SIGNAL PROCESSING

SCHOOL OF ELECTRICAL ENGINEERING

**Digital Signal Processing** EQ2300 / 2E1340

Final Examination 2015–04–08, 14.00–19.00 Examples of Solutions

**1.** a) **iii**, as $y[m-k] = x[2(m-k)] \neq x[2m-k]$

b) We can factor the polynomial $z^N \sum_n h[n]z^{-n}$ and use the factors to form shorter FIR filters that are implemented in series.

c) We have computed the periodogram for normalized frequencies $\nu = \nu_k = k/N$, as

$$\hat{P}_x(\nu) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x[n]e^{-j2\pi\nu n}\right| .$$

d) By noting that $Y[k] = H[k]X[k] \Rightarrow y[n] = h[n] \circledast_8 x[n]$ we get

$$y[n] = \{\underset{\uparrow}{8}, 7, 6, 9, 8, 7, 6, 9\}$$

e) You store the 8 values for $x[n]$ is a vector, *zero pad* to increase the length to $N = 16$, and apply the FFT algorithm to get $X[k]$. Then $X(\nu_k) = X[k]$.

**2.** a) We can solve the problem by first computing a non-causal symmetric ideal filter $g[n]$ and then windowing and shifting it to be causal. Let $g[n]$ have a DTFT given by

$$G(\nu) = \begin{cases} 0 & |\nu| < \frac{1}{3} \\ 1 & \frac{1}{3} \leq |\nu| \leq \frac{1}{2} \end{cases} ,$$

i.e., such that $|G(\nu)| = |H_{\mathrm{I}}(\nu)|$ where $H_{\mathrm{I}}(\nu)$ is the ideal frequency response shown in the figure. Taking the inverse DTFT of $G(\nu)$ yields

$$\begin{aligned} g[n] &= \int_{-\frac{1}{2}}^{\frac{1}{2}} G(\nu)e^{j2\pi\nu n}d\nu \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{j2\pi\nu n}d\nu - \int_{-\frac{1}{3}}^{\frac{1}{3}} e^{j2\pi\nu n}d\nu \\ &= \delta[n] - \frac{\sin(\pi\frac{2}{3}n)}{\pi n} = \delta[n] - \frac{3}{2}\mathrm{sinc}\left(\frac{2}{3}n\right) . \end{aligned}$$

The requested FIR approximation is then obtained as

$$h[n] = w[n]g[n - (N-1)/2] ,$$

for $n = 0, \ldots, N-1$.

b) The complexity (in terms of number of complex valued multiplications) of overlap add for a filter of length $M$ and an FFT of length $N$ is

$$C = \frac{N \log_2 2N}{N - M + 1} .$$

We simply need to find the largest $M$ such that there is an $N = 2^p \geq M$ for which $C \leq 12$. Selecting $M = 86$ and $N = 512$ yields $C = 11.99$, while $M = 87$ will yield a smallest $C$ of 12.0085 (obtained by selecting $N = 2^{10} = 1024$). Thus, we can not implement filters longer than $M = 86$ taps, in order not to violate the complexity constraint. The longest Type I FIR filter we can have is of length $M = 85$ since $M$ needs to be an odd number.

**3.** a) Because spectral leakage (side-lobes) from the strong sinus-components hides the lower amplitude sinusoid.

b) Due to lowered spectral resolution the two close-by peaks are merged into one by the main-lobe of the Chebyshev window.

c) They are $\nu_1 \approx 0.1$, $\nu_2 \approx 0.103$ as can be seen in the periodogram-figure, and $\nu_3 \approx 0.125$ as can be seen in the figure for the modified periodogram. Note that the exact number $\nu_2 \approx 0.103$ is hard to tell from the figure, but as the two peaks at $\nu_1$ and $\nu_2$ are resolvable by the priodogram they should be at least $0.89/512 \approx 0.0017$ apart, and since they are not resolvable using the Chebychew windowed modified periodogram they should be no more than $1.85/512 \approx 0.0036$ apart.

d) As the widening of the sinus peaks by the main lobe of the Chebyshev window in Welch's method is at most $4.6/128 \approx 0.04$ away from the center frequency before getting to the sidelobes which are 100 dB below the main peak, we know that the squigly line in Welch's method which is only 60-70 dB below the main peak has to be the noise floor. Thus, we can tell from the Welch figure that $\sigma^2$ is at -50 dB or equivalently $\sigma^2 = 10^{-5}$.

