SOLUTIONS

E 93 **Digital Signalbehandling**, 2E1340

Final Examination 2004–04–23, 08.00–13.00

1. a) The Yule-Walker equations give

1st order predictor: $a_1(1) = -\frac{\hat{r}_{xx}(1)}{\hat{r}_{xx}(0)} = -\frac{1}{\sqrt{2}}$; $\epsilon_1 = \sqrt{2}(1-\frac{1}{2}) = \frac{1}{\sqrt{2}}$.

2nd order predictor:

$$\begin{bmatrix}\sqrt{2} & 1 \\ 1 & \sqrt{2}\end{bmatrix}\begin{bmatrix}a_2(1) \\ a_2(2)\end{bmatrix} = -\begin{bmatrix}1 \\ 0\end{bmatrix}$$

Hence $a_2(1) = -\sqrt{2}$ and $a_2(2) = 1$. $\epsilon_2 = \epsilon_1(1-1) = 0$, note that this means perfect prediction.

3rd order predictor: since the 2nd order prediction is perfect, $a_3(1) = a_2(1) = -\sqrt{2}$, $a_3(2) = a_2(2) = 1$ and $a_3(3) = 0$; $\epsilon_3 = \epsilon_2 = 0$.

b) Use the Yule-Walker equations to calculate the autocorrelation from the model parameters, $\hat{r}_{xx}(3) = -a_2(1)\hat{r}_{xx}(2) - a_2(2)\hat{r}_{xx}(1) = -1$.

c) This is a 2nd order AR process, since $\epsilon_2 = 0$.

2. a) Using the indirect method for calculating the periodogram, we get

$$\hat{P}_{xx}(f) = \mathcal{F}\{\hat{r}_{xx}(k)\} = \sum_{k=-2}^{2} \hat{r}_{xx}(k)e^{-j2\pi fk} = e^{j2\pi f} + \sqrt{2} + e^{-j2\pi f} = \sqrt{2} + 2\cos(2\pi f) \ ,$$

see Figure 1 for a sketch of the function.

Note that something is wrong here, since a periodogram can never be negative. The explanation is that the estimate of the autocorrelation was not calculated using the biased estimator, $\hat{r}_{xx}(k) = 1/N\sum x(n)x^*(n-k)$, that should be used in the indirect method for the periodogram (this observation gives one bonus point).

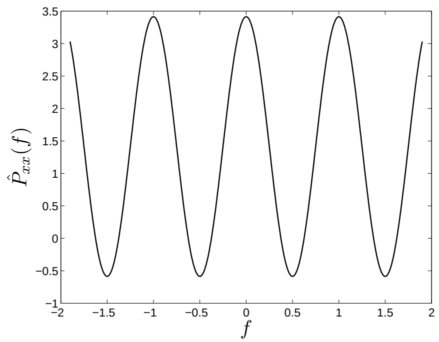

Figure 1: Estimated periodogram

b) Since the periodogram only has one main lobe at $f = 0$, we have too few data samples to be able to even resolve the positive and negative frequencies of this single sinusoid.

c) The Blackman-Tukey method applies an additional window on the estimated autocorrelation function, which will reduce the resolution further (but will improve other aspects, such as the variance). So, if the resolution is too low with the periodogram, it will also be too low using Blackman-Tukey.

d) The poles of the AR model are found from $z^2 + a_2(1)z + a_2(2) = 0$, which gives the poles $z_p = e^{\pm j\frac{\pi}{4}}$. This corresponds to the frequency $f_0 = \frac{\pi}{4}/2\pi = \frac{1}{8}$

3. a) Proper decimation is done using low-pass filtering,

$$z(n) = x(n) * h(n) = \sum_k x(n-k)h(k)$$

followed by down-sampling,

$$y(m) = z(2m) = \sum_k x(2m-k)h(k) \ .$$

This corresponds to implementation 1.

Note that implementation 4 corresponds to half a polyphase implementation, (only the term with the $p_0(m)$ filter).

b) In a), we assumed that $x(n) = 0$ for $n < 0$. In the described real-time system, this is only reasonable for the first buffer. For the other buffers, the first output values will depend also on the final values of the previous input buffer. Two solutions to overcome the problem are:

- Overlap-save: Use a working buffer $w(n)$ of length $M+N-1$, containing the $M-1$ last values of the previous input buffer followed by the N values of the current input buffer. Modify the algorithm to use $w(n)$ as input instead of $x(n)$ and add an offset of $M-1$ to the index of $w(n)$.
- Overlap-add: Run the outer for loop $\lfloor (N+M-1)/2 \rfloor$ iterations instead of $\lfloor N/2 \rfloor$. Save the last $(M-1)/2$ output values and add them to the first output values of the next buffer.

c) The sampled signal has normalized frequency $f_x = F/F_s = \pm 0.055$. After the decimation by two, the normalized frequency is doubled, $f_y = \pm 0.11$. The highest peaks of the periodogram will appear at frequency bins $k = (\pm 0.11 \cdot 512) \ _{\text{mod } 512} = \{56, 456\}$.

4. The solution to this problem is in CD-player terminology known as “Oversampling”. Early CD-players often had the oversampling rate clearly written on the front panel.

