SOLUTIONS

E 85 **Digital Signalbehandling**, 2E1340

Final Examination 2001–08–31, 14.00–18.00

1. a) $u(n) = \sum x((n-k) \ {}_{\bmod 5})y(k)$ corresponds to

$$\mathbf{C}_x = \begin{bmatrix} 1 & 5 & 4 & 3 & 2 \\ 2 & 1 & 5 & 4 & 3 \\ 3 & 2 & 1 & 5 & 4 \\ 4 & 3 & 2 & 1 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix}$$

b) $v(n) = \sum x(n-k)y(k)$ corresponds to

$$\mathbf{L}_x = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 \\ 4 & 3 & 2 & 1 & 0 \\ 5 & 4 & 3 & 2 & 1 \\ 0 & 5 & 4 & 3 & 2 \\ 0 & 0 & 5 & 4 & 3 \\ 0 & 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

c) In general,

$$x(n) \textcircled{N} y(n) = \underbrace{\begin{bmatrix} x(0) & x(N-1) & \ldots & x(1) \\ x(1) & x(0) & \ddots & x(2) \\ \vdots & \ddots & \ddots & \vdots \\ x(N-1) & \ldots & x(1) & x(0) \end{bmatrix}}_{\mathbf{C}_x} \mathbf{y}$$

$$x(n) * y(n) = \underbrace{\begin{bmatrix} x(0) & 0 & \ldots & 0 \\ x(1) & x(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ x(N-1) & \ddots & \ddots & x(0) \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & x(N-1) \end{bmatrix}}_{\mathbf{L}_x} \mathbf{y}$$

2. Quantization noise will occur at both multiplicators. In the standard stochastic model, two quantization noise signals with variance σ_q, will be added, directly before and directly after the decimator, respectively.

The variance of the quantization noise from each of the multiplicators may be written: $\sigma_q^2 = \frac{2^{-2b}}{12}$. The decimation does not change the variance, since the noise is white (a sequence of independent variables with the same variance), consequently it could just as well be added after the decimator.

The resulting quantization noise with variance $2\sigma_q^2$, will pass through the second stage of the filter. The impulse response of the second stage of the filter is $h_2(n) = (0.5)^n u(n)$, where $u(n)$ is the unit step function. Thus the output quantization noise variance $\sigma_y^2 = 2\sigma_q^2 \sum_{-\infty}^{+\infty} h_2^2(n) = 2\frac{2^{-2b}}{12}\frac{4}{3} = \frac{2}{9}2^{-2b}$

3. a)

i	$x_i(n)$	$x_i(n)w(n)$	$\text{FFT}[x_i(n)w(n)]$	$\lvert\text{FFT}[x_i(n)w(n)]\rvert^2$
0	$\{5, 2, -1, 0\}$	$\{5, 6, -3, 0\}$	$\{8, 8-6j, -4, 8+6j\}$	$\{64, 100, 16, 100\}$
1	$\{-1, 0, 0, -4\}$	$\{-1, 0, 0, -4\}$	$\{-5, -1-4j, 3, -1+4j\}$	$\{25, 17, 9, 17\}$
2	$\{0, -4, -5, 1\}$	$\{0, -12, -15, 1\}$	$\{-26, 15+13j, -4, 15-13j\}$	$\{676, 394, 16, 394\}$
3	$\{-5, 1, 6, 2\}$	$\{-5, 3, 18, 2\}$	$\{18, -23-j, 8, -23+j\}$	$\{324, 530, 64, 530\}$

The normalization factor for the window is $U = \frac{1}{4}\sum w^2(n) = 5$ and averaging and normalizing the $L = 4$ segments gives

$$P_{xx}^{W}(k) = \frac{1}{LMU}\sum_{i=0}^{L-1}\left|\sum_{n=0}^{M-1} x_i(n)w(n)e^{-j2\pi\frac{k}{M}}\right|^2$$
$$= \frac{1}{80}\{1089, 1041, 105, 1041\} \approx \{13.6, 13.0, 1.3, 13.0\}$$

which is plotted in Figure 1.

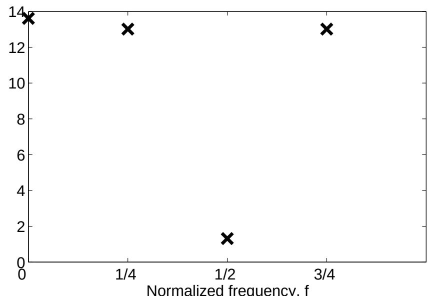

Figure 1:

b) The resolution is worse than the periodogram since each segment is shorter. It is also worse than the Bartlett spectrum since the window has a wider main lobe. On the other hand, the Welch spectrum estimate has lower variance than both other methods and lower leakage (lower sidelobes of the window) than Bartlett.

Of course, the resolution in this specific example is too poor to be useful anyhow, since only 10 samples are available, no matter which method is used.

