Möbius Transformations and the Complex Projective Line

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A Möbius transformation is a function on complex numbers

$$f(z) = \frac{az+b}{cz+d},$$

where a, b, c, and d are complex. Before continuing, watching this video will give a good idea of how these transformations look on the complex plane. They include translation, rotation, scaling, and *inversion*, a strange-looking process which turns the plane inside out.

There are two points usually made about these transformations. First, the parameters a, b, c, and d are restricted so that the output of the function is not constant (when it is defined). For example,

$$g(z) = \frac{2z+1}{6z+3} = 3$$

for all z except $-\frac{1}{2}$, where $g(z) = \frac{0}{0}$, which is undefined. Thus, g is not a Möbius transformation. In general, $az+b$ and $cz+d$ can't be multiples of each other. It can be shown that this is equivalent to the statement

$$ad - bc \neq 0.$$

When these functions map $\mathbb{C}$ to $\mathbb{C}$ they are usually *almost* bijective: most of them are undefined for a single input (where the denominator is 0), and most miss the single output $\frac{a}{b}$, since the constant terms b and d prevent the numerator and denominator from having the right ratio. For this reason, the transformation's range is often extended to the *Riemann sphere*, the set of complex numbers with ∞. (This is what the narrator of the video means by "taking a cue from Bernhard Riemann.") ∞ is, roughly speaking, any nonzero number divided by zero. ($\frac{0}{0}$ is still undefined.) Then, $f(z) = \infty$ when $cz + d = 0$, i.e. when z is $-\frac{d}{c}$ if $c \neq 0$ and ∞ if $c = 0$. (Since $az + b$ must not be a multiple of $cz + d$, $\frac{0}{0}$ never occurs.) Also, $f(\infty)$ is $\frac{a}{b}$ if $b \neq 0$ and ∞ if $b = 0$, since it may be thought of as $\lim_{z\to\infty} f(z)$.

There are many edge cases to consider involving ∞ and division by 0, but doing so makes the Möbius transformations (where $ad - bc \neq 0$) bijective over the Riemann sphere. To handle these edge cases more naturally, it is useful to consider the Riemann sphere as the complex projective line, which is made of pairs of complex numbers $(s, t) \neq (0, 0)$. Two such pairs are considered to be equivalent if one is a multiple of the other.

Let $(s, t) \in$ proj. line: value is ∞ when $t = 0$ and s/t otherwise
Condition: $0/0$ is undefined, so $(s, t) \neq (0, 0)$

$$\frac{a\frac{s}{t}+b}{c\frac{s}{t}+d} = \frac{as+bt}{cs+dt}$$

(this also works for edge case of ∞)
Applying f to (s, t) gives $(as + bt, cs + dt)$, which is equiv. to matrix multiplication:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} as+bt \\ cs+dt \end{bmatrix}$$

Condition on f is equiv. to invertibility of matrix, condition on (s, t) is equiv. to $\neq \vec{0}$
Invertibility Theorem: $(as + bt, cs + dt) \neq \vec{0}$, so output is always $\in$ proj. line