A Generating Function Problem

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Problem:

$$f(x, 0) = \frac{e^x - 1}{x}$$

$$f(x, y) = \frac{\partial f}{\partial x}(x, y) + \frac{\partial f}{\partial y}(x, y)$$

Solution: Define

$$A(j, k) = \frac{\partial^{j+k} f}{\partial x^j \partial y^k}(0, 0)$$

Get Taylor series of base case:

$$f(x, 0) = \frac{1}{x}\left(\sum_{i=0}^{\infty} \frac{x^i}{i!} - 1\right) = \frac{1}{x}\left(\sum_{i=1}^{\infty} \frac{x^i}{i!}\right) = \frac{1}{x}\left(\sum_{i=0}^{\infty} \frac{x^{i+1}}{(i+1)!}\right) = \sum_{i=0}^{\infty} \frac{x^i}{(i+1)!}$$

Find partial derivatives wrt x:

$$A(j, 0) = \frac{\partial^j f}{\partial x^j} \sum_{i=0}^{\infty} \frac{x^i}{(i+1)!} = \frac{\partial^j f}{\partial x^j} \frac{x^j}{(j+1)!} = \frac{j!}{(j+1)!} = \frac{1}{j+1}$$

Use diffeq to establish recurrence relation on A:

$$\begin{aligned}
A(j, k) &= \frac{\partial^{j+k} f}{\partial x^j \partial y^k}(0, 0) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial^{j+k} f}{\partial x^j \partial y^k}\right)(0, 0) + \frac{\partial}{\partial y}\left(\frac{\partial^{j+k} f}{\partial x^j \partial y^k}\right)(0, 0) \\
&= \frac{\partial^{j+k+1} f}{\partial x^{j+1} \partial y^k}(0, 0) + \frac{\partial^{j+k+1} f}{\partial x^j \partial y^{k+1}}(0, 0) \\
&= A(j+1, k) + A(j, k+1)
\end{aligned}$$

Rearrange to compute higher values of k:

$$A(j, k+1) = A(j, k) - A(j+1, k)$$

These are sufficient to determine all $A(j, k)$, which may be found by computation and guess-and-check:

$$A(j, k) = \frac{j!k!}{(j+k+1)!} = B(j+1, k+1)$$

where B is the beta function. Construct 2D Taylor series:

$$f(x,y) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} A(j,k) \frac{x^j y^k}{j!k!}$$

$$= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{j!k!}{(j+k+1)!} \frac{x^j y^k}{j!k!}$$

$$= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{x^j y^k}{(j+k+1)!}$$

$$= \sum_{n=0}^{\infty} \sum_{j+k=n} \frac{x^j y^k}{(j+k+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \sum_{j+k=n} x^j y^k$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \frac{x^{n+1} - y^{n+1}}{x-y}$$

$$= \frac{1}{x-y} \left(\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} - \sum_{n=0}^{\infty} \frac{y^{n+1}}{(n+1)!} \right)$$

$$= \frac{1}{x-y} ((e^x - 1) - (e^y - 1))$$

$$= \frac{e^x - e^y}{x-y}$$

Check answer:

$$f(x,0) = \frac{e^x - e^0}{x-0} = \frac{e^x - 1}{x}$$

$$\frac{\partial f}{\partial x} = \frac{e^x(x-y) - (e^x - e^y)}{(x-y)^2}$$

$$\frac{\partial f}{\partial y} = \frac{-e^y(x-y) + (e^x - e^y)}{(x-y)^2}$$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = \frac{(e^x - e^y)(x-y)}{(x-y)^2}$$

$$= \frac{e^x - e^y}{(x-y)}$$

$$= f(x,y)$$

3D variant:
$$A(j, k, l) = \frac{j!k!l!}{(j + k + l + 2)!} = B(j + 1, k + 1, l + 1) = \frac{\partial^{j+k+l} f}{\partial x^j \partial y^k \partial z^l}$$

Diffeq:
$$f(x, y, z) = \frac{\partial f}{\partial x}(x, y, z) + \frac{\partial f}{\partial y}(x, y, z) + \frac{\partial f}{\partial z}(x, y, z)$$

