Unit-5 → Functions of complex variable

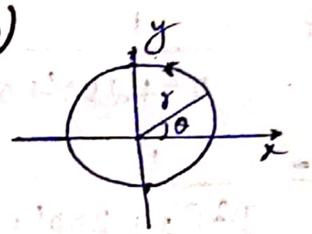
Complex Numbers :→

– A number of the form $z = a+ib$ is called complex number, (where a and b are real numbers.

– Also $\bar{z} = a-ib$ is called conjugate to $z = a+ib$

"A pair of complex no. $a+ib$ and $a-ib$ are said to be conjugate of each other.

$\boxed{z = x+iy}$ (Cartesian form)



put $x = r\cos\theta$

$y = r\sin\theta$

$z = r\cos\theta + ir\sin\theta$

or $\boxed{z = r(\cos\theta + i\sin\theta)}$ (polar form) polar plane

Exponential form → $\boxed{z = re^{i\theta}}$ $(\because e^{i\theta} = \cos\theta + i\sin\theta)$

Modulus and Argument :→

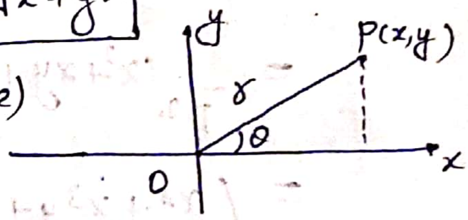
Let $x+iy$ be a complex number

put $x = r\cos\theta$ and $y = r\sin\theta$ } — ①

Squaring and adding, we get $x^2+y^2 = r^2(\cos^2\theta + \sin^2\theta)$

or $x^2+y^2 = r^2$

or $\boxed{r = \sqrt{x^2+y^2}}$



The "r" is called modulus (absolute value) of the complex no. $x+iy$.

from Equation ① (dividing)

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta}$$

$\Rightarrow \tan\theta = y/x$

or $\boxed{\theta = \tan^{-1}(y/x)}$

The angle "θ" is called argument (amplitude) of the complex no. $x+iy$.

Complex Variable Function

Definition: →

$f(z)$ is a function of a complex variable z and is denoted by

$w = f(z)$

or $w = u + iv$

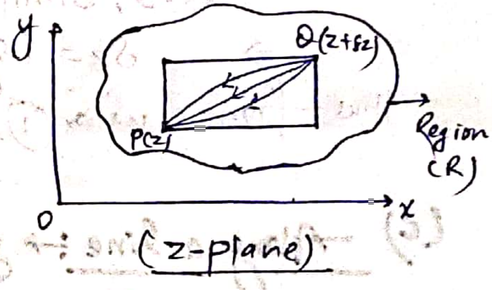
Real part of $f(z)$ (u) — Imaginary part of $f(z)$ (v)

*** Differentiability:** →

Let $f(z)$ be a single valued function of the variable z, then

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

provided that the limit exists and it is independent of the path along which $\delta z \to 0$



Ex. 1 Consider the function

$$f(z) = 4x + y + i(-x + 4y)$$

and discuss differentiability.

Sol: → Here $f(z) = 4x + y + i(-x+4y)$ —— (1)

Comparing it $f(z) = u + iv$

(where $u = 4x + y$ and $v = -x + 4y$)

thus $$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} \quad \text{—— (2)}$$

$$f(z+\delta z) = 4(x+\delta x) + (y+\delta y) - i(x+\delta x) + 4i(y+\delta y)$$

and $\delta z = \delta x + i\delta y$ ($\because z = x + iy$)

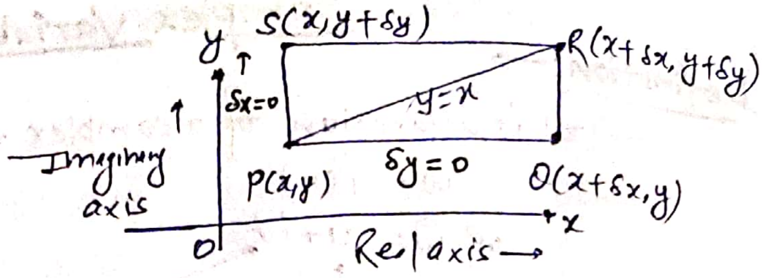
put this value in Eq. (2), we get

$$f'(z) = \frac{\delta f}{\delta z} = \frac{[4(x+\delta x) + (y+\delta y) - i(x+\delta x) + 4i(y+\delta y)] - [4x + y + i(-x+4y)]}{\delta x + i\delta y}$$

$$\Rightarrow \frac{\delta f}{\delta z} = \frac{\cancel{4x} + 4\delta x + \cancel{y} + \delta y - \cancel{ix} - i\delta x + \cancel{4iy} + 4i\delta y - \cancel{4x} - \cancel{y} + \cancel{ix} - \cancel{4iy}}{\delta x + i\delta y}$$

$$\Rightarrow \frac{\delta f}{\delta z} = \frac{4\delta x + \delta y - i\delta x + 4i\delta y}{\delta x + i\delta y} \quad \text{—— (3)}$$

(a) Along real axis :→



If Q is taken on the horizontal line through $P(x,y)$ and Q then approaches P along this line, we have $\delta y=0$ and then $\delta z=\delta x$ } putting these values in Eq ③ thus from equation ③, we get

$$\frac{\delta f}{\delta z} = \frac{4\delta x - i\delta x}{\delta x} = 4-i$$

(b) Along imaginary axis :→ If S is taken on the vertical line through P and then Q approaches P along this line, we have

$x=0$, similarly $\delta x=0$

$\Rightarrow \quad \delta z = i\delta y$, put δz in equation ③,

thus from equation ③, we get $\frac{\delta f}{\delta z} = \frac{\delta y + 4i\delta y}{i\delta y} = \frac{1}{i}(1+4i) = 4-i$

(c) Along a line :→ $y=x$:→

then $z = x+iy$

or $z = x+ix = (1+i)x$

On putting these values in Eq. ③, we get

$$\frac{\delta f}{\delta z} = \frac{4\delta x + \delta x - i\delta x + 4i\delta x}{\delta x + i\delta x} = \frac{4+1-i+4i}{1+i} = \frac{5+3i}{1+i}$$

$$\text{or } \frac{\delta f}{\delta z} = \frac{(5+3i)(1-i)}{(1+i)(1-i)} = 4-i$$

In all these three paths}, we get same value $\frac{\delta f}{\delta z} = 4-i$ approaching from P }→

Analytic function :→

Def: → A function $f(z)$ is said to be analytic at a point z_0 if "f" is differentiable not only at z_0 but at every point of some neighbourhood of z_0.

– A function $f(z)$ is analytic in a domain if it is analytic at every point of the domain.

The point at which the function is not differentiable is called a singular point of the function.

– An analytic function is also know as "holomorphic" function.

The Necessary and sufficient conditions for $f(z)$ to be analytic :-

→ The necessary condition for a function $f(z) = u + iv$ to be analytic at all the points in a region R are

$$\left.\begin{array}{ll} \text{(i)} & \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \text{(ii)} & \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{array}\right\} \text{known as "Cauchy Riemann (CR)" equations}$$

provided $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ exist.

The sufficient condition for a function $f(z) = u + iv$ to be analytic at all the points in a region R are

(i) $\partial u/\partial x = \partial v/\partial y$

(ii) $\partial u/\partial y = -\partial v/\partial x$

and $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$ and $\partial v/\partial y$ are continuous function of x and y in region R.

Note: (i) C-R conditions are necessary but not sufficient for analytic funⁿ.

(ii) C-R conditions are sufficient if partial derivatives are continuous.

Ex. ① Show that the function $f(z) = e^x(\cos y + i\sin y)$ is an analytic function and find it's derivative.

Sol:→ Given $f(z) = e^x(\cos y + i\sin y)$

or $f(z) = \underbrace{e^x\cos y}_{\text{Real part of } f(z)} + i\underbrace{e^x\sin y}_{\text{Imaginary part of } f(z)}$

Comparing it with $f(z) = u + iv$

so $u = e^x\cos y$ and $v = e^x\sin y$

then $\frac{\partial u}{\partial x} = e^x\cos y$, $\frac{\partial v}{\partial x} = e^x\sin y$

Similarly $\frac{\partial u}{\partial y} = -e^x\sin y$, $\frac{\partial v}{\partial y} = e^x\cos y$

Here we see that $\partial u/\partial x = \partial v/\partial y$

and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

that is, C-R equations are satisfied and partial derivatives are continuous. Hence $f(z) = e^x(\cos y + i\sin y)$ is analytic.

thus derivative $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= e^x\cos y + ie^x\sin y$

$= e^x(\cos y + i\sin y)$

$= e^x \cdot e^{iy}$

$= e^{x+iy}$

$f'(z) = e^z$ **Ans.**

Ex.② Determine whether $\frac{1}{z}$ is analytic or not?

let $\omega = \frac{1}{z}$

or $\omega = \frac{1}{x+iy}$

or $\omega = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2}$

or $\omega = \underbrace{\frac{x}{x^2+y^2}}_{\text{Real part of } \omega} - i\underbrace{\frac{y}{x^2+y^2}}_{\text{Imaginary part of } \omega}$

Thus $u = \frac{x}{x^2+y^2}$ and $v = \frac{-y}{x^2+y^2}$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{(x^2+y^2).1 - x.(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$

$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}$ and $\frac{\partial v}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Thus C-R equations are satisfied, also partial derivatives are continuous except at $(0,0)$. Thus $(1/z)$ is analytic everywhere except $z=0$. Also $\frac{d\omega}{dz} = \frac{-1}{z^2}$

C-R equations in polar form:—

$$\boxed{\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{r}\frac{\partial v}{\partial \theta} \\ \frac{\partial u}{\partial \theta} &= -r\frac{\partial v}{\partial r} \end{aligned}}$$

Ex. 1 Show that $f(z) = \frac{1}{z}$ is analytic except $z = 0$.

Sol:→ given $f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$

or $f(z) = \frac{1}{r}e^{-i\theta} = \frac{1}{r}(\cos\theta - i\sin\theta)$

$$\Rightarrow f(z) = \underbrace{\frac{\cos\theta}{r}}_{u} - i\underbrace{\frac{\sin\theta}{r}}_{v}$$

So $\frac{\partial u}{\partial r} = -\frac{\cos\theta}{r^2}$ and $\frac{\partial u}{\partial \theta} = -\frac{\sin\theta}{r}$

also $\frac{\partial v}{\partial \theta} = \frac{\sin\theta}{r^2}$ and $\frac{\partial v}{\partial \theta} = -\frac{\cos\theta}{r}$

$\Rightarrow \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$

Hence Cauchy-Riemann (CR) equations are satisfied, then $f(z) = 1/z$ is analytic except $z=0$.

Harmonic function

— Any function f(z) which satisfies the Laplace's equation is known as a harmonic function.

i.e. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ or $\nabla^2 u = 0$

Similarly $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ or $\nabla^2 v = 0$

Therefore both u and v are harmonic function.

→ Such function u, v are called conjugate harmonic function, if $u+iv$ is also analytic function.

Definition of Harmonic conjugate of a function :→

let $u(x,y)$ be a harmonic function. Then a function $v(x,y)$ is said to be a Harmonic conjugate of $u(x,y)$ if :→

(i) $v(x,y)$ is harmonic

(ii) C-R equations satisfied (ie. $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$)

Questions based on Harmonic function :→

Q1 :- prove that the following functions are Harmonic

(i) $u = 2x - x^3 + 3xy^2$

(ii) $u = x^3 - 3xy^2$ is Harmonic and find the corresponding analytic function f(z).

Sol :→ (i) Given $u = 2x - x^3 + 3xy^2$

then $\frac{\partial u}{\partial x} = 2 - 3x^2 + 3y^2$, $\frac{\partial^2 u}{\partial x^2} = -6x$

and $\frac{\partial u}{\partial y} = 6xy$, $\frac{\partial^2 u}{\partial y^2} = 6x$

Then by Laplace Equation

$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6x + 6x = 0$

$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Hence given function $u = 2x - x^3 + 3xy^2$ is a harmonic function.

(ii) Given $u = x^3 - 3xy^2$

$\Rightarrow \frac{\partial u}{\partial x} = 3x^2 - 3y^2, \ \frac{\partial^2 u}{\partial x^2} = 6x$

and $\frac{\partial u}{\partial y} = -6xy, \ \frac{\partial^2 u}{\partial y^2} = -6x$

According to Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0$

$\Rightarrow \nabla^2 u = 0$

Hence $u = x^3 - 3xy^2$ is harmonic function;

Now we have to find $f(z)$

Since $f(z) = u + iv$

and $f'(z) = u_x + iv_x$ $\left(\text{or } f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)$

or $f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ $\left(\because \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}; \text{Acc to C-R Equation}\right)$

$\Rightarrow f'(z) = (3x^2 - 3y^2) - i(-6xy)$

$= 3x^2 - 3y^2 + i6xy$

$= 3(x^2 - y^2 + 2ixy)$

$= 3(x + iy)^2$

$f'(z) = 3z^2$

$\int f'(z)\,dz = 3\int z^2 dz + c_1$

$\Rightarrow f(z) = \frac{\cancel{3}z^3}{\cancel{3}} + c_1$

or $f(z) = z^3 + c_1$ **Ans.**

Alternate method:→ Given $u = x^3 - 3xy^2$

$\partial u/\partial x = 3x^2 - 3y^2, \ \partial^2 u/\partial x^2 = 6x$

$\partial u/\partial y = -6xy, \ \partial^2 u/\partial y^2 = -6x$

Then $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$

or $dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$ $\left[\because \text{Acc. to CR equation } \partial v/\partial x = -\partial u/\partial y \text{ and } \partial v/\partial y = \partial u/\partial x\right]$

$\Rightarrow dv = +6xy\,dx + (3x^2 - 3y^2)\,dy$

$\int dv = 6\int \underset{(y\text{-as const.})}{xy\,dx} + \int (3x^2 - 3y^2)dy$, ignoring terms of x → (Same as Exact diff. Equation)

$v = \cancel{6}\frac{x^2}{\cancel{2}}y + \int -3y^2 dy + c_1$

or $v = 3x^2y - \cancel{3}\frac{y^3}{\cancel{3}} + c_1$

$v = 3x^2y - y^3 + c_1$

Thus $f(z) = u + iv$

$= x^3 - 3xy^2 + i(3x^2y - y^3 + c_1)$

$= (x+iy)^3 + c_1$

$f(z) = z^3 + c_1$ **Ans.**

Milne-Thomson Method (To construct an analytic function)

By this method $f(z)$ is directly construct without finding v and the method is given below:→

Since we know :→ $z = x+iy$ and $\bar{z} = x - iy$

$\Rightarrow x = \frac{z+\bar{z}}{2}$ → $\left\{ \because z+\bar{z} = x+\cancel{iy}+x-\cancel{iy} \Rightarrow z+\bar{z} = 2x \Rightarrow \frac{z+\bar{z}}{2} = x \right\}$

and $y = \frac{z-\bar{z}}{2i}$ → $\left\{ \because z-\bar{z} = \cancel{x}+iy-\cancel{x}+iy,\ z-\bar{z} = 2iy \Rightarrow \frac{z-\bar{z}}{2i} = y \right\}$

We know $f(z) = u(x,y) + iv(x,y)$ — (1)

$f(z) \equiv u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2}\right)$

Replacing $\bar{z}$ by z, we get

$f(z) = u(z,0) + iv(z,0)$

* **Case I:→** If u is given :→

then $\boxed{f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz + c}$

Where $\phi_1(x,y) = \frac{\partial u}{\partial x}$ and $\phi_2(x,y) = \frac{\partial u}{\partial y}$

→ $\phi_1(x,y) = \frac{\partial u}{\partial x}$ * $\phi_2(x,y) = \frac{\partial u}{\partial y}$

Case II: → If v is given:→

$$\boxed{f(z) = \int \psi_1(z,0)\,dz + i\int \psi_2(z,0)\,dz + c}$$

(where $\psi_{1(x,y)} = \frac{\partial v}{\partial y}$ and $\psi_{2(x,y)} = \frac{\partial v}{\partial x}$)

Ex. 1 If $u = x^2 - y^2$, find a corresponding analytic function

$$\frac{\partial u}{\partial x} = 2x = \phi_1(x,y)$$

and $\frac{\partial u}{\partial y} = -2y = \phi_2(x,y)$

Replacing x by z and y by 0, we get →

$\phi_1(z,0) = 2z$

$\phi_2(z,0) = 0$

According to Milne-Thomson Method $\Big\}$ $f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz + c$

$$= \int 2z\,dz - i\int 0\,dz + c$$

$$f(z) = \frac{\cancel{2}z^2}{\cancel{2}} + c$$

$$\Rightarrow f(z) = z^2 + c$$ **Ans.**

Ex. 2 Find the analytic function for which $e^x(x\cos y - y\sin y)$ is imaginary part.

