Optimal Bump Functions with Lipschitz n^{th} Derivatives

Aresh Pourkavoos

November 26, 2024

Theorem 0.1. *Let $n \in \mathbb{N}^{>0}$, $I = (t_0, t_n)$ be an interval, $p : I \to \mathbb{R}^{>0}$, $q : I \to \mathbb{R}^{<0}$. Define*

$$\mathcal{F} = \left\{ f : I \to \mathbb{R} \mid q \le f \le p, \forall i \in \{1, \dots, n-1\} \left(\int_I f(t)(t_n - t)^{i-1} dt = 0 \right) \right\}.$$

Then if there exist $t_1 < \ldots < t_{n-1} \in I$ s.t. $f^* \in \mathcal{F}$, where

$$f^*(t) = \begin{cases} p(t) & t_{2i} \le t < t_{2i+1} \\ q(t) & t_{2i+1} \le t < t_{2i+2}, \end{cases}$$

then

1. *f^* maximizes $\int_I f(t)(t_n - t)^{n-1} dt$ over $f \in \mathcal{F}$, and*
2. *f^* is the unique maximizer up to a set of measure 0.*

Proof. (Found with the help of Xander Heckett and Robert Trosten) Suppose f^* exists as described. Let $f \in \mathcal{F}$. Then $f = f^*(1 - \eta)$ for some $\eta : I \to \mathbb{R}^{\ge 0}$. For 1., suffices to show that

$$\int_I f^*(t)(t_n - t)^{n-1} dt \ge \int_I f(t)(t_n - t)^{n-1} dt,$$

i.e. that

$$\int_I f^*(t)\eta(t)(t_n - t)^{n-1} dt \ge 0.$$

Let $a(t) = (t_1 - t) \cdots (t_{n-1} - t)$. Then $f^*(t)a(t) > 0$ everywhere except at t_i, so

$$\int_I f^*(t)\eta(t)a(t) dt \ge 0.$$

Thus, suffices to show

$$\int_I f^*(t)\eta(t)((t_n - t)^{n-1} - a(t)) dt = 0.$$

Note $(t_n - t)^{n-1} - a(t) \in \operatorname{span}\{(t_n - t)^{i-1} | i \in \{1, \dots, n-1\}\}$, so suffices to show

$$\int_I f^*(t)\eta(t)(t_n - t)^{i-1} dt = 0$$

for all $i \in \{1, \dots, n-1\}$. Since $f^*\eta = f^* - f$ and $f, f^* \in \mathcal{F}$,

$$\int_I f^*(t)\eta(t)(t_n - t)^{i-1} dt = \int_I (f^*(t) - f(t))(t_n - t)^{i-1} dt = \int_I f^*(t)(t_n - t)^{i-1} dt - \int_I f(t)(t_n - t)^{i-1} dt = 0.$$

For 2., suppose $f \ne f^*$ on a set of positive measure, so $\eta > 0$ on a set of positive measure, so inequality becomes strict. $\square$

Lemma 0.1. *Let $n \in \mathbb{N}^{>0}$, $I = (0, 4)$, $p(t) = 1$, and $q(t) = -1$ under the conditions of the previous theorem. Then the values $t_i = 4\cos^2\left(\frac{i\pi}{2n}\right)$ yield an $f^* \in \mathcal{F}$. Furthermore, $\int_I f^*(t)(t_n - t)^{n-1}dt = 4$.*

Proof. Note

$$\begin{aligned} f^*(t) &= \begin{cases} 1 & t_{2i} \le t < t_{2i+1} \\ -1 & t_{2i+1} \le t < t_{2i+2} \end{cases} \\ &= \sum_{i=0}^{n-1} (-1)^i \left([t \ge t_i] - [t \ge t_{i+1}]\right), \end{aligned}$$

where [] denotes the Iverson bracket. We want to determine $\int_0^4 f^*(t)(4-t)^{k-1}dt$ for all $k \in \{1, \ldots, n\}$, which should be 4 for $k = n$ and 0 otherwise. Note

$$\int_0^4 [t \ge t_i](4-t)^{k-1}dt = \int_{t_i}^4 (4-t)^{k-1}dt = -\frac{1}{k}(4-t)^k\Bigg|_{t_i}^4 = \frac{1}{k}(4-t_i)^k = \frac{1}{k}t_{n-i}^k,$$

