SOLUTIONS

E 82 **Digital Signalbehandling**, 2E1340

Final Examination 2000–09–01, 1400–1800

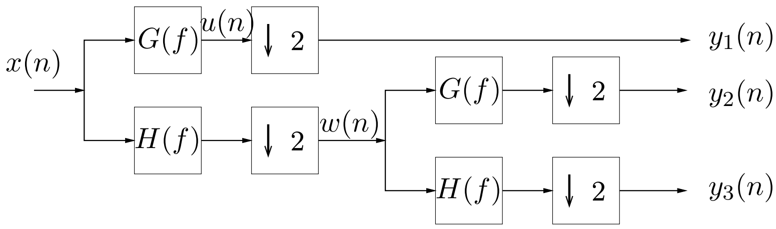
1. Define the vectors $u(n) = [\underset{\uparrow}{1}, 2, 3, 4]$ and $U = \mathrm{DFT}\{u\}$.

a) Conjugation in the frequency domain corresponds to time reversal and conjugation in the time domain, therefore $x(n) = u^c((-n)_4) = [\underset{\uparrow}{1}, 4, 3, 2]$, so the answer is iii).

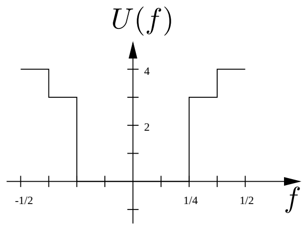
b) Reversing the order in time gives a reversed order also in frequency, thus $x(n) = u((-n)_4) = [\underset{\uparrow}{1}, 4, 3, 2]$, so the answer is iii).

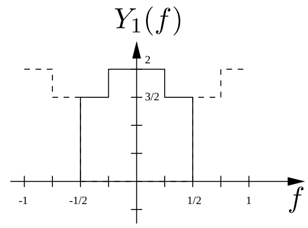
c) Compared to b), the DFT sequence is circularly shifted one step to the right, which corresponds to $x_c(n) = e^{-j2\pi n/4} x_b(n) = [\underset{\uparrow}{1}, -4j, -3, 2j]$ and the answer is iv).

2. Introduce the intermediate signals $u(n)$ and $w(n)$:

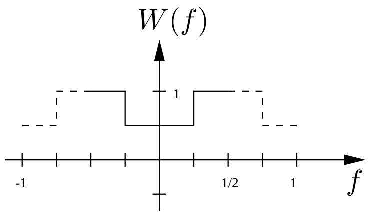


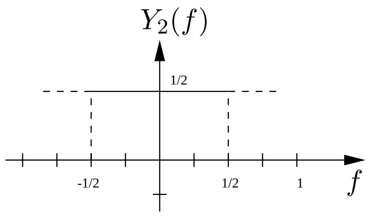
Then, $U(f)$ contains the part of $X(f)$ with frequencies higher than 1/4 and $Y_1(f) = 1/2(U(f/2) + U((f-1)/2)$, which gives:

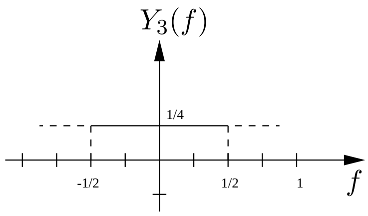




Likewise, $W(f)$ contain the low-frequency part of $X(f)$ scaled in amplitude and frequency and repeated in frequency. The operation from $W(f)$ to $Y_2(f)$ and $Y_3(f)$ is the same as from $X(f)$ to $Y_1(f)$ and $W(f)$, which gives:







3. a) The plots look different, since we have only 64 sample values and use no zero-padding in the FFT. The DFT of the signal $x(n) = \cos(2\pi f n)$ is

$$X(k) = \sum_{n=0}^{N-1} \frac{1}{2}(e^{j2\pi fn} + e^{-j2\pi fn})e^{-j2\pi kn/N} = \dots$$
$$= e^{j\pi \frac{N-1}{N}(Nf-k)} \frac{\cos(\pi(Nf-k))}{\cos(\pi(f-\frac{k}{N}))} + e^{-j\pi \frac{N-1}{N}(Nf+k)} \frac{\cos(-\pi(Nf+k))}{\cos(-\pi(f+\frac{k}{N}))}$$