Solution:

$$
\begin{aligned}
f(x, y, z) &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{x^j y^k z^l}{(j + k + l + 2)!} \\
&= \sum_{n=0}^{\infty}\sum_{j+k+l=n} \frac{x^j y^k z^l}{(j + k + l + 2)!} \\
&= \sum_{n=0}^{\infty} \frac{1}{(n + 2)!} \sum_{j+k+l=n} x^j y^k z^l \\
&= \sum_{n=0}^{\infty} \frac{1}{(n + 2)!} \frac{(y - z)x^{n+2} + (z - x)y^{n+2} + (x - y)z^{n+2}}{(z - y)(x - z)(y - x)} \\
&= \frac{1}{(z - y)(x - z)(y - x)}\left((y - z)\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!} + (z - x)\sum_{n=0}^{\infty}\frac{y^{n+2}}{(n+2)!} + (x - y)\sum_{n=0}^{\infty}\frac{z^{n+2}}{(n+2)!}\right) \\
&= \frac{1}{(z - y)(x - z)(y - x)}((y - z)(e^x - 1 - x) + (z - x)(e^y - 1 - y) + (x - y)(e^z - 1 - z)) \\
&= \frac{e^x(y - z) + e^y(z - x) + e^z(x - y)}{(z - y)(x - z)(y - x)} \\
&= -\frac{e^x}{(x - z)(y - x)} - \frac{e^y}{(z - y)(y - x)} - \frac{e^z}{(z - y)(x - z)} \\
&= \frac{e^x}{(x - y)(x - z)} + \frac{e^y}{(y - x)(y - z)} + \frac{e^z}{(z - x)(z - y)}
\end{aligned}
$$

Check answer:

$$
\begin{aligned}
\frac{\partial f}{\partial x}(x, y, z) &= \frac{\partial}{\partial x}\left(\frac{e^x}{(x - y)(x - z)} + \frac{e^y}{(y - x)(y - z)} + \frac{e^z}{(z - x)(z - y)}\right) \\
&= \frac{\partial}{\partial x}\frac{e^x}{(x - y)(x - z)} + \frac{\partial}{\partial x}\frac{e^y}{(y - x)(y - z)} + \frac{\partial}{\partial x}\frac{e^z}{(z - x)(z - y)} \\
&= \frac{\left(\frac{\partial}{\partial x}e^x\right)(x - y)(x - z) - e^x\left(\frac{\partial}{\partial x}((x - y)(x - z))\right)}{(x - y)^2(x - z)^2 w} + \frac{e^y}{(y - x)^2(y - z)} + \frac{e^z}{(z - x)^2(z - y)} \\
&= \frac{e^x(x - y)(x - z) - e^x((x - y) + (x - z))}{(x - y)^2(x - z)^2} + \frac{e^y}{(y - x)^2(y - z)} + \frac{e^z}{(z - x)^2(z - y)} \\
&= \frac{e^x}{(x - y)(x - z)} - \frac{e^x}{(x - y)(x - z)^2} - \frac{e^x}{(x - y)^2(x - z)} + \frac{e^y}{(y - x)^2(y - z)} + \frac{e^z}{(z - x)^2(z - y)} \\
\frac{\partial f}{\partial y}(x, y, z) &= \frac{e^x}{(x - y)^2(x - z)} + \frac{e^y}{(y - x)(y - z)} - \frac{e^y}{(y - x)(y - z)^2} - \frac{e^y}{(y - x)^2(y - z)} + \frac{e^z}{(z - x)(z - y)^2} \\
\frac{\partial f}{\partial z}(x, y, z) &= \frac{e^x}{(x - y)(x - z)^2} + \frac{e^y}{(y - x)(y - z)^2} + \frac{e^z}{(z - x)(z - y)} - \frac{e^z}{(z - x)(z - y)^2} - \frac{e^z}{(z - x)^2(z - y)}
\end{aligned}
$$

Each term with quadratic denominator only appears once across all partials
Each term with cubic denominator appears twice across all partials, once positive and once negative
When added, cubic terms cancel and quadratics remain to form $f(x, y, z)$

ND solution:

$$f(x_1, \ldots, x_N) = \sum_{I=1}^{N} \frac{e^{x_I}}{\prod_{J \neq I} (x_I - x_J)}$$

$I \neq K$: Define

$$T(I, K) = \frac{\partial}{\partial x_K} \frac{e^{x_I}}{\prod_{J \neq I} (x_I - x_J)} = \frac{e^{x_I}}{(x_I - x_K) \prod_{J \neq I} (x_I - x_J)}$$

