Notes on Geometric Algebra

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1 Introduction

Geometric algebras (or real Clifford algebras) provide a unified framework for diverse topics in math and physics, mostly centered on transformations in n-dimensional space. These include but are not limited to the familiar reflections, rotations, and translations of Euclidean space; the hyperbolic rotations, AKA Lorentz boosts, which appear in special relativity; the circle inversions used in compass-and-straightedge constructions and their higher-dimensional analogs; and the perspective transformations which find applications in computer vision. They can represent area, volume, and beyond in the same way that vectors represent length. When Clifford algebras are constructed using complex numbers instead of reals, they capture the unitary transformations and spinors of quantum mechanics. When combined with calculus, geometric algebra encompasses the theory of differential forms.

1.1 Linear algebra review

Since geometric algebra is an extension of linear algebra, a review of the basic ideas of the latter will be helpful in understanding the former. For this section, scalars are in italics and vectors are in bold. However, geometric algebra adds many other objects to scalars and vectors (and in fact, scalars are themselves represented as vectors), so the distinction between them becomes less meaningful. For this reason, all variables will be italicized in later sections.

A vector space consists of a set of scalars F and a set of vectors V. F must be a field, meaning that scalars may be added, subtracted, multiplied, or divided (except by zero) to produce another scalar. The functions of addition and multiplication, as well as the additive and multiplicative inverses which make subtraction and division possible, must obey certain properties for F to be a field, e.g. $a + b = b + a$ for all scalars a and b. These notes will focus mostly on the cases where $F = \mathbb{R}$, the field of real numbers. For V to form a vector space, vectors must be able to be added to one another or multiplied by a scalar to produce another vector. These functions of vector addition and scalar multiplication, too, must obey certain properties, e.g. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ for all scalars a and vectors $\mathbf{u}$ and $\mathbf{v}$.

Every vector space has a basis, a set of vectors which may be multiplied by scalars and added to each other to uniquely produce any vector in V. The size of the basis is called the dimension of V. These notes are concerned with vector spaces with finite dimension n, so for a given basis $\mathbf{e_1}, \dots, \mathbf{e_n}$, every vector $\mathbf{v}$ can be uniquely represented as $v_1\mathbf{e_1} + \dots + v_n\mathbf{e_n}$, where $v_1, \dots, v_n$ are scalars. For short, $\mathbf{v}$ is often written with a column listing these scalars:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

Vectors are basic objects in linear algebra, along with matrices, which represent ways to transform vectors.

2 Definition

Despite everything than can be shown about vector spaces alone, they do not define a way to multiply two vectors. This lack gives rise to the idea of an algebra: a vector space equipped with an additional operation which multiplies two vectors to produce another vector. This vector product is constrained by only one rule: it must be linear with respect to both arguments, i.e. when one argument is fixed, it must respect vector addition and scalar multiplication in the other argument. In general, it does not need to be commutative, associative, etc. However, thanks to the constraint of linearity, it is only necessary to define the products of all the basis vectors with each other, and

2.1 Example: the exterior algebra

3 Properties

3.1 Conic sections: a tangent

4 Representing transformations

4.1 Reflections, rotations, and translations

4.2 Symmetries of spacetime

4.2.1 Classical mechanics

Classical (Newtonian) mechanics relies on three principles governing the symmetries of space and time. For this section, "frames of reference" refer specifically to inertial frames of reference, namely those of observers which are not accelerating.

1. Shifts between frames of reference are performed by linear transformations.

2. The laws of physics remain the same for all frames of reference.

3. The passage of time is absolute for all frames of reference.

4.2.2 Special relativity

4.3 Conformal transformations

5 From $\mathbb{R}$ to $\mathbb{C}$ to $\mathbb{H}$

So far, the components of multivectors have been real numbers, and the Clifford algebra has been generated by a quadratic form. With a few tricks related to representations, we can extend what we know about Clifford algebras over the real numbers $\mathbb{R}$ to related algebras over the complex numbers $\mathbb{C}$ and algebra-like objects over the quaternions $\mathbb{H}$.

5.1 $\mathbb{R}$ to $\mathbb{C}$

A complex number $a + bi$ can be represented by the 2×2 real matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ so that complex addition, multiplication, and inverses map to the corresponding operations on matrices. For example,

$$(a + bi)(c + di) = (ac - bd) + (bc + ad)i,$$

$$\text{and} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac - bd & -(bc + ad) \\ bc + ad & ac - bd \end{bmatrix}.$$

This matrix can also be thought of as a linear transformation of the plane, namely a scaling and rotation.

Likewise, an $m \times n$ complex matrix can be represented as a $2m \times 2n$ real matrix, with each 2×2 block of the real matrix representing a single entry of the complex matrix. Since breaking down matrices into blocks, multiplying them as units, then adding the results together is a valid method of matrix multiplication, the complex matrix product is preserved with this real representation.

5.2 $\mathbb{C}$ to $\mathbb{H}$

6 Lie groups and Lie algebras