Paper: Discrete Fourier transform methods in the theory of equations, BR Neelley, 1992, https://ttu-ir.tdl.org/bitstream/handle/2346/60037/31295007093692.pdf?sequence=1

Solving Quadratics, Cubics, and Quartics with the Discrete Fourier Transform

Quadratics (Po-Shen Loh)

$$\begin{aligned}
&\boldsymbol{Ax^2 + Bx + C = 0, A \neq 0}\\
&\boldsymbol{b = \frac{B}{A}, c = \frac{C}{A}}\\
&x^2 + bx + c = 0\\
&\boldsymbol{x = x_1, x_2}\\
&\boldsymbol{x_1 = p + q, x_2 = p - q}\\
&\boldsymbol{p = -\frac{b}{2}}, b = -2p\\
&y = x - p, x = y + p
\end{aligned}$$

$$\begin{aligned}
&0 = (y+p)^2 - 2p(y+p) + c = y^2 + (c - p^2)\\
&\boldsymbol{c' = c - p^2}
\end{aligned}$$

$$0 = (x - x_1)(x - x_2) = (x - (p+q))(x - (p-q)) = (y-q)(y+q) = y^2 - q^2$$

$$\begin{aligned}
&c' = -q^2\\
&q^2 = -c'\\
&\boldsymbol{q = (-c')^{\frac{1}{2}}}
\end{aligned}$$

Cubics

$$\boldsymbol{Ax^3+Bx^2+Cx+D=0, A\neq 0}$$
$$\boldsymbol{b=\frac{B}{A}, c=\frac{C}{A}, d=\frac{D}{A}}$$
$$x^3+bx^2+cx+d=0$$
$$\boldsymbol{x=x_1,x_2,x_3}$$
$$\boldsymbol{\omega=\frac{\sqrt{3}i-1}{2}}$$
$$\omega^2+\omega+1=0, \omega^2+\omega=-1, \omega^3=1$$
$$\boldsymbol{x_1=p+q+r, x_2=p+\omega q+\omega^2 r, x_3=p+\omega^2 q+\omega r}$$
$$\boldsymbol{p=-\frac{b}{3}}$$
$$y=x-p, x=y+p$$

$$\begin{aligned}0&=(y+p)^3-3p(y+p)^2+c(y+p)+d\\&=y^3+(c-3p^2)y+(d-2p^3+pc)\\ \boldsymbol{c'}&\boldsymbol{=c-3p^2, d'=d-2p^3+pc}\end{aligned}$$

$$\begin{aligned}0&=(x-x_1)(x-x_2)(x-x_3)\\&=(x-(p+q+r))(x-(p+\omega q+\omega^2 r))(x-(p+\omega^2 q+\omega r))\\&=(y-(q+r))(y-(\omega q+\omega^2 r))(y-(\omega^2 q+\omega r))\\&=y^3-3qry-(q^3+r^3)\end{aligned}$$

$$c'=-3qr, d'=-(q^3+r^3)$$
$$qr=-\frac{c'}{3}$$
$$u_1=q^3, u_2=r^3$$
$$\boldsymbol{u=u_1,u_2}$$
$$0=(u-u_1)(u-u_2)=u^2-(u_1+u_2)u+u_1u_2=u^2-(q^3+r^3)u+q^3r^3=u^2-(q^3+r^3)u+(qr)^3$$
$$\boldsymbol{u^2+d'u-\left(\frac{c'}{3}\right)^3=0}$$
$$\boldsymbol{q=u_1^{\frac{1}{3}}}$$
$$q=0:\boldsymbol{r=u_2^{\frac{1}{3}}}$$
$$q\neq 0:\boldsymbol{r=-\frac{c'}{3q}}$$

Quartics

$$\boldsymbol{Ax^4+Bx^3+Cx^2+Dx+E=0, A\neq 0}$$
$$\boldsymbol{b=\frac{B}{A}, c=\frac{C}{A}, d=\frac{D}{A}, e=\frac{E}{A}}$$
$$x^4+bx^3+cx^2+dx+e=0$$
$$\boldsymbol{x=x_1,x_2,x_3,x_4}$$
$$\boldsymbol{x_1=p+q+r+s}$$
$$\boldsymbol{x_2=p+qi-r-si}$$
$$\boldsymbol{x_3=p-q+r-s}$$
$$\boldsymbol{x_4=p-qi-r+si}$$
$$\boldsymbol{p=-\frac{b}{4}}$$
$$y=x-p, x=y+p$$

$$\begin{aligned} 0 &= (y+p)^4-4p(y+p)^3+c(y+p)^2+d(y+p)+e \\ &= y^4+(c-6p^2)y^2+(d-8p^3+2pc)y+(e-3p^4+p^2c+pd) \\ & \boldsymbol{c'=c-6p^2, d'=d-8p^3+2pc, e'=e-3p^4+p^2c+pd} \end{aligned}$$

$$\begin{aligned} 0 &= (x-x_1)(x-x_2)(x-x_3)(x-x_4) \\ &= (x-(p+q+r+s))(x-(p+qi-r-si))(x-(p-q+r-s))(x-(p-qi-r+si)) \\ &= (y-(q+r+s))(y-(qi-r-si))(y-(-q+r-s))(y-(-qi-r+si)) \\ &= y^4-(2r^2+4qs)y^2-4r(q^2+s^2)y-(q^2+s^2)^2+r^4-4qr^2s \end{aligned}$$

$$c'=-(2r^2+4qs), d'=-4r(q^2+s^2), e'=-(q^2+s^2)^2+r^4-4qr^2s$$
$$t=q^2+s^2, u=4qs$$
$$qs=\frac{u}{4}$$
$$v_1=q^2, v_2=s^2$$
$$\boldsymbol{v=v_1,v_2}$$
$$0=(v-v_1)(v-v_2)=v^2-(v_1+v_2)v+v_1v_2=v^2-(q^2+s^2)v+q^2s^2=v^2-(q^2+s^2)v+(qs)^2$$
$$\boldsymbol{v^2-tv+\left(\frac{u}{4}\right)^2=0}$$
$$\boldsymbol{q=v_1^{\frac{1}{2}}}$$
$$q=0:\boldsymbol{s=v_2^{\frac{1}{2}}}$$
$$q\neq 0:\boldsymbol{s=\frac{u}{4q}}$$

$$c' = -(2r^2 + u), d' = -4rt, e' = -t^2 + r^4 - r^2u$$
$$u = -(2r^2 + c')$$
$$-\frac{d'}{4} = rt$$
$$w = r^2$$
$$\left(\frac{d'}{4}\right)^2 = r^2t^2 = t^2w$$
$$e' = -t^2 + r^4 - r^2(-2r^2 + c') = 3r^4 - c'r^2 - t^2 = 3w^2 - c'w - t^2$$
$$e'w = 3w^3 - c'w^2 - t^2w = 3w^3 - c'w^2 - \left(\frac{d'}{4}\right)^2$$
$$\boldsymbol{3w^3 - c'w^2 - e'w - \left(\frac{d'}{4}\right)^2 = 0}$$
$$\boldsymbol{r = w^{\frac{1}{2}}}$$
$$\boldsymbol{t = -\frac{d'}{4r}}$$
$$\boldsymbol{u = -(2w + c')}$$