SOLUTIONS

E 87 **Digital Signalbehandling**, 2E1340

Final Examination 2002–04–03, 9.00–13.00

1. a) The DTFTs of the signals are related by $U(f_u) = \frac{1}{D}\sum_{k=0}^{D-1} X(\frac{f_u-k}{D})$ and $V(f_u) = H(f_u)U(f_u)$. This gives $Y(f_y) = V(f_yI) = \frac{1}{D}H(f_yI)\sum_{k=0}^{D-1} X(\frac{f_yI-k}{D})$, where $D = 3$ and $I = 2$.

b) Since an AR signal is a stochastic process, we should use the corresponding relationships between the spectral densities. This is a bit tricky, especially for interpolation, since an interpolated signal typically isn't stationary. A strict treatment of these results is described in "Complementary Reading in Digital Signal Processing", but the results can be summarized in the following figures.

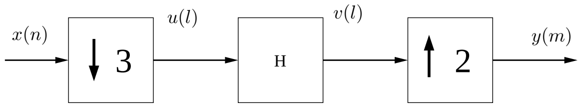

Figure 1:

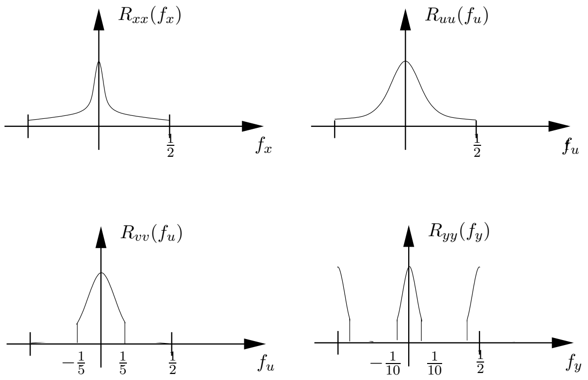

Figure 2:

2. a)–c) Consider the noise-free signal $x(n) = e^{j2\pi f_1} + e^{j2\pi f_2+\phi}$, which means that $X(f) = \delta(f-f_1) + e^{j\phi}\delta(f-f_2)$. The averaging does not contribute to the resolution, so it is sufficient to study one segment. The length l of the window should match the length of each segment. Then, the Welch spectral estimate is $P(f) = |W(f-f_1) + e^{j\phi}W(f-f_2)|^2$ which gives $P(f_1) \approx P(f_2) \approx |W(0)|^2$ and $P(\frac{f_1+f_2}{2}) \leq |2W(B)|^2$ if $B = \frac{f_1-f_2}{2}$. This gives

a) ii) $M = l$

b) ii) $|W(B)| = \frac{|W(0)|}{2}$

c) iii) $|f_1 - f_2| > 2B$

d) iii) $l = 2N-1$, since we can estimate the autocorrelation function at most from $r_{xx}(-N+1)$ through $r_{xx}(N-1)$ using the available N data samples. Mostly, a much shorter window is used since the estimated $\hat{r}_{xx}(k)$ are inaccurate unless $k \ll N$.

e) iii) $|f_1 - f_2| > 2B_{\text{comb}}$ based on the same reasoning as in c).

3. Using the standard stochastic approximation of the quantization error, the system can

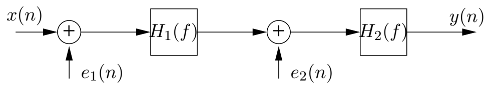

Figure 3: Equivalent model

equivalently be described by Figure 3, where $e_1(n)$ and $e_2(n)$ are independent white noise sources with power $\sigma_e^2 = \frac{2^{-2(b-1)}}{12} = \frac{2^{-2b}}{3}$ (quantization to $b-1$ bits excluding the sign bit). The total quantization noise power at the output is

$$P_{\text{quant}} = \underbrace{\int_{-1/2}^{1/2} \sigma_e^2 |H_1(f)H_2(f)|^2 df}_{\text{contrib. from } e_1(n)} + \underbrace{\int_{-1/2}^{1/2} \sigma_e^2 |H_2(f)|^2 df}_{\text{contrib. from } e_2(n)}$$

$$= 2\sigma_e^2 \int_0^{1/3} ((1-2f)^2 + 1) df = \frac{80\sigma_e^2}{81} = \frac{80}{243} 2^{-2b}$$

