Notes on Geometric Algebra

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1 Introduction

Geometric algebras (or real Clifford algebras) provide a unified framework for diverse topics in math and physics, mostly centered on transformations in n-dimensional space. These include but are not limited to the familiar reflections, rotations, and translations of Euclidean space; the hyperbolic rotations, AKA Lorentz boosts, which appear in special relativity; the circle inversions used in compass-and-straightedge constructions and their higher-dimensional analogs; and the perspective transformations which find applications in computer vision. They can represent area, volume, and beyond in the same way that vectors represent length. When Clifford algebras are constructed using complex numbers instead of reals, they capture the unitary transformations and spinors of quantum mechanics. When combined with calculus, geometric algebra encompasses the theory of differential forms.

1.1 Linear algebra review

Since geometric algebra is an extension of linear algebra, a review of the basic ideas of the latter will be helpful in understanding the former. For this section, scalars are in italics and vectors are in bold. However, geometric algebra adds many other objects to scalars and vectors (and in fact, scalars are themselves represented as vectors), so the distinction between them becomes less meaningful. For this reason, all variables will be italicized in later sections.

A vector space consists of a set of scalars F and a set of vectors V. F must be a field, meaning that scalars may be added, subtracted, multiplied, or divided (except by zero) to produce another scalar. The functions of addition and multiplication, as well as the additive and multiplicative inverses which make subtraction and division possible, must obey certain properties for F to be a field, e.g. $a + b = b + a$ for all scalars a and b. These notes will focus mostly on the cases where $F = \mathbb{R}$, the field of real numbers. For V to form a vector space, vectors must be able to be added to one another or multiplied by a scalar to produce another vector. These functions of vector addition and scalar multiplication, too, must obey certain properties, e.g. $a(\mathbf{u}+\mathbf{v}) = a\mathbf{u}+a\mathbf{v}$ for all scalars a and vectors $\mathbf{u}$ and $\mathbf{v}$. (This property can be stated as "the scalar product respects vector addition".)

Every vector space has a basis, a set of vectors which may be multiplied by scalars and added to each other to uniquely produce any vector in V. The size of the basis is called the dimension of V. These notes are concerned with vector spaces with finite dimension n, so for a given basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, every vector $\mathbf{v} \in V$ can be uniquely represented as $v_1\mathbf{e}_1 + \ldots + v_n\mathbf{e}_n$, where $v_1, \ldots, v_n \in F$. The vector space is thus called F^n, or specifically $\mathbb{R}^n$ in the case of geometric algebras. For short, $\mathbf{v}$ is often written as a column listing these scalars:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

Vectors are basic objects in linear algebra, along with matrices, which represent ways to transform vectors in a way that respects vector addition and scalar multiplication.

2 Definition

2.1 Algebras

Despite everything than can be shown about vector spaces alone, they do not define a way to multiply two vectors. This lack gives rise to the idea of an algebra: a vector space equipped with an additional operation which multiplies two vectors to produce another vector. This vector product is constrained by only one rule: it must be linear with respect to both arguments, i.e. when one argument is fixed, it must respect vector addition and scalar multiplication in the other argument. In general, the vector product does not need to be commutative, associative, etc. However, thanks to the constraint of linearity, it is only necessary to define the products of the basis vectors with each other, and all other values of the product become defined. For example, consider a two-dimensional vector space with a basis $\{e_1, e_2\}$ and two vectors within that space

$u = \begin{bmatrix} a \\ b \end{bmatrix}$ and $v = \begin{bmatrix} c \\ d \end{bmatrix}$. [1] If e_1e_1, e_1e_2, e_2e_1, and e_2e_2 are known, then

$$uv = \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = (ae_1 + be_2)(ce_1 + de_2) = ace_1e_1 + ade_1e_2 + bce_2e_1 + bde_2e_2.$$

Scalars must commute with the vector product, hence ace_1e_1 is written above instead of ae_1ce_1. However, e_1e_2 and e_2e_1 are distinguished and not combined as like terms, since the vector product itself is not necessarily commutative.

2.1.1 The cross product

For example, the cross product $\times$ forms an algebra over $\mathbb{R}^3$, and it returns the following for all pairs of basis vectors, where 0 is the null vector:

$\times$	e_1	e_2	e_3
e_1	0	e_3	$-e_2$
e_2	$-e_3$	0	e_1
e_3	e_2	$-e_1$	0

As shown in the table, the cross product is anticommutative rather than commutative, i.e. $u \times v = -v \times u$ for all basis vectors and, by extension, for all vectors u and v. This vector product is widely used in physics to calculate quantities related to rotation, such as angular momentum. However, it cannot be generalized to higher dimensions, but it can be restricted to work in two by omitting the last row and column of the table. Notice, though, that the result will simply be a scalar multiple of e_3, so the cross product in 2D effectively returns a scalar instead of a vector. The cross product, then, has the peculiar property that it returns a scalar in two dimensions, a vector in three dimensions, and nothing in any other dimension. Additionally, since the components of vectors usually contain a unit of length, e.g. meters or meters/second, the cross product of two such vectors has units of area.

Geometric algebra defines an alternative to the cross product which works in any number of dimensions, which always returns the same type of object, and which has a more natural geometric interpretation. This will be discussed in later sections.

2.2 Quadratic forms

2.2.1 Signature

3 Properties

3.1 Conic sections: a tangent

4 Representing transformations

4.1 Reflections, rotations, and translations

4.2 Symmetries of spacetime

4.2.1 Classical mechanics

Classical (Newtonian) mechanics relies on three principles governing the symmetries of space and time. For this section, "frames of reference" refer specifically to inertial frames of reference, namely those of observers which are not accelerating.

1. Shifts between frames of reference are performed by linear transformations.

2. The laws of physics remain the same for all frames of reference.

3. The passage of time is absolute for all frames of reference.

[1] As mentioned in the review of linear algebra, vectors will be italicized like other variables from this point forward.

4.2.2 Special relativity

4.3 Conformal transformations

5 From $\mathbb{R}$ to $\mathbb{C}$ to $\mathbb{H}$

So far, the components of multivectors have been real numbers, and the Clifford algebra has been generated by a quadratic form. With a few tricks related to representations, we can extend what we know about Clifford algebras over the real numbers $\mathbb{R}$ to related algebras over the complex numbers $\mathbb{C}$ and algebra-like objects over the quaternions $\mathbb{H}$.

5.1 $\mathbb{R}$ to $\mathbb{C}$

A complex number $a + bi$ can be represented by the 2×2 real matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ so that complex addition, multiplication, and inverses map to the corresponding operations on matrices. For example,

$$(a + bi)(c + di) = (ac - bd) + (bc + ad)i,$$
$$\text{and} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac - bd & -(bc + ad) \\ bc + ad & ac - bd \end{bmatrix}.$$

This matrix can also be thought of as a linear transformation of the plane, namely a scaling and rotation.

Likewise, an $m \times n$ complex matrix can be represented as a $2m \times 2n$ real matrix, with each 2×2 block of the real matrix representing a single entry of the complex matrix. Since breaking down matrices into blocks, multiplying them as units, then adding the results together is a valid method of matrix multiplication, the complex matrix product is preserved with this real representation.

5.2 $\mathbb{C}$ to $\mathbb{H}$

6 Lie groups and Lie algebras