KTH, INFORMATION SCIENCE AND ENGINEERING

SCHOOL OF ELECTRICAL ENGINEERING AND COMPUTER SCIENCE

Digital Signal Processing EQ2300 / 2E1340

Final Examination 2019–01–11, 08:00–13:00 Examples of Solutions

1. a) i. The complexity of overlap-add in terms of complex valued multiplications per sample is

$$C_N = \frac{N \log_2 2N}{N - M + 1}$$

where M is the filter length and where N of the form $N = 2^p$ is the FFT length. Length's that minimized the complexity are tabulated in the summary notes, and the minimum complexity when $M = 20$is given by $N = 128$, for which $C_N = 9.39$. $N = 128$ is the most appropriate length. (2p)

ii. Each block takes in $N - M + 1 = 109$ new signal samples. (1p). As $500/109 \approx 4.58$ we have to use 5 blocks, where 5 is the smallest integer greater than of equal to $500/109$. (1p)

b) i. The resolution $\Delta\nu$ of the modified periodogram with $N = 100$ and a Blackman window is $1.68/100 = 0.0168$ (1p). The frequency of the first to sinusoids are $\nu_1 = 0.1$ ($\omega_1 = 2\pi\nu_1$) and $\nu_2 = 0.15$, and as $0.15 - 0.1 = 0.05 > 0.0168$ we will have distinct spectrum peaks (1p).

ii. The third sinusoid has a power that is $20 \log_{10}(0.01) = -40$ dB below the main peaks (1p). As the Blackman window has a side-lobe level of -58 dB, the spectral peak from $x_3[n]$ should be visible in the spectrum. (1p)

c) The spectrum estimator is a Welch estimator, based on averaging modified periodograms with Blackman windows at 50% overlap which can be seen by the line $D = L/2$. (2p)

2. a) A model order of $M = 1$ is justified, because $b_{0,M}^2$already reaches it minimum value for $M = 1$, indication that a model of order $M = 1$ is sufficient to generate the observed data. (1p)

b) From the image we see that $b_0^2 = 1$. From the text we have that $\hat{r}_x[0] = 4/3$, and $\hat{r}_x[1] = 2/3$ from the text. From the Yule-Walker equations we have for $k = 0$ that

$$r_x[0] + a_1 r_x[1] = b_0^2$$

and for $k = 1$ that

$$r_x[k] + a_1 r_x[0] = 0\,.$$

Substituting $r_x[k]$ with $\hat{r}_x[k]$ we get $2/3 + a_1 4/3 = 0$ or $a_1 = -1/2$ (2p). Inserting this in the first equation (for $k = 0$) yields $4/3 - 1/2 \times 2/3 = 1 = b_0^2$ in the first place, confirming what we could tell from the image. One could also obtain b_0^2 this way if one did not read it off the figure. The spectrum estimate becomes

$$\hat{P}_{x,\mathrm{AR}}(\nu) \stackrel{(1p)}{=} \frac{|b_0|^2}{|1 + a_1 e^{-j2\pi\nu}|^2} = \frac{b_0^2}{1 + a_1^2 + 2a_1 \cos(2\pi\nu)} \stackrel{(1p)}{=} \frac{1}{\frac{5}{4} - \cos(2\pi\nu)}\,.$$

Explicitly computing some of the values of the estimate yields $\hat{P}_{x,\mathrm{AR}}(0) = 4$ and $\hat{P}_{x,\mathrm{AR}}(\frac{1}{2}) = \frac{4}{9}$ which seems to match the numerical illustration.

c) For $M = 1$ the Blackman-Tukey setimate with a rectangular window is just the discrete time Fourier transform (DTFT) of

$$\{r_x[-1],\, \underset{\uparrow}{r_x[0]}, r_x[1]\} = \{\tfrac{2}{3},\, \underset{\uparrow}{\tfrac{4}{3}},\, \tfrac{2}{3}\}$$

where we use $r_x[-1] = r_x[1]$ (1p). The DTFT becomes (2p)

$$\hat{P}_{x,\mathrm{BT}}(\nu) = \frac{2}{3}e^{j2\pi\nu} + \frac{4}{3} + \frac{2}{3}e^{-j2\pi\nu} = \frac{4}{3} + \frac{4}{3}\cos(2\pi\nu)$$