4. (a) The transfer function $H(z) = Y(z)/X(z)$ can be written as $H(z) = H_1(z) + H_2(z)$, where H_1 and H_2 correspond to the upper and lower halves of the circuit, respectively. Clearly, $H_2(z) = \frac{1}{1-Bz^{-1}}$. The upper half of the circuit is a standard lattice implementation of an all-pole IIR filter with reflection coefficients $K_1 = A$ and $K_2 = 0.5$. According to section 7.3.5 in the text book, $H_1(z) = \frac{1}{A_2(z)}$, where the lattice recursion gives $A_0(z) = 1$, $A_1(z) = A_0(z) + K_1 z^{-1} A_0(z^{-1}) = 1 + Az^{-1}$ and $A_2(z) = A_1(z) + K_2 z^{-2} A_1(z^{-1}) = 1 + \frac{3A}{2}z^{-1} + \frac{1}{2}z^{-2}$.
Thus, $H(z) = \frac{1}{1+\frac{3A}{2}z^{-1}+\frac{1}{2}z^{-2}} + \frac{1}{1-Bz^{-1}}$.

(b) $H(z)$ is stable iff $H_1(z)$ and $H_2(z)$ are stable. $H_1(z)$ has a pole in B and is stable iff $|B| < 1$. For $H_2(z)$, we use the Schur-Cohn stability test, i.e. transform the denominator $A_2(z) = 1 + \frac{3A}{2}z^{-1} + \frac{1}{2}z^{-1}$ of $H_2(z)$ to lattice form. However, we already know from a) that the reflection coefficients are given by $K_1 = A$ and $K_2 = 0.5$. According to Schur-Cohn, $H_2(z)$ is stable iff $|K_n| < 1$, i.e. $|A| < 1$.
To conclude, the filter is stable iff $|A| < 1$ and $|B| < 1$.

5. a) Define

$$\mathbf{x} = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix} \quad \mathbf{s}_k = \begin{bmatrix} 1 \\ e^{j\frac{2\pi k}{N}} \\ e^{j\frac{2\pi 2k}{N}} \\ \vdots \\ e^{j\frac{2\pi(N-1)k}{N}} \end{bmatrix}$$

Since $\|\mathbf{y}\|^2 = \sum |\mathbf{y}_k|^2$ for all vectors, it follows directly that the two expressions are equivalent.

b) $\mathbf{W}$ is given by $\mathbf{W}_{k,l} = e^{-j\frac{2\pi kl}{N}}$ and it is easy to see, as in the text book, that $\mathbf{W}\mathbf{W}^* = \mathbf{W}^*\mathbf{W} = N\mathbf{I}$. Thus,

$$\|\mathbf{W}\mathbf{y}\|^2 = (\mathbf{W}\mathbf{y})^*(\mathbf{W}\mathbf{y}) = \mathbf{y}^*\mathbf{W}^*\mathbf{W}\mathbf{y} = N\mathbf{y}^*\mathbf{y} = N\|\mathbf{y}\|^2$$

for all vectors $\mathbf{y}$. In particular, $\|\mathbf{W}(\mathbf{x} - \alpha\mathbf{s}_k)\|^2 = N\|\mathbf{x} - \alpha\mathbf{s}_k\|^2$ and the same α minimizes both expressions.

c) Denote the DFT of $\mathbf{x}$ by $\mathbf{X}$ and note that $\mathbf{X} = \mathbf{W}\mathbf{x}$. Also, it is easy to see that $\mathbf{W}\mathbf{s}_k$ is a vector with all zeros, except for element k which is N (note for example that $\mathbf{s}_k$ is the kth column of $\mathbf{W}^*$). Therefore, the LS problem can be written

$$\min_\alpha \|\mathbf{W}(\mathbf{x} - \alpha\mathbf{s}_k)\|^2 = \min_\alpha \left\| \begin{bmatrix} X(0) \\ \vdots \\ X(k-1) \\ X(k) \\ X(k+1) \\ \vdots \\ X(N-1) \end{bmatrix} - \alpha \begin{bmatrix} 0 \\ \vdots \\ 0 \\ N \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\|^2 = \min_\alpha |X(k) - N\alpha|^2 + \text{const.}$$

and it follows directly that the least squares estimate of α is $\hat{\alpha} = X(k)/N$.