SIGNAL PROCESSING

DEPARTMENT OF ELECTRICAL ENGINEERING

Digital Signal Processing EQ2300/ 2E1340

Final Examination 2010–12–16, 8.00–13.00
Sample Solutions

1. (a) As the upper and lower branches are identical in structure, we can first treat the upper branch and then repeat the same process for the lower branch. The round-off (or quantization) noise at the output of Q_0 is modeled as white additive noise with an ACF of $r_{e_0}(k) = \sigma_{e_0}^2 \delta(k)$ where

$$\sigma_{e_0}^2 = \frac{2^{-2B_0}}{12} .$$

The noise is added at the position of Q_0 in the schematics. Amplification by a_0 yields an ACF of $r_z(k) = \sigma_{e0}^2 a_0^2 \delta(k)$. After up-sampling by a factor 2 (assuming the random delay), the ACF of the noise is (cf. complementary reading)

$$\frac{1}{2} r_z(2k) = \frac{\sigma_0^2 a_0^2}{2} \delta(k) ,$$

i.e., the noise is still white and the noise power at the output of the filter is therefore given by

$$\sigma_{\epsilon_0}^2 = \frac{\sigma_0^2 a_0^2}{2} \sum_{n=-\infty}^{\infty} g_0^2(n) = \frac{\sigma_0^2 a_0^2}{2} \frac{3}{2} = \frac{3a_0^2 2^{-2B_0}}{48} .$$

Using the same argument, the noise in the lower branch becomes

$$\sigma_{\epsilon_1}^2 = \frac{\sigma_1^2 a_1^2}{2} \sum_{n=-\infty}^{\infty} g_1^2(n) = \frac{\sigma_1^2 a_0^2}{2} \frac{23}{32} = \frac{23a_1^2 2^{-2B_1}}{768} .$$

Assuming the that the noise components in the two branches are independent, the total noise in the output is given by

$$P_Q = \sigma_{\epsilon_0}^2 + \sigma_{\epsilon_1}^2 = \frac{3a_0^2 2^{-2B_0}}{48} + \frac{23a_1^2 2^{-2B_1}}{768} ,$$

an expression that could be optimized over B_0 and B_1 to yield the optimal bit-allocation.

(b) The approximation is based on the assumption that the noise components introduced by the two quantizers are mutually independent and temporally white. The whiteness assumption depends on the input signals being sufficiently wide band, and B_0 and B_1 being reasonable large. The assumption that the noise components are mutually independent is also not exactly true (as the quantizer inputs are correlated) but will also be reasonable if B_0 and B_1 are large.

2. (a) We choose $p = 2$, and a transfer function of the AR model according to

$$H(z) = \frac{b_0}{1 + \sum_{k=1}^{p} a_k z^{-k}} .$$

as the the two peaks in the Periodogram can then be accurately modeled by the two poles in the transfer function.

(b) The AR parameters are given by solving the Yule-Walker equations:

$$\begin{bmatrix} \hat{r}_x(0) & \hat{r}_x^*(1) \\ \hat{r}_x(1) & \hat{r}_x(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = - \begin{bmatrix} \hat{r}_x(1) \\ \hat{r}_x(2) \end{bmatrix}$$

and $|b(0)|^2$ (the power of the driving noise) is obtained as

$$|b_0|^2 = \epsilon_2 = \hat{r}_x(0) + a_1\hat{r}_x(1) + a_2\hat{r}_x(2) \,.$$

In the $p = 2$ case we can also quite easily obtain a_1 and a_2 explicitly by solving the equation system above.

(c) When $p = 3$, $x(n)$ is modeled as AR(3) process given by

$$\bar{H}(z) = \frac{\bar{b}_0}{1 + \sum_{k=1}^{3} \bar{a}_k z^{-k}} \,.$$

The Levinson-Durbin Recursion become

- $\gamma_2 = r_s(3) + a_1 r_s(2) + a_2 r_s(1)$
  $\Gamma_3 = -\gamma_2/\epsilon_2$
- $\bar{a}_1 = a_1 + \Gamma_3 a_2^*$
  $\bar{a}_2 = a_2 + \Gamma_3 a_1^*$
  $\bar{a}_3 = \Gamma_3$
- $|\bar{b}_0|^2 = \epsilon_3 = \epsilon_2\,(1 - |\Gamma_3|^2)$

