SOLUTIONS

E 84 **Digital Signalbehandling**, 2E1340

Final Examination 2001–04–18, 09.00–13.00

1. a) Answer: (ii). The only value of ($\pm\frac{12\,800}{8\,000}$ mod 1) within the requested interval is 0.4.

b) Answer: (ii) If we use a 300-point FFT, the 300 points will correspond to 1000 MHz frequency spectrum, i.e. each FFT value represents 1000/300 MHz. Since the useful bandwidth is 200 MHz, 200/(1000/300) = 60 points should be saved.

c) Answer: (i)–(B), (ii)–(C), (iii)–(A) To solve this question, we could consider the resolution, the sidelobe level, and the largest magnitude of each plot. Since A has lower sidelobes than B and C, it corresponds to the Hamming window method. Both the resolution and the scaling of the vertical axis indicates that B is calculated with fewer points than C.

2. a) His processing of the signal, corresponds to a multiplication of the DFT of the signal by

$$H(k) = \begin{cases} 1 & k = -1, 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

This low-pass filter preserves normalized frequencies up to $B = 1/N = 1/64 \approx 0.0156$.

b) Multiplication of the DFT by $H(k)$ corresponds to a *circular* convolution of each segment by $h(n) = \text{IFFT}[H(k)]$. Since it is a low-pass filter, the circular convolution will give a result where the first and last elements in each segment are almost identical, as can be seen in the middle figure. The proper way to implement a filter using FFT is to use overlap-save or overlap-add.

c) Time-domain filtering by a filter of length $M = 10$ requires 10 multiplications for each output value.

His method requires one 64 point FFT and one IFFT for each segment. Each N point FFT/IFFT requires $\frac{N}{2}\log_2 N = 192$ complex valued multiplications. Since he uses all 64 output values from each segment and each complex valued multiplication requires 4 real valued multiplications, this corresponds to $4\frac{192+192}{64} = 24$ real multiplications for each output value (can be reduced by a factor 2 since the signal is real valued).

Overlap-save, finally, requires one FFT, N complex valued multiplications and one IFFT to produce $N + 1 - M = 55$ output values for each segment. This gives a total of $4\frac{192+64+192}{55} \approx 32.6$ real valued multiplications for each output value. For longer filters and longer segments, though, this method would be significantly faster than the time-domain implementation.

3. In the desired system, the output is given by

$$Y(F) = P(F)\frac{1}{T}\sum_{m=-\infty}^{\infty} X(F - mF_s) = \begin{cases} \sum_{m=-\infty}^{\infty} X(F - mF_s) & |F| \leq \frac{F_s}{2} \\ 0 & \text{otherwise} \end{cases}$$

If $Y(F)$ is bandlimited to $[-2F_s, 2F_s]$, then only the terms with $m = -2, -1, 0, 1, 2$ will be non-zero.

In the oversampled system, the output is given by (assuming that $Y(F)$ is bandlimited to $[-2F_s, 2F_s]$):

$$Y(F) = Q(F)H\left(\frac{F}{4F_s}\right)G\left(\frac{F}{F_s}\right)\frac{1}{4}\sum_{k=0}^{3} F\left(\frac{F}{4F_s} - \frac{k}{4}\right)\sum_{m=-\infty}^{\infty} X(F - kF_s - 4mF_s)$$

If we set $F(F) = G(F) = 1$, and introduce the summation variable $l = k + 4m$, this reduces to

$$Y(F) = Q(F)H\left(\frac{F}{4F_s}\right)\frac{1}{4}\sum_{l=-\infty}^{\infty} X(F - lF_s)$$

and we will get the desired output if $\frac{1}{4}Q(F)H(\frac{F}{4F_s}) = P(F)$, i.e.

$$H(f) = \frac{4P(fF_s)}{Q(fF_s)} = \begin{cases} 1 & |f| \leq \frac{1}{8} \\ 0 & \text{otherwise} \end{cases}$$

4. a) Quantization noise will occur at both multiplicators and equivalently, two quantization noise signals with variances σ_q, will add into the second adder. Assume B bits excluding sign bit, in the following.

