SOLUTIONS

E 97 **Digital Signalbehandling**, 2E1340

Final Examination 2005–08–25, 08.00–13.00

1. A partial fractions expansion of $H(z)$ results in something of the form

$$H(z) = A + \frac{B}{1 - 3/2z^{-1}} + \frac{C}{1 - 1/3z^{-1}}$$

Since both poles have a magnitude smaller than 2, the relevant geometric series expansion is

$$\frac{1}{1 - az^{-1}} = \sum_{n=0}^{\infty} a^n z^{-1}$$

(which in general holds as long as $|z| > |a|$), so the impulse response has the form

$$h(n) = \begin{cases} A\delta(n) + B(3/2)^n + C(1/3)^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

This shows that

a) The system is unstable

b) The system is causal

c) Since $x(n)$ is real valued, the DFT has the following symmetry property: $X((-k)_{\mod 32}) = X^c(k)$. So, it is sufficient to know the 16 values $X(0), \ldots, X(15)$.

d) From the definition, $X(0) = \sum_0^{31} x(n)$, so the average is $X(0)/32$ is positive.
Nothing can be said about the symmetry around $n = 16$. Consider, for example, the signals $x_1(0) = 10$, $x_1(n) = 0$, $n > 0$ compared with $x_2(15) = x_2(16) = 5$, $x_2(n) = 0$ otherwise.

2. a) The Blackman-Tukey estimate can be written:

$$P_{\rm BT}(f) = \sum_{n=-3}^{3} \hat{r}_x(n) w(n) e^{-j2\pi fn},$$

where

$$\hat{r}_x(n) = \begin{cases} \frac{1}{N}\sum_{m=0}^{N-1-n} x(m+n)x(m), & n = 0, \ldots, N-1, \\ \hat{r}_x(-n), & n < 0. \end{cases}$$

$w(n)$ should be triangular and non-zero from $w(-3)$ through $w(3)$. To get the correct normalization we need $\int_{-1/2}^{1/2} W(f)df = 1 \iff w(0) = 1$. $\hat{r}_x(n)w(n)$ is symmetric around zero, so we can rewrite $P_{\rm BT}(f)$ as

$$\begin{aligned} P_{\rm BT}(f) &= \hat{r}_x(0)w(0) + \sum_{n=1}^{3} \hat{r}_x(n)w(n)(e^{-j2\pi fn} + e^{j2\pi fn}) \\ &= \hat{r}_x(0)w(0) + 2\sum_{n=1}^{3} \hat{r}_x(n)w(n)\cos(2\pi fn). \end{aligned}$$

$f = \frac{k}{8}$, $k = 0, \ldots, 7 \implies$

$$P_{\rm BT}(k) = \hat{r}_x(0)w(0) + 2\sum_{n=1}^{3} \hat{r}_x(n)w(n)\cos\frac{\pi kn}{4}$$

n	0	1	2	3
$\hat{r}_x(n)$	$74/10$	$33/10$	$-8/10$	$-14/10$
$w(n)$	$4/4$	$3/4$	$2/4$	$1/4$
$\hat{r}_x(n)w(n)$	$296/40$	$99/40$	$-16/40$	$-14/40$

k	0	1	2	3	4	5	6	7
$P_{\rm BT}(k)$	$\frac{439}{40}$	$\frac{1}{40}(296 + 113\sqrt{2})$	$\frac{238}{40}$	$\frac{1}{40}(296 - 113\sqrt{2})$	$\frac{94}{40}$	$P_{\rm BT}(3)$	$P_{\rm BT}(2)$	$P_{\rm BT}(1)$
$\approx$	10.85	11.40	8.20	3.40	2.35	3.40	8.2	11.40

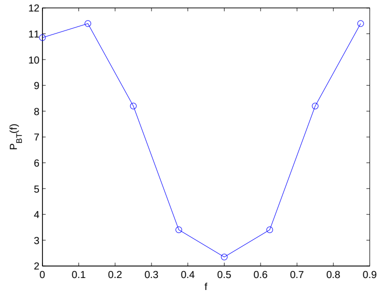

Figure 1: Plot of spectrum estimate

b) Since the size of the window is reduced compared to the periodogram, the resolution capabilities of this estimate are reduced as well. However, this spectrum estimate has lower variance than a standard periodogram. On the other hand, when the number of samples is as low as 10, it is difficult to get any useful results, no matter what method is used.

3. We want to determine the missing parameters, M, A and B. Study first for which frequencies the output $X(f)$ is zero and non-zero. No matter what $W(f)$ is, the output will be zero in the intervals $[(k - B)/M, (k + B)/M]$ for any integer k since $X(f) = W(Mf)$, see the figure. Comparing to the provided shape of $X(f)$, we first see that $M = 2$ since the signal is non-zero around $1/3$, $1/4$, ... Then, we can conclude that $B/M = 1/12$, which gives $B = 1/6$. Next, from

$$V(f) = \frac{1}{2}\left(X\left(\frac{f}{2}\right) + X\left(\frac{f-1}{2}\right)\right)$$

we plot $V(f_v)$ and recognize that the corners at $f_v = 1 - 2A$ and $f_v = 2A$ must correspond to the corners of $X(f_x)$ at $f_x = 1/6$ and $f_x = 1/3$, respectively. However, since $f_x = 2f_x$, we get $A = 1/3$.