In this case the numerical illustration in the original exam was unfortunately incorrectly plotted in the original exam. No point have been deducted due to errors caused by this misleading piece of information.

d) We have for the BT spectrum estimator that

$$\mathrm{Var}\left\{\hat{P}_{x,\mathrm{BT}}(\nu)\right\} \approx \left[\frac{1}{N}\sum_{k=-M}^{M} w^2[k]\right\} P_x^2(nu)\,.$$

We obtain

$$\frac{1}{N}\sum_{k=-M}^{M} w^2[k] = \frac{3}{1024}$$

and as $\hat{P}_{x,\mathrm{AR}} \approx P_x(\nu)$ we get (1p)

$$\mathrm{Var}\left\{\hat{P}_{x,\mathrm{BT}}(\nu)\right\} \approx \frac{3}{1024}\left[\frac{1}{\frac{5}{4} - \cos(2\pi\nu)}\right]^2$$

which is quite small, e.g., a factor $3/1024$ smaller than the variance of the Peridogram. This said, in both the case with the correct BT spectrum or the incorrectly plotted spectrum, the BT spectrum is not an appropriate spectrum estimator. This is because the bias deviates greatly from the (known korrect) AR model spectrum (1p).

3. a) For each cases we have:

- Type I for $M = 2$:

  $$H(\nu) = h[0] + h[1]e^{-j2\pi\nu} + h[2]e^{-j4\pi\nu} = e^{-j2\pi\nu}\left(2h_0\cos(2\pi\nu) + h_1\right). \qquad (1p)$$

  Hence, from the diagram of the filter, we obtain

  $$\begin{cases} |H(0)| = |2h_0 + h_1| = 1, \\ |H\left(\frac{1}{4}\right)| = |h_1| = 1, \\ |H\left(\frac{1}{2}\right)| = |-2h_0 + h_1| = 0. \end{cases} \qquad (1p)$$

  The equations does not have an answer because it follows from the first and the last ones that $|h_1| = 1/2$, which contradicts with the second one. (1p).

- Type III and IV:

  $$h[n] = -h[M-n] \Longrightarrow H(\nu) = -e^{-j2\pi\nu M}H(-\nu). \qquad (1p)$$

  By considering the case $\nu = 0$, we obtain that

  $$H(0) = -H(0) \Longrightarrow H(0) = 0. \qquad (1p)$$

  However, as we can see in the lowpass filter diagram, $|H(0)| = 1 > 0$, which is a contradiction.

b) For Type II FIR filter we have that

$$
\begin{aligned}
H(\nu) &= h_0 + h_1 e^{-j2\pi\nu} + h_1 e^{-j4\pi\nu} + h_0 e^{-j6\pi\nu} \qquad \text{(1p)} \\
&= 2e^{-j3\pi\nu} \left(h_0 \cos(3\pi\nu) + h_1 \cos(\pi\nu)\right). \qquad \text{(0.5p)}
\end{aligned}
$$

Hence, the unknown variables are $h_0 := h[0]$ and $h_1 := h[1]$. The equations are

$$
\begin{aligned}
|H(0)| = 1 &\Longrightarrow |h_0 + h_1| = \frac{1}{2}, \qquad \text{(1p)} \\
\left|H\left(\frac{1}{4}\right)\right| = 1 &\Longrightarrow |h_0 - h_1| = \frac{1}{\sqrt{2}}, \qquad \text{(1p)}
\end{aligned}
$$

