SIGNAL PROCESSING

DEPARTMENT OF ELECTRICAL ENGINEERING

E 103 **Digital Signalbehandling** EQ2300/ 2E1340

Final Examination 2008–06–02, 14.00–19.00
Solutions

1. (a) Let $\underline{X} = [X(0), X(1), X(2), X(3)]^T$ and $\underline{x} = [x(0), x(2)]^T$. Exploiting the matrix-vector product of obtaining DFT, we have,

$$\begin{aligned} \underline{X} &= \mathbf{F}\underline{x} \\ \mathbf{F} &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \end{aligned}$$

- Now, $[X(0), X(1)]^T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} [x(0), x(2)]^T$. Since the transformation matrix is invertible, it is *possible* to recover $\{x(0), x(2)\}$.
- Now, $[X(0), X(2)]^T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} [x(0), x(2)]^T$. Since the transformation matrix is not invertible, it is *impossible* to recover $\{x(0), x(2)\}$.

(b)

$$\begin{aligned} X(z^2) &\leftrightarrow \{\cdots, x(-1), 0, \underset{\uparrow}{x(0)}, 0, x(1), \cdots\} \\ X(z^{-2}) &\leftrightarrow \{\cdots, x(1), 0, \underset{\uparrow}{x(0)}, 0, x(-1), \cdots\} \\ Y(z^2) &\leftrightarrow \{\cdots, y(-1), 0, \underset{\uparrow}{y(0)}, 0, y(1), \cdots\} \\ Y(-z^2) &\leftrightarrow \{\cdots, -y(-1), 0, \underset{\uparrow}{y(0)}, 0, -y(1), \cdots\} \\ z^{-1}Y(-z^2) &\leftrightarrow \{\cdots, 0, -y(-1), \underset{\uparrow}{0}, y(0), 0, \cdots\} \\ X(z^2) + z^{-1}Y(-z^2) &\leftrightarrow \{\cdots, y(-2), x(1), -y(-1), \underset{\uparrow}{x(0)}, y(0), x(-1), -y(1), \cdots\} \end{aligned}$$

(c) The problem involves incorporating negative lags and introducing sufficient zeros in order to compute the 1024 FFT. Consider,

$$\underline{r} = [r(0), r(1), \ldots, r(127), 0, 0, \ldots, 0, 0, r^*(127), r^*(126), \ldots, r^*(1)]^T$$

where sufficient zeros are padded so that $\underline{r}$ is 1024×1 vector. Use the modulo property of DFT and the conjugate symmetry of autocorrelation to show that the 1024 point FFT of $\underline{r}$ does give the required power spectrum.

2. (a) The autocorrelation lags are computed as,

$$\begin{aligned} r_{yy}(k) &= E\{y(n+k)[y(n)]^*\} \\ &= 2E\{\sin(\omega_0 n + \phi)\sin(\omega_0(n+k) + \phi)\} \\ &= \cos(\omega_0 k) - E\{\cos(\omega_0(2n+k) + 2\phi)\} \quad \text{(Using trigonometric identities)} \\ r_{yy}(k) &= \cos(\omega_0 k) \end{aligned}$$

where the last equality follows from uniform distribution of ϕ and the periodicity of $\cos(\omega_0(2n+k)+2\phi)$.

(b) The values a_1 and a_2 can be solved using the standard Yule-Walker's equations to obtain

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = -\begin{pmatrix} r_{yy}(0) & r_{yy}(-1) \\ r_{yy}(1) & r_{yy}(0) \end{pmatrix}^{-1} \begin{pmatrix} r_{yy}(1) \\ r_{yy}(2) \end{pmatrix}$$

This yields $a_1 = -2\cos(\omega_0)$ and $a_2 = 1$. Using the standard equations, we can also find the modelling error as

$$\begin{aligned} \sigma_e^2 &= r_{yy}(0) + a_1 r_{yy}(1) + a_2 r_{yy}(2) \\ &= 0 \end{aligned}$$

(c) Choosing a higher order predictor will not improve the accuracy as the error in modelling is zero and cannot be reduced further. It can also be intuitively seen from the fact that $\sin(\cdot)$ can be obtained from a two pole filter with poles at $e^{j\omega_0}$ and $e^{-j\omega_0}$. Also note that the random process under consideration is a *predictable* random process.

3. (a) Equivalence of two systems:

- When $D = 3, I = 3$, the two systems are not equivalent. It should be shown that,

$$\begin{aligned} y(n) &= \{\cdots, x(-3), 0, 0, \underset{\uparrow}{x(0)}, 0, 0, x(3), \cdots\} \\ z(n) &= x(n) \end{aligned}$$

- When $D = 3, I = 2$, it can be shown (should be shown in exam) that

$$\begin{aligned} y(n) &= \{\cdots, x(-3), 0, \underset{\uparrow}{x(0)}, 0, x(3), \cdots\} \\ z(n) &= \{\cdots, x(-3), 0, \underset{\uparrow}{x(0)}, 0, x(3), \cdots\} \end{aligned}$$

Hence, the two systems are equivalent.

