MC Estimation of a Downward Closed Set of Subsets

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$V = \{v_1, \ldots, v_n\}$ finite set, T predicate on subsets of V
$A \subseteq B, T(B) \to T(A)$ (T downward closed)
Notation:

- $\binom{V}{k} := \{B \subseteq V \mid |B| = k\}$
- $S_n :=$ set of permutations on $\{1, \ldots, n\}$
- $V_{\pi(:k)} := \left\{v_{\pi(i)} \middle| 1 \leq i \leq k\right\}$ for $\pi \in S_n$
- $[T(B)]$ is the Iverson bracket: 1 if $T(B)$ is true, 0 otherwise
- Expectations over e.g. $\pi \in S_n$ are uniform

Want to estimate

$$\begin{aligned}
&|\{B \subseteq V | T(B)\}| \\
&= \sum_{k=0}^{n} \left| \left\{ B \in \binom{V}{k} \middle| T(B) \right\} \right| \\
&= \sum_{k=0}^{n} \binom{n}{k} \mathbb{E}_{B \in \binom{V}{k}}([T(B)]) \\
&= \sum_{k=0}^{n} \binom{n}{k} \mathbb{E}_{\pi \in S_n} \left(\left[T\left(V_{\pi(:k)}\right) \right] \right) \\
&= \mathbb{E}_{\pi \in S_n} \left(\sum_{k=0}^{n} \binom{n}{k} \left[T\left(V_{\pi(:k)}\right) \right] \right)
\end{aligned}$$

Simple MC method: sample random permutation of V, add $\binom{n}{k}$ for each $V_{\pi(:k)}$ satisfying T
Since T downward closed, first (smallest) value of k such that $\neg T(V_{\pi(:k)})$ makes all subsequent terms 0: formalized later
Next, WTS this is equivalent to choosing new elements of the permutation "on the fly"
Define x_B to be the expected sum remaining after $B \subseteq V$ has already been chosen as the prefix:

$$x_B := \begin{cases} 0 & \text{if } \neg T(B) \\ 1 & \text{if } T(B) \text{ and } B = V \\ \binom{n}{|B|} + \mathbb{E}_{v \in V \setminus B}\left(x_{B \cup \{v\}}\right) & \text{if } T(B) \text{ and } B \neq V \end{cases}$$

Lemma 1. *For all $B \subseteq V$, $x_B = \mathbb{E}_{\pi \in S_n} \left(\sum_{k=|B|}^{n} \binom{n}{k} \left[T\left(V_{\pi(:k)}\right) \right] \middle| B = V_{\pi(:|B|)} \right)$.*

Proof. Induct backwards on $|B|$.

- Base case: $|B| = n$, so $B = V$
  $x_B = [T(B)]$ by first two cases of def
  RHS is the same: condition is always true, sum has 1 term, $\binom{n}{n} = 1$

- Inductive step: $|B| < n$, so $B \neq V$
  IH holds for all B' where $|B'| = |B| + 1$
  Case on $T(B)$

  - If $\neg T(B)$, then $x_B = 0$; WTS RHS $= 0$
    Given that $B = V_{\pi(:|B|)}$, if $k \geq |B|$, then $B \subseteq V_{\pi(:k)}$, so $\neg T(V_{\pi(:k)})$
    Thus sum is 0 for all relevant π, so expectation is 0
  - If $T(B)$, then consider $\Pi := \left\{\pi \in S_n \middle| B = V_{\pi(:|B|)}\right\}$
    For all j where $v_j \in V \setminus B$, define $B_j := B \cup \{v_j\}$
    May partition Π into $\Pi_j := \left\{\pi \in S_n \middle| B_j = V_{\pi(:|B|+1)}\right\}$
    All Π_j are of equal size because this is equivalent to partitioning by $\pi(|B| + 1)$
    IH holds for B_j, i.e.

$$\begin{aligned} x_{B_j} &= \mathbb{E}_{\pi \in S_n}\left(\sum_{k=|B|+1}^{n} \binom{n}{k} \left[T\left(V_{\pi(:k)}\right)\right] \middle| B_j = V_{\pi(:|B|+1)}\right) \\ &= \mathbb{E}_{\pi \in \Pi_j}\left(\sum_{k=|B|+1}^{n} \binom{n}{k} \left[T\left(V_{\pi(:k)}\right)\right]\right) \end{aligned}$$

