# SIGNAL PROCESSING

# SCHOOL OF ELECTRICAL ENGINEERING

## Digital Signal Processing EQ2300 / 2E1340

Final Examination 2014–03–11, 14.00–19.00 Examples of Solutions

**1.** Lets call the DFT function $\mathcal{F}$, i.e., $X[k] = \mathcal{F}\{x[n]\}$, in what follows.

a) The simplest solution is to just apply the DFT circuit to the time reversed sequence $x[15-n]$, i.e., $A[k] = \mathcal{F}\{a[n]\}$ where $a[n] = x[15-n]$. We can also from first principles obtain

$$A[k] = \sum_{n=0}^{15} x[15-n]e^{-j2\pi\frac{nk}{N}} = e^{-j2\pi\frac{15k}{N}} \sum_{m=0}^{15} x[m]e^{j2\pi\frac{mk}{N}} = e^{-j2\pi\frac{15k}{N}} X[(-k)_{16}]$$

where $(\cdot)_{16}$ denotes the modulo 16 operation.

b) This can be done with zero-padding. More precisely, if we let $y[n] = x[n]$ for $n = 0, \ldots, 7$ and $y[n] = 0$ for $n = 8, \ldots, 15$, and let $Y[n] = \mathcal{F}\{y[n]\}$, then $B[k] = Y[2k]$ for $k = 0, \ldots, 7$.

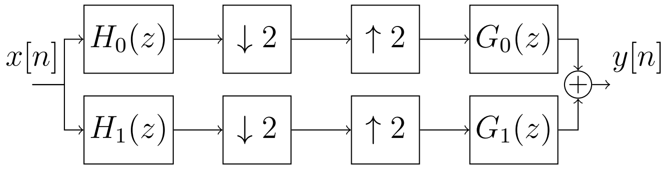
c) One can use the recursive nature of the FFT algorithm to break the $N = 64$ point sequence into two 32 point sequences and than these to four 16 sequences. The only difference with the standard radix-2 FFT is that we, using the DFT circuit for 16-point DFTs, stop at 16 point sequences instead of further splitting the sequences down to length 1. Each "layer" FFT algorithm require $N/2$ multiplications, so to combine the four 16 point sequences to two 32 point sequences require $N/2 = 32$ multiplications, and to combine the two 32 point sequences to one 64 point sequence require another $N/2 = 32$ multiplications, yielding a total of $N = 64$ multiplications in addition to the 4 applications of the DFT circuits.

In math (which is not necessary to get full points), let the computations of the DFT circuit be given by

$$X_{16,i}[k] = \mathcal{F}\{x[4n+i]\}$$

for $i = 0, \ldots, 3$ and $k = 0, \ldots, 15$. Then we compute $A_{32,i}[k] = W_{32}^k X_{16,2i+1}[k]$ for $i = 0, 1$ and $k = 0, \ldots, 15$ using at most 32 complex valued multiplications and form $X_{32,i}[k] = X_{16,2i}[k] + A_{32,i}[k]$ for $k = 0, \ldots, 15$ and $X_{32,i}[k] = X_{16,2i}[k] - A_{32,i}[k]$ for $k = 0, \ldots, 15$ with no additional multiplications. Then we form $A_{64}[k] = W_{64}^k X_{32,1}[k]$ for $k = 0, \ldots, 31$ at a cost of at most 32 complex valued multiplications, and finally form $X[k] = X_{32,0}[k] + A_{64}[k]$ for $k = 0, \ldots, 31$ and $X[k] = X_{32,0}[k] - A_{64}[k]$ for $k = 0, \ldots, 31$ with no further multiplications.

**2.** a) Consider first the simplified system given by



and note that as

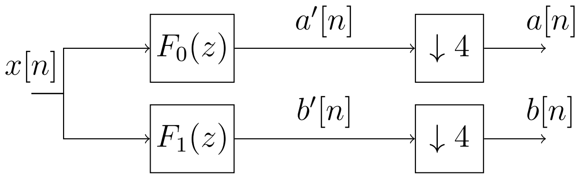
$$
\begin{aligned}
&G_0(z)H_0(-z) + G_1(z)H_1(-z)\\
=&\frac{1}{\sqrt{2}}(1+z^{-1})\frac{1}{\sqrt{2}}(1+(-z)^{-1}) + \frac{1}{\sqrt{2}}(-1+z^{-1})\frac{1}{\sqrt{2}}(1-(-z)^{-1})\\
=&\frac{1}{2}(1+z^{-1})(1-z^{-1}) - \frac{1}{2}(1-z^{-1})(1+z^{-1}) = 0
\end{aligned}
$$

we do not have any aliasing in the reconstruction for this system. We also have

$$
\begin{aligned}
&G_0(z)H_0(z) + G_1(z)H_1(z)\\
=&\frac{1}{\sqrt{2}}(1+z^{-1})\frac{1}{\sqrt{2}}(1+z^{-1}) + \frac{1}{\sqrt{2}}(-1+z^{-1})\frac{1}{\sqrt{2}}(1-z^{-1})\\
=&\frac{1}{2}(1+z^{-1})^2 - \frac{1}{2}(1-z^{-1})^2 = 2z^{-1}
\end{aligned}
$$

which implies perfect reconstruction with a delay of $l = 1$. The system in the problem consists of two nested such system, with the inner system operating at half the rate. The system given in the problem thus ensures perfect reconstruction with a delay of $L = l + 2l = 3$.

