SPOTD 18

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Theorem. *For* $x \in (0,1)$, *define*

$$f(x) := \sum_{i=0}^{\infty} (-1)^i x^{2^i}.$$

Then $\lim_{x \to 1^-} f(x)$ *does not exist.*

Proof. First, for $n \in \mathbb{N}$, define

$$x_n := 2^{-2^{-n}}.$$

Then $x_n < (0,1)$, so we may also define $y_n := f(x_n)$. Also, $\lim_{n\to\infty} x_n = 1$, so it suffices to show that $\lim_{n\to\infty} y_n$ does not exist. Note

$$\begin{aligned}
y_n &= \sum_{i=0}^{\infty} (-1)^i x_n^{2^i} \\
&= \sum_{i=0}^{\infty} (-1)^i \left(2^{-2^{-n}}\right)^{2^i} \\
&= \sum_{i=0}^{\infty} (-1)^i 2^{(-2^{-n})(2^i)} \\
&= \sum_{i=0}^{\infty} (-1)^i 2^{-2^{i-n}} \\
&= \sum_{i=-n}^{\infty} (-1)^{i+n} 2^{-2^i}.
\end{aligned}$$

It suffices to show that $\lim_{k\to\infty} y_{2k} < \frac{1}{2} < \lim_{k\to\infty} y_{2k+1}$ (if both exist).

By the previous,

$$\begin{aligned}
y_{2k} &= \sum_{i=-2k}^{\infty} (-1)^i 2^{-2^i} \\
&= \sum_{i=-2k}^{-1} (-1)^i 2^{-2^i} + \sum_{i=0}^{\infty} (-1)^i 2^{-2^i} \\
&= \sum_{i=1}^{2k} (-1)^i 2^{-2^{-i}} + y_0 \\
&= \sum_{i=1}^{2k} (-1)^i x_i + y_0 \\
&= \sum_{i=1}^{k} (x_{2i} - x_{2i-1}) + y_0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
y_{2k+1} &= \sum_{i=-(2k+1)}^{\infty} (-1)^{i+1} 2^{-2^i} \\
&= \sum_{i=-(2k+1)}^{-2} (-1)^{i+1} 2^{-2^i} + \sum_{i=-1}^{\infty} (-1)^{i+1} 2^{-2^i} \\
&= \sum_{i=2}^{2k+1} (-1)^{i+1} 2^{-2^{-i}} + y_1 \\
&= \sum_{i=2}^{2k+1} (-1)^{i+1} x_i + y_1 \\
&= \sum_{i=1}^{k} (x_{2i+1} - x_{2i}) + y_1.
\end{aligned}$$

Since $x_{n+1} - x_n > 0$ for all n, $(y_{2k})_k$ and $(y_{2k+1})_k$ are both (strictly) increasing.

First, we show that $\lim_{k\to\infty} y_{2k+1} > \frac{1}{2}$. Since the sequence is increasing, it suffices to find such a k. (In fact, this alone shows that $\lim_{x\to 1^-} f(x) \neq \frac{1}{2}$, per the original problem.)

Take $k = 2$, so $y_5 = y_{2k+1} =$

$\square$