Collection of Formulas in Signal Processing

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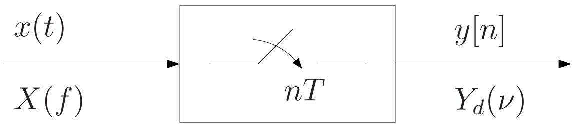
Kungliga Tekniska Högskolan
School of Electrical Engineering and Computer Science
Information science and engineering
SE-100 44 Stockholm

Contents

1 Definitions and Relations for Deterministic Signals

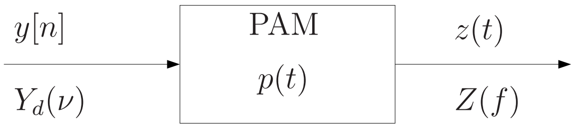
	Time continuous	Time discrete
	Fundamentals	
Spectrum	$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$	$X_d(\nu) = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi\nu n}$
Inverse	$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$	$x[n] = \int_{-1/2}^{1/2} X_d(\nu)e^{j2\pi\nu n}d\nu$
Energy Spectrum	$S_X(f) = \|X(f)\|^2$	$S_X(\nu) = \|X_d(\nu)\|^2$
Total Energy	$\int_{-\infty}^{\infty} \|x(t)\|^2dt = \int_{-\infty}^{\infty} \|X(f)\|^2df$	$\sum_{n=-\infty}^{\infty} \|x[n]\|^2 = \int_{-1/2}^{1/2} \|X_d(\nu)\|^2d\nu$
	Linear Filtering	
Output Signal	$y(t) = h(t) * x(t)$	$y[n] = h[n] * x[n]$
	$= \int_{-\infty}^{\infty} h(u)x(t-u)du$	$= \sum_{m=-\infty}^{\infty} h[m]x[n-m]$
Output Spectrum	$Y(f) = H(f)X(f)$	$Y_d(\nu) = H_d(\nu)X_d(\nu)$
Frequency response	$H(f) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}dt$	$H_d(\nu) = \sum_{n=-\infty}^{\infty} h[n]e^{-j2\pi\nu n}$
	Transfer function (causal systems)	
	$H(s) = \int_{0}^{\infty} h(t)e^{-st}dt$	$H(z) = \sum_{n=0}^{\infty} h[n]z^{-n}$
	Transfer function (non-causal systems)	
	$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$	$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$

Sampling



$$y[n] = x(nT) \qquad \Longrightarrow \qquad Y_d(\nu) = \frac{1}{T} \sum_m X\left(\frac{\nu - m}{T}\right)$$

Pulse amplitude modulation



$$z(t) = \sum_n y[n]p(t - nT) \qquad \Longrightarrow \qquad Z(f) = P(f)Y_d(fT)$$

Reconstruction – continuous time

$$\hat{x}(t) = \sum_n x(nT)h(t - nT) \qquad\qquad E_\epsilon = \int_{-\infty}^{\infty} |\hat{x}(t) - x(t)|^2 \, dt$$

$$E_\epsilon = \int_{-\infty}^{\infty} \left| \left(\frac{H(f)}{T} - 1\right) X(f) + \sum_{m \neq 0} \frac{H(f)}{T} X(f - m/T) \right|^2 df$$

The sampling theorem

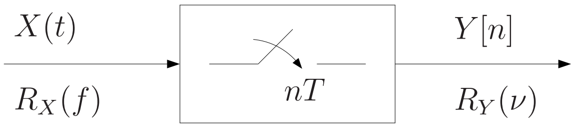
If $X(f) = 0$ for $|f| \geq \frac{1}{2T}$, then

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT) \frac{\sin(\pi(t - nT)/T)}{\pi(t - nT)/T}$$

