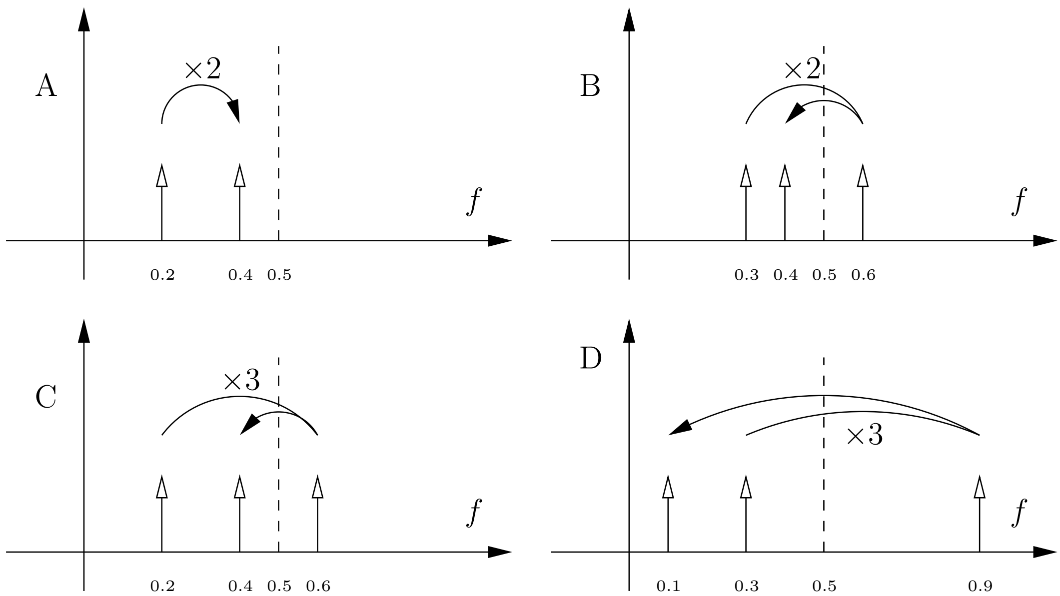
SIGNAL PROCESSING

DEPARTMENT OF ELECTRICAL ENGINEERING

Digital Signal Processing EQ2300/ 2E1340

Final Examination 2012–06–08, 8.00–13.00
Sample Solutions

1. When downsamling $x(n)$ by a factor D, the frequency f_x is increased by a multiplicative factor D and becomes Df_x. However, if the frequency falls outside of the range $f = [0, 1/2]$, there will be aliasing, i.e., the frequency will be equivalent a frequency $f = k \pm Df_x \in [0, 1/2]$ where $k \in \mathbb{Z}$. In the problem there are two possible frequencies f_x which would give $f_y = 0.4$, these are $f_x = 0.2$ which would lead to $f_y = 0 + 2 \times 0.2 = 0.4$ directly and without alias (as shown in case A in the figure below), and $f_x = 0.3$ which would lead to $f_y = 1 - 2 \times 0.3 = 0.4$ after aliasing (case B). However, if $f_x = 0.2$ then we would have $f_z = 1 - 3 \times 0.2 = 0.4$ (case C). Hence, the only valid solution is $f_z = 1 - 3 \times 0.3 = 0.1$ (case D) which implies that $f_x = 0.3$ is the correct answer to the problem.



2. We introduce additive noise sources after the multipliers to model round-off errors, as usual. We examine the two realizations next:

- (I): In that case, the noise from b_1, b_2 and a is filtered by the transfer function $H_1(z) = \dfrac{1}{1 + az^{-1}}$, therefore the round-off noise variance at the output will be given by $\sigma_I^2 = 3\sigma_e^2 \sum_{m=0}^{\infty} h_1^2(m)$, where $\{h_1(m)\}_{m=0}^{\infty}$ is the impulse response corresponding to the transfer function $H_1(z)$ and we have assumed that the noise sources after the multiplications are uncorrelated. We have that $h_1(m) = (-a)^m u(m)$, so the round-off noise variance at the output is

$$\sigma_I^2 = \frac{3}{1 - a^2}\sigma_e^2 \tag{1}$$

- (II): In that case, the noise from a is filtered by $H_2(z) = \frac{b_1 + b_2 z^{-1}}{1 + a z^{-1}}$ and the noise from b_1 and b_2 goes directly to the output, therefore the round-off noise variance at the output will be given by

$$\sigma_{II}^2 = 2\sigma_e^2 + \sigma_e^2 \sum_{m=0}^{\infty} h_2^2(m) = \sigma_e^2 \left[2 + \sum_{m=0}^{\infty} h_2^2(m) \right], \tag{2}$$

where $\{h_2(m)\}_{m=0}^{\infty}$ is the impulse response corresponding to the transfer function $H_2(z)$ and we have assumed again that the noise sources after the multiplications are uncorrelated with variance σ_e^2. Since

$$h_2(m) = b_1(-a)^m u(m) + b_2(-a)^{m-1} u(m-1),$$

we have that

$$\sum_{m=0}^{\infty} h_2^2(m) = b_1^2 + \sum_{m=1}^{\infty} \left[b_1(-a)^m + b_2(-a)^{m-1} \right]^2 =$$

$$= b_1^2 + \left(b_1 - \frac{b_2}{a} \right)^2 \left[\sum_{m=0}^{\infty} (-a)^{2m} - 1 \right] = \tag{3}$$

$$= b_1^2 + \frac{(b_2 - b_1 a)^2}{1 - a^2},$$

and therefore the round-off noise variance at the output of (II) is given by

$$\sigma_{II}^2 = \sigma_e^2 \left[2 + b_1^2 + \frac{(b_2 - b_1 a)^2}{1 - a^2} \right] \tag{4}$$