Partial derivative of remaining terms ($I = K$):

$$
\begin{aligned}
\frac{\partial}{\partial x_K} \frac{e^{x_K}}{\prod_{J \neq K} (x_K - x_J)} &= \frac{\frac{\partial}{\partial x_K} e^{x_K}}{\prod_{J \neq K} (x_K - x_J)} - \frac{e^{x_K} \frac{\partial}{\partial x_K} \prod_{J \neq K} (x_K - x_J)}{\prod_{J \neq K} (x_K - x_J)^2} \\
&= \frac{e^{x_K}}{\prod_{J \neq K} (x_K - x_J)} - \frac{e^{x_K} \sum_{L \neq K} \prod_{J \notin \{K, L\}} (x_K - x_J)}{\prod_{J \neq K} (x_K - x_J)^2} \\
&= \frac{e^{x_K}}{\prod_{J \neq K} (x_K - x_J)} - \sum_{L \neq K} \frac{e^{x_K}}{(x_K - x_L) \prod_{J \neq K} (x_K - x_J)} \\
&= \frac{e^{x_K}}{\prod_{J \neq K} (x_K - x_J)} - \sum_{L \neq K} T(K, L)
\end{aligned}
$$

Overall derivative for given x_K:

$$
\begin{aligned}
\frac{\partial f}{\partial x_K}(x_1, \ldots, x_N) &= \frac{\partial}{\partial x_K} \sum_{I=1}^{N} \frac{e^{x_I}}{\prod_{J \neq I} (x_I - x_J)} \\
&= \frac{\partial}{\partial x_K} \frac{e^{x_K}}{\prod_{J \neq K} (x_K - x_J)} + \sum_{I \neq K} \frac{\partial}{\partial x_K} \frac{e^{x_I}}{\prod_{J \neq I} (x_I - x_J)} \\
&= \frac{e^{x_K}}{\prod_{J \neq K} (x_K - x_J)} - \sum_{L \neq K} T(K, L) + \sum_{I \neq K} T(I, K)
\end{aligned}
$$

Sum of all partials:

$$
\begin{aligned}
\sum_{K=1}^{N} \frac{\partial f}{\partial x_K}(x_1, \ldots, x_N) &= \sum_{K=1}^{N} \left(\frac{e^{x_K}}{\prod_{J \neq K} (x_K - x_J)} - \sum_{L \neq K} T(K, L) + \sum_{I \neq K} T(I, K) \right) \\
&= \sum_{K=1}^{N} \frac{e^{x_K}}{\prod_{J \neq K} (x_K - x_J)} - \sum_{K=1}^{N} \sum_{L \neq K} T(K, L) + \sum_{K=1}^{N} \sum_{I \neq K} T(I, K) \\
&= \sum_{K=1}^{N} \frac{e^{x_K}}{\prod_{J \neq K} (x_K - x_J)} \\
&= f(x_1, \ldots, x_N)
\end{aligned}
$$

Remains to be shown that derivative is the multivariate beta function:

$$\frac{\partial^{k_1 + \ldots + k_K} f}{\partial x_1^{k_1} \ldots \partial x_N^{k_n}} = \frac{k_1! \ldots k_N!}{(k_1 + \ldots + k_N + N - 1)!}$$

Lemma 1:

$$\prod_{1 \leq i < j \leq n} (x_j - x_i) = \sum_{\sigma \in S_n} \left(\mathrm{sgn}(\sigma) \prod_{k=1}^{n} x_k^{\sigma_k - 1} \right),$$

where S_n is the set of permutations of the sequence $(1, \ldots, n)$. Ex for $n = 3$:

$$(x_2 - x_1)(x_3 - x_1)(x_3 - x_2) = x_1^0 x_2^1 x_3^2 - x_1^0 x_2^2 x_3^1 - x_1^1 x_2^0 x_3^2 + x_1^1 x_2^2 x_3^0 - x_1^2 x_2^0 x_3^1 + x_1^2 x_2^1 x_3^0$$
$$= x_2 x_3^2 - x_2^2 x_3 - x_1 x_3^2 + x_1 x_2^2 - x_1^2 x_3 + x_1^2 x_2$$

Proof (sketch): Setting $x_i = x_j$ for any $1 \leq i < j \leq n$ makes the RHS 0 because the permutations may be placed into canceling pairs where σ_i and σ_j switch values. The product is the same for each, but the sign is flipped. Thus the RHS is divisible by $x_j - x_i$. Since $x_j - x_i$ share no factors with each other, the RHS is divisible by their product, i.e. the LHS. Since the LHS and RHS are both of degree $\binom{n}{2}$, they can only differ by a constant factor. On the RHS, the identity permutation has the term $\prod_{k=1}^{n} x_k^k$ with coefficient 1 (since the identity permutation is even).