KTH, SIGNAL PROCESSING LAB

SCHOOL OF ELECTRICAL ENGINEERING

Digital Signal Processing EQ2300/ 2E1340

Final Examination 2011–12–20, 14.00–19.00

Literature:

- Hayes: *Statistical Digital Signal Processing and Modeling*
  or
  Proakis, Manolakis: *Digital Signal Processing*
- Bengtsson: *Complementary Reading in Digital Signal Processing*
- Begtsson and Jaldén: *Summary slides*
- *Beta – Mathematics Handbook*
- *Collection of Formulas in Signal Processing, KTH*
- Unprogrammed pocket calculator.

Notice:

- Answer in English or Swedish.
- At most one problem should be treated per page.
- Answers without motivation/justification carry no rewards.
- Write your name and *personnummer* on each page.
- Write the number of solution pages on the cover page.

The exam consists of five problems with a maximum of 10 points each. For a passing grade, 24 points are normally required.

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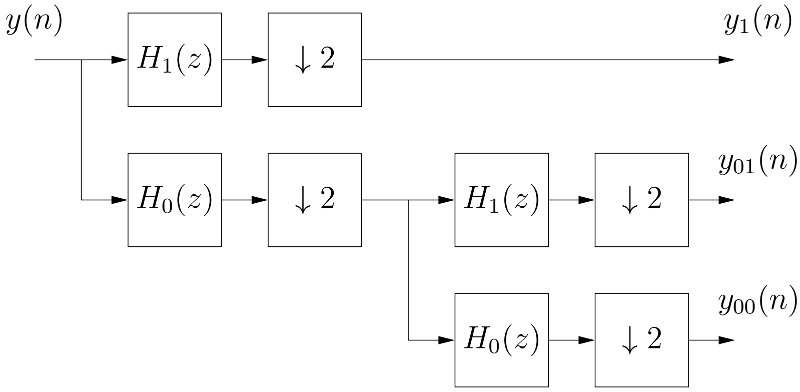
Results: Will be reported within three working weeks on "My pages".

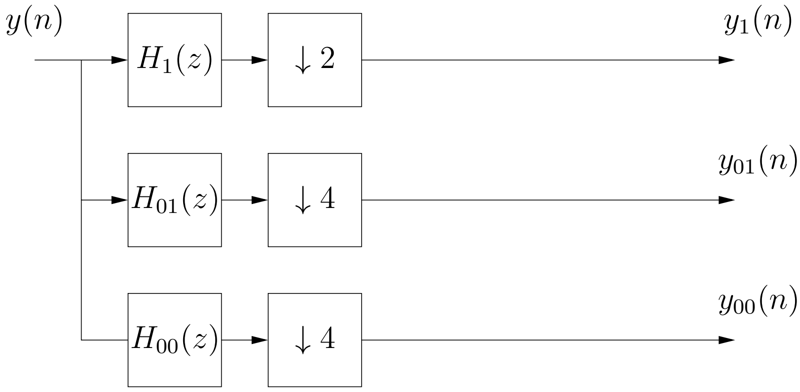
Solutions: Will be available on the course homepage after the exam.

Good luck!

1. In project 2 you used the cascaded 2-step analysis filter bank below to split the input signal $y(n)$ into three different frequency bands.

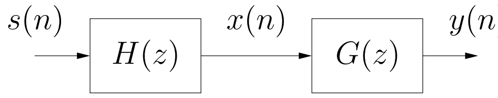


The filters in the filter bank were given by $H_0(z) = -\frac{1}{8} + \frac{1}{4}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{4}z^{-3} - \frac{1}{8}z^{-4}$ and $H_1(z) = \frac{1}{2} - z^{-1} + \frac{1}{2}z^{-2}$. An alternative implementation of the analysis filter bank is as a 3 branch 1-step filter bank as given below, where $H_{01}(z)$ and $H_{00}(z)$ are suitably chosen filters.



- Obtain the filter $H_{01}(z)$ so that the two implementations are equivalent, i.e., so that $y_{01}(n)$ is the same in both cases. (4p)
- Considering direct implementations of all filters, how many multiplications are required to obtain a single sample of $y_{01}(n)$ in each of the two implementations? Which implementation is most efficient? (Only consider the multiplications required to obtain $y_{01}(n)$, not $y_0(n)$ and $y_{00}(n)$) (3p)
- Show how to implement the filter $H_{01}(n)$ as a polyphase filter. Specify all polyphase component filters and compute the number of multiplications required to compute a singe sample of $y_{01}(n)$. How do the number of multiplications per sample compare to the direct implementations? (3p)

2. Consider the following communication link: The transmitter sends signal $s(n)$, which is distorted by the channel $H(z)$ during transmission. The received signal is $x(n)$, and the data sequence after equalizer $G(z)$ is denoted as $y(n)$.



