

A Generating Function Problem

Aresh Pourkavoos

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Problem:

$$f(x, 0) = \frac{e^x - 1}{x}$$
$$f(x, y) = \frac{\partial f}{\partial x}(x, y) + \frac{\partial f}{\partial y}(x, y)$$

Solution: Define

$$A(j, k) = \frac{\partial^{j+k} f}{\partial x^j \partial y^k}(0, 0)$$

Get Taylor series of base case:

$$f(x, 0) = \frac{1}{x} \left(\sum_{i=0}^{\infty} \frac{x^i}{i!} - 1 \right) = \frac{1}{x} \left(\sum_{i=1}^{\infty} \frac{x^i}{i!} \right) = \frac{1}{x} \left(\sum_{i=0}^{\infty} \frac{x^{i+1}}{(i+1)!} \right) = \sum_{i=0}^{\infty} \frac{x^i}{(i+1)!}$$

Find partial derivatives wrt x :

$$A(j, 0) = \frac{\partial^j f}{\partial x^j} \sum_{i=0}^{\infty} \frac{x^i}{(i+1)!} = \frac{\partial^j f}{\partial x^j} \frac{x^j}{(j+1)!} = \frac{j!}{(j+1)!} = \frac{1}{j+1}$$

Use diffeq to establish recurrence relation on A :

$$\begin{aligned} A(j, k) &= \frac{\partial^{j+k} f}{\partial x^j \partial y^k}(0, 0) \\ &= \frac{\partial}{\partial x} \left(\frac{\partial^{j+k} f}{\partial x^j \partial y^k} \right)(0, 0) + \frac{\partial}{\partial y} \left(\frac{\partial^{j+k} f}{\partial x^j \partial y^k} \right)(0, 0) \\ &= \frac{\partial^{j+k+1} f}{\partial x^{j+1} \partial y^k}(0, 0) + \frac{\partial^{j+k+1} f}{\partial x^j \partial y^{k+1}}(0, 0) \\ &= A(j+1, k) + A(j, k+1) \end{aligned}$$

Rearrange to compute higher values of k :

$$A(j, k+1) = A(j, k) - A(j+1, k)$$

These are sufficient to determine all $A(j, k)$, which may be found by computation and guess-and-check:

$$A(j, k) = \frac{j!k!}{(j+k+1)!} = B(j+1, k+1)$$

where B is the beta function. Construct 2D Taylor series:

$$\begin{aligned}
f(x, y) &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} A(j, k) \frac{x^j y^k}{j! k!} \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{j! k!}{(j+k+1)!} \frac{x^j y^k}{j! k!} \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{x^j y^k}{(j+k+1)!} \\
&= \sum_{n=0}^{\infty} \sum_{j+k=n} \frac{x^j y^k}{(j+k+1)!} \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \sum_{j+k=n} x^j y^k \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \frac{x^{n+1} - y^{n+1}}{x - y} \\
&= \frac{1}{x - y} \left(\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} - \sum_{n=0}^{\infty} \frac{y^{n+1}}{(n+1)!} \right) \\
&= \frac{1}{x - y} ((e^x - 1) - (e^y - 1)) \\
&= \frac{e^x - e^y}{x - y}
\end{aligned}$$

Check answer:

$$\begin{aligned}
f(x, 0) &= \frac{e^x - e^0}{x - 0} = \frac{e^x - 1}{x} \\
\frac{\partial f}{\partial x} &= \frac{e^x(x - y) - (e^x - e^y)}{(x - y)^2} \\
\frac{\partial f}{\partial y} &= \frac{-e^y(x - y) + (e^x - e^y)}{(x - y)^2} \\
\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} &= \frac{(e^x - e^y)(x - y)}{(x - y)^2} \\
&= \frac{e^x - e^y}{(x - y)} \\
&= f(x, y)
\end{aligned}$$

3D variant:

$$A(j, k, l) = \frac{j!k!l!}{(j+k+l+2)!} = B(j+1, k+1, l+1)$$

Diffeq:

$$f(x, y, z) = \frac{\partial f}{\partial x}(x, y, z) + \frac{\partial f}{\partial y}(x, y, z) + \frac{\partial f}{\partial z}(x, y, z)$$