2 Definitions and relations for wide-sense stationary processes

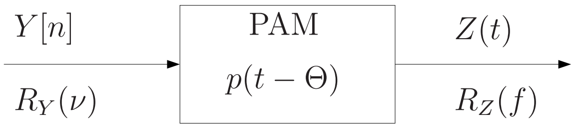
	Continuous time	Discrete time
	Fundamentals	
acf	$r_X(\tau) = \mathrm{E}[X(t+\tau)X(t)]$	$r_X(k) = \mathrm{E}[X(n+k)X(n)]$
PSD	$R_X(f) = \int_{-\infty}^{\infty} r_X(\tau)e^{-j2\pi f\tau}d\tau$	$R_X(\nu) = \sum_{k=-\infty}^{\infty} r_X(k)e^{-j2\pi\nu k}$
Inverse	$r_X(\tau) = \int_{-\infty}^{\infty} R_X(f)e^{j2\pi ft}df$	$r_X(k) = \int_{-1/2}^{1/2} R_X(\nu)e^{j2\pi\nu k}d\nu$
Total power	$\int_{-\infty}^{\infty} R_X(f)df = \mathrm{E}[X(t)^2]$	$\int_{-1/2}^{1/2} R_X(\nu)d\nu = \mathrm{E}[X(n)^2]$
	Linear filtering	
Filtered signal	$Y(t) = h(t) * X(t)$ $= \int_{-\infty}^{\infty} h(u)X(t-u)du$	$Y(n) = h(n) * X(n)$ $= \sum_{m=-\infty}^{\infty} h(m)X(n-m)$
Expected value	$m_Y = m_X \int_{-\infty}^{\infty} h(u)du$ $= m_X H(0)$	$m_Y = m_X \sum_{m=-\infty}^{\infty} h(m)$ $= m_X H(0)$
acf	$r_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)h(v)\times$ $r_X(\tau-u+v)dudv$	$r_Y(k) = \sum_{\ell=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} h(\ell)h(m)\times$ $r_X(k-\ell+m)$
PSD	$R_Y(f) = \lvert H(f)\rvert^2 R_X(f)$	$R_Y(\nu) = \lvert H(\nu)\rvert^2 R_X(\nu)$

Sampling



$$Y[n] = X(nT) \qquad \Longrightarrow \qquad R_Y(\nu) = \frac{1}{T}\sum_m R_X\left(\frac{\nu - m}{T}\right)$$

Pulse amplitude modulation



$$Z(t) = \sum_n Y[n]p(t - nT - \Theta) \qquad \Longrightarrow \qquad \begin{cases} R_Z(f) & = \frac{1}{T}|P(f)|^2 R_Y(fT) \\ \mathrm{E}[Z(t)] & = \frac{1}{T}P(0)\,\mathrm{E}[Y(n)] \end{cases}$$

Reconstruction – contionuous time

$$\hat{X}(t) = \sum_n X(nT + \Theta)h(t - nT - \Theta) \qquad P_\epsilon = \mathrm{E}[(\hat{X}(t) - X(t))^2]$$

$$P_\epsilon = \int_{-\infty}^{\infty} \left|\left(\frac{H(f)}{T} - 1\right)\right|^2 R_X(f) + \sum_{m\neq 0}\left|\frac{H(f)}{T}\right|^2 R_X(f - m/T)\,df$$

3 Some common distributions

Uniform distribution Re(a, b)

$$\begin{aligned} f_X(x) &= \begin{cases} \dfrac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases} \\ m_X &= \mathrm{E}[X] = \frac{a+b}{2} \\ \sigma^2 &= \mathrm{E}[(X-m_X)^2] = \frac{(b-a)^2}{12} \end{aligned}$$

Rayleigh distribution

$$\begin{aligned} f_X(x) &= \begin{cases} \dfrac{x}{a^2} e^{-\frac{x^2}{2a^2}} & x \geq 0 \\ 0 & x < 0 \end{cases} \\ m_X &= \mathrm{E}[X] = a\sqrt{\frac{\pi}{2}} \\ \sigma^2 &= \mathrm{E}[(X-m_X)^2] = a^2\,(2 - \pi/2) \end{aligned}$$

One-sided exponential distribution

$$\begin{aligned} f_X(x) &= \begin{cases} a e^{-ax} & x \geq 0 \\ 0 & x < 0 \end{cases} \\ m_X &= \sigma = \frac{1}{a} \end{aligned}$$

Two-sided exponential distribution

$$\begin{aligned} f_X(x) &= \frac{a}{2} e^{-a|x|} \\ m_X &= 0, \qquad \sigma^2 = \frac{2}{a^2} \end{aligned}$$