Study the first term (which is the only one that contributes in the region $0 \le f \le 1/2$). If Nf is an integer, then the DFT will be zero for all $k \ne Nf$, just as in the plot for $x_1(n)$. If Nf is not an integer, then all DFT values will be non-zero, just as in the plot for $x_2(n)$. For $F = 4$kHz, $Nf = 64 \cdot 4/16 = 16$ which is an integer, whereas for $F = 4125$Hz, $Nf = 64 \cdot 4125/16000 = 16.5$. Thus, x_1 has frequency $F = 4$kHz and x_2 has frequency $F = 4125$Hz. If we had used zero-padding, both plots would have shown the side-lobes caused by the short data sample.

b) The three peaks in the spectrum correspond to signal components of the form $e^{j2\pi ft}$ with normalized frequencies, $f = 0.3$, 0.2 and -0.2 (corresponding to the peak at 0.8). Since the spectrum is not symmetric, the signal is not real valued, which excludes the alternatives x_1 and x_2. x_4 is impossible since it would give 2+1+1=4 peaks in the spectrum. The spectrum of the sampled version of x_3 is $X_3(f) = j/2(\delta(f - f_1) - \delta(f + f_1)) + \delta(f - f_2)$ with peaks at $\pm f_1$ and f_2, thus $f_1 = 0.2$ and $f_2 = 0.3$ and the corresponding continuous time frequencies are $F_1 = f_1 F_s = 2$kHz and $F_2 = f_2 F_s = 3$kHz.

4. The quantization noise for each muliplicator is approximately white with power $\sigma_e^2 = \frac{2^{-2b}}{12}$. In the eqivalent model this noise is added in each summator following a multiplicator. The equivalent quantization noise power at the input is $\sigma_\epsilon^2 = (2 + k_2^2)\sigma_e^2 = 2.098\sigma_e^2$. The transfer function for the filter is

$$H(z) = \frac{1}{1 + k_1(1+k_2)z^{-1} + k_2 z^{-2}} = \frac{1}{1 + \frac{1}{8}z^{-1} - \frac{5}{16}z^{-2}} = \frac{\frac{4}{9}}{1 - \frac{1}{2}z^{-1}} + \frac{\frac{5}{9}}{1 + \frac{5}{8}z^{-1}}$$

which gives the impulse response

$$h(n) = \left(\frac{4}{9}\left(\frac{1}{2}\right)^n + \frac{5}{9}\left(-\frac{5}{8}\right)^n\right) u(n)$$

The quantization noise power at the output is then given by

$$\sigma_{output}^2 = \sigma_\epsilon^2 \sum_{n=0}^{\infty} h^2(n) = (2 + k_2^2)\frac{2^{-2b}}{12} \sum_{n=0}^{\infty} h^2(n)$$

where

$$\sum_{n=0}^{\infty} h^2(n) = \sum_{n=0}^{\infty} \left(\frac{4}{9}\left(\frac{1}{2}\right)^n + \frac{5}{9}\left(-\frac{5}{8}\right)^n\right)^2 = \frac{16 \cdot 4}{81 \cdot 3} + \frac{2 \cdot 20 \cdot 16}{81 \cdot 21} + \frac{25 \cdot 64}{81 \cdot 39} = 1.146$$

The SNR is then

$$\text{SNR} = \frac{P_s}{\sigma_{output}^2} = \frac{0.001}{2.098 \cdot 1.146 \cdot \frac{2^{-2b}}{12}}$$

Sufficient SNR is then obtained when using at least $b = 6$ bits.

