

# The Generalized Hockey-stick Identity

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**Generalized Hockey-stick Identity.** For all  $J, k_1, k_2 \in \mathbb{N}$ ,

$$\sum_{j_1+j_2=J} \binom{j_1+k_1}{j_1} \binom{j_2+k_2}{j_2} = \binom{J+k_1+k_2+1}{J}.$$

*Proof.* Let  $J, k_1, k_2 \in \mathbb{N}$ . Define

$$f(j, k) = \binom{j+k}{k} = \binom{j+k}{j} = \frac{(j+k)!}{j!k!}$$

so that the identity may be rewritten

$$\sum_{j_1+j_2=J} f(j_1, k_1) f(j_2, k_2) = f(J, k_1 + k_2 + 1).$$

Note  $f(0, k) = f(j, 0) = 1$  and  $f(j+1, k+1) = f(j, k+1) + f(j+1, k)$  for all  $j, k \in \mathbb{N}$ .

Proof by induction over  $J$  and  $k_2$ :

1.  $J = 0$  is trivial:

$$\begin{aligned} \sum_{j_1+j_2=0} f(j_1, k_1) f(j_2, k_2) &= f(0, k_1 + k_2 + 1) \\ f(0, k_1) f(0, k_2) &= f(0, k_1 + k_2 + 1) \\ 1 \times 1 &= 1 \end{aligned}$$

2.  $k_2 = 0$  reduces to the normal hockey stick identity:

$$\begin{aligned} \sum_{j_1+j_2=J} f(j_1, k_1) f(j_2, 0) &= f(J, k_1 + 0 + 1) \\ \sum_{j_1=0}^J f(j_1, k_1) &= f(J, k_1 + 1) \end{aligned}$$

3.  $J = J' + 1, k_2 = k'_2 + 1$ : Assume

$$\begin{aligned} \sum_{j_1+j_2=J'} f(j_1, k_1) f(j_2, k_2) &= f(J', k_1 + k_2 + 1), \\ \sum_{j_1+j_2=J} f(j_1, k_1) f(j_2, k'_2) &= f(J, k_1 + k'_2 + 1). \end{aligned}$$

Then

$$\begin{aligned}
& \sum_{j_1+j_2=J} f(j_1, k_1) f(j_2, k_2) \\
&= \sum_{j_1=0}^J f(j_1, k_1) f(J-j_1, k_2) \\
&= \sum_{j_1=0}^{J'} f(j_1, k_1) f(J-j_1, k_2) + f(J, k_1) \\
&= \sum_{j_1=0}^{J'} f(j_1, k_1) (f(J'-j_1, k_2) + f(J-j_1, k_2')) + f(J, k_1) \\
&= \sum_{j_1=0}^{J'} (f(j_1, k_1) f(J'-j_1, k_2) + f(j_1, k_1) f(J-j_1, k_2')) + f(J, k_1) \\
&= \sum_{j_1=0}^{J'} f(j_1, k_1) f(J'-j_1, k_2) + \sum_{j_1=0}^{J'} f(j_1, k_1) f(J-j_1, k_2') + f(J, k_1) \\
&= \sum_{j_1=0}^{J'} f(j_1, k_1) f(J'-j_1, k_2) + \sum_{j_1=0}^J f(j_1, k_1) f(J-j_1, k_2') \\
&= \sum_{j_1+j_2=J'} f(j_1, k_1) f(j_2, k_2) + \sum_{j_1+j_2=J} f(j_1, k_1) f(j_2, k_2') \\
&= f(J', k_1 + k_2 + 1) + f(J, k_1 + k_2' + 1) \\
&= f(J, k_1 + k_2 + 1).
\end{aligned}$$

□