Poisson distribution Discrete integer distribution

$$P(X = k) \triangleq p_k = e^{-a} \frac{a^k}{k!} \qquad \text{for } k = 0, 1, 2, \ldots$$

$$m_X = \sigma^2 = a$$

Normal distribution One-dimensional, $N(m_X, \sigma)$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m_X)^2}{2\sigma^2}}$$

Two-dimensional, $N(m_X, m_Y, \sigma_X, \sigma_Y, \rho)$

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{g(x,y)}{2(1-\rho^2)}}$$

where

$$g(x,y) = \left(\frac{x-m_X}{\sigma_X}\right)^2 - 2\rho\frac{x-m_X}{\sigma_X}\frac{y-m_Y}{\sigma_Y} + \left(\frac{y-m_Y}{\sigma_Y}\right)^2$$

and

$$\rho = \rho(X,Y) = \frac{\mathrm{E}[XY] - m_X m_Y}{\sigma_X\sigma_Y}$$

If A, B, C and D are all Normal distributed, then

$$\mathrm{E}[ABCD] = \mathrm{E}[AB]\,\mathrm{E}[CD] + \mathrm{E}[AC]\,\mathrm{E}[BD] + \mathrm{E}[AD]\,\mathrm{E}[BC] - 2\,\mathrm{E}[A]\,\mathrm{E}[B]\,\mathrm{E}[C]\,\mathrm{E}[D]$$

The so-called Q-function can also be convenient to use. If X is $N(m_X, \sigma)$, then $\Pr[X > a] = Q\left(\frac{a-m_X}{\sigma}\right)$ where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-u^2/2} du$$

For $x > 1$,

$$\frac{1-x^{-2}}{x\sqrt{2\pi}} e^{-x^2/2} < Q(x) < \frac{1}{x\sqrt{2\pi}} e^{-x^2/2}$$

4 Continuous Fourier transform

Properties

$x(t)$	$X(f)$	
$x(t)$	$\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\, dt$	(4.1)
$\int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\, df$	$X(f)$	(4.2)
$cx(t) + dy(t)$	$cX(f) + dY(f)$	(4.3)
$x(ct), \quad (c \neq 0)$	$\frac{1}{\lvert c\rvert}X(\frac{f}{c})$	(4.4)
$x(-t)$	$X(-f)$	(4.5)
$x^*(t)$	$X^*(-f)$	(4.6)
$X(t)$	$x(-f)$	(4.7)
$x(t-P)$	$e^{-j2\pi Pf}X(f)$	(4.8)
$e^{j2\pi f_0 t}x(t)$	$X(f-f_0)$	(4.9)
$\left(\frac{\partial}{\partial t}\right)^n x(t)$	$(j2\pi f)^n X(f)$	(4.10)
$(-j2\pi t)^n x(t)$	$\left(\frac{\partial}{\partial f}\right)^n X(f)$	(4.11)
$x(t) * y(t)$	$X(f)Y(f)$	(4.12)
$x(t)y(t)$	$X(f) * Y(f)$	(4.13)

Parseval's theorem

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |X(f)|^2\, df \tag{4.14}$$

Generalized Parseval's theorem

$$\int_{-\infty}^{\infty} x(t)y^*(t)\, dt = \int_{-\infty}^{\infty} X(f)Y^*(f)\, df \tag{4.15}$$

Common transform pairs (Continuous Fourier transform)