so

$$\begin{aligned} \int_0^4 f^*(t)(4-t)^{k-1}dt &= \int_0^4 \left(\sum_{i=0}^{n-1}(-1)^i\left([t \ge t_i] - [t \ge t_{i+1}]\right)\right)(4-t)^{k-1}dt \\ &= \sum_{i=0}^{n-1}(-1)^i\left(\int_0^4 [t \ge t_i](4-t)^{k-1}dt - \int_0^4 [t \ge t_{i+1}](4-t)^{k-1}dt\right) \\ &= \sum_{i=0}^{n-1}(-1)^i\left(\frac{1}{k}t_{n-i}^k - \frac{1}{k}t_{n-(i+1)}^k\right) \\ &= \frac{1}{k}\sum_{i=0}^{n-1}(-1)^i\left(t_{n-i}^k - t_{n-(i+1)}^k\right), \end{aligned}$$

so

$$\begin{aligned} k\int_0^4 f^*(t)(4-t)^{k-1}dt &= \sum_{i=0}^{n-1}(-1)^i\left(t_{n-i}^k - t_{n-(i+1)}^k\right) \\ &= \sum_{i=0}^{n-1}(-1)^i t_{n-i}^k - \sum_{i=0}^{n-1}(-1)^i t_{n-(i+1)}^k \\ &= \sum_{i=0}^{n-1}(-1)^i t_{n-i}^k + \sum_{i=0}^{n-1}(-1)^{i+1} t_{n-(i+1)}^k \\ &= \sum_{i=0}^{n-1}(-1)^i t_{n-i}^k + \sum_{i=0}^{n-1}(-1)^{n-i} t_i^k \\ &= \sum_{i=0}^{n-1}((-1)^i t_{n-i}^k + (-1)^{n-i}t_i^k). \end{aligned}$$

which we WTS is $4k = 4n$ for $k = n$ and 0 otherwise.

Define $\omega = \exp\left(\frac{\pi}{n}\sqrt{-1}\right)$, so

$$t_i = 4\cos^2\left(\frac{i\pi}{2n}\right) = \left(2\cos\left(\frac{i\pi}{2n}\right)\right)^2 = (\omega^{i/2} + \omega^{-i/2})^2,$$

so

$$
\begin{aligned}
t_i^k &= \left(\omega^{i/2} + \omega^{-i/2}\right)^{2k} \\
&= \sum_{j=0}^{2k} \binom{2k}{j} \omega^{(i/2)(2k-j)} \omega^{(-i/2)j} \\
&= \sum_{j=0}^{2k} \binom{2k}{j} \omega^{(k-j)i} \\
&= \sum_{j=-k}^{k} \binom{2k}{k-j} \omega^{ji}.
\end{aligned}
$$

Then

$$
k \int_0^4 f^*(t)(4-t)^{k-1} dt = \sum_{i=0}^{n-1} \left((-1)^i \sum_{j=-k}^{k} \binom{2k}{k-j} \omega^{j(n-i)} + (-1)^{n-i} \sum_{j=-k}^{k} \binom{2k}{k-j} \omega^{ji} \right).
$$

The rest is trig bashing, which I will copy from an earlier iteration of the writeup. There seems to be a missing factor of $(-1)^n$ which I cannot account for at the moment (and which is relevant since the result is nonzero in the $k = n$ case), but it is what it is.

