A Generating Function Problem

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Problem:

$$f(x,0) = \frac{e^x - 1}{x}$$
$$f(x,y) = \frac{\partial f}{\partial x}(x,y) + \frac{\partial f}{\partial y}(x,y)$$

Solution: Define

$$A(j,k) = \frac{\partial^{j+k} f}{\partial x^j \partial y^k}(0,0)$$

Get Taylor series of base case:

$$f(x,0) = \frac{1}{x}\left(\sum_{i=0}^{\infty} \frac{x^i}{i!} - 1\right) = \frac{1}{x}\left(\sum_{i=1}^{\infty} \frac{x^i}{i!}\right) = \frac{1}{x}\left(\sum_{i=0}^{\infty} \frac{x^{i+1}}{(i+1)!}\right) = \sum_{i=0}^{\infty} \frac{x^i}{(i+1)!}$$

Find partial derivatives wrt x:

$$A(j,0) = \frac{\partial^j f}{\partial x^j} \sum_{i=0}^{\infty} \frac{x^i}{(i+1)!} = \frac{\partial^j f}{\partial x^j} \frac{x^j}{(j+1)!} = \frac{j!}{(j+1)!} = \frac{1}{j+1}$$

Use diffeq to establish recurrence relation on A:

$$\begin{aligned}
A(j,k) &= \frac{\partial^{j+k} f}{\partial x^j \partial y^k}(0,0) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial^{j+k} f}{\partial x^j \partial y^k}\right)(0,0) + \frac{\partial}{\partial y}\left(\frac{\partial^{j+k} f}{\partial x^j \partial y^k}\right)(0,0) \\
&= \frac{\partial^{j+k+1} f}{\partial x^{j+1} \partial y^k}(0,0) + \frac{\partial^{j+k+1} f}{\partial x^j \partial y^{k+1}}(0,0) \\
&= A(j+1,k) + A(j,k+1)
\end{aligned}$$

Rearrange to compute higher values of k:

$$A(j,k+1) = A(j,k) - A(j+1,k)$$

These are sufficient to determine all $A(j,k)$, which may be found by computation and guess-and-check:

$$A(j,k) = \frac{j!k!}{(j+k+1)!} = B(j+1,k+1)$$

where B is the beta function. Construct 2D Taylor series:

$$
\begin{aligned}
f(x,y) &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} A(j,k)\frac{x^j y^k}{j!k!} \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{j!k!}{(j+k+1)!}\frac{x^j y^k}{j!k!} \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{x^j y^k}{(j+k+1)!} \\
&= \sum_{n=0}^{\infty}\sum_{j+k=n} \frac{x^j y^k}{(j+k+1)!} \\
&= \sum_{n=0}^{\infty}\frac{1}{(n+1)!}\sum_{j+k=n} x^j y^k \\
&= \sum_{n=0}^{\infty}\frac{1}{(n+1)!}\frac{x^{n+1}-y^{n+1}}{x-y} \\
&= \frac{1}{x-y}\left(\sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)!} - \sum_{n=0}^{\infty}\frac{y^{n+1}}{(n+1)!}\right) \\
&= \frac{1}{x-y}((e^x-1)-(e^y-1)) \\
&= \frac{e^x-e^y}{x-y}
\end{aligned}
$$

Check answer:

$$
f(x,0) = \frac{e^x-e^0}{x-0} = \frac{e^x-1}{x}
$$

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{e^x(x-y)-(e^x-e^y)}{(x-y)^2} \\
\frac{\partial f}{\partial y} &= \frac{-e^y(x-y)+(e^x-e^y)}{(x-y)^2} \\
\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y} &= \frac{(e^x-e^y)(x-y)}{(x-y)^2} \\
&= \frac{e^x-e^y}{(x-y)} \\
&= f(x,y)
\end{aligned}
$$

3D variant:

$$A(j,k,l) = \frac{j!k!l!}{(j+k+l+2)!} = B(j+1,k+1,l+1) = \frac{\partial^{j+k+l} f}{\partial x^j \partial y^k \partial z^l}$$