If the sampled signal is upsampled and interpolated by digital signal processing, the margin to the Nyquist frequency can be increased. Figure 2 shows the principle in the case of 2x upsampling.

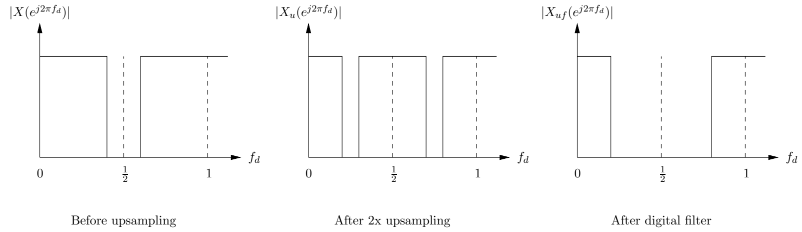

Figure 2: Interpolation by a factor 2

Assume that the signal is upsampled a factor I. Then a zero order hold D/A converter with sampling rate IF_s must be used. The resulting output is shown in Figure 3. To be able to use the cheap analog filter, the alias component should be moved into the stopband of the analog filter. We get

$$
\begin{aligned}
IF_s - 20\text{kHz} &\geq 332.8\text{kHz} \\
I &\geq \frac{352.8\text{kHz}}{44.1\text{kHz}} \\
I &\geq 8.
\end{aligned}
$$

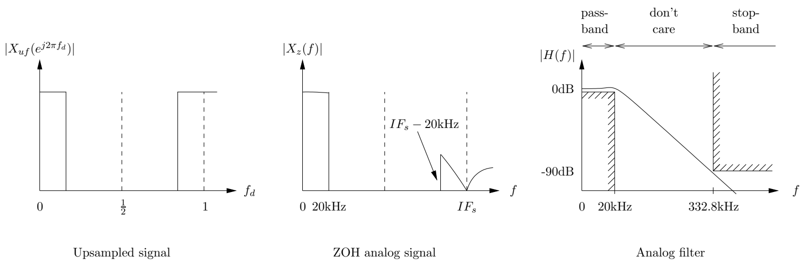

Figure 3: Zero order hold D/A conversion of upsampled signal

Using an oversampling rate of $I = 8$, we get the block diagram in Figure 4. The digital filter should be a lowpass filter with passband for $f_d < 20/(8 \cdot 44.1) = 0.056$, and stopband for $f_d > (20 + 4.1)/(8 \cdot 44.1) = 0.068$. This sharp filter can easily be implemented in a DSP.

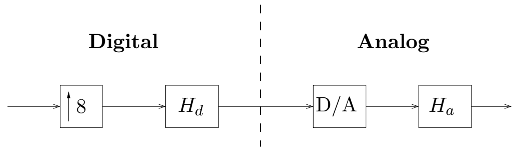

Figure 4: Block diagram of final system

5. Introduce the signal $w(n)$ before the right-most summation. Then, $w(n) = aw(n-1) + x(n-1) + cy(n-2)$ and $y(n) = w(n) + by(n-1)$, which gives

$$
\begin{cases}
W(z) = az^{-1}W(z) + z^{-1}X(z) + cz^{-2}Y(z) \\
Y(z) = W(z) + bz^{-1}Y(z)
\end{cases}
$$

$$\Longrightarrow$$

$$
\begin{cases}
W(z) = \frac{z^{-1}X(z)+cz^{-2}Y(z)}{1-az^{-1}} \\
Y(z) = \frac{W(z)}{1-bz^{-1}} = \frac{z^{-1}X(z)}{(1-az^{-1})(1-bz^{-1})(1-\frac{cz^{-2}}{(1-az^{-1})(1-bz^{-1})})}
\end{cases}
$$

which gives

$$
H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1-(a+b)z^{-1}+(ab-c)z^{-2}} = \frac{z^{-1}}{1+0.4z^{-1}-0.21z^{-2}}
$$

From the usual approximation of round-off errors as independent white noise sources, the multiplications by a and c introduce two noise sources that add to the input signal $x(n)$. No round-offs are introduced by the $b = -1$ multiplication. The corresponding transfer functions from the noise sources to the output are

$$
H_a(z) = H_c(z) = H(z) = \frac{z^{-1}}{1+0.4z^{-1}-0.21z^{-2}} = \frac{1}{1-0.3z^{-1}} - \frac{1}{1+0.7z^{-1}}
$$

which gives the impulse response $h_a(n) = h_c(n) = h(n) = 0.3^n - (-0.7)^n$, $n \geq 0$. Since 15 bits are used, each quantization noise source has power $\sigma_e^2 = 2^{-30}/12$ and the total noise power at the output is

$$
\begin{aligned}
P_q =& 2\sigma_e^2 \sum_{n=0}^{\infty} h^2(n) = \frac{2^{-30}}{6} \sum_{n=0}^{\infty} 0.09^n - 2 \cdot (-0.21)^n + 0.49^n \\
=& \frac{2^{-30}}{6} \left(\frac{1}{1-0.09} - \frac{2}{1+0.21} + \frac{1}{1-0.49} \right) \approx 2.2 \cdot 10^{-10}
\end{aligned}
$$