MLE for Poisson Process with Increasing Rate

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November 27, 2024

Observe a time range $[0, T)$, obtain events at times $\mathbf{t} = (t_1, \ldots, t_n)$, where $n \geq 1$, $0 < t_1 < \ldots < t_n < T$
Events are generated by Poisson process with rate function $\lambda : [0, T) \to [0, \infty)$
Log-likelihood of observation: $\ell(\lambda; \mathbf{t}) = \sum_{i=1}^{n} \ln(\lambda(t_i)) - \int_0^T \lambda(t)\ \mathrm{d}t$
If λ is unrestricted, can obtain arbitrarily high $\ell(\lambda; \mathbf{t})$ by concentrating value around the observations
Instead, assume λ is (non-strictly) increasing: how to maximize $\ell(\lambda; \mathbf{t})$ (equivalently, minimize $-\ell(\lambda; \mathbf{t})$)?
Given increasing λ, we may define increasing λ' as follows:

$$\lambda'(t) = \begin{cases} 0 & t \in [0, t_1) \\ \lambda(t_i) & t \in [t_i, t_{i+1}) \text{ for } 1 \leq i \leq n \text{ (convention: } t_{n+1} = T) \end{cases}$$

Intuition: replace the rate before the first event with 0, extend the rate at each event to the right until the next event
Then $\lambda'(t) \leq \lambda(t)$ for all t, so $\int_0^T \lambda'(t)\ \mathrm{d}t \leq \int_0^T \lambda(t)\ \mathrm{d}t$,
and $\lambda'(t_i) = \lambda(t_i)$ for all i, so $\sum_{i=1}^{n} \ln(\lambda'(t_i)) = \sum_{i=1}^{n} \ln(\lambda(t_i))$
Thus $\ell(\lambda'; \mathbf{t}) \geq \ell(\lambda; \mathbf{t})$, so the search space may be restricted to all such λ':
if the MLE among λ' does not exist, it does not exist in general, and if it does, it is the MLE in general
From here, λ is assumed to be of this form
λ may be parameterized by $\lambda_1 = \lambda(t_1), \ldots, \lambda_n = \lambda(t_n)$: $\lambda \in \mathbb{R}^n$

Optimization problem: minimize

$$-\ell(\lambda; \mathbf{t}) = \sum_{i=1}^{n} (t_{i+1} - t_i)\lambda_i - \sum_{i=1}^{n} \ln(\lambda_i)$$

subject to

$$0 < \lambda_1 \leq \ldots \leq \lambda_n$$

Slight generalization: introduce weights $x_1 < \ldots < x_{n+1}$, minimize

$$\sum_{i=1}^{n} (t_{i+1} - t_i)\lambda_i - \sum_{i=1}^{n} (x_{i+1} - x_i) \ln(\lambda_i)$$

with the same constraints on λ
To recover the original problem, take $x_i = i$
Def: $s_{j,k} = \frac{x_k - x_j}{t_k - t_j}$ for $j < k$
Intuition: represent the problem as points (t_i, x_i); $s_{j,k}$ is the slope of the segment between points j and k
Def: i is an *interior* point if there are $j < i$, $k > i$ s.t. $s_{j,i} > s_{i,k}$
Otherwise, i is a *hull* point
Intuition: hull points are part of the underside of the convex hull of all (t_i, x_i)
Ex: 1 and $n + 1$ are vacuously hull points

Lemma 1. *If there are no interior points, then the optimal values of λ are $\lambda_i = s_{i,i+1}$.*

Proof. Function to minimize may be rewritten

$$\sum_{i=1}^{n} \left((t_{i+1} - t_i)\lambda_i - (x_{i+1} - x_i)\ln(\lambda_i)\right),$$

so term i depends only on λ_i
Term i is minimized when $\lambda_i = \frac{x_{i+1}-x_i}{t_{i+1}-t_i} = s_{i,i+1}$
For all $i \in \{2, \ldots, n\}$, take $j = i - 1$, $k = i + 1$ in the definition of hull points:

$$\lambda_{i-1} = s_{i-1,i} \leq s_{i,i+1} = \lambda_i,$$

so the constraints are satisfied. □

Lemma 2. *If there is an interior point, then $s_{i-1,i} > s_{i,i+1}$ for some $i \in \{2, \ldots, n\}$.*

Proof. By contraposition, assume $s_{i-1,i} \leq s_{i,i+1}$ for all $i \in \{2, \ldots, n\}$, WTS every point i is a hull point
Let $j < i < k$, WTS $s_{j,i} \leq s_{i,k}$
Note

$$s_{j,i} = \frac{x_j - x_i}{t_j - t_i} = \frac{\sum_{l=j}^{i-1}(x_{l+1} - x_l)}{t_j - t_i} = \sum_{l=j}^{i-1} \frac{x_{l+1} - x_l}{t_j - t_i} = \sum_{l=j}^{i-1} \left(\frac{t_{l+1} - t_l}{t_j - t_i} \frac{x_{l+1} - x_l}{t_{l+1} - t_l}\right) = \sum_{l=j}^{i-1} \left(\frac{t_{l+1} - t_l}{t_j - t_i} s_{l,l+1}\right)$$

Sum of coefficients $\sum_{l=j}^{i-1} \frac{t_{l+1}-t_l}{t_j-t_i} = 1$ and each one is positive
Thus $s_{j,i}$ is a convex combination of $\{s_{l,l+1} \mid l \in \{j, \ldots, i-1\}\}$
So $s_{j,i} \leq \max\{s_{l,l+1} \mid l \in \{j, \ldots, i-1\}\} = s_{i-1,i}$
Similarly, $s_{i,k}$ is a convex combination of $\{s_{l,l+1} \mid l \in \{i, \ldots, k-1\}\}$
So $s_{i,k} \geq \min\{s_{l,l+1} \mid l \in \{i, \ldots, k-1\}\} = s_{i,i+1}$
Then $s_{j,i} \leq s_{i-1,i} \leq s_{i,i+1} \leq s_{i,k}$, so i is a hull point □

Theorem 1. *The optimal solution is $\lambda_i = s_{j,k}$, where j is the largest hull point $\leq i$ and k is the smallest hull point $> i$.*

Proof. Induct on the number of interior points:

- Base case: If there are no interior points, by Lemma 1, the global minimum $\lambda_i = s_{i,i+1}$ is inside the constraint region, so it is also the constrained minimum
  Also, since all i are hull points, $j = i$ and $k = i + 1$ are the correct values given i

- Inductive step: if there is an interior point, by Lemma 2, the global minimum $(s_{1,2}, \ldots, s_{n,n+1})$ is outside the constraint region
  Draw a line segment between the global min and the constrained min
  Where the segment intersects the boundary of the constraint region, $\lambda_{i-1} = \lambda_i$ for some i such that $s_{i-1,i} > s_{i,i+1}$ (so i is an interior point)
  This intersection is also a constrained minimum because the function being optimized is convex
  Thus we may restrict the search space to $\lambda_{i-1} = \lambda_i$, which is equivalent to removing the point (t_i, x_i) from the problem: the terms $(t_i - t_{i-1})\lambda_i + (t_{i+1} - t_i)\lambda_{i+1}$ become $(t_{i+1} - t_{i-1})\lambda_i$, and similarly for the x terms
  Removing this point does not change the set of hull points
  By the IH, the constrained minimum of the new problem in 1 fewer variable is given by the hull points as described in the theorem statement
  Thus, so is the minimum in the original problem

□