INFORMATION SCIENCE AND ENGINEERING

SCHOOL OF ELECTRICAL ENGINEERING

Digital Signal Processing EQ2300 / 2E1340

Final Examination 2017–04–12, 14.00–19.00 Examples of Solutions

1. a) We have in general that

$$y[n] = x[n] \textcircled{4} h[n] \sum_{m=0}^{3} = x[m]h[(n-m) \mod 4] \,,$$

so

$$y[0] = x[0]h[0] + x[1]h[3] + x[2]h[2] + x[3]h[1] = 3 \times 0 + 2 \times 3 + 1 \times 2 + 0 \times 1 = 8 \,,$$

$$y[1] = x[0]h[1] + x[1]h[0] + x[2]h[3] + x[3]h[2] = 3 \times 1 + 2 \times 0 + 1 \times 3 + 0 \times 2 = 6 \,,$$

$$y[2] = x[0]h[2] + x[1]h[1] + x[2]h[0] + x[3]h[1] = 3 \times 2 + 2 \times 1 + 1 \times 0 + 0 \times 3 = 8 \,,$$

and

$$y[3] = x[0]h[3] + x[1]h[2] + x[2]h[1] + x[3]h[0] = 3 \times 3 + 2 \times 2 + 1 \times 1 + 0 \times 0 = 14 \,.$$

In other words,

$$\{3,\ 2,\ 1,\ 0\} \textcircled{4} \{0,\ 1,\ 2,\ 3\} = \{8,\ 6,\ 8,\ 14\} \,.$$

b) Fourier transforms are real valued if the signal is symmetric around $n = 0$ (or $t = 0$ in the continuous-time case). For the DFT one has to interpret this symmetry in the right way, and view the specified part as part of an 5-periodic (infinite sequence). If we rewrite the sequences with $n = 0$ in the middle. we get

i) $\tilde{x}[n] = \{1,\ 1,\ \underset{\uparrow}{1},\ 1,\ 1\}$

ii) $\tilde{x}[n] = \{0,\ 0,\ \underset{\uparrow}{1},\ 1,\ 1\}$

iii) $\tilde{x}[n] = \{1,\ 1,\ \underset{\uparrow}{1},\ 1,\ 0\}$

iv) $\tilde{x}[n] = \{0,\ 1,\ \underset{\uparrow}{1},\ 1,\ 0\}$

v) $\tilde{x}[n] = \{0,\ 0,\ \underset{\uparrow}{1},\ 0,\ 0\}$

and we immediately see that the correct answer is i, iv, and v. We can also directly compute the DFT according to

$$\sum_{n=0}^{4} x[n] e^{-j2\pi\frac{kn}{5}}$$

and note that

$$e^{-j2\pi\frac{k1}{5}} + e^{-j2\pi\frac{k4}{5}} = e^{-j2\pi\frac{k1}{5}} + e^{-j2\pi\frac{k(4-5)}{5}} = e^{-j2\pi\frac{k1}{5}} + e^{j2\pi\frac{k1}{5}} = 2\cos\left(2\pi\frac{k}{5}\right)$$

and

$$e^{-j2\pi\frac{k2}{5}} + e^{-j2\pi\frac{k3}{5}} = e^{-j2\pi\frac{k2}{5}} + e^{-j2\pi\frac{k(3-5)}{5}} = e^{-j2\pi\frac{k2}{5}} + e^{j2\pi\frac{k2}{5}} = 2\cos\left(2\pi\frac{2k}{5}\right) \,.$$

c) Note that $X_1(\nu) = 1$, so if $y_1[n] = x[2n]$, it follows that

$$Y_1(\nu) = \frac{1}{2}\sum_{k=0}^{1} X_1\left(\frac{\nu - k}{2}\right) = \frac{1}{2}(1+1) = 1 \neq Y(\nu).$$

Both

$$\frac{1}{2}\sum_{k=0}^{1} X_2\left(\frac{\nu - k}{2}\right) = 1 = Y(\nu) \quad \text{and} \quad \frac{1}{2}\sum_{k=0}^{1} X_3\left(\frac{\nu - k}{2}\right) = 1 = Y(\nu),$$

which can be checked graphically, so the right answer is $x_1[n]$.