b) Using the nobel identities we can express $a[n]$ and $b[n]$ using the equivalent system




where

$$F_0(z) = H_0(z)H_0(z^2) = \frac{1}{2}\left(1 + z^{-1} + z^{-2} + z^{-3}\right)$$

and

$$F_1(z) = H_0(z)H_1(z^2) = \frac{1}{2}\left(1 + z^{-1} - z^{-2} - z^{-3}\right).$$

As downsampling a WSS stochastic process does not reduce the power, we can compare the power in $a'[n]$ and $b'[n]$ instead. These are proportional to the gain of $F_0(z)$ and $F_1(z)$ at $z = z_0 \triangleq e^{j\pi/3} = \frac{1}{2} + j\frac{\sqrt{3}}{2}$ which implies that $z^{-1} = \frac{1}{2} - j\frac{\sqrt{3}}{2}$. By inserting this into the above equations we obtain

$$F_0(z_0) = j\frac{\sqrt{3}}{2} \quad \Rightarrow \quad |F_0(z_0)| = \frac{\sqrt{3}}{2}$$

and

$$F_1(z_0) = 3 \quad \Rightarrow \quad |F_1(z_0)| = 3\,.$$

Consequently, $b[n]$ has higher power than $a[n]$ as the gain of $F_1(\nu)$ is higher than $F_0(\nu)$at $\nu = \frac{1}{6}$.

**3.** a) We can answer this by studying the side lobes, which are most easily displayed around the spectral peak at $\nu_3 = 0.433$. The side lobes in figure B are approximately 13 dB below the main lobe and match the Rectangular window. The side lobes in figure C are approximately 27 dB below the main lobe and match the Bartlett (triangular) window. For figure $A$ the side lobes are masked by the noise at $-40$ dB but comparing the levels of the peaks and the noise floor we see that the side lobes must have at least a 50 dD suppression with respect to the main lobe and this leaves only the Blackman window.

b) A would be most appropriate, as the noise floor in B and to a large extent in C is masked by the side lobes of the window. However, in $A$ the noise level is above the side lobes and can be clearly read from the figure.

c) We would need a 3 dB bandwith that match the required resolution. The resolution is set by the two closest peaks which for the block length $L$ requires that

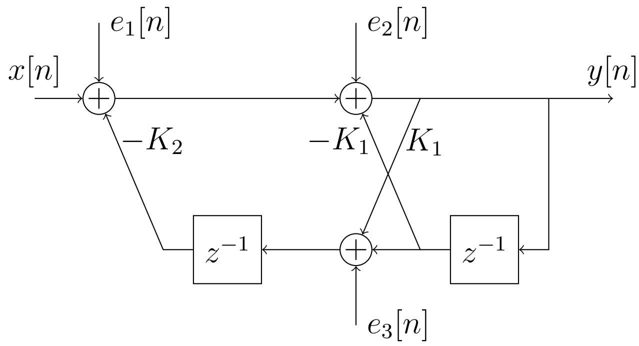
$$\Delta\nu_{3\mathrm{dB}} = \frac{1.68}{L} \leq \nu_2 - \nu_1 = 0.011 \quad \Rightarrow \quad L \geq 153\,.$$

where we use that the number of samples per block must be an integer. With $M = 20$ blocks the number of samples require at 50% overlap is given by

$$N = \frac{M+1}{2} L \geq 1606$$

where the division by 2 is due to the 50% overlap (we get twice the number of blocks as compared to Bartlett's method) and where the $M+1$ expression is to account for edge effects. We also here round up to the next even integer as the number of samples must be an integer.

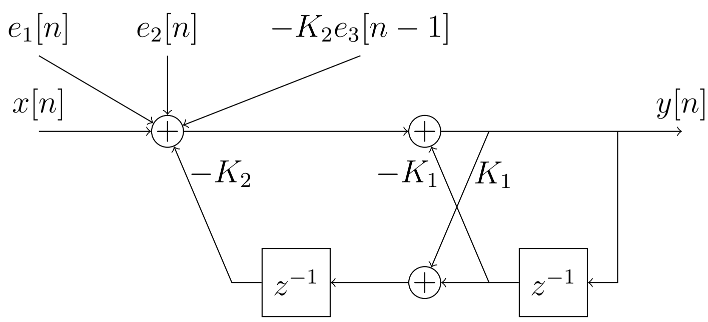
**4.** Using the standard quantization noise analysis yields