$x(t),\quad a>0$	$X(f),\quad (\omega = 2\pi f)$	
$\delta(t)$	1	(4.16)
1	$\delta(f)$	(4.17)
$\mathrm{rect}_P(t) = \begin{cases} 1 & \text{for } \lvert t\rvert \le \frac{P}{2} \\ 0 & \text{for } \lvert t\rvert > \frac{P}{2} \end{cases}$	$P\mathrm{sinc}(fP) = \dfrac{\sin(\pi f P)}{\pi f}$	(4.18)
$B\mathrm{sinc}(Bt) = \dfrac{\sin(\pi B t)}{\pi t}$	$\mathrm{rect}_B(f) = \begin{cases} 1 & \text{for } \lvert f\rvert \le \frac{B}{2} \\ 0 & \text{for } \lvert f\rvert > \frac{B}{2} \end{cases}$	(4.19)
$e^{j2\pi f_0 t}$	$\delta(f - f_0)$	(4.20)
$\sin(2\pi f_0 t)$	$\dfrac{1}{2j}(\delta(f-f_0) - \delta(f+f_0))$	(4.21)
$\cos(2\pi f_0 t)$	$\dfrac{1}{2}(\delta(f-f_0) + \delta(f+f_0))$	(4.22)
$u(t) = \begin{cases} 1, & t>0 \\ 1/2, & t=0 \\ 0, & t<0 \end{cases}$	$\dfrac{1}{j2\pi f} + \dfrac{1}{2}\delta(f)$	(4.23)
$e^{-at}u(t)$	$\dfrac{1}{a + j2\pi f}$	(4.24)
$e^{at}u(-t)$	$\dfrac{1}{a - j2\pi f}$	(4.25)
$e^{-a\lvert t\rvert}$	$\dfrac{2a}{a^2 + (2\pi f)^2}$	(4.26)
$e^{-at}\sin(2\pi f_0 t)u(t)$	$\dfrac{\omega_0}{(j\omega + a)^2 + \omega_0^2} \qquad (\omega_0 = 2\pi f_0)$	(4.27)
$e^{-at}\cos(2\pi f_0 t)u(t)$	$\dfrac{j\omega + a}{(j\omega + a)^2 + \omega_0^2}$	(4.28)
$e^{-a\lvert t\rvert}\sin(2\pi f_0 \lvert t\rvert)$	$\dfrac{2\omega_0(a^2 + \omega_0^2 - \omega^2)}{(a^2 + \omega_0^2 - \omega^2)^2 + (2a\omega)^2}$	(4.29)
$e^{-a\lvert t\rvert}\cos(2\pi f_0 t)$	$\dfrac{2a(a^2 + \omega_0^2 + \omega^2)}{(a^2 + \omega_0^2 - \omega^2)^2 + (2a\omega)^2}$	(4.30)
e^{-at^2}	$\sqrt{\dfrac{\pi}{a}}\,e^{-(\pi f)^2/a}$	(4.31)

5 The Laplace transform

Properties of the two-sided Laplace transform

$x(t)$	$X(s)$	
$x(t)$	$\int_{-\infty}^{\infty} x(t)e^{-st}\,dt$	(5.1)
$\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds$	$X(s)$	(5.2)
$cx(t)+dy(t)$	$cX(s)+dY(s)$	(5.3)
$x(ct), \quad (c \neq 0)$	$\frac{1}{\lvert c\rvert}X(\frac{s}{c})$	(5.4)
$x(t-P)$	$e^{-sP}X(s)$	(5.5)
$e^{-at}x(t)$	$X(s+a)$	(5.6)
$t^n x(t)$	$(-1)^n\frac{\partial^n X(s)}{\partial s^n}$	(5.7)
$\frac{\partial^n x(t)}{\partial t^n}$	$s^n X(s)$	(5.8)
$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	(5.9)
$\big(x(t)u(t)\big) * \big(y(t)u(t)\big)$	$X(s)Y(s)$	(5.10)

Region of convergence for (5.10): $\mathrm{ROC}\big\{X(s)Y(s)\big\} = \mathrm{ROC}\big\{X(s)\big\} \cap \mathrm{ROC}\big\{Y(s)\big\}$

Properties of the one-sided Laplace transform

$x(t)$	$X(s)$	
$x(t)$	$\int_{0^-}^{\infty} x(t)e^{-st}\,dt$	(5.11)
$\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds$	$X(s)$	(5.12)
$cx(t)+dy(t)$	$cX(s)+dY(s)$	(5.13)
$x(ct)$	$\frac{1}{c}X(\frac{s}{c}), \quad c>0$	(5.14)
$x(t-P)u(t-P)$	$e^{-sP}X(s), \quad P>0$	(5.15)
$e^{-at}x(t)$	$X(s+a)$	(5.16)
$t^n x(t)$	$(-1)^n \frac{\partial^n X(s)}{\partial s^n}$	(5.17)
$\frac{\partial^n x(t)}{\partial t^n}$	$s^n X(s) - \sum_{i=0}^{n-1} s^{n-1-i} \left.\frac{\partial^i x(t)}{\partial t^i}\right\vert_{t=0^-}$	(5.18)
$\int_{0^-}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	(5.19)
$x(t)*y(t)$	$X(s)Y(s)$	(5.20)

