KTH, INFORMATION SCIENCE AND ENGINEERING

SCHOOL OF ELECTRICAL ENGINEERING AND COMPUTER SCIENCE

Digital Signal Processing EQ2300 / 2E1340

Final Examination 2019–01–11, 08:00-13:00

Literature:

- Jaldén: *Summary notes for EQ2300* (30 pages printed material).
- *Beta – Mathematics Handbook*
- *Collection of Formulas in Signal Processing, KTH.*
- *One A4 of your own notes.* You may write on both sides, and it does not have to be hand written, but cannot contain full solutions to tutorial problems or previous exam problems.
- An unprogrammed pocket calculator.

Notice:

- Answer in English or Swedish.
- At most one problem should be treated per page.
- Answers without motivation/justification carry no rewards.
- Write your name and *personnummer* on each page.
- Write the number of solution pages on the cover page.

The exam consists of five problems with a maximum of 10 points each. 24 points is required for a passing grade (E).

Contact: Joakim Jaldén, Dept. of Information Science and Engineering, 08-790 77 88

Results: Will be reported within three working weeks on “My pages”.

Solutions: Will be available on the course homepage after the exam.

Good luck!

1. We begin with a few mixed shorter questions, **and remember to motivate your answers. . .**

a) Assume that $x[n]$ is a sequence of length 500, i.e., $x[n] = 0$ for $n < 0$ and $n \geq 500$, and that $h[n]$ is the impulse response of an FIR filter of length 20. We wish to filter $x[n]$ with the FIR filter using overlap-add.

i. Propose the most appropriate FFT length N for the implementation, and specify the number of complex valued multiplications per sample?. (2p)

ii. How many blocks should $x[n]$ be divided into? (2p)

b) Assume that you have a block of $N = 100$ signal samples from the stochastic process

$$x[n] = \underbrace{\cos(0.2\pi n + \theta_1)}_{x_1[n]} + \underbrace{\cos(0.3\pi n + \theta_2)}_{x_2[n]} + \underbrace{0.01\cos(0.6\pi n + \theta_3)}_{x_3[n]},$$

where θ_1, θ_2, and θ_3 are independent random variables uniformly distributed over $[0, 2\pi)$. You estimate the power spectrum of $x[n]$ using a modified periodogram with a Blackman window, using all signal samples.

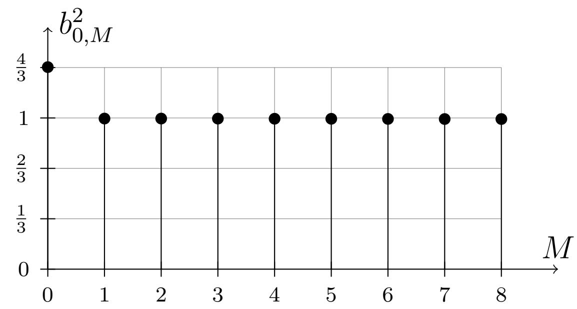
i. Will $x_1[n]$ and $x_2[n]$ appear as distinct peaks in the spectrum estimate? (2p)

ii. Will the spectral peak resulting from $x_3[n]$ be clearly visible in the spectrum estimate, when plotted on a dB scale? (2p)

c) What spectrum estimator is implemented in the code below? (2p)

```
w = window('blackman',L)';
D = L/2;
U = 1/L*sum(abs(w).^2);
Ph = 0;
for k=0:K-1
    xk = x(k*D+1:k*D+L);
    Xk = fft(xk.*w,R);
    Ph = Ph + 1/(K*L*U)*abs(Xk).^2;
end;
```

2. A stochastic process $x[n]$ is suspected to come from an AR-process. To verify this assumption, AR models of different order are fitted to $N = 1024$ samples of the process, and the constant gain parameter $b_{0,M}$ is computed and plotted for different model orders M in the range $M = 0, \ldots, 8$. The results are shown in the figure below.



