A Generating Function Problem

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Problem:

$$f(x, 0) = \frac{e^x - 1}{x}$$

$$f(x, y) = \frac{\partial f}{\partial x}(x, y) + \frac{\partial f}{\partial y}(x, y)$$

Solution: Define

$$A(j, k) = \frac{\partial^{j+k} f}{\partial x^j \partial y^k}(0, 0)$$

Get Taylor series of base case:

$$f(x, 0) = \frac{1}{x}\left(\sum_{i=0}^{\infty} \frac{x^i}{i!} - 1\right) = \frac{1}{x}\left(\sum_{i=1}^{\infty} \frac{x^i}{i!}\right) = \frac{1}{x}\left(\sum_{i=0}^{\infty} \frac{x^{i+1}}{(i+1)!}\right) = \sum_{i=0}^{\infty} \frac{x^i}{(i+1)!}$$

Find partial derivatives wrt x:

$$A(j, 0) = \frac{\partial^j f}{\partial x^j} \sum_{i=0}^{\infty} \frac{x^i}{(i+1)!} = \frac{\partial^j f}{\partial x^j} \frac{x^j}{(j+1)!} = \frac{j!}{(j+1)!} = \frac{1}{j+1}$$

Use diffeq to establish recurrence relation on A:

$$A(j, k) = \frac{\partial^{j+k} f}{\partial x^j \partial y^k}(0, 0)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial^{j+k} f}{\partial x^j \partial y^k}\right)(0, 0) + \frac{\partial}{\partial y}\left(\frac{\partial^{j+k} f}{\partial x^j \partial y^k}\right)(0, 0)$$

$$= \frac{\partial^{j+k+1} f}{\partial x^{j+1} \partial y^k}(0, 0) + \frac{\partial^{j+k+1} f}{\partial x^j \partial y^{k+1}}(0, 0)$$

$$= A(j+1, k) + A(j, k+1)$$

Rearrange to compute higher values of k:

$$A(j, k+1) = A(j, k) - A(j+1, k)$$

These are sufficient to determine all $A(j, k)$, which may be found by computation and guess-and-check:

$$A(j, k) = \frac{j! k!}{(j+k+1)!} = B(j+1, k+1)$$

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where B is the beta function. Construct 2D Taylor series:

$$f(x,y) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} A(j,k)\frac{x^j y^k}{j!k!}$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{j!k!}{(j+k+1)!}\frac{x^j y^k}{j!k!}$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{x^j y^k}{(j+k+1)!}$$

$$= \sum_{n=0}^{\infty}\sum_{j+k=n} \frac{x^j y^k}{(j+k+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\sum_{j+k=n} x^j y^k$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\frac{x^{n+1}-y^{n+1}}{x-y}$$

$$= \frac{1}{x-y}\left(\sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)!} - \sum_{n=0}^{\infty}\frac{y^{n+1}}{(n+1)!}\right)$$

$$= \frac{1}{x-y}((e^x-1)-(e^y-1))$$

$$= \frac{e^x - e^y}{x-y}$$

Check answer:

$$f(x,0) = \frac{e^x - e^0}{x-0} = \frac{e^x - 1}{x}$$

$$\frac{\partial f}{\partial x} = \frac{e^x(x-y)-(e^x-e^y)}{(x-y)^2}$$

$$\frac{\partial f}{\partial y} = \frac{-e^y(x-y)+(e^x-e^y)}{(x-y)^2}$$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = \frac{(e^x-e^y)(x-y)}{(x-y)^2}$$

$$= \frac{e^x - e^y}{(x-y)}$$

$$= f(x,y)$$