Diffeq:

$$f(x,y,z) = \frac{\partial f}{\partial x}(x,y,z) + \frac{\partial f}{\partial y}(x,y,z) + \frac{\partial f}{\partial z}(x,y,z)$$

Solution:

$$\begin{aligned}
f(x,y,z) &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{x^j y^k z^l}{(j+k+l+2)!} \\
&= \sum_{n=0}^{\infty} \sum\sum_{j+k+l=n} \frac{x^j y^k z^l}{(j+k+l+2)!} \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+2)!} \sum\sum_{j+k+l=n} x^j y^k z^l \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+2)!} \frac{(y-z)x^{n+2} + (z-x)y^{n+2} + (x-y)z^{n+2}}{(z-y)(x-z)(y-x)} \\
&= \frac{1}{(z-y)(x-z)(y-x)} \left((y-z)\sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} + (z-x)\sum_{n=0}^{\infty} \frac{y^{n+2}}{(n+2)!} + (x-y)\sum_{n=0}^{\infty} \frac{z^{n+2}}{(n+2)!} \right) \\
&= \frac{1}{(z-y)(x-z)(y-x)} ((y-z)(e^x - 1 - x) + (z-x)(e^y - 1 - y) + (x-y)(e^z - 1 - z)) \\
&= \frac{e^x(y-z) + e^y(z-x) + e^z(x-y)}{(z-y)(x-z)(y-x)} \\
&= -\frac{e^x}{(x-z)(y-x)} - \frac{e^y}{(z-y)(y-x)} - \frac{e^z}{(z-y)(x-z)} \\
&= \frac{e^x}{(x-y)(x-z)} + \frac{e^y}{(y-x)(y-z)} + \frac{e^z}{(z-x)(z-y)}
\end{aligned}$$

Check answer:

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y,z) &= \frac{\partial}{\partial x}\left(\frac{e^x}{(x-y)(x-z)} + \frac{e^y}{(y-x)(y-z)} + \frac{e^z}{(z-x)(z-y)} \right) \\
&= \frac{\partial}{\partial x}\frac{e^x}{(x-y)(x-z)} + \frac{\partial}{\partial x}\frac{e^y}{(y-x)(y-z)} + \frac{\partial}{\partial x}\frac{e^z}{(z-x)(z-y)} \\
&= \frac{\left(\frac{\partial}{\partial x} e^x\right)(x-y)(x-z) - e^x\left(\frac{\partial}{\partial x}((x-y)(x-z))\right)}{(x-y)^2(x-z)^2 w} + \frac{e^y}{(y-x)^2(y-z)} + \frac{e^z}{(z-x)^2(z-y)} \\
&= \frac{e^x(x-y)(x-z) - e^x((x-y)+(x-z))}{(x-y)^2(x-z)^2} + \frac{e^y}{(y-x)^2(y-z)} + \frac{e^z}{(z-x)^2(z-y)} \\
&= \frac{e^x}{(x-y)(x-z)} - \frac{e^x}{(x-y)(x-z)^2} - \frac{e^x}{(x-y)^2(x-z)} + \frac{e^y}{(y-x)^2(y-z)} + \frac{e^z}{(z-x)^2(z-y)} \\
\frac{\partial f}{\partial y}(x,y,z) &= \frac{e^x}{(x-y)^2(x-z)} + \frac{e^y}{(y-x)(y-z)} - \frac{e^y}{(y-x)(y-z)^2} - \frac{e^y}{(y-x)^2(y-z)} + \frac{e^z}{(z-x)(z-y)^2} \\
\frac{\partial f}{\partial z}(x,y,z) &= \frac{e^x}{(x-y)(x-z)^2} + \frac{e^y}{(y-x)(y-z)^2} + \frac{e^z}{(z-x)(z-y)} - \frac{e^z}{(z-x)(z-y)^2} - \frac{e^z}{(z-x)^2(z-y)}
\end{aligned}$$

