Symmetries of the Multinomial and Hypergeometric Functions

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You've probably come across the binomial formula:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!},$$

where $n \in \mathbb{N}$ and $0 \leq k \leq n$, representing **the number of different k-element subsets of an n-element set**. The formula generates entries of Pascal's Triangle:

$$\begin{array}{ccccccccccccc}
&&&&&&1&&&&&&\\
&&&&&1&&1&&&&&\\
&&&&1&&2&&1&&&&\\
&&&1&&3&&3&&1&&&\\
&&1&&4&&6&&4&&1&&\\
&1&&5&&10&&10&&5&&1&\\
1&&6&&15&&20&&15&&6&&1
\end{array}$$

where n is the row number and k is the position within the row. One of the most striking features of Pascal's Triangle is its mirror symmetry: two entries on opposite sides of the same row are the same. This makes sense in terms of the subset definition: for every k-element subset, there is exactly one $n-k$-element subset, namely its complement. Expressed in terms of the binomial formula, though, this symmetry isn't immediately obvious.

$$\binom{n}{n-k} = \frac{n!}{(n-(n-k))!(n-k)!} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}$$

However, with a slight change of variables, we can express this symmetry much more clearly. Let

$$j = n-k, n = j+k,$$

so that

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{(j+k)!}{j!k!} = f(j,k).$$

This way, finding the entry on the opposite side of the row is as easy as switching the order of the arguments, which does not affect the output thanks to commutativity:

$$f(k,j) = \frac{(k+j)!}{k!j!} = \frac{(j+k)!}{j!k!} = f(j,k).$$

Note also that the restriction on the value of k is gone: the function returns a result for all $j, k \in \mathbb{N}$. But, how do we interpret j and this new formula in general? In terms of the traditional formula, j is the number of elements *not* chosen. Instead, we'll place the complementary subsets with sizes j and k on equal footing: the formula returns **the number of ways to split an appropriately-sized (namely, $(j+k)$-element) set into j- and k-element subsets**. This interpretation generalizes well to more subsets. For example, the number of ways to split a $(j+k+l)$-element set into $j-$, $k-$, and $l-$element subsets is

$$f(j,k,l) = \frac{(j+k+l)!}{j!k!l!}$$

In the general case, with m subsets which have k_1 through k_m elements, the formula becomes

$$f(k_1, \ldots, k_m) = \frac{\left(\sum\limits_{i=1}^{m} k_m\right)!}{\prod\limits_{i=1}^{m} k_m!},$$

known as the multinomial coefficient. To me, both its notation and its pronunciation, “the factorial of the sum over the product of the factorials,” rhyme in a satisfying way.

$$\frac{\binom{N-K}{n-k}\binom{K}{k}}{\binom{N}{n}}$$

	Red	Green	Total
Not chosen	j_1	j_2	J
Chosen	k_1	k_2	K
Total	n_1	n_2	N

$$\frac{(j_1+j_2)!(k_1+k_2)!(j_1+k_1)!(j_2+k_2)!}{j_1!j_2!k_1!k_2!(j_1+j_2+k_1+k_2)!} = \frac{J!K!n_1!n_2!}{j_1!j_2!k_1!k_2!N!}$$

Args of F have symmetry group D_4: symmetries of a square

What is this thing? *Generated by Artbreeder*

$$F\left(\begin{array}{cc} j_1, & j_2, \\ k_1, & k_2 \end{array}\right) = \frac{f(j_1,j_2)f(k_1,k_2)}{f(j_1+k_1,j_2+k_2)} = \frac{\binom{j_1+j_2}{j_1}\binom{k_1+k_2}{k_1}}{\binom{j_1+j_2+k_1+k_2}{j_1+k_1}} = \frac{\binom{J}{j_1}\binom{K}{k_1}}{\binom{N}{n_1}}$$

$$F\left(\begin{array}{cc} j_2, & j_1, \\ k_2, & k_1 \end{array}\right) = \frac{f(j_2,j_1)f(k_2,k_1)}{f(j_2+k_2,j_1+k_1)} = \frac{\binom{j_2+j_1}{j_2}\binom{k_2+k_1}{k_2}}{\binom{j_2+j_1+k_2+k_1}{j_2+k_2}} = \frac{\binom{J}{j_2}\binom{K}{k_2}}{\binom{N}{n_2}}$$

$$F\left(\begin{array}{cc} k_1, & k_2, \\ j_1, & j_2 \end{array}\right) = \frac{f(k_1,k_2)f(j_1,j_2)}{f(k_1+j_1,k_2+j_2)} = \frac{\binom{k_1+k_2}{k_1}\binom{j_1+j_2}{j_1}}{\binom{k_1+k_2+j_1+j_2}{k_1+j_1}} = \frac{\binom{K}{k_1}\binom{J}{j_1}}{\binom{N}{n_1}}$$

$$F\left(\begin{array}{cc} k_2, & k_1, \\ j_2, & j_1 \end{array}\right) = \frac{f(k_2,k_1)f(j_2,j_1)}{f(k_2+j_2,k_1+j_1)} = \frac{\binom{k_2+k_1}{k_2}\binom{j_2+j_1}{j_2}}{\binom{k_2+k_1+j_2+j_1}{k_2+j_2}} = \frac{\binom{K}{k_2}\binom{J}{j_2}}{\binom{N}{n_2}}$$

$$F\left(\begin{array}{cc} j_1, & k_1, \\ j_2, & k_2 \end{array}\right) = \frac{f(j_1,k_1)f(j_2,k_2)}{f(j_1+j_2,k_1+k_2)} = \frac{\binom{j_1+k_1}{j_1}\binom{j_2+k_2}{j_2}}{\binom{j_1+k_1+j_2+k_2}{j_1+j_2}} = \frac{\binom{n_1}{j_1}\binom{n_2}{j_2}}{\binom{N}{J}}$$

$$F\left(\begin{array}{cc} k_1, & j_1, \\ k_2, & j_2 \end{array}\right) = \frac{f(k_1,j_1)f(k_2,j_2)}{f(k_1+k_2,j_1+j_2)} = \frac{\binom{k_1+j_1}{k_1}\binom{k_2+j_2}{k_2}}{\binom{k_1+j_1+k_2+j_2}{k_1+k_2}} = \frac{\binom{n_1}{k_1}\binom{n_2}{k_2}}{\binom{N}{K}}$$

$$F\left(\begin{array}{cc} j_2, & k_2, \\ j_1, & k_1 \end{array}\right) = \frac{f(j_2,k_2)f(j_1,k_1)}{f(j_2+j_1,k_2+k_1)} = \frac{\binom{j_2+k_2}{j_2}\binom{j_1+k_1}{j_1}}{\binom{j_2+k_2+j_1+k_1}{j_2+j_1}} = \frac{\binom{n_2}{j_2}\binom{n_1}{j_1}}{\binom{N}{J}}$$

$$F\left(\begin{array}{cc} k_2, & j_2, \\ k_1, & j_1 \end{array}\right) = \frac{f(k_2,j_2)f(k_1,j_1)}{f(k_2+k_1,j_2+j_1)} = \frac{\binom{k_2+j_2}{k_2}\binom{k_1+j_1}{k_1}}{\binom{k_2+j_2+k_1+j_1}{k_2+k_1}} = \frac{\binom{n_2}{k_2}\binom{n_1}{k_1}}{\binom{N}{K}}$$