KTH, INFORMATION SCIENCE AND ENGINEERING

SCHOOL OF ELECTRICAL ENGINEERING AND COMPUTER SCIENCE

Digital Signal Processing EQ2300 / 2E1340

Final Examination 2020–01–11, 09:00–14:00 Examples of Solutions

1. a) We have

$$\{1,\ 2,\ 3,\ 0\} \circledast_4 \{\ 1,\ 1,\ 1,\ 0\} = \{1,\ 2,\ 3, 0\}+\{0,\ 1,\ 2,\ 3\}+\{3,\ 0,\ 1,\ 2\} = \{4,\ 3,\ 6, 5\}\,.$$

b) It is easiest if we note that the signal $x[n]$ can be written as

$$x[n] = (-1)^n = e^{j\pi n} = e^{j2\pi\frac{4n}{8}}$$

from which it immediately follows that $X_8[k] = 8\delta[k-4]$ for $k = 0, \ldots, 7$.

c) The Yule-Walker equations state that

$$r_x[0] + a_1 r_x[0] = b_0^2$$

and

$$r_x[1] + a_1 r_x[0] = 0\,.$$

From the second equation it follows that $a_1 = -\frac{r_x[1]}{r_x[0]} = -\frac{1}{2}$. Inserted into the first equation yields $b_0^2 = \frac{3}{4}$. The AR1 filter has frequency response

$$H(\nu) = \frac{b_0}{1 + a_1 e^{-j2\pi\nu}}$$

which implies that

$$P_x(\nu) = |H(\nu)|^2 = \frac{b_0^2}{1 + a_1^2 + 2a_1\cos(2\pi\nu)} = \frac{\frac{3}{4}}{\frac{5}{4} - \cos(2\pi\nu)} = \frac{3}{5 - 4\cos(2\pi\nu)}\,.$$

d) Assume that $h_1[n]$ has length $N_1 = M_1 + 1$ (order M_1) and $h_2[n]$ has length $N_2 = M_2 + 1$ (order M_2), where the length is defined as the number of non-zero taps. Then $h[n]$ will have length $N_1+N_2-1 = (M_1+1)+(M_2+1)-1 = M_1+M_2+1$ (order $M = M_1 + M_2$). It is thus an FIR filter. Since $h_1[n]$ and $h_2[n]$ are Type I FIR filters, we have that $h_1[n] = h_1[M_1 - n]$ and $h_2[n] = h_2[M_2 - n]$. It follows that

$$h[n] = \sum_{m=-\infty}^{\infty} h_1[m]h_2[n-m]$$

and

$$\begin{aligned}
h[M-n] &= \sum_{m=-\infty}^{\infty} h_1[m]h_2[M-n-m] = \sum_{m=-\infty}^{\infty} h_1[m]h_2[M_1+M_2-n-m] \\
&= \left[k = M_1 - m\right] = \sum_{k=-\infty}^{\infty} h_1[M_1-k]h_2[M_1+M_2-n-(M_1-k)] \\
&= \sum_{k=-\infty}^{\infty} h_1[M_1-k]h_2[M_2-(n-k)] = \sum_{k=-\infty}^{\infty} h_1[k]h_2[n-k] = h[n]
\end{aligned}$$

which establishes that $h[n]$ has the symmetry required to be a Type I FIR filter.

2. a) i. If $N = 256$ and $L = 16$, then we get $K = N/L = 16$ blocks to average over. Reading from the summary notes we can see that the variance of Bartlett's method is

$$\mathrm{Var}\{\hat{P}_x(\nu)\} \approx \frac{1}{K} P_x^2(\nu) = \frac{1}{16} P_x^2(\nu)\,.$$

ii. Since Bartlett's method uses rectangular windows, and since the block length is $L = 16$, we get a resolution of

$$\Delta\nu = \frac{0.89}{L} = \frac{0.89}{16} = 0.0556\,.$$

b) The bias of Bartlett's method is

$$\mathrm{E}\{\hat{P}_x^{\mathrm{B}}(\nu)\} = P_x(\nu) \circledast \frac{1}{L} |W_{\mathrm{R}}^{(L)}(\nu)|^2$$