3. (a) We have

$$\begin{aligned} H(f) &= \sum_{n=-\infty}^{\infty} h[n] e^{-j2\pi f n} \\ &= \sum_{n=-2}^{2} h[n] e^{-j2\pi f n} \\ &= e^{j4\pi f} + 2e^{j2\pi f} + 3 + 2e^{-j2\pi f} + e^{-j4\pi f} \\ &= 2\text{cos}(4\pi f) + 4\text{cos}(2\pi f) + 3 \end{aligned}$$

(b) To answer this part, we apply the fact that $u[n] = e^{j2\pi f_0 n} \leftrightarrow U(f) = \delta(f - f_0)$. The input sequence can be rewritten as

$$x[n] = \left[e^{j\pi/4n} + e^{-j\pi/4n}\right] + \frac{1}{2}\left[e^{j\pi n} + e^{-j\pi n}\right]$$

1. $\Rightarrow X(f) = \delta(f - 1/8) + \delta(f + 1/8) + \frac{1}{2}\delta(f - 1/2) + \frac{1}{2}\delta(f + 1/2)$. Thus,

$$\begin{aligned} Y(f) &= X(f)H(f) \\ &= H(\frac{1}{8})\delta(f - \frac{1}{8}) + H(\frac{-1}{8})\delta(f + \frac{1}{8}) + \frac{1}{2}H(\frac{1}{2})\delta(f - \frac{1}{2}) + \frac{1}{2}H(\frac{1}{2})\delta(f - \frac{1}{2}) + \frac{1}{2}H(\frac{-1}{2}) \end{aligned}$$

Since $H(f)$ is an even function, therefore

$$\begin{aligned} &H(\frac{1}{8}) = H(\frac{-1}{8}) = 2\sqrt{2} + 3 \\ &H(\frac{1}{2}) = H(\frac{-1}{2}) = 1 \\ &\Rightarrow Y(f) = (2\sqrt(2) + 3)[\delta(f - \frac{1}{8}) + \delta(f + \frac{1}{8})] + \frac{1}{2}[\delta(f - \frac{1}{2}) + \delta(f + \frac{1}{2})] \end{aligned}$$

2. Finally, the output sequence can be stated as

$$
\begin{aligned}
y[n] &= (2\sqrt{2}+3)\left[e^{j2\pi n/8}+e^{-j2\pi n/8}\right]+\frac{1}{2}\left[e^{j2\pi n/2}+e^{-j2\pi n/2}\right] \\
&= 2(2\sqrt{2}+3)\cos(2\pi n/8)+\cos(2\pi n/2)
\end{aligned}
$$

(c) If $f^\star$ is at the null of $H(f)$, then $y[n] = y^\star[n]$. Furthermore,

$$
\begin{aligned}
H(f) &= 4\cos^2(2\pi f)+4\cos(2\pi f)+1 \\
&= (\cos(2\pi f)+\frac{1}{2})^2
\end{aligned}
$$

where the last equality follows from the property $\cos(2\alpha) = 2\cos^2(\alpha)-1$. Hence, $H(f)=0$ results in $\cos(2\pi f^\star) = \frac{-1}{2} \implies f^\star = \frac{1}{3}$.

4. The thing to realize here is that the problem is very similar to nonparametric spectrum estimation. In particular, truncation of the impulse response corresponds to multiplication with a rectangular window $w_M(n) = 1$ for $|n| \leq M$ and $w_M(n) = 0$ for $|n| > M$. The TDFT of the input response is given by

$$
W_M(f) = \frac{\sin((2M+1)\pi f)}{\sin(\pi f)}\,.
$$

(a) Multiplication in the time-domain corresponds to convolution in the frequency domain. Thus,

$$
H_M(f) = H(f) * W_M(f) = \int_{\tau=-1/2}^{1/2} H(\tau)W_M(f-\tau)d\tau
$$

$$
= \int_{\tau=-1/4}^{1/4} \frac{\sin((2M+1)\pi(f-\tau))}{\sin(\pi(f-\tau))}d\tau\,.
$$

(b) Increasing M leads to a narrowing of the main-lobe and the side-lobes, but not a reduction of their magnitude (cf. spectrum estimation). Thus, the answer is: No, increasing M will not reduce the magnitude of the side lobes.

(c) One way (cf. spectrum estimation) is to use another window than the rectangular window, e.g., the Bartlet, Hamming, Hanning, or Blackman window. This will reduce the magnitude of the side-lobes, but also a widening of the main-lobe (and the pass-band of the windowed filter). Note here that there are also windows specifically designed for shortening FIR filters. One such family of windows are the Kaiser windows.

