The Generalized Hockey-stick Identity

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Generalized Hockey-stick Identity. *For all natural variables,*

$$\sum_{j_1+j_2=J} \binom{j_i+k_1}{j_1}\binom{j_2+k_2}{j_2} = \binom{J+k_1+k_2+1}{J}.$$

Equivalently, define

$$f(j,k) = \frac{(j+k)!}{j!k!} = \binom{j+k}{k} = \binom{j+k}{j}$$

so that the identity may be rewritten

$$\sum_{j_1+j_2=J} f(j_1,k_1)f(j_2,k_2) = f(J,k_1+k_2+1).$$

Note $f(0,k) = f(j,0) = 1$ *and* $f(j+1,k+1) = f(j,k+1) + f(j+1,k)$.

Proof. (based on combinatorics) The left and right sides of the equation can be interpreted as cardinalities of sets, and a bijection between these sets is sufficient to show that their cardinalities are equal. $f(j,k)$ becomes the set of ways to arrange j red and k blue objects in a line, multiplication becomes the Cartesian product of sets, and summation becomes the disjoint union. Then the left side represents the number of pairs of arrangements of j_1 red and k_1 blue objects and arrangements of j_2 red and k_2 blue objects such that $j_1 + j_2 = J$. Likewise, the right side is the number of arrangements of J red and $k_1 + k_2 + 1$ blue objects.

Left$\rightarrow$right: Take the first arrangement in the given pair (with $j_1 + k_1$ objects), add a blue object to its right as a separator, and add the second arrangement (with $j_2 + k_2$ objects) to the right of that to obtain an arrangement of $j_1 + j_2 = J$ red and $k_1 + k_2 + 1$ blue objects.

Right$\rightarrow$left: Take the $(k_1 + 1)^{\text{th}}$ blue object in the given arrangement (with J red and $k_1 + k_2 + 1$ blue objects) to act as a separator, so the arrangement to its left has k_1 blue objects and the arrangement to its right has k_2 blue objects. The number of red objects on the left and right (j_1 and j_2, respectively) are not determined by the split, but they must still add to J.

Note that these functions are inverses and thus form a bijection. $\square$

Proof. (based on polynomial expansion) Let $p(x) = 1 + x + x^2 + \ldots$, the Taylor series expansion of $(1-x)^{-1}$. Then $f(j,k)$ is the degree-j coefficient in the expansion of $p(x)^{k+1}$ (to prove later). Trivially, $p(x)^{k_1+1}p(x)^{k_2+1} = p(x)^{(k_1+k_2+1)+1}$. Since polynomial multiplication may be done by convolution over the coefficients, the degree-J coefficient of $p(x)^{(k_1+k_2+1)+1}$ is the sum of products of the appropriate coefficients in $p(x)^{k_1+1}$ and $p(x)^{k_2+1}$, namely those whose degrees j_1 and j_2 add to J. $\square$

Proof. (based on Pascal's rule) Proof by induction over J and k_2:

1. $J = 0$ is trivial:

$$\sum_{j_1+j_2=0} f(j_1,k_1)f(j_2,k_2) = f(0,k_1+k_2+1)$$
$$f(0,k_1)f(0,k_2) = f(0,k_1+k_2+1)$$
$$1 \times 1 = 1$$

2. $k_2 = 0$ reduces to the normal hockey stick identity:

$$\sum_{j_1+j_2=J} f(j_1,k_1)f(j_2,0) = f(J,k_1+0+1)$$
$$\sum_{j_1=0}^{J} f(j_1,k_1) = f(J,k_1+1)$$

3. $J = J'+1$, $k_2 = k_2'+1$: Assume

$$\sum_{j_1+j_2=J'} f(j_1,k_1)f(j_2,k_2) = f(J',k_1+k_2+1),$$
$$\sum_{j_1+j_2=J} f(j_1,k_1)f(j_2,k_2') = f(J,k_1+k_2'+1).$$

Then

$$\begin{aligned}
&\sum_{j_1+j_2=J} f(j_1,k_1)f(j_2,k_2)\\
&=\sum_{j_1=0}^{J} f(j_1,k_1)f(J-j_1,k_2)\\
&=\sum_{j_1=0}^{J'} f(j_1,k_1)f(J-j_1,k_2)+f(J,k_1)\\
&=\sum_{j_1=0}^{J'} f(j_1,k_1)(f(J'-j_1,k_2)+f(J-j_1,k_2'))+f(J,k_1)\\
&=\sum_{j_1=0}^{J'} (f(j_1,k_1)f(J'-j_1,k_2)+f(j_1,k_1)f(J-j_1,k_2'))+f(J,k_1)\\
&=\sum_{j_1=0}^{J'} f(j_1,k_1)f(J'-j_1,k_2)+\sum_{j_1=0}^{J'} f(j_1,k_1)f(J-j_1,k_2')+f(J,k_1)\\
&=\sum_{j_1=0}^{J'} f(j_1,k_1)f(J'-j_1,k_2)+\sum_{j_1=0}^{J} f(j_1,k_1)f(J-j_1,k_2')\\
&=\sum_{j_1+j_2=J'} f(j_1,k_1)f(j_2,k_2)+\sum_{j_1+j_2=J} f(j_1,k_1)f(j_2,k_2')\\
&=f(J',k_1+k_2+1)+f(J,k_1+k_2'+1)\\
&=f(J,k_1+k_2+1).
\end{aligned}$$

$\square$