Paper: Discrete Fourier transform methods in the theory of equations, BR Neelley, 1992, https://ttu-ir.tdl.org/bitstream/handle/2346/60037/31295007093692.pdf?sequence=1

Solving Quadratics, Cubics, and Quartics with the Discrete Fourier Transform

1. Quadratics (Po-Shen Loh)

$$\boldsymbol{Ax^2 + Bx + C = 0, A \neq 0}$$

$$\boldsymbol{b = \frac{B}{A}, c = \frac{C}{A}}$$

$$x^2 + bx + c = 0$$

$$\boldsymbol{x = x_1, x_2}$$

$$(x - x_1)(x - x_2) = 0$$

$$\boldsymbol{x_1 = p + q, x_2 = p - q}$$

$$0 = (x - (p + q))(x - (p - q)) = (x - p - q)(x - p + q) = (x - p)^2 - q^2 = x^2 - 2px + p^2 - q^2$$

$$b = -2p, c = p^2 - q^2$$

$$\boldsymbol{p = -\frac{b}{2}}$$

$$q^2 = p^2 - c$$

$$\boldsymbol{q = (p^2 - c)^{\frac{1}{2}}}$$

2. Cubics

$$\boldsymbol{Ax^3+Bx^2+Cx+D=0, A\neq 0}$$

$$\boldsymbol{b=\frac{B}{A}, c=\frac{C}{A}, d=\frac{D}{A}}$$

$$x^3+bx^2+cx+d=0$$

$$\boldsymbol{x=x_1,x_2,x_3}$$

$$(x-x_1)(x-x_2)(x-x_3)=0$$

$$\boldsymbol{\omega=\frac{\sqrt{3}i-1}{2}}$$

$$\omega^2+\omega+1=0, \omega^2+\omega=-1, \omega^3=1$$

$$\boldsymbol{x_1=p+q+r, x_2=p+\omega q+\omega^2 r, x_3=p+\omega^2 q+\omega r}$$

$$y=x-p$$

$$\begin{aligned}
0 &= (x-(p+q+r))(x-(p+\omega q+\omega^2 r))(x-(p+\omega^2 q+\omega r)) \\
&= ((x-p)-(q+r))((x-p)-(\omega q+\omega^2 r))((x-p)-(\omega^2 q+\omega r)) \\
&= (y-(q+r))(y-(\omega q+\omega^2 r))(y-(\omega^2 q+\omega r)) \\
&= (y-(q+r))(y^2-(\omega^2 q+\omega r)y-(\omega q+\omega^2 r)y+(\omega q+\omega^2 r)(\omega^2 q+\omega r)) \\
&= (y-(q+r))(y^2-(\omega^2 q+\omega r+\omega q+\omega^2 r)y+(\omega^3 q^2+\omega^2 qr+\omega^4 qr+\omega^3 r^2)) \\
&= (y-(q+r))(y^2-(\omega^2 q+\omega r+\omega q+\omega^2 r)y+(q^2+\omega^2 qr+\omega qr+r^2)) \\
&= (y-(q+r))(y^2-((\omega^2+\omega)q+(\omega+\omega^2)r)y+(q^2+(\omega^2+\omega)qr+r^2)) \\
&= (y-(q+r))(y^2-(-q-r)y+(q^2-qr+r^2)) \\
&= (y-(q+r))(y^2+(q+r)y+(q^2-qr+r^2)) \\
&= y^3+(q+r)y^2+(q^2-qr+r^2)y-(q+r)y^2-(q+r)^2y-(q+r)(q^2-qr+r^2) \\
&= y^3+((q^2-qr+r^2)-(q+r)^2)y-(q+r)(q^2-qr+r^2) \\
&= y^3+((q^2-qr+r^2)-(q^2+2qr+r^2))y-(q^3-q^2r+qr^2+q^2r-qr^2+r^3) \\
&= y^3-3qry-(q^3+r^3)
\end{aligned}$$

$$c'=-3qr, d'=-(q^3+r^3)$$

$$\begin{aligned}
0 &= y^3+c'y+d' \\
&= (x-p)^3+c'(x-p)+d' \\
&= (x^3-3px^2+3p^2x-p^3)+c'(x-p)+d' \\
&= x^3-3px^2+3p^2x-p^3+c'x-pc'+d' \\
&= x^3-3px^2+(3p^2+c')x-p^3-pc'+d' \\
&= x^3+bx^2+cx+d
\end{aligned}$$

$$b=-3p, c=3p^2+c', d=-p^3-pc'+d'$$

$$\boldsymbol{p=-\frac{b}{3}}$$

$$\begin{aligned}
c' &= c-3p^2 \\
&= -3qr
\end{aligned}$$

$$d'=d+pc'+p^3$$

$$= d + p(c - 3p^2) + p^3$$
$$= d + pc - 3p^3 + p^3$$
$$= d + pc - 2p^3$$
$$= -(q^3 + r^3)$$
$$qr = -\frac{c'}{3} - \frac{1}{3}(c - 3p^2) = \frac{3p^2 - c}{3} = p^2 - \frac{c}{3}$$
$$q^3 r^3 = \left(p^2 - \frac{c}{3}\right)^3$$
$$u_1 = q^3, u_2 = r^3$$
$$\boldsymbol{u = u_1, u_2}$$
$$0 = (u - u_1)(u - u_2)$$
$$= u^2 - (u_1 + u_2)u + u_1 u_2$$
$$= u^2 - (q^3 + r^3)u + q^3 r^3$$
$$= u^2 + (2p^3 - pc - d)u + \left(p^2 - \frac{c}{3}\right)^3$$
$$\boldsymbol{u^2 + (2p^3 - pc - d)u + \left(p^2 - \frac{c}{3}\right)^3 = 0}$$

