Paper: Discrete Fourier transform methods in the theory of equations, BR Neelley, 1992, https://ttu-ir.tdl.org/bitstream/handle/2346/60037/31295007093692.pdf?sequence=1

Solving Quadratics, Cubics, and Quartics with the Discrete Fourier Transform

Quadratics (Po-Shen Loh)

$$\begin{aligned}
&\boldsymbol{Ax^2+Bx+C=0, A\neq 0}\\
&\boldsymbol{b=\frac{B}{A}, c=\frac{C}{A}}\\
&x^2+bx+c=0\\
&\boldsymbol{x=x_1,x_2}\\
&\boldsymbol{x_1=p+q, x_2=p-q}\\
&\boldsymbol{p=-\frac{b}{2}}, b=-2p\\
&y=x-p, x=y+p
\end{aligned}$$

$$\begin{aligned}
&0=(y+p)^2-2p(y+p)+c=y^2+(c-p^2)\\
&\boldsymbol{c'=c-p^2}
\end{aligned}$$

$$0=(x-x_1)(x-x_2)=(x-(p+q))(x-(p-q))=(y-q)(y+q)=y^2-q^2$$

$$\begin{aligned}
&c'=-q^2\\
&q^2=-c'\\
&\boldsymbol{q=(-c')^{\frac{1}{2}}}
\end{aligned}$$

Cubics

$$\boldsymbol{Ax^3+Bx^2+Cx+D=0, A\neq 0}$$
$$\boldsymbol{b=\frac{B}{A}, c=\frac{C}{A}, d=\frac{D}{A}}$$
$$x^3+bx^2+cx+d=0$$
$$\boldsymbol{x=x_1, x_2, x_3}$$
$$\boldsymbol{\omega=\frac{\sqrt{3}i-1}{2}}$$
$$\omega^2+\omega+1=0, \omega^2+\omega=-1, \omega^3=1$$
$$\boldsymbol{x_1=p+q+r, x_2=p+\omega q+\omega^2 r, x_3=p+\omega^2 q+\omega r}$$
$$\boldsymbol{p=-\frac{b}{3}}$$
$$y=x-p, x=y+p$$

$$\begin{aligned}
0 &= (y+p)^3-3p(y+p)^2+c(y+p)+d \\
&= y^3+(c-3p^2)y+(d-2p^3+pc) \\
&\boldsymbol{c'=c-3p^2, d'=d-2p^3+pc}
\end{aligned}$$

$$\begin{aligned}
0 &= (x-x_1)(x-x_2)(x-x_3) \\
&= (x-(p+q+r))(x-(p+\omega q+\omega^2 r))(x-(p+\omega^2 q+\omega r)) \\
&= (y-(q+r))(y-(\omega q+\omega^2 r))(y-(\omega^2 q+\omega r)) \\
&= y^3-3qry-(q^3+r^3)
\end{aligned}$$

$$c'=-3qr, d'=-(q^3+r^3)$$
$$qr=-\frac{c'}{3}$$
$$u_1=q^3, u_2=r^3$$
$$\boldsymbol{u=u_1, u_2}$$
$$0=(u-u_1)(u-u_2)=u^2-(u_1+u_2)u+u_1u_2=u^2-(q^3+r^3)u+q^3r^3=u^2-(q^3+r^3)u+(qr)^3$$
$$\boldsymbol{u^2+d'u-\left(\frac{c'}{3}\right)^3=0}$$
$$\boldsymbol{q=u_1^{\frac{1}{3}}}$$
$$q=0: \boldsymbol{r=u_2^{\frac{1}{3}}}$$
$$q\neq 0: \boldsymbol{r=-\frac{c'}{3q}}$$

Quartics

$$\boldsymbol{Ax^4 + Bx^3 + Cx^2 + Dx + E = 0, A \neq 0}$$
$$\boldsymbol{b = \frac{B}{A}, c = \frac{C}{A}, d = \frac{D}{A}, e = \frac{E}{A}}$$
$$x^4 + bx^3 + cx^2 + dx + e = 0$$
$$\boldsymbol{x = x_1, x_2, x_3, x_4}$$
$$\boldsymbol{x_1 = p + q + r + s}$$
$$\boldsymbol{x_2 = p + qi - r - si}$$
$$\boldsymbol{x_3 = p - q + r - s}$$
$$\boldsymbol{x_4 = p - qi - r + si}$$
$$\boldsymbol{p = -\frac{b}{4}}$$
$$y = x - p, x = y + p$$

$$\begin{aligned} 0 &= (y+p)^4 - 4p(y+p)^3 + c(y+p)^2 + d(y+p) + e \\ &= y^4 + (c - 6p^2)y^2 + (d - 8p^3 + 2pc)y + (e - 3p^4 + p^2c + pd) \\ & \boldsymbol{c' = c - 6p^2, d' = d - 8p^3 + 2pc, e' = e - 3p^4 + p^2c + pd} \end{aligned}$$

$$\begin{aligned} 0 &= (x - x_1)(x - x_2)(x - x_3)(x - x_4) \\ &= (x - (p + q + r + s))(x - (p + qi - r - si))(x - (p - q + r - s))(x - (p - qi - r + si)) \\ &= (y - (q + r + s))(y - (qi - r - si))(y - (-q + r - s))(y - (-qi - r + si)) \\ &= y^4 - (2r^2 + 4qs)y^2 - 4r(q^2 + s^2)y - (q^2 + s^2)^2 + r^4 - 4qr^2s \end{aligned}$$

$$c' = -(2r^2 + 4qs), d' = -4r(q^2 + s^2), e' = -(q^2 + s^2)^2 + r^4 - 4qr^2s$$
$$t = q^2 + s^2, u = 4qs$$
$$qs = \frac{u}{4}$$
$$v_1 = q^2, v_2 = s^2$$
$$\boldsymbol{v = v_1, v_2}$$
$$0 = (v - v_1)(v - v_2) = v^2 - (v_1 + v_2)v + v_1v_2 = v^2 - (q^2 + s^2)v + q^2s^2 = v^2 - (q^2 + s^2)v + (qs)^2$$
$$\boldsymbol{v^2 - tv + \left(\frac{u}{4}\right)^2 = 0}$$
$$\boldsymbol{q = v_1^{\frac{1}{2}}}$$
$$q = 0 : \boldsymbol{s = v_2^{\frac{1}{2}}}$$
$$q \neq 0 : \boldsymbol{s = \frac{u}{4q}}$$

$$
\begin{aligned}
c' &= -(2r^2+u), d' = -4rt, e' = -t^2+r^4-r^2u \\
u &= -(2r^2+c') \\
-\frac{d'}{4} &= rt \\
w &= r^2 \\
\left(\frac{d'}{4}\right)^2 &= r^2t^2 = t^2w \\
e' &= -t^2+r^4-r^2(-2r^2+c') = 3r^4-c'r^2-t^2 = 3w^2-c'w-t^2 \\
e'w &= 3w^3-c'w^2-t^2w = 3w^3-c'w^2-\left(\frac{d'}{4}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&\boldsymbol{3w^3-c'w^2-e'w-\left(\frac{d'}{4}\right)^2=0} \\
&\boldsymbol{r=w^{\frac{1}{2}}} \\
&\boldsymbol{t=-\frac{d'}{4r}} \\
&\boldsymbol{u=-(2w+c')}
\end{aligned}
$$