d) If we write out the transfer functions for $a = \frac{1}{4}$, we get

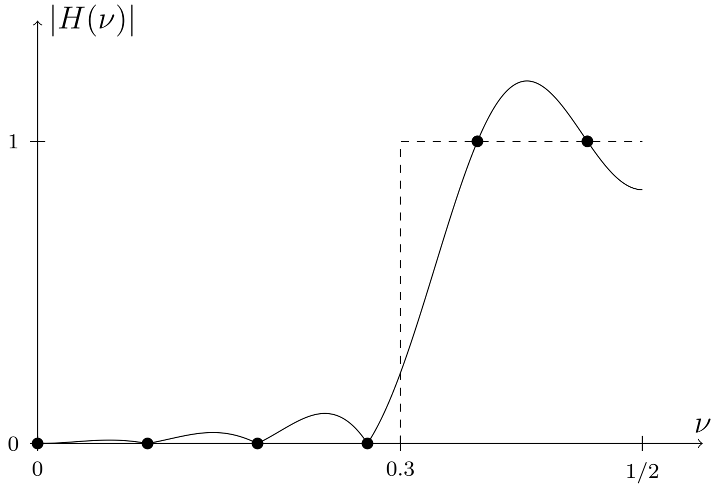
$$H_{\mathrm{A}}(z) = \frac{z^2}{z^2 - z + \frac{1}{4}} = \frac{z^2}{(z - \frac{1}{2})^2}$$

and

$$H_{\mathrm{B}}(z) = \frac{z^2}{z^2 - \frac{1}{4}} = \frac{z^2}{(z - \frac{1}{2})(z + \frac{1}{2})}.$$

Thus, we see that $H_{\mathrm{A}}(z)$ has a double pole (at $\frac{1}{2}$), while $H_{\mathrm{B}}(z)$ does not. $H_{\mathrm{A}}(z)$ is more sensitive to perturbations because filters with poles close together are generally more sensitive to perturbations of the coefficients.

2. a) The filter length N is the same as the number of points where we require that $H(\nu)$ is equal to $H_{\mathrm{I}}(\nu)$. With equal spacing in the frequency domain, these are spaces $\frac{1}{N}$ appart, or $\frac{1}{M+1}$ where M is the filter order, and $N = M + 1$. These points are shown below for the filter at hand.



Note however that we only see the range $\nu \in [0, \frac{1}{2})$. If we display the full range $\nu \in (-\frac{1}{2}, \frac{1}{2})$, we would count a total of $N = 11$ points (we see 6 points in $[0, \frac{1}{2})$, and do not duplicate the point at $\nu = 0$). Note here that N has to odd, given that the filter is a Type I filter, and this is consistent with the answer.

b) Proceeding form the Summary notes, we have

$$h[n] = \frac{2}{M+1}\left[A[0] + 2\sum_{k=1}^{M/2}(-1)^k A[k] \cos\frac{\pi k(1+2n)}{M+1}\right],$$

where

$$A[k] = 0 \quad \text{for} \quad k = 0, \ldots, 3$$

and

$$A[k] = 1 \quad \text{for} \quad k = 4, \ldots, 5\,.$$

This can be seen by just reading of the values at $\nu_k = \frac{k}{N} = \frac{k}{M+1}$ in the figure above. Simplifying the expression for $h[n]$ above yields

$$\begin{aligned}
h[n] =& \frac{2}{M+1} \sum_{k=4}^{5} (-1)^k \cos\left(\pi \frac{k(1+2n)}{M+1}\right) \\
=& \frac{2}{11}\left[\cos\left(\pi\frac{4(1+2n)}{11}\right) - \cos\left(\pi\frac{5(1+2n)}{11}\right)\right] \\
=& \frac{2}{11}\left[\cos\left(\pi\frac{8(n-5)}{11}\right) + \cos\left(\pi\frac{10(n-5))}{11}\right)\right]\,,
\end{aligned}$$

which holds for $n = 0, \ldots, 10$. The last line has symmetrized the impulse response around the group delay $\tau = 5$. It can be directly verified that the last two lines are the same, using standard cosine rules, and that the filter should be symmetric around $\tau = \frac{M}{2} = 5$ follows as it is a Type I filter.

c) The signal $h[n]$ has the form

$$h[n] = \frac{2}{11}\cos(2\pi\nu_1(n-5)) + \frac{2}{11}\cos(2\pi\nu_2(n-5))\,,$$

for $n = 0, \ldots, 10$, and $h[n] = 0$ otherwise. In the above, $\nu_1 = \frac{8}{22}$ and $\nu_2 = \frac{10}{22}$. Consider first the simpler signal

$$x[n] = \frac{2}{11}\cos(2\pi\nu_1 n) + \frac{2}{11}\cos(2\pi\nu_2 n)\,,$$

for all $n \in \mathbb{Z}$, and note that $h[n] = x[n-5]w[n]$, where $w[n]$ is a rectangular window function for which $w[n] = 1$ when $n = 0, \ldots, 10$ and $w[n] = 0$ elsewhere. The simple signal $x[n]$ has a discrete-time Fourier transform given by