- $u_1 = 0$: $\boldsymbol{q = 0, r = u_2^{\frac{1}{3}}}$
- $u_1 \neq 0$: $\boldsymbol{q = u_1^{\frac{1}{3}}, r = \frac{3p^2 - c}{3q}}$

3. Quartics

$$\boldsymbol{Ax^4+Bx^3+Cx^2+Dx+E=0, A\neq 0}$$

$$\boldsymbol{b=\frac{B}{A}, c=\frac{C}{A}, d=\frac{D}{A}, e=\frac{E}{A}}$$

$$x^4+bx^3+cx^2+dx+e=0$$

$$\boldsymbol{x=x_1,x_2,x_3,x_4}$$

$$\boldsymbol{x_1=p+q+r+s, x_2=p+qi-r-si, x_3=p-q+r-s, x_4=p-qi-r+si}$$

$$y=x-p$$

$$\begin{aligned}
0&=(x-x_1)(x-x_2)(x-x_3)(x-x_4)\\
&=(x-(p+q+r+s))(x-(p+qi-r-si))(x-(p-q+r-s))(x-(p-qi-r+si))\\
&=((x-p)-(q+r+s))((x-p)-(qi-r-si))((x-p)-(q+r-s))((x-p)-(-qi-r+si))\\
&=(y-(q+r+s))(y-(qi-r-si))(y-(-q+r-s))(y-(-qi-r+si))\\
&=((y-r)-(q+s))((y+r)-(q-s)i)((y-r)+(q+s))((y+r)+(q-s)i))\\
&=((y-r)-(q+s))((y-r)+(q+s))((y+r)-(q-s)i)((y+r)+(q-s)i))\\
&=((y-r)^2-(q+s)^2)((y+r)^2-((q-s)i)^2)\\
&=((y-r)^2-(q+s)^2)((y+r)^2+(q-s)^2)\\
&=(y^2-2ry+r^2-q^2-2qs-s^2)(y^2+2ry+r^2+q^2-2qs+s^2)\\
&=((y^2+r^2-2qs)-(2ry+q^2+s^2))((y^2+r^2-2qs)+(2ry+q^2+s^2))\\
&=(y^2+r^2-2qs)^2-(2ry+q^2+s^2)^2
\end{aligned}$$

$$u_1=r^2-2qs, u_2=q^2+s^2$$

$$\begin{aligned}
0&=(y^2+u_1)^2-(2ry+u_2)^2\\
&=(y^4+2u_1y^2+u_1^2)-(4r^2y^2+4ru_2y+u_2^2)\\
&=y^4+2u_1y^2+u_1^2-4r^2y^2-4ru_2y-u_2^2\\
&=y^4+(2u_1-4r^2)y^2-4ru_2y+u_1^2-u_2^2
\end{aligned}$$

$$c'=2u_1-4r^2, d'=-4ru_2, e'=u_1^2-u_2^2$$

$$\begin{aligned}
0&=y^4+c'y^2+d'y+e'\\
&=(x-p)^4+c'(x-p)^2+d'(x-p)+e'\\
&=(x^4-4px^3+6p^2x^2-4p^3x+p^4)+c'(x^2-2px+p^2)+d'(x-p)+e'\\
&=x^4-4px^3+6p^2x^2-4p^3x+p^4+c'x^2-2pc'x+p^2c'+d'x-pd'+e'\\
&=x^4-4px^3+(6p^2+c')x^2+(-4p^3-2pc'+d')x+p^4+p^2c'-pd'+e'\\
&=x^4+bx^3+cx^2+dx+e
\end{aligned}$$

$$b=-4p, c=6p^2+c', d=-4p^3-2pc'+d', e=p^4+p^2c'-pd'+e'$$

$$\boldsymbol{p=-\frac{b}{4}}$$

$$\begin{aligned}
c'&=c-6p^2\\
&=2u_1-4r^2
\end{aligned}$$

$$\begin{aligned}
d'&=d+4p^3+2pc'\\
&=d+4p^3+2p(c-6p^2)
\end{aligned}$$

$$= d + 4p^3 + 2pc - 12p^3$$
$$= d - 8p^3 + 2pc$$
$$= -4ru_2$$
$$e' = e - p^4 - p^2c' + pd'$$
$$= e - p^4 - p^2(c - 6p^2) + p(d - 8p^3 + 2pc)$$
$$= e - p^4 - p^2c + 6p^4 + pd - 8p^4 + 2p^2c$$
$$= e - 3p^4 + p^2c + pd$$
$$= u_1^2 - u_2^2$$
$$v = 2r^2$$
$$\boldsymbol{r = (\frac{v}{2})^{\frac{1}{2}}}$$
$$2u_1 = 4r^2 + c'$$
$$u_1 = 2r^2 + \frac{c'}{2}$$
$$= v + \frac{c'}{2}$$
$$d'^2 = 16u_2^2r^2$$
$$\frac{d'^2}{8} = 2u_2^2r^2$$
$$= u_2^2v$$
$$e'v = u_1^2v - u_2^2v$$
$$= (v + \frac{c'}{2})^2v - \frac{d'^2}{8}$$
$$= (v^2 + c'v + \frac{c'^2}{4})v - \frac{d'^2}{8}$$
$$= v^3 + c'v^2 + \frac{c'^2}{4}v - \frac{d'^2}{8}$$
$$\boldsymbol{v^3 + c'v^2 + (\frac{c'^2}{4} - e')v - \frac{d'^2}{8} = 0}$$
$$\boldsymbol{u_1 = v + \frac{c'}{2}}$$
$$\boldsymbol{u_2 = \frac{d'^2}{8v}}$$