5. It is convenient to rewrite the data model in matrix form:

$$
\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \cos(2\pi f_0) & \sin(2\pi f_0) \\ \vdots & \vdots \\ \cos(2\pi f_0(N-1)) & \sin(2\pi f_0(N-1)) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} e(0) \\ e(1) \\ \vdots \\ e(N-1) \end{bmatrix}.
$$

Define $\mathbf{x} = [x(0), x(1), \cdots, x(N-1)]^{\mathrm{T}}$, $\mathbf{h}_1 = [1, \cos(2\pi f_0), \cdots, \cos(2\pi f_0(N-1))]^{\mathrm{T}}$ and $\mathbf{h}_2 = [0, \sin(2\pi f_0), \cdots, \sin(2\pi f_0(N-1))]^{\mathrm{T}}$. Also, let $\mathbf{H} = [\mathbf{h}_1\ \mathbf{h}_2]$.

Finding the LSE corresponds to computing

$$\begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = (\mathbf{H}^{\mathrm{T}}\mathbf{H})^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{x},$$

since the inverse of $\mathbf{H}^{\mathrm{T}}\mathbf{H}$ exists. To explicitly show this, recall that $\mathbf{H} = [\mathbf{h}_1 \ \mathbf{h}_2]$ and thus

$$\mathbf{H}^{\mathrm{T}}\mathbf{H} = \begin{bmatrix} \mathbf{h}_1^{\mathrm{T}} \\ \mathbf{h}_2^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{h}_1^{\mathrm{T}}\mathbf{h}_1 & \mathbf{h}_1^{\mathrm{T}}\mathbf{h}_2 \\ \mathbf{h}_2^{\mathrm{T}}\mathbf{h}_1 & \mathbf{h}_2^{\mathrm{T}}\mathbf{h}_2 \end{bmatrix}, \text{ with}$$

$$\begin{aligned}\mathbf{h}_1^{\mathrm{T}}\mathbf{h}_1 &= \sum_{n=0}^{N-1} \cos^2(2\pi f_0 n) = \sum_{n=0}^{N-1} \left(\frac{1}{2} + \frac{1}{2}\cos(4\pi f_0 n)\right) = \frac{N}{2} + \frac{1}{2}\sum_{n=0}^{N-1} \cos(4\pi f_0 n) = \\ &= \frac{N}{2} + \frac{1}{2}\Re\left(\sum_{n=0}^{N-1} e^{j4\pi \frac{k}{N} n}\right) = \frac{N}{2} + \frac{1}{2}\Re\left(\frac{1-e^{j4\pi k}}{1-e^{j4\pi \frac{k}{N}}}\right) = \frac{N}{2},\end{aligned}$$

$$\mathbf{h}_2^{\mathrm{T}}\mathbf{h}_2 = \sum_{n=0}^{N-1} \sin^2(2\pi f_0 n) = N - \sum_{n=0}^{N-1} \cos^2(2\pi f_0 n) = \frac{N}{2}, \text{ and}$$

$$\mathbf{h}_1^{\mathrm{T}}\mathbf{h}_2 = \sum_{n=0}^{N-1} \cos(2\pi f_0 n)\sin(2\pi f_0 n) = \frac{1}{2}\sum_{n=0}^{N-1} \sin(4\pi f_0 n) = \frac{1}{2}\Im\left(\sum_{n=0}^{N-1} e^{j4\pi \frac{k}{N} n}\right) = 0.$$

Hence, the LSE can be expressed as

$$\begin{aligned}\begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} &= \begin{bmatrix} \frac{N}{2} & 0 \\ 0 & \frac{N}{2} \end{bmatrix}^{-1} \mathbf{H}^{\mathrm{T}}\mathbf{x} = \\ &= \begin{bmatrix} \frac{2}{N} & 0 \\ 0 & \frac{2}{N} \end{bmatrix} \begin{bmatrix} \mathbf{h}_1^{\mathrm{T}} \\ \mathbf{h}_2^{\mathrm{T}} \end{bmatrix} \mathbf{x} = \frac{2}{N} \begin{bmatrix} \mathbf{h}_1^{\mathrm{T}}\mathbf{x} \\ \mathbf{h}_2^{\mathrm{T}}\mathbf{x} \end{bmatrix},\end{aligned}$$

which gives

$$\begin{aligned}\hat{\alpha} &= \frac{2}{N}\sum_{n=0}^{N-1} x(n)\cos(2\pi f_0 n), \qquad \text{and also} \\ \hat{\beta} &= \frac{2}{N}\sum_{n=0}^{N-1} x(n)\sin(2\pi f_0 n).\end{aligned}$$