Simple Proof of Almost Complex Result

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July 24, 2024

WTS

$$\Pr(|X-Y| \le b) < \left(2\left\lceil\frac{b}{a}\right\rceil + 1\right)\Pr(|X-Y| \le a).$$

(Complex result has -1 instead of $+1$.) Define $n := \lceil \frac{b}{a} \rceil$, so $\frac{b}{a} \le n$, i.e. $b \le na$. Suffices to show

$$\Pr(|X-Y| \le na) < (2n+1)\Pr(|X-Y| \le a)$$

since $\Pr(|X-Y| \le b) \le \Pr(|X-Y| \le na)$. Define

$$p_z := \Pr(za \le X < (z+1)a) = \Pr(za \le Y < (z+1)a)$$

since $X \sim Y$. Then

$$\begin{aligned}
&(2n+1)\Pr(|X-Y| \le a) \\
\ge&(2n+1)\Pr\left(\bigvee_{z\in\mathbb{Z}} za \le X < (z+1)a \wedge za \le Y < (z+1)a\right) && \text{(plot region of } X,\, Y \text{ in } \mathbb{R}^2\text{)} \\
=&(2n+1)\sum_{z\in\mathbb{Z}}\Pr(za \le X < (z+1)a)\Pr(za \le Y < (z+1)a) && \text{(disjoint squares, } X \perp Y\text{)} \\
=&(2n+1)\sum_{z\in\mathbb{Z}} p_z^2 && \text{(def. of } p_z\text{)} \\
=&\sum_{z\in\mathbb{Z}} p_z^2 + \sum_{i=1}^{n} 2\sum_{z\in\mathbb{Z}} p_z^2 \\
=&\sum_{z\in\mathbb{Z}} p_z^2 + \sum_{i=1}^{n}\sum_{z\in\mathbb{Z}}(p_z^2 + p_{z+i}^2) && \text{(pair squares } i \text{ apart)} \\
>&\sum_{z\in\mathbb{Z}} p_z^2 + \sum_{i=1}^{n}\sum_{z\in\mathbb{Z}} 2p_z p_{z+i} && \text{(strict because } p_z \text{ not all equal)} \\
=&\sum_{z\in\mathbb{Z}} p_z^2 + \sum_{i=1}^{n}\left(\sum_{z\in\mathbb{Z}} p_z p_{z+i} + \sum_{z\in\mathbb{Z}} p_z p_{z-i}\right) && \text{(shift one copy by } -i\text{)} \\
=&\sum_{z\in\mathbb{Z}} p_z p_{z+0} + \sum_{i=1}^{n}\sum_{z\in\mathbb{Z}} p_z p_{z+i} + \sum_{i=-n}^{-1}\sum_{z\in\mathbb{Z}} p_z p_{z+i} \\
=&\sum_{i=-n}^{n}\sum_{z\in Z} p_z p_{z+i} \\
=&\sum_{i=-n}^{n}\sum_{z\in Z}\Pr(za \le X < (z+1)a)\Pr((z+i)a \le Y < (z+i+1)a) && \text{(def. of } p_z\text{)} \\
=&\Pr\left(\bigvee_{i=-n}^{n}\bigvee_{z\in Z} za \le X < (z+1)a \wedge (z+i)a \le Y < (z+i+1)a\right) && \text{(disjoint, } \perp\text{)} \\
\ge&\Pr(|X-Y| \le na). && \text{(plot region)}
\end{aligned}$$