SIGNAL PROCESSING

DEPARTMENT OF ELECTRICAL ENGINEERING

E 104 **Digital Signalbehandling** EQ2300/ 2E1340

Final Examination 2008–12–17, 14.00–19.00
Sample Solutions

1. (a) **F T F F T F**

(b) One of the important properties of an autocorrelation function is the positive semi-definiteness that leads to a positive power spectral density. Computing the power spectral density using the given $r_{yy}(k)$, we have,

$$P_y(\omega) = 9 - 18\alpha\cos(\omega)$$

Since $P_y(\omega) \geq 0, \forall\omega$, it follows that

$$\alpha\cos(\omega) < 0.5, \quad \forall\omega.$$

This leads to $\alpha \leq \frac{1}{2}$. Note $|r_{yy}(1)| < r_{yy}(0)$ is also satisfied.

(c) The signal is $v(t) = 1000e^{j230\pi t}$. The magnitude follows from noting the peak of the main lobe and comparing it with the standard result. Further, the frequency should occur on or around the fourth DFT index. If the frequency had occurred at the 4th DFT index, we would have only one peak and there will be no spread of energy. Since the plot depicts a spread in energy (non-negligible) around the fourth DFT index, the frequency cannot be 125 Hz. This leads to the choice of the signal made.

2. (a) From filter 1 we find

$$y(n) = 2x(n) + 1.1y(n-1) - 0.3y(n-2)$$

which gives

$$\begin{aligned} H(z) &= \frac{2}{1 - 1.1z^{-1} + 0.3z^{-2}} \\ &= \frac{2}{(1-0.6z^{-1})(1-0.5z^{-1})} \end{aligned}$$

(b) As in (a) we can find

$$\begin{aligned} Y1(z) &= -\frac{10}{1-az^{-1}}X(z) \\ Y2(z) &= \frac{12}{1-bz^{-1}}X(z) \end{aligned}$$

and

$$\begin{aligned} Y(z) &= Y1(z) + Y2(z) \\ &= \frac{2 + (10b - 12a)z^{-1}}{(1-bz^{-1})(1-az^{-1})}X(z). \end{aligned}$$

Choosing $b = .6$ and $a = .5$ will give the same transfer function for both filters.

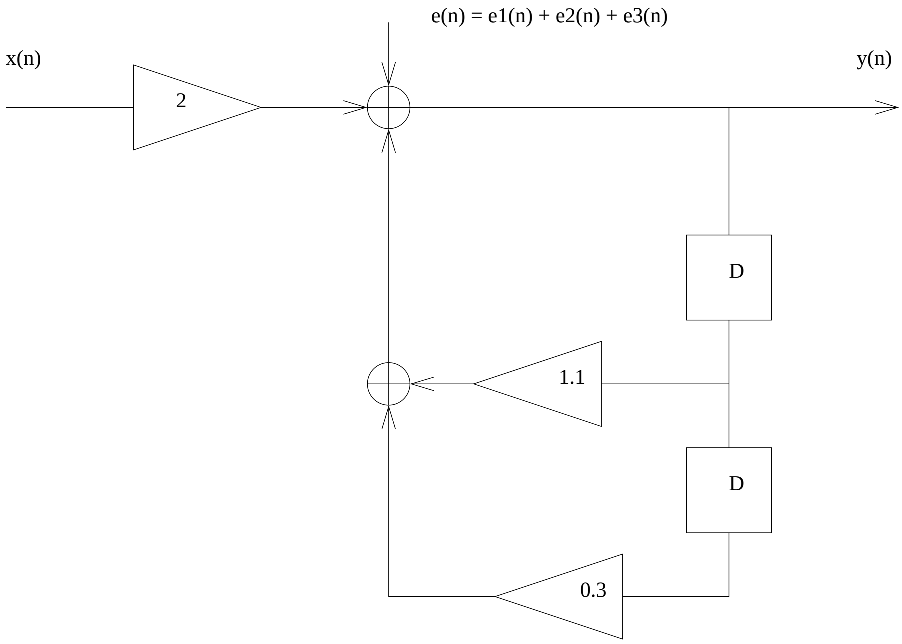

Figure 1: System corresponding to filter 1.

