Extending Pascal's Triangle

Aresh Pourkavoos

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Pascal's triangle contains binomial coefficients:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}, n \in \mathbb{N}, k \in \mathbb{N}, k \leq n$$

Satisfies Pascal's identity for all arguments in range:

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$$

How to extend to all integers, i.e. fill the grid with numbers?
Entries where $n \geq 0$ are unambiguous: all numbers outside of triangle are 0
Factorials of negative integers are undefined, so the usual binomial formula cannot be used
$n = -1$: can choose a single number freely, after which the rest of the row is forced
Same goes for all remaining negative rows
Negative rows are thus uniquely determined by $\binom{n}{0}$ for all $n < 0$
Setting $\binom{n}{0} = 1$ creates "rotated" Pascal's triangle with sign changes: Taylor polynomials of $(1+x)^n$
Rows $n = 0$ through $n = -9$, inclusive:

0	0	0	0	0	1	0	0	0	0	0	
0	0	0	0	0	0	1	-1	1	-1	1	-1
0	0	0	0	0	0	1	-2	3	-4	5	
0	0	0	0	0	0	0	1	-3	6	-10	15
0	0	0	0	0	0	0	1	-4	10	-20	
0	0	0	0	0	0	0	0	1	-5	15	-35
0	0	0	0	0	0	0	0	1	-6	21	
0	0	0	0	0	0	0	0	0	1	-7	28
0	0	0	0	0	0	0	0	0	1	-8	
0	0	0	0	0	0	0	0	0	0	1	-9

Setting $\binom{n}{0} = \frac{1}{n+1}$ gives the beta function for $k \in [n, 0]$: symmetric along same axis as Pascal's triangle
However, values outside the range of the beta function are also given

$\frac{\circ}{1!}$	0	0	0	0	0	1	0	0	0	0	0	
$\frac{\circ}{2!}$	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1
$\frac{\circ}{3!}$	14	-11	8	-5	2	1	2	-5	8	-11	14	
$\frac{\circ}{4!}$	158	-102	58	-26	6	2	2	6	-26	58	-102	158
$\frac{\circ}{5!}$	-954	444	-154	24	6	4	6	24	-154	444	-954	
$\frac{\circ}{6!}$	-9432	3708	-1044	120	24	12	12	24	120	-1044	3708	-9432
$\frac{\circ}{7!}$	33984	-8028	720	120	48	36	48	120	720	-8028	33984	
$\frac{\circ}{8!}$	341136	-69264	5040	720	240	144	144	240	720	5040	-69264	341136
$\frac{\circ}{9!}$	-663696	40320	5040	1440	720	576	720	1440	5040	40320	-663696	
$\frac{\circ}{10!}$	-6999840	362880	40320	10080	4320	144	144	4320	10080	40320	362880	-6999840