4. a) Assuming the input sequence $\ldots, x(-1), x(0), x(1), x(2), \ldots$, will yield the upsampled sequence $\ldots, x(-1), 0, x(0), 0, x(1), 0, x(2), 0, \ldots$. We want to keep the non-zero values and insert $\frac{1}{2}\left[x\left(\frac{n-1}{2}\right) + x\left(\frac{n+1}{2}\right)\right]$ at each zero. It is easy to see that the impulse response $h(-1) = .5$, $h(0) = 1$ and $h(1) = .5$ gives the desired output.

b) $H(\omega) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} = 1 + \cos(\omega)$

c) Both filters will be non-causal low-pass filters. The ideal filter will preserve the spectral shape of the input while the linear interpolation filter will change it.

5. a) Since the phase ϕ is a non-linear parameter, we reparameterize the problem to get linear parameters.

$$y(n) = A\sin(\omega n + \phi) + v(n) = \underbrace{A\cos(\phi)}_{a_1}\sin(\omega n) + \underbrace{A\sin(\phi)}_{a_2}\cos(\omega n) + v(n)$$

note that each value of the pair $\{a_1, a_2\}$ is in one-to-one correspondence with a value of $\{A, \phi\}$ (except when $A = 0$). In matrix notation, the signal is given by

$$\underbrace{\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} \sin(\omega) & \cos(\omega) \\ \sin(2\omega) & \cos(2\omega) \\ \vdots & \vdots \\ \sin(N\omega) & \cos(N\omega) \end{bmatrix}}_{\mathbf{F}} \underbrace{\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}}_{\mathbf{a}} + \underbrace{\begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(N) \end{bmatrix}}_{\mathbf{v}} =$$

The Least Squares estimate of $\mathbf{a}$ is found by completing the squares

$$\begin{aligned} \min_{A,\phi} \sum_{n=1}^{N} |y(n) - A\sin(\omega n + \phi)|^2 &= \min_{\mathbf{a}} \|\mathbf{y} - \mathbf{F}\mathbf{a}\|^2 = \min_{\mathbf{a}} \mathbf{y}^T\mathbf{y} - \mathbf{y}^T\mathbf{F}\mathbf{a} - \mathbf{a}^T\mathbf{F}^T\mathbf{y} + \mathbf{a}^T\mathbf{F}^T\mathbf{F}\mathbf{a} \\ &= \min_{\mathbf{a}} (\mathbf{a} - (\mathbf{F}^T\mathbf{F})^{-1}\mathbf{F}^T\mathbf{y})^T\mathbf{F}^T\mathbf{F}(\mathbf{a} - (\mathbf{F}^T\mathbf{F})^{-1}\mathbf{F}^T\mathbf{y}) + \mathbf{y}^T(\mathbf{I} - \mathbf{F}(\mathbf{F}^T\mathbf{F})^{-1}\mathbf{F}^T)\mathbf{y} \\ &= \mathbf{y}^T(\mathbf{I} - \mathbf{F}(\mathbf{F}^T\mathbf{F})^{-1}\mathbf{F}^T)\mathbf{y} \end{aligned}$$

with equality iff $\mathbf{a} = (\mathbf{F}^T\mathbf{F})^{-1}\mathbf{F}^T\mathbf{y}$. Therefore, the LS estimate is given by

$$\begin{aligned} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \end{bmatrix} &= (\mathbf{F}^T\mathbf{F})^{-1}\mathbf{F}^T\mathbf{y} \\ \hat{A} &= \sqrt{\hat{a}_1^2 + \hat{a}_2^2} \\ \hat{\phi} &= \arctan\left(\frac{\hat{a}_2}{\hat{a}_1}\right) \quad (+\pi \text{ if } \hat{a}_1 < 0) \end{aligned}$$

An alternative linear parameterization is obtained from Eulers formula

$$y(n) = \underbrace{\frac{A}{2j}e^{j\phi}}_{\alpha_1} e^{j\omega n} - \underbrace{\frac{A}{2j}e^{-j\phi}}_{\alpha_2} e^{-j\omega n}$$

b) Once the signal parameters have been estimated, the noise can be estimated by

$$\hat{v}(n) = y(n) - \hat{A}\sin(\omega n + \hat{\phi})$$

and the noise variance estimate is given by

$$\hat{\sigma}_v^2 = \frac{1}{N}\sum_{n=1}^{N} \hat{v}^2(n) = \frac{1}{N}\|\mathbf{y} - \mathbf{F}\hat{\mathbf{a}}\|^2 = \frac{1}{N}\min_{\mathbf{a}}\|\mathbf{y} - \mathbf{F}\mathbf{a}\|^2 = \frac{1}{N}\mathbf{y}^T(\mathbf{I} - \mathbf{F}(\mathbf{F}^T\mathbf{F})^{-1}\mathbf{F}^T)\mathbf{y}$$