Cycle Relations

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Problem. Let S be a set. A ternary relation $\circlearrowleft$ on S is a *cycle relation* when:

(1) For all $a \in S$, ${}_{a}\overset{a}{\circlearrowleft}_{a}$.

(2) For all $a, b, c \in S$, ${}_{b}\overset{a}{\circlearrowleft}_{c}$ implies ${}_{a}\overset{c}{\circlearrowleft}_{b}$.

(3) For all $a \in S$, if we define the relation $\overset{a}{\leq}$ on $S \setminus \{a\}$ so that $b \overset{a}{\leq} c$ when ${}_{b}\overset{a}{\circlearrowleft}_{c}$, then $\overset{a}{\leq}$ is a partial order.

Let $z \in S$, and let $\preceq$ be a partial order on $S \setminus \{z\}$.

(i) Does there exist a cycle relation $\circlearrowleft$ on S such that $\overset{z}{\leq}$ as defined above is equal to $\preceq$?

(ii) Assuming that such a $\circlearrowleft$ exists, is it unique?

Solution.

(i) We show that a cycle relation exists. Define $\circlearrowleft$ as follows:

(A) For all $a, b \in S$, ${}_b\overset{a}{\circlearrowleft}_b$.

(B) For all $w, x \in S \setminus \{z\}$ where $w \preceq x$, ${}_w\overset{z}{\circlearrowleft}_x$.

(C) For all $w, x, y \in S \setminus \{z\}$ where $w \preceq x \preceq y$, ${}_x\overset{w}{\circlearrowleft}_y$.

(D) All cycles of the above satisfy $\circlearrowleft$, e.g. ${}_z\overset{x}{\circlearrowleft}_w$ and ${}_x\overset{w}{\circlearrowleft}_z$ where $w \preceq x$, and no other triples do.

By (A) with $b = a$, $\circlearrowleft$ satisfies property (1), and by (D), it also satisfies (2).

To show (3), we find the relations $\overset{a}{\leq}$ for $a \in S$. In the first case, $a = z$. Neither (A) nor (C) contributes to $\overset{z}{\leq}$, even when cycled by (D). However, by (B), for all $w, x \in S \setminus \{z\}$ where $w \preceq x$, $w \overset{z}{\leq} x$, and the cycles of (B) again contribute nothing. Thus $\overset{z}{\leq} = \preceq$, so $\overset{z}{\leq}$ is a partial order (and has the desired agreement with $\preceq$).

In the second case, let $w \in S \setminus \{z\}$. Then $\overset{w}{\leq}$ is defined as follows:

(I) By (A) with $a = w$, for all $b \in S \setminus \{w\}$, $b \overset{w}{\leq} b$.

(II) By (B), for all $x \in S \setminus \{w, z\}$, if $w \preceq x$, then ${}_w\overset{z}{\circlearrowleft}_x$, so ${}_x\overset{w}{\circlearrowleft}_z$, so $x \overset{w}{\leq} z$.
Also, if $x \preceq w$, then ${}_x\overset{z}{\circlearrowleft}_w$, so ${}_z\overset{w}{\circlearrowleft}_x$, so $z \overset{w}{\leq} x$.

(III) By (C), for all $x, y \in S \setminus \{w, z\}$, if $w \preceq x \preceq y$, then ${}_x\overset{w}{\circlearrowleft}_y$, so $x \overset{w}{\leq} y$.
Also, if $y \preceq w \preceq x$, then ${}_w\overset{y}{\circlearrowleft}_x$, so ${}_x\overset{w}{\circlearrowleft}_y$, so $x \overset{w}{\leq} y$.
Finally, if $x \preceq y \preceq w$, then ${}_y\overset{x}{\circlearrowleft}_w$, so ${}_x\overset{w}{\circlearrowleft}_y$, so $x \overset{w}{\leq} y$.

We show that $\overset{w}{\leq}$ is a partial order. By (I), $\overset{w}{\leq}$ is reflexive. To show that it is antisymmetric, let $a, b \in S \setminus \{w\}$, and suppose $a \overset{w}{\leq} b$ and $b \overset{w}{\leq} a$. We case on the values of a and b.

- If both are z, $a = b$ trivially.
- If one is z and the other is not, this contradicts the goal $a = b$, so we want to show that this case cannot occur. WLOG $a = z$ and $b \in S \setminus \{w, z\}$, so $z \overset{w}{\leq} b$ and $b \overset{w}{\leq} z$. This must arise from (II), so $b \preceq w$ and $w \preceq b$, respectively. Since $\preceq$ is antisymmetric, $b = w$, which contradicts $b \in S \setminus \{w, z\}$.
- If neither is z, i.e. $a, b \in S \setminus \{w, z\}$, then $a \overset{w}{\leq} b$ and $b \overset{w}{\leq} a$ must arise from either (I), which would immediately imply that $a = b$, or (III). For the latter, since $a \overset{w}{\leq} b$, one of the following is true:
  - $w \preceq a \preceq b$
  - $b \preceq w \preceq a$
  - $a \preceq b \preceq w$

  Likewise, since $b \overset{w}{\leq} a$, one of the following is true:
  - $w \preceq b \preceq a$
  - $a \preceq w \preceq b$
  - $b \preceq a \preceq w$

  From here, it may be verified that any of the $3 \times 3 = 9$ cases implies one of the following:
  - $a \preceq b$ and $b \preceq a$, so $a = b$ because $\preceq$ is antisymmetric.
  - $a \preceq w$ and $w \preceq a$, so $a = w$, which contradicts $a \in S \setminus \{w, z\}$.
  - $b \preceq w$ and $w \preceq b$, so $b = w$, which contradicts $b \in S \setminus \{w, z\}$.