Initial-value theorem

If $X(s)$ is rational, i.e., $X(s)=\frac{P(s)}{Q(s)}$, where order $P(s) <$ order $Q(s)$, then

$$\lim_{t\to 0^+} x(t) = \lim_{s\to\infty} sX(s) \tag{5.21}$$

Final-value theorem

If $sX(s)$ has all poles in the left halfplane, then

$$\lim_{t\to +\infty} x(t) = \lim_{s\to 0} sX(s) \tag{5.22}$$

Common transform pairs (one-/two-sided Laplace transfrom)

$x(t)$	$X(s)$	ROC	
$\delta(t)$	1	$\forall\, s$	(5.23)
$u(t) = \begin{cases} 1, & t > 0 \\ 1/2, & t = 0 \\ 0, & t < 0 \end{cases}$	$\dfrac{1}{s}$	$\mathrm{Re}\{s\} > 0$	(5.24)
$tu(t)$	$\dfrac{1}{s^2}$	$\mathrm{Re}\{s\} > 0$	(5.25)
$\dfrac{t^n}{n!}u(t)$	$\dfrac{1}{s^{n+1}}$	$\mathrm{Re}\{s\} > 0$	(5.26)
$e^{-at}u(t)$	$\dfrac{1}{s+a}$	$\mathrm{Re}\{s\} > -a$	(5.27)
$\dfrac{t^n e^{-at}}{n!}u(t)$	$\dfrac{1}{(s+a)^{n+1}}$	$\mathrm{Re}\{s\} > -a$	(5.28)
$\sin(\omega_0 t)u(t)$	$\dfrac{\omega_0}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\} > 0$	(5.29)
$\cos(\omega_0 t)u(t)$	$\dfrac{s}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\} > 0$	(5.30)
$e^{-at}\sin(\omega_0 t)u(t)$	$\dfrac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\} > -a$	(5.31)
$e^{-at}\cos(\omega_0 t)u(t)$	$\dfrac{s+a}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\} > -a$	(5.32)
$e^{-at}\left(\cos\omega_0 t - \dfrac{a}{\omega_0}\sin\omega_0 t\right)u(t)$	$\dfrac{s}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\} > -a$	(5.33)
$\left[1 - e^{-at}\left(\cos\omega_0 t + \dfrac{a}{\omega_0}\sin\omega_0 t\right)\right]u(t)$	$\dfrac{a^2+\omega_0^2}{s\left[(s+a)^2+\omega_0^2\right]}$	$\mathrm{Re}\{s\} > -a$	(5.34)

*Common **anti-causal** transform pairs (two-sided Laplace transform)*

$x(t)$	$X(s)$	ROC	
$-u(-t)$	$\frac{1}{s}$	$\mathrm{Re}\{s\} < 0$	(5.35)
$-tu(-t)$	$\frac{1}{s^2}$	$\mathrm{Re}\{s\} < 0$	(5.36)
$-\frac{t^n}{n!}u(-t)$	$\frac{1}{s^{n+1}}$	$\mathrm{Re}\{s\} < 0$	(5.37)
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\mathrm{Re}\{s\} < -a$	(5.38)
$-\frac{t^n e^{-at}}{n!}u(-t)$	$\frac{1}{(s+a)^{n+1}}$	$\mathrm{Re}\{s\} < -a$	(5.39)

Common transform pairs (one-/two-sided Laplace transfrom)

