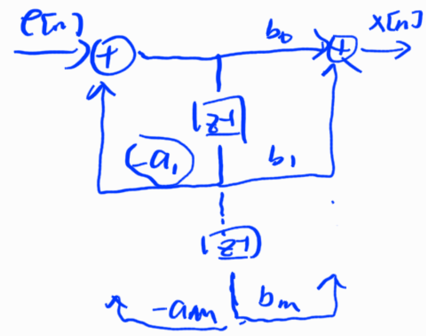
AR | MA



ARMA system. $e[n] \longrightarrow \boxed{H(z)} \longrightarrow x[n]$

$m+h$.

$$x[n] + \sum_{m=1}^{M} a_m x[n-m] = \sum_{m=0}^{M} b_m e[n-m]$$

$\times\, x[n-k] \rightarrow$ Yule-Walker

$$\underbrace{E(x[n]x[n-k])}_{r_x[k]} + \sum_{m=1}^{M} a_m \underbrace{E\{x[n-m]x[n-k]\}}_{r_x[m-k]} = \sum_{m=0}^{M} b_m E\{\underbrace{e[n-m]x[n-k]}_{r_{ex}[m-k]}\}$$

$$r_{ex}[\underline{m-k}] = h[m-k]$$

$$\begin{aligned} r_{ex}[k] &= E[x[n]e[n-k]] \\ &= E_n\{\underbrace{\sum_{q=-\infty}^{+\infty} h[q]e[n-q]}_{x[n]=h[n]*e[n]} \cdot e[n-k]\}. \\ &= \sum_{q=-\infty}^{+\infty} h[q]\, E\{\underbrace{e[n-q]\cdot e[n-k]}_{r_e[k-q]}\}. \\ &= \underbrace{\sum_{q=-\infty}^{+\infty} h[q]\cdot \underline{r_e[k-q]}} = h[k] * r_e[k] \end{aligned}$$

since. $e[n]$. white zero mean. unit variance.

$$r_e[k] = \begin{cases} 1 & k=0 \\ 0 & o.w \end{cases} = \delta[k]$$

$$\underline{= h[k]}.$$

Yule-Walker equation.

$$r_x[k] + \sum_{m=1}^{M} a_m r_x[m-k] = \sum_{m=0}^{M} b_m h[m-k]$$

$$a_m,\ b_m,\ \underbrace{h[m-k]}$$

Parametric Spectral Estimation.

Given $x[n]$. $n = 0, 1, 2, \cdots, N-1$.

① estimate $\hat{r}_x[k]$

② solve Yule-Walker equation.

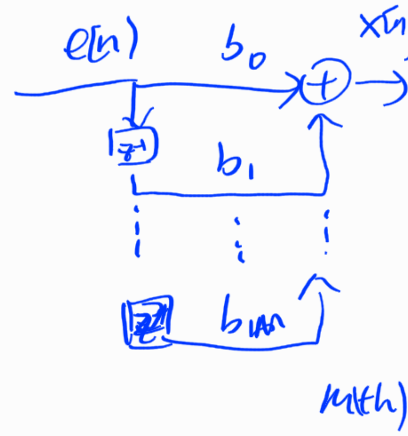
* not easy! a_m b_m.

③ $P_x(\nu) = \left| H(\nu) \right|^2$.

$$= \left| \frac{\sum b_m e^{-j2\pi\nu m}}{1 + \sum a_m e^{-j2\pi\nu m}} \right|^2.$$

MA : $x[n] = \sum_{m=0}^{M} b_m e[n-m]$

[FIR]



impulse response.

$$h[n] = \begin{cases} b_n & n=0,\dots,M. \\ 0. & o.w. \end{cases}$$

Yule-Walker equation,

$$E\{x[n]x[n-k]\} = \sum_0^M b_m E\{e[n-m]x[n-k]\}.$$

$$\Leftrightarrow \quad r_x[k] = \sum_0^M b_m r_{ex}[m-k] = \sum_0^M b_m h[m-k]$$

$$= \sum_{m=k}^{M} b_m b_{m-k}$$

$0 \le k \le M$

Solve. $\{b_m\}_0^M$

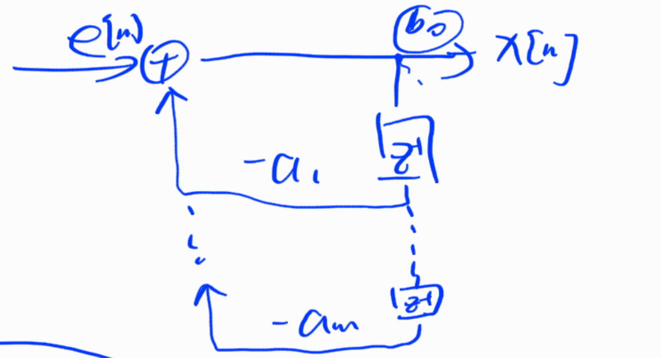
$$P_x(\nu) = |H(\nu)|^2 = \left|\sum_m b_m e^{-j2\pi\nu m}\right|^2.$$

