Möbius Transformations and the Complex Projective Line

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A Möbius transformation is a function on complex numbers

$$f(z) = \frac{az+b}{cz+d},$$

where a, b, c, and d are complex. Before continuing, watching this video will give a good idea of how these transformations look on the complex plane. They include translation, rotation, scaling, and *inversion*, a strange-looking process which turns the plane inside out.

There are two points usually made about these transformations. First, the parameters a, b, c, and d are restricted so that the output of the function is not constant (when it is defined). For example,

$$g(z) = \frac{2z+1}{6z+3} = 3$$

for all z except $-\frac{1}{2}$, where $g(z) = \frac{0}{0}$, which is undefined. Thus, g is not a Möbius transformation. In general, $az+b$ and $cz+d$ can't be multiples of each other. It can be shown that this is equivalent to the statement

$$ad - bc \neq 0.$$

Next, note that when the denominator $cz+d = 0$, the output is not defined within the complex numbers. For this reason, the transformation's range is often extended to the *Riemann sphere*, the set of complex numbers with ∞. (This is what the narrator of the video means by "taking a cue from Bernhard Riemann.") ∞, roughly speaking, is any nonzero number divided by zero. ($\frac{0}{0}$ is still undefined.) Then, $f(z) = \infty$ when $cz+d=0$. (Since $az+b$ must not be a multiple of $cz+d$, $\frac{0}{0}$ never occurs.)

Let $(s,t) \in$ proj. line: value is ∞ when $t=0$ and s/t otherwise
Condition: $0/0$ is undefined, so $(s,t) \neq (0,0)$

$$\frac{a\frac{s}{t}+b}{c\frac{s}{t}+d} = \frac{as+bt}{cs+dt}$$

(this also works for edge case of ∞)
Applying f to (s,t) gives $(as+bt, cs+dt)$, which is equiv. to matrix multiplication:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} as+bt \\ cs+dt \end{bmatrix}$$

Condition on f is equiv. to invertibility of matrix, condition on (s,t) is equiv. to $\neq \vec{0}$
Invertibility Theorem: $(as+bt, cs+dt) \neq \vec{0}$, so output is always $\in$ proj. line