SIGNAL PROCESSING

DEPARTMENT OF ELECTRICAL ENGINEERING

E 102 **Digital Signalbehandling** EQ2300/ 2E1340

Solutions to Final Examination 2007–12–15, 14.00–19.00

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1. (a) i. Let $s(n) = x(n)④h(n) = \sum_{k=0}^{3} x(k)h((n-k)_N)$ where $(a)_N$ refers to a mod N. It then follows that,

$$\begin{aligned} s(0) &= x(0)h(0) + x(1)h(3) + x(2)h(2) + x(3)h(1) \\ s(1) &= x(1)h(0) + x(2)h(3) + x(3)h(2) + x(0)h(1) \\ s(2) &= x(2)h(0) + x(3)h(3) + x(0)h(2) + x(1)h(1) \\ s(3) &= x(3)h(0) + x(0)h(3) + x(1)h(2) + x(2)h(1) \end{aligned}$$

where we define $h(3) = 0$. On the other hand, $y(n) = z_3 * h(n) = \sum_{k=0}^{3} h(k)z_3(n-k), n = 0, 1, \ldots, 10$. Evaluating the linear convolution for $n = 3, 4, 5, 6$ and using the definition of z_3, it follows that,

$$\begin{aligned} y(3) &= x(0)h(0) + x(3)h(1) + x(2)h(2) \\ y(4) &= x(1)h(0) + x(0)h(1) + x(3)h(2) \\ y(5) &= x(2)h(0) + x(1)h(1) + x(0)h(2) \\ y(6) &= x(3)h(0) + x(2)h(1) + x(1)h(2) \end{aligned}$$

Noting that $h(3) = 0$ in the calculation of $s(n)$, it follows that, $\{y(3), y(4), y(5), y(6)\} = x(n)④h(n)$.

ii. • $z_1 * h(n)$: Cannot extract $x(n)④h(n)$.

• $z_2 * h(n)$: Yes, we can extract $x(n)④h(n)$.

For the present example, the 4 point circular convolution $(x(n)④h(n))$ differs from the linear convolution $(x(n) * h(n))$ in the first 2 entries (and of course in the number of samples as well). In other words, the first entry of linear convolution uses only $x(0)$ and the second entry uses $x(0), x(1)$. On the other hand, the circular convolution uses $x(0), x(1), x(2)$ in all the entries. Hence, it suffices to check if the first two entries of $s(n)$ can be extracted from $z_1 * h(n)$ and $z_2 * h(n)$. It turns out that $s(0)$ cannot be extracted from $z_1 * h(n)$ while $s(0), s(1)$ can be extracted from $z_2 * h(n)$ (in fact, $\{y(2), y(3), y(4), y(5)\} = x(n)④h(n)$). The solution follows.

(b) Clearly, $Y(z) = X(z)X(z)$. If $y(n)$ is the sequence whose z-transform is $Y(z)$, then, it follows that $y(n) = x(n) * x(n)$, where $*$ denotes linear convolution (multiplication - convolution duality). Hence, we have,

$$\begin{aligned} y(n) &= \sum_k x(k)x(n-k), n \geq 0 \\ &= \sum_k \rho^k \rho^{n-k} u(n-k)u(k) \\ &= \rho^n \sum_k u(n-k)u(k) \\ &= \rho^n \sum_{k=0}^{n} u(n-k)u(k) = (n+1)\rho^n, n \geq 0 \end{aligned} \tag{1}$$

Hence, the solution is $y(n) = (n+1)\rho^n, n \geq 0$. You could also solve it using the derivative property of z transform.

(c) Let $x(n) = \{\underset{\uparrow}{a}, b, c, d, e\}$ and $X(k) = \sum_{n=0}^{4} x(n)e^{\frac{-j2\pi nk}{5}}$ (DFT). Hence, $X(k) = \{\underset{\uparrow}{A}, B, C, D, E\}$. We are interested in,

$$Y(m) = \sum_{k=0}^{4} X(k)e^{\frac{-j2\pi mk}{5}}, m = 0, 1, \ldots, 4 \tag{2}$$