It remains to show that $\overset{w}{\leq}$ is transitive. Let $a, b, c \in S \setminus \{w\}$, and suppose $a \overset{w}{\leq} b$ and $b \overset{w}{\leq} c$. If any pair of a, b, and c are equal, then $a \overset{w}{\leq} c$ follows immediately, so we assume from here that all three are distinct. Then at most one of them can be z.

- If $a = z$, then $b, c \in S \setminus \{w, z\}$. Also, $z = a \overset{w}{\leq} b$ must arise from (II), so $b \preceq w$, and $b \overset{w}{\leq} c$ must arise from (III), so one of the following is true:
  - $w \preceq b \preceq c$
  - $c \preceq w \preceq b$
  - $b \preceq c \preceq w$

  In all cases, $w \preceq b$ or $c \preceq w$. If $w \preceq b$, then $b = w$, which contradicts $b \in S \setminus \{w, z\}$. If $c \preceq w$, then by (II), $z \overset{w}{\leq} c$, i.e. $a \overset{w}{\leq} c$.

- If $b = z$, then $a, c \in S \setminus \{w, z\}$. Also, $a \overset{w}{\leq} b = z$ must arise from (II), so $w \preceq a$, and $z = b \overset{w}{\leq} c$ must arise from (II), so $c \preceq w$. Thus $c \preceq w \preceq a$, so by (III), $a \overset{w}{\leq} c$.

- If $c = z$, then $a, b \in S \setminus \{w, z\}$. Also, $b \overset{w}{\leq} c = z$ must arise from (II), so $w \preceq b$, and $a \overset{w}{\leq} b$ must arise from (III), so one of the following is true:
  - $w \preceq a \preceq b$
  - $b \preceq w \preceq a$
  - $a \preceq b \preceq w$

  In all cases, $b \preceq w$ or $w \preceq a$. If $b \preceq w$, then $b = w$, which contradicts $b \in S \setminus \{w, z\}$. If $w \preceq a$, then by (II), $a \overset{w}{\leq} z$, i.e. $a \overset{w}{\leq} c$.

- If none of them is z, then $a, b, c \in S \setminus \{w, z\}$. $a \overset{w}{\leq} b$ must arise from (III), so one of the following is true:
  - $w \preceq a \preceq b$
  - $b \preceq w \preceq a$
  - $a \preceq b \preceq w$

  $b \overset{w}{\leq} c$ must also arise from (III), so one of the following is true:
  - $w \preceq b \preceq c$
  - $c \preceq w \preceq b$
  - $b \preceq c \preceq w$

  From here, it may be verified that any of the $3 \times 3 = 9$ cases implies one of the following:
  - $b \preceq w$ and $w \preceq b$, so $b = w$, contradicting $b \in S \setminus \{w, z\}$.
  - $w \preceq a \preceq c$, so by (III), $a \overset{w}{\leq} c$.
  - $c \preceq w \preceq a$, so by (III), $a \overset{w}{\leq} c$.
  - $a \preceq c \preceq w$, so by (III), $a \overset{w}{\leq} c$.

Since $\overset{w}{\leq}$ is transitive, $\overset{w}{\leq}$ is a partial order on $S \setminus \{w\}$, so $\circlearrowleft$ is a cycle relation.

(ii) We show that the cycle relation is not unique if it exists. Let $S = \{w, x, y, z\}$, and let $\preceq = \{(w, w), (x, x), (y, y)\}$. Define $\circlearrowleft$ as follows:

(A) For all $a, b \in S$, ${}_{b}\overset{a}{\circlearrowleft}_{b}$.

(B) ${}_{x}\overset{w}{\circlearrowleft}_{y}$.

(C) All cycles of the above satisfy $\circlearrowleft$, and no other triples do.

Then by (A) with $a = b$, $\circlearrowleft$ satisfies property (1), and by (C), it also satisfies (2). To show (3), we find the relations $\overset{a}{\leq}$ for $a \in S$:

- $\overset{w}{\leq} = \{(x, x), (y, y), (z, z), (x, y)\}$
- $\overset{x}{\leq} = \{(w, w), (y, y), (z, z), (y, w)\}$
- $\overset{y}{\leq} = \{(w, w), (x, x), (z, z), (w, x)\}$
- $\overset{z}{\leq} = \{(w, w), (x, x), (y, y)\}$

All of these are partial orders, so (3) is satisfied. Thus $\circlearrowleft$ is a cycle relation. Also, $\overset{z}{\leq} = \preceq$.

Next, define $\circlearrowright$ (with the direction of the arrow reversed) so that ${}_{b}\overset{a}{\circlearrowright}_{c}$ when ${}_{c}\overset{a}{\circlearrowleft}_{b}$. Since $\circlearrowleft$ satisfies (1) and (2), so does $\circlearrowright$. Also, each relation $\overset{a}{\geq}$ for $\circlearrowright$ is the reverse of the corresponding $\overset{a}{\leq}$ for $\circlearrowleft$. Since the reverse of a partial order is also a partial order, $\circlearrowright$ is a cycle relation. Also, $\overset{z}{\geq} = \overset{z}{\leq} = \preceq$.

Finally, ${}_{x}\overset{w}{\circlearrowleft}_{y}$, but ${}_{x}\overset{w}{\not\circlearrowright}_{y}$, so the cycle relation is not unique.