where $W_{\mathrm{R}}^{(L)}(\nu)$ is the discrete Fourier transform of the rectangular window of length L, the window given by $w_{\mathrm{R}}[n] = 1$ for $n = 0, \ldots, L-1$ and $w_{\mathrm{R}}[n] = 0$ elsewhere. The bias of Blackman-Tuley's method is

$$\mathrm{E}\{\hat{P}_x^{\mathrm{BT}}(\nu)\} \approx P_x(\nu) \circledast W^{(2M+1)}(\nu)$$

where $W^{(2M+1)}(\nu)$ is the discrete Fourier transform of a triangular (Bartlett) window of length $2M+1$ that is also symmetric around zero, i.e., the window given by

$$w[n] = \frac{M+1-|n|}{M+1}$$

for $|n| \le M$ and $w[n] = 0$ elsewhere. We thus wish to choose M such that $W^{(2M+1)}(\nu) = \frac{1}{L}|W_{\mathrm{R}}^{(L)}(\nu)|^2$. In the time domain this statement is equal to

$$w[n] = \frac{1}{L} w_{\mathrm{R}}^{(L)}[n] * w_{\mathrm{R}}^{(L)}[-n]\,,$$

i.e., the statement that the triangular window is obtained by the convolution of a rectangular window with itself. If the rectangular window has length L the triangular window will be non-zero for $|n| < L$. In other words, the max lag is given by $M = L - 1$. It is straightforward to verify that the normalisation will also be correct for this choice of M.

c) The variance of Blackman-Tukey's method with a triangular window and max-lag M is given in the summary notes as

$$\mathrm{Var}\{\hat{P}_x^{\mathrm{BT}}(\nu) \approx \frac{2M}{3N} P_x^2(\nu)\,.$$

With $M = L - 1 \approx L = 16$, we get

$$\mathrm{Var}\{\hat{P}_x^{\mathrm{BT}}(\nu)\} \approx \frac{2}{3L} P_x^2(\nu) = \frac{2}{3 \times 16} P_x^2(\nu)$$

which is a reduction of $\frac{2}{3}$ over Bartlett's method.

d) In Bartlett's method we have that

$$P_x^{\mathrm{B}}(\nu) = \frac{1}{K} \sum_{k=0}^{K-1} \frac{1}{L} \left|\mathcal{F}\{x_k[n]\}\right|^2 = \frac{1}{K} \sum_{k=0}^{K-1} \hat{P}_{x,k}(\nu)$$

where $\hat{P}_{x,k}(\nu)$ is the periodogram estimate computed over the kth block. Thus,

$$\hat{r}_x^{\mathrm{B}}[m] = \frac{1}{K}\sum_{k=0}^{K-1}\hat{r}_{x,k}[m]$$

where $\hat{r}_{x,k}[m]$ is the periodogram autocorrelation estimate computed over the kth block. Note that

$$\hat{r}_{x,k}[0] = \frac{1}{L}\sum_{l=0}^{L-1}x_k^2[l]$$

where $x_k[l] = x[l+kL]$, which implies that

$$\hat{r}_x^{\mathrm{B}}[0] = \frac{1}{K}\sum_{k=0}^{K-1}\frac{1}{L}\sum_{l=0}^{L-1}x_k^2[l] = \frac{1}{N}\sum_{n=0}^{N-1}x^2[n]\,.$$

For Blackman-Tukey's method we have

$$\hat{r}_x^{\mathrm{BT}}[k] = w[k]\frac{1}{N}\sum_{n=0}^{N-|k|}x[n]x[n+|k|]\,.$$

Since $w[0]=1$, it follows that $\hat{r}_x^{\mathrm{B}}[m] = \hat{r}_x^{\mathrm{BT}}[k]$ which is what we needed to prove. What this essentially states is that both estimators obtain the same estimate for the total power. Most well designed estimators will. However, the estimate of the power spectral density at any one specific frequency can be different, and is generally worse of for Bartlett's method in comparison with Blackman-Tukey's method.