Sol: → Here imaginary part means "v" is given such that

$$v = e^x(x\cos y - y\sin y)$$

$$\frac{\partial v}{\partial x} = e^x(x\cos y - y\sin y) + e^x\cos y = \psi_2(x,y) \quad \text{--- (1)}$$

$$\frac{\partial v}{\partial y} = e^x(-x\sin y - y\cos y - \sin y) = \psi_1(x,y) \quad \text{--- (2)}$$

We need $\psi_1(z,0)$ and $\psi_2(z,0)$ then put $x=z$ and $y=0$ in Eq. ① and ②, We get

$$\psi_2(z,0) = e^z(z\cos 0 - 0\sin 0) + e^z\cos 0$$

then $\psi_2(z,0) = e^z(z) + e^z$

and $\psi_1(z,0) = e^z(-z\sin 0 - 0\cos 0 - \sin 0)$

$$= e^z(0) = 0$$

and $\psi_1 = (z,0) = 0$

Then by Milne-Thomson Method

$$f(z) = \int \psi_1(z,0)\,dz + i\int \psi_2(z,0)\,dz + c$$

$$= \int 0\,dz + i\int (ze^z + e^z)\,dz + c$$

$$= i(ze^z - \cancel{e^z} + \cancel{e^z}) + c$$

$$= \underline{ize^z + c} \quad \text{Ans.}$$

Unit ⑤ :→ Assignment No. 1

Q.1 Find the analytic function whose real part is $e^{2x}(x\cos 2y - y\sin 2y)$

Q.2 Show that the function $u(x,y) = x^4 - 6x^2y^2 + y^4$ is harmonic.

Q.3 Check the analyticity of the function $w = \log z$.

Q.4 (a) Find the values of c_1 and c_2 such that the function

$$f(z) = x^2 + c_1y^2 - 2xy + i(c_2x^2 - y^2 + 2xy)$$ is analytic,

also find $f'(z)$.

Q.4 (b) Show that the function $f(z)$ defined by

$$f(z) = \frac{x^3y^5(x+iy)}{x^6+y^{10}},\ z \neq 0,\ f(0) = 0$$ is not

analytic at the origin even though it satisfies

Cauchy-Riemann equation at the origin.

Q.5 Prove that $u = x^2 - y^2$ and $v = \frac{y}{x^2+y^2}$ are harmonic function

of (x,y) but are not harmonic conjugates.

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Solutions of Assignment No. 1 (Complex Analysis)

Q. (1) Ans: → (Sol): →

Given real part of analytic function, is $e^{2x}(x\cos 2y - y\sin 2y)$

So $u = e^{2x}(x\cos 2y - y\sin 2y)$

Now from Milne-Thomson Method: →

$$\frac{\partial u}{\partial x} = e^{2x}(\cos 2y) + (x\cos 2y - y\sin 2y)\,2e^{2x} = \phi_1(x,y) \quad \text{—(1)}$$

and $\frac{\partial u}{\partial y} = e^{2x}[-2x\sin 2y - (y\cos 2y \cdot 2 + \sin 2y)]$

$$\text{or } \frac{\partial u}{\partial y} = -2xe^{2x}\sin 2y - e^{2x}(2y\cos 2y + \sin 2y) = \phi_2(x,y) \quad \text{—(2)}$$

Replacing x by z and y by 0 in Eq (1) and (2), we get

$$\phi_1(z,0) = e^{2z}[\cos 2(0)] + [z\cos 2(0) - 0\sin 2(0)]\,2e^{2z}$$

$$\text{or } \phi_1(z,0) = e^{2z} + (z)\,2e^{2z}$$

Also $\phi_2(z,0) = -2ze^{2z}\sin 2(0) - e^{2z}[2\times 0\cos 2(0) + \sin 2(0)]$

$$\text{or } \phi_2(z,0) = 0$$

So According to Milne-Thomson Method: →

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz + C$$

$$f(z) = \int (e^{2z} + 2ze^{2z})\,dz - i\int 0 + C$$

$$\text{or } f(z) = \frac{e^{2z}}{2} + 2\left[\frac{ze^{2z}}{2} - \frac{e^{2z}}{4}\right] + C$$

$$\text{or } f(z) = \cancel{\frac{e^{2z}}{2}} + ze^{2z} - \cancel{\frac{e^{2z}}{2}} + C$$

$$\Rightarrow \boxed{f(z) = ze^{2z} + C} \quad \textbf{Ans.}$$

Q.(2) Sol: → Given $u = x^4 - 6x^2y^2 + y^4$

Thus $\frac{\partial u}{\partial x} = 4x^3 - 12xy^2$, $\frac{\partial^2 u}{\partial x^2} = 12x^2 - 12y^2$

and $\frac{\partial u}{\partial y} = -12x^2y + 4y^3$, $\frac{\partial^2 u}{\partial y^2} = -12x^2 + 12y^2$

Therefore $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 12x^2 - 12y^2 - 12x^2 + 12y^2 = 0$

$\Rightarrow \nabla^2 u = 0$ (Harmonic function) ($\because$ it satisfies Laplace Equation)

Q.3 Sol:→ Given $\omega = \log z$

$$= \log(re^{i\theta}) \quad (\text{polar form of } z = re^{i\theta})$$

$$= \log[r(\cos\theta + i\sin\theta)] \quad (\because e^{i\theta} = \cos\theta + i\sin\theta)$$

$$= \log r + \log e^{i\theta}$$

$$= \log r + i\theta$$

or $$\omega = \underbrace{\log\sqrt{x^2+y^2}}_{u} + i\underbrace{\tan^{-1}\left(\frac{y}{x}\right)}_{v} \quad \left\{\because r = \sqrt{x^2+y^2} \text{ and } \theta = \tan^{-1}(y/x)\right\}$$

Here $u = \log\sqrt{x^2+y^2}$ and $v = \tan^{-1}\left(\frac{y}{x}\right)$

Therefore $\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{x^2+y^2}(2x) = \frac{x}{x^2+y^2}$

and $\frac{\partial v}{\partial y} = \frac{1}{\left(1+\frac{y^2}{x^2}\right)}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$

$$\Rightarrow \partial u/\partial x = \partial v/\partial y \quad \text{---} (1)$$

Again $\frac{\partial u}{\partial y} = \frac{1}{2}\frac{1}{x^2+y^2}(2y) = \frac{y}{x^2+y^2}$

and $\frac{\partial v}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \quad \text{---} (2)$$

Eq. (1) and (2) are C-R equations, and all partial derivatives are continuous. Hence $\omega = \log z$ is an analytic function except zero.

Reason → except zero $\left\{ \text{when } x^2+y^2=0 \Rightarrow x=y=0 \Rightarrow x+iy=0 \Rightarrow z=0 \right\}$

Q.4(a) Sol:→ Given $f(z) = \underbrace{x^2+c_1y^2-2xy}_{u} + i(\underbrace{c_2x^2-y^2+2xy}_{v})$ is analytic

Here $u = x^2+c_1y^2-2xy$
$v = c_2x^2-y^2+2xy$ } If $f(z)$ is analytic, thus it should have satisfied C-R Equation.

$\Rightarrow \frac{\partial u}{\partial x} = 2x - 2y \;,\; \frac{\partial v}{\partial x} = 2c_2 x + 2y$

and $\frac{\partial u}{\partial y} = 2c_1 y - 2x \;,\; \frac{\partial v}{\partial y} = -2y + 2x$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ (✓) (true) $\Rightarrow \underbrace{2x - 2y = -2y + 2x}_{\text{Identically satisfied}}$

Similarly $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\Rightarrow (2c_1 y - 2x) = -(2c_2 x + 2y)$

$\Rightarrow 2c_1 y - 2x = -2c_2 x - 2y$

Comparing both side $c_1 = -1$ ✓

and $c_2 = 1$ ✓ **Ans.**

put c_1 and c_2 in function, we get

$f(z) = x^2 - y^2 - 2xy + i(x^2 - y^2 + 2xy)$

and Also $f'(z) = \partial u/\partial x + i\,\partial v/\partial x$

$= (2x - 2y) + i(2x + 2y)$

$\Rightarrow 2(x-y) + 2i(x+y) \Rightarrow 2[(x-y) + i(x+y)]$

$= 2[x - y + ix + iy]$

$\Rightarrow 2[(x+iy) + i(x+iy)]$

$\Rightarrow 2[(x+iy)(1+i)]$

Hence $f'(z) = 2z(1+i)$ **Ans.**

Q.4(b) Sol:→ Given $f(z) = \frac{x^3 y^5 (x+iy)}{x^6 + y^{10}}, \; z \neq 0, \; f(0) = 0$

$\Rightarrow f(z) = \underset{u}{\underbrace{\frac{x^4 y^5}{x^6 + y^{10}}}} + i \underset{v}{\underbrace{\frac{x^3 y^6}{x^6 + y^{10}}}}$

So $u = \frac{x^4 y^5}{x^6 + y^{10}}, \; v = \frac{x^3 y^6}{x^6 + y^{10}}$

$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(0+h, 0) - u(0,0)}{h} = \lim_{h \to 0} \frac{0/h^6}{h} = \lim_{h \to 0} \frac{0}{h} = 0$

$\frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{u(0, 0+k), u(0,0)}{k} = \lim_{k \to 0} \frac{0/k^{10}}{k} = \lim_{k \to 0} \frac{0}{k} = 0$

$\frac{\partial v}{\partial x} = \lim_{h \to 0} \frac{v(0+h, 0) - v(0,0)}{h} = \lim_{h \to 0} \frac{(0/h^6)}{h} = \lim_{h \to 0} \frac{0}{h} = 0$

and $\frac{\partial v}{\partial y} = \lim_{k \to 0} \frac{v(0, 0+k) - v(0,0)}{k} = \lim_{k \to 0} \frac{(0/k^{10})}{k} = \lim_{k \to 0} \frac{0}{k} = 0$

$\Rightarrow \partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$

$\Rightarrow$ C-R Equations are satisfied at origin.

↓

$\Rightarrow$ But $f'(0) = \lim_{z \to 0} \dfrac{f(0+z) - f(0)}{z} = \lim_{\substack{x\to 0\\ y\to 0}} \dfrac{\left[\dfrac{x^3y^5(x+iy)}{x^6+y^{10}} - 0\right]}{x+iy}$

$$f'(0) = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3y^5}{x^6+y^{10}} \quad \text{—— (1)}$$

let $z \to 0$ along the radius vector $y = mx$, then Eq. (1) becomes:→

$$f'(0) = \lim_{x\to 0} \frac{x^3(mx)^5}{x^6+(mx)^{10}} = \lim_{x\to 0} \frac{x^8m^5}{x^6+m^{10}x^{10}}$$

$$= \lim_{x\to 0} \frac{x^2m^5}{1+m^{10}x^4} = \frac{0}{1+0} = \frac{0}{1} = 0 \quad \text{—— (2)}$$

Again let $z \to 0$ along the curve $y^5 = x^3$; Eq. (1) becomes:→

$$f'(0) = \lim_{x\to 0} \frac{x^3 \cdot x^3}{x^6+x^6} = \lim_{x\to 0} \frac{x^6}{x^6+x^6} = \frac{1}{2} \quad \text{—— (3)}$$

Eq. (2) and (3) shows that $f'(0)$ does not exist. Hence $f(z)$ is not analytic at origin although C-R Equations are satisfied there.

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Q. 5 Prove that $u = x^2 - y^2$ and $v = \frac{y}{x^2+y^2}$ are harmonic function of (x,y) but are not harmonic conjugates.

Sol. of Q. 5 :→ for function to be harmonic, it should satisfies Laplace-equations

Here $u = x^2 - y^2$ and $v = \frac{y}{x^2+y^2}$

$\Rightarrow \frac{\partial u}{\partial x} = 2x$

also $\frac{\partial^2 u}{\partial x^2} = 2$

$\partial u/\partial y = -2y$

$\partial^2 u/\partial y = -2$

$\frac{\partial v}{\partial x} = \frac{-2xy}{(x^2+y^2)^2}$

$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2+y^2)^2(-2y) + 2xy[2(x^2+y^2)2x]}{(x^2+y^2)^4} = \frac{-2y}{(x^2+y^2)^2} + \frac{8x^2y}{(x^2+y^2)}$

also $\frac{\partial v}{\partial y} = \frac{x^2-y^2}{(x^2+y^2)^2}$

and $\frac{\partial^2 v}{\partial y^2} = \frac{-6y}{(x^2+y^2)^2} + 8y^3(x^2+y^2)^{-3}$

Here $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ (both u and v, satisfy Laplace equation)

but $\partial u/\partial x \neq \partial v/\partial y$ and $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$

⇒ " If function is harmonic it doesn't mean it is Analytic function

⇒ Thus "If function is Analytic ⇒ function is harmonic."

Line Integral:→

In case of real variable, the path of integration of $\int_a^b f(x)\,dx$ is always along the x-axis from $x=a$ to $x=b$

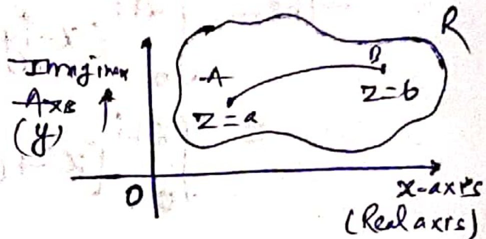
But in case of complex function $f(z)$ the path of definite integral $\int_a^b f(z)\,dz$ can be along any curve from $z=a$ to $z=b$

→ As we know $z = x+iy$

$$\Rightarrow dz = dx + i\,dy \quad -(1)$$

if $y=0$, then $dz = dx \quad -(2)$

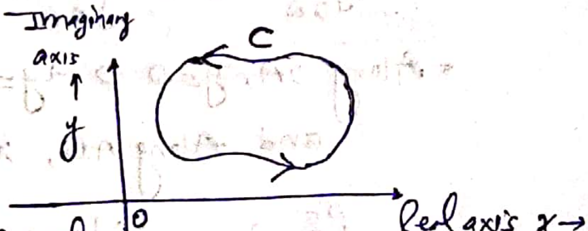
and if $x=0$, then $dz = i\,dy \quad -(3)$



In (1), (2), (3) the direction of dz are different. Its value depends upon the path (curve) of integration. But the value of integral from a to b remains the same along any regular curve from a to b.

Contour Integral:→

→ In case the initial point and final point coincide, so that c is a closed curve, then this integral is called contour integral and denoted by $\oint_c f(z)\,dz$



$$= \oint_c (u+iv)(dx+i\,dy)$$

$$= \oint_c u\,dx + iu\,dy + iv\,dx - v\,dy$$

$$= \oint_c (u\,dx - v\,dy) + i\oint_c (v\,dx + u\,dy)$$

Thus $\oint_c f(z)\,dz = \oint_c (u\,dx - v\,dy) + \oint_c i(v\,dx + u\,dy)$

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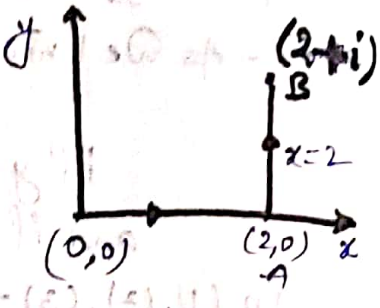
Q.(7) Evaluate $\int_0^{2+i} (\bar{z})^2 dz$ along the real axis from $z=0$ to $z=2$ and then along a line parallel to y-axis from $z=2$ to $z=2+i$

(We know $\bar{z} = x-iy$ and $dz = dx+idy$

then $\int_0^{2+i} (x-iy)^2 (dx+idy)$



$= \int_0^{2+i} [x^2+(iy)^2-2ixy][dx+idy]$

$= \int_0^{2+i} (x^2-y^2-2ixy)(dx+idy)$

$\sim \int_0^{2+i} (x^2 dx + i(2-iy)^2 dy$

$\Rightarrow \int_{OA} x^2 dx + \int_{AB} i(2-iy)^2 dy$

Along OA, $y=0 \Rightarrow dy=0$, so x varies from 0 to 2.

and Along AB, $x=2$, $dx=0$, so y varies from 0 to 1

$\int_0^2 x^2 dx + \int_0^1 i(2-iy)^2 dy$

$= \left[\frac{x^3}{3}\right]_0^2 + i\int_0^1 (4-4iy-y^2)dy$

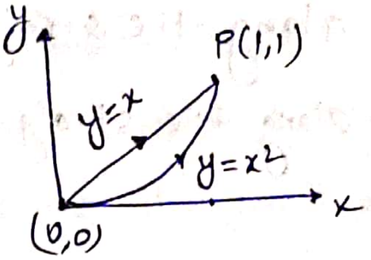
$= \frac{8}{3} + i\left[4y-2iy^2-\frac{y^3}{3}\right]_0^1$

$= \frac{8}{3} + i\left[4-2i-\frac{1}{3}\right] = \frac{8}{3} + \frac{i}{3}(11-6i)$