5. a) Least squares problem which can be solved by, for example, completing the squares.

$$\begin{aligned}\sum_{t=1}^{N}(\boldsymbol{x}(t)-\boldsymbol{a}s(t))^*(\boldsymbol{x}-\boldsymbol{a}s(t)) &= \sum_{t=1}^{N}(\boldsymbol{x}^*(t)\boldsymbol{x}(t)-s^*(t)\boldsymbol{a}^*\boldsymbol{x}(t)-\boldsymbol{x}(t)^*\boldsymbol{a}s(t)+s^*(t)\boldsymbol{a}^*\boldsymbol{a}s(t))\\ &= \sum_{t=1}^{N}(\boldsymbol{x}^*(t)\boldsymbol{x}(t)-s^*(t)\boldsymbol{a}^*\boldsymbol{x}(t)-\boldsymbol{x}(t)^*\boldsymbol{a}s(t)+ms^*(t)s(t))\end{aligned}$$

where $\boldsymbol{a}=\boldsymbol{a}(\theta_0)$. Minimizing this expression is equivalent to minimizing

$$\sum_{t=1}^{N}\left(s(t)-\frac{1}{m}\boldsymbol{a}^*\boldsymbol{x}(t)\right)^*\left(s(t)-\frac{1}{m}\boldsymbol{a}^*\boldsymbol{x}(t)\right)$$

and thus $\hat{s}(t)=\frac{1}{m}\boldsymbol{a}^*\boldsymbol{x}(t),\quad t=1,\dots,N.$

b) Similarly we have

$$\begin{aligned}\sum_{t=1}^{N}(\boldsymbol{x}(t)-\boldsymbol{A}\boldsymbol{s}(t))^*(\boldsymbol{x}-\boldsymbol{A}\boldsymbol{s}(t)) &= \sum_{t=1}^{N}(\boldsymbol{x}^*(t)\boldsymbol{x}(t)-\boldsymbol{s}^*(t)\boldsymbol{A}^*\boldsymbol{x}(t)-\boldsymbol{x}(t)^*\boldsymbol{A}\boldsymbol{s}(t)+s^*(t)\boldsymbol{A}^*\boldsymbol{A}s(t))\\ &= \sum_{t=1}^{N}\Big(\left(\boldsymbol{s}(t)-(\boldsymbol{A}^*\boldsymbol{A})^{-1}\boldsymbol{A}^*\boldsymbol{x}(t)\right)^*\boldsymbol{A}^*\boldsymbol{A}\left(\boldsymbol{s}(t)-(\boldsymbol{A}^*\boldsymbol{A})^{-1}\boldsymbol{A}^*\boldsymbol{x}(t)\right)\\ &\quad+\boldsymbol{x}^*(t)\boldsymbol{x}(t)-\boldsymbol{x}^*(t)\boldsymbol{A}(\boldsymbol{A}^*\boldsymbol{A})^{-1}\boldsymbol{A}^*\boldsymbol{x}(t)\Big)\end{aligned}$$

Since $\boldsymbol{A}^*\boldsymbol{A}$ is positive-semi definite, this is minimized when

$$\hat{\boldsymbol{s}}(t)=(\boldsymbol{A}^*\boldsymbol{A})^{-1}\boldsymbol{A}^*\boldsymbol{x}(t),\quad t=1,\dots,N.$$

c)

$$\mathrm{E}\{\hat{\boldsymbol{s}}(t)\}=(\boldsymbol{A}^*\boldsymbol{A})^{-1}\boldsymbol{A}^*(\boldsymbol{A}\boldsymbol{s}(t)+\mathrm{E}\{\boldsymbol{n}(t)\})=\boldsymbol{s}(t)$$

The estimator is unbiased.

$$\begin{aligned}\mathrm{E}\{(\hat{\boldsymbol{s}}(t)-\boldsymbol{s}(t))(\hat{\boldsymbol{s}}(t)-\boldsymbol{s}(t))^* &= \mathrm{E}\{\hat{\boldsymbol{s}}(t)\hat{\boldsymbol{s}}^*(t)\}-\boldsymbol{s}(t)\boldsymbol{s}^*(t)\\ &= (\boldsymbol{A}^*\boldsymbol{A})^{-1}\boldsymbol{A}^*\mathrm{E}\{\boldsymbol{n}(t)\boldsymbol{n}^*(t)\}\boldsymbol{A}(\boldsymbol{A}^*\boldsymbol{A})^{-1}\\ &= \sigma^2(\boldsymbol{A}^*\boldsymbol{A})^{-1}\end{aligned}$$

Thus, the covariance is proportional to the noise variance and decreases with increasing number of sensors.