Note that the equation $H(1/2) = 0$ (0.5p) is always satisfied for Type II FIR filters. Therefore, the valid answers are

$$
\begin{aligned}
(h_0, h_1) = &\boxed{\left(\frac{1+\sqrt{2}}{4}, \frac{1-\sqrt{2}}{4}\right)}, \boxed{\left(\frac{1-\sqrt{2}}{4}, \frac{1+\sqrt{2}}{4}\right)}, \\
&\boxed{\left(\frac{-1+\sqrt{2}}{4}, \frac{-1-\sqrt{2}}{4}\right)}, \boxed{\left(\frac{-1-\sqrt{2}}{4}, \frac{-1+\sqrt{2}}{4}\right)}. \qquad \text{(1p)}
\end{aligned}
$$

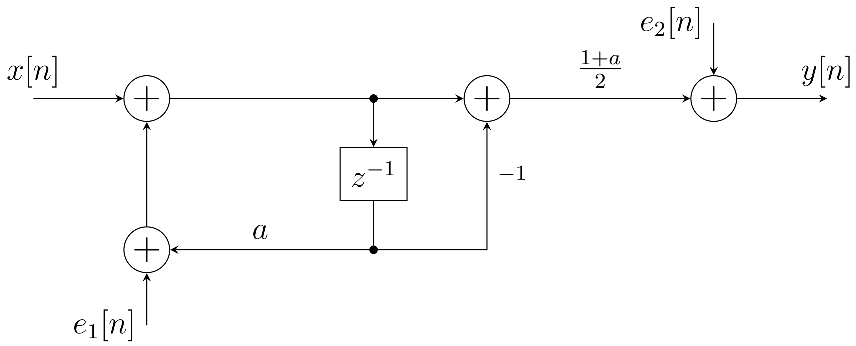
4. a) Each multiplication introduces quantization noise into the system. Both noise sources $e_1[n]$ and $e_2[n]$ are white and uncorrolated to each other, and to the signal $x[n]$. (0.5p) Hence, for $i = 1, 2$,

$$
r_{e_i e_i}[m] := \mathbb{E}[y[n]y[n-m]] = \sigma_i^2 \delta[m].
$$

where the power of noise is

$$
\sigma_e^2 := \sigma_{e_1}^2 = \sigma_{e_2}^2 = \frac{2^{-2B}}{12}. \qquad \text{(0.5p)}
$$

The diagram is as following: (1p)



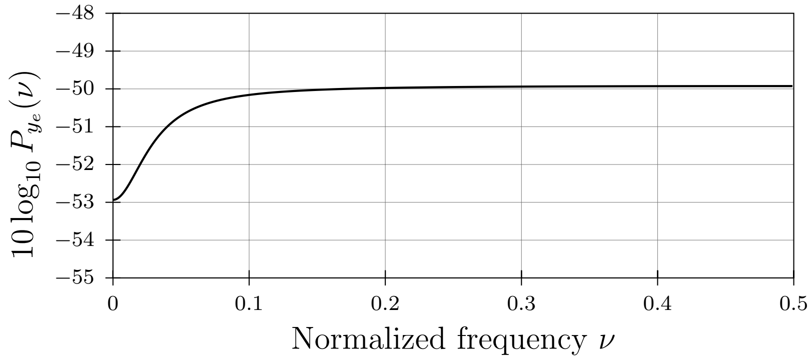
b) The noise $e_1[n]$ goes through the same system that $x[n]$ does, while $e_2[n]$ goes directly to the output. Hence, we have that the noise part of $y[n]$ is

$$
y_e[n] = e_1[n] * h[n] + e_2[n], \qquad \text{(0.5p)}
$$

where $h[n]$ is the impulse response of the sysem. The power spectrum of the noise at the output is thus given by

$$
P_{y_e}(\nu) = |H(\nu)|^2 \sigma_{e_1}^2 + \sigma_{e_2}^2 = \left[|H(\nu)|^2 + 1\right] \frac{2^{-2B}}{14}. \qquad \text{(0.5p)}
$$