$= \frac{1}{3}(8+11i+6) = \frac{1}{3}(14+11i)$

Q.① Evaluate $\int_0^{1+i}(x^2-iy)\,dz$ along the path (a) $y=x$ (b) $y=x^2$

Sol:- (a) – Along the path $y=x$ —①



then, we know $z=x+iy$

$dz = dx + i\,dy$ —②

from Eq.① $y=x$, then $dy=dx$, put this value in ② we get

$$dz = dx + i\,dx$$

or $dz = (1+i)\,dx$

Now limit of x :→ $0\le x\le 1$, then

$$\int_0^{1+i}(x^2-iy)\,dz = \int_0^1 (x^2-ix)(1+i)\,dx$$

$$= (1+i)\int_0^1 (x^2-ix)\,dx$$

$$= (1+i)\left[\frac{x^3}{3}-\frac{ix^2}{2}\right]_0^1 = (1+i)\left[\frac{1}{3}-\frac{i}{2}\right]$$

$$= \frac{(1+i)(2-3i)}{6} = \frac{5}{6}-\frac{1}{6}i \quad \text{Ans.}$$

(b) – Along the curve $y=x^2$ (parabola)

$$\Rightarrow dy = 2x\,dx$$

and we know $dz = dx + i\,dy$

Now $dz = dx + i(2x\,dx)$

or $dz = (1+2ix)\,dx$

Now limit of x ⟶ $0\le x\le 1$

$$\Rightarrow \int_0^{1+i}(x^2-iy)\,dz = \int_0^1 (x^2-ix^2)(1+2ix)\,dx$$

$$= \int_0^1 x^2(1-i)(1+2ix)\,dx$$

$$= (1-i)\int_0^1 x^2(1+2ix)\,dx$$

$$= (1-i)\int_0^1 (x^2+2ix^3)\,dx$$

$$= (1-i)\left[\frac{x^3}{3}+2i\frac{x^4}{4}\right]_0^1$$

$$= (1-i)\left[\frac{1}{3}+\frac{1}{2}i\right] = \frac{(1-i)(2+3i)}{6}$$

$$= \frac{1}{6}(2+3i-2i+3) = \frac{5}{6}+i\frac{1}{6} \quad \text{Ans.}$$

Q② Find the value of the integral $\int_0^{1+i} (x-y+ix^2)\,dz$

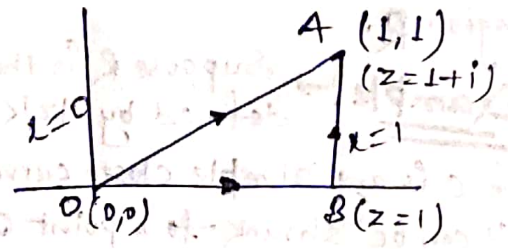
(a) along the straight line from $z=0$ to $z=1+i$

(b) along the real axis from $z=0$ to $z=1$ and then along a line parallel to imaginary axis from $z=1$ to $z=1+i$

Sol:→ (a) We know: $z=x+iy$

thus $dz=dx+idy$

Here OA is the straight line joining $z=0$ to $z=1+i$



$$x+iy=0 \Rightarrow (x=0,\,y=0) \quad\Big|\quad x+iy=1+i \Rightarrow (x=1,\,y=1)$$

also OA is a line, which is $y=x$

then $dy=dx$

Hence $\int_{OA} (x-y+ix^2)\,dz = \int_0^1 (\cancel{x}-\cancel{x}+ix^2)(dx+idx)$

$= \int_0^1 ix^2(1+i)\,dx = i(1+i)\int_0^1 x^2\,dx$

$= i(1+i)\left[\frac{x^3}{3}\right]_0^1 = \frac{i(1+i)}{3} = \frac{i-1}{3}$ **Ans.**

(b) The line OB from $z=0$ to $z=1$ is real axis, i.e. $y=0$

$$x+iy=0 \;(0,0) \qquad x+iy=1+i0 \;(1,0)$$

Since $z=x+iy$

from condition $z=x$

$\Rightarrow dz=dx$

$\int_{OB} (x-y+ix^2)\,dz = \int_0^1 (x-0+ix^2)\,dx = \left[\frac{x^2}{2}+\frac{ix^3}{3}\right]_0^1$

$= \frac{1}{2}+\frac{i}{3}$

also BA is the line parallel to imaginary axis from $z=1$ to $z=1+i$

∴ BA is the line $x=1$, $dx=0$

$\therefore$ $z = 1 + iy$

$dz = i\,dy$

$$\int_{BA} (x - y + ix^2)\,dz = \int_0^1 (1 - y + i)\, i\,dy$$

$$= i\int_0^1 [(1+i) - y]\,dy = \left[i(1+i)y - \frac{iy^2}{2}\right]_0^1$$

$$= i - 1 - \frac{i}{2} = \frac{i}{2} - 1$$

$$\therefore \int_{OBA} (x - y + ix^2)\,dz = \int_{OB} (x - y + ix^2)\,dz + \int_{BA} (x - y + ix^2)\,dz$$

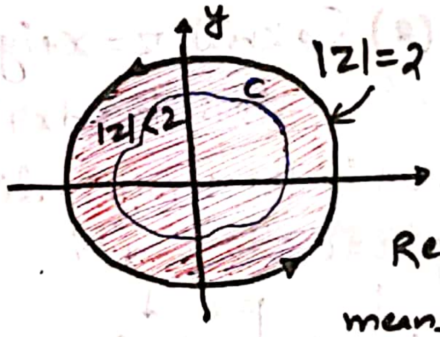
$$= \frac{1}{2} + \frac{i}{3} + \frac{i}{2} - 1 = \underline{\underline{-\frac{1}{2} + \frac{5i}{6}}} \quad \textbf{Ans.}$$

Important Definitions:→

(1) Simply connected Region:→ A connected region is said to be a simply connected if all the interior points of a closed curve C drawn in the region R are the points of the region R.

for Example:→ Suppose R is the region defined by $|z|<2$

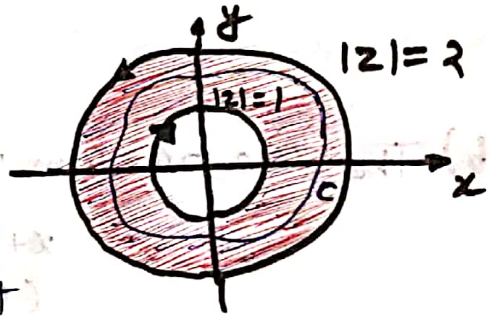
If C is any simple closed curve lying in R, it can be shrunk to a point which lies in R, and thus does not leave R, so that R is simply-connected.



Remember $|z|=r$ means, a circle with radius r, and centre O.

(2) Multi-connected region:→ Multiconnected region is bounded by more than one curve. (OR A region R which is not simply-connected is called multi-connected.)

for Example:→ If R is the region defined by $1<|z|<2$, then there is a simple closed curve C lying in R which can not possibly be shrunk to a point without leaving R, so R is multi-connected region.



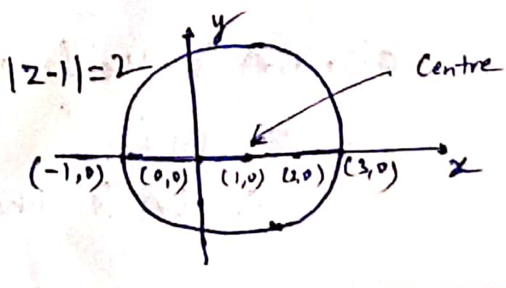
Here Region $1<|z|<2$

Conclusion:→ A simply-connected region is one which does not have any "holes" in it.

(3) Notations:→ $|z-a|=r$ means, a circle with radius r and centre a.

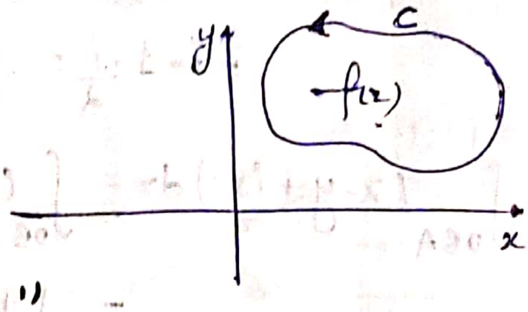
Example:→ Draw $|z-1|=2$

Here centre is 1 and radius is 2



Cauchy's Integral Theorem:→

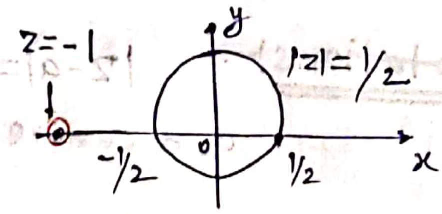
If a function $f(z)$ is analytic and its derivative $f'(z)$ continuous at all points inside and on a simple closed curve c, then $\int_c f(z)dz = 0$.



⇒ "If there is no pole inside & on the contour, then value of integral is zero."

Q.1:→ find the integral of $\int_c \frac{3z^2+7z+1}{z+1} dz$, where c is circle $|z| = 1/2$

Sol:→ given $\int_c \frac{3z^2+7z+1}{(z+1)}$



Now pole(s) is, $z+1=0$

$\Rightarrow z=-1$

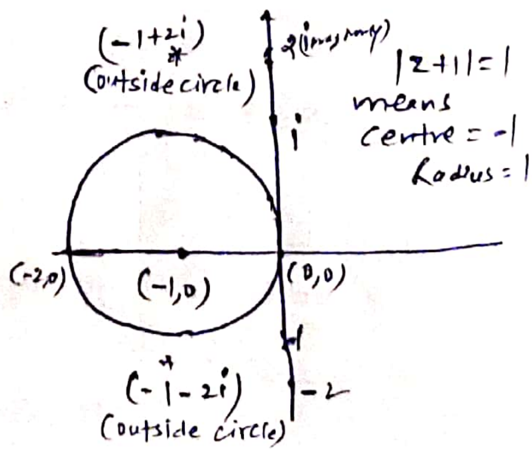
(which is outside the circle $|z| = 1/2$)

Hence circle $|z| = 1/2$ does not enclose any singularity (or pole) for given $f(z)$, then $\int_c \frac{3z^2+7z+1}{z+1} = 0$

(By Cauchy's Integral theorem)

Q.2 find the integral of $\int_c \frac{z+4}{z^2+2z+5} dz$, where c is circle $|z+1| = 1$

Sol:→ Checking poles:→ $z^2+2z+5=0$



$\Rightarrow z = \frac{-2 \pm \sqrt{4-20}}{2}$

$z = \frac{-2 \pm 4i}{2}$

or $z = -1 \pm 2i$

or $z = (-1+2i, -1-2i)$

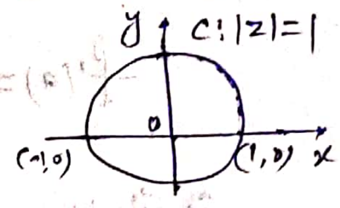
Hence both poles do not lie inside circle $|z+1|=1$, so According to Cauchy's Integral theorem $\int_C \frac{z+4}{z^2+2z+5} = 0.$

Q.3:→ Evaluate $\int_C z^2 dz$ using Cauchy's Integral theorem where $C: |z|=1$.

Sol:→ Here, no denominator to find poles. So we use the definition (statement) of Cauchy's Integral theorem, i.e. we shall show $f(z)=z^2$ is analytic and $f'(z)$ is continuous.

Now checking Analyticity of $f(z)$.

Region ↓



Here $f(z)=z^2$

$= (x+iy)^2 = x^2+(iy)^2+2xiy$

$= \underbrace{x^2-y^2}_{u} + i\underbrace{2xy}_{v}$

Now $u = x^2-y^2$ $\quad v = 2xy$

$\frac{\partial u}{\partial x} = 2x$ $\quad \frac{\partial v}{\partial x} = 2y$

$\frac{\partial u}{\partial y} = -2y$ $\quad \frac{\partial v}{\partial y} = 2x$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

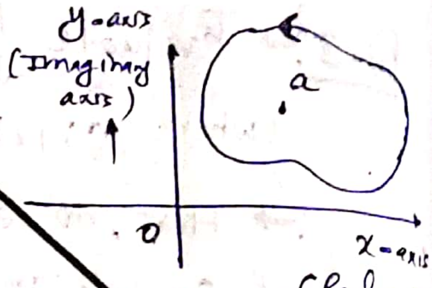
$\Rightarrow$ C-R Equations are satisfied $\Rightarrow f(z)$ is analytic

$\Rightarrow \int_C z^2 dz = 0$ (Acc to Cauchy's Int theorem)

Cauchy Integral formula :→

If $f(z)$ is analytic within and on a closed curve c and if "a" is any point within C, then

$$\boxed{f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz}$$



$$\boxed{\int_C \frac{f(z)}{(z-a)}\,dz = 2\pi i\, f(a)}$$

* Cauchy Integral formula for the "derivate" of any analytic function :→

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz$$

Similarly $$f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}\,dz$$

and $$f'''(a) = \frac{3!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^4}$$

⋮

Therefore

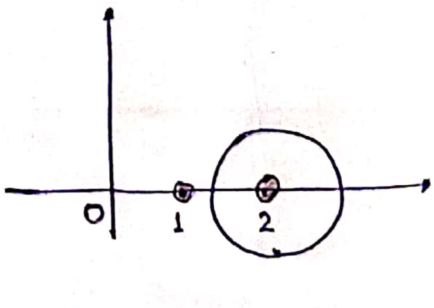
General formula → $$\boxed{f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{n+1}}}$$

* **Note** :→ If, in denominator $(z-a)=0$, then it means $z=a$ is a simple pole
  - → If in denominator $(z-a)^2=0$ then it means at $z=a$ is a double pole (pole of order 2)
  - Similarly ⟶ $(z-a)^3=0$, means at $z=a$, there is a pole of order three
  - → Thus, in denominator $(z-a)^n=0$, means at $z=a$, there is a pole of order n.

Q. 1 :→ Use Cauchy's integral formula to evaluate $\int_C \frac{z}{(z^2-3z+2)}\,dz$ (where c is the circle $|z-2| = 1/2$.

Sol :→ Given $\int_C \frac{z}{z^2-3z+2}\,dz$

Now finding poles, by putting the denominator equal to zero,



i.e. $z^2-3z+2=0$

$\Rightarrow (z-1)(z-2)=0$

$\Rightarrow z = 1, 2$ (simple pole at $z=1$ and $z=2$)

So, there are two poles at $z=1$ and $z=2$,
there is only one pole at $z=2$ inside the given circle.

Thus $\int_c \frac{z}{(z^2-3z+2)} dz = \int_c \frac{z}{(z-1)(z-2)} dz$

$$= \int \frac{\overbrace{\left(\frac{z}{z-1}\right)}^{f(z)}}{(z-2)} dz$$

$$= 2\pi i \left[\frac{z}{z-1}\right]_{z=2}$$

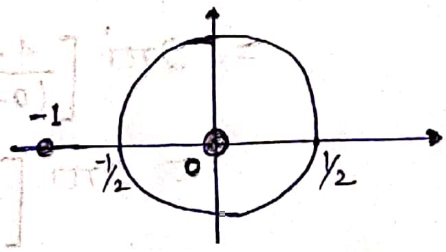
Remember the formula:→
$\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$

$$= 2\pi i \left(\frac{2}{2-1}\right)$$

Therefore $\int_C \frac{z}{z^2-3z+2} = \underline{\underline{4\pi i}}$ **Ans.**

Q.(2) Use Cauchy's integral formula to calculate
$\int_C \frac{2z+1}{z^2+z} dz$ Where c is $|z| = \frac{1}{2}$

Sol:→ Checking poles $z^2+z=0$
$z(z+1)=0$
$\Rightarrow z=0,-1$



There is only one pole at $z=0$ inside the given circle.

(Remember $|z|=1/2$ means, a circle at origin and radius $= 1/2$)

$\therefore \int_C \frac{2z+1}{z^2+z} dz = \int_C \frac{2z+1}{z(z+1)} dz$

$$= \int_C \frac{\overbrace{\left(\frac{2z+1}{z+1}\right)}^{f(z)}}{z} dz$$

$$= 2\pi i \left[\frac{2z+1}{z+1}\right]_{z=0}$$

$$= 2\pi i \left[\frac{0+1}{0+1}\right] = \underline{\underline{2\pi i}}$$ **Ans.**

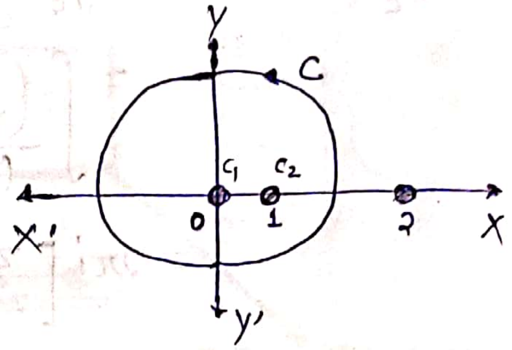
Q.3 Evaluate the following integral using Cauchy integral formula

$$\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz, \text{ where } c \text{ is the circle } |z| = \frac{3}{2}$$

Sol:→ Checking poles, by putting the denominator equal to zero

$\Rightarrow z(z-1)(z-2) = 0$

$\Rightarrow z = 0, 1, 2$ (Simple poles)



There are two poles at $z=0$ and $z=1$ inside the circle

{ $|z| = 3/2$ means, a circle with centre at $z=0$ and radius $= \frac{3}{2}(1.5)$ }

$\therefore$

$$\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz = \int_{C_1} \frac{\frac{4-3z}{(z-1)(z-2)}}{z}\,dz + \int_{C_2} \frac{\frac{4-3z}{z(z-2)}}{z-1}\,dz$$

$$= 2\pi i \left[\frac{4-3z}{(z-1)(z-2)}\right]_{z=0} + 2\pi i \left[\frac{4-3z}{z(z-2)}\right]_{z=1}$$

$$= 2\pi i \left[\frac{4-3(0)}{(0-1)(0-2)}\right] + 2\pi i \left[\frac{4-3(1)}{1(1-2)}\right]$$

$$= 2\pi i \left[\frac{4}{2}\right] + 2\pi i \left[\frac{1}{-1}\right]$$

$$= 4\pi i - 2\pi i$$

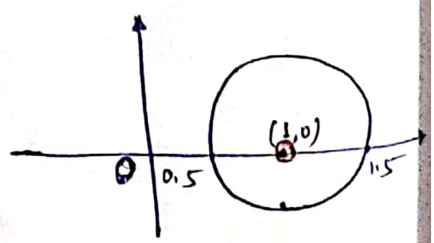
$$= 2\pi i \text{ Ans.}$$

Q.4:→ Evaluate $\int_C \frac{\log z}{(z-1)^3}\,dz$, using Cauchy's integral formula, where c is $|z-1| = \frac{1}{2}$

Sol:→ Checking poles :→ $(z-1)^3 = 0$

$\Rightarrow z = 1, 1, 1$

[one pole of order 3 at $z = 1$]



$$\because \int \frac{f(z)}{(z-a)^3}\,dz = \frac{2\pi i}{2!} f''(a)$$

$$\Rightarrow \int \frac{f(z)}{(z-a)^3}\,dz = \pi i\, f''(a)$$

$f(z)$

Thus according to Question $\displaystyle\int_c \frac{\log z}{(z-1)^3}\,dz = \frac{2\pi i}{2!}\left[\frac{d^2}{dz^2}\log z\right]_{z=1}$

$$= \pi i\left[\frac{d}{dz}\left(\frac{1}{z}\right)\right]_{z=1}$$

$$= \pi i\left[\frac{-1}{z^2}\right]_{z=1}$$

$$\Rightarrow -\pi i$$

Zero of Analytic Function:→

A zero of analytic function $f(z)$ is the value of z for which $f(z)=0$

Ex:→ $f(z)=(z-5)^4=0$

So $f(z)$ has a zero at $z=5$

{for finding zeros, just put "zero" on numerator} i.e. $f(z)=0$ (on numerator)

and order of zero is the power of the given function at given zero.