Solution:

$$\begin{aligned} f(x, y, z) &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{x^j y^k z^l}{(j+k+l+2)!} \\ &= \sum_{n=0}^{\infty} \sum_{j+k+l=n} \frac{x^j y^k z^l}{(j+k+l+2)!} \\ &= \sum_{n=0}^{\infty} \frac{1}{(n+2)!} \sum_{j+k+l=n} x^j y^k z^l \\ &= \sum_{n=0}^{\infty} \frac{1}{(n+2)!} \frac{(y-z)x^{n+2} + (z-x)y^{n+2} + (x-y)z^{n+2}}{(z-y)(x-z)(y-x)} \\ &= \frac{1}{(z-y)(x-z)(y-x)} \left((y-z) \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} + (z-x) \sum_{n=0}^{\infty} \frac{y^{n+2}}{(n+2)!} + (x-y) \sum_{n=0}^{\infty} \frac{z^{n+2}}{(n+2)!} \right) \\ &= \frac{1}{(z-y)(x-z)(y-x)} ((y-z)(e^x - 1 - x) + (z-x)(e^y - 1 - y) + (x-y)(e^z - 1 - z)) \\ &= \frac{e^x(y-z) + e^y(z-x) + e^z(x-y)}{(z-y)(x-z)(y-x)} \\ &= -\frac{e^x}{(x-z)(y-x)} - \frac{e^y}{(z-y)(y-x)} - \frac{e^z}{(z-y)(x-z)} \\ &= \frac{e^x}{(x-y)(x-z)} + \frac{e^y}{(y-x)(y-z)} + \frac{e^z}{(z-x)(z-y)} \end{aligned}$$

Check answer:

$$\begin{aligned} \frac{\partial f}{\partial x}(x, y, z) &= \frac{\partial}{\partial x} \left(\frac{e^x}{(x-y)(x-z)} + \frac{e^y}{(y-x)(y-z)} + \frac{e^z}{(z-x)(z-y)} \right) \\ &= \frac{\partial}{\partial x} \frac{e^x}{(x-y)(x-z)} + \frac{\partial}{\partial x} \frac{e^y}{(y-x)(y-z)} + \frac{\partial}{\partial x} \frac{e^z}{(z-x)(z-y)} \\ &= \frac{\left(\frac{\partial}{\partial x} e^x \right) (x-y)(x-z) - e^x \left(\frac{\partial}{\partial x} ((x-y)(x-z)) \right)}{(x-y)^2(x-z)^2} + \frac{e^y}{(y-x)^2(y-z)} + \frac{e^z}{(z-x)^2(z-y)} \\ &= \frac{e^x(x-y)(x-z) - e^x((x-y) + (x-z))}{(x-y)^2(x-z)^2} + \frac{e^y}{(y-x)^2(y-z)} + \frac{e^z}{(z-x)^2(z-y)} \\ &= \frac{e^x}{(x-y)(x-z)} - \frac{e^x}{(x-y)(x-z)^2} - \frac{e^x}{(x-y)^2(x-z)} + \frac{e^y}{(y-x)^2(y-z)} + \frac{e^z}{(z-x)^2(z-y)} \\ \frac{\partial f}{\partial y}(x, y, z) &= \frac{e^x}{(x-y)^2(x-z)} + \frac{e^y}{(y-x)(y-z)} - \frac{e^y}{(y-x)(y-z)^2} - \frac{e^y}{(y-x)^2(y-z)} + \frac{e^z}{(z-x)(z-y)^2} \\ \frac{\partial f}{\partial z}(x, y, z) &= \frac{e^x}{(x-y)(x-z)^2} + \frac{e^y}{(y-x)(y-z)^2} + \frac{e^z}{(z-x)(z-y)} - \frac{e^z}{(z-x)(z-y)^2} - \frac{e^z}{(z-x)^2(z-y)} \end{aligned}$$

Each term with quadratic denominator only appears once across all partials Each term with cubic denominator appears twice across all partials, once positive and once negative

When added, cubic terms cancel and quadratics remain to form $f(x, y, z)$

ND solution(?):

$$f(x_1, x_2, \dots, x_N) = \frac{e^{x_1}}{(x_1 - x_2) \cdots (x_1 - x_N)} + \frac{e^{x_2}}{(x_2 - x_1) \cdots (x_2 - x_N)} + \cdots + \frac{e^{x_N}}{(x_N - x_1)(x_N - x_2) \cdots}$$

Partial derivative:

$$\begin{aligned} \frac{\partial f}{\partial x_1}(x_1, x_2, \dots, x_N) &= \frac{e^{x_1} \frac{\partial f}{\partial x_1}((x_1 - x_2) \cdots (x_1 - x_N)) - e^{x_1}(x_1 - x_2) \cdots (x_1 - x_N)}{(x_1 - x_2)^2 \cdots (x_1 - x_N)^2} + \frac{e^{x_2}}{(x_2 - x_1)^2 \cdots (x_2 - x_N)} + \cdots \\ &\quad + \frac{e^{x_N}}{(x_N - x_1)^2 (x_N - x_2) \cdots} \end{aligned}$$