(c) Assume that each multiplication gives rise to an additive noise term which is zero mean and Gaussian distributed with variance σ^2. Then filter 1 is equivalent to the system in Figure 1, where $e(n) = e1(n) + e2(n) + e3(n)$ is the total quantization noise added. The quantization noise sees the filter

$$H_1(z) = \frac{1}{1 - 1.1z^{-1} + 0.3z^{-2}} = \frac{6}{1 - 0.6z^{-1}} - \frac{5}{1 - 0.5z^{-1}},$$

and the total noise power at the output is then

$$\begin{aligned}
\sigma_e^2 &= 3\sigma^2 \sum_{n=0}^{\infty} |h_1(n)|^2 = 3\sigma^2 \sum_{n=0}^{\infty} (6 \cdot 0.6^n - 5 \cdot .5^n)^2 \\
&= 3\sigma^2 \sum_{n=0}^{\infty} (36 \cdot 0.36^n + 25 \cdot 0.25^n - 60 \cdot 0.3^n) \\
&= 3\sigma^2 (\frac{36}{1 - 0.36} + \frac{25}{1 - 0.25} - \frac{60}{1 - 0.3}) \approx 12\sigma^2
\end{aligned}$$

Filter 2 is equivalent to the system seen in Figure 2. The two noise contributions see the filters

$$H_2(z) = \frac{1}{1 - 0.5z^{-1}}$$

and

$$H_3(z) = \frac{1}{1 - 0.6z^{-1}}$$

respectively. Thus the total noise power at the output is

$$\begin{aligned}
\sigma_e^2 &= 2\sigma^2 \sum_{n=0}^{\infty} (|h_2(n)|^2 + |h_3(n)|^2) \\
&= 2\sigma^2 \sum_{n=0}^{\infty} (0.5^n)^2 + (0.6^n)^2 = 2\sigma^2 \sum_{n=0}^{\infty} (0.25^n) + (0.36^n) \\
&= 2\sigma^2 (\frac{1}{1 - 0.25} + \frac{1}{1 - 0.36}) \approx 6\sigma^2.
\end{aligned}$$

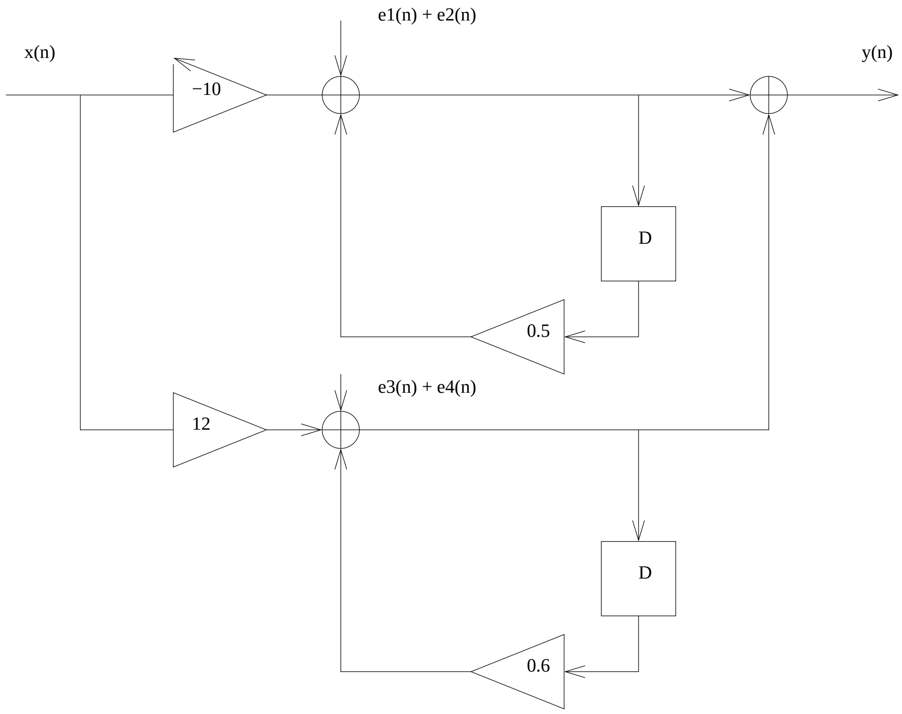

Figure 2: System corresponding to filter 2.