e) In order to tell what the amplitudes are, we need the amplitude of a known window at $\nu = 0$. The easiest one (and the only one we know) is the rectangular window used for the periodogram. In particular, we have

$$W(0) = \sum_{n=0}^{511} e^{-j2\pi 0 n} = 512,$$

and $|W(0)|^2 = 512^2$. The (expected value) of the periodogram is given by

$$E\{\hat{P}_x(\nu)\} = P_x(\nu) \circledast \frac{1}{N}|W(\nu)|^2 .$$

As the power spectrum of a single sinus-component with amplitude $a_k$ and frequency $\nu_k$ is given by

$$P_x(\nu) = \frac{a_k^2}{4}\Big(\delta(\nu - \nu_k) + \delta(\nu + \nu_k)\Big)$$

we can expect the height of the peak due to a single sinusoid in the peridogram to be

$$\frac{a_k^2}{4} \times \frac{512^2}{512} .$$

As the peaks for the first two sinusoids are at $20\,\text{dB} = 10^2 = 100$, we get

$$\frac{a_1^2}{4} \times 512 \approx 100 \Rightarrow a_1 \approx \sqrt{\frac{4 \times 100}{512}} \approx 1$$

and $a_2 \approx 1$. We cannot see $a_3$ in the periodogram, but we can see from the modified periodogram that $a_3^2$ is about 40 dB below $a_1^2$ which implies that $a_3^2 \approx 10^{-4} a_1^2$ or $a_3 \approx 10^{-2} a_1 \approx 0.01$.

**4.** The two circuits are familiar from the classes. Although it is not commonly done in exams, it is a good practice to verify that both circuits are indeed implementing the same transfer functions. Let us call $H_a(z)$ and $H_b(z)$ the transfer functions defined by the circuits in Figures 1 and 3. Follow the derivations in Table 1 to verify how $H(z)$ can be derived from each of the circuits. There, every signal is noted $_a$ or $_b$ according to where the analysis is being done.

| | |
|---|---|
| $R_a(z) = \frac{1}{2}X_a(z) + \frac{1}{2}z^{-1}X_a(z) =$ | $R_b(z) = X_b(z) + \frac{1}{3}z^{-1}R_b(z) \Rightarrow$ |
| $= \frac{1}{2}X_a(z)\left(1+z^{-1}\right)$ | $\Rightarrow R_b(z) = \frac{X_b(z)}{1-\frac{1}{3}z^{-1}}$ |
| $Y_a(z) = R_a(z) + \frac{1}{3}z^{-1}Y_a(z) \Rightarrow$ | $Y_b(z) = \frac{1}{2}R_b(z) + \frac{1}{2}z^{-1}R_b(z) =$ |
| $\Rightarrow Y_a(z) = \frac{R_a(z)}{1-\frac{1}{3}z^{-1}} \Rightarrow$ | $= \frac{1}{2}R_b(z)\left(1+z^{-1}\right) \Rightarrow$ |
| $\Rightarrow Y_a(z) = \frac{1}{2}\frac{1+z^{-1}}{1-\frac{1}{3}z^{-1}}X_a(z) \Rightarrow$ | $\Rightarrow Y_b(z) = \frac{1}{2}\frac{1+z^{-1}}{1-\frac{1}{3}z^{-1}}X_b(z) \Rightarrow$ |
| $\Rightarrow H_a(z) = \frac{1}{2}\frac{1+z^{-1}}{1-\frac{1}{3}z^{-1}}$ | $\Rightarrow H_b(z) = \frac{1}{2}\frac{1+z^{-1}}{1-\frac{1}{3}z^{-1}}$ |

Table 1: Derivation of the transfer function $H(z) = H_a(z) = H_b(z)$ from both circuits, proving that Figures 1 and 3 represent the same system.

In order to evaluate the effect of quantization in each case, we will use the usual model (see video-lectures on quantization and fixed-point implementation for more detailed explanations). Thus, we will assume that every one of the $M$ multipliers the circuit includes introduces a white, independent, uniformly distributed, zero-mean, additive noise $e_i[n], i = \{1, \ldots, M\}$. Then, because independence implies:

$$E\left\{\left(\sum_{i=1}^{M} h_{e_i}[n] * e_i[n]\right)^2\right\} = \sum_{i=1}^{M} E\left\{(h_{e_i}[n] * e_i[n])^2\right\},$$

we will compute the power of each of these noises' contributions to the output. Note that $h_{e_i}[n]$ here represents the impulse response that represents the path from the entry point of the noise $e_i[n]$ to the circuit's output. Moreover, because the noises are white, we know that

