

Infinite lists

An ordinal is either 0 or an infinite list of ordinals written right to left, which is all 0 after some point $\gamma = (\dots, 0, 0, \gamma_n, \dots, \gamma_0)$, so $\gamma_k = 0$ for all $k > n$
 For all i (including n and larger), $\gamma_i < (\dots, \gamma_{i+3}, \gamma_{i+2}, \gamma_{i+1} + 1, 0, \dots, 0)$
 Should match the behavior of finitary Veblen functions while retaining lexicographical order
 $(\dots, 0) = 1$, $(\dots, 0, 1) = 2$, $(\dots, 0, 1, 0) = \omega$

Finite lists

Leading 0s may be removed from infinite list and length of remaining list put in front: $(n : \gamma_{n-1}, \dots, \gamma_0)$
 Requires construction of naturals first, unlike infinite lists
 Still lexicographic since lengths are compared first
 No need for base case since list may be empty: could set $(0 :) = 0$, $(1 : (0 :)) = (1 : 0) = 1$, $(1 : 1) = 2$, etc.
 This would offset naturals by 1 from infinite list representation, but otherwise identical

Ordinal-indexed lists

“Redundant” finite lists: place index of each element before it: $(n : \gamma_n, \dots, \gamma_0) \rightarrow (n : \gamma_n, \dots, 0 : \gamma_0)$
 “Sparse” finite lists: remove all 0 entries from list (along with their indices),
 turn all entries $\alpha \rightarrow -1 + \alpha$ to fill gap: $(2 : 1, 1 : 0, 0 : \omega) \rightarrow (2 : 0, 0 : \omega)$
 Ordinal-indexed lists: allow indices to be ordinals themselves: $(\omega : 0) = \sup(\{(0 : 0), (1 : 0), (2 : 0), \dots\})$
 No longer requires naturals to be constructed first

Formal definition

Ordinal is a finite list of pairs $(\beta_n : \gamma_n, \beta_{n-1} : \gamma_{n-1}, \dots, \beta_0 : \gamma_0)$, ordered lexicographically
 Indices are strictly decreasing: $\beta_n > \beta_{n-1} > \dots > \beta_0$
 For all i , if $i < n$ and $\beta_{i+1} = \beta_i + 1$, then $\gamma_i < (\beta_n : \gamma_n, \dots, \beta_{i+1} : \gamma_{i+1} + 1)$
 Otherwise, $\gamma_i < (\beta_n : \gamma_n, \dots, \beta_{i+1} : \gamma_{i+1}, \beta_i + 1 : 0)$

Examples

$$\begin{aligned}
 () &= 0 \\
 (0 : 0) &= 1 \\
 (0 : 1) &= 2 \\
 (0 : n) &= 1 + n, & n < (1 : 0) \\
 (1 : 0) &= \omega \\
 (1 : 0, 0 : \gamma) &= \omega + (1 + \gamma), & \gamma < (1 : 1) \\
 (1 : 1) &= \omega^2 \\
 (1 : \gamma) &= \omega^{1+\gamma}, & \gamma < (2 : 0) \\
 (1 : \gamma_1, 0 : \gamma_0) &= \omega^{1+\gamma_1} + (1 + \gamma_0), & \gamma_1 < (2 : 0), \gamma_0 < (1 : \gamma_1 + 1) = \omega^{1+\gamma_1+1} \\
 (2 : 0) &= \varepsilon_0 \\
 (2 : 0, 1 : \gamma_1) &= \varepsilon_0 \omega^{1+\gamma_1} & \gamma_1 < (2 : 1) \\
 (2 : 0, 1 : \gamma_1, 0 : \gamma_0) &= \varepsilon_0 \omega^{1+\gamma_1} + (1 + \gamma_0) & \gamma_1 < (2 : 1), \gamma_0 < (2 : 0, 1 : \gamma_1 + 1) = \varepsilon_0 \omega^{1+\gamma_1+1} \\
 (3 : 0) &= \zeta_0 \\
 (\beta : \gamma) &= \varphi_{-1+\beta}(\gamma) & \beta \geq 2
 \end{aligned}$$

Relation to Veblen functions

The notation goes up to $\Gamma_0 = \varphi(1, 0, 0)$, since each lower- β term essentially applies another two-variable Veblen function, except for the last, which is added to the result. It is closely related to Veblen normal form, but restricted to two arguments and preserving lexicographical order.