A Generating Function Problem

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$$f(x,y) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{(j+k+1)!} x^j y^k$$

$$\frac{d^{j+k}f}{dx^j dy^k}(0,0) = \frac{j!k!}{(j+k+1)!} = B(j+1,k+1)$$

$$B(j+1,k+1) = B(j+1,k+2) + B(j+2,k+1)$$

$$\begin{aligned}
\frac{d^{j+k}f}{dx^j dy^k}(0,0) &= \frac{d^{j+k+1}f}{dx^{j+1} dy^k}(0,0) + \frac{d^{j+k+1}f}{dx^j dy^{k+1}}(0,0) \\
&= \frac{d}{dx}\left(\frac{d^{j+k}f}{dx^j dy^k}\right)(0,0) + \frac{d}{dy}\left(\frac{d^{j+k}f}{dx^j dy^k}\right)(0,0)
\end{aligned}$$

$$f = \frac{df}{dx} + \frac{df}{dy}$$

$$B(j+1,1) = \frac{1}{j+1} = \frac{d^j f}{dx^j}(0,0)$$

$$\begin{aligned}
f(x,0) &= \sum_{j=0}^{\infty} \frac{1}{j+1}\frac{x^j}{j!} \\
&= \sum_{j=0}^{\infty} \frac{x^j}{(j+1)!} \\
&= \frac{1}{x}\sum_{j=0}^{\infty} \frac{x^{j+1}}{(j+1)!} \\
&= \frac{1}{x}\left(0\sum_{j=0}^{\infty}\left(\frac{x^j}{j!}\right) - \frac{x^0}{0!}\right) \\
&= \frac{e^x - 1}{x}
\end{aligned}$$

$$
\begin{aligned}
f(x,y) &= \sum_{n=0}^{\infty} \sum_{j+k=n} \frac{1}{(j+k+1)!} x^j y^k \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \sum_{j+k=n} x^j y^k \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \frac{x^{n+1} - y^{n+1}}{x-y} \\
&= \frac{1}{x-y} \left(\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} - \sum_{n=0}^{\infty} \frac{y^{n+1}}{(n+1)!} \right) \\
&= \frac{1}{x-y} ((e^x - 1) - (e^y - 1)) \\
&= \frac{e^x - e^y}{x-y}
\end{aligned}
$$

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\begin{aligned}
\frac{df}{dx} &= \frac{e^x(x-y) - (e^x - e^y)}{(x-y)^2} \\
\frac{df}{dy} &= \frac{-e^y(x-y) + (e^x - e^y)}{(x-y)^2} \\
\frac{df}{dx} + \frac{df}{dy} &= \frac{(e^x - e^y)(x-y)}{(x-y)^2} \\
&= \frac{e^x - e^y}{(x-y)} \\
&= f(x,y)
\end{aligned}
$$