Further, since $x(n)$ is the IDFT of $X(k)$, we have,

$$x(n) = \frac{1}{5}\sum_{k=0}^{4} X(k)e^{\frac{j2\pi mk}{5}}, m = 0, 1, \ldots, 4 \tag{3}$$

Comparing equations 2 and 3, it follows that,

$$Y(0) = 5x(0) \tag{4}$$

Using the fact that $e^{\frac{-j2\pi mk}{5}} = e^{\frac{j2\pi(5-m)k}{5}}$, we can write the equation 2 as,

$$Y(m) = \sum_{k=0}^{4} X(k)e^{\frac{j2\pi(5-m)k}{5}}, m = 0, 1, \ldots, 4 \tag{5}$$

Comparing equations 5 and 3, it follows that,

$$Y(m) = 5x(5-m), m = 1, 2, 3, 4 \tag{6}$$

Thus, the expected answer is $\{5\underset{\uparrow}{a}, 5e, 5d, 5c, 5b\}$

2. Solution: Define $a(f_0) = [1, e^{j2\pi f_0}]^T$. Following the definition of the correlation matrix (Summary slides), we have,

$$\mathbf{R} = A^2 a(f_0)[a(f_0)]^* + \sigma_w^2 \mathbf{I}_2 \tag{7}$$

where $[a(f_0)]^*$ is the complex conjugate transpose of $a(f_0)$ and $\mathbf{I}_2$ is a 2×2 identity matrix. Also note that,

$$\mathbf{R} = \begin{bmatrix} r_{yy}(0) & [r_{yy}(1)]^* \\ r_{yy}(1) & r_{yy}(0) \end{bmatrix} \tag{8}$$

Let λ_i be the i^{th} eigen value of $\mathbf{R}$ and $\underline{v}_i$ be the corresponding eigen vector. Using the standard procedure, we can obtain the following,

$$\begin{aligned} \lambda_1 &= 6.03 \\ \lambda_2 &= 0.03 \\ \underline{v}_1 &= [0.6124 - 0.3536i, 0.7071]^T \\ \underline{v}_2 &= [0.6124 - 0.3536i, -0.7071]^T \end{aligned}$$

From the theory leading to Pisarenko method, it follows that the noise variance, σ_w^2, is the smallest eigen value of $\mathbf{R}$. Hence,

$$\sigma_w^2 = 0.03 \tag{9}$$

Further, it also follows that, f_0 can be obtained as the solution of $[\underline{v}_2]^* a(f_0) = 0$. Since, $[\underline{v}_2]^* a(f_0) = 0.6124 + 0.3536i - 0.7071 e^{j2\pi f_0} = 0$, we have,

$$f_0 = 1/12 = 0.0833 \tag{10}$$

Finally, we see that $r_{yy}(0) = A^2 + \sigma_w^2$. Using the data available and the derived value for σ_w^2, it follows that,

$$A = \sqrt{(}3) \tag{11}$$

3. Solution:

Reparameterization of the signal model:

$$\begin{aligned} A\sin(\omega_0 n + \phi) &= A\,\sin(\phi)\,\cos(\omega_0 n) + A\,\cos(\phi)\,\sin(\omega_0 n) \\ &= \alpha\cos(\omega_0 n) + \beta\sin(\omega_0 n) \end{aligned}$$

where

$$\begin{aligned} \alpha &= A\sin(\phi) \\ \beta &= A\cos(\phi) \end{aligned}$$

and the A and ϕ can be found from the new parameters as

$$\begin{aligned} A &= \sqrt{\alpha^2 + \beta^2} \\ \phi &= \arctan(\alpha/\beta). \end{aligned}$$

Introducing the following vectors and matrices

$$\begin{aligned} \mathbf{y} &= \begin{bmatrix} y(0) & \dots & y(N-1) \end{bmatrix}^T \\ \mathbf{w} &= \begin{bmatrix} w(0) & \dots & w(N-1) \end{bmatrix}^T \\ \mathbf{U} &= \begin{bmatrix} 0 & \sin(\omega_0) & \dots & \sin(\omega_0\,(N-1)) \\ 1 & \cos(\omega_0) & \dots & \cos(\omega_0\,(N-1)) \end{bmatrix}^T \\ \theta &= \begin{bmatrix} \alpha & \beta \end{bmatrix}^T \end{aligned}$$

