Minimal Polynomials of Real Parts of Roots of Unity

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What are the coordinates of the vertices of a regular n-gon of radius 1? The simple answer comes from trigonometry: the vertices are points on the unit circle, separated by angles of $\frac{\tau}{n}$ (where τ is the number of radians in a full turn), so their coordinates are

$$\left(\cos\left(\frac{m}{n}\tau\right), \sin\left(\frac{m}{n}\tau\right)\right)$$

for all $0 \leq m < n$. However, this formula makes symbolic manipulation unwieldy when trying to reduce expressions involving these terms to their simplest form. For example, when $n = 5$ and $m = 1$, the x-coordinate of the point is $\cos\left(\frac{1}{5}\tau\right) = \frac{1}{2}(1 - \varphi)$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio, the positive solution to $\varphi^2 = \varphi + 1$. Writing the coordinate in terms of φ is easier than the trigonometric way, since products involving it can easily be simplified by replacing φ^2 with $\varphi + 1$.

$$\begin{aligned}
r &= 2\cos\left(\frac{m}{n}\tau\right) \\
p(r) &= 0 \\
z &= \operatorname{cis}\left(\frac{m}{n}\tau\right) \\
z^{-1} &= \operatorname{cis}\left(-\frac{m}{n}\tau\right) \\
r &= z + z^{-1} \\
q(z) &= 0
\end{aligned}$$

$$\begin{aligned}
&n = 7 \\
&0 = z^3 + z^2 + z + 1 + z^{-1} + z^{-2} + z^{-3} \\
&\quad = a + br + cr^2 + dr^3 \\
&\quad = a + b(z + z^{-1}) + c(z + z^{-1})^2 + d(z + z^{-1})^3 \\
&\quad = a + b(z + z^{-1}) + c(z^2 + 2 + z^{-2}) + d(z^3 + 3z + 3z^{-1} + z^{-3}) \\
&\quad = dz^3 + cz^2 + (b + 3d)z + (a + 2c) + (b + 3d)z^{-1} + cz^{-2} + dz^{-3}
\end{aligned}$$

$$\begin{bmatrix}1 & 0 & 2 & 0\\0 & 1 & 0 & 3\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

$$\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}a\\c\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

$$\begin{bmatrix}1 & 3\\0 & 1\end{bmatrix}\begin{bmatrix}b\\d\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

$$\begin{bmatrix}a\\c\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix}$$

$$\begin{bmatrix}b\\d\end{bmatrix} = \begin{bmatrix}-2\\1\end{bmatrix}$$

$$r^3 + r^2 - 2r - 1 = 0$$

1	$r-2$
2	$r+2$
3	$r+1$
4	r
5	r^2+r-1
6	$r-1$
7	r^3+r^2-2r-1
8	r^2-2
9	r^2-3r+1
10	$r^2 \pm r-1$