Recurrence Relations and Differential Equations

Aresh Pourkavoos

March 2, 2022

What do these two have in common?

$$f''(x) = f'(x) + f(x)$$
$$F_{n+2} = F_{n+1} + F_n$$

The first is a differential equation, which relates the value of a function to its derivatives, and the other is a recurrence relation, a rule for building a number sequence by looking at previous entries. Beyond their visual similarities, their solutions both involve the golden ratio and its conjugate,

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$$
$$\overline{\varphi} = \frac{1-\sqrt{5}}{2} \approx -0.618,$$

the two solutions of the polynomial $x^2 = x + 1$. Specifically, the general solution to the first equation is

$$f(x) = a\exp(\varphi x) + b\exp(\overline{\varphi} x),$$

and the second is

$$F_n = a\varphi^n + b\overline{\varphi}^n.$$

Both have two degrees of freedom, which makes sense considering the equations themselves: in the differential equation, $f(0)$ and $f'(0)$ may be chosen freely, and $f''(0)$ and everything else are uniquely determined. In the recurrence relation, F_0 and F_1 may be chosen freely, and F_2 and all other terms follow from them. Additionally, both equations are linear, meaning that it is possible to multiply a solution by a number to obtain another solution, and solutions may also be added together. Given the two degrees of freedom and the linearity, all we need to do in both cases is to find two solutions which are not multiples of each other, and all other solutions may be made from them through multiplication and addition. In linear algebra terms, the solutions form a two-dimensional vector space, so any independent set of two vectors forms a basis of this vector space, of which all other vectors are linear combinations.