SOLUTIONS

E 92 **Digital Signalbehandling**, 2E1340

Final Examination 2003–12–18, 08.00–13.00

1. We solve the general case directly.

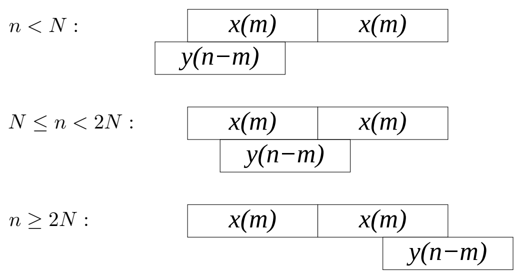


Figure 1: Different cases for the convolution.

c) The convolution of the duplicated x signal with y can be divided into three main cases, see Figure 1. When $n < N$, there's no difference compared to $x(n) * y(n)$. Similarly, when $n \ge 2N-1$, the convolution looks the same as $x(n) * y(n)$ with $n \ge N-1$, i.e. the last N values of $z(n)$ is the same as the last N values of $x(n) * y(n)$ (notice that $z(N-1) = z(2N-1)$). This gives

$$x(n) * y(n) = \{\underset{\uparrow}{z(0)}, z(1), \ldots, z(N-1) = z(2N-1), z(2N), z(2N+1), \ldots, z(3N-2)\}\,.$$

When $N \le n < 2N$:, $y(n-m)$ overlaps two repetitions of $x(n)$, which corresponds to circular convolution. The first value of $x(n) \circledS y(n)$ will correspond to $z(N)$, so

$$x(n) \circledS y(n) = \{\underset{\uparrow}{z(N)}, z(N+1), \ldots, z(2N-1)\}$$

a) From above, we see

$$x(n) * y(n) = \{\underset{\uparrow}{z(0)}, \ldots, z(3) = z(7), z(8), \ldots, z(3N-2)\} = \{\underset{\uparrow}{6}, 7, 14, 28, -12, 29, -21\}$$

b) and

$$x(n) \circledS y(n) = \{\underset{\uparrow}{z(N)}, z(N+1), \ldots, z(2N-1)\} = \{\underset{\uparrow}{-6}, 36, -7, 28\}\,.$$

2. In all the questions, the most limiting case is to be able to distinguish between the two tones with smallest separation, C at 262Hz and C$^\sharp$ at 277Hz. The frequency difference is $\Delta F = 277 - 262 = 15$Hz, corresponding to a separation of $\Delta f = 15/8000 \approx 0.0019$ in normalized frequency for the sampled signal.

If the signal to noise ratio is high enough, the frequency of a single sinusoid can be determined using as few as 2 samples, so the number of samples is not limiting in subproblems a), c), e) or g). However, we need at least $1/\Delta f \approx 533$ values in the frequency domain, to be able to see the difference between the tones.

In subproblems b), d), f) and h), the limiting factor is the resolution, which is of the order of $1/M$, where M is the number of samples used in each periodogram.

This gives the following results

a) No, since we get only $128 \ll 533$ values in the frequency domain.

b) No, from a).

c) Yes, since $2048 \gg 533$.

d) No, since $1/128 \gg 0.0019$.

e) Yes, since $2048 \gg 533$.

f) Yes, since $1/2048 \ll 0.0019$.

g) Yes, since the zero-padding is sufficient.

h) No, since each segment has length 205 and $1/205 \gg 0.0019$.

3. The quantization noise from the multiplications all enter at the same place in the system. The transfer function from these quantization noise sources to the output is given by

$$H(z) = \frac{1}{1 - dz^{-1} - ez^{-2}} = \frac{1}{1 - 0.5z^{-1} + 0.1z^{-2}}$$

This gives the impulse response $h(n) = (\alpha\beta^n + \alpha^*(\beta^*)^n)$, where $\beta = \frac{1}{4} + j\sqrt{\frac{3}{80}} = 0.25 + j\sqrt{0.0375} \approx 0.25 + j0.19$ and $\alpha = \frac{\beta}{\beta - \beta^*} = \frac{1}{2} - j\sqrt{\frac{5}{12}} \approx 0.5 - j0.65$.

The noise power at the output is

$$P_Q = \sigma_{e,\text{tot}}^2 \sum_{n=0}^{\infty} |h(n)|^2 = \sigma_{e,\text{tot}}^2 \sum_{n=0}^{\infty} 2|\alpha|^2|\beta|^{2n} + 2\,\mathrm{Re}[\alpha^2\beta^{2n}]$$
$$= 2\sigma_{e,\text{tot}}^2 \left(\frac{|\alpha|^2}{1-|\beta|^2}) + \mathrm{Re}\left[\frac{\alpha^2}{1-\beta^2}\right]\right) = \frac{275}{216}\sigma_{e,\text{tot}}^2 \approx 1.27\sigma_{e,\text{tot}}^2$$

An alternative is to use formula (6.28) from the "Collection of Formulas in Signal Processing" and look up the resulting infinite sum in Beta.