3. a) From the table in the summary notes, we can see that a length $M = 1000$ filter can be implemented with $C_N = 15.95 \approx 16$ complex valued multiplications per sample. This implies that it can be implemented with $16 \times 4 = 64 < 100$ real valued multiplications per sample. Thus, the platform is capable fo implementing a filter of length $M = 1000$ as it can spend up to 100 real valued multiplications per sample.

b) For a given FFT length N and filter length M we have a total number of complex valued multiplications given by

$$C_N = \frac{N\log_2 2N}{N-M+1}\,.$$

As there are 4 real valued multiplications per complex valued multiplication, and we can spend at most 100 real valued multiplications per sample, we get

$$4 \times \frac{N\log_2 2N}{N-M+1} \leq 100\,.$$

From this expression we get

$$M \leq N - 4\times\frac{N\log_2 2N}{100} + 1 = 1024 - \frac{4\times 1024\times 11}{100} + 1 \approx 574\,.$$

The filter can thus not be longer than 574 taps.

c) With $N = 1024$ and $M = 525$ we collect $N - M + 1 = 500$ new samples per block. With a sample rate of 1 kHz the overlap save block processing is called two times every second, and since each block required two FFT operations we get 4 FFT operations per second.

d) We have to wait for the buffer to fill up. This can take up to 0.5second as we need to collect 500 samples at a rate of 1 kHz to fill up the buffer. After this the total number of real valued multiplication that need to be performed are

$$4 \times N \log_2 2N = 4 \times 1024 \times 11 \approx 4 \times 10^4 .$$

As the platform can do 10^5 multiplications per second, this will take up to 0.4 s. Thus, the total latency will be around 0.9 s for the overlap save. This is significantly longer than the time between incoming and outgoing sample which only 1 ms. This is the penalty we have for filtering in the frequency domain.

4. a) Let $v[n]$ be the signal at the output of the delay (z^{-1}). Then we have that

$$V(z) = z^{-1}\big(X(z) - \gamma_1 V(z)\big)$$

and

$$Y(z) = \gamma_1\big(X(z) - \gamma_1 V(z)\big) + V(z) = \gamma_1 X(z) + (1-\gamma_1^2)V(z) .$$

From the first equation we have

$$V(z) = \frac{z^{-1}}{1+\gamma_1 z^{-1}} X(z) .$$

Plugging this into the second equation yields

$$Y(z) = \left(\gamma_1 + \frac{(1-\gamma_1^2)z^{-1}}{1+\gamma_1 z^{-1}}\right) X(z) = \frac{\gamma_1 + \gamma_1^2 z^{-1} + z^{-1} - \gamma_1^2 z^{-1}}{1+\gamma_1 z^{-1}} X(z)$$

and the transfer function

$$H(z) = \frac{\gamma_1 + z^{-1}}{1+\gamma_1 z^{-1}} = \frac{z^{-1}A(z^{-1})}{A(z)}$$

where $A(z) = 1 + \gamma_1 z^{-1}$. It is thus clear that the filter is an all-pass filter. To get the impulse response we can write the transfer function as

$$H(z) = \frac{\gamma_1}{1+\gamma_1 z^{-1}} + \frac{z^{-1}}{1+\gamma_1 z^{-1}} = \gamma_1 \frac{z}{z+\gamma_1} + z^{-1}\frac{z}{z+\gamma_1} .$$

Via tables it follows that

$$h[n] = \gamma_1(-\gamma_1)^n u[n] + (-\gamma_1)^{n-1}u[n-1] = \gamma_1\delta[n] + (1-\gamma_1^2)(-\gamma_1)^{n-1}u[n-1] .$$

b) The noise created by the multiplication with γ_1 directly pass to the output. The noise created by the multiplication by $-\gamma_1$ must pass through the entire system before reaching the output. To deal with this we begin by evaluating

$$\sum_{n=-\infty}^{\infty} h^2[n] = \gamma_1^2 + \big(1-\gamma_1^2\big)^2 \sum_{n=1}^{\infty}(\gamma_1^2)^{n-1} = \gamma_1^2 + \big(1-\gamma_1^2\big)^2 \frac{1}{1-\gamma_1^2} = 1 .$$

This implies that the noise will not change in power from where it is inserted to when it reach the output. This can retroactively be understood by noting that the defining property of the system is that

$$\big|H(\nu)\big| = 1 ,$$

which means that the power spectral density of the noise is not changed as it passes though the system. Thus, the total amount of noise at the output will thus be

$$2 \times \frac{2^{-2B}}{12}$$

which corresponds to the noise added from the multiplications with γ_1 and $-\gamma_1$.