$$E\left\{(h_{e_i}[n] * e_i[n])^2\right\} = E\{e_i^2[n]\} \sum_{n=-\infty}^{+\infty} h_{e_i}^2[n] = \sigma_e^2 \sum_{n=-\infty}^{+\infty} h_{e_i}^2[n].$$

a) Figure 2 describes our model of the effects of quantization noise on the circuit from Figure 1. We can easily see that the three noises will travel through the same path to get to the circuit's output , i.e. $h_e[n] \triangleq h_{e_1}[n] = h_{e_2}[n] = h_{e_3}[n]$. Similarly to what we did to derive the overall transfer function $H(z)$, we analyze the relation between $y_{e_i}[n]$ and $e_i[n]$, i.e. between the noises and their contribution to the output, to obtain $H_e(z)$.

$$Y_{e_i}(z) = E_i(z) + \frac{1}{3}z^{-1}Y_{e_i}(z) \Rightarrow Y_{e_i}(z) = \frac{1}{1-\frac{1}{3}z^{-1}}E_i(z) \Rightarrow H_e(z) = \frac{1}{1-\frac{1}{3}z^{-1}}$$

Using the inverse Z-transform, we obtain:

$$h_e[n] = \frac{1}{3^n} u[n], \text{ where } u[n] = \begin{cases} 0 \text{ if } n < 0 \\ 1 \text{ if } n \geq 0 \end{cases} .$$

Thus, the power of each noise's contribution to the output will be:

$$E\left\{(h_{e_i}[n] * e_i[n])^2\right\} = \sigma_e^2 \sum_{n=-\infty}^{+\infty} h_{e_i}^2[n] = \sigma_e^2 \sum_{n=0}^{+\infty} \frac{1}{9^n} = \frac{0-1}{\frac{1}{9}-1}\sigma_e^2 = \frac{9}{8}\sigma_e^2 .$$

Therefore, the total power of the quantization noise at the output will be:

$$E\left\{\left(\sum_{i=1}^{3} h_{e_i}[n] * e_i[n]\right)^2\right\} = 3\,\frac{9}{8}\sigma_e^2 = \frac{27}{8}\sigma_e^2 \approx 3.357\sigma_e^2 .$$

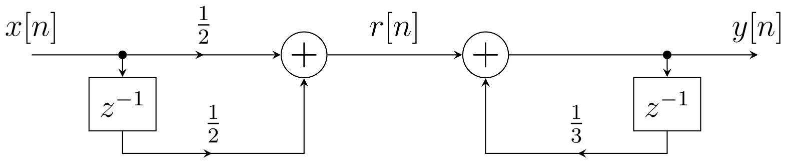



Figure 1: Circuit from a), with an intermediate signal $r[n]$ defined.

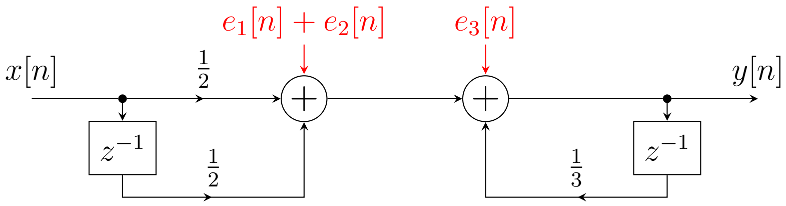



Figure 2: Modeling of the effect of quantization noise on the circuit from Figure 1.

b) Figure 4 describes our model of the effects of quantization noise on the circuit from Figure 3. On one hand, both $e_2[n]$ and $e_3[n]$ are directly inserted at the output, and thus $h_{e_2}[n] = h_{e_3}[n] = \delta[n]$, where $\delta[n]$ is the Kronecker delta. On the other hand, $e_1[n]$ is inserted at the circuit's input, and thus, has to travel through the whole circuit, and thus $h_{e_1}[n] = h[n]$. In Table 1 we have derived the expression for $H(z)$. Therefore, we have that:

$$H(z) = \frac{1}{2}\frac{1+z^{-1}}{1-\frac{1}{3}z^{-1}} = \frac{1}{2}\frac{1}{1-\frac{1}{3}z^{-1}} + \frac{1}{2}z^{-1}\frac{1}{1-\frac{1}{3}z^{-1}} ,$$

which, through the inverse Z-transform yields:

$$h_{e_3}[n] = \frac{1}{2}\frac{1}{3^n}u[n] + \frac{1}{2}\frac{1}{3^{n-1}}u[n-1] .$$