Each term with quadratic denominator only appears once across all partials
Each term with cubic denominator appears twice across all partials, once positive and once negative
When added, cubic terms cancel and quadratics remain to form $f(x,y,z)$

ND solution:

$$f(x_1,\ldots,x_N) = \sum_{I=1}^{N} \frac{e^{x_I}}{\prod\limits_{J\neq I}(x_I - x_J)}$$

$I \neq K$: Define

$$T(I,K) = \frac{\partial}{\partial x_K} \frac{e^{x_I}}{\prod\limits_{J\neq I}(x_I - x_J)} = \frac{e^{x_I}}{(x_I - x_K)\prod\limits_{J\neq I}(x_I - x_J)}$$

Partial derivative of remaining terms ($I = K$):

$$\begin{aligned}
\frac{\partial}{\partial x_K} \frac{e^{x_K}}{\prod\limits_{J\neq K}(x_K - x_J)} &= \frac{\frac{\partial}{\partial x_K} e^{x_K}}{\prod\limits_{J\neq K}(x_K - x_J)} - \frac{e^{x_K}\frac{\partial}{\partial x_K}\prod\limits_{J\neq K}(x_K - x_J)}{\prod\limits_{J\neq K}(x_K - x_J)^2} \\
&= \frac{e^{x_K}}{\prod\limits_{J\neq K}(x_K - x_J)} - \frac{e^{x_K}\sum\limits_{L\neq K}\prod\limits_{J\notin\{K,L\}}(x_K - x_J)}{\prod\limits_{J\neq K}(x_K - x_J)^2} \\
&= \frac{e^{x_K}}{\prod\limits_{J\neq K}(x_K - x_J)} - \sum_{L\neq K}\frac{e^{x_K}}{(x_K - x_L)\prod\limits_{J\neq K}(x_K - x_J)} \\
&= \frac{e^{x_K}}{\prod\limits_{J\neq K}(x_K - x_J)} - \sum_{L\neq K} T(K,L)
\end{aligned}$$

Overall derivative for given x_K:

$$\begin{aligned}
\frac{\partial f}{\partial x_K}(x_1,\ldots,x_N) &= \frac{\partial}{\partial x_K}\sum_{I=1}^{N}\frac{e^{x_I}}{\prod\limits_{J\neq I}(x_I - x_J)} \\
&= \frac{\partial}{\partial x_K}\frac{e^{x_K}}{\prod\limits_{J\neq K}(x_K - x_J)} + \sum_{I\neq K}\frac{\partial}{\partial x_K}\frac{e^{x_I}}{\prod\limits_{J\neq I}(x_I - x_J)} \\
&= \frac{e^{x_K}}{\prod\limits_{J\neq K}(x_K - x_J)} - \sum_{L\neq K} T(K,L) + \sum_{I\neq K} T(I,K)
\end{aligned}$$

Sum of all partials:

$$\begin{aligned}
\sum_{K=1}^{N}\frac{\partial f}{\partial x_K}(x_1,\ldots,x_N) &= \sum_{K=1}^{N}\left(\frac{e^{x_K}}{\prod\limits_{J\neq K}(x_K - x_J)} - \sum_{L\neq K} T(K,L) + \sum_{I\neq K} T(I,K)\right) \\
&= \sum_{K=1}^{N}\frac{e^{x_K}}{\prod\limits_{J\neq K}(x_K - x_J)} - \sum_{K=1}^{N}\sum_{L\neq K} T(K,L) + \sum_{K=1}^{N}\sum_{I\neq K} T(I,K) \\
&= \sum_{K=1}^{N}\frac{e^{x_K}}{\prod\limits_{J\neq K}(x_K - x_J)} \\
&= f(x_1,\ldots,x_N)
\end{aligned}$$

Remains to be shown that derivative is the multivariate beta function:

$$\frac{\partial^{k_1+\ldots+k_K} f}{\partial x_1^{k_1}\ldots\partial x_N^{k_n}} = \frac{k_1!\ldots k_N!}{(k_1+\ldots+k_N+N-1)!}$$