$x(t), \quad x(t) = 0 \, \forall \, t < 0$	$X(s)$	
$\frac{a^2 e^{-bt} - b^2 e^{-at}}{a-b} + abt - a - b$	$\frac{a^2 b^2}{s^2 (s+a)(s+b)}$	(5.40)
$\frac{1}{ab^2} - \frac{1}{a^2+b^2}\left(\frac{\sin bt}{b} + \frac{a\cos bt}{b^2} + \frac{e^{-at}}{a}\right)$	$\frac{1}{s(s+a)\left(s^2+b^2\right)}$	(5.41)
$2 - 2\cos at - at\sin at$	$\frac{2a^4}{s\left(s^2+a^2\right)^2}$	(5.42)
$e^{-at}(\sin bt - bt\cos bt)$	$\frac{2b^3}{\left[(s+a)^2+b^2\right]^2}$	(5.43)
$\cos at - \cos bt$	$\frac{\left(b^2-a^2\right)s}{(s^2+a^2)(s^2+b^2)}$	(5.44)
$\frac{\cos bt}{b^2} - \frac{\cos at}{a^2} + \frac{1}{a^2} - \frac{1}{b^2}$	$\frac{\left(b^2-a^2\right)}{s(s^2+a^2)(s^2+b^2)}$	(5.45)
$\frac{ce^{-bt}}{(a-b)\left(b^2+c^2\right)} - \frac{ce^{-at}}{(a-b)\left(a^2+c^2\right)} + - \frac{c(a+b)\cos ct + \left(c^2-ab\right)\sin ct}{\left(a^2+c^2\right)\left(b^2+c^2\right)}$	$\frac{c}{(s+a)(s+b)\left(s^2+c^2\right)}$	(5.46)

The convergence region for above transform pairs is the half plane to the right of the rightmost pole.

6 z-Transform

Properties of the two-sided z-transform

$x[n]$	$X(z)$	
$x[n]$	$\sum_{n=-\infty}^{\infty} x[n]z^{-n}$	(6.1)
$\frac{1}{2\pi j} \oint_{\lvert z\rvert=r} X(z)z^{n-1}\,dz$	$X(z)$	(6.2)
$cx[n] + dy[n]$	$cX(z) + dY(z)$	(6.3)
$x[-n]$	$X(z^{-1})$	(6.4)
$x^*[n]$	$X^*(z^*)$	(6.5)
$x[n-k]$	$z^{-k}X(z)$	(6.6)
$a^n x[n]$	$X(z/a)$	(6.7)
$x[n] * y[n]$	$X(z)Y(z)$	(6.8)
$nx[n]$	$-z\frac{\partial X(z)}{\partial z}$	(6.9)

Region of convergence for (6.8): $\mathrm{ROC}\{X(z)Y(z)\} = \mathrm{ROC}\{X(z)\} \cap \mathrm{ROC}\{Y(z)\}$

Properties of the one-sided z-transform

$x[n]$	$X(z)$	
$x[n]$	$\sum_{n=0}^{\infty} x[n]z^{-n}$	(6.10)
$\frac{1}{2\pi j}\oint_{\lvert z\rvert=r} X(z)z^{n-1}\,dz$	$X(z)$	(6.11)
$cx[n]+dy[n]$	$cX(z)+dY(z)$	(6.12)
$x^*[n]$	$X^*(z^*)$	(6.13)
$x[n-k], k>0$	$z^{-k}X(z)+\sum_{m=1}^{k} x[-m]z^{m-k}$	(6.14)
$x[n+k], k>0$	$z^{k}X(z)-\sum_{m=0}^{k-1} x[m]z^{k-m}$	(6.15)
$a^n x[n]$	$X(z/a)$	(6.16)
$x[n]*y[n]$	$X(z)Y(z)$	(6.17)
$nx[n]$	$-z\frac{\partial X(z)}{\partial z}$	(6.18)

Initial-value theorem

If $X(z)$ is rational, i.e., $X(z)=\frac{P(z)}{Q(z)}$ where order $P(z) \leq$ order $Q(z)$, then

$$x[0]=\lim_{z\to\infty} X(z) \tag{6.19}$$

Final-value theorem

If $(z-1)X(z)$ has all poles strictly inside the unit circle, then

$$\lim_{n\to\infty} x[n]=\lim_{z\to 1}(z-1)X(z) \tag{6.20}$$

Common transform pairs (one-/two-sided z-transform)