By the hint, we have that $10\log_{10}\sigma_e^2 \approx -53\text{dB}$. For $\nu = 0$ we have $|H(\nu)|^2 = 0$ and $P_{y_e}(\nu) \approx -53\text{dB}$ (0.5p). For $0.1 < \nu < 0.5$ we have $|H(\nu)|^2 \approx 1$ (by the figure), so $|H(\nu)|^2 + 1 \approx 2 = 3\text{dB}$, and $P_{y_e}(\nu) \approx -50\text{dB}$ (1p). Using the plot of the filter gain curve in the problem, we get the following spectrum:



c) Based on the proof of the previous parts, we obtain that

$$\begin{aligned} r_{y_e y_e}[m] &:= \mathbb{E}[y[n]y[n-m]] \\ &= (r_{e_1 e_1}[m]) * h[m] * h[-m] + r_{e_2 e_2}[m] \\ &= \sigma_e^2 h[m] * h[-m] + \sigma_e^2. \qquad \text{(1p)} \end{aligned}$$

The power of the noise is

$$\sigma_{y_e}^2 := r_{y_e y_e}[0] = \sigma_e^2 \left(\sum_{m=-\infty}^{+\infty} h^2[m] + 1 \right). \qquad \text{(1p)}$$

Note that, the impulse response in the frequency domain is

$$H(\nu) = \frac{1+a}{2} \frac{1 - e^{-j2\pi\nu}}{1 - ae^{-j2\pi\nu}}. \qquad \text{(0.5p)}$$

Therefore, we have that

$$h[m] = \frac{1+a}{2}(a^m u[m] - a^{m-1} u[m-1]). \qquad \text{(0.5p)}$$

Now, we calculate the following summation:

$$\begin{aligned} \sum_{m=-\infty}^{+\infty} h^2[m] &= \frac{(1+a)^2}{4} \left(1 + \sum_{m=0}^{\infty} (a^{m+1} - a^m)^2 \right) \\ &= \frac{(1+a)^2}{4} \left(1 + \sum_{m=0}^{\infty} a^{2(m+1)} + a^{2m} - 2a^{2m+1} \right) \\ &= \frac{(1+a)^2}{4} \left(1 + (a^2 + 1 - 2a) \sum_{m=0}^{\infty} a^{2m} \right) \\ &= \frac{(1+a)^2}{4} \left(1 + (a-1)^2 \frac{1}{1-a^2} \right) \\ &= \frac{1+a}{2}. \qquad \text{(1.5p)} \end{aligned}$$

Thus, we obtain that

$$\sigma_{y_e}^2 = \frac{2^{-2B}}{12} \left(\frac{1+a}{2} + 1 \right) = \boxed{\frac{2^{-2B}}{12} \frac{a+3}{2}}. \qquad \text{(0.5p)}$$

5. a) From diagram, we should have

$$y[m] = \sum_{n=-\infty}^{+\infty} h[m-n] z[n] = \sum_{k=-\infty}^{+\infty} h[m-2k] x[k]. \qquad \text{(1p)}$$

Further, we know that

$$y[m] = y\left(\frac{m}{2}\right) = \sum_{k=-\infty}^{+\infty} p\left(\frac{m}{2} - k\right) x[k]. \qquad \text{(0.5p)}$$

Hence,

$$\boxed{h[m] = p\left(\frac{m}{2}\right) = \delta[m] + \frac{3}{4}(\delta[m-1] + \delta[m+1]) - \frac{1}{4}(\delta[m-3] + \delta[m+3])}. \qquad \text{(0.5p)}$$

is a valid answer.

b) We can write $h[n]$ as following

$$h[n] = h_0[n] + h_1[n],$$

where

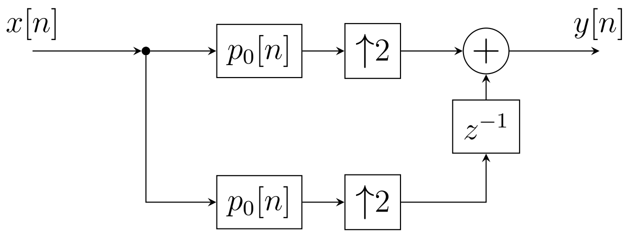
$$h_0[n] = \begin{cases} h[n] & n \text{ is even,} \\ 0 & n \text{ is odd,} \end{cases} \qquad h_1[n] = \begin{cases} h[n] & n \text{ is odd,} \\ 0 & n \text{ is even.} \end{cases}$$

