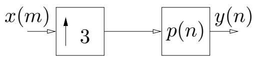
SOLUTIONS

E 99 **Digital Signalbehandling**, 2E1340

Final Examination 2006–06–05, 14.00–19.00

**1.** a) The direct implementation consist of an upsampling block followed by a linear filter. This is shown in the figure below.




What needs to be done is to select $p(n)$ such that $y(n)$ corresponds to linear interpolation. This is done by selecting

$$p(n) = \begin{cases} 1 & n = 0 \\ \frac{2}{3} & |n| = 1 \\ \frac{1}{3} & |n| = 2 \\ 0 & |n| \geq 3. \end{cases}$$

b) The output $y(n)$ can be computed directly according to

$$\begin{aligned} y(3m) &= x(m) \\ y(3m+1) &= \tfrac{2}{3}x(m) + \tfrac{1}{3}x(m+1) \\ y(3m+2) &= \tfrac{1}{3}x(m) + \tfrac{2}{3}x(m+1). \end{aligned}$$

This is the polyphase filter bank solution which is a more efficient implementation, see e.g. the complementary reading material.

c) For the general implementation we have a direct implementation with filter

$$p(n) = \begin{cases} \frac{k-|n|}{k} & |n| < k \\ 0 & |n| \geq k. \end{cases}$$

Note that this is equal to the answer to a) in the special case where $k = 3$.

**2.** a) Both the periodogram and the Bartlett method will on the average give a correct level for the variance of the white noise, so the noise level will stay around 0dB, even though the variations will be smaller with Bartlett than in the periodogram, since the Bartlett estimate has a smaller variance.

From the solution of problem 1 from the previous exam, we know that a sinusoid with amplitude $a$ will contribute to the periodogram with a peak of height $a^2N/4$ if $N$ is the number of samples. The Bartlett spectrum is just the average of a number of periodograms, so the height of the peak will be $a^2L/4$ if $L$ is the number of samples in each data segment. Here, we used $512/128 = 4$ segments, so the peaks will be a factor 4 lower, corresponding to $10\log_{10}(4) \approx 6$dB lower values. So the peaks at $\pm 0.1$ will be approximately at $30 - 6 = 24$dB and the peaks at $\pm 0.3$ will be approximately at $20 - 6 = 14$dB. Also, the width of the peaks will be 4 times higher. See Figure 1.

b) Zero-padding will only add more points on the curve, so at least when the plotted figure is so small as it is here, we will not notice much of a difference at all. However, if we zoom in on a small part of the figure, we will be able to see more details if we use zero-padding. For example, it will be possible determine the location and height of the peaks more accurately from the zero-padded plot.



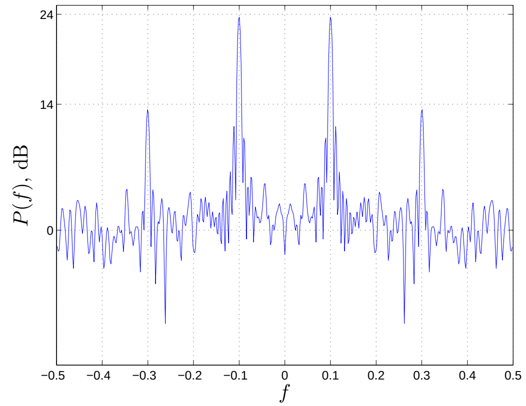



Figure 1: Bartlett estimate of the signal.

c) You could for example consider

**Welch method:** Gives lower variance than the periodogram (slightly lower than the Bartlett method) so it will be possible to estimate the noise level somewhat more accurately.

**Blackman-Tukey:** Same argument as for Welch.

**Pisarenko:** The method is specialized for this situation where you have a known number of sinusoids in white noise. You get numeric values directly for the frequencies and noise power. However, for well-separated frequencies like here, you will probably get more accurate frequency estimates from the periodogram.

**MUSIC:** Gives better accuracy of the frequency estimates than Pisarenko. If the frequencies are more closely located, MUSIC and Pisarenko are better than the non-parametric methods since it has higher resolution. However, in this particular example that is not an issue.