So here order is $\Rightarrow 4$

Explain:→ Given $f(z)=(z-5)^4 \Rightarrow f(5)=0$

$$f'(z)=4(z-5)^3 \Rightarrow f'(5)=0$$

$$f''(z)=12(z-5)^2 \Rightarrow f''(5)=0$$

$$f'''(z)=24(z-5) \Rightarrow f'''(5)=0$$

and $f^{IV}(z)=24 \Rightarrow f^{IV}(5)\neq 0$

thus $f(z)$ has a zero of order ~~4~~ (4) at $z=5$.

Singularity of the function:→

If a function $f(z)$ is not analytic at a point $z=a$, then $f(z)$ has a singularity at $z=a$.

OR

$z=a$ is a singular point, or it is singularity of the function

Ex:→ $f(z)=\frac{1}{z-5}$

Here $z=5$ is a singular point of $f(z)$

OR $f(z)$ has a singular point at $z=5$. ($\because$ at $z=5$, $f(z)$ becomes undefined)

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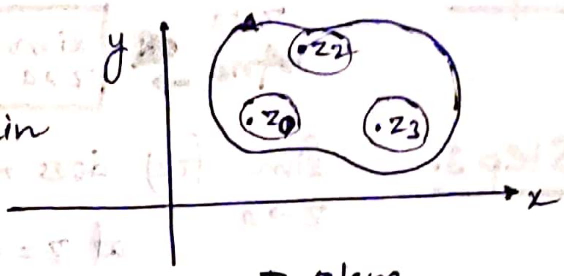
* Types of Singularity :→

There are mainly two types of singularity

1. Isolated Singularity
2. Non- Isolated singularity

1. Isolated Singularity :→

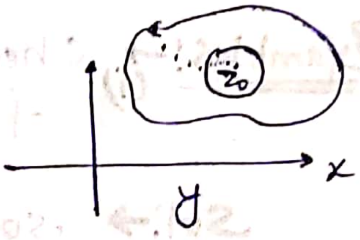
If $z=z_0$ is a singularity of $f(z)$ and if there is no other singularity within a small circle surronding the point $z=z_0$, then $z=z_0$ is said to be an isolated singularity of the function $f(z)$, otherwise it is called non-isolated.



z-plane

2. Non-Isolated Singularity :→

If $f(z)$ is non analytic at $z=z_0$ as well as their neighbourhoods, is known as non-isolated singularity.



* Examples :→

(i) $f(z)=\dfrac{1}{(z-1)(z-3)}$ has two isolated singular points namely, $z=1$ and $z=3$.

(ii) $f(z)=\log z$, has non-isolated singularity at $z=0$.

* Types of Isolated Singularities :→

1. Removable singularity
2. Poles
3. Essential singularity

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Working Rule to finding singularity :→

Step 1. If $\lim_{z \to a} f(z)$ exists and finite, then $f(z)$ has removable singularity at $z=a$

Step 2. If $\lim_{z \to a} f(z)$ infinite, then $f(z)$ has a pole at $z=a$

And → $\boxed{\lim_{z \to a} (z-a)^m f(z) = \text{finite and exist}}$

Step 3. $\lim_{z \to a} f(z)$ does not exist, then $f(z)$ has essential singularity at $z=a$

Examples:→ (i) Check the nature of singularity of function $f(z) = \frac{\sin z}{z}$ at point $z=0$

Sol:→ So $\lim_{z \to 0} f(z) = \lim_{z \to 0} \left(\frac{\sin z}{z}\right) = \lim_{z \to 0} \left(\frac{\cos z}{1}\right)$ [L-hospital Rule]

$= 1$ (Exists and finite),

Hence $f(z) = \frac{\sin z}{z}$ has removable singularity at $z=0$.

(ii) Check the nature of singularity of the function

$$f(z) = \frac{5z+1}{(z-2)^3(z+3)(z+2)}$$

Sol:→ function $f(z)$ has a pole of order 3 at $z=2$ and simple poles at $z=-3$ and $z=-2$.

$\{ \because \lim_{z \to 2} (z-2)^3 \frac{5z+1}{(z-2)^3(z+3)(z+2)} = \frac{11}{20}$ (finite exist)

$\lim_{z \to -3} (z+3) \frac{5z+1}{(z-2)^3(z+3)(z+2)} = \frac{-14}{(125)(-1)}$ (finit Exist)

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(iii) Check the nature of singularity of function $f(z) = e^{1/(z-3)}$ at the point $z=3$.

Sol:→ $\lim_{z \to 3} f(z) = \lim_{z \to 3} e^{1/(z-3)}$ = does not exist, then $f(z)$ has an essential singularity at $z=3$.

Assignment No. 2

Q.1:→ Evaluate the Integral. $\int_C |z|\,dz$, Where c is the contour the straight line from $z=-i$ to $z=i$.

Q.2:→ Evaluate $\oint_C \frac{z-3}{z^2+2z+5}\,dz$, Where c is the circle:→

(i) $|z|=1$ (ii) $|z+1-i|=2$

Q.3:→ Evaluate $\int_C \frac{12z-7}{(z-1)^2(2z+3)}\,dz$, Where c is the circle:→

(i) $|z|=2$ (ii) $|z+i|=\sqrt{3}$

Q.4:→ Evaluate $\int_C \frac{z}{z^2+1}\,dz$, Where $c=\left|z+\frac{1}{2}\right|=2$

Q.5:→ $\int_C \frac{e^{2z}}{(z+1)^4}\,dz$, Where c is the circle $|z|=3$

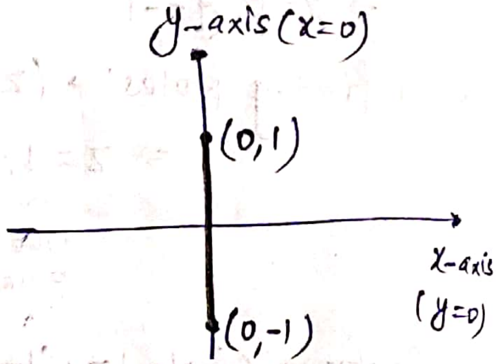
(Hint:→ Use Cauchy Integral formula of the derivative
$$f'''(a)=\frac{3!}{2\pi i}\int \frac{f(z)\,dz}{(z-a)^4}\;)$$

Solutions of Assignment No. 2
(Complex Analysis)

Q.1:- Ans:→ Evaluate integral $\int_c |z|\,dz$, where c is contour the straight line from $z=-i$ to $z=i$

first of all $|z| = \sqrt{x^2+y^2}$ —① and $dz = dx + i\,dy$

and limit of x and y are:→



given $z = -i$ and $z = i$

$\Rightarrow x+iy = -i$ and $x+iy = i$

or $x+iy = 0-i$ and $x+iy = 0+i$

on comparing $\Rightarrow x=0, y=-1$ and $x=0, y=1$

So it is a line $x=0$ varies from -1 to 1 on y-axis, then

$\Rightarrow dx = 0$

Equation ① becomes $|z| = \sqrt{0+y^2}$ ($\because x=0$)

and $dz = dx + i\,dy$

$\Rightarrow dz = i\,dy$ ($\because dx = 0$)

So ultimately $|z| = \sqrt{y^2} = y$

and $dz = i\,dy$

Hence $\int_c |z|\,dz = \int_{-1}^{1} y\,(i\,dy) = i\int_{-1}^{1} y\,dy$

$= i\left[\frac{y^2}{2}\right]_{-1}^{1} = i\left[\frac{(1)^2}{2} - \frac{(-1)^2}{2}\right] = i\left[\frac{1}{2} - \frac{1}{2}\right] = 0$ **Ans.**

$\Rightarrow \int_c |z|\,dz = 0$ **Ans.**

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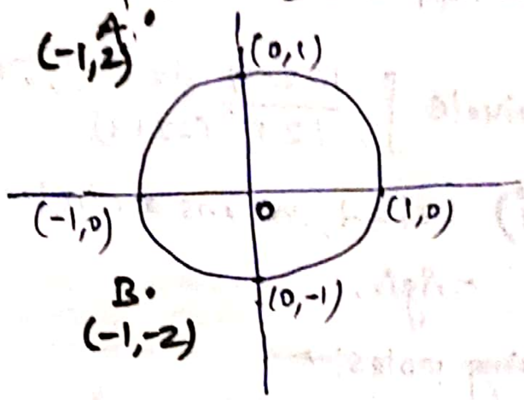
Q.2 Ans:→ Evaluate $\oint_C \frac{z-3}{z^2+2z+5}dz$, where c is the circle

(i) $|z|=1$ (ii) $|z+1-i|=2$

(i) for $|z|=1$ means, a circle of radius 1 and centre at 0

Now finding poles:→



$z^2+2z+5=0$

$\Rightarrow z = \frac{-2 \pm \sqrt{4-4\times5}}{2}$

$\Rightarrow z = -1 \pm 2i$

$\Rightarrow z = -1+2i, -1-2i$
$(-1,2) \quad (-1,-2)$

Both poles outside the circle $|z|=1$, So according to Cauchy's Integral theorem $\oint_C \frac{z-3}{z^2+2z+5}dz = 0$

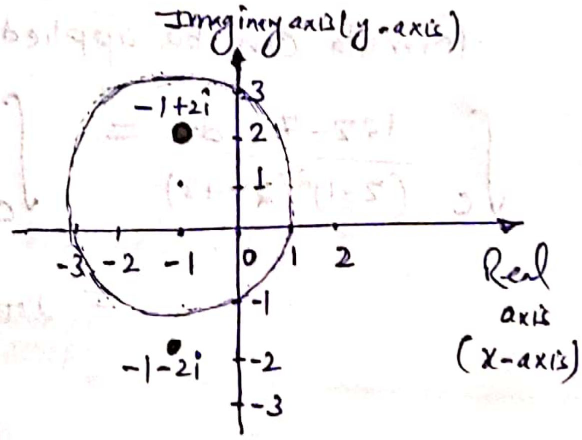
(ii) for $|z+(1-i)| = 2$ means, a circle of radius 2 and centre at $(-1,1)$

you can compare it with $|z-a|=r$

here $a=-(1-i)$ and $r=2$

$\Rightarrow a=-1+i$ and $r=2$

$\Rightarrow a=(-1,1)$ and Radius $=2$



Poles are same as we know

$z=-1+2i$ and $-1-2i$

Here pole $(-1+2i)$ inside the circle,

So using Cauchy's integral formula:↳

$$\oint_C \frac{z-3}{z^2+2z+5}dz = \oint_C \frac{z-3 \quad dz}{[z-(-1+2i)][z-(-1-2i)]}$$

$$= \int_C \frac{\left[\frac{z-3}{z-(-1-2i)}\right]}{z-(-1+2i)} dz = \int \frac{\left(\frac{z-3}{z+1+2i}\right)}{(z+1-2i)} dz$$

$$= 2\pi i \left[f(z)\right]_{z=-1+2i}$$

$$= 2\pi i \left[\frac{z-3}{z+1+2i}\right]_{z=-1+2i} = 2\pi i\left[\frac{-1+2i-3}{\cancel{-1}+2i+\cancel{1}+2i}\right] \Rightarrow$$

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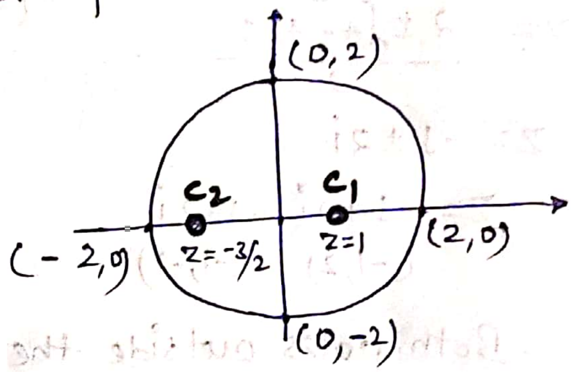
$$= 2\pi i\left[\frac{-4+2i}{+4i}\right]$$

$$= \frac{4\pi i[-2+i]}{+4i} = +\pi[-2+i] = \underline{\underline{\pi[i-2]}} \ \underline{\underline{\text{Ans.}}}$$

Q.3:→ Evaluate $\int_c \frac{12z-7\,dz}{(z-1)^2(2z+3)}$ where c is the circle:→

Sol:→ **for (i)** $|z|=2$, means a circle of radius 2 and centre at origin.



Now finding poles:→

$(z-1)^2(2z+3)=0$

$\Rightarrow (z-1)^2=0 \Rightarrow z=1,1$

and $(2z+3)=0 \Rightarrow z=-\frac{3}{2}$

It means it has a pole at $z=1$ of order 2 and other pole is at $z=-3/2$ (simple pole)

Both poles lie inside the circle $|z|=2$, so Cauchy's Integral formula can be applied:→

$$\int_c \frac{12z-7}{(z-1)^2(2z+3)}dz = \int_{c_1}\frac{\overbrace{\left(\frac{12z-7}{2z+3}\right)}^{f(z)}}{(z-1)^2}dz + \int_{c_2}\frac{\overbrace{\frac{12z-7}{(z-1)^2}}^{f(z)}dz}{(2z+3)}$$

$$= 2\pi i\left[f'(z)\right]_{z=1} + 2\pi i\left[f(z)\right]_{z=-3/2}$$

Why $f'(z)$ here?

[∵ pole has order 2]

$$= 2\pi i\left[\frac{(2z+3)\frac{d}{dz}(12z-7)-(12z-7)\frac{d}{dz}(2z+3)}{(2z+3)^2}\right]_{z=1} + 2\pi i\left[\frac{12z-7}{(z-1)^2}\right]$$

$$= 2\pi i\left[\frac{(2z+3)12-(12z-7)2}{(2z+3)^2}\right]_{z=1} + 2\pi i\left[\frac{12\left(-\frac{3}{2}\right)-7}{\left(-\frac{3}{2}-1\right)^2}\right]$$

$$= 2\pi i\left[\frac{\cancel{24z}+36-\cancel{24z}+14}{(2z+3)^2}\right]_{z=1} + 2\pi i\left[\frac{-18-7}{+\left(\frac{-3-2}{2}\right)^2}\right]$$

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$$= 2\pi i \left[\frac{50}{(2z+3)^2}\right]_{z=1} + 2\pi i\left[\frac{-25}{\left(-\frac{5}{2}\right)^2}\right]$$

$$= 2\pi i\left[\frac{50}{(2\times1+3)^2}\right] + 2\pi i\left[\frac{-25}{\left(\frac{25}{4}\right)}\right]$$

$$= 2\pi i \times \frac{50}{(5)^2} - \frac{50\pi i}{(25/4)}$$

$$= \frac{100\pi i}{25} - \frac{50\pi i \times 4}{25}$$

$$= 4\pi i - 8\pi i = \underline{\underline{-4\pi i}} \quad \underline{\underline{\text{Ans.}}}$$

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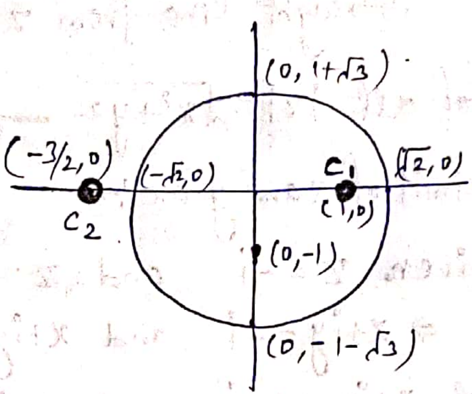
3(ii) Solution:→ $|z+i| = \sqrt{3}$ means a circle with centre at $(0,-1)$ and radius $\sqrt{3}$.

