Recurrence Relations and Differential Equations

Aresh Pourkavoos

February 28, 2022

What do these two have in common?

$$f''(x) = f'(x) + f(x)$$
$$F_{n+2} = F_{n+1} + F_n$$

The first is a differential equation, which relates the value of a function to its derivatives, and the other is a recurrence relation, a rule for building a number sequence by looking at previous entries. Beyond their visual similarities, their solutions both involve the golden ratio and its conjugate,

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$$
$$\overline{\varphi} = \frac{1-\sqrt{5}}{2} \approx -0.618,$$

the two solutions of the polynomial $x^2 = x + 1$. Specifically, the general solution to the first equation is

$$f(x) = a\exp(\varphi x) + b\exp(\overline{\varphi} x),$$

and the second is

$$F_n = a\varphi^n + b\overline{\varphi}^n.$$

Both have two degrees of freedom, which makes sense considering the equations themselves: in the differential equation, $f(0)$ and $f'(0)$ may be chosen freely, and $f''(0)$ and everything else are uniquely determined. In the recurrence relation, F_0 and F_1 may be chosen freely, and F_2 and all other terms follow from them.