Deriving a Trig Identity

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In acoustics, there is a phenomenon known as "beating," which occurs when two tones with similar frequencies are played together. Instead of two separate notes, we hear a single note whose frequency is the average of the given notes and which periodically becomes quieter and louder. The phenomenon may be explained by the following trigonometric identity:

$$\frac{1}{2}(\cos(ax)+\cos(bx)) = \cos\left(\frac{a+b}{2}x\right)\cos\left(\frac{a-b}{2}x\right)$$

The left side describes a waveform as a function of x (time) as an average of two waves with (angular) frequencies a and b. The right side, on the other hand, is a product of functions, the first of which has the average frequency of the two tones and the second of which describes the beating: if a and b are close, this function has a low frequency and may be heard not as a pitch, but as a gradual change in volume of the note. To avoid fractions, we may double a, b, and both sides of this identity to obtain:

$$\begin{aligned}\cos(2ax)+\cos(2bx) &= 2\cos((a+b)x)\cos((a-b)x)\\ \sin(2ax)+\sin(2bx) &= 2\sin((a+b)x)\cos((a-b)x)\end{aligned}$$

An analogous identity for sines exists, which is also written above.

$$(\cos(2a)+\cos(2b))+i(\sin(2a)+\sin(2b)) = 2\cos(a+b)\cos(a-b)+2i\sin(a+b)\cos(a-b)$$

$$\text{cis}(2a)+\text{cis}(2b) = 2\text{cis}(a+b)\cos(a-b)$$

$$\text{cis}(2a)+\text{cis}(2b) = \text{cis}(a+b)(\text{cis}(a-b)+\text{cis}(b-a))$$

Envelope of $c\cos(ax)+d\cos(bx)$: $\sqrt{c^2+2cd\cos((a-b)x)+d^2}$