Why 8?

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This leans further toward the garden-variety recreational math involving specific numbers rather than very general formulas, and it answers the question: why does

$$\frac{987654321}{123456789} \approx 8.0000000729?$$

Since base 10 is a bit unwieldy to work with in this case, we will instead note that similar identities seem to hold in other bases. To avoid confusion, numbers which are not in decimal will be given a subscript denoting the base, which is itself written in decimal. The exception is single-digit numbers, which are unambiguous and never have subscripts.

Instead of the decimal identity, we will show its analog in seximal, or base 6:

$$\frac{54321_6}{12345_6} \approx 4.000325_6$$

It might be hard to see what's special about a number whose digits are increasing (or decreasing), but they also appear in another place, namely the repeating decimal (or seximal) expansions of some fractions. For example,

$$\frac{1}{81} = 0.\overline{012345679}$$
$$\frac{1}{41_6} = 0.\overline{01235}_6$$

Already there are some tenuous links to the original problem: we see the 8 and the 4 appear on the left-hand side in both decimal and seximal. The expansion doesn't look quite right, though: it skips the same digit as it increases. Also, it raises the further question of why *this* is even true. The next thing to see is that both of the denominators are square and their roots are 1 less than the base: $81 = 9^2, 41_6 = 25 = 5^2$.

Thus

$$\frac{1}{25} = \left(\frac{1}{5}\right)^2 = (0.\overline{1}_6)^2.$$

To understand why $\frac{1}{5} = 0.\overline{1}\ldots_6$, multiply both sides by 5: $1 = 0.\overline{5}\ldots_6$, the seximal version of the identity $1 = 0.\overline{9}\ldots$. Given this, we can find the seximal expansion of $(0.\overline{1}_6)^2$ as follows. $0.\overline{1}_6 = 0.1_6 + 0.01_6 + 0.001_6 + \ldots$, so each term can be multiplied by $0.\overline{1}_6$:

$$0.\overline{1}_6^2 = 0.0\overline{1}_6 + 0.00\overline{1}_6 + 0.000\overline{1}_6 + \ldots$$

To see what this converges to, we can look at the partial sums:

$$\begin{aligned} & 0.0\overline{1}_6 \\ +0.00\overline{1}_6 &= 0.01\overline{2}_6 \\ +0.000\overline{1}_6 &= 0.012\overline{3}_6 \\ +0.0000\overline{1}_6 &= 0.0123\overline{4}_6 \\ +0.00000\overline{1}_6 &= 0.01234\overline{5}_6 \end{aligned}$$

At this point, though, something interesting happens. since $0.\bar{5}_6 = 1$, The repeating 5 carries over a 1 to the previous place, which happens to have a 4 in it. Thus the fifth partial sum is just 0.01235_6. In the next term to add, the first 1 is in the 7^{th} place after the decimal point, so it comes 2 places after the 5 and leaves a 0 in between:

$$\begin{aligned} & 0.01235_6 \\ +0.0000001\bar{1}_6 &= 0.0123501\bar{1}_6 \\ +0.00000001\bar{1}_6 &= 0.0123501\bar{2}_6 \\ & \dots \end{aligned}$$

From here, we can see that the limit of this sum is $0.\overline{01235}_6$, where each 5 new terms added turns a 4 into a 5 and resets the repeating portion to 0.

$$\frac{(b-1)^3}{b^b - (b^2 - b + 1)}$$

$$\frac{1}{b^b - (b^2 - b + 1)} = \sum_{n=0}^{\infty} \frac{(b^2 - b + 1)^n}{b^{b(n+1)}}$$