the signal model can then be written as

$$\mathbf{y} = \mathbf{U}\,\theta + \mathbf{w}$$

and the LS-problem formulated as

$$\hat{\theta} = \min_{\theta}(||\mathbf{y} - \mathbf{U}\,\theta||^2).$$

The solution to the least square problem is given by

$$\hat{\theta} = (\mathbf{U}^T\mathbf{U})^{-1}\mathbf{U}^T\,\mathbf{y}$$

When $\omega_0 = \pi/2$ and N is even, it follows that

$$\mathbf{U}^T\mathbf{U} = \begin{pmatrix} \frac{N}{2} & 0 \\ 0 & \frac{N}{2} \end{pmatrix} \tag{12}$$

When $\omega_0 = \pi/2$ and N is odd, we have

$$\mathbf{U}^T\mathbf{U} = \begin{pmatrix} \frac{N-1}{2} & 0 \\ 0 & \frac{N+1}{2} \end{pmatrix} \tag{13}$$

Based on these simplifications, the Least squares estimate can be approximated for large N as,

$$\hat{\theta} \approx \frac{2}{N} \begin{bmatrix} \sum_{n=0}^{N-1} y(n)\sin(\omega_0\, n) \\ \sum_{n=0}^{N-1} y(n)\cos(\omega_0\, n) \end{bmatrix}.$$

We could also calculate the estimates, as,

$$\hat{\alpha} \approx \frac{2}{N}\text{Imag}\left(\sum_{n=0}^{N-1} y(n)exp^{(j\omega_0\, n)}\right)$$
$$\hat{\beta} \approx \frac{2}{N}\text{Real}\left(\sum_{n=0}^{N-1} y(n)exp^{(j\omega_0\, n)}\right)$$

Can you sense the periodogram?

4. (a)

$$Y(z) = kY(z)z^{-1} + X(z) \quad \Longrightarrow$$
$$H(z) = \frac{1}{1 - kz^{-1}}$$

(b) A periodogram evaluated at $\omega = \omega_0$ with $L = 1000$ samples is

$$\hat{P}(e^{j\omega_0}) = \frac{1}{L}\left|\sum_{n=0}^{L-1} y(n+n_0)e^{-j\omega_0 n}\right|^2,$$

where n_0 denotes the time index at the start of the sample buffer. At the frequency of interest, the sinusoidal power is (by inspection) approximately 45 dB stronger than the noise power. Hence, we can safely disregard the noise term in $y(n)$. Sinusoid input gives a sinusoid output, and consequently

$$y(n+n_0) \approx \left|H(e^{j\omega_0})\right| \cos(\omega_0 n + \theta),$$

where the phase term $\theta = \phi + \arg\left(H(e^{j\omega_0})\right) + \omega_0 n_0$. Using Euler's rule we have

$$\hat{P}(e^{j\omega_0}) \approx \frac{\left|H(e^{j\omega_0})\right|^2}{L}\left|\sum_{n=0}^{L-1} \frac{e^{j\omega_0 n + j\theta} + e^{-j\omega_0 n - j\theta}}{2} e^{-j\omega_0 n}\right|^2 =$$

$$\frac{\left|H(e^{j\omega_0})\right|^2}{4L}\left|\sum_{n=0}^{L-1} e^0 + e^{-j2\omega_0 n - j2\theta}\right|^2 =$$

$$\frac{L\left|H(e^{j\omega_0})\right|^2}{4}\left|1 + \frac{1}{L}\sum_{n=0}^{L-1} e^{-j2\omega_0 n - j2\theta}\right|^2.$$

If L is large, $\omega_0 \neq 0$, and $\omega_0 \neq \pi$, then

$$\left| \frac{1}{L} \sum_{n=0}^{L-1} e^{-j2\omega_0 n - j2\theta} \right| \ll 1,$$

and hence

$$\hat{P}(e^{j\omega_0}) \approx \frac{L\left|H(e^{j\omega_0})\right|^2}{4}.$$

This value is independent of the shift n_0, and hence the averaged periodogram will have the same peak value.

