\} \cup \bigcup_{\alpha \in [0, \varepsilon_0)} [\omega^\alpha, \omega^{\alpha+1}) \\
[\omega^\alpha, \omega^{\alpha+1}) &= \{\omega^\alpha + \beta \mid \beta \in [0, \omega^{\alpha+1})\} \\
[0, \omega^{\alpha+1}) &= \{0\} \cup \bigcup_{\beta \in [0, \alpha+1)} [\omega^\beta, \omega^{\beta+1}) \\
[0, 0+1) &= \{0\} \\
[0, \omega^\alpha + \beta + 1) &= [0, \omega^\alpha) \cup [\omega^\alpha, \omega^\alpha + \beta + 1) \\
[\omega^\alpha, \omega^\alpha + \beta + 1) &= \{\omega^\alpha + \gamma \mid \gamma \in [0, \beta + 1)\} \\
[0, \omega^\alpha) &= \{0\} \cup \bigcup_{\beta \in [0, \alpha)} [\omega^\beta, \omega^{\beta+1}) \\
[0, 0) &= \varnothing \\
[0, \omega^\alpha + \beta) &= [0, \omega^\alpha) \cup [\omega^\alpha, \omega^\alpha + \beta) \\
[\omega^\alpha, \omega^\alpha + \beta) &= \{\omega^\alpha + \gamma \mid \gamma \in [0, \beta)\}
\end{aligned}
$$

May cast β from $[0, \omega^{\alpha+1})$ to $[0, \varepsilon_0)$ so $[0, \beta + 1)$ may be defined