$\because |z+i| = \sqrt{3}$

Centre at $z = -i$, Radius $= \sqrt{3}$

$\Rightarrow x+iy = 0-i$

$\Rightarrow x = 0,\ y = -1$



Now finding poles:→ $(z-1)^2(2z+3) = 0$

$\Rightarrow z = \underbrace{1, 1}_{\text{double poles}}, \underbrace{-3/2}_{\text{simple pole}}$

So from circle $|z+i| = \sqrt{3}$, the pole $z = -3/2$ is outside the circle and only $z = 1$ inside it, so according to Cauchy's Integral formula:→

$$\int_C \frac{12z-7}{(z-1)^2(2z+3)}\,dz = \int_{C_1} \frac{\overbrace{\frac{12z-7}{(2z+3)}}^{f(z)}}{(z-1)^2}\,dz$$

$$= 2\pi i\left[f'(z)\right]_{z=1}$$

$$= 2\pi i\left[\frac{50}{(2z+3)^2}\right]_{z=1}$$

$$= 2\pi i \times \frac{50}{(2+3)^2} = \frac{2\pi i \times 50}{(5)^2}$$

$$= \frac{10\cancel{0}\,\pi i}{2\cancel{5}}$$

$$= 4\pi i \quad \textbf{Ans.}$$

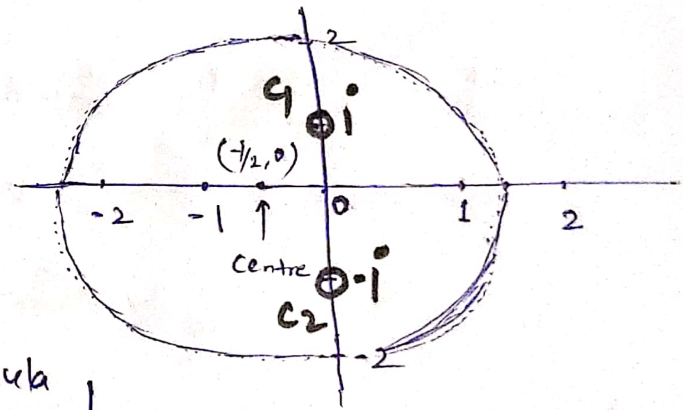
Q.4:→ Evaluate $\int_C \frac{z}{z^2+1} dz$, where $C: \left|z+\frac{1}{2}\right| = 2$

Here $\left|z+\frac{1}{2}\right| = 2$ means a circle of radius 2 and centre at $\left(-\frac{1}{2}, 0\right)$

Now finding poles :→ $z^2+1=0$

$\Rightarrow z^2 = -1$

$\Rightarrow \underline{z = \pm i}$



poles at $z=i$ and $z=-i$

Both poles lie inside circle

then using Cauchy's Integral formula ↓

$$\int_C \frac{z}{z^2+1}\,dz = \int_C \frac{z}{(z+i)(z-i)}\,dz$$

$$= \int_{C_1} \frac{\overbrace{\left(\frac{z}{z+i}\right)}^{f(z)}}{(z-i)}\,dz + \int_{C_2} \frac{\overbrace{\frac{z}{(z-i)}}^{f(z)}}{(z+i)}\,dz$$

According to Cauchy's Integral formula

$$= 2\pi i\,[f(z)]_{z=i} + 2\pi i\,[f(z)]_{z=-i}$$

$$= 2\pi i\left[\frac{z}{z+i}\right]_{z=i} + 2\pi i\left[\frac{z}{z-i}\right]_{z=-i}$$

$$= 2\pi i\left[\frac{i}{i+i}\right] + 2\pi i\left[\frac{-i}{-i-i}\right]$$

$$= 2\pi i\left[\frac{i}{2i}\right] + 2\pi i\left[\frac{-i}{-2i}\right]$$

$$\Rightarrow \pi i + \pi i$$

$$= \underline{\underline{2\pi i}}$$ **Ans.**

Q.5:→ $\int_C \frac{e^{2z}}{(z+1)^4}\,dz$, where c is the circle $|z|=3$

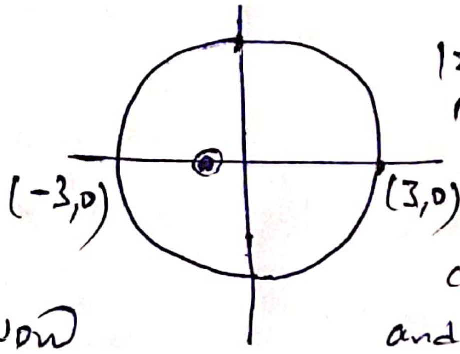
(Hint:→ Use Cauchy Integral formula of the derivative
$$f'''(a) = \frac{3!}{2\pi i}\int \frac{f(z)\,dz}{(z-a)^4} \;)$$

Solution:→ Checking poles:→

$(z+1)^4 = 0$

$\Rightarrow z = \underbrace{-1,-1,-1,-1}$

pole at $z=-1$ of order 4,



$|z|=3$ Means, a circle with centre O and radius 3.

Since all poles are inside the circle, Now using Cauchy's Integral formula (derivative form)

$$f'''(a) = \frac{3!}{2\pi i}\int \frac{f(z)\,dz}{(z-a)^4}$$

{∵ pole has more than 1 order{ Here $a=-1$, and $f(z)=e^{2z}$ }

$$\Rightarrow \frac{2\pi i}{3!} f'''(a) = \int \frac{f(z)\,dz}{(z-a)^4}$$

Here $f(z) = e^{2z}$
then $f'(z) = 2e^{2z}$
$f''(z) = 4e^{2z}$
$f'''(z) = 8e^{2z}$
then $f'''(-1) = 8e^{-2}$

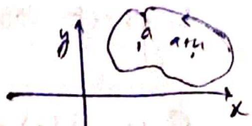
then $\frac{2\pi i}{6} f'''(-1) = \int \frac{e^{2z}}{(z+1)^4}\,dz$

$\Rightarrow \frac{\cancel{2}\pi i}{\cancel{6}\,3}\, 8e^{-2} = \int \frac{e^{2z}}{(z+1)^4}\,dz$

$\Rightarrow \int \frac{e^{2z}}{(z+1)^4}\,dz = \pi i \frac{8}{3} e^{-2}$ **Ans.**

Analysis of singularities through Laurent's Series (Only for isolated singularity)

Taylor's Series: Let $f(z)$ be analytic inside and on a simple closed curve C. let a and $a+h$ be two point inside C, then



$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^n}{n!} f^n(a)$$

or writting $z = a+h$, so $h = z-a$

so $$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \dots + \frac{f^n(a)}{n!}(z-a)^n$$

This is called "Taylor series or expansion for $f(a+h)$ or $f(z)$

or It can be written as

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n, \text{ where } a_n = \frac{f^n(a)}{n!}}$$

$$\boxed{\text{or } a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-a)^{n+1}}}$$

where the integration is taken counter-clock wise along the path C (closed curve) that contains "a" inside it.

$$\left\{ \because f^n(a) = \frac{n!}{2\pi i} \int \frac{f(z)\,dz}{(z-a)^{n+1}} \right\}$$

Laurent's Series: → Let $f(z)$ be a function having isolated singularity at $z=a$, then

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-a)^n}_{\text{Regular Part}} + \underbrace{\sum_{n=1}^{\infty} b_n (z-a)^{-n}}_{\text{principal part}}$$

where $a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz$ and $b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-a)^{-n+1}}$

Note: → (1) If there is no principal part, then $z=a$ is either "**Removable Singularity**" or "Regular point."

i.e. $b_n = 0 \Rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$

Example: → find singularity of $f(z) = \frac{\sin z}{z}$

$$= \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots\right]$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} \Rightarrow$$ No principal part (i.e. no negative Power of z) at $z=0$

Hence $f(z) = \frac{\sin z}{z}$ has removable singularity at $z=0$.

Note (2):→ If the principal part of the Laurent's expansion contains the finite number of terms, then $z=a$ is the "**pole**" and highest power of $\frac{1}{z-a}$ is defined as the order of pole.

Example:→ find the singularity of $f(z) = \frac{e^{z-1}}{(z-1)^2}$

Sol:→ $\frac{e^{z-1}}{(z-1)^2} = \frac{1}{(z-1)^2}\left[1+(z-1)+\frac{(z-1)^2}{2!}+\frac{(z-1)^3}{3!}+\dots\right]$

$\left(\because e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\right)$

$$= \frac{1}{(z-1)^2} + \frac{1}{(z-1)} + \frac{1}{2!} + \frac{(z-1)}{3!}$$

OR

$$= (z-1)^{-2} + (z-1)^{-1} + \frac{1}{2!} + \frac{(z-1)}{3!}$$

Highest power of $\left(\frac{1}{z-1}\right)$ is 2.

Thus function has a pole at $z=1$ of order 2

Ans.

Note (3):→ If the principal part of the Laurent expansion contains infinite no. of terms, then $z=a$ is **Essential Singularity.**

Example:→ find the singularity of $f(z) = e^{1/(z-3)}$

Sol:→ $e^{1/(z-3)} = 1 + \frac{1}{(z-3)} + \frac{1}{2!}\frac{1}{(z-3)^2} + \frac{1}{3!}\frac{1}{(z-3)^3} + \dots + \infty$

Here laurent expansion contains infinite no. of terms at $z=3$. Thus $f(z)$ has Essensial singularity at $z=3$.

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Side by side comparison of finding the singularity Rules (Only for Isolated singularity)

Limit Method

1. Removable singularity

If at $f(a)$ is not defined and $\lim_{z\to a} f(z)$ exist and finite, then $f(z)$ has removable singularity at $z=a$

⇒ for $f(z) = \frac{\sin z}{z}$

at $z=0$, $f(0) = \frac{\sin 0}{0} = \frac{0}{0}$ (Not define)

but $\lim_{z\to 0} f(z) = \lim_{z\to 0}\left(\frac{\sin z}{z}\right) = 1$ (Exists and finite)

⇒ $f(z)$ has removable singularity at $z=0$

2. Pole:

If $\lim_{z\to a} f(z) \to \infty$ and $\lim_{z\to a}(z-a)^m f(z) =$ finite and Exist

⇒ $f(z)$ has a pole at $z=a$ of order m.

⇒ for example $f(z) = \frac{e^{z-1}}{(z-1)^2}$

at $z=1$, $f(z)\to\infty$

(i.e. $\lim_{z\to 1}\frac{e^{z-1}}{(z-1)^2}\to\infty$)

and $\lim_{z\to 1}(z-1)^2 f(z) = \lim_{z\to 1}\cancel{(z-1)^2}\frac{e^{z-1}}{\cancel{(z-1)^2}} = 1$ (finite and Exist)

⇒ $f(z)$ has a pole at $z=1$ of order 2

3. Essential Singularity:

If $\lim_{z\to a} f(z)$ does not exist, then $f(z)$ has essential singularity at $z=a$

⇒ Example $f(z) = e^{1/z}$ at $z=0$

$\lim_{z\to 0} f(z) = \lim_{z\to 0} e^{1/z}$ ⇒ does not exist

then $f(z)$ has singularity at $z=0$ (Essensial Singularity)

Laurent's series Expansion Method

1. Removable singularity

If there is no principal part then $z=a$ is removable singularity for the $f(z)$.

for $f(z) = \frac{\sin z}{z}$

⇒ $= \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots\right]$

$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!}$

(No principal part of Larent series at $z=0$)

⇒ $f(z)$ has removable singularity at $z=0$

2. Pole

If the principal part of Laurent's expression contains finite no. of terms, then $z=a$ is the pole and highest power of $\frac{1}{(z-a)}$ is defined as order of pole

⇒ Same example $f(z) = \frac{e^{z-1}}{(z-1)^2}$

thus $\frac{e^{z-1}}{(z-1)^2} = \frac{1}{(z-1)^2}\left[1 + (z-1) + \frac{(z-1)^2}{2!} + \ldots\right]$

OR $= \frac{1}{(z-1)^2} + \frac{1}{(z-1)} + \frac{1}{2!} + \ldots$

Highest power of $\frac{1}{(z-1)}$ is 2

⇒ $f(z)$ has a pole at $z=1$ of order 2.

3. Essensial Singularity

If the principal part of Laurent's expansion contains infinite no. of terms, then $z=a$ is Essensial Singularity

⇒ Same example $f(z) = e^{1/z}$

$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \ldots$

Hence series contains infinite no. of terms at $z=0$, then $f(z)$ has Essensial Singularity at $z=0$.

27-April-2020

Residue at pole:→

Let $z=a$ be a pole of order m of a function $f(z)$ and c_1 circle of radius r with centre at $z=a$ which does not contain any other singularities except $z=a$, then According to Laurent's series

$$f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n+\sum_{n=1}^{\infty} b_n(z-a)^{-n}$$

where

$$a_n=\frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)^{n+1}}dz \qquad b_n=\frac{1}{2\pi i}\int_{c_1}\frac{f(z)}{(z-a)^{-n+1}}dz$$

Particularly: $b_1=\frac{1}{2\pi i}\int_{c_1} f(z)\,dz$

The coefficient b_1 is called residue of $f(z)$ at the pole $z=a$, It is denoted as $Res(z=a)$

* Method of finding Residues:→

(a) **Residue at simple pole:→** If $f(z)$ has a pole at $z=a$, then

$$\boxed{Res(\text{at } z=a)=\lim_{z\to a}(z-a)f(z)}$$

[**Example:→** Evaluate the residue of $\frac{z^2}{(z-1)(z-2)(z-3)}$ of function]

Example (i) - Find the order of each pole and residue of

$$f(z)=\frac{1-2z}{z(z-1)(z-2)}$$

Sol:→ finding poles first,

Here poles are $z(z-1)(z-2)=0$

$\Rightarrow z=0,1,2$

(All are simple poles)

Now (i) Residue of $f(z)$ at $z=0$

$\therefore Res(\text{at } z=0)=\lim_{z\to 0}(z-0)f(z)$

$$Res(\text{at } z=0) = \lim_{z\to 0} \cancel{(z-0)} \frac{1-2z}{\cancel{z}(z-1)(z-2)} = \frac{1-2(0)}{(0-1)(0-2)} = \frac{1}{2}$$

(ii) Residue of $f(z)$ at $z=1$

$$\therefore Res(\text{at } z=1) = \lim_{z\to 1} \cancel{(z-1)} \frac{1-2z}{z\cancel{(z-1)}(z-2)} = \frac{1-2(1)}{1(1-2)} = 1$$

(iii) Residue at $f(z)$ at $z=2$

$$\therefore Res(\text{at } z=2) = \lim_{z\to 2} \cancel{(z-2)} \frac{1-2z}{z(z-1)\cancel{(z-2)}} = \frac{1-2(2)}{2(2-1)} = -3/2$$

(b) Residue at pole of order m :→ If $f(z)$ has a pole of order m at $z=a$, then

$$\boxed{Res(\text{at } z=a) = \frac{1}{(m-1)!} \left\{ \frac{d^{m-1}}{dz^{m-1}} [(z-a)^m f(z)] \right\}_{z=a}}$$

Example!→ find the residue of a function at it's poles :→

for $f(z) = \dfrac{z^2}{(z+1)^2(z-2)}$

Sol:→ first find the poles :→ $(z+1)^2(z-2) = 0$

$\Rightarrow z = \underbrace{-1,-1}_{\text{double pole at } z=-1}, \; 2 \rightarrow$ simple pole at $z=2$

(i) Now Residue of $f(z)$ at $z=2$

$$\therefore Res(\text{at } z=2) = \lim_{z\to 2} (z-2) f(z)$$

$$= \lim_{z\to 2} \cancel{(z-2)} \frac{z^2}{(z+1)^2\cancel{(z-2)}} = \frac{(2)^2}{(2+1)^2} = \frac{4}{9}$$

(ii) Residue of $f(z)$ at $z=-1$

$\because$ at $z=-1$ is double pole (pole of order 2)

By formula $\Rightarrow$ $$Res(\text{at } z=-1) = \lim_{z\to -1} \frac{1}{(2-1)!} \left[\frac{d}{dz} \left\{ \cancel{(z+1)^2} \frac{z^2}{\cancel{(z+1)^2}(z-2)} \right\} \right]$$

$$= \lim_{z\to -1} \left[\frac{d}{dz} \left(\frac{z^2}{z-2} \right) \right] = \lim_{z\to -1} \left[\frac{(z-2)2z - z^2 \cdot 1}{(z-2)^2} \right]$$

$$= \lim_{z\to -1} \left[\frac{z^2 - 4z}{(z-2)^2} \right] = \frac{(-1)^2 - 4(-1)}{(-1-2)^2} = \frac{1+4}{9} = \frac{5}{9}$$

Ex.(2) find the residue of $f(z) = \frac{1}{(z^2+1)^3}$ at it's poles, only at $z=i$.