Then, the constituent filters are defined as following

$$p_0[n] = h_0[2n] = \boxed{\delta[n]}, \qquad \text{(1p)}$$

$$p_1[n] = h_1[2n+1] = \boxed{\frac{3}{4}\delta[n] + \frac{3}{4}\delta[n+1] - \frac{1}{4}\delta[n-1] - \frac{1}{4}\delta[n+2]}. \qquad \text{(1p)}$$

Further, $y[n]$ can be implementd as following (1p)



c) Becaue $z[n]$ is the upsampled version of $x[n]$, its Fourier transform, $Z(\nu)$, is

$$Z(\nu) = X(2\nu). \qquad \text{(1p)}$$

Using the fact that $\delta(a\nu) = (1/|a|)\delta(\nu)$ (0.5p), we obtaine that for $\nu \in [-1/2, 1/2)$

$$\begin{aligned} Z(\nu) &= \frac{1}{2}\delta(2\nu - \nu_0) + \frac{1}{2}\delta(2\nu + \nu_0) + \frac{1}{2}\delta(2\nu + 1 - \nu_0) + \frac{1}{2}\delta(2\nu - 1 + \nu_0) \\ &= \frac{1}{4}\delta\left(\nu - \frac{\nu_0}{2}\right) + \frac{1}{4}\delta\left(\nu + \frac{\nu_0}{2}\right) + \frac{1}{4}\delta\left(\nu + \frac{1-\nu_0}{2}\right) + \frac{1}{4}\delta\left(\nu - \frac{1-\nu_0}{2}\right). \ \text{(1p)} \end{aligned}$$

Moreover, from part 5a we obtain that the Fourier transform of $h[n]$ is

$$\begin{aligned} H(\nu) &= 1 + \frac{3}{4}\left(e^{-j2\pi\nu} + e^{j2\pi\nu}\right) - \frac{1}{4}\left(e^{-j6\pi\nu} + e^{j6\pi\nu}\right) \\ &= 1 + \frac{3}{2}\cos(2\pi\nu) - \frac{1}{2}\cos(6\pi\nu). \qquad \text{(1p)} \end{aligned}$$

Hence,

$$
\begin{aligned}
Y(\nu) &= Z(\nu)H(\nu) \qquad \textcolor{blue}{(0.5\text{p})}\\
&= \frac{1}{4}\left(\delta\left(\nu-\frac{\nu_0}{2}\right)+\delta\left(\nu+\frac{\nu_0}{2}\right)\right)\left(1+\frac{3}{2}\cos(\pi\nu_0)-\frac{1}{2}\cos(3\pi\nu_0)\right)\\
&\quad +\frac{1}{4}\left(\delta\left(\nu+\frac{1-\nu_0}{2}\right)+\delta\left(\nu-\frac{1-\nu_0}{2}\right)\right)\\
&\qquad \times\left(1+\frac{3}{2}\cos(\pi(1-\nu_0))-\frac{1}{2}\cos(3\pi(1-\nu_0))\right) \qquad \textcolor{blue}{(0.5\text{p})}\\
&= \frac{1}{4}\left(\delta\left(\nu-\frac{\nu_0}{2}\right)+\delta\left(\nu+\frac{\nu_0}{2}\right)\right)\left(1+\frac{3}{2}\cos(\pi\nu_0)-\frac{1}{2}\cos(3\pi\nu_0)\right)\\
&\quad +\frac{1}{4}\left(\delta\left(\nu+\frac{1-\nu_0}{2}\right)+\delta\left(\nu-\frac{1-\nu_0}{2}\right)\right)\left(1-\frac{3}{2}\cos(\pi\nu_0)+\frac{1}{2}\cos(3\nu_0)\right). \ \textcolor{blue}{(0.5\text{p})}
\end{aligned}
$$