$$= \left(\sum_m b_m e^{-j2\pi\nu m}\right)\left(\sum_n b_m e^{j2\pi\nu m}\right)$$

$$= \sum_m \sum_n b_n b_m e^{-j2\pi\nu(m-n)} = \underbrace{\sum_k \sum_n b_n b_{n-k} e^{j2\pi\nu k}}$$

$$= \underline{F_\nu(r_x(k))}$$

AR system (IIR),



$$H(z) = \frac{b_0}{1 + a_1 z^{-1} + \cdots a_m z^{-m}}$$

$$h[n] = Z^{-1}[H(z)]$$

$$x[n] + \sum_{m=1}^{M} a_m x[n-m] = b_0 e[n]$$

$$r_{ex}[k] = h[k]$$

$$\Leftrightarrow r_x[k] + \sum_{m=1}^{M} a_m r_x[m-k] = b_0 \underbrace{E[e[n]x[n-k]]} \cdots$$

x[n] depends on e[n]

$x[n-m]$ $m = 1 \cdots M$

$$E[e[n]x[n-k]] = E[e[n]]\,E[x[n-k]] = \begin{cases} 0 & k > 0 \\ b_0 & k = 0 \end{cases}$$

not depends on future $e[n+\tau]$ $x[n+\tau]$ $\tau > 0$

$$\Leftrightarrow r_x[k] + \sum_{m=1}^{M} a_m r_x[k-m] = b_0^2 \delta[k]$$

② $k = 0$, $r_x[0] + \sum_{m=1}^{M} a_m r_x[m] = b_0^2$

① $k = 1 \dots M$

$$\sum_{m=1}^{M} a_m r_x[k-m] = -r_x[k]$$

$$\underbrace{\begin{bmatrix} r_x[0] & r_x[1] & \cdots & r_x[m-1] \\ r_x[1] & r_x[0] & \cdots & r_x[m-2] \\ \vdots & & & \vdots \\ r_x[M-1] & r_x[m-2] & \cdots & r_x[0] \end{bmatrix}}_{\bar{R}_x} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}}_{\bar{a}} = - \underbrace{\begin{bmatrix} r_x[1] \\ \vdots \\ r_x[m] \end{bmatrix}}_{\bar{r}_x}$$

$$a = [\bar{R}_x]^{-1} \cdot \bar{r}_x$$

9.1. $s[n] = A\cos(2\pi f n + \phi)$ $\quad \phi \sim U[0, 2\pi]$

$e[n] = s[n] + a s[n-1] + b s[n-2]$ $\quad$ AR(2).

Yule-Walker equation.

$$r_s[k] + a r_s[k-1] + b r_s[k-2] = r_{ex}[k] \quad \begin{cases} 1 & k=0 \\ 0 & \text{o.w} \end{cases}$$

$$\begin{bmatrix} r_s[0] & r_s[1] \\ r_s[1] & r_s[0] \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = - \begin{bmatrix} r_s[1] \\ r_s[2] \end{bmatrix}$$

$r_s[k]$. $\quad s[n] = A\cos(2\pi f n + \phi)$

$$r_s[k] = E\{A\cos(2\pi f n + \phi) \cdot A\cos(2\pi f(n-k) + \phi)\}$$

$$= E\{A^2 \underbrace{\cos(2\pi f n + \phi)\cos(2\pi f(n-k) + \phi)}\}$$

$$\cos A \cdot \cos B = \frac{\cos(A+B) + \cos(A-B)}{2}$$

$$= E_n\left\{\frac{A^2}{2}\left[\cos(2\pi f(2n-k) + 2\phi) + \underbrace{\cos(2\pi f k)}\right]\right\}$$

$$= \frac{A^2}{2} \cdot \underbrace{E\{\cos(2\pi f(2n-k) + 2\phi)\}}_{\sim U[0, 2\pi]} + \frac{A^2}{2}\cos(2\pi f k).$$

$$= \lim_{N \to \infty} \frac{1}{N} \cdot \int_0^{2\pi} \underbrace{\frac{1}{2\pi} \cdot \cos(2\pi f(2n-k) + 2\phi)\, d\phi}.$$

$$= 0$$

$$= \frac{A^2}{2}\cos(2\pi f k) = r_s[k]$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = - \begin{bmatrix} r_s(0) & r_s(1) \\ r_s(1) & r_s(0) \end{bmatrix}^{-1} \begin{bmatrix} r_s(1) \\ r_s(2) \end{bmatrix}$$

$$= \begin{bmatrix} -2\cos(2\pi f) \\ 1 \end{bmatrix}.$$

9.2. $x[n] + a_1 x[n-1] + a_2 x[n-2] = b_{x,0}\, \underline{w[n]}$. AR(2)

white

$y[n] = x[n] + v[n]$

$q[n]$

$v[n] = b_0 q[n] + b_1 q[n-1]$. MA(1)

$\hat{P}_v(w) = 3 + 2\cos(w)$

$\hat{r}_y[k] = [5 \quad 2 \quad 0 \quad -1 \quad 0.5]$.