We then have that:

$$
\begin{aligned}
\sum_{n=-\infty}^{+\infty} h_{e_1}^2[n] &= \sum_{n=-\infty}^{+\infty} \left( \frac{1}{2}\frac{1}{3^n}u[n] + \frac{1}{2}\frac{1}{3^{n-1}}u[n-1] \right)^2 \\
&= \frac{1}{4}\left( \underset{n<0}{0} + \underset{n=0}{1} + \sum_{n=1}^{+\infty}\left( \frac{1}{3^n} + \frac{1}{3^{n-1}} \right)^2 \right) \\
&= \frac{1}{4} + \frac{1}{4}\sum_{n=1}^{+\infty}\left( \frac{1+3}{3^n} \right)^2 = \frac{1}{4} + 4\sum_{n=1}^{+\infty}\frac{1}{9^n} \\
&= \frac{1}{4} + 4\frac{0-\frac{1}{9}}{\frac{1}{9}-1} = \frac{1}{4} + 4\frac{1}{8} = \frac{3}{4}
\end{aligned},
$$

which implies that $E\left\{(h_{e_1}[n] * e_1[n])^2\right\} = \sigma_e^2 \frac{3}{4}$. Thus,

$$
E\left\{ \left( \sum_{i=1}^{3} h_{e_i}[n] * e_i[n] \right)^2 \right\} = \left( 2\sum_{n=-\infty}^{+\infty} \delta^2[n] + \frac{3}{4} \right)\sigma_e^2 = \left( 2 + \frac{3}{4} \right)\sigma_e^2 = 2.75\sigma_e^2\,.
$$

We then conclude that the circuit in a) (Figure 1) yields more fixed point quantization noise than the circuit in b) (Figure 3), because $3.357\sigma_e^2 > 2.75\sigma_e^2$.

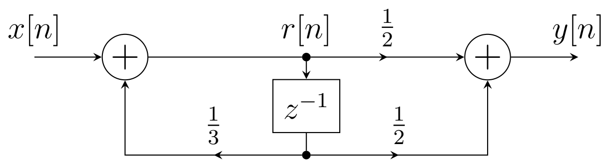



Figure 3: Circuit from b), with an intermediate signal $r[n]$ defined.

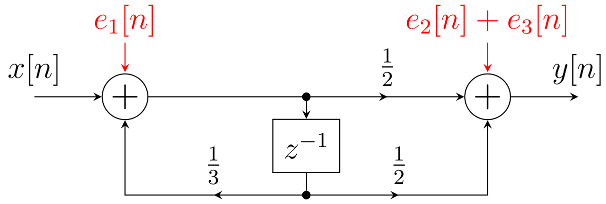
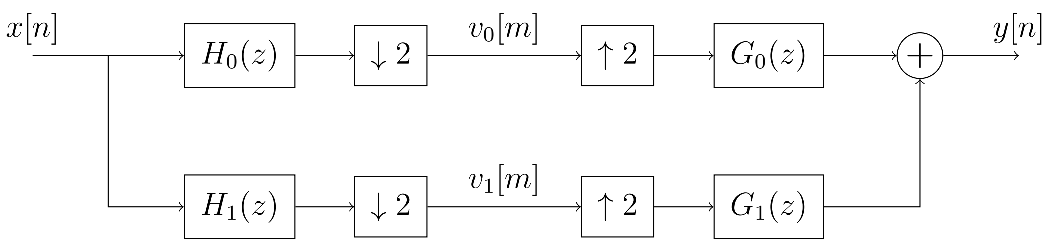



Figure 4: Modeling of the effect of quantization noise on the circuit from Figure 3.

**5.** a) The circuit is the polyphase filter implementation of the filterbank




where $H_0(z) = \frac{1}{\sqrt{2}}(1+z^{-1})$, $G_0(z) = \frac{1}{\sqrt{2}}(1+z^{-1})$, $H_1(z) = \frac{1}{\sqrt{2}}(1-z^{-1})$ and $G_1(z) = \frac{1}{\sqrt{2}}(-1+z^{-1})$. Using these filters we see that

$$
G_0(z)H_0(-z) + G_1(z)H_1(-z) = 0
$$

and

$$G_0(z)H_0(z) + G_1(z)H_1(z) = 2z^{-1}$$

which implies that perfect reconstruction is achieved with delay $l = 1$.

b) The problem is the same as in the course project assignment. As the inner system operates at half the sample rate it adds a delay of $2l$, while the outer system adds a delay of $l = 1$ on its own. The total delay becomes $L = 2l + l = 3$.