A Generating Function Problem

Aresh Pourkavoos

April 8, 2022

Problem:

$$f(x,0) = \frac{e^x - 1}{x}$$
$$f(x,y) = \frac{\partial f}{\partial x}(x,y) + \frac{\partial f}{\partial y}(x,y)$$

Solution: Define

$$A(j,k) = \frac{\partial^{j+k} f}{\partial x^j \partial y^k}(0,0)$$

Get Taylor series of base case:

$$f(x,0) = \frac{1}{x}\left(\sum_{i=0}^{\infty} \frac{x^i}{i!} - 1\right) = \frac{1}{x}\left(\sum_{i=1}^{\infty} \frac{x^i}{i!}\right) = \frac{1}{x}\left(\sum_{i=0}^{\infty} \frac{x^{i+1}}{(i+1)!}\right) = \sum_{i=0}^{\infty} \frac{x^i}{(i+1)!}$$

Find partial derivatives wrt x:

$$A(j,0) = \frac{\partial^j f}{\partial x^j} \sum_{i=0}^{\infty} \frac{x^i}{(i+1)!} = \frac{\partial^j f}{\partial x^j} \frac{x^j}{(j+1)!} = \frac{j!}{(j+1)!} = \frac{1}{j+1}$$

Use diffeq to establish recurrence relation on A:

$$\begin{aligned}
A(j,k) &= \frac{\partial^{j+k} f}{\partial x^j \partial y^k}(0,0) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial^{j+k} f}{\partial x^j \partial y^k}\right)(0,0) + \frac{\partial}{\partial y}\left(\frac{\partial^{j+k} f}{\partial x^j \partial y^k}\right)(0,0) \\
&= \frac{\partial^{j+k+1} f}{\partial x^{j+1} \partial y^k}(0,0) + \frac{\partial^{j+k+1} f}{\partial x^j \partial y^{k+1}}(0,0) \\
&= A(j+1,k) + A(j,k+1)
\end{aligned}$$

Rearrange to compute higher values of k:

$$A(j,k+1) = A(j,k) - A(j+1,k)$$

These are sufficient to determine all $A(j,k)$, which may be found by computation and guess-and-check:

$$A(j,k) = \frac{j!k!}{(j+k+1)!} = B(j+1,k+1)$$

where B is the beta function. Construct 2D Taylor series:

$$
\begin{aligned}
f(x,y) &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} A(j,k)\frac{x^j y^k}{j!k!} \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{j!k!}{(j+k+1)!}\frac{x^j y^k}{j!k!} \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{x^j y^k}{(j+k+1)!} \\
&= \sum_{n=0}^{\infty}\sum_{j+k=n} \frac{x^j y^k}{(j+k+1)!} \\
&= \sum_{n=0}^{\infty}\frac{1}{(n+1)!}\sum_{j+k=n} x^j y^k \\
&= \sum_{n=0}^{\infty}\frac{1}{(n+1)!}\frac{x^{n+1}-y^{n+1}}{x-y} \\
&= \frac{1}{x-y}\left(\sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)!} - \sum_{n=0}^{\infty}\frac{y^{n+1}}{(n+1)!}\right) \\
&= \frac{1}{x-y}((e^x-1)-(e^y-1)) \\
&= \frac{e^x-e^y}{x-y}
\end{aligned}
$$

Check answer:

$$
f(x,0) = \frac{e^x - e^0}{x-0} = \frac{e^x-1}{x}
$$

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{e^x(x-y)-(e^x-e^y)}{(x-y)^2} \\
\frac{\partial f}{\partial y} &= \frac{-e^y(x-y)+(e^x-e^y)}{(x-y)^2} \\
\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} &= \frac{(e^x-e^y)(x-y)}{(x-y)^2} \\
&= \frac{e^x-e^y}{(x-y)} \\
&= f(x,y)
\end{aligned}
$$

3D variant:

$$A(j,k,l) = \frac{j!k!l!}{(j+k+l+2)!} = B(j+1,k+1,l+1)$$

$$\begin{aligned}
f(x,y,z) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{x^j y^k z^l}{(j+k+l+2)!} \\
&= \sum_{n=0}^{\infty} \sum\sum_{j+k+l=n} \frac{x^j y^k z^l}{(j+k+l+2)!} \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+2)!} \sum\sum_{j+k+l=n} x^j y^k z^l \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+2)!} \frac{(y-z)x^{n+2} + (z-x)y^{n+2} + (x-y)z^{n+2}}{(z-y)(x-z)(y-x)} \\
&= \frac{1}{(z-y)(x-z)(y-x)} \left((y-z)\sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} + (z-x)\sum_{n=0}^{\infty} \frac{y^{n+2}}{(n+2)!} + (x-y)\sum_{n=0}^{\infty} \frac{z^{n+2}}{(n+2)!} \right) \\
&= \frac{1}{(z-y)(x-z)(y-x)} ((y-z)(e^x - 1 - x) + (z-x)(e^y - 1 - y) + (x-y)(e^z - 1 - z)) \\
&= \frac{e^x(y-z) + e^y(z-x) + e^z(x-y)}{(z-y)(x-z)(y-x)} \\
&= -\frac{e^x}{(x-z)(y-x)} - \frac{e^y}{(z-y)(y-x)} - \frac{e^z}{(z-y)(x-z)} \\
&= \frac{e^x}{(x-y)(x-z)} + \frac{e^y}{(y-x)(y-z)} + \frac{e^z}{(z-x)(z-y)}
\end{aligned}$$