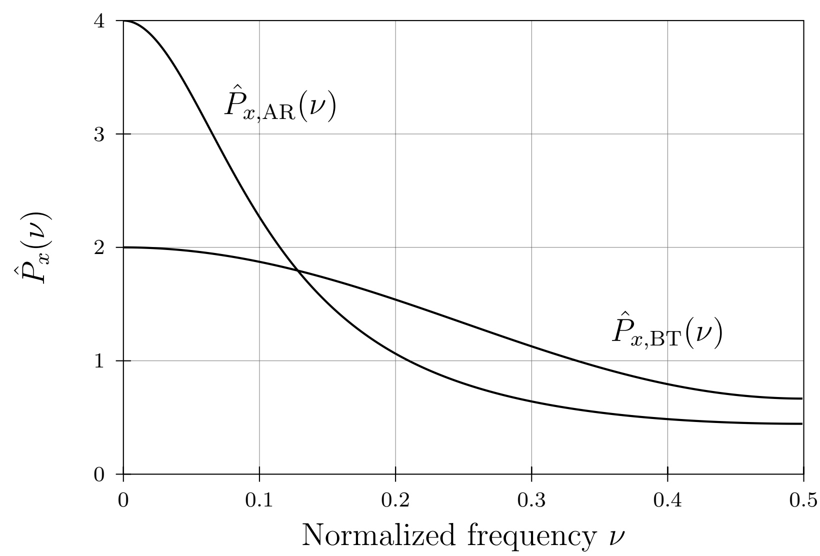
Based on this, a model order of $M = 1$ is selected. The autocorrelation estimate is obtained as $\hat{r}_x[0] = 4/3$ and $\hat{r}_x[1] = 2/3$, based on which an AR1 based spectrum estimator is constructed. A Blackman-Tukey spectrum estimator, with a rectangular window and a max-lag of $M = 1$ is also computed for reference.

a) Explain why the choice of $M = 1$ was justified for the AR model. (1p)

b) Provide a closed form expression for the AR1 model based estimate $\hat{P}_{x,\mathrm{AR}}(\nu)$ of the power spectrum $P_x(\nu)$ of $x[n]$. (4p)

c) Provide a closed form expression for the Blackman-Tukey estimate $\hat{P}_{x,\mathrm{BT}}(\nu)$ of the power spectrum $P_x(\nu)$ of $x[n]$. (3p)

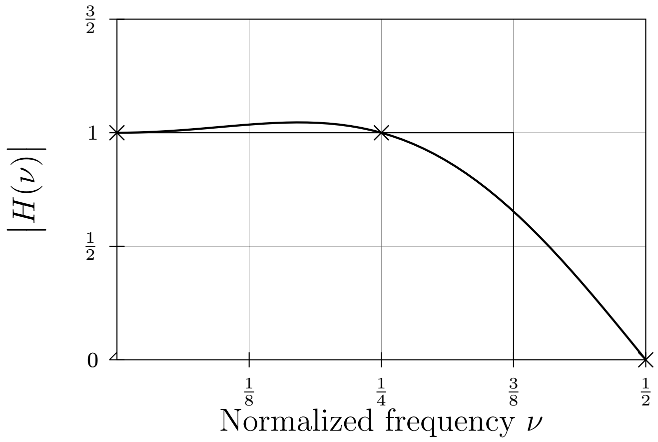
d) Assuming that $\hat{P}_{x,\mathrm{AR}}(\nu) \approx P_x(\nu)$, what is then the variance of the spectrum estimator $\hat{P}_{x,\mathrm{BT}}(\nu)$? Is Blackman-Tukey with a rectangular window and max-lag $M = 1$ an appropriate spectrum estimator in this case? Refer to the figure below for a numerical illustration of the spectrum estimates. (2p)



3. In this problem you shoud to design a causal linear phase low-pass FIR filter of maximal length $N = 4$ (model order $M = 3$), i.e., a filter for which $h[n] = 0$ when $n < 0$ and $n > 3$. The aim is to approximate the ideal frequency response, with magnitude given by

$$\left|H_{\text{Ideal}}(\nu)\right| = \begin{cases} 1 & 0 \le |\nu| \le \frac{3}{8} \\ 0 & \frac{3}{8} < |\nu| \le \frac{1}{2} \end{cases},$$

and your filter design need to match the ideal response exactly at $\nu = 0$, $\nu = \frac{1}{4}$, and $\nu = \frac{1}{2}$ as shown in the image below.



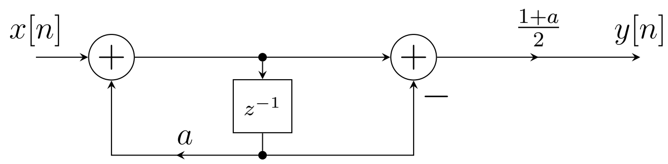
The defining properties of the four different linear phase FIR filter types are listed in the following table, see also the summary notes for further information.