(c) At the peak the periodogram is 20 dB $= 10^2$.

$$10^2 = \frac{L\left|H(e^{j\omega_0})\right|^2}{4} = \frac{10^3}{4|1-ke^{-j\pi/3}|^2} \Rightarrow$$

$$1 + k^2 - 2k\cos(\pi/3) = \frac{10}{4} \Rightarrow (k-1/2)^2 = \frac{7}{4} \Rightarrow$$

$$k = \frac{1}{2} \pm \frac{\sqrt{7}}{2}.$$

Only one root results in a causal and stable filter, namely

$$k = (1-\sqrt{7})/2 \approx -0.82.$$

(d) Quantization noise is introduced in the A/D converter as well as in the multiplication with k. The noise can be assumed to be white, zero-mean with a total power of $\sigma^2 = 2\Delta^2/12$. At the output the quantization noise has been colored by the filter $H(z)$. There are no sinusoidal peaks close to $\omega = \pi$ (the spectrum is smooth), hence the averaged periodogram can be approximated by the true spectrogram of the process as

$$\hat{P}(-1) \approx P(-1) = |H(-1)|^2\, \sigma^2.$$

Due to periodogram averaging, the error variance of the approximation can be regarded as small. Insert $\hat{P}(-1) = -12$ dB $= 1/16$, and compute Δ as

$$\Delta \approx \big(1 - k(-1)\big)\sqrt{\frac{6}{16}} = \frac{(3-\sqrt{7})\sqrt{6}}{8} \approx 0.11 \text{ [Volts]}.$$

5. **Solution :** The following is a simple illustration of the solution.

From Figure 1 it is clear that in order to cause a periodogram of the down sampled signal, where all signal components are located at the same frequency it must hold that

$$2\, f_{x,0} = \frac{1}{2\,D} \text{ for } f_{x,0} = \frac{1}{20} \Longrightarrow D = 5$$

Now, we have assumed that $Df_{x,0}$ lies in $[0, 0.5]$. Note that a similar result can be obtained if $Df_{x,0}$ lies in $[m, m+0.5]$ for some integer m so that it aliases to $[0, 0.5]$ (periodicity). Hence, we get different solutions for D as we vary m. Further, it is also shown in the figure that $f_{y,0} < 0.5$. Combining these, it can be shown that the output signals have the same frequency ($f_{y,0}, f_{y,0} < 0.5$) when D is a odd multiple of 5.

You could also use the equation for $y(m)$ in time domain to reach a similar conclusion.

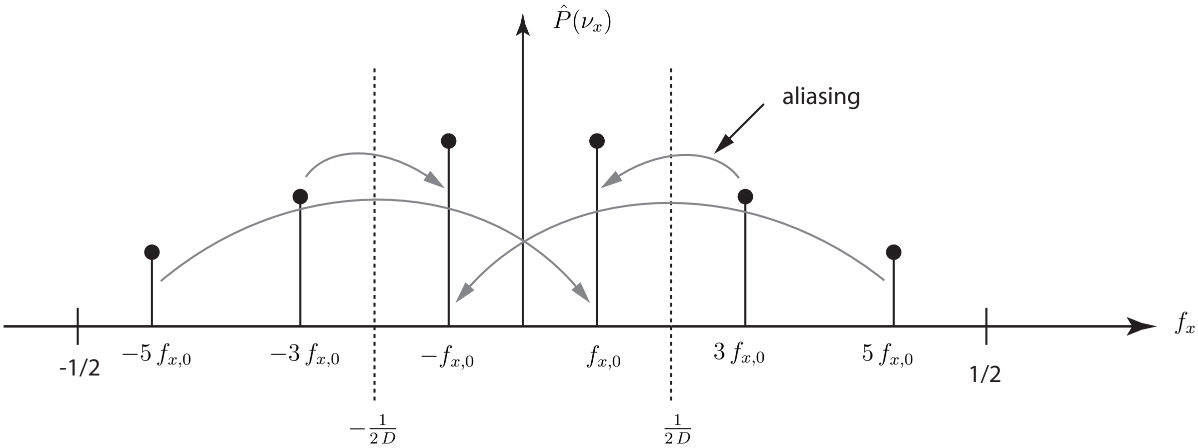

Figure 1: Spectrum of $x(n)$ and illustration of the aliasing effects due to the down sampling with a factor D.