At first sight, it seems that we have 5 noise sources, since there are 5 multiplications. However, if we are more careful, we notice that multiplications by 1 or 2 does not give any round-off errors, which reduces the number of noise sources to 2. This gives $\sigma_{e,\text{tot}}^2 = 2 \cdot 2^{-2B)}/12 = \frac{1}{6}2^{-2B}$ and $P_Q \approx 0.21 \cdot 2^{-2B}$.

(Note: An even more careful solver of the problem would notice that multiplication by 0.5 corresponds to shifting out the least significant bit, which gives a slightly different noise power than the uniform quantization noise resulting from a general multiplication.)

4. a) Since $w(n)$ is a white noise process and is uncorrelated with $s(n)$, the cross-correlation between $s(n) + w(n)$ and $w(n)$ is the same as $r_{ww}(k)$ so the Wiener-Hopf equations for the optimal Wiener solution can be written as

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Hence $b_0 = 1$ and $b_1 = 0$.

b) By doing the inverse Fourier transform on $P_w(e^{jw}) = 3+2\cos(w)$, we obtain the autocorrelation function of $w(n)$, $r_{ww}(n) = 3\delta(n)+\delta(n+1)+\delta(n-1)$. Hence, the Wiener-Hopf equation becomes

$$\begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

The solution is $b_0 = 1$, $b_1 = 0$, $b_2 = 0$ and $b_3 = 0$.

Note: It is easy to see from the figure that the best solution is to subtract away the noise $w(n)$ which leaves the signal $s(n)$ in the output. Thus, the optimal filter is $B(z) = 1$, no matter how long filters we allow or what kind of signal or noise processes we have. However, the result will not hold if we have any correlation between $s(n)$ and $w(n)$.

5. a) The quantization noise is modeled as white with variance

$$\sigma_e^2 = \frac{\Delta^2}{12}.$$

$x(n)$ is assumed to take on values between -1 and 1. The number of quantization intervals is 2^b, where $b = 12$. This gives $\Delta = \frac{1-(-1)}{2^{12}} = 2^{-11} \quad \Rightarrow \quad \sigma_e^2 = \frac{2^{-22}}{12}$.

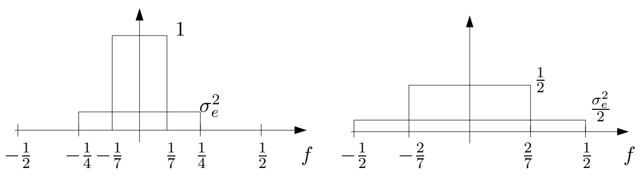
σ_x^2 can be found from the relation

$$\sigma_x^2 = r_{xx}(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} P_{xx}(f)\, df = \int_{-\frac{1}{7}}^{\frac{1}{7}} df = \frac{2}{7}$$

Thus, the SQNR is $\frac{\sigma_x^2}{\sigma_e^2} = \frac{12 \cdot 2^{23}}{7} \approx 1.44 \cdot 10^6$ (or 71.6 dB)

b) The spectral density after ideal filtering and decimation by 2 is given by (Complementary Reading, Section 2.4)

$$P_{yy}(f) = \frac{1}{2} P_{xx}(\frac{f}{2}), \ |f| < \frac{1}{2}$$



(a) Filtered signals (b) Decimated signals

Figure 2: Power spectral density of filtered and decimated signals

The variance of the decimated signal is $\sigma_y^2 = \frac{1}{2}\int_{-\frac{2}{7}}^{\frac{2}{7}} df = \frac{2}{7} = \sigma_x^2$

The variance of the decimated quantization noise $\varepsilon(n)$ is $\sigma_\varepsilon^2 = \frac{\sigma_e^2}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} df = \frac{\sigma_e^2}{2}$

Finally, the SQNR after decimation is $\frac{\sigma_y^2}{\sigma_\varepsilon^2} = 2\frac{\sigma_x^2}{\sigma_e^2} = \frac{12 \cdot 2^{24}}{7} \approx 2.88 \cdot 10^6$ (or 74.6 dB)

We see that the SQNR is doubled by the decimation, which is intuitive since half the noise power is removed by the anti-alias filter, while the band-limited signal is preserved.