Sol:→ finding poles :→ $(z^2+1)^3 = 0$

$\Rightarrow [(z+i)(z-i)]^3 = 0$

$\Rightarrow (z+i)^3(z-i)^3 = 0$

$\Rightarrow z = \pm i, \pm i, \pm i$

OR $z = \underbrace{i, i, i}_{\text{order of pole is 3 at } z=i}$ and $\underbrace{-i, -i, -i}_{\text{order of pole is 3 at } z=-i}$

∴ Residue at $z=i$

$$\text{Res}(\text{at } z=i) = \lim_{z \to i} \frac{1}{(3-1)!}\left\{\frac{d^{3-1}}{dz^{3-1}}\left[\cancel{(z-i)^3}\frac{1}{(z+i)^3\cancel{(z-i)^3}}\right]\right\}$$

$$= \lim_{z\to i} \frac{1}{2!}\left\{\frac{d^2}{dz^2}\left(\frac{1}{z+i}\right)^3\right\}$$

$$= \lim_{z\to i} \frac{1}{2}\left[\frac{3\times 4}{(z+i)^5}\right] = \frac{1}{2}\times\frac{12}{(i+i)^5} = \frac{6}{32i} = \frac{3}{16i}$$

$$= \frac{-3i}{16} \quad \underline{\text{Ans.}}$$

(c) Residue at $z = \infty$

$$\boxed{= \lim_{z\to\infty}[-z f(z)]}$$

Example:→ Evaluate the residue of $f(z) = \frac{z^2}{(z-1)(z-2)(z-3)}$ at infinity (∞)

Sol:→ Residue of $f(z)$ at $(z=\infty)$

$$= \lim_{z\to\infty} -z f(z) = \frac{-z(z^2)}{(z-1)(z-2)(z-3)}$$

$$= \lim_{z\to\infty} \frac{-z^3}{(z-1)(z-2)(z-3)}$$

$$= \lim_{z\to\infty} \frac{-z^3}{z^3\left[1-\frac{1}{z}\right]\left[1-\frac{2}{z}\right]\left[1-\frac{3}{z}\right]} = -1$$

$$= \underline{-1} \quad \underline{\text{Ans.}}$$

More questions based on Residue (only finding Residue at pole):→

Q.1:→ find the residue of $\frac{z^3}{(z-1)^2(z-2)(z-3)}$ at a pole of order 2

Sol:→ (We have only asked for finding residue at pole of order 2,

Here order two pole at $z=1$. (Remember order is higher than 1)

so using the formula to finding residue

$$\text{Res}(at\ z=a) = \frac{1}{(m-1)!}\left\{\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]\right\}_{z=a} \quad \text{—(1)}$$

Here $a=1$ and $m=2$, put it on formula (1), we get

$$\text{Res}(at\ z=1) = \frac{1}{(2-1)!}\left\{\frac{d^{2-1}}{dz^{2-1}}\left[(z-1)^2 f(z)\right]\right\}_{at\ z=1}$$

$$= \frac{1}{1!}\left\{\frac{d}{dz}\left[\frac{\cancel{(z-1)^2}\cdot z^3}{\cancel{(z-1)^2}(z-2)(z-3)}\right]\right\}_{at\ z=(a=1)} \quad \left(\because f(z) = \frac{z^3}{(z-1)^2(z-2)(z-3)}\right)$$

$$= \frac{d}{dz}\left[\frac{z^3}{(z-2)(z-3)}\right]_{z=1}$$

$$\Rightarrow \left[\frac{(z-2)(z-3)\frac{d}{dz}z^3 - z^3\frac{d}{dz}(z^2-5z+6)}{[(z-2)(z-3)]^2}\right]_{z=1}$$

$$\Rightarrow \left[\frac{(z-2)(z-3)\times 3z^2 - z^3(2z-5)}{(z-2)^2(z-3)^2}\right]_{z=1}$$

$$\Rightarrow \frac{(1-2)(1-3)\times 3(1)^2 - 1^3(2\times 1-5)}{(1-2)^2(1-3)^2}$$

$$= \frac{(-1)(-2)\times 3 - (2-5)}{(1)(4)} = \frac{6-(-3)}{4} = \frac{9}{4} \quad \text{Ans.}$$

Q.2:→ find the residue of $f(z) = \frac{1}{z^2(z-i)}$ at $z=i$

(We have asked for finding residue of $f(z)$ only at $z=i$ (simple pole)

So According to formula :→ $\text{Res}(at\ z=a) = \lim\limits_{z\to a}(z-a)f(z)$

Here $a=i$

$$\text{Thus Res}(at\ z=i) = \lim_{z\to i}\cancel{(z-i)}\frac{1}{z^2\cancel{(z-i)}} = \lim_{z\to i}\frac{1}{z^2}$$

$$= \frac{1}{i^2} = \frac{1}{-1} = -1 \quad \text{Ans.}$$

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Assignment No. 3 (Complex Analysis)

Q.1: find out the zeros and discuss the nature of the singularities of $f(z) = \frac{z-2}{z^2}\sin\left(\frac{1}{z-1}\right)$.

Q.2: find the residue of $f(z) = \frac{z^3}{(z-2)(z-3)}$ at each pole.

Q.3: find the residue of $\frac{1}{(z^2+9)^2}$ at $z = 3i$

Q.4: find the residue of $f(z) = \frac{z^2}{z^2-1}$ at $z \to \infty$

Q.5: find the singularity of the functions:

(a) $f(z) = \sin\frac{1}{z}$ (b) $f(z) = \frac{e^{1/z}}{z^2}$

Q.6: find the residue of $f(z) = \frac{ze^z}{(z-a)^3}$ at it's pole.

Solutions of Assignment No. 3

Q.1:- Ans:→ We shall find out the zero and discuss the nature of singularities of $f(z) = \frac{(z-2)}{z^2}\sin\left(\frac{1}{z-1}\right)$

We know that zero of $f(z)$ is given by equating to zero, i.e. $f(z)=0$

$$\Rightarrow f(z) = 0$$

$$\Rightarrow \frac{(z-2)}{z^2}\sin\left(\frac{1}{z-1}\right) = 0$$

$$\Rightarrow \text{either } z-2=0 \quad \text{or} \quad \sin\left(\frac{1}{z-1}\right)=0$$

$$\Rightarrow \underline{\underline{z=2}} \quad \text{and} \quad \sin\left(\frac{1}{z-1}\right) = \sin n\pi \left(\text{Where } n \in I_+\right)$$

$$\text{and} \quad \frac{1}{z-1} = n\pi$$

$$\Rightarrow \underline{\underline{z = \frac{1}{n\pi}}}, \text{ Where } (n \in \text{Integers})$$

Both are simple zero, at $z=2$ and $z = 1/n\pi$

Now for poles, equating to zero the denominator of $f(z)$,

We get $z^2 = 0 \Rightarrow z = 0, 0$ (double pole)

$\Rightarrow$ $\underline{\underline{f(z) \text{ has a pole at } z=0 \text{ of order } 2}}$ **Ans.**

Sol. (2) Given $f(z) = \frac{z^3}{(z-2)(z-3)}$

Now its poles are $z=2$ and $z=3$

$\therefore$ Residue (at $z=2$) $= \lim_{z\to 2} \cancel{(z-2)} \frac{z^3}{\cancel{(z-2)}(z-3)} = \lim_{z\to 2} \frac{z^3}{(z-3)} = \frac{8}{-1} = \underline{\underline{-8}}$

And for pole $z=3$,

Residue (at $z=3$)

$$= \lim_{z\to 3} \cancel{(z-3)} \frac{z^3}{\cancel{(z-3)}(z-2)}$$

$$= \lim_{z\to 3} \frac{(z)^3}{(z-2)}$$

$$= \frac{(3)^3}{(3-2)} = \underline{\underline{27}}$$

Sol:→ Q.3:→ Given $f(z) = \frac{1}{(z^2+3^2)^2}$, (We shall find the residue at $z=3i$

$f(z)$ can be Written as :→ $f(z) = \frac{1}{[(z+3i)(z-3i)]^2}$

$$= \frac{1}{(z+3i)^2(z-3i)^2}$$

(order of both poles are two)

Using Residue formula:→ $\text{Res}(\text{at } z=a) = \lim_{z\to a} \frac{1}{(m-1)!} \left\{ \frac{d^{m-1}}{dz^{m-1}} [(z-a)^m f(z)] \right\}$

Here order $m=2$ and $a=3i$

$\Rightarrow$ $\text{Res}(\text{at } z=3i) = \lim_{z\to 3i} \frac{1}{(2-1)!} \left\{ \frac{d^{2-1}}{dz^{2-1}} \left[\cancel{(z-3i)^2} \times \frac{1}{(z+3i)^2 \cancel{(z-3i)^2}} \right] \right\}$

$$= \text{Res}\,(\text{at } z=3i) = \lim_{z\to 3i} \frac{d}{dz}\left(\frac{1}{(z+3i)^2}\right)$$

$$= \lim_{z\to 3i}\left[\frac{(z+3i)^2\frac{d}{dz}(1) - 1\,\frac{d}{dz}(z+3i)^2}{(z+3i)^4}\right]$$

OR $$\Rightarrow \left[\frac{-2\cancel{(z+3i)}}{(z+3i)^4}\right]_{z=3i}$$

$$\Rightarrow \left[\frac{-2}{(z+3i)^3}\right]_{z=3i} = \frac{-2}{(3i+3i)^3} = \frac{-2}{(6i)^3}$$

$$= \frac{-2}{216\, i^3} = \frac{-\cancel{2}}{\cancel{216}(-i)} \quad [\because i^3 = -i]$$

$$= \frac{\cancel{-}1}{\cancel{-}108\, i} = \frac{1}{108\, i} = \frac{-i}{108}$$ **Ans.**

Q.4. Solution:→ We shall find the residue of $f(z) = \frac{z^2}{z^2-1}$ at $z \to \infty$

Since, we know residue at infinity is

$$\Rightarrow \lim_{z\to\infty} -z\, f(z)$$

$$\Rightarrow \lim_{z\to\infty} -z \left[\frac{z^2}{z^2-1}\right] = \lim_{z\to\infty} -z \left[\frac{\cancel{z^2}}{\cancel{z^2}(1-1/z^2)}\right]$$

$$= \lim_{z\to\infty} \frac{-z}{(1-1/z^2)} = -\infty$$ **Ans.**

Q.5:- Sol:→ We have to discuss the singularity of **(a)** $f(z) = \sin(1/z)$

By Method 1:- limit method:→ We know at $z=0$, $f(z) = \sin(1/z)$ is not analytic. Thus $\lim_{z\to 0} f(z) = \lim_{z\to 0} \sin\left(\frac{1}{z}\right) =$ does not exist, then $f(z)$ has an essential singularity at $z=0$.

By Method 2:→ Laurent's series Method:→

Expanding $\sin\left(\frac{1}{z}\right)$, we get

$$\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{3!\,z^3} + \frac{1}{5!\,z^5} + \cdots + (-1)^n \frac{1}{(2n+1)!\,z^{2n+1}} + \cdots$$

$$= z^{-1} - \frac{1}{3!} z^{-3} + \frac{1}{5!} z^{-5} + \cdots$$

{Negative power, it means that principal part of laurent's Series contains infinite number of terms at $z=0$, then $f(z)$ has an essential singularity at $z=0$}

(b) for $f(z) = \dfrac{e^{1/z}}{z^2}$

Now from expanding we get

$$\frac{e^{1/z}}{z^2} = \frac{1}{z^2}\left(1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \frac{1}{4!\,z^4} + \cdots\right)$$

$$= \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{2!\,z^4} + \frac{1}{3!\,z^5} + \cdots = z^{-2} + z^{-3} + \frac{1}{2!} z^{-4} + \frac{1}{3!} z^{-5} + \cdots$$

According to Laurent's Series, $f(z)$ has infinite no. of terms in negative power of z.
Hence $f(z)$ has essential singularity at $z=0$.

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Q.6:→ Sol:→ We shall find the residue of $f(z) = \frac{ze^z}{(z-a)^3}$ at it's pole.

Now finding pole, $(z-a)^3 = 0$

$\Rightarrow z = a, a, a$ (pole of order 3)

Hence by Residue theorem:→

$$\lim_{z \to a} \frac{1}{(m-1)!} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z) \right] \right\}$$

Here $m = 3$ (order), and $\boxed{a = a}$

$$\Rightarrow \lim_{z \to a} \frac{1}{(3-1)!} \left\{ \frac{d^{3-1}}{dz^{3-1}} \left[\cancel{(z-a)^3} \times \frac{ze^z}{\cancel{(z-a)^3}} \right] \right\}$$

$$= \lim_{z \to a} \frac{1}{2!} \left\{ \frac{d^2}{dz^2} (ze^z) \right\}$$

$$= \lim_{z \to a} \frac{1}{2!} \left[\frac{d}{dz} (ze^z + e^z) \right]$$

$$= \lim_{z \to a} \frac{1}{2!} \left[ze^z + 2e^z \right]$$

$$= \lim_{z \to a} \frac{1}{2!} \left[e^z (z+2) \right] = \frac{1}{2!} e^a (a+2)$$

$$= e^a \left[\frac{a}{2} + 1 \right]$$ **Ans.**

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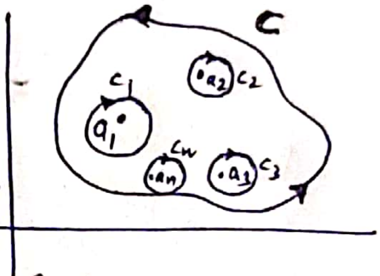
* Cauchy's Residue Theorem:→

If $f(z)$ is analytic in a closed curve C, except at a finite number of poles within C, then

$$\boxed{\int_C f(z)\,dz = 2\pi i(\text{sum of residue at poles within } C)}$$



formula can be written as

$$\int_C f(z)\,dz = 2\pi i\left[\text{Res} f(a_1) + \text{Res} f(a_2) + \text{Res} f(a_3) + \ldots \text{Res} f(a_n)\right]$$

$$= 2\pi i \times \sum_{m=1}^{n} \text{Res} f(a_m)$$

* "Cauchy's Residue theorem is helpful to finding Complex Integration around the closed curve."

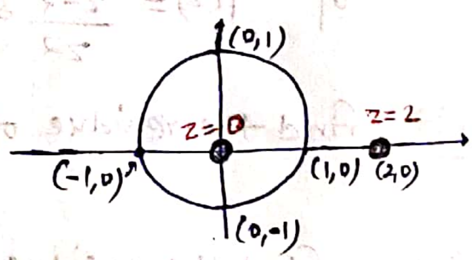
Example:→

↓

Evaluate the following Integral using residue theorem:→

$$\int_C \frac{1+z}{z(2-z)}\,dz, \text{ Where } C \text{ is the circle } |z| = 1$$

Solution:→ first finding the poles:→



$\Rightarrow \quad z(2-z) = 0$

$\Rightarrow \quad z = 0, 2$ (Simple poles)

Here pole $z=2$ is outside the circle, but $z=0$ inside the circle. So we only deals with inside one.

Thus $\text{Res}(\text{at } z=0) = \lim_{z\to 0} (z-0) f(z)$

$$= \lim_{z\to 0} \cancel{(z-0)} \frac{1+z}{\cancel{z}(2-z)} = \lim_{z\to 0} \frac{1+z}{2-z} = \frac{1}{2}$$

$\Rightarrow \text{Res}(\text{at } z=0) = 1/2$

Hence by Residue theorem:– $\displaystyle\int_C \frac{1+z}{z(2-z)}\,dz = 2\pi i\left[\text{Sum of residue at poles within } C\right]$

$$= \cancel{2}\pi i \times \frac{1}{\cancel{2}}$$

$$= \pi i \text{ Ans.}$$

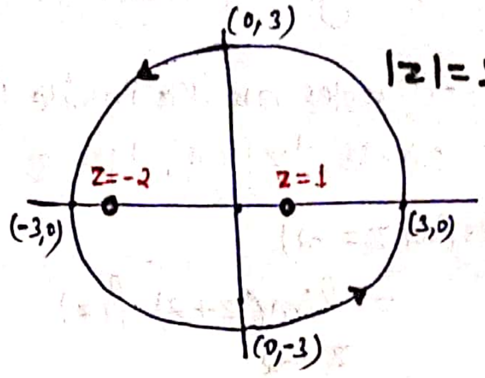
Example ②:→ Evaluate the following using residue theorem

$$\int_C \frac{z^2}{(z-1)(z+2)}\,dz \text{ , where } C: |z|=3$$

Sol:→ first finding poles:→

$(z-1)(z+2)=0$

$\Rightarrow z = 1, -2$ (simple poles)



Here poles $z=1$ and $z=-2$ both lie inside the circle.

So, we shall find residue at both poles.