Replacing a, b_1 and b_2 in equations (1) and (4) respectively yields

$$\sigma_I^2 = \frac{3}{1 - \cos^2\theta} \sigma_e^2 = \frac{3}{\sin^2\theta} \sigma_e^2$$

and also

$$\sigma_{II}^2 = \sigma_e^2 \left[2 + 4 + \frac{1}{1 - \cos^2\theta} \right] = 6\sigma_e^2 + \frac{\sigma_e^2}{\sin^2\theta}.$$

So, we have that

$$\sigma_I^2 < \sigma_{II}^2 \Longleftrightarrow \frac{1}{\sin^2\theta} < 3 \Longleftrightarrow |\sin\theta| > \frac{\sqrt{3}}{3}$$

therefore (I) is preferable over (II) in case where $\sin\theta \in \left(-1, -\frac{\sqrt{3}}{3} \right) \cup \left(\frac{\sqrt{3}}{3}, 1 \right)$.

Also, $\sigma_I^2 \geq \sigma_{II}^2 \Longleftrightarrow |\sin\theta| \leq \frac{\sqrt{3}}{3}$ so (II) is preferable only when $\sin\theta \in \left[-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right]$.

3. (a) We can proceed as in the fast Fourier transform (FFT) algorithm and divide $y(n)$ into its odd and even parts, and then combine the DFTs of these sequences into $Y_N(k)$, the DFT of $y(n)$. The combination required $N/2$ complex multiplications. In the standard FFT such subdivision would continue recursively in $\log_2 N$ steps until one reached DFTs of size 1. This is what yields the complexity $N/2 \log_2 N$ as derived in the complementary reading. However, after only $\log_2 N/M$ steps (subdivisions) we would have N/M size M blocks for which we could apply our circuit. We wold thus have to apply the circuit N/M times, and use a total of $N/2 \log_2 N/M$ complex multiplications to compute the full N-point DFT.

(b) There are more than one way of doing this, but suppose that we zero pad $y(n)$ to have length M, and call the zero padded sequence $x(n)$, we would have

$$Y_N(k) = \sum_{n=0}^{N-1} y(n) e^{-2\pi \frac{kn}{N}} = \sum_{n=0}^{M-1} x(n) e^{-2\pi \frac{kn}{N}} = \sum_{n=0}^{M-1} x(n) e^{-2\pi \frac{(kM/N)n}{M}} = X_M(kM/N)\,,$$

and we could thus read the values of $Y_N(k)$ from every M/Nth value of $X_M(k)$.

4. (a) To find the AR coefficients $a(1)$ and $a(2)$, we will apply the Yule-Walker method for which we first need to estimate the autocorrelation sequence as follows

$$\hat{r}_x(k) = \frac{1}{N} \sum_{n=0}^{N-1-k} x(n+k) x^*(n) \quad ; \qquad k = 0, 1, ..., N-1.$$

The Yule-Walker equations for an AR(2), in a matrix form, are given by

$$\begin{bmatrix} \hat{r}_x(0) & \hat{r}_x(1) \\ \hat{r}_x(1) & \hat{r}_x(0) \end{bmatrix} \begin{bmatrix} a(1) \\ a(2) \end{bmatrix} = - \begin{bmatrix} \hat{r}_x(1) \\ \hat{r}_x(2) \end{bmatrix}.$$

So, the autocorrelation values $\hat{r}_x(0)$, $\hat{r}_x(1)$ and $\hat{r}_x(2)$ are obtained as

$$\hat{r}_x(0) = 31/6 = 5.1667, \quad \hat{r}_x(1) = 14/6 = 2.3333, \quad \hat{r}_x(2) = 9/6 = 1.5.$$

Now, the AR coefficients a(1) and a(2) will be calculated as

$$\begin{bmatrix} a(1) \\ a(2) \end{bmatrix} = - \begin{bmatrix} 31/6 & 14/6 \\ 14/6 & 31/6 \end{bmatrix}^{-1} \begin{bmatrix} 14/6 \\ 9/6 \end{bmatrix} = \begin{bmatrix} -0.4026 \\ -0.1085 \end{bmatrix}$$

The poles of the filter $H(z)$, denoted by p_i for $i = 1, 2$, are the zeros of the polynomial $A(z) = 1 + a(1)z^{-1} + a(2)z^{-2}$, then

$$A(z) = 0 \quad \Rightarrow \quad p_1 = 0.5873 \quad , \quad p_2 = -0.1847.$$

(b) To have a stable filter, all the poles should lie inside the unit circle, then according to the magnitude of the pair of poles p_i which is

$$|p_i| < 1, \quad i = 1, 2$$

it is concluded that the all-pole filter $H(z)$ is stable.

5. (a) From figure 1, we can see two peaks. The resolution for the periodogram $\frac{0.89}{N} < f_{p1} - f_{p0}$, and the sidelobe level -13dB $< P_1(f_{p1}) - P_1(f_{p0})$. We can conclude $f_0 = f_{p0} = 0.2, f_1 = f_{p1} = 0.21$.
Bartlett's method in figure 2 doesn't provide enough resolution ($\frac{0.89}{N/L} > f_{p1} - f_{p0}$) to distinguish the interference from signal, the second highest peak is the side lobe of rectangular window.

(b) From the definition of Periodogram

$$P(f) \;=\; \frac{1}{N} \left| x(n) e^{-j2\pi f n} \right|^2 \approx \frac{NA^2}{4}$$

$P(f_0) = \frac{NA_0^2}{4} \Rightarrow A_0 = 1;$
$P(f_1) = \frac{NA_1^2}{4} \Rightarrow A_1 = 0.5;$
$\sigma_v^2 = P - \frac{A_0^2}{2} - \frac{A_1^2}{2} = 0.1.$