$$
\begin{aligned}
k\int_0^4 f^*(t)(4-t)^{k-1}dt =& \sum_{i=0}^{n-1}\left((-1)^i\sum_{j=-k}^{k}\binom{2k}{k-j}\omega^{ji} + (-1)^{n-i}\sum_{j=-k}^{k}\binom{2k}{k-j}\omega^{j(n-i)}\right)\\
=& \sum_{i=0}^{n-1}\left(\sum_{j=-k}^{k}\binom{2k}{k-j}(-1)^i\omega^{ji} + \sum_{j=-k}^{k}\binom{2k}{k-j}(-1)^{n-i}\omega^{j(n-i)}\right)\\
=& \sum_{i=0}^{n-1}\left(\sum_{j=-k}^{k}\binom{2k}{k-j}(-1)^i\omega^{ji} + \sum_{j=-k}^{k}\binom{2k}{k+j}(-1)^{n-i}\omega^{-j(n-i)}\right)\\
=& \sum_{i=0}^{n-1}\left(\sum_{j=-k}^{k}\binom{2k}{k-j}\omega^{ni}\omega^{ji} + \sum_{j=-k}^{k}\binom{2k}{k-j}\omega^{-n(n-i)}\omega^{-j(n-i)}\right)\\
=& \sum_{i=0}^{n-1}\left(\sum_{j=-k}^{k}\binom{2k}{k-j}\omega^{(n+j)i} + \sum_{j=-k}^{k}\binom{2k}{k-j}\omega^{-(n+j)(n-i)}\right)\\
=& \sum_{i=0}^{n-1}\left(\sum_{j=-k}^{k}\binom{2k}{k-j}\omega^{(n+j)i} + \sum_{j=-k}^{k}\binom{2k}{k-j}\omega^{(n+j)(i-n)}\right)\\
=& \sum_{i=0}^{n-1}\left(\sum_{j=-k}^{k}\binom{2k}{k-j}\left(\omega^{(n+j)i} + \omega^{(n+j)(i-n)}\right)\right)\\
=& \sum_{j=-k}^{k}\left(\binom{2k}{k-j}\sum_{i=0}^{n-1}\left(\omega^{(n+j)i} + \omega^{(n+j)(i-n)}\right)\right)\\
=& \sum_{j=-k}^{k}\left(\binom{2k}{k-j}\left(\sum_{i=0}^{n-1}\omega^{(n+j)i} + \sum_{i=0}^{n-1}\omega^{(n+j)(i-n)}\right)\right)\\
=& \sum_{j=-k}^{k}\left(\binom{2k}{k-j}\left(\sum_{i=0}^{n-1}\omega^{(n+j)i} + \sum_{i=-n}^{-1}\omega^{(n+j)i}\right)\right)\\
=& \sum_{j=-k}^{k}\left(\binom{2k}{k-j}\sum_{i=-n}^{n-1}\omega^{(n+j)i}\right)\\
=& \sum_{j=-k}^{k}\left(\binom{2k}{k-j}[2n|n+j]2n\right)\\
=& \sum_{j=-k}^{k}\left(\binom{2k}{k-j}[k=n]([j=-k]+[j=k])2n\right)\\
=& [k=n]2n\sum_{j=-k}^{k}\left(\binom{2k}{k-j}([j=-k]+[j=k])\right)\\
=& [k=n]2n\left(\binom{2k}{k-(-k)}+\binom{2k}{k-k}\right)\\
=& [k=n]2n(2)\\
=& [k=n]4n,
\end{aligned}
$$

which is $4n$ when $k=n$ and 0 otherwise, as desired. $\square$

Theorem 0.2. *Let $n \in \mathbb{N}^{>1}$, $g : \mathbb{R} \to \mathbb{R}$. Suppose g is supported on an open interval $I = (a, b)$, g is $n-2$ times differentiable, and $g^{(n-2)}$ is Lipschitz with constant L. Then*

$$\int_{\mathbb{R}} g(t)dt \leq \frac{1}{4^{n-1}(n-1)!} L(b-a)^n.$$

Furthermore, this bound is tight.

Proof. Since g is supported on I, so is $g^{(n-2)}$. Since $g^{(n-2)}$ is also Lipschitz, there is some f supported on I such that $g^{(n-2)}(t) = \int_a^t f(t)dt$ and $-L \leq f \leq L$. By the Cauchy formula for repeated integration, and since $g(t) = 0$ for $t \leq a$,

$$g^{(n-2-k)}(t) = \frac{1}{k!} \int_a^t f(s)(t-s)^k ds$$

for $k \in \{0, \ldots, n-1\}$, where $g^{(-1)}(t) = \int_{-\infty}^t g(s)ds = \int_a^t g(s)ds$. Let f_{norm} be f normalized to fit the previous lemma, i.e. $f_{norm}(s) = \frac{1}{L} f\left(\frac{b-a}{4}s + a\right)$, so $f(t) = L f_{norm}\left(\frac{4}{b-a}(t-a)\right)$. Then

$$\int_0^4 f_{norm}(s)(4-s)^{n-1} ds \leq 4,$$

so substituting $t = b$, $k = n-1$,

$$\begin{aligned}
\int_{\mathbb{R}} g(t)dt &= \frac{1}{(n-1)!} \int_a^b f(t)(b-t)^{n-1} dt \\
&= \frac{1}{(n-1)!} \int_a^b L f_{norm}\left(\frac{4}{b-a}(t-a)\right)(b-t)^{n-1} dt \\
&= \frac{1}{(n-1)!} \int_0^4 L f_{norm}(s) \left(\frac{b-a}{4}(4-s)\right)^{n-1} \frac{b-a}{4} ds \\
&= \frac{1}{4^n(n-1)!} L(b-a)^n \int_0^4 f_{norm}(s)(4-s)^{n-1} ds \\
&\leq \frac{1}{4^{n-1}(n-1)!} L(b-a)^n.
\end{aligned}$$

The bound may be shown tight by transforming f^* from the previous lemma and repeatedly integrating to get g: $f(t) = L f^*\left(\frac{4}{b-a}(t-a)\right)$, $g(t) = \frac{1}{(n-2)!} \int_a^t f(s)(t-s)^{k-2} ds$. $\square$