Type	M	Symmetry
I	even	$h[n] = h[M-n], \quad n = 0, \dots, M$
II	odd	$h[n] = h[M-n] \quad n = 0, \dots, M$
III	even	$h[n] = -h[M-n] \quad n = 0, \dots, M$
IV	odd	$h[n] = -h[M-n] \quad n = 0, \dots, M$

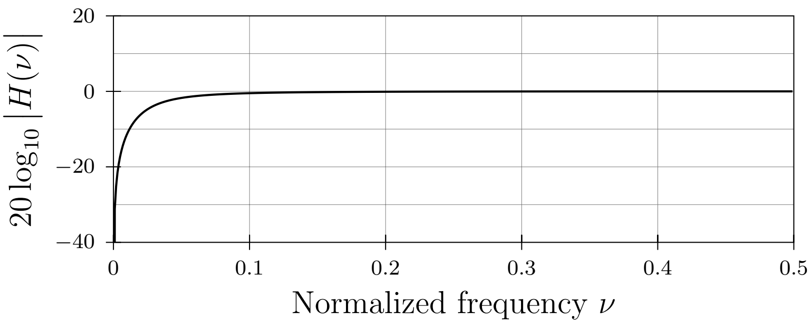
a) Explain why the Type II filter is the only filter type that can satisfy the specifications given, i.e., explain why Type I, Type III or Type IV cannot be used, even if, say, $M = 2 < 3$ is used for Type I and Type III to make M even. (5p)

b) Obtain a closed form solution for a filter $h[n]$ that satisfies the specifications. Simplify your solution as far as possible. (5p)

4. The filter below is a first order so-called notch filter designed to remove the DC component of the input signal $x[n]$, while maintaining the amplitude of other frequencies.



The parameter $a \in (0, 1)$ is a parameter that can be tuned to determine the sharpness of the filter, and the magnitude of the filter's frequency response is shown below for the case where $a = 0.8$. The filter gets sharper as a tends to 1.



We now wish to implement the filter in fixed point arithmetics using a B+1 signed bit magnitude representation of the range $[-1, 1]$.

a) Specify the statistical model for the quantization error in the circuit above. (2pt)

b) Sketch the power spectrum of the quantization noise at the output of the circuit when $B = 7$ and $a = 0.8$. Label you axes (see hint below). (3pt)

c) Obtain a closed form expression for the total power of the quantization noise at the output of the circuit, for a generic number of bits B and a generic value for the parameter $a \in (0, 1)$. (5pt)

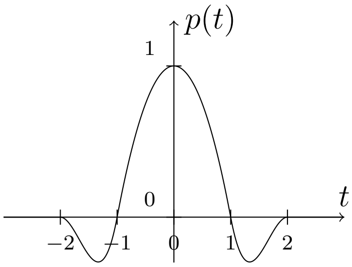
Hint: For part b), you may for the labeling of axes be helped by knowing that

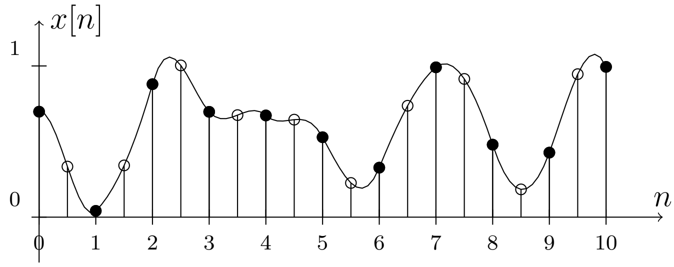
$$10 \log_{10} \frac{2^{-14}}{12} \approx -53\,.$$

5. Consider the piece-wise cubic spline interpolation scheme given by

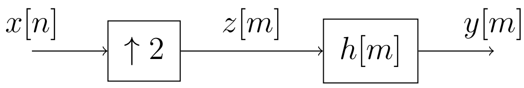
$$y(t) = \sum_{n=-\infty}^{\infty} p(t-n)x[n]\,, \quad \text{where} \quad p(t) = \begin{cases} 1-|t|^2 & |t| \leq 1 \\ 2(2-|t|)^3 - 2(2-|t|)^2 & 1 < |t| \leq 2 \\ 0 & 2 < |t| \end{cases}.$$

One of the salient features of this interpolation scheme is that it yields a continuous curve $y(t)$, where $y(t)|_{t=n} = x[n]$, with continuous first derivatives. The pulse $p(t)$ is illustrated below, together with an example of an interpolated curve and interpolated samples at a factor of 2.





We wish to use the circuit below to obtain these interpolated samples (shown using non-filled circles). Here, we will have that $y[m] = x[m/2]$ for even m and $y[m] = y(t)|_{t=m/2}$ for every m.



a) Obtain the impulse response $h[m]$ of the discrete time interpolation filter. (2p)

b) Draw the polyphase implementation of the interpolation circuit, and specify the constituent filters $p_k[n]$ for $k = 0, 1$. (3p)

c) Assume that $x[n] = \cos(2\pi\nu_0 n)$, where $\nu_0 \in [0, 1/2)$, such that

$$X(\nu) = \frac{1}{2}\delta(\nu - \nu_0) + \frac{1}{2}\delta(\nu + \nu_0)$$

for $\nu \in (-1/2, 1/2)$, and obtain $Y(\nu)$ for $\nu \in (-1/2, 1/2)$. (5p)