$x[n]$	$X(z)$	ROC	
$\delta[n]$	1		(6.21)
$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$	$\dfrac{z}{z-1}$	$\|z\| > 1$	(6.22)
$nu[n]$	$\dfrac{z}{(z-1)^2}$	$\|z\| > 1$	(6.23)
$n^2u[n]$	$\dfrac{z(z+1)}{(z-1)^3}$	$\|z\| > 1$	(6.24)
$a^n u[n]$	$\dfrac{z}{z-a}$	$\|z\| > \|a\|$	(6.25)
$na^n u[n]$	$\dfrac{za}{(z-a)^2}$	$\|z\| > \|a\|$	(6.26)
$n^2a^n u[n]$	$\dfrac{z(z+a)a}{(z-a)^3}$	$\|z\| > \|a\|$	(6.27)
$a^n \sin(\alpha n)u[n]$	$\dfrac{za\sin(\alpha)}{z^2 - 2za\cos(\alpha) + a^2}$	$\|z\| > \|a\|$	(6.28)
$a^n \cos(\alpha n)u[n]$	$\dfrac{z(z - a\cos(\alpha))}{z^2 - 2za\cos(\alpha) + a^2}$	$\|z\| > \|a\|$	(6.29)

Common ***non-causal*** *transform pairs (two-sided z-transform)*

$x[n]$	$X(z)$	ROC	
$a^{-n}u[-n]$	$\dfrac{1}{1-az}$	$\|z\| < \dfrac{1}{\|a\|}$	(6.30)
$a^{\|n\|}$	$\dfrac{(a^2-1)z}{az^2 - (1+a^2)z + a}$	$\|a\| < \|z\| < \dfrac{1}{\|a\|}$	(6.31)

7 Discrete Time Fourier Transform

Properties $\circledast$ *circular convolution*

$x[n]$	$X_d(\nu)$	
	$X_d(\nu) = X_d(\nu + 1)$	(7.1)
$x[n]$	$\displaystyle\sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi\nu n}$	(7.2)
$\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} X_d(\nu)e^{j2\pi\nu n}\, d\nu$	$X_d(\nu)$	(7.3)
$cx[n] + dy[n]$	$cX_d(\nu) + dY_d(\nu)$	(7.4)
$x[-n]$	$X_d(-\nu)$	(7.5)
$x^*[n]$	$X_d^*(-\nu)$	(7.6)
$x[n-k]$	$e^{-j2\pi k\nu}X_d(\nu)$	(7.7)
$e^{j2\pi\nu_0 n}x[n]$	$X_d(\nu - \nu_0)$	(7.8)
$x[n] * y[n]$	$X_d(\nu)Y_d(\nu)$	(7.9)
$x[n]y[n]$	$X_d(\nu) \circledast Y_d(\nu)$	(7.10)
$nx[n]$	$\displaystyle -\frac{1}{j2\pi}\frac{\partial X_d(\nu)}{\partial \nu}$	(7.11)

Parseval's theorem

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} |X(\nu)|^2\, d\nu \tag{7.12}$$

Generalized Parseval's theorem

$$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} X(\nu)Y^*(\nu)\, d\nu \tag{7.13}$$

Common transform pairs (Discrete Time Fourier Transform)

$x[n], \quad \lvert a\rvert < 1$	$X_d(\nu) \quad \nu \in (-\frac{1}{2}, \frac{1}{2}]$	
$\delta[n]$	1	(7.14)
1	$\delta(\nu)$	(7.15)
$\mathrm{rect}_K[n] = \begin{cases} 1 & \text{for } \lvert n\rvert \leq K \\ 0 & \text{for } \lvert n\rvert > K \end{cases}$	$\Sigma_{1,(2K+1)}(\nu) = \dfrac{\sin(\pi\nu(2K+1))}{\sin(\pi\nu)}$	(7.16)
$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$	$\dfrac{1}{1-e^{-j2\pi\nu}} + \dfrac{1}{2}\delta(\nu)$	(7.17)
$e^{j2\pi\nu_0 n}$	$\delta(\nu - \nu_0)$	(7.18)
$a^n u[n]$	$\dfrac{1}{1-ae^{-j2\pi\nu}}$	(7.19)
$a^{-n}u[-n]$	$\dfrac{1}{1-ae^{j2\pi\nu}}$	(7.20)
$a^{\lvert n\rvert}$	$\dfrac{1-a^2}{1+a^2-2a\cos(2\pi\nu)}$	(7.21)
$a^n \sin(2\pi\nu_0 n)u[n]$	$\dfrac{a\sin(2\pi\nu_0)e^{-j2\pi\nu}}{1-2a\cos(2\pi\nu_0)e^{-j2\pi\nu}+a^2e^{-j4\pi\nu}}$	(7.22)
$a^n \cos(2\pi\nu_0 n)u[n]$	$\dfrac{1-a\cos(2\pi\nu_0)e^{-j2\pi\nu}}{1-2a\cos(2\pi\nu_0)e^{-j2\pi\nu}+a^2e^{-j4\pi\nu}}$	(7.23)