(b) Use partial fractions to show that,

$$\begin{aligned} H(z) &= \frac{1}{(1-az^{-1})(1-bz^{-1})} \\ &= \frac{a}{(a-b)(1-az^{-1})} + \frac{b}{(b-a)(1-bz^{-1})} \end{aligned}$$

Using inverse $z-$ transforms or by inspection, it follows that,

$$\begin{aligned} H(z) &= H_0(z^2) + z^{-1}H_1(z^2) \\ H_0(z) &= \frac{a}{(a-b)(1-a^2z^{-1})} + \frac{b}{(b-a)(1-b^2z^{-1})} \\ H_1(z) &= \frac{a^2}{(a-b)(1-a^2z^{-1})} + \frac{b^2}{(b-a)(1-b^2z^{-1})} \end{aligned}$$

They can be further simplified as,

$$\begin{aligned} H_0(z) &= \frac{1+abz^{-1}}{(1-a^2z^{-1})(1-b^2z^{-1})} \\ H_1(z) &= \frac{a+b}{(1-a^2z^{-1})(1-b^2z^{-1})} \end{aligned}$$

The specific filters are obtained by choosing $a = -0.9$ and $b = 0.8$.

4. (a) The resolution does not vary with i. The resolution is determined by the periodogram computation ($\hat{P}_i(f)$) and is approximately of the order $\frac{1}{N}$.

(b) It can be shown that,

$$\begin{aligned} \hat{F}_2(f) &= \alpha(1-\alpha)\hat{P}_1(f) + (1-\alpha)\hat{P}_2(f) \\ \sigma^2 - E(\hat{F}_2(f)) &= \alpha^2\sigma^2 \end{aligned}$$

The last equality follows from $E(\hat{P}_i(f)) = \sigma^2$. Hence the bias in $\hat{F}_2(f)$ is $\alpha^2\sigma^2$. Using the expression for $\hat{F}_2(f)$ presented above and noting that the signal is a white Gaussian noise, we can write,

$$\begin{aligned} \text{Variance}(\hat{F}_2(f)) &= \alpha^2(1-\alpha)^2\text{Variance}(\hat{P}_1(f)) + (1-\alpha)^2\text{Variance}(\hat{P}_2(f)) \\ &= (1+\alpha^2)(1-\alpha)^2\sigma^4 \end{aligned}$$

The last equality follows from variance properties of the periodogram.

(c) The tuples, $(\alpha, \text{Variance}, \text{Bias}, MSE)$ are given below:

- $(0.3, 0.5341\sigma^2, 0.09\sigma^4, 0.5422\sigma^4)$
- $(0.6, 0.2176\sigma^2, 0.36\sigma^4, 0.3472\sigma^4)$
- $(0.9, 0.0181\sigma^2, 0.81\sigma^4, 0.6742\sigma^4)$

Clearly, we see a Bias variance tradeoff. The Bias increases with α while the variance reduces. It is also interesting to note the variations in MSE computed above ($MSE = Bias^2 + Variance$).

5. (a) Cascade implementation: The minimum error is obtained by implementing $\frac{1}{1-0.95z^{-1}}$ first followed by $\frac{1}{1-0.75z^{-1}}$ and then the multiplier. The resulting noise variance calculation is detailed below:

- Contribution to noise from $\frac{1}{1-0.95z^{-1}}$: $\sigma_e^2(\sum_{n=0}^{\infty} h^2(n))$, where $h(n)$ is the impulse response of $H(z)$.
- Contribution to noise from $\frac{1}{1-0.75z^{-1}}$: $\sigma_e^2(0.2)^2\sum_{n=0}^{\infty} h_1^2(n)$, where $h_1(n)$ is the impulse response of $\frac{1}{1-0.75z^{-1}}$. Note that we include the attenuation by (0.2).
- Contribution to noise by the multiplier : σ_e^2.

The total noise contribution, is therefore,

$$N_1 = \sigma_e^2\left(\sum_{n=0}^{\infty} h^2(n) + (0.2)^2\sum_{n=0}^{\infty} h_1^2(n) + 1\right) \tag{1}$$

From the given data, $\sum_{n=0}^{\infty} h^2(n) = 5.5856$ and $\sum_{n=0}^{\infty} h_1^2(n) = 2.2857$. Hence the total noise for cascade implementation is $N_1 = (5.5856 + (0.2)^2 2.2857 + 1)\sigma_e^2$ or $N_1 = 6.6770\sigma_e^2$

(b) Now

$$H(z) = \frac{-0.75}{1-0.75z^{-1}} + \frac{0.95}{1-0.95z^{-1}}$$

As in the cascade method, the quantization noise for the parallel implementation is minimized when the multipliers -0.75 and 0.95 appear after the first order sections. In other words, implement $\frac{1}{1-0.75z^{-1}}$ first followed by the multiplier on one of the branches. Perform a similar operation for the second branch. The resulting noise calculation is detailed below:

- Contribution to noise from $\frac{1}{1-0.95z^{-1}}$: $(0.95)^2\sigma_e^2(\sum_{n=0}^{\infty} h_2^2(n))$, where $h_2(n)$ is the impulse response of $\frac{1}{1-0.95z^{-1}}$. The effect of the multiplier is also absorbed.
- Contribution to noise from $\frac{1}{1-0.75z^{-1}}$: $(0.75)^2\sigma_e^2(\sum_{n=0}^{\infty} h_1^2(n))$. Note that we include the attenuation by 0.75.
- Contribution to noise by the multipliers : $2\sigma_e^2$.

The total noise contribution, is therefore,

$$N_2 = \sigma_e^2 \left((0.95)^2 \sum_{n=0}^{\infty} h_1^2(n) + (0.75)^2 \sum_{n=0}^{\infty} h_2^2(n) + 2 \right) \tag{2}$$

This can be simplified to $N_2 = 12.5421\sigma_e^2$. Thus cascade implementation seems better. ´