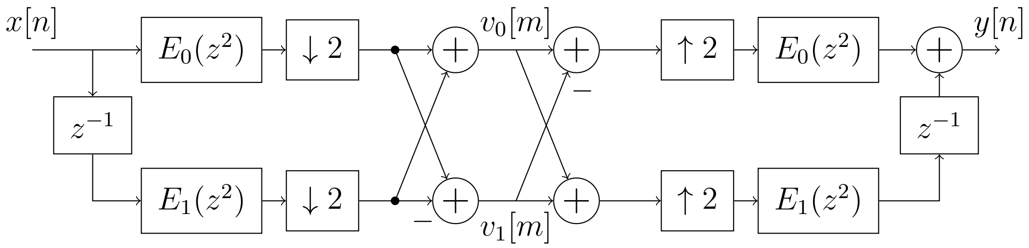
c) If one has made the realisation made at end of problem b), this problem is straight forwards. The noise introduced by multiplication with $-\gamma_2$ will also pass a system with frequency response

$$\big|H(\nu)\big| = 1\,.$$

Thus, the amount of noise power it contributes at the output is also

$$\frac{2^{-2B}}{12}\,.$$

5. a) By using the noble identities we can rewrite the system as follows.



Further interchanging the order of the up and down sampling and the summations yields that

$$\begin{aligned}
H_0(z) &= E_0(z^2) + z^{-1}E_1(z^2)\,,\\
H_1(z) &= E_0(z^2) - z^{-1}E_1(z^2)\,,\\
G_0(z) &= E_0(z^2) + z^{-1}E_1(z^2)\,,\\
G_1(z) &= -E_0(z^2) + z^{-1}E_1(z^2)\,.
\end{aligned}$$

The components $E_0(z)$ and $E_1(z)$ are nothing more than the polyphase components of $H_0(z)$, and will thus have the relation to $H_0(z)$ specified by the polyphase decomposition.

b) Note that since $G_0(z) = H_1(-z)$ and $G_1(z) = -H_0(-z)$, we will not have any aliasing. Further,

$$Y(z) = \frac{1}{2}\Big[G_0(z)H_0(z) + G_1(z)H_1(z)\Big]X(z).$$

Evaluating the expression in the brackets yield

$$\Big(E_0(z^2)+z^{-1}E_1(z^2)\Big)\Big(E_0(z^2)+z^{-1}E_1(z^2)\Big)+\Big(-E_0(z^2)+z^{-1}E_1(z^2)\Big)\Big(E_0(z^2)-z^{-1}E_1(z^2)\Big)$$

$$= 4z^{-1}E_0(z^2)E_1(z^2)$$

and

$$T(z) = 2z^{-1}E_0(z^2)E_1(z^2)$$

as was to be proved.

c) We have

$$E_0(z^2) = \frac{1}{\sqrt{2}} \times \frac{c_0 + z^{-2}}{1 + c_0 z^{-2}}$$

and

$$E_1(z^2) = \frac{1}{\sqrt{2}} \times \frac{c_1 + z^{-1}}{1 + c_1 z^{-1}}\,.$$

We have that

$$H_0(z) = E_0(z^2)+z^{-1}E_1(z^2) = \frac{1}{\sqrt{2}} \frac{(c_0 + z^{-2})(1 + c_1 z^{-1}) + z^{-1}(c_1 + z^{-1})(1 + c_0 z^{-2})}{(1 + c_0 z^{-2})(1 + c_1 z^{-1})}$$

$$= \frac{1}{\sqrt{2}} \frac{c_0 + c_1 z^{-1} + (1 + c_0 c_1) z^{-2} + (1 + c_0 c_1) z^{-3} + c_1 z^{-4} + c_0 z^{-5}}{1 + (c_0 + c_1) z^{-2} + (c_0 + c_1) z^{-4}} .$$

From this expression the coefficients can be directly read, and the filter order can be seen to be $N = 5$. It is not obvious how to select c_0 and c_1 so that one obtain a good low pass filter, but it can be done as shown in the example.