Now $\text{Res}(\text{at } z=1) = \lim_{z\to 1}(z-1)f(z)$

$$= \lim_{z\to 1}\cancel{(z-1)}\frac{z^2}{\cancel{(z-1)}(z+2)} = \lim_{z\to 1}\frac{z^2}{(z+2)} = \frac{1}{3}$$

and $\text{Res}(\text{at } z=-2) = \lim_{z\to -2}(z+2)f(z)$

$$= \lim_{z\to -2}\cancel{(z+2)}\frac{z^2}{(z-1)\cancel{(z+2)}} = \lim_{z\to -2}\frac{z^2}{(z-1)} = \frac{(-2)^2}{(-2-1)} = \frac{4}{-3}$$

Hence by Residue theorem

$$\int_C \frac{z^2}{(z-1)(z+2)}\,dz = 2\pi i\,[\text{Sum of residue at poles within } C]$$

$$= 2\pi i\,[\text{Res}(z=1) + \text{Res}(z=-2)]$$

$$= 2\pi i\left[\frac{1}{3} + \left(-\frac{4}{3}\right)\right]$$

$$= 2\pi i\left[\frac{1-4}{3}\right] = 2\pi i\left(\frac{-\cancel{3}}{\cancel{3}}\right) = -2\pi i$$

Ans.

$$\Rightarrow \int_C \frac{z^2}{(z-1)(z+2)} = -2\pi i$$

$\downarrow$

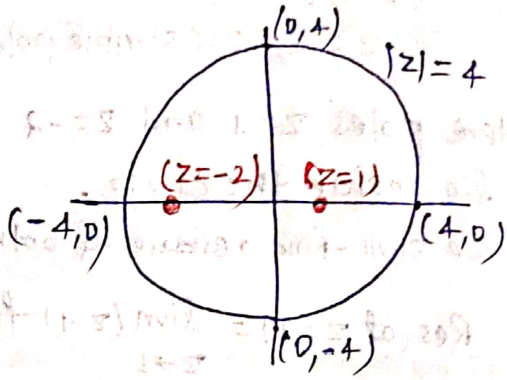
$|z|=3$

* **Do yourself:→** Using Residue theorem, Evaluate $\int_C \frac{z^2}{(z-1)^2(z+2)}\,dz$, where C is a circle $|z|=4$.

Solution:→ Given $\int_c \frac{z^2 \, dz}{(z-1)^2(z+2)}$, $c: |z|=4$

finding all poles :→ $(z-1)^2(z+2)=0 \Rightarrow z = \underbrace{1,1}_{\text{double poles}}, \underbrace{-2}_{\text{simple Pole}}$

- All poles are lie inside the circle $|z|=4$, Hence



Res (at $z=-2$)

$$= \lim_{z\to -2} (z+2) f(z)$$

$$= \lim_{z\to -2} \cancel{(z+2)} \frac{z^2}{(z-1)^2 \cancel{(z+2)}}$$

$$= \frac{(-2)^2}{(-2-1)^2} = \frac{4}{9}$$

Now, Res (at $z=1$) $= \frac{1}{(2-1)!} \left\{ \frac{d^{2-1}}{dz^{2-1}} \left[\cancel{(z-1)^2} \frac{z^2}{\cancel{(z-1)^2}(z+2)} \right] \right\}_{z=1}$

Using formula for (higher order poles)

$$= \frac{d}{dz} \left\{ \frac{z^2}{(z+2)} \right\}_{z=1}$$

$$\text{or } \lim_{z\to 1} \left[\frac{2(z+2)z - z^2}{(z+2)^2} \right]$$

$$= \lim_{z\to 1} \left[\frac{(z+2)2z - z^2}{(z+2)^2} \right]$$

$$= \lim_{z\to 1} \left[\frac{2z^2+4z-z^2}{(z+2)^2} \right] = \lim_{z\to 1} \left[\frac{z^2+4z}{(z+2)^2} \right]$$

$$= \frac{(1)^2+4(1)}{(1+2)^2} = \frac{5}{9}$$

Hence According to Residue theorem

$$\int_c \frac{z^2}{(z-1)^2(z+2)} dz = 2\pi i \sum_{i=1}^{2} \text{Res}(z_i)$$

$$= 2\pi i \left[\text{Res}(\text{at } z=-2) + \text{Res}(\text{at } z=1) \right]$$

$$= 2\pi i \left[\frac{4}{9} + \frac{5}{9} \right] = 2\pi i \quad \textbf{Ans.}$$

Assignment No. 4

Q.1:→ Apply calculus of residues to evaluate $\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta$.

(UTU: 2011-12)

Q.2:→ By Cauchy residue theorem solve the integral

$\oint_C \frac{3z^2+z+1}{(z^2-1)(z+3)}\,dz$, where $C: |z|=2$. (UTU: 2013-14)

Q.3:→ Use residue calculus to evaluate the following integral

$\int_0^{2\pi} \frac{1}{5-4\sin\theta}\,d\theta$. (UTU: 2014-15)

Q.4:→ Evaluate the contour integration $\int_0^{2\pi} \frac{1}{a+b\sin\theta}\,d\theta$, where $a>b$.

(UTU: 2014-15, BP)

Q.5:→ Using "Cauchy integral formula" to evaluate

$\int_C \frac{e^{2z}}{z-\pi i}\,dz$, where c is the ellipse $|z-2|+|z+2|=6$

(UTU: 2015-16)

Q.6:→ Using Cauchy residue theorem to evaluate the integral

$\int_C \frac{z^2-2z}{(z+1)^2(z^2+4)}\,dz$, where c is the circle $|z|=5$.

(UTU: 2015-16)

Solutions of Assignment No. 4

Q1 Apply calculus of residue to evaluate $\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta$

Sol:→ let $I = \int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta = \frac{1}{2}\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta$ ($\because$ Even function)

$$= \text{Real part of } \frac{1}{2}\int_0^{2\pi} \frac{1+2e^{i\theta}}{5+4\cos\theta}\,d\theta \quad (\because e^{i\theta} = \cos\theta + i\sin\theta)$$

$$= \text{R.P. of } \frac{1}{2}\int_0^{2\pi} \frac{1+2e^{i\theta}\,d\theta}{5+\frac{4}{2}(e^{i\theta}+e^{-i\theta})} \quad \text{—(1)}$$

Now put $e^{i\theta} = z$, $d\theta = dz/iz$, where c is the unit circle $|z|=1$ in Eq.(1)

$$= \text{R.P. of } \frac{1}{2}\int_c \frac{(1+2z)}{5+2\left(z+\frac{1}{z}\right)} \times \frac{dz}{iz} = \text{R.P. of } \frac{1}{2}\int \frac{-i(1+2z)\,dz}{2z^2+5z+2}$$

Now finding the poles, $2z^2+5z+2=0$

$\Rightarrow (2z+1)(z+2)=0$

$\Rightarrow z = -1/2, -2$

This term is cancelled with numerator $(1+2z)$

$$\text{R.P. of } \frac{1}{2}\int \frac{-i\cancel{(1+2z)}\,dz}{\cancel{(2z+1)}(z+2)}$$

$$\text{R.P. of } \frac{1}{2}\int \frac{-i\,dz}{(z+2)}$$

So only pole is

$(z+2)=0$

$\Rightarrow z=-2$

(outside the circle $|z|=1$)

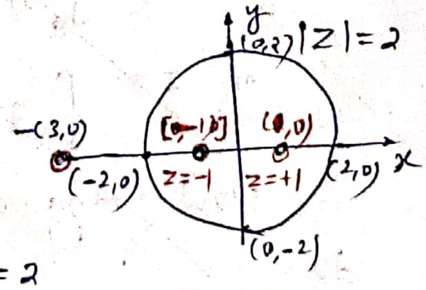
So pole $z=-2$ is outside the circle, Thus there is no pole of $f(z)$ inside the unit circle $C: |z|=1$,

Hence $\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta = 0$ Ans.

Q.(2) By Cauchy residue theorem Solve the integral

$$\oint_c \frac{3z^2+z+1}{(z^2-1)(z+3)}\,dz, \text{ where } |z|=2$$

Sol:→ Given $I = \oint_c \frac{3z^2+z+1}{(z^2-1)(z+3)}\,dz$



Now checking poles:— $(z^2-1)(z+3)=0$

$\Rightarrow z = \pm 1, -3$

Here poles $z = \pm 1$ inside the circle $|z|=2$

thus Res at $z=1 \Rightarrow \lim_{z\to 1} \cancel{(z-1)} \frac{3z^2+z+1}{(z+1)\cancel{(z-1)}(z+3)} = \frac{3+1+1}{(1+1)(1+3)} = 5/8$

Res at $z=-1$ ⇒ $Res(z=-1) = \lim_{z\to -1} \cancel{(z+1)} \frac{3z^2+z+1}{\cancel{(z+1)}(z-1)(z+3)} = \frac{3(-1)^2+(-1)+1}{(-1-1)(-1+3)}$

$= \underline{-3/4}$

By Residue theorem

$$\int_C \frac{3z^2+z+1}{(z^2-1)(z+3)} dz = 2\pi i \sum_{i=1}^{n} Res(at\ z_i)$$

$$= 2\pi i \left[\frac{5}{8} - \frac{3}{4}\right] = \cancel{2}\pi i \left[\frac{5-6}{\cancel{8}}\right] = \frac{-\pi i}{4}$$ Ans

Q. (3) Use residue theorem to calculate the following integral

$$\int_0^{2\pi} \frac{1}{5-4\sin\theta} d\theta$$

Sol:→ let $I = \int_0^{2\pi} \frac{1}{5-4\sin\theta} d\theta$ —— (1)

We know $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$, then Eq (1) becomes :→

$$I = \int_0^{2\pi} \frac{1\ d\theta}{5 - \frac{\cancel{4}}{\cancel{2}i}(e^{i\theta} - e^{-i\theta})} = \int_0^{2\pi} \frac{1\ d\theta}{5+2i(e^{i\theta} - e^{-i\theta})}$$

Now put $e^{i\theta} = z$

So $ie^{i\theta}d\theta = dz$

$\Rightarrow iz\,d\theta = dz$

$\Rightarrow d\theta = \frac{dz}{iz}$, and where $C: |z| = 1$,

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Now $I = \int_C \frac{1}{[5+2i(z-1/z)]} \frac{dz}{iz} = \frac{1}{i}\int_C \frac{dz}{\left[5+2i\left(\frac{z^2-1}{z}\right)\right]z}$

$= \frac{1}{i}\int_C \frac{dz}{\left[\frac{5z+2i(z^2-1)}{\cancel{z}}\right]\cancel{z}} = \frac{1}{i}\int \frac{dz}{(5z+2iz^2-2i)}$

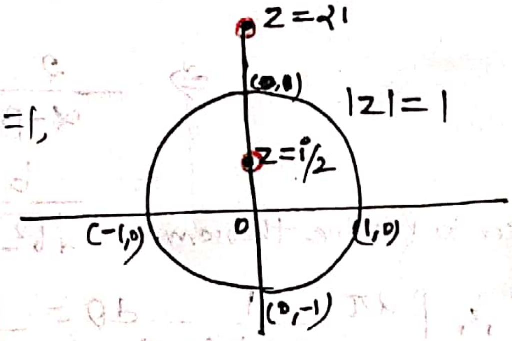
$= \int_C \frac{dz}{5iz-2z^2+2}$

Now finding poles:→ $-2z^2+5iz+2=0$

So $z = \frac{-5i \pm \sqrt{(5i)^2-4(-2)(2)}}{2(-2)} = \frac{-5i\pm\sqrt{-25+16}}{-4} = \frac{-5i\pm\sqrt{-9}}{-4}$

or $z = \frac{-5i\pm 3i}{-4} = \frac{-5i+3i}{-4}, \frac{-5i-3i}{-4}$

$\Rightarrow z = 2i, i/2$



Pole at $z = i/2$ is inside the $C: |z|=1$,

So Res (at $z = i/2$)

$= \lim_{z\to i/2} (z-i/2)\frac{1}{5iz-2z^2+2}$

$= \lim_{z\to i/2} \left(\frac{\cancel{2z-i}}{2}\right)\frac{1}{(\cancel{2z-i})(2i-z)} = \lim_{z\to i/2}\frac{1}{2(2i-z)} = \frac{1}{2\left[2i-\frac{i}{2}\right]}$

$= \frac{1}{\cancel{2}\left[\frac{4i-i}{\cancel{2}}\right]} = \frac{1}{3i}$

—According to Residue theorem

$\int_C \frac{dz}{5iz-2z^2+2} = 2\pi i\sum_{i=1}^{n} Res(z_i) = 2\pi \cancel{i} \times \frac{1}{3\cancel{i}} = \frac{2\pi}{3}$ **Ans.**

Q ④ Evaluate the contour integration $\int_0^{2\pi}\frac{1}{a+b\sin\theta}d\theta$, where $a>b$.

Sol:→ let $I = \int_0^{2\pi}\frac{d\theta}{a+b\sin\theta}$

here $\sin\theta = \left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)$, $e^{i\theta}=z$, $d\theta = dz/iz$

So $I = \int_C \frac{1}{a+\frac{b}{2i}(z-1/z)}\frac{dz}{iz}$, here $C: |z|=1$

$$= \int_c \frac{1}{2iaz + bz^2 - b}\,dz = \frac{1}{b}\int_c \frac{2}{z^2 + \frac{2aiz}{b} - 1}\,dz$$

$$= \frac{1}{b}\int_c \frac{2}{(z-\alpha)(z-\beta)}\,dz \quad \left[bz^2 + 2aiz - b = b\left\{z^2 + \frac{2aiz}{b} - 1\right\}\right]$$

(Where $\alpha + \beta = -\frac{2ai}{b}$ $\left\{\because \text{Sum of zero} = -\frac{\text{Coefficient of } z}{\text{Coefficient of } z^2}\right\}$

and $\alpha\beta = -1$ $\left\{\because \text{product of zero} = \frac{\text{Constant}}{\text{Coefficient of } z^2}\right\}$

Remember $a > b$ [$\because \alpha < 1$ & $\beta > 1$]

here pole lies at $z = \alpha$ in the $c : |z| = 1$

So Residue at $z = \alpha \Rightarrow \lim_{z\to\alpha} (z-\alpha)\frac{2}{(z-\alpha)(z-\beta)} = \frac{2}{\alpha - \beta}$

$$\Rightarrow \frac{2}{\alpha-\beta}$$

$$= \frac{b}{\sqrt{b^2 - a^2}} = \frac{b}{i\sqrt{a^2 - b^2}}$$

$\left\{(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta = -\frac{4a^2}{b^2} + 4 \Rightarrow \alpha - \beta = \frac{2\sqrt{b^2 - a^2}}{b}\right\}$

Acc. to Residue theorem

$$\therefore \int_0^{2\pi} \frac{1}{a + b\sin\theta}\,d\theta = \frac{1}{b}\int_c \frac{2}{z^2 + 2\left(\frac{aiz}{b}\right) - 1}\,dz = 2\pi i \times \frac{b}{\cancel{b}\,i\sqrt{a^2 - b^2}} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

Ans.

Q. 5 Using Cauchy integral formula to evaluate $\int_c \frac{e^{2z}}{z - \pi i}\,dz$, where c is the ellipse $|z-2| + |z+2| = 6$

Sol:→ Given $I = \int_c \frac{e^{2z}}{z - \pi i}\,dz$, so poles are $z - \pi i = 0$

$$\Rightarrow z = \pi i$$

Also, here the closed contour is an ellipse

$$|z-2| + |z+2| = 6$$

$$\sqrt{(x+2)^2 + y^2} + \sqrt{(x-2)^2 + y^2} = 6$$

or $\sqrt{(x+2)^2 + y^2} = 6 - \sqrt{(x-2)^2 + y^2}$

Sqr. both side

$$(x+2)^2 + \cancel{y^2} = 36 + (x-2)^2 + \cancel{y^2} - 12\sqrt{(x-2)^2 + y^2}$$

$$\Rightarrow (x+2)^2 = 36 + (x-2)^2 - 12\sqrt{(x-2)^2 + y^2}$$

or $12\sqrt{(x-2)^2 + y^2} = (x-2)^2 - (x+2)^2 + 36$

$$\Rightarrow 12\sqrt{(x-2)^2 + y^2} = [(x-2) + (x+2)][(x-2) - (x+2)] + 36$$

or $12\sqrt{(x-2)^2 + y^2} = 2x(-4) + 36$

or $12\sqrt{(x-2)^2+y^2} = 36-8x$

$144[(x-2)^2+y^2] = (36-8x)^2$

or $\cancel{144}[(x-2)^2+y^2] = \cancel{16}(9-2x)^2$

or $9[x^2+4-4x+y^2] = 81+4x^2-36x$

$\Rightarrow 9x^2+36-\cancel{36x}+9y^2 = 81+4x^2-\cancel{36x}$

$\Rightarrow 9x^2+9y^2-4x^2 = 81-36$

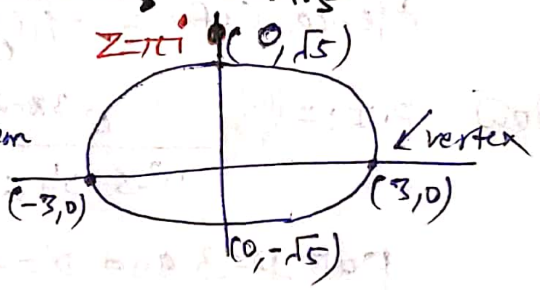
$\Rightarrow 5x^2+9y^2 = 45$

$\Rightarrow \dfrac{x^2}{9}+\dfrac{y^2}{5} = 1$

Here, Ellipse has $(3,0)$ and $(0,\sqrt{5})$ as a ~~foci~~ an major and minor axis ÷ vertices

Here the pole $z=\pi i$, outside the Ellipse, thus by Cauchy Integral theorem

$$\Rightarrow \int_C \frac{e^{z}\,dz}{z-\pi i} = 0$$ Ans.