$$\begin{aligned}
X(\nu) =& \frac{1}{11}\Big[\delta(\nu-\nu_1) + \delta(\nu+\nu_1) + \delta(\nu-\nu_2) + \delta(\nu+\nu_2)\Big] \\
=& \frac{1}{11}\Big[\delta(\nu - \frac{8}{22}) + \delta(\nu + \frac{8}{22}) + \delta(\nu + \frac{10}{22}) + \delta(\nu + \frac{10}{22})\Big]\,,
\end{aligned}$$

for $\nu \in [-\frac{1}{2}, \frac{1}{2}]$. The expression $h[n] = x[n-5]w[n]$ yields

$$H(\nu) = \Big[e^{-j2\pi 5\nu}X(\nu)\Big] \circledast W(\nu) = \Big[e^{-j10\pi\nu}X(\nu)\Big] \circledast W(\nu)\,,$$

where $X(\nu)$ is given above, and where

$$\begin{aligned}
W(\nu) =& \sum_{n=0}^{N-1} e^{-j2\pi\nu n} = \frac{1-e^{-j2\pi\nu N}}{1-e^{-j2\pi\nu}} \\
=& \frac{e^{-j\pi\nu N}}{e^{j\pi\nu_0}}\left[\frac{e^{j\pi\nu N} - e^{-j\pi\nu N}}{e^{j\pi\nu} - e^{-j\pi\nu}}\right] \\
=& e^{-j\pi\nu(N-1)}\frac{\sin(\pi\nu N)}{\sin(\pi\nu)} = e^{-j10\pi\nu}\frac{\sin(11\pi\nu)}{\sin(\pi\nu)}\,.
\end{aligned}$$

This is sufficient as an answer to the problem. Note however that, even though not required by the problem, it is actually possible to evaluate $H(\nu)$ in closed form, given the special structure of $X(\nu)$ as a sequence of Dirac delta functions. To end, note that

$$\left[e^{-j10\pi\nu}X(\nu)\right] \circledast W(\nu) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j10\pi\tau}X(\tau)W(\nu-\tau)d\tau$$
$$= \frac{e^{-j\frac{10\times 8}{22}\pi}}{11} e^{-j10\pi(\nu-\frac{8}{22})} \frac{\sin(11\pi(\nu-\frac{8}{22}))}{\sin(\pi(\nu-\frac{8}{22}))} + \frac{e^{j\frac{10\times 8}{22}\pi}}{11} e^{-j10\pi(\nu+\frac{8}{22})} \frac{\sin(11\pi(\nu+\frac{8}{22}))}{\sin(\pi(\nu+\frac{8}{22}))}$$
$$+ \frac{e^{-j\frac{10\times 10}{22}\pi}}{11} e^{-j10\pi(\nu-\frac{10}{22})} \frac{\sin(11\pi(\nu-\frac{10}{22}))}{\sin(\pi(\nu-\frac{10}{22}))} + \frac{e^{j\frac{10\times 10}{22}\pi}}{11} e^{-j10\pi(\nu+\frac{10}{22})} \frac{\sin(11\pi(\nu+\frac{10}{22}))}{\sin(\pi(\nu+\frac{10}{22}))}.$$

3. a) Welch's method. You can see this as the code averages modified periodograms taken from 50% overlapped blocks of data. While the windows used for the modified periodograms is the Bartlett-window, it is not Bartlett's method.

b) The frequency resolution for the Bartlett (triangular) window used is

$$\Delta\nu = \frac{1.28}{L} = 0.0025$$

where $L = 512$ is the length of the window.

c) By zero-padding the sequence to length $M = 2048$ before taking the FFT, the code computes the discrete time Fourier transform at discrete frequencies

$$\nu_k = \frac{k}{M} = \frac{k}{2048}$$

where $k = 0, \ldots, M-1$.

d) The variance of Welch's method, with a triangular window and 50% overlap, is

$$\mathrm{Var}\{\hat{P}_y(\nu)\} = \frac{8}{9K} P_y^2(\nu)$$

where $K$is the number of blocks. For the stated requirement, we need that

$$\frac{9}{8K} \leq \frac{1}{10} \quad \Leftrightarrow \quad K \geq \frac{90}{8} = 11.25\,,$$

which implies that $K \geq 12$ as K have to be an integer. We can see from the code that

$$K \leq \frac{N-L}{D} + 1$$

for

$$N \geq (K-1)D + L = (12-1) \times \frac{512}{2} + 512 = 3328\,.$$

4. (a) Note that if $H(z) = H_0(z)$, it follows that $H_1(z) = H_0(-z)$ if $H_1(z) = H(-z)$. The results follows by noting that

$$H_0(-z) = \frac{1}{4}(1 - 3z^{-1} + 3z^{-2} - z^{-3}) \neq \frac{1}{4}(-1 - 3z^{-1} + 3z^{-2} + z^{-3}) = H_1(z)\,.$$

(b) Filters are finite impulse response linear phase filters if and only if they satisfy certain symmetry or antisymmetry properties, as listed in the summary notes. In particular, a Type II linear phase FIR filter satisfies

$$h[n] = h[M-n]$$

for an odd filter order M. In our case $M = 3$, and we see that $H_0(z)$ is a Type II linear phase FIR filter. Similarly, a Type IV linear phase FIR filter satisfies

$$h[n] = -h[M-n]$$

which is consistent with $H_1(z)$.

