SOLUTIONS

E 89 **Digital Signalbehandling**, 2E1340

Final Examination 2002–12–17, 14.00–19.00

1. a) Answer: (iv) (or possibly (iii)). The only reasonable possibility is a difference in relative phase between the two signals (another possibility is to change the sign of A_1, which is equivalent to increasing ϕ_1 by π). Changing M or the window would affect the side-lobes, the zero-padding only affects the resolution of the curves and it clearly looks as if the two sinusoids have the same amplitude (at least in magnitude).

b) Answer: (iii). $k = 841$ corresponds to the normalized frequency $f = k/N \approx 0.82$. Thus, F/F_s or $-F/F_s$ should be approximately $0.82 + k$ for some integer k. The only possibility is $F_s = 11\,025$ which gives $-F/F_s \approx 0.82 - 2$.

c) Answer: (i). From the Parseval formula for DFTs,

$$17\,000 = \hat{P}_{\text{tot}} = \sum_{k=0}^{1023} \hat{P}_{\text{per}}(k) = \sum_{k=0}^{1023} |X(k)|^2 = \sum_{k=0}^{1023} |x(n)|^2 = 1024|A|^2 \ ,$$

so $A = \sqrt{17\,000/1\,024} \approx 4.07$.

2. Quantization noise with variance $\sigma_q^2 = \frac{2^{-2b}}{12}$ will occur at four multipliers.

The noise generated in the a multiplier will will pass through the complete circuit, with transfer function $H(z) = \frac{b_0 + b_1 z^{-1}}{(1-az^{-1})(1-cz^{-1})}$, while the three noise components generated in the b_0, b_1 and c multipliers will add in the adders to the right (which are equivalent) and thus pass through $H_c(z) = \frac{1}{(1-cz^{-1})}$.

With the given numerical values and after partition fractioning, the transfer functions and impulse responses are given as:

$$H_c(z) = \frac{1}{(1 - \frac{1}{5}z^{-1})} \Leftrightarrow h_c(n) = (\frac{1}{5})^n u(n)$$

and

$$H(z) = \frac{2 - z^{-1}}{(1 - \frac{4}{5}z^{-1})(1 - \frac{1}{5}z^{-1})} = \frac{1}{1 - \frac{4}{5}z^{-1}} + \frac{1}{1 - \frac{1}{5}z^{-1}} \Leftrightarrow h_c(n) = (\frac{4}{5})^n u(n) + (\frac{1}{5})^n u(n)$$

where $u(n)$ is the unit step function.

Thus the output quantization noise variance $\sigma_y^2 = \sigma_q^2 \sum_{-\infty}^{+\infty} h^2(n) + 3\sigma_q^2 \sum_{-\infty}^{+\infty} h_c^2(n) \approx (6.20 + 3 \cdot 1.04)\frac{2^{-2b}}{12}$

3. (a) Let $H_I(z)$ denote the filter part on the left, and $H_C(z)$ the part on the right side of the decimation. Evaluating each expression separately gives

$$H_I(z) = \frac{1}{1 - z^{-1}}$$

$$H_C(z) = 1 - z^{-1}$$

The first Noble identity allows us to move $H_C(z)$ before the decimation as follows

$$H(z) = H_I(z) H_C(z^R) = \frac{1 - z^{-R}}{1 - z^{-1}}$$

(b) The expression above can be identified as a geometric sum, thus

$$H(z) = \sum_{n=0}^{R-1} z^{-n}$$

i.e. the CIC-filter can be seen as an FIR-filter of length R, with all coefficients equal to unity. The magnitude response is the same as that of a rectangular window

$$|H(e^{2\pi f})| = \left| \frac{\sin(\pi R f)}{\sin(\pi f)} \right|$$

(c) Three reasons for placing $H_C(z)$ after the decimation are

- The number of delay elements in $H_C(z)$ are reduced from R to one, saving hardware resources.
- The filter implementation becomes independent of the rate change R. This allows the decimation rate R to be changed dynamically, without having to change the filter.
- Half the filter can use a slower clock, allowing slower, and cheaper, hardware to be used.

4. We want the output

$$w(n) = y(n) = \sum_{k=0}^{M-1} h(k) x(n-k)$$

for $n = 0, 1, \ldots, L-1$. Because of the DFT properties, the FFT implementation will give $w(n) = v(n) \circledN h(n)$. This circular convolution can be viewed as linear convolution of $h(n)$ with a periodically repeated version of $x(n)$, see Figure 1. In order to avoid any delay in the output we should have , $v(n) = x(n)$, $n = 0, 1, \ldots, L-1$. Since $v(n)$ begins at $n = 0$, The initial state, $x(-k)$, $k = 1, 2, \ldots, M-1$ has to be placed at positions $(-k)_{\text{ mod } N} = N - k$, resulting in

$$v(n) = \{x(0), x(1), \ldots, x(L-1), \underbrace{0, \ldots, 0}_{N-L-M+1 \text{ zeros}}, x(-M+1), \ldots, x(-1)\}$$

The FFT length, should be $N \geq M + L - 1$.

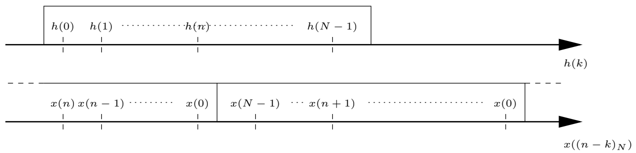

Figure 1: Illustration of circular convolution.

5. a) If $F(z)H(z) = 1$, then $e(n) \equiv 0$, so $P_e = 0$ which clearly is the minimum.

b) The Z-transform of $h(n)$ is

$$H(z) = 1 - \alpha z^{-1}.$$

Therefore,

$$F(z) = \frac{1}{1 - \alpha z^{-1}}\ ,$$

which gives

$$f(n) = \alpha^n u(n),$$

where $u(n)$ is the unit step function.

c) The combination of $h(n)$ and $(f(n)$ has the total impulse response

$$g(n) = \{1, -\alpha\} * \{f_0, f_1\} - \delta(n) = (f_0 - 1)\delta(n) + (-\alpha f_0 + f_1)\delta(n-1) - \alpha f_1 \delta(n-2)\ .$$

Assume the variance of the input white noise is σ^2, then P_e is given by

$$P_e = \sigma^2 \sum_{n=0}^{\infty} g^2(n).$$

Obviously, minimizing P_e is equivalent to minimizing $\sum_{n=0}^{\infty} g^2(n)$. It is easy to show that

$$\sum_{n=0}^{\infty} g^2(n) = (f_0 - 1)^2 + (-\alpha f_0 + f_1)^2 + (\alpha f_1)^2.$$

Calculating the derivative of the above expression and solving the system of equations, we obtain

$$f_0 = \frac{1 + \alpha^2}{1 + \alpha^2 + \alpha^4}$$

and

$$f_1 = \frac{\alpha}{1 + \alpha^2 + \alpha^4}.$$