Extending Pascal's Triangle

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Pascal's triangle contains binomial coefficients:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}, n \in \mathbb{N}, k \in \mathbb{N}, k \leq n$$

Satisfies Pascal's identity for all arguments in range:

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$$

How to extend to all integers, i.e. fill the grid with numbers?
Entries where $n \geq 0$ are unambiguous: all numbers outside of triangle are 0
Factorials of negative integers are undefined, so the usual binomial formula cannot be used
$n = -1$: can choose a single number freely, after which the rest of the row is forced
Same goes for all remaining negative rows
Negative rows are thus uniquely determined by $\binom{n}{0}$ for all $n < 0$
Setting $\binom{n}{0} = 1$ creates "rotated" Pascal's triangle with sign changes: Taylor polynomials of $(1+x)^n$
Rows $n = 0$ through $n = -9$, inclusive:

	0		0		0		0		0		1		0		0		0		0		0	
0		0		0		0		0		0		1		-1		1		-1		1		-1
	0		0		0		0		0		0		1		-2		3		-4		5	
0		0		0		0		0		0		0		1		-3		6		-10		15
	0		0		0		0		0		0		0		1		-4		10		-20	
0		0		0		0		0		0		0		0		1		-5		15		-35
	0		0		0		0		0		0		0		0		1		-6		21	
0		0		0		0		0		0		0		0		0		1		-7		28
	0		0		0		0		0		0		0		0		0		1		-8	
0		0		0		0		0		0		0		0		0		0		1		-9

Setting $\binom{n}{0} = \frac{1}{n+1}$ gives the beta function for $k \in [n, 0]$: symmetric along same axis as Pascal's triangle
However, values outside the range of the beta function are also given

$\frac{\circ}{1!}$		0		0		0		0		0		1		0		0		0		0		0	
$\frac{\circ}{2!}$	-1		1		-1		1		-1		1		1		-1		1		-1		1		-1
$\frac{\circ}{3!}$		14		-11		8		-5		2		1		2		-5		8		-11		14	
$\frac{\circ}{4!}$	158		-102		58		-26		6		2		2		6		-26		58		-102		158
$\frac{\circ}{5!}$		-954		444		-154		24		6		4		6		24		-154		444		-954	
$\frac{\circ}{6!}$	-9432		3708		-1044		120		24		12		12		24		120		-1044		3708		-9432
$\frac{\circ}{7!}$		33984		-8028		720		120		48		36		48		120		720		-8028		33984	
$\frac{\circ}{8!}$	341136		-69264		5040		720		240		144		144		240		720		5040		-69264		341136
$\frac{\circ}{9!}$		-663696		40320		5040		1440		720		576		720		1440		5040		40320		-663696	
$\frac{\circ}{10!}$	-6999840		362880		40320		10080		4320		144		144		4320		10080		40320		362880		-6999840