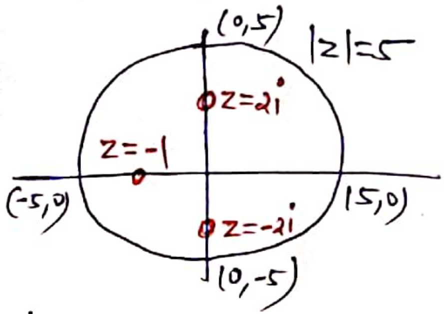


Q.(6) Using Cauchy's Residue theorem evaluate the integral $\int_c \frac{z^2-2z}{(z+1)^2(z^2+4)}dz$ where c is the circle $|z|=5$

Let $I = \int_c \frac{z^2-2z}{(z+1)^2(z^2+4)}dz$, Here poles are $(z+1)^2(z^2+4)=0$

$\Rightarrow z = \underbrace{-1,-1}_{\text{double pole at } z=-1}, \underbrace{\pm 2i}_{\text{simple pole at } z=2i}$

All the poles lie inside the circle $C: |z|=5$



Thus Res (at $z=-1$)

$$= \lim_{z\to -1} \frac{1}{(2-1)!}\left\{\frac{d^{2-1}}{dz^{2-1}}\left[\cancel{(z+1)^2}\frac{z^2-2z}{\cancel{(z+1)^2}(z^2+4)}\right]\right\}$$

$$\Rightarrow \lim_{z\to -1} \frac{d}{dz}\left\{\frac{z^2-2z}{(z^2+4)}\right\}$$

$$\lim_{z\to -1} \frac{(z^2+4)\frac{d}{dz}(z^2-2z)-(z^2-2z)\frac{d}{dz}(z^2+4)}{(z^2+4)^2} = \lim_{z\to -1}\frac{(z^2+4)(2z-2)-(z^2-2z)\times 2z}{(z^2+4)^2}$$

$$\text{or } \lim_{z\to -1} \frac{2[z^3-z^2+4z-4]-2z^3+4z^2}{(z^2+4)^2} = \lim_{z\to -1}\frac{\cancel{2z^3}-2z^2+8z-8-\cancel{2z^3}+4z^2}{(z^2+4)^2}$$

$$\Rightarrow \lim_{z\to -1}\frac{2z^2+8z-8}{(z^2+4)^2} = \frac{2(-1)^2+8(-1)-8}{(1+4)^2} = \frac{-14}{25}$$

Also Res (at $z=2i$) $= \lim_{z\to 2i} \frac{\cancel{(z-2i)}\, z^2-2z}{(z+1)^2(z+2i)\cancel{(z-2i)}} = \frac{(2i)^2-2(2i)}{(2i+1)^2(2i+2i)}$

$$= \frac{-4-4i}{(-4+1+4i)(4i)} = \frac{-\cancel{4}(1+i)\times(i)}{\cancel{4}i(-3+4i)\times(i)}$$

$$= \frac{i(1+i)}{(4i-3)} = \frac{i+i^2}{4i-3} = \frac{i-1}{4i-3}\times\frac{4i+3}{4i+3} = \frac{-4+3i-4i-3}{-16-9}$$

$$= \frac{-7-i}{-25} = \frac{7+i}{25}$$

and Res (at $z=-2i$) $= \lim_{z\to -2i} \frac{\cancel{(z+2i)}\, z^2-2z}{(z+1)^2\cancel{(z+2i)}(z-2i)} = \frac{(-2i)^2-2(-2i)}{(-2i+1)^2(-4i)}$

$$= \frac{-4+4i}{(-4+1-4i)(-4i)} = \frac{-4(1-i)}{-4i(-3-4i)}\times\frac{i}{i}$$

$$= \frac{i(1-i)}{-i(-3-4i)} = \frac{i+1}{3+4i}$$

$$\Rightarrow \frac{i+1}{3+4i}\times\frac{3-4i}{3-4i} = \frac{3i+4+3-4i}{9-(-16)} = \frac{7-i}{25}$$

Acc. to Residue theorem $\Rightarrow 2\pi i \sum_{i=1}^{3} \text{Res (at } z_i) = 2\pi i\left(\frac{-14}{25}+\frac{7+i}{25}+\frac{7-i}{25}\right) = 2\pi i\left[\frac{-14+7+i+7-i}{25}\right]$

$$= 2\pi i \times 0 = \underline{0} \text{ Ans.}$$

Integration about unit circle :→

Evaluation of Real Integral (Unit circle)

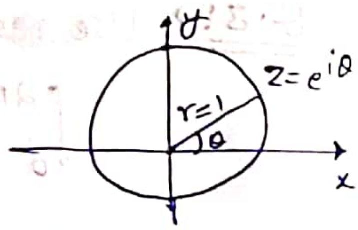
Type

(a) Integration round unit circle of the type :→

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$$

(Where $f(\cos\theta, \sin\theta)$ is rational function of $\cos\theta$ and $\sin\theta$.

Working Rule to evaluate :→



Step ① Convert $\sin\theta$ (and $\cos\theta$) into z.

Considering a unit circle

(Where $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \left(\dfrac{z - 1/z}{2i}\right)$

and $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2} = \left(\dfrac{z + 1/z}{2}\right)$

$\because z = re^{i\theta}$ (polar form)

Here unit circle means $r = 1$

$\Rightarrow z = re^{i\theta} = 1 \cdot e^{i\theta}$

$z = e^{i\theta}$

Hence put $\sin\theta = \dfrac{1}{2i}\left[z - \dfrac{1}{z}\right]$ and $\cos\theta = \dfrac{1}{2}\left[z + \dfrac{1}{z}\right]$

Step ② We know $z = e^{i\theta}$ $(\because r = 1)$

$\Rightarrow dz = ie^{i\theta} d\theta$

$\Rightarrow dz = iz\, d\theta$

$\Rightarrow \dfrac{dz}{iz} = d\theta$

$\Rightarrow$ so put $d\theta = \dfrac{dz}{iz}$

So Integrand is converted into a function of z

Step ③ then apply Cauchy's residue theorem to evaluate the integral.

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Example (i):→ Evaluate the integral $\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta}$

Sol:→ Remember, We shall convert $\cos\theta$ and $d\theta$ in the function of z.

So put $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$

$$\Rightarrow \int_0^{2\pi} \frac{d\theta}{5-3\cos\theta} = \int_0^{2\pi} \frac{d\theta}{5-3\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)}$$

$$= \int_0^{2\pi} \frac{d\theta}{\left(\frac{10-3e^{i\theta}-3e^{-i\theta}}{2}\right)}$$

$$= \int_0^{2\pi} \frac{2\,d\theta}{10-3e^{i\theta}-3e^{-i\theta}}$$

$$= \int_C \frac{2\,(dz/iz)}{10-3z-3\left(\frac{1}{z}\right)} \qquad \left[\because d\theta = \frac{dz}{iz},\ e^{i\theta}=z\right]$$

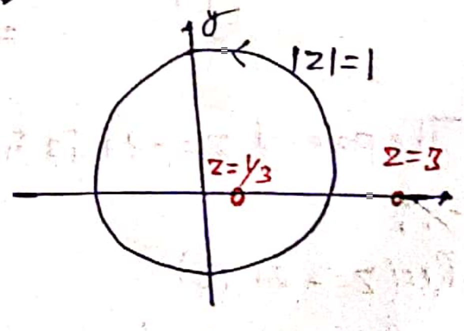
$$= \int_C \frac{2\,dz}{\left(10-3z-\frac{3}{z}\right)iz} = \frac{1}{i}\int_C \frac{2\,dz}{(10z-3z^2-3)}$$

{C is unit circle $|z|=1$}

$$\Rightarrow -\frac{2}{i}\int_C \frac{dz}{3z^2-10z+3} = -\frac{2}{i}\int_C \frac{dz}{(3z-1)(z-3)} = 2i\int_C \frac{dz}{(3z-1)(z-3)}$$

Now finding poles:→ $(3z-1)(z-3)=0$

$\Rightarrow z = 1/3,\ 3$ (simple poles)



Only pole of $z=1/3$ inside the circle $|z|=1$

$\therefore$ Res at $\left(z=\frac{1}{3}\right) = \lim_{z\to 1/3}\left(z-\frac{1}{3}\right)f(z)$

$$= \lim_{z\to 1/3}(z-1/3)\frac{1\times 2i}{(3z-1)(z-3)} = \lim_{z\to 1/3}\frac{\cancel{(3z-1)}\times 2i}{3\cancel{(3z-1)}(z-3)}$$

$$= \lim_{z\to 1/3} 2i\,\frac{1}{3(z-3)} = \frac{2i}{3\left(\frac{1}{3}-3\right)} = \frac{-2i}{8} = -i/4$$

By Cauchy Residue theorem $\int_0^{2\pi}\frac{d\theta}{5-3\cos\theta} = 2\pi i\left(\text{sum of Residues inside } C\right) = 2\pi i\left(\frac{-i}{4}\right) = \pi/2$ **Ans.**

Ex. (2) :→ Evaluate $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$ using complex variable technique.

Sol :→ We know $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$, So put it on Integrand,

$$\text{So } \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_0^{2\pi} \frac{d\theta}{2+\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)}$$

$$= \int_0^{2\pi} \frac{2\,d\theta}{4+e^{i\theta}+e^{-i\theta}} \quad \text{——— (1)}$$

Now put $e^{i\theta} = z$

so that $e^{i\theta}(i\,d\theta) = dz$

$\Rightarrow iz\,d\theta = dz$

$\Rightarrow d\theta = \frac{dz}{iz}$ in Equation (1), we get

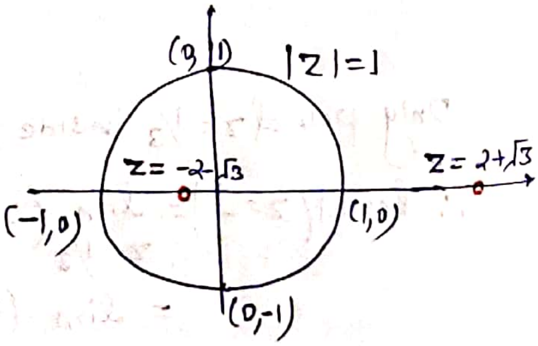
$$\int_0^{2\pi} \frac{2\,d\theta}{4+e^{i\theta}+e^{-i\theta}} = \int_C \frac{2\left(\frac{dz}{iz}\right)}{4+z+\frac{1}{z}} = \frac{1}{i}\int_C \frac{2\,dz}{z^2+4z+1}$$

Here C is always a unit circle (i.e. $|z|=1$)

Now checking poles :→ $z^2+4z+1=0$

$$\Rightarrow z = \frac{-4\pm\sqrt{16-4}}{2} = \frac{-4\pm 2\sqrt{3}}{2} = -2\pm\sqrt{3}$$

The pole at $z=-2+\sqrt{3}$ inside the circle.



$\therefore$

$$\text{Res}(z=-2+\sqrt{3}) = \lim_{z\to(-2+\sqrt{3})} \frac{1}{i}\frac{\cancel{(z+2-\sqrt{3})}\,2}{\cancel{(z+2-\sqrt{3})}(z+2+\sqrt{3})}$$

$$= \lim_{z\to(-2+\sqrt{3})} \frac{2}{i(z+2+\sqrt{3})} = \frac{2}{i[(-2+\sqrt{3})+2+\sqrt{3}]}$$

$$= 1/\sqrt{3}\,i$$

Hence by Cauchy's Residue theorem $\int_0^{2\pi}\frac{d\theta}{2+\cos\theta} = 2\pi i\left[\begin{matrix}\text{Sum of residue}\\ \text{inside } C\end{matrix}\right] = 2\pi i \times \frac{1}{\sqrt{3}\,i} = \frac{2\pi}{\sqrt{3}}$

Type b:→ $\int_0^{2\pi} f(\cos n\theta, \sin n\theta)\, d\theta$

Ex.(1) Prove that $\int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{5+4\cos\theta} = \frac{\pi}{6}$

let $I = \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta$

or $I =$ Real part of $\int_0^{2\pi} \frac{\cos 2\theta + i\sin 2\theta}{5+4\cos\theta} d\theta$

or $I =$ Real part of $\int_0^{2\pi} \frac{e^{i2\theta}}{5+4\cos\theta} d\theta$ $\quad (\because e^{i2\theta} = \cos 2\theta + i\sin 2\theta)$

Now we know from substitution

put $z = e^{i\theta}$

$d\theta = dz/iz$

and here curve(closed) is a unit circle $|z| = 1$.

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So we get $I = R.p.d. \int_0^{2\pi} \frac{e^{i2\theta}}{5+4\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)}\, d\theta$

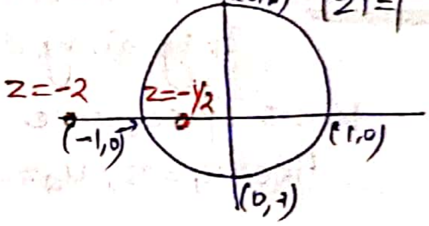
$$I = R.p.\ of \int_C \frac{z^2}{5+\cancel{4}\left(\frac{z+1/z}{\cancel{2}}\right)}\frac{dz}{i\cancel{z}} = R.p.d. \int_C \frac{z\,dz}{i\left[\frac{5z+2z^2+2}{z}\right]}$$

$$= R.p.\ of \int_C \frac{1}{i}\times\frac{z^2}{(2z^2+5z+2)}\,dz = R.p.\ of\ \frac{1}{i}\int_C \frac{z^2\,dz}{(2z+1)(z+2)}$$

Now checking the poles; $(2z+1)(z+2)=0 \Rightarrow z = -1/2, -2$

Only $z=-1/2$ inside the circle $|z|=1$, so

$$Res(z=-1/2) = \lim_{z\to -1/2}(z+1/2)f(z)$$



$$= \lim_{z\to -1/2}\left(\frac{\cancel{2z+1}}{2}\right)\times\frac{z^2}{(\cancel{2z+1})(z+2)}$$

$$= \lim_{z\to -1/2}\frac{z^2}{2(z+2)} = \frac{(-1/2)^2}{2[-1/2+2]} = \frac{(1/4)}{\cancel{2}[3/\cancel{2}]} = \frac{1}{12}$$

$\therefore I =$ Real part of $\frac{1}{i}\int_C\frac{z^2}{(2z+1)(z+2)}\,dz = \frac{1}{\cancel{i}}\times 2\pi\cancel{i}$(sum of Residue)

$$= 2\pi\left(\frac{1}{12}\right) = \pi/6 \quad \textbf{Proved}$$

Ex. 2: Evaluate $\int_0^{2\pi}\frac{\sin^2\theta\,d\theta}{5-4\cos\theta}$.

let $I = \int_0^{2\pi}\frac{\sin^2\theta\,d\theta}{5-4\cos\theta}$

or $I = \frac{1}{2}\int_0^{2\pi}\frac{1-\cos2\theta}{5-4\cos\theta}\,d\theta$

or $I =$ Real part of $\frac{1}{2}\int_0^{2\pi}\frac{1-\cos2\theta-i\sin2\theta}{5-4\cos\theta}\,d\theta$

$= R.p.\ of\ \frac{1}{2}\int_0^{2\pi}\frac{1-e^{2i\theta}}{5-4\cos\theta}\,d\theta$

$= R.p.\ of\ \frac{1}{2}\int_C\frac{1-z^2}{5-2\left(z+\frac{1}{z}\right)}\frac{dz}{iz}$ $\left[\because \cos\theta = \frac{1}{2}(z+1/z)\right.$ and $z=e^{i\theta}$, $\left.d\theta = dz/iz\right]$

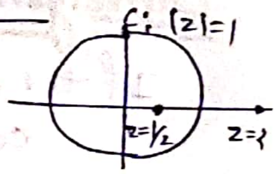
$= R.p.\ of\ \frac{1}{2i}\int_C\frac{1-z^2}{5z-2z^2-2}\,dz$

$= R.p.\ of\ \frac{1}{2i}\int_C\frac{z^2-1}{2z^2-5z+2}\,dz$

$I = R.p.\ of\ \frac{1}{2i}\int_C\frac{z^2-1}{(2z-1)(z-2)}\,dz$

Now poles $z = 1/2, 2$

Only pole $z=1/2$ lies inside $|z|=1$, thus



$$Res(z=1/2) = \lim_{z\to 1/2}(z-1/2)\times\frac{z^2-1}{(2z-1)(z-2)}\,dz$$

$$= \lim_{z\to 1/2}\frac{(\cancel{2z-1})}{2}\times\frac{(z^2-1)}{(\cancel{2z-1})(z-2)}$$

$$= \frac{1/4-1}{2(1/2-2)} = 1/4$$

$I = R.p.\ of\ \frac{1}{2i}\int_C\frac{z^2-1}{(2z-1)(z-2)}\,dz$

$= \frac{1}{2i}\times 2\pi i$[sum of Residues]

$= \frac{1}{2i}\times 2\pi i\times\frac{1}{4} = \frac{\pi}{4}$ **Ans.**