Assume we know that the channel distortion can be modeled as

$$H(z) = 1 + 0.5z^{-1}.$$

(a) Design a causal equalizer $g(n)$ so that $y(n) = s(n)$. (1p)

(b) Assume we have received $N = 200$ samples at the receiver. The FIR filter $\hat{g}(n)$ is an 20-tap FIR filter which approximates $g(n)$ with its first 20 taps (i.e., $\hat{g}(n)$ is equal to $g(n)$ for $n = 0, \ldots, 19$, and is zero elsewhere). Now we want to filter $x(n)$ with $\hat{g}(n)$.

i. A X-point DFT is performed on $x(n)$ and $\hat{g}(n)$ respectively, then the two DFTs are multiplied and an inverse is performed to compute their circular convolution. How do you need to choose X to make sure the circular convolution is the same as the linear convolution of $x(n)$ and $\hat{g}(n)$? (1p)

ii. If we instead filter $x(n)$ with $\hat{g}(n)$ using the overlap-save method. What is then the best choice of the FFT block-length M? (Consider only the case $M = 2^p$, where p is a positive integer). (3p)

(c) Now we want to estimate the power spectrum of the data sequence $y(n)$, $n = 0, \cdots, 199$. The estimation should fullfill the following requirements:

i. 3 dB resolution bandwidth (BW) $\Delta f \leq 0.013$, where $f \in [-\frac{1}{2}, \frac{1}{2}]$ is the normalized frequency.

ii. Sidelode level ≤ -20dB

iii. The variance $\text{Var}(\hat{P}(f)) \leq \frac{1}{2}P^2(f)$, where $P(f)$ is the true power spectrum.

How do you choose your windowing and estimation method? (5p)

Window	Sidelobe Level (dB)	3 dB BW Δf
Rectangular	-13	$0.89/N$
Bartlett	-27	$1.28/N$
Hanning	-32	$1.44/N$
Hamming	-43	$1.30/N$
Blackman	-58	$1.68/N$

Table 1: Properties of a few commonly used windows. Each window is assumed to be of length N.

3. The autocorrelation sequence of an autoregressive (AR) process $x(n)$ is given by

$$r_x(k) = \alpha^{|k|}, \qquad |\alpha| < 1.$$

(a) Using the Levinson-Durbin recursion, determine the order of the AR model. Find the transfer function of an all-pole filter which is applied to unit variance white noise to generate this AR process, i.e.,

$$H(z) = \frac{b(0)}{A_p(z)} = \frac{b(0)}{1 + \sum_{k=1}^{p} a_p(k) z^{-k}}$$

where $b(0) = r_x(0) + \sum_{l=1}^{p} a_p(l) r_x(l)$. (4p)

(b) Suppose that the available measurements of $x(n)$ are noisy and the observed process $y(n)$ is

$$y(n) = x(n) + w(n)$$

where $w(n)$ is the white noise with variance σ_w^2 and uncorrelated with $x(n)$. It is here known that $y(n)$ is a second-order AR process. Find the AR parameters for $y(n)$, i.e., $\mathbf{a}_2 = [1 \;\; a_2(1) \;\; a_2(2)]^T$. (4p)

(c) How will the AR parameters change if $\sigma_w^2 \to 0$? Interpret your answer. (2p)

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4. Consider an AR(1) process $d(n)$, described by the relation $d(n) = 0.3d(n-1) + e(n)$, where $e(n)$ is white noise with variance $\sigma_e^2 = 1$.