The variance of the quantization noise from one of the multiplicators may be written: $\sigma_q^2 = \frac{2^{-2B}}{12}$. As there are two multipliers, the quantization noise variance added is $2\sigma_q^2$

The quantization noise generated, will pass through the second stage of the filter. The impulse response of the second stage of the filter, $h_2(n) = a^n u(n)$, where $u(n)$ is the unit step function. Thus the output quantization noise variance $\sigma_y^2 = 2\sigma_q^2\sum_{-\infty}^{+\infty} h_2^2(n) = 2\frac{2^{-2B}}{12}\frac{1}{1-a^2}$

b) The white quantization noise will be filtered through the second half of the filter, $h_2(n)$ and will thus be colored. The transfer function $H_2(f) = \frac{1}{1-ae^{-j2\pi f}}$. The output spectrum due to quantization noise may then be written $R_Y(f) = R_q(f)|H(f)|^2 = 2\sigma_q^2\frac{1}{1-2a\cos(2\pi f)+a^2}$, where $R_q(f) = 2\sigma_q^2$. The spectrum $R_Y(f)$ is plotted in Figure 1.

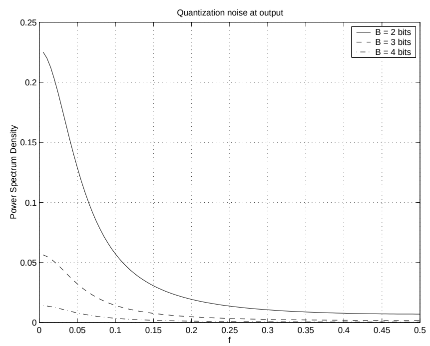

Figure 1: Noise Power Spectrum Density

5. a) If we define the decimated signal $y(n) = x(2n)$, the given model is equivalent to the standard AR(2) model

$$y(n) + a_1 y(n-1) + a_2 y(n-2) = v(n)$$

where $v(n) = e(2n)$, which gives $\sigma_v^2 = \sigma_e^2$. The autocorrelation function of $y(n)$ is given by $r_y(k) = r_x(2k)$. Standard Yule-Walker gives

$$\begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \end{bmatrix} = -\begin{bmatrix} r_y(0) & r_y(1) \\ r_y(1) & r_y(0) \end{bmatrix}^{-1} \begin{bmatrix} r_y(1) \\ r_y(2) \end{bmatrix} = \begin{bmatrix} 75.5 & -63.4 \\ -63.4 & 75.5 \end{bmatrix}^{-1} \begin{bmatrix} -63.4 \\ 37.0 \end{bmatrix} \approx \begin{bmatrix} 1.45 \\ 0.73 \end{bmatrix}$$

and

$$\hat{\sigma}_e^2 = \hat{\sigma}_y^2 = r_y(0) + \hat{a}_1 r_y(1) + \hat{a}_2 r_y(2) \approx 10.4$$

b) The AR model for the decimated signal $y(n)$ has poles given by the solution of

$$z^2 + \hat{a}_1 z + \hat{a}_2 = 0$$

i.e. $\hat{z}_{1,2} \approx -0.726 \pm 0.450i \approx 0.854e^{\pm 2.59j}$. These poles will give peaks in the spectrum at $\hat{f}_y \approx \frac{\arg[\hat{z}_{1,2}]}{2\pi} \approx \pm 0.412$. Since $y(n)$ is decimated by a factor of two, compared to $x(n)$, this will correspond to the frequency $\hat{f}_x \approx \frac{0.412}{2} = 0.206$ for $x(n)$ (it could also correspond to an aliased frequency at $0.5 - 0.206$, but then $r_x(1)$ would have been negative).