

# A Generating Function Problem

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Problem:

$$f(x, 0) = \frac{e^x - 1}{x}$$
$$f(x, y) = \frac{\partial f}{\partial x}(x, y) + \frac{\partial f}{\partial y}(x, y)$$

Solution: Define

$$A(j, k) = \frac{\partial^{j+k} f}{\partial x^j \partial y^k}(0, 0)$$

Get Taylor series of base case:

$$f(x, 0) = \frac{1}{x} \left( \sum_{i=0}^{\infty} \frac{x^i}{i!} - 1 \right) = \frac{1}{x} \left( \sum_{i=1}^{\infty} \frac{x^i}{i!} \right) = \frac{1}{x} \left( \sum_{i=0}^{\infty} \frac{x^{i+1}}{(i+1)!} \right) = \sum_{i=0}^{\infty} \frac{x^i}{(i+1)!}$$

Find partial derivatives wrt  $x$ :

$$A(j, 0) = \frac{\partial^j f}{\partial x^j} \sum_{i=0}^{\infty} \frac{x^i}{(i+1)!} = \frac{\partial^j f}{\partial x^j} \frac{x^j}{(j+1)!} = \frac{j!}{(j+1)!} = \frac{1}{j+1}$$

Use diffeq to establish recurrence relation on  $A$ :

$$\begin{aligned} A(j, k) &= \frac{\partial^{j+k} f}{\partial x^j \partial y^k}(0, 0) \\ &= \frac{\partial}{\partial x} \left( \frac{\partial^{j+k} f}{\partial x^j \partial y^k} \right)(0, 0) + \frac{\partial}{\partial y} \left( \frac{\partial^{j+k} f}{\partial x^j \partial y^k} \right)(0, 0) \\ &= \frac{\partial^{j+k+1} f}{\partial x^{j+1} \partial y^k}(0, 0) + \frac{\partial^{j+k+1} f}{\partial x^j \partial y^{k+1}}(0, 0) \\ &= A(j+1, k) + A(j, k+1) \end{aligned}$$

Rearrange to compute higher values of  $k$ :

$$A(j, k+1) = A(j, k) - A(j+1, k)$$

These are sufficient to determine all  $A(j, k)$ , which may be found by computation and guess-and-check:

$$A(j, k) = \frac{j!k!}{(j+k+1)!} = B(j+1, k+1)$$

where B is the beta function. Construct 2D Taylor series:

$$\begin{aligned}
f(x, y) &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} A(j, k) \frac{x^j y^k}{j! k!} \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{j! k!}{(j+k+1)!} \frac{x^j y^k}{j! k!} \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{x^j y^k}{(j+k+1)!} \\
&= \sum_{n=0}^{\infty} \sum_{j+k=n} \frac{x^j y^k}{(j+k+1)!} \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \sum_{j+k=n} x^j y^k \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \frac{x^{n+1} - y^{n+1}}{x - y} \\
&= \frac{1}{x - y} \left( \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} - \sum_{n=0}^{\infty} \frac{y^{n+1}}{(n+1)!} \right) \\
&= \frac{1}{x - y} ((e^x - 1) - (e^y - 1)) \\
&= \frac{e^x - e^y}{x - y}
\end{aligned}$$

Check answer:

$$\begin{aligned}
f(x, 0) &= \frac{e^x - e^0}{x - 0} = \frac{e^x - 1}{x} \\
\frac{\partial f}{\partial x} &= \frac{e^x(x - y) - (e^x - e^y)}{(x - y)^2} \\
\frac{\partial f}{\partial y} &= \frac{-e^y(x - y) + (e^x - e^y)}{(x - y)^2} \\
\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} &= \frac{(e^x - e^y)(x - y)}{(x - y)^2} \\
&= \frac{e^x - e^y}{(x - y)} \\
&= f(x, y)
\end{aligned}$$