Recurrence Relations and Differential Equations

Aresh Pourkavoos

April 7, 2022

What do these three have in common?

$$\begin{aligned} f''(x) &= f'(x) + f(x) \\ F_{n+2} &= F_{n+1} + F_n \\ \lambda v &= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} v \end{aligned}$$

The first is a differential equation, which relates the value of a function to its derivatives. The second other is a recurrence relation, a rule for building a number sequence by looking at previous entries. The third is a linear equation whose solutions λ and x are the eigenvalues and eigenvectors of the given 2×2 matrix. Beyond their visual similarities, their solutions both involve the golden ratio and its conjugate,

$$\begin{aligned} \varphi &= \frac{1+\sqrt{5}}{2} \approx 1.618 \\ \overline{\varphi} &= \frac{1-\sqrt{5}}{2} \approx -0.618, \end{aligned}$$

the two solutions of the polynomial $x^2 = x + 1$. Specifically, the general solution to the first equation is

$$f(x) = a\exp(\varphi x) + b\exp(\overline{\varphi} x),$$

and the second is

$$F_n = a\varphi^n + b\overline{\varphi}^n.$$

Both have two degrees of freedom, which makes sense considering the equations themselves: in the differential equation, $f(0)$ and $f'(0)$ may be chosen freely, and $f''(0)$ and everything else are uniquely determined. In the recurrence relation, F_0 and F_1 may be chosen freely, and F_2 and all other terms follow from them. Additionally, both equations are linear, meaning that it is possible to multiply a solution by a number to obtain another solution, and solutions may also be added together. Given the two degrees of freedom and the linearity, all we need to do in both cases is to find two solutions which are not multiples of each other, and all other solutions may be made from them through multiplication and addition. In linear algebra terms, the solutions form a two-dimensional vector space, so any independent set of two vectors forms a basis of this vector space, of which all other vectors are linear combinations. The question then becomes: how do we find such a basis?

There is an easy way to find a basis which gives little information about the actual solution: set one of the degrees of freedom to 1 and the other to 0, extrapolate to get a full solution, repeat for all degrees of freedom. For example, in the case of F, the two solutions starting with $1, 0$ and $0, 1$ are $1, 0, 1, 1, 2, 3, \ldots$ and $0, 1, 1, 2, 3, 5, \ldots$. Both of these happen to be shifted copies of the Fibonacci sequence, and we can create any other sequence that follows the recurrence relation by adding multiples of these two together. For example, the Lucas numbers begin with $2, 1$, so they may be formed by adding 2 times the first sequence to the second:

1	0	1	1	2	3
0	1	1	2	3	5
2	1	3	4	7	11

So going across the third row, $2+1=3$, $1+3=4$, $3+4=7$, etc, and going down the columns, $2\times1+0=2$, $0\times2+1=1$, $2\times1+1=3$, etc. However, we still do not have a closed form (i.e. non-recursive equation) for the Fibonacci numbers themselves, which would be necessary to find a closed form of the Lucas numbers using this approach. Instead, we need to guess what type of equation might satisfy this relation.

The Fibonacci numbers grow exponentially in the limit, and the ratio between consecutive numbers approaches