    Then

$$\begin{aligned} x_B &= \binom{n}{|B|} + \mathbb{E}_{v_j \in V \setminus B}\left(x_{B_j}\right) \\ &= \binom{n}{|B|} + \mathbb{E}_{v_j \in V \setminus B}\left(\mathbb{E}_{\pi \in \Pi_j}\left(\sum_{k=|B|+1}^{n} \binom{n}{k} \left[T\left(V_{\pi(:k)}\right)\right]\right)\right) \\ &= \binom{n}{|B|} + \mathbb{E}_{\pi \in \Pi}\left(\sum_{k=|B|+1}^{n} \binom{n}{k} \left[T\left(V_{\pi(:k)}\right)\right]\right) \\ &= \mathbb{E}_{\pi \in \Pi}\left(\binom{n}{|B|} + \sum_{k=|B|+1}^{n} \binom{n}{k} \left[T\left(V_{\pi(:k)}\right)\right]\right) \\ &= \mathbb{E}_{\pi \in \Pi}\left(\sum_{k=|B|}^{n} \binom{n}{k} \left[T\left(V_{\pi(:k)}\right)\right]\right) \\ &= \mathbb{E}_{\pi \in S_n}\left(\sum_{k=|B|}^{n} \binom{n}{k} \left[T\left(V_{\pi(:k)}\right)\right] \middle| B = V_{\pi(:|B|)}\right) \end{aligned}$$

$\square$

Finally, want to define the desired MC method and show it has the correct expectation
For $B \subseteq V$, define $C_B = \{v \in V \setminus B | T(B \cup \{v\})\}$
Define RVs X_B by

$$X_B := \begin{cases} 0 & \text{if } \neg T(B) \\ 1 & \text{if } T(B) \text{ and } B = V \\ \binom{n}{|B|} + \frac{|C_B|}{|V \setminus B|} X_{B \cup \{v\}} & \text{if } T(B) \text{ and } B \neq V\text{, for random } v \in C_B \end{cases}$$

Lemma 2. *For all $B \subseteq V$, $\mathbb{E}(X_B) = x_B$.*

Proof. Induct backwards on $|B|$.

- Base case: $|B| = n$, so $B = V$
  $\mathbb{E}(X_B) = [T(B)] = x_B$

- Inductive step: $|B| < n$, so $B \neq V$
  IH holds for all B' where $|B'| = |B| + 1$
  If $\neg T(B)$, then $\mathbb{E}(X_B) = 0 = x_B$
  If $T(B)$, then

$$
\begin{aligned}
&\mathbb{E}(X_B) \\
&= \mathbb{E}_{v \in C_B}\left(\binom{n}{|B|} + \frac{|C_B|}{|V \setminus B|} X_{B \cup \{v\}}\right) \\
&= \binom{n}{|B|} + \frac{|C_B|}{|V \setminus B|} \mathbb{E}_{v \in C_B}\left(X_{B \cup \{v\}}\right) \\
&= \binom{n}{|B|} + \frac{1}{|V \setminus B|} \sum_{v \in C_B} \left(X_{B \cup \{v\}}\right) \\
&= \binom{n}{|B|} + \frac{1}{|V \setminus B|} \sum_{v \in V \setminus B} \left(X_{B \cup \{v\}}\right) \\
&= \binom{n}{|B|} + \mathbb{E}_{v \in V \setminus B}\left(X_{B \cup \{v\}}\right) \\
&= \binom{n}{|B|} + \mathbb{E}_{v \in V \setminus B}\left(x_{B \cup \{v\}}\right) \\
&= x_B
\end{aligned}
$$

□

Theorem 1. $\mathbb{E}(X_{\{\}}) = |\{B \subseteq V | T(B)\}|$.

Proof.

$$
\begin{aligned}
\mathbb{E}(X_{\{\}}) &= x_{\{\}} \\
&= \mathbb{E}_{\pi \in S_n}\left(\sum_{k=|\{\}|}^{n} \binom{n}{k} \left[T\left(V_{\pi(:k)}\right)\right] \middle| \{\} = V_{\pi(:|\{\}|)}\right) \\
&= \mathbb{E}_{\pi \in S_n}\left(\sum_{k=0}^{n} \binom{n}{k} \left[T\left(V_{\pi(:k)}\right)\right]\right) \\
&= |\{B \subseteq V | T(B)\}|
\end{aligned}
$$

□