(c) Just as in the project, we can note that aliasing is cancelled if we choose $G_0(z) = H_1(-z)$ and $G_1(z) = -H_0(-z)$, i.e.,

$$G_0(z) = \frac{1}{4}(-1 + 3z^{-1} + 3z^{-2} - z^{-3})$$

and

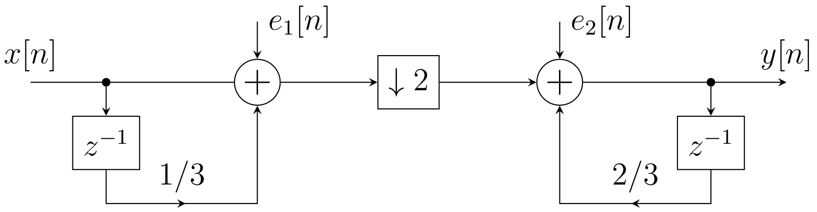
$$G_1(z) = \frac{1}{4}(-1 + 3z^{-1} - 3z^{-2} + z^{-3})\,.$$

It is not immediately obvious that this choice of filter also provide perfect reconstruction, but by noting that

$$\begin{aligned}
&H_0(z)G_0(z) + H_1(z)G_1(z)\\
=&\frac{1}{16}(1 + 3z^{-1} + 3z^{-2} + z^{-3}))(-1 + 3z^{-1} + 3z^{-2} - z^{-3})\\
+&\frac{1}{16}(-1 - 3z^{-1} + 3z^{-2} + z^{-3})(-1 + 3z^{-1} - 3z^{-2} + z^{-3})\\
=&\frac{1}{16}\Big(-1 + 9z^{-2} + 16z^{-3} + 9z^{-4} - z^{-6}\Big)\\
+&\frac{1}{16}\Big(1 - 9z^{-2} + 16z^{-3} - 9z^{-4} + z^{-6}\Big)\\
=&2z^{-3}
\end{aligned}$$

it follows that we have perfect reconstruction with a delay of $l = 3$.

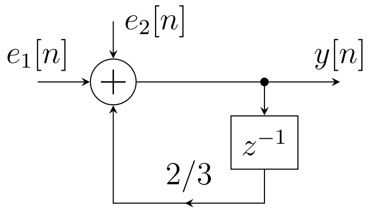
5. a) The quantization noise is modeled by adding noise sources according to the figure below.



As the noise process $e_1[n]$ is assumed white, it will have the same spectral density after downsampling. This can also be seen noting that $P_{e_1}(\nu) = 1$ which implies that

$$\frac{1}{2}\sum_{k=0}^{1} P_{e_1}\left(\frac{\nu - k}{D}\right) = \frac{1}{2}(1+1) = 1 = P_{e_1}(\nu)\,.$$

Compare this to the result of problem 1c. Thus, to compute the quantization noise at the output, we can equivalently study the following circuit.



This circuit has impulse response

$$h[n] = \left(\frac{2}{3}\right)^n u[n]$$

so

$$\sum_{n=0}^{\infty} h^2[n] = \sum_{n=0}^{\infty} \left(\frac{4}{9}\right)^n = \frac{1}{1-\frac{4}{9}} = \frac{9}{5}\,.$$

The total noise at the output thus becomes

$$\sigma^2 = 2\frac{2^{-2B}}{12}\frac{9}{5} = \frac{3}{10}2^{-2B}\,.$$

b) The filter in the second figure above have a frequency response given by

$$H(\nu) = \frac{1}{1-\frac{2}{3}e^{-j2\pi\nu}}\,.$$

This implies that

$$\begin{aligned}|H(\nu)|^2 =& H^*(\nu)H(\nu) = \frac{1}{(1-\frac{2}{3}e^{j2\pi\nu})((1-\frac{2}{3}e^{-j2\pi\nu}))}\\ =& \frac{1}{\frac{13}{9}-\frac{12}{9}\cos(2\pi\nu)} = \frac{9}{13-12\cos(2\pi\nu)}\,.\end{aligned}$$

The power spectral density at the output is given by

$$|H(\nu)|^2(P_{e_1}(\nu)+P_{e_2}(\nu)) = \frac{2\times 9}{12}\frac{1}{13-12\cos(2\pi\nu)}2^{-2B}\,.$$

The resulting power spectral density is shown below (in dB scale) for $B = 4$.