Infinite lists

An ordinal is either 0 or an infinite list of ordinals written right to left, which is all 0 after some point $\gamma = (\ldots, 0, 0, \gamma_n, \ldots, \gamma_0)$, so $\gamma_k = 0$ for all $k > n$
For all i (including n and larger), $\gamma_i < (\ldots, \gamma_{i+3}, \gamma_{i+2}, \gamma_{i+1} + 1, 0, \ldots, 0)$

Finite lists

Leading 0s may be removed from infinite list and length of remaining list put in front: $(n : \gamma_{n-1}, \ldots, \gamma_0)$
Requires construction of naturals first, unlike infinite lists
Still lexicographic since lengths are compared first
No need for base case since list may be empty: could set $(0 :) = 0$, $(1 : (0 :)) = (1 : 0) = 1$, $(1 : 1) = 2$, etc.
This would offset naturals by 1 from infinite list representation, but otherwise identical

Ordinal-indexed lists

"Redundant" finite lists: place index of each element before it: $(n : \gamma_n, \ldots, \gamma_0) \to (n : \gamma_n, \ldots, 0 : \gamma_0)$
"Sparse" finite lists: remove all 0 entries from list (along with their indices),
turn all entries $\alpha \to -1 + \alpha$ to fill gap: $(2 : 1, 1 : 0, 0 : \omega) \to (2 : 0, 0 : \omega)$
Ordinal-indexed lists: allow indices to be ordinals themselves: $(\omega : 0) = \sup(\{(0 : 0), (1 : 0), (2 : 0), \ldots\})$
No longer requires naturals to be constructed first

Formal definition

Ordinal is a finite list of pairs $(\beta_n : \gamma_n, \beta_{n-1} : \gamma_{n-1}, \ldots, \beta_0 : \gamma_0)$, ordered lexicographically
Indices are strictly decreasing: $\beta_n > \beta_{n-1} > \ldots > \beta_0$
For all i, if $i < n$ and $\beta_{i+1} = \beta_i + 1$, then $\gamma_i < (\beta_n : \gamma_n, \ldots, \beta_{i+1} : \gamma_{i+1} + 1)$
Otherwise, $\gamma_i < (\beta_n : \gamma_n, \ldots, \beta_{i+1} : \gamma_{i+1}, \beta_i + 1 : 0)$

Examples

$$
\begin{aligned}
() &= 0 \\
(0:0) &= 1 \\
(0:1) &= 2 \\
(0:n) &= 1+n, && n < (1:0) \\
(1:0) &= \omega \\
(1:0, 0:\gamma) &= \omega + (1+\gamma), && \gamma < (1:1) \\
(1:1) &= \omega^2 \\
(1:\gamma) &= \omega^{1+\gamma}, && \gamma < (2:0) \\
(1:\gamma_1, 0:\gamma_0) &= \omega^{1+\gamma_1} + (1+\gamma_0), && \gamma_1 < (2:0), \gamma_0 < (1:\gamma_1+1) = \omega^{1+\gamma_1+1} \\
(2:0) &= \varepsilon_0 \\
(2:0, 1:\gamma_1) &= \varepsilon_0 \omega^{1+\gamma_1} && \gamma_1 < (2:1) \\
(2:0, 1:\gamma_1, 0:\gamma_0) &= \varepsilon_0 \omega^{1+\gamma_1} + (1+\gamma_0) && \gamma_1 < (2:1), \gamma_0 < (2:0, 1:\gamma_1+1) = \varepsilon_0 \omega^{1+\gamma_1+1} \\
(3:0) &= \zeta_0 \\
(\beta:\gamma) &= \varphi_{-1+\beta}(\gamma) && \beta \geq 2
\end{aligned}
$$

Relation to Veblen functions

The notation goes up to $\Gamma_0 = \varphi(1, 0, 0)$, since each lower-$\beta$ term essentially applies another two-variable Veblen function, except for the last, which is added to the result. It is closely related to Veblen normal form, but restricted to two arguments and preserving lexicographical order.

Further extensions

Allow colons to separate more entries than pairs

Veblen functions discard initial 0 arguments, but we want the restrictions on values to be only from above

As such, $(0 : 0 : 0) = \Gamma_0$

Lengths of lists within list representing a given ordinal should be in non-strictly decreasing order

Should also allow 1- and 0-entry colon-separated lists

$1 + \beta : \gamma \to \beta : \gamma$, $0 : 1 + \gamma \to \gamma$, $0 : 0 \to$