Assume that $d(n)$ goes through a filter with impulse response $h(n) = \delta(n) - \delta(n-1)$ and that the output, say $f(n)$, is corrupted by an additive noise source $w(n)$ producing the signal

$$x(n) = f(n) + w(n) = d(n) * h(n) + w(n)$$

The noise source $w(n)$ is assumed white, zero-mean with variance $\sigma_w^2 = 1$ (independent of $e(n)$ and $d(n)$). We would like to build a one-step predictor for $d(n)$, of the form

$$\hat{d}(n+1) = c + w_0 x(n) + w_1 x(n-1)$$

so as to minimize the the MSE

$$J = \mathbb{E}\left\{ \left[\hat{d}(n+1) - d(n+1) \right]^2 \right\}$$

with respect to c, w_0 and w_1. (Note that c is a constant, but we need to design it as well).

a) Find the autocorrelation function (ACF) of $x(n)$ (3p)

b) Use the previous calculation to find the optimal values for c, w_0, w_1, i.e., the values that minimize J (7p)

5. Consider a continuous–time signal $x_a(t)$ with power spectrum

$$X_a(F) = \begin{cases} 1 - |F|/F_N, & |F| \le F_N \\ 0, & |F| > F_N \end{cases}$$

Suppose that we oversample $x_a(t)$ by a factor of M, i.e., assume that we sample $x_a(t)$ with frequency $F_s = 2F_N M$ (which leads to a sampling period $T = \frac{1}{2F_N M}$). Finally, suppose that we pass the resulting discrete–time signal $x(n) = x_a(nT)$ through a quantizer Q and a proper decimator, as shown in Figure 1.

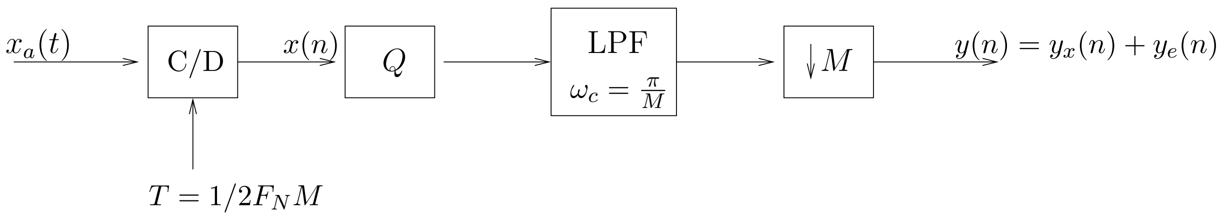

Figure 1: Oversampling followed by direct quantization and decimation

The continuous-to-discrete time converter (C/D) is ideal, i.e., implemented with infinite precision. The quantizer Q is scalar and uniform, with the quantization step Δ sufficiently small in order for the usual assumptions regarding the quantization error $e(n)$ to be valid. The filter LPF is an ideal lowpass filter with frequency response

$$H_D(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \pi/M \\ 0, & \pi/M < |\omega| \le \pi \end{cases}$$

a) Calculate the signal-to-quantization noise ratio (SQNR) at the output $y(n)$ (5p)

b) Assume that we add a feedback loop around the quantizer, as shown in Figure 2 below, where we choose $H(z) = 1/(1 - z^{-1})$. Calculate the new SQNR at the output $y(n)$. Assume that M is sufficiently large so that $\sin\left(\frac{\omega}{2M}\right) \approx \frac{\omega}{2M}$ to simplify the calculation of the integrals. Hint: You may also find the following trigonometric identity useful: $2\sin^2(\omega/2) = 1 - \cos(\omega)$ (3p)

c) What have we accomplished by adding the feedback loop in Figure 2? If you cannot solve part (b), try to answer part (c) by reasoning (2p)

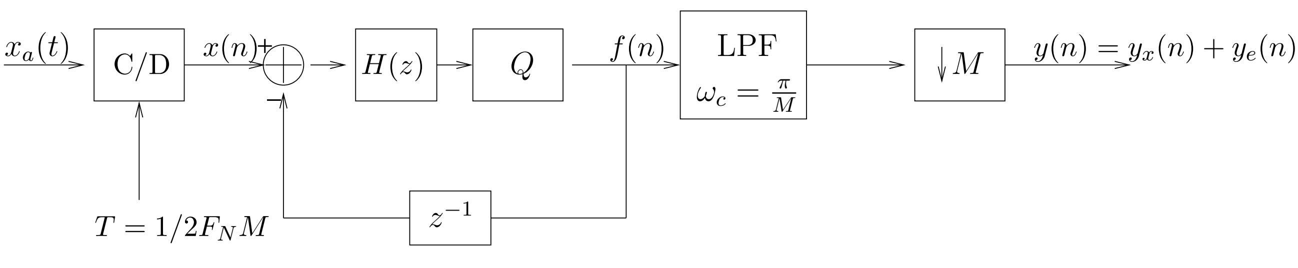

Figure 2: Adding a feedback loop around the quantizer