where $e_i[n]$ for $i = 1, \ldots, 3$ is white noise with a variance of

$$\sigma_e^2 = \frac{2^{-2B}}{12}\,.$$

We can however equivalently write this as




which as time delays do not affect the whiteness of the noise implies that we can treat all quantization noise as coming from a single source with variance

$$\sigma^2 = (1 + 1 + K_2^2)\frac{2^{-2B}}{12} = \frac{73}{432} 2^{-2B}\,.$$

By considering the $z$-transform for the input-output relation of the entire circuit we see that

$$Y(z) = X(z) - K_1K_2z^{-1}Y(z) - K_2z^{-2}Y(z) - K_1z^{-1}Y(z)$$

which implies that the transfer function of the entire circuit is given by

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 + K_1(1+K_2)z^{-1} + K_2z^{-2}} = \frac{1}{1 + \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}\,.$$

Finding the roots and splitting the fraction yields

$$H(z) = \frac{3}{1 + \frac{1}{2}z^{-1}} - \frac{2}{1 + \frac{1}{3}z^{-1}}$$

and the impulse response

$$h[n] = 3\left(\tfrac{1}{2}\right)^n - 2\left(\tfrac{1}{3}\right)^n$$

for $n \geq 0$. We obtain

$$|h[n]|^2 = 9\left(\tfrac{1}{4}\right)^n - 12\left(\tfrac{1}{6}\right)^n + 4\left(\tfrac{1}{9}\right)^n$$

and

$$\sum_{n=0}^{\infty} \left|h[n]\right|^2 = \frac{9}{1-\frac{1}{4}} - \frac{12}{1-\frac{1}{6}} + \frac{4}{1-\frac{1}{9}} = \frac{21}{10}\,.$$

The total noise power at the output will thus be

$$\sigma^2 \sum_{n=0}^{\infty} \left|h[n]\right|^2 = \frac{73}{432}2^{-2B} \times \frac{21}{10} = \frac{511}{1440}2^{-2B}\,.$$

**5.** a) The coefficients of the AR-2 model are obtained by solving the Yule-Walker equations given by

$$\begin{bmatrix} \hat{r}_x(0) & \hat{r}_x(1) \\ \hat{r}_x(1) & \hat{r}_x(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -\begin{bmatrix} \hat{r}_x(1) \\ \hat{r}_x(2) \end{bmatrix}\,.$$

Plugging in the given numbers yield

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -\begin{bmatrix} 1 & \frac{5}{7} \\ \frac{5}{7} & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{5}{7} \\ \frac{3}{7} \end{bmatrix} = -\frac{1}{1-\frac{25}{49}} \begin{bmatrix} 1 & -\frac{5}{7} \\ -\frac{5}{7} & 1 \end{bmatrix} \begin{bmatrix} \frac{5}{7} \\ \frac{3}{7} \end{bmatrix} = \begin{bmatrix} -\frac{5}{6} \\ \frac{1}{6} \end{bmatrix},$$

and finally

$$b_0^2 = \hat{r}_x(0) + a_1\hat{r}_x(1) + a_2\hat{r}_x(2) = \frac{10}{21}\,.$$

The AR-2 spectrum estimate is thus given by

$$\hat{P}_x^{\mathrm{AR}}(\nu) = |H_{\mathrm{AR}}(\nu)|^2 = \frac{b_0^2}{|1 + a_1e^{-j2\pi\nu} + a_2e^{-j4\pi\nu}|^2} = \frac{\frac{10}{21}}{|1 - \frac{5}{6}e^{-j2\pi\nu} + \frac{1}{6}e^{-j4\pi\nu}|^2}$$

which can potentially be simplified using cosine functions.

b) The Blackman-Tukey estimate is straightforwardly obtained from its definition according to

$$\begin{aligned}\hat{P}_x^{\mathrm{BT}} &= \sum_{k=-\infty}^{\infty} w[k]\hat{r}_x[k]e^{-j2\pi\nu} = 1 + \frac{2}{3} \times \frac{5}{7} \times 2\cos(2\pi\nu) + \frac{1}{3} \times \frac{3}{7} \times 2\cos(4\pi\nu) \\ &= 1 + \frac{20}{21}\cos(2\pi\nu) + \frac{2}{7}\cos(4\pi\nu)\,.\end{aligned}$$

c) The easiest way to differentiate the spectra is to consider $\hat{P}_x^{\mathrm{AR}}(\nu)$ and $\hat{P}_x^{\mathrm{BT}}(\nu)$ at $\nu = 0$, i.e.,

$$\hat{P}_x^{\mathrm{AR}}(0) = \frac{\frac{10}{21}}{|1 - \frac{5}{6} + \frac{1}{6}|^2} = \frac{30}{7}$$

and

$$\hat{P}_x^{\mathrm{BT}}(0) = 1 + \frac{20}{21} + \frac{2}{7} = \frac{47}{21} < \frac{30}{7}\,.$$

Figure A, for which $\hat{P}_x(0)$ is the largest, therefore corrspond to the AR based estimate, and Figure B correspond to the Blackman-Tukey estimate.