We see that filter 2 gives rise to less quantization noise than filter 1.

3. (a) Let $P_y(f)$ and $P_w(f)$ respectively denote the power spectral densities of $y(n)$ and $w(n)$. Then, using the standard relation between input and output power spectral densities, we have,

$$P_y(f) = \frac{P_w(f)}{|H_1(f)|^2} \tag{1}$$

$$P_w(f) = \frac{\sigma^2}{|H_2(f)|^2} \tag{2}$$

where $H_1(f) = 1 + \sum_{k=1}^{N} \alpha_k e^{-j2\pi fk}$ and $H_2(f) = 1 + \sum_{k=1}^{M} \beta_k e^{-j2\pi fk}$. Combining equations 1 and 2, we have,

$$P_y(f) = \frac{\sigma^2}{|H_1(f)H_2(f)|^2} \tag{3}$$

Denoting $H(f) = H_1(f)H_2(f)$ and letting $h(n)$ to be the impulse response of $H(f)$, we see that $h(n)$ is the convolution of sequences $\{1, \alpha_1, \ldots, \alpha_N\}$ and $\{1, \beta_1, \ldots, \beta_M\}$. Thus $h(n)$ is a length $M + N + 1$ filter with $h(0) = 1$. This immediately leads to the fact that $y(n)$ is $AR(M + N)$.

(b) By differentiating $E\left[(x(n) - \widehat{x}(n))^2\right]$ w. r. t γ and equating the result to zero, we see that,

$$\gamma = \frac{r_{xx}(L)}{r_{xx}(0)}$$

and the resulting error is,

$$\sigma_e^2 = r_{xx}(0) - \gamma r_{xx}(L)$$

Hence, it follows that,

$$\sigma_e^2 = r_{xx}(0) - \frac{r_{xx}^2(L)}{r_{xx}(0)}$$

Since $r_{xx}(0) > 0$, σ_e^2 is minimized by choosing an $L > 0$ with maximum $|r_{xx}(L)|$. From the given values, it follows that one should choose, $L = 3$ and correspondingly, ,

$$\begin{aligned}\gamma &= -0.6 \\ \sigma_e^2 &= 0.64\end{aligned}$$

You should show that γ chosen above indeed minimizes and not maximizes the cost function to get full credits.

4. (a) Let $\mathbf{x} = [x(0), x(1), \ldots, x(N-1)]^T$, $\mathbf{f}_{n_k}$ denote the n_kth column of the IDFT matrix, $\mathbf{c} = [c_1, c_2, \ldots, c_p]^T$, $\mathbf{A} = [\mathbf{f}_{n_1}, \mathbf{f}_{n_2}, \ldots, \mathbf{f}_{n_p}]$ and $\mathbf{w} = [w(0), w(1), \ldots, w(N-1)]^T$. Then,

$$\mathbf{x} = \mathbf{A}\mathbf{c} + \mathbf{w}$$

wherein we have exploited the fact that $\omega_k = \frac{2\pi n_k}{N}$. Since the columns of $\mathbf{A}$ are columns of IDFT matrix, operating by DFT matrix on both sides yields,

$$\mathbf{W}\mathbf{x} = N\mathbf{v} + \eta$$

where, $\mathbf{W}$ is the DFT matrix, η is the transformed noise vector, $\mathbf{v}$ is a $N \times 1$ vector with n_kth element being c_k and the rest being zeros. Letting $\mathbf{X} = \mathbf{W}\mathbf{x}$, it follows from negligible noise component that,

$$\begin{aligned}|X(m)|^2 &\approx N|c_k|^2, \text{if} \quad m = n_k \quad \text{for some} \quad k = 1, 2, \ldots, p \\ |X(m)|^2 &= |\eta(m)|^2 \quad \text{else}\end{aligned}$$

