The Plastic Field

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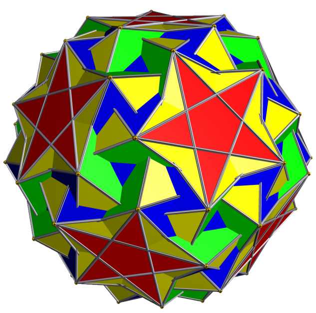
The plastic ratio ρ is the real solution to $\rho^3 = \rho + 1$, with an approximate value of 1.3247. Its definition puts it into the set of numbers that gives the golden ratio many of its interesting (and, to some, mystical) properties: the algebraic integers. A few quick facts:

- Contrary to first impressions, the word "plastic" in this constant's name refers not to the material but to the idea of plasticity, or flexibility.
- It is the smallest of the PV numbers, which are the reals greater than 1 whose powers approach integers. The powers of ρ approach a sequence known as the Perrin numbers. (The golden ratio is also a PV number, and its powers approach the Lucas numbers, which are closely related to the Fibonacci numbers.)

My most recent encounter with this number came from the uniform polyhedron known as the snub icosidodecadodecahedron, or "sided" for short.



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Like many polyhedra related to the icosahedron, the coordinates of its vertices involve the golden ratio. What surprised to me, however, was the inclusion of the plastic ratio as well, which does not appear in any other uniform polyhedra. At a certain scale, its vertices may be written as follows, where φ is the golden ratio:

$$
\begin{aligned}
&(2\alpha, 2\beta, 2\gamma),\\
&(\alpha + \beta/\varphi + \gamma\varphi, -\alpha\varphi + \beta + \gamma/\varphi, \alpha/\varphi + \beta\varphi - \gamma),\\
&(-\alpha/\varphi + \beta\varphi + \gamma, -\alpha + \beta/\varphi - \gamma\varphi, \alpha\varphi + \beta - \gamma/\varphi),\\
&(-\alpha/\varphi + \beta\varphi - \gamma, \alpha - \beta/\varphi - \gamma\varphi, \alpha\varphi + \beta + \gamma/\varphi),\\
&(\alpha + \beta/\varphi - \gamma\varphi, \alpha\varphi - \beta + \gamma/\varphi, \alpha/\varphi + \beta\varphi + \gamma),
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha &= \rho + 1,\\
\beta &= \varphi^2(\rho^2 + \rho) + \varphi,\\
\gamma &= \rho(\varphi + \rho).
\end{aligned}
$$

These 5 sets of coordinates form a single pentagrammic face of sided. The 3 coordinates of each vertex are rotated to create 10 new triplets, e.g. $(2\alpha, 2\beta, 2\gamma)$ yields $(2\beta, 2\gamma, 2\alpha)$ and $(2\gamma, 2\alpha, 2\beta)$. This represents a 120-degree rotation about the axis $x = y = z$, and the image above is almost looking down on such a threefold symmetry axis, running through the center of 3 pentagrams. Next, even sign changes are applied, meaning that that the 4 possible ways of negating none or two of the coordinates represent vertices of sided, bringing the total from 15 to 60, the actual number of vertices. The even sign changes represent 180-degree rotations, and the twofold symmetry axes around which these rotations are performed may also be seen above, running between adjacent pentagrams.

$$
\begin{aligned}
a + b\rho + c\rho^2 \cong \begin{bmatrix} a & c & b \\ b & a+c & b+c \\ c & b & a+c \end{bmatrix} &= M \\
M^{-1} &= \frac{\mathrm{adj}(M)}{|M|} \\
\mathrm{adj(M)} &= \begin{bmatrix} d & & \\ e & \dots & \\ f & & \end{bmatrix} \cong d + e\rho + f\rho^2 \\
d &= (a+c)^2 - b(b+c) \\
e &= c^2 - ab \\
f &= b^2 - (a+c)c \\
|M| &= ad + ce + bf \\
\frac{|M|}{a + b\rho + c\rho^2} &= d + e\rho + f\rho^2 \\
&= (a + b\rho_2 + c\rho_2^2)(a + b\rho_3 + c\rho_3^2) \\
&= \frac{1}{4}\left((2a - b\rho + c(2-\rho^2))^2 + (b - c\rho)^2(3\rho^2 - 4)\right) \\
&= K \\
a, b, c \in \mathbb{R} : &K \geq 0, \\
&K = 0 \iff a\rho^2 = b = c\rho \\
a, b, c \in \mathbb{Q} : &K = 0 \iff a = b = c = 0, \\
&\mathrm{sgn}(|M|) = \mathrm{sgn}(a + b\rho + c\rho^2)
\end{aligned}
$$

In the plastic ratio base, the only digits are zero and one since $\rho < 2$, and all ones must have at least 4 zeros between them. This is because a pair of ones with 3 zeroes between them would be $1 + \rho^4 = \rho^5$ (times some power of ρ overall), and any closer ones would also carry over into the next power of ρ (or even the one after that). The carryover rules are as follows:

010001	0100100000	0101000	0011	00200000
100000	1000000001	1000001	1000	10000001

The integer place values are 1, 2, 3, 4, 5, 6, 8, 11, 15, 20, 25, 31, ..., where $P_n = n + 1$ for $n = 0, \ldots, 4$ and $P_{n+5} = P_n + P_{n+4}$. It follows the same constraints as the real base, and in fact produces all terminating sequences that follow these constraints (with a finite number of 1s) in lexicographical order.