④ ③

? $\hat{P}_x(w) \leftarrow a_1\ a_2\ b_{x,0} \leftarrow$ Yule-Walker $\hat{r}_x[k]$

② ①

$P(v)$ $\cancel{b_0\ b_1}$ $\longrightarrow r_y(k),\ r_v[k]$

① . $\hat{P}_v(w) = 3 + 2\cos(w) = 3 + 2 \cdot \dfrac{e^{jw} + e^{-jw}}{2}$.

$$= 3 + e^{jw} + e^{-jw}$$

$$3\delta[k] \quad \delta[k+1] \quad \delta[k-1]$$

$r_v[k] = F^{-1}(\hat{P}_v(w)) = 3\delta[k] + \delta[k+1] + \delta[k-1]$

$$\begin{array}{cccccc} k= & 0 & 1 & 2 & 3 & 4 \\ = & [3 & 1 & 0 & 0 & 0] \end{array}$$

② . $r_v \quad r_y \rightarrow r_x$. $\quad v[n] = MA(1)$

$= b_0 q + b_1 q$.

$y[n] = \underline{x[n]} + \underline{v[n]}$

uncorrelated .

$q[n] \rightarrow \boxed{LTI} \rightarrow v[n]$

$\underset{=0}{E[q]} \longrightarrow \underset{=0}{E[v]}$

$$r_y[n] = E\{y[n]\, y[n-k]\}$$

$$= E\{(\underline{x[n]} + \underline{v[n]})(\underline{x[n-k]} + \underline{v[n-k]})\}$$

$$= r_x[\underline{k}] + r_v[k] + \cancel{E[x[n]]\, \underbrace{E[v[n-k]]}_{0}} + \cancel{E[x[n-k]]\, \underbrace{E[v[n]]}_{0}}$$

$$r_y[k] = r_x[k] + r_v[k]$$

$$\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \\ 0.5 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

③ . $x[n] + a_1 x[n-1] + a_2 x[n-2] = b_{x,0}\, w[n]$

$$\underbrace{\begin{bmatrix} r_x[0] & r_x[1] \\ r_x[1] & r_x[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = - \begin{bmatrix} r_x[1] \\ r_x[2] \end{bmatrix}}$$

$\rightarrow a_1 = -\frac{2}{3}$; $a_2 = \frac{1}{3}$.

$k=0$: $b_{x,0}^2 = 2 + a_1 = \frac{4}{3}$

④. $P_x(\nu) = |H(\nu)|^2$

$$= \left| \frac{b_{x,0}}{1 + a_1 e^{-j2\pi\nu} + a_2 e^{-j2\pi 2\nu}} \right|^2$$

$y[n] = [2 \quad 1 \quad 2]$ for $n = 0, 1, 2$.

$k=0 \qquad k=1 \qquad k=2$. $\qquad r[0] \qquad r[2]$

unb. $\hat{r}_y[k] = \frac{9}{3} = 3$. $\quad \frac{2+2}{2} = 2$, $\quad 4 \quad [3, 2, 4]$

$\frac{1}{N-k}$

b. $\hat{r}_y[k] = \frac{9}{3} = 3 \quad \frac{2+2}{3} = \frac{4}{3}$. $\quad \frac{4}{3} \quad [3, \frac{4}{3}, \frac{4}{3}]$

$\frac{1}{N}$

$\left|\frac{N-k}{N}\right| = 1$

Yule-Walker equation.

$$\begin{bmatrix} r_y[0] & r_y[1] \\ r_y[1] & r_y[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = - \begin{bmatrix} r_y[1] \\ r_y[2] \end{bmatrix}$$

9.3.

① $\hat{r}_y[k]$ unbiased $\frac{1}{14-k}$

$a_1 = \frac{2}{5} \quad a_2 = -\frac{8}{5}$

$$H_{①} = \frac{b_0}{\underbrace{1 + \frac{2}{5}z^{-1} - \frac{8}{5}z^{-2}}_{=0}}$$

ax^2+bx+c

$$z_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

$$P^{①}_{1,2} = -\frac{1}{5} \pm \sqrt{\frac{41}{5}}$$

$\approx -1.48 ; 1.08.$

$|P| > 1$

unstable.

② $\hat{r}_y[k]$: biased $\frac{1}{14}$

$a_1 = -\frac{4}{13} \quad a_2 = -\frac{4}{13}.$

$$H_{②} = \frac{b_0'}{\underbrace{1 - \frac{4}{13}z^{-1} - \frac{4}{13}z^{-2}}_{=0}}$$

$$P^{②}_{1,2} = \frac{2}{13} \pm \sqrt{\frac{4}{169} + \frac{4}{13}}$$

$\approx -0.42 ; 0.73.$

$|P| < 1$

stable.