Based on these, a simple algorithm would be,

- Obtain $|X(k)|^2$ and choose p largest entries.
- The locations of these entries yield n_k's and $\omega_k = \frac{2\pi n_k}{N}$. Further, $\frac{|X(k)|^2}{N}$ is a good approximation of $|c_k|^2$ as noise is assumed negligible.

The extension to ω_k not having the earlier form does not have any simple frequency estimation algorithm.

(b) If $S(k), T(k)$ and $Z(k)$ denote the 6 point DFT of $s(n), t(n)$ and $z(n)$, then,

$$Z(k) = S(k)T(k).$$

Since $s(n)$ and $z(n)$ are known, one can evaluate $T(k) = Z(k)/S(k)$. The sequence $t(n)$ is nothing but the 6 point IDFT of $T(k)$. However, this works out only if $S(k) \neq 0, \forall k$. Since $s(n)$ is periodic with period 3, it follows that,

$$S(k) = s(0)\left(1 + e^{-jk\pi}\right) + s(1)e^{\frac{-2jk\pi}{6}}\left(1 + e^{-jk\pi}\right) + s(2)e^{\frac{-j4k\pi}{6}}\left(1 + e^{-jk\pi}\right)$$

Clearly $S(k) = 0$ for all odd k and hence it is not possible to recover $T(k)$ from $Z(k)$.

5. (a) We have $F_1(z) = \sum_{m=0}^{L}(-1)^{m+1}f_0(L-m)z^{-m}$. Substituting $n = L - m$, we have,

$$F_1(z) = -\sum_{n=0}^{L}(-1)^{L-n}f_0(n)z^{n-L} = -(-1)^L z^{-L}\sum_{n=0}^{L}f_0(n)(-z)^n$$

Since L is odd, it follows that,

$$F_1(z) = z^{-L} F_0(-z^{-1})$$

Since $h_i(n) = f_i(L-n), \quad i = 0, 1$, we have, $H_i(z) = z^{-L} F_i(z^{-1})$. Then we have,

$$\begin{aligned} F_1(z) &= z^{-L} F_0(-z^{-1}) \\ H_0(z) &= z^{-L} F_0(z^{-1}) \\ H_1(z) &= F_0(-z) \end{aligned}$$

which yields the required relation.

(b) Since $r(n) = f_0(n) * f_0(-n)$, $R(z) = F_0(z)F_0(z^{-1})$. Let $s(n) = r(2n)$. Clearly $s(n) = \delta(n)$ and hence $S(z) = 1$. However, since $s(n) = r(2n)$, we have,

$$S(z) = \frac{1}{2}\left[R(z^{\frac{1}{2}}) + R(-z^{\frac{1}{2}})\right]$$

Using $S(z) = 1$ and $R(z) = F_0(z)F_0(z^{-1})$, we have,

$$F_0(z^{\frac{1}{2}})F_0(z^{\frac{-1}{2}}) + F_0(-z^{\frac{1}{2}})F_0(-z^{\frac{-1}{2}}) = 2$$

Since the earlier equation holds for all z, replacing z by z^2 yields the necessary result.

(c) Writing out the various filter functions in terms of $F_1(z)$, it can be shown that,

$$\begin{aligned} F_0(z) &= H_1(-z) \\ F_1(z) &= -H_0(-z) \\ F_0(z)H_0(z) - F_0(-z)H_0(-z) &= 2z^{-L} \end{aligned}$$

These conditions yield a perfect reconstructing filter bank.

(d) Evaluating $F_0(z)F_0(z) + F_0(-z)F_0(-z) = 2$ on the unit circle yields $|F_0(\omega)|^2 + |F_0(\omega + \pi)|^2 = 2$. However, the given filter does not satisfy this condition.