**3.** a) The use of the buffering technique will introduce a delay corresponding to two full buffers, i.e. the delay is $2BT$, where $L$ is the buffer size in samples and $T$ is the sampling period. Thus

$$\frac{2L}{16000} < 0.05s$$

gives $L < 400$ samples.

b) The filter used is linear, so the overlap-add method can be used. For each data segment, we need a working buffer and an FFT of length $L + P - 1 = 199 + P$ or more.

c) We need one FFT of length N to calculate the DFT of the filter coefficients. For each input segment, we need one FFT plus one IFFT plus N complex valued multiplications. The FFT/IFFT implementations require $\frac{N}{4}\log_2 N$ complex multiplications (since the input to the FFT and the output of the IFFT are real valued). Also, half of the multiplications between signal and filter FFTs can be saved because of the symmetry properties of FFTs for real valued signals. Since 10 buffers are processed in total, the cost is

$$\frac{N}{4}\log_2 N + 10\left(\frac{N}{4}\log_2 N + \frac{N}{2} + \frac{N}{4}\log_2 N\right),$$

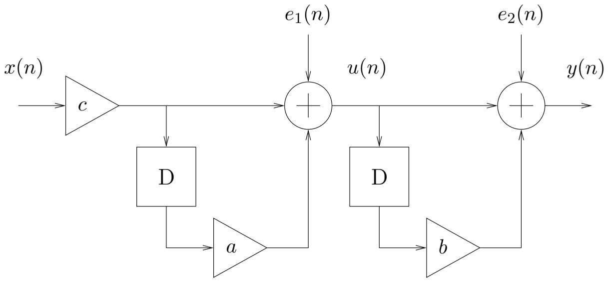



Figure 2: Cascade implementation of the FIR filter.

complex valued multiplications. In this case $N = 199 + P$ and each complex multiplication is four real so the cost is four times larger in terms of real valued multiplications.

The direct convolution method requires $P$ real multiplications for each input sample, i.e. a total of $2000P$ real valued multiplications.

**4.** a) Figure 2 shows the implementation. The noise source $e_1(n)$ corresponds to round-off noise from the multiplications by $c$ and $a$ and $e_2(n)$ is the round-off noise from the multiplication by $b$ (if you place the multiplication by $c$ somewhere else in the system, the corresponding round-off noise might contribute to $e_2(n)$ instead, but that will not affect the argument below).

The coefficient $a$ should be chosen as one of the zeros $z_i$ and $b$ should be chosen as the other one. As can be seen from the figure, only the noise from $e_1(n)$ will be affected by this choice. The transfer function from $e_1(n)$ to the output is $G(z)$, so the contribution to the noise power at the output is

$$\sigma_{e_1}^2 \sum |g(n)|^2 = \sigma_{e_1}^2 (1 + |b|^2)$$

which is minimized if $|b|^2$ is chosen as small as possible. Therefore, we should use,

$$\begin{aligned} a &= z_1, b = z_2 \qquad \text{if } |z_1| > |z_2| \\ a &= z_2, b = z_1 \qquad \text{if } |z_1| < |z_2| \end{aligned}$$

b) The implementation of the all-pole filter is shown in Figure 3. This time, $e_1(n)$ models the round-off noise from the multiplications by $C$ and $A$ whereas $e_2(n)$ corresponds to the multiplication by $B$. The noise from $e_1(n)$ passes through the full filter, so it does not matter how we choose $A$ and $B$. The noise from $e_2(n)$, on the other hand will only pass through the second link of the filter, $K(z)$ (with impulse response $k(n) = B^n$) and the contribution from $e_2(n)$ at the output is

$$\sigma_{e_2}^2 \sum |k(n)|^2 = \sigma_{e_2}^2 \sum |B|^{2n} = \frac{\sigma_{e_2}^2}{1 - |B|^2}$$

This expression is minimized if $|B|^2$ is chosen as small as possible. Therefore, we should choose,

$$\begin{aligned} A &= p_1, B = p_2 \qquad \text{if } |p_1| > |p_2| \\ A &= p_2, B = p_1 \qquad \text{if } |p_1| < |p_2| \end{aligned}$$



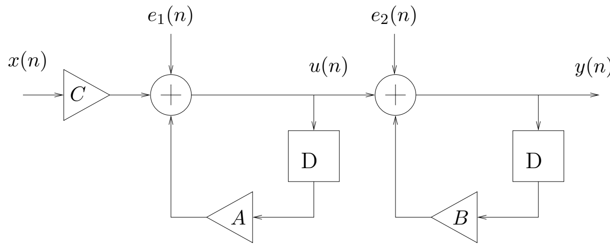



Figure 3: Cascade implementation of the all-pole filter.

