Deriving a Trig Identity

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In acoustics, there is a phenomenon known as "beating," which occurs when two tones with similar frequencies are played together. Instead of two separate notes, we hear a single note whose frequency is the average of the given notes and which periodically becomes quieter and louder. The phenomenon may be explained by the following trigonometric identity:

$$\frac{1}{2}(\cos(ax) + \cos(bx)) = \cos\left(\frac{a+b}{2}x\right)\cos\left(\frac{a-b}{2}x\right)$$

The left side describes a waveform as a function of x (time) as an average of two waves with (angular) frequencies a and b. The right side, on the other hand, is a product of functions, the first of which has the average frequency of the two tones and the second of which describes the beating: if a and b are close, this function has a low frequency and may be heard not as a pitch, but as a gradual change in volume of the note. To avoid fractions, we may double a, b, and both sides of this identity to obtain:

$$\cos(2ax) + \cos(2bx) = 2\cos((a+b)x)\cos((a-b)x)$$

Since x only exists when multiplied by a or b, We may replace ax with a and bx with b to obtain the equivalent identity

$$\cos(2a) + \cos(2b) = 2\cos(a+b)\cos(a-b)$$

An analogous identity for sines also exists:

$$\sin(2a) + \sin(2b) = 2\sin(a+b)\cos(a-b)$$

Both may be derived simultaneously by interpreting them as the real and imaginary parts of a unified equation:

$$(\cos(2a) + \cos(2b)) + i(\sin(2a) + \sin(2b)) = 2\cos(a+b)\cos(a-b) + 2i\sin(a+b)\cos(a-b)$$

Multiple parts of the formula are now in the form $\cos(\theta) + i\sin(\theta)$ for some value θ, which is sometimes written $\mathrm{cis}(\theta)$ (<u>c</u>osine + <u>i</u> times <u>s</u>ine):

$$\mathrm{cis}(2a) + \mathrm{cis}(2b) = 2\mathrm{cis}(a+b)\cos(a-b)$$

The last cos, since it is multiplied by 2, can also be written in terms of cis:

$$2\cos(\theta) = \cos(\theta) + i\sin(\theta) + \cos(\theta) - i\sin(\theta) = \cos(\theta) + i\sin(\theta) + \cos(-\theta) + i\sin(-\theta) = \mathrm{cis}(\theta) + \mathrm{cis}(-\theta)$$

Making this substitution,

$$\mathrm{cis}(2a) + \mathrm{cis}(2b) = \mathrm{cis}(a+b)(\mathrm{cis}(a-b) + \mathrm{cis}(b-a))$$

Finally, $\mathrm{cis}(\theta) = e^{i\theta}$, so

$$e^{2ia} + e^{2ib} = e^{i(a+b)}(e^{i(a-b)} + e^{i(b-a)}),$$

which is true by algebra.

Envelope of $c\cos(ax) + d\cos(bx)$: $\sqrt{c^2 + 2cd\cos((a-b)x) + d^2}$

$$(c\cos(a) + d\cos(b))^2 = c^2\cos^2(a) + 2cd\cos(a)\cos(b) + d^2\cos^2(b)$$
$$(c\cos(a) + d\cos(b))^2 = c^2\cos^2(a) + 2cd\cos(a)\cos(b) + d^2\cos^2(b)$$