8 Discrete Fourier Transform (DFT)

Properties — $\textcircled{\scriptsize N}$ = circular convolution with N values

$x[n]$	$X[m]$	
$x[n]$	$\displaystyle\sum_{n=0}^{N-1} x[n]e^{-j2\pi mn/N}$	(8.1)
$\displaystyle\frac{1}{N}\sum_{m=0}^{N-1} X[m]e^{j2\pi mn/N}$	$X[m]$	(8.2)
$cx[n] + dy[n]$	$cX[m] + dY[m]$	(8.3)
$x^*[N-n]$	$X^*[m]$	(8.4)
$x^*[n]$	$X^*[N-m]$	(8.5)
$x[n-k]$	$e^{-j2\pi km/N}X[m]$	(8.6)
$e^{j2\pi nk/N}x[n]$	$X[m-k]$	(8.7)
$N^{-1}X[n]$	$x[-m]$	(8.8)
$x[n] \textcircled{\scriptsize N} y[n]$	$X[m]Y[m]$	(8.9)
$x[n]y[n]$	$N^{-1}X[m] \textcircled{\scriptsize N} Y[m]$	(8.10)

Parseval's theorem

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{m=0}^{N-1} |X[m]|^2 \tag{8.11}$$

Generalized Parseval's theorem

$$\sum_{n=0}^{N-1} x[n]y^*[n] = \frac{1}{N}\sum_{m=0}^{N-1} X[m]Y^*[m] \tag{8.12}$$

9 Fourier Series

Properties T = Fundamental period

Signal: $x(t) = x(t+T)$	Fourier Coefficient: c_k	
$x(t)$	$\frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-j2\pi kt/T}\,dt$	(9.1)
$\sum_{k=-\infty}^{\infty} c_k e^{j2\pi kt/T}$	c_k	(9.2)
$ax(t) + by(t)$	$ac_{x,k} + bc_{y,k}$	(9.3)
$x(-t)$	c_{-k}	(9.4)
$x^*(t)$	c^*_{-k}	(9.5)
$x(t-\tau)$	$e^{-j2\pi k\tau/T}c_k$	(9.6)
$e^{j2\pi mt/T}x(t)$	c_{k-m}	(9.7)
$\frac{\partial^n x(t)}{\partial t^n}$	$\left(\frac{j2\pi k}{T}\right)^n c_k$	(9.8)
$\int_{t_0}^{t}\big(x(\tau) - c_0\big)\,d\tau$	$\begin{cases}\frac{c_k}{j2\pi k/T}, & k \neq 0\\ \text{use (9.1) for } k = 0\end{cases}$	(9.9)
$\int_{t_0}^{t_0+T} x(\tau)y(t-\tau)\,d\tau$	$c_{x,k}c_{y,k}$	(9.10)
$x(t)y(t)$	$\sum_{m=-\infty}^{\infty} c_{x,m}c_{y,k-m}$	(9.11)

Parseval's theorem

$$\frac{1}{T}\int_{t_0}^{t_0+T} |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |c_k|^2 \tag{9.12}$$

Generalized Parseval's theorem

$$\frac{1}{T}\int_{t_0}^{t_0+T} x(t)y^*(t)\,dt = \sum_{k=-\infty}^{\infty} c_{x,k}c^*_{y,k} \tag{9.13}$$

Real valued Fourier series

$$x(t) = a_0 + \sum_{k=1}^{\infty}\big[a_k\cos(2\pi kt/T) + b_k\sin(2\pi kt/T)\big] \tag{9.14}$$

$$a_0 = c_0, \quad 2c_k = a_k - jb_k, \quad 2c_{-k} = a_k + jb_k, \quad k \geq 1 \tag{9.15}$$