MLE for Poisson Process with Increasing Rate

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Observe time range $[0, T)$, obtain events at times $\mathbf{t} = (t_1, \ldots, t_n)$, where $n \geq 1$, $0 < t_1 < \ldots < t_n < T$
Events generated by Poisson process with rate function $\lambda : [0, T) \to [0, \infty)$
Log-likelihood of observation: $\ell(\lambda; \mathbf{t}) = \sum_{i=1}^{n} \ln(\lambda(t_i)) - \int_0^T \lambda(t)\ \mathrm{d}t$
If λ unrestricted, can obtain arbitrarily high $\ell(\lambda; \mathbf{t})$ by concentrating value around observations
Instead, assume λ (non-strictly) increasing: how to maximize $\ell(\lambda; \mathbf{t})$ (equivalently, minimize $-\ell(\lambda; \mathbf{t})$)?
Given increasing λ, may define increasing λ' as follows:

$$\lambda'(t) = \begin{cases} 0 & t \in [0, t_1) \\ \lambda(t_i) & t \in [t_i, t_{i+1}) \text{ for } 1 \leq i \leq n \text{ (convention: } t_{n+1} = T) \end{cases}$$

Intuition: replace rate before first event with 0, extend rate at each event to the right until next event
Then $\lambda'(t) \leq \lambda(t)$ for all t, so $\int_0^T \lambda'(t)\ \mathrm{d}t \leq \int_0^T \lambda(t)\ \mathrm{d}t$,
and $\lambda'(t_i) = \lambda(t_i)$ for all i, so $\sum_{i=1}^{n} \ln(\lambda'(t_i)) = \sum_{i=1}^{n} \ln(\lambda(t_i))$
Thus $\ell(\lambda'; \mathbf{t}) \geq \ell(\lambda; \mathbf{t})$, so search space may be restricted to all such λ':
if MLE among λ' does not exist, it does not exist in general, and if it does, it is MLE in general
From here, λ assumed to be of this form
λ may be parameterized by $\lambda_1 = \lambda(t_1), \ldots, \lambda_n = \lambda(t_n)$
Then $\ell(\lambda; \mathbf{t}) = \sum_{i=1}^{n} \ln(\lambda_i) - \sum_{i=1}^{n} s_i \lambda_i$, where $s_i = t_{i+1} - t_i > 0$
λ increasing, so $0 < \lambda_1 \leq \ldots \leq \lambda_n$

Lemma: Let f be a convex function over convex $C \subseteq \mathbb{R}^n$, let $D \subseteq \mathbb{R}^n$ be closed and convex, and suppose $C \cap D$ is nonempty. If f attains min over C at $\mathbf{x} \notin D$, then f attains min over $C \cap D$ at some $\mathbf{w} \in \partial D$.

Proof: Let $\mathbf{y}$ be a point in $C \cap D$ where f is minimized. Then the segment between $\mathbf{x}$ and $\mathbf{y}$ intersects ∂D (if $\mathbf{y} \in \partial D$, at $\mathbf{y}$ itself). Call this intersection $\mathbf{z}$. Then $f(\mathbf{z}) \leq f(\mathbf{y})$ by convexity.

Apply lemma with $C = (\mathbb{R}^+)^n$, $D = \{\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \mid \lambda_1 \leq \ldots \leq \lambda_n\}$

WTS: if unconstrained problem has $\lambda_i > \lambda_{i+1}$ for some i, then constrained problem has $\lambda_i = \lambda_{i+1}$ (for one of those i, not an arbitrary one at first)
Next, WTS

Integral of MLE is underside of convex hull of $\{(t_i, i) \mid 0 \leq i \leq n+1\}$ $(t_0 = 0)$
May help find bias/variance