One possible problem formulation is *"Figure xxx shows a multirate system and the DTFT $U(f)$ of the input signal $u(n)$. Determine and plot the DTFT of the output signal $x(n)$ if $M = 2$, $A = 1/3$ and $B = 1/6$."*

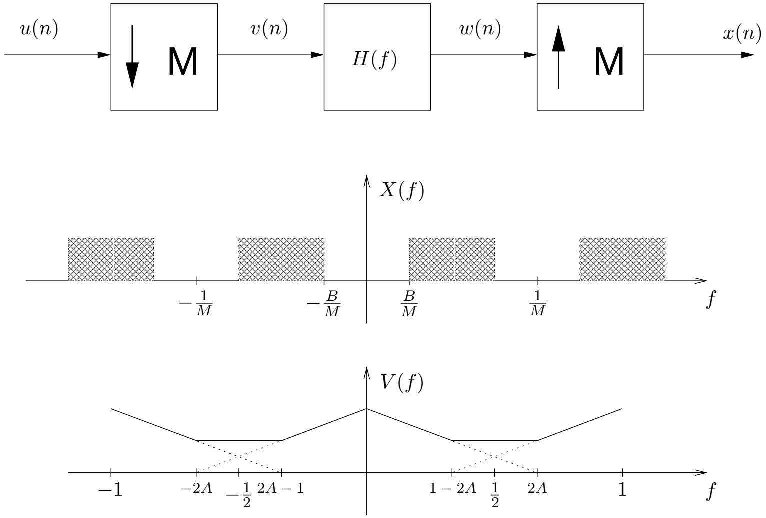

Figure 2:

4. a) We want to prove $y_1 * h_2 = y_2 * h_1$. We'll use that convolution is both associative and commutative. We know that:

$$y_1 = x * h_1$$

$$y_2 = x * h_2$$

Thus:

$$y_1 * h_2 = (x * h_1) * h_2 = x * h_2 * h_1 = y_2 * h_1 \tag{1}$$

b) Rewrite (1) as:

$$\sum_{k=0}^{1} h_2(k) y_1(n-k) - \sum_{k=0}^{1} h_1(k) y_2(n-k) = 0$$

expand the summations to get:

$$h_2(0)y_1(n) + h_2(1) * y_1(n-1) - h_1(0) * y_2(n) - h_1(1)y_2(n-1) = 0$$

We now use that $h_1(0) = 1$ to get.

$$h_2(0)y_1(n) + h_2(1) * y_1(n-1) - h_1(1)y_2(n-1) = y_2(n)$$

Writing this in matrix structure we get:

$$\begin{bmatrix} y_1(0) & 0 & 0 \\ y_1(1) & y_1(0) & -y_2(0) \\ \vdots & & \vdots \\ y_1(N) & y_1(N-1) & -y_2(N-1) \end{bmatrix} \begin{bmatrix} h_2(0) \\ h_2(1) \\ h_1(1) \end{bmatrix} = \begin{bmatrix} y_2(0) \\ y_2(1) \\ \vdots \\ y_2(N) \end{bmatrix}$$

Which for $n > 2$ is an overdetermined equation system, easily solved by least squares.

5. We assume a floating point representation similar to the IEEE 754 standard, where numbers are represented as $y = (-1)^S m 2^e$, where the exponent e is determined so that the mantissa is in the range $1/2 \leq m < 1$. Here, the results of the multiplication are in the range $[2\pi, 3\pi] \approx [6.3, 9.4]$ so the only values used for the exponent is $e = 3$, i.e. $y = 8m$, for values in the range $[6.3, 8[$ and $e = 4$, i.e. $y = 16m$, for values in the range $[8, 9.4]$. Assume that the mantissa (which is a fixed point number) is represented by B_m bits. Then, the quantization error in the mantissa is uniformly distributed in $[-\frac{2^{-B_m}}{2}, \frac{2^{-B_m}}{2}]$. This will correspond to a quantization error in y in the range $[-8\frac{2^{-B_m}}{2}, 8\frac{2^{-B_m}}{2}]$ for y values where $e = 3$ and $[-16\frac{2^{-B_m}}{2}, 16\frac{2^{-B_m}}{2}]$ when $e = 4$. The total distribution of the quantization errors will be a combination of these uniform distributions, which gives the staircase shape shown in the figure. From the figure, we see that the maximum quantization error is $1.91 \cdot 10^{-6} \approx 2^{-19}$, which must correspond to $16\frac{2^{-B_m}}{2}$, which gives $B_m = 22$. Since we only study the system for a small range of values and do not observe any overflow, there is no information available to tell how many bits are used for the exponent.

Since the mantissa is known to be in the range $1/2 \leq m < 1$, the most significant bit will always be 1, so this bit is actually never stored. If we take this trick into account, the number of bits used in the actual representation of the mantissa is actually $B_m = 21$.

To summarize:

a) $B_m = 22$ (or $B_m = 21$ depending on how the mantissa is stored) bits are used for the mantissa.

b) We do not have enough information to tell how many bits are used for the exponent.