CRF Diminishings of the 600-Cell

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Definitions

- **CRF**: convex regular-faced
  - **Convex**: may be formed as the face lattice of a bounded convex set, i.e. all elements are strictly convex
  - **Face**: 2D element (not of any dimension, as is used elsewhere)
- **600-cell (ex)**: the convex regular polychoron with 120 vertices and 600 tetrahedral cells, 5 to an edge
- **Diminishing of ex**: convex hull of a subset of the vertices of ex which are not all on the same hyperplane
  - Same hyperplane $\rightarrow$ lower dimension

CRF Polyhedra

- 5 Platonic solids
- 13 Archimedean solids
- ∞ prisms
  - $n \in [3, \infty)$ sides, $n = 4 \rightarrow$ cube
- ∞ antiprisms
  - $n \in [3, \infty)$ sides, $n = 3 \rightarrow$ octahedron
- 92 Johnson solids[1][2]

[1] Norman Johnson, 1966
[2] Victor Zalgaller, 1969

Blind Polytopes

- **Blind polytope**: convex polytope with regular facets
  - Achtung: short “i” sound!
- Blind polychora have regular (Platonic solid) cells
- There are 314248357 Blind polychora(!)
- All but 13 of them are **special cuts** of ex

Special Cuts

- A special cut is ex, diminished by a (nonempty) independent set of vertices
- Diminishing a vertex replaces 20 tetrahedra with an icosahedron
- Diminishing two adjacent vertices would mean the new cells are no longer icosahedral: remember this for later
- There are 314248344 nonempty independent sets of ex up to symmetry, so there are the same number of special cuts
  - One independent set, the vertices of an inscribed 24-cell, creates a uniform special cut, the snub 24-cell

CRF Polychora

- The other main way to generalize Johnson solids (or the family they complete) to 4D
- Too many to count! At least 10^{hundreds}, maybe $10^{\text{thousands}}$
- This class contains even more ex diminishings than the special cuts
- But how many?

Edge Length

- CRFs have all edges the same length
- 8 different distances between verts of ex: possible edge lengths
- Longest edges have no CRFs: too cramped
- Intermediate edges have some CRFs
  - Diminishings of inscribed polychora e.g. 24-cell
- Vast majority of CRFs have same edge length as ex
  - Focus of this presentation

Faces and Cells

- Faces: triangles, pentagons, decagons
- Cells, in order of increasing circumradius:
  - Tetrahedron
  - Icosahedron, its 4 "shallow" diminishings, and the pentagonal pyramid (its "deep" diminishing)
  - Dodecahedron
  - Icosidodecahedron and its "half", the pentagonal rotunda
- Higher circumradius $\rightarrow$ deeper cut, up to icosidodecahedron at the equator
- Can also make cuts deeper than the equator
- Most CRF diminishings of ex consist of only the shallowest cuts, which diminish 1 vertex each
  - Intuition: deeper cuts remove more vertices, so there are fewer options for where to place the remaining shallow cuts

Shallow Cuts

- Diminishing 1 vertex creates an icosahedron, as in the special cuts
- Diminishing 2 adjecent vertices creates 2 diminished icosahedra, joined at the pentagon
- Diminishing a triangle creates icosahedra with 2 adjacent vertices removed, which have trapezoidal faces: not allowed
- Otherwise, any triangle-free set may be diminished to produce a CRF
  - Other diminishings of the icosahedron may be created by diminishing a vertex and an independent set of its neighbors