**5.** a) The MSE is given by

$$\begin{aligned} \text{MSE} &= \mathrm{E}[|\hat{s}(n) - s(n)|^2] = \mathrm{E}[|\mathbf{a}^H(\mathbf{h}bs(n) + \boldsymbol{\nu}(n)) - s(n)|^2] = \Big/ \begin{array}{c} s(n), \boldsymbol{\nu}(n) \\ \text{independent} \end{array} \Big/ \\ &= \mathrm{E}[|(\mathbf{a}^H\mathbf{h}b - 1)s(n)|^2] + \mathrm{E}[|\mathbf{a}^H\boldsymbol{\nu}(n)|^2] \\ &= |\mathbf{a}^H\mathbf{h}b - 1|^2 \underbrace{\mathrm{E}[|s(n)|^2]}_{=1} + \mathbf{a}^H \underbrace{\mathrm{E}[\boldsymbol{\nu}\boldsymbol{\nu}^H]}_{=\sigma^2\mathbf{I}} \mathbf{a} = |\mathbf{a}^H\mathbf{h}b - 1|^2 + \sigma^2\mathbf{a}^H\mathbf{a} \end{aligned}$$

b) The second term is independent of $b$ and the first term is minimized when $\mathbf{a}^H\mathbf{h}b - 1 = 0$, i.e.

$$b = \frac{1}{\mathbf{a}^H\mathbf{h}}$$

c) Rewrite the MSE and complete the squares, with respect to $\mathbf{a}$:

$$\begin{aligned} \text{MSE} &= (\mathbf{a}^H\mathbf{h}b - 1)(\mathbf{a}^H\mathbf{h}b - 1)^H + \sigma^2\mathbf{a}^H\mathbf{a} = \mathbf{a}^H(\mathbf{h}\mathbf{h}^H|b|^2 + \sigma^2\mathbf{I})\mathbf{a} - \mathbf{a}^H\mathbf{h}b - \mathbf{h}^H\mathbf{a}b^* + 1 \\ &= \left(\mathbf{a} - (\mathbf{h}\mathbf{h}^H|b|^2 + \sigma^2\mathbf{I})^{-1}\mathbf{h}b\right)^H (\mathbf{h}\mathbf{h}^H|b|^2 + \sigma^2\mathbf{I}) \left(\mathbf{a} - (\mathbf{h}\mathbf{h}^H|b|^2 + \sigma^2\mathbf{I})^{-1}\mathbf{h}b\right) + \text{const.} \end{aligned}$$

This expression is minimized when $\mathbf{a} - (\mathbf{h}\mathbf{h}^H|b|^2 + \sigma^2\mathbf{I})^{-1}\mathbf{h}b = 0$, i.e. when

$$\mathbf{a} = (\mathbf{h}\mathbf{h}^H|b|^2 + \sigma^2\mathbf{I})^{-1}\mathbf{h}b$$

Comment: This is an example of a common situation, where it is difficult or impossible to find the optimal value of all parameters jointly, but where it is easy to optimize some parameters at a time. A common trick, then, is to iterate between these solutions. For example, we could start with some arbitrary value of $b$, use that to find $\mathbf{a}$, which in turn is used to find a new value for $b$. Normally, such an iteration will quickly converge to a solution that hopefully is the global optimum, but it may also get stuck in a local optimum or diverge if we are unlucky.

Additional comment: If you insert the optimal $b$ in the MSE expression, you get MSE= $\sigma^2\|\mathbf{a}\|^2$, which clearly is minimized when both elements of the vector $\mathbf{a}$ go to zero. However, then $b$ will tend to infinity, which is an unreasonable solution. Physically, this would correspond to using a very high transmit power to obtain an arbitrarily high signal to noise ratio at the receiver. In practice, we have to add a constraint on the maximum permitted power level at the transmitter to avoid such unrealistic solutions.