SOLUTIONS

E 98 **Digital Signalbehandling**, 2E1340

Final Examination 2005–12–19, 14.00–19.00

1. We have the sampled signal

$$\begin{aligned} x[n] &= a_1 \sin(2\pi f_1 n) + a_2 \sin(2\pi f_2 n) + w(n) \\ &= \frac{a_1}{2j}\left(e^{j2\pi f_1 n} - e^{-j2\pi f_1 n}\right) + \frac{a_2}{2j}\left(e^{j2\pi f_2 n} - e^{-j2\pi f_2 n}\right) + w(n) \end{aligned}$$

The periodogram can be expressed as:

$$\hat{P}(f) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n) e^{-j2\pi f n}\right|^2$$

Looking at the figure we directly see that the sinusoid with the higher amplitude has to be the one at $f_d = 0.1$.
Now we use the fact that the spectrum of a sinusoid is only a peak at the specific frequency and that the spectrum of white noise is a constant. Furthermore we use the assumption that it is only the sinusoids that contribute to the periodogram at frequencies f_1 and f_2 respectively. When looking at the peak at $f_d = 0.1$, we have

$$\begin{aligned} \hat{P}(f_2) &= \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n) e^{-j2\pi f_2 n}\right|^2 \\ &\approx \{\text{neglecting the noise and the other sinusoid}\} \\ &\approx \frac{1}{N}\left|\sum_{n=0}^{N-1}\left(\frac{a_2}{2j}\left(e^{j2\pi f_2 n} - e^{-j2\pi f_2 n}\right)\right) e^{-j2\pi f_2 n}\right|^2 \\ &= \frac{1}{N}\left|\sum_{n=0}^{N-1}\frac{a_2}{2}\left(1 - e^{-j4\pi f_2 n}\right)\right|^2 \\ &\approx \frac{1}{N} N^2 \frac{a_2^2}{4} = \frac{a_2^2 N}{4} \end{aligned}$$

Similarly, for the peak at $f_d = 0.3$, we get.

$$\hat{P}(f_1) \approx \frac{a_1^2 N}{4}$$

Reading from the figure we get that $\hat{P}(f_1) = 20dB$ and $\hat{P}(f_2) = 30dB$. Using this we get.

$$\frac{a_1^2 N}{4} = 10^{\frac{20}{10}} \quad \Rightarrow \quad a_1 \approx 0.8839$$

$$\frac{a_2^2 N}{4} = 10^{\frac{30}{10}} \quad \Rightarrow \quad a_2 \approx 2.7951$$

We now use the fact that $a_2 = K a_1 \Rightarrow K = 3$ (3.1623), since K is an integer.
For the noise power we have, assuming no contribution from the sinusoids outside frequencies $f_d = 0.1$ and $f_d = 0.3$

$$E[\hat{P}(f)] = P_{xx}(f) = \{\text{White noise}\} = \sigma_w^2 \tag{1}$$

From the figure we read the average noise power to be 0dB.
Thus we get, $\sigma_w^2 = 10^0 = 1$
Finally looking at the frequencies we have $f_1 = 0.3 \quad \Rightarrow \quad f_2 = 0.9$. f_2 is aliased in on $f = 0.9 - 1 = -0.1$ and $f = -0.9 + 1 = 0.1$.
There is no information in the problem that tells us about aliasing of the lower frequency, thus for a complete answer we need to state that the lower frequency is $f_1 = \pm 0.3 + F$ for some integer F and thus $f_2 = \pm 0.9 + 3F$

2. a) This is a somewhat subjective question. One can argue that downsampling is a special case of the general decimation structure, we have studied, but with a bad choice of anti-aliasing filter ($H(f) = 1$). Similarly, the upsampling is a special case of the interpolation structure, again with the same bad choice of filter. The downsampling will work correctly for low-pass signals and the low-frequency part of the upsampled signal will also be correct in that case. The estimate of the spectral density, on the other hand, has a scaling error, compared to the periodogram, which is significant if you want to determine a signal level or a noise level, for example. Still, if you instead only are interested in a signal to noise ratio or the frequency of the highest peak, then the absolute scaling is irrelevant.

b) Better solutions are, for example, to

- decimate the signal, using a low-pass filter with cut-off frequency at $f = 1/4$ as anti-aliasing filter.
- interpolate the signal, using a low-pass filter with cut-off frequency at $f = 1/4$ and correct scaling, as reconstruction filter.
- use the periodogram with correct scaling, or even better to use Bartlett or Welch or Blackman-Tukey to reduce the variance.

c) **sinusoid with frequency** $f < 1/4$: No difference in the decimated signal, $z(n)$, since the original signal should pass the low-pass filter undisturbed.
The interpolated signal $w(n)$, will only contain a signal at the original frequency, whereas the upsampled signal also contains an "aliased" frequency at $1/2 - f$.

sinusoid with frequency $f > 1/4$: With proper anti-aliasing filtering in the decimation, both $y(n)$ and thereby also $w(n)$ will be zero, since this frequency cannot be represented using the sampling rate of the decimated signal. Of course, if you know that the input signal has only high frequency components, then you can use high-pass filters in the decimation and interpolation to recover the signal for frequencies $f > 1/4$, compare to the high-pass branch of a filter bank.
The naïve downsampling/upsampling, on the other hand, will lead to a $w(n)$ which contains both the original signal and an aliased sinusoid at $1/2 - f$.

3. (a) See Figure 1.

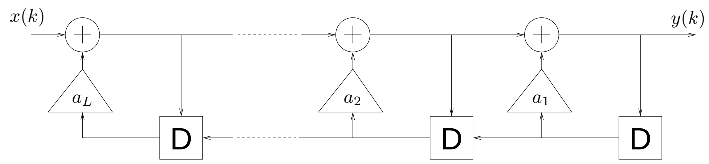

Figure 1: Answer to 3 a)

(b) The SQNR is usually defined as the ratio (assuming that all signals are zero mean)

$$\frac{E[y(n)^2]}{E[(y(n)-\hat{y}(n))^2]},$$

where $y(n)$ is the output of the ideal system (without round-off effects) and $\hat{y}(n)$ is the output of the real-world system that includes the round-off effects. Inherent in the definition is thus the assumption that the error introduced by the round-off (in this case, by each multiplication) is a stochastic process. Typically round-off is modeled as additive white noise uncorrelated with the signal in the system, that is injected after each multiplication. It can be seen from the figure that the round-off noise from each of the L multiplications could equivalently be added at the input. Let $e_i(k)$ denote the round-off noise caused by the multiplication $a_i y(k-i)$ and denote its variance by σ_i^2. Then, assuming that the noise signals are mutually uncorrelated, the aggregated equivalent noise that acts on the input is white with variance

$$\sigma_e^2 = \sum_{i=1}^{L} \sigma_i^2.$$

Thus by defining $g(n) = y(n) - \hat{y}(n)$ the quantization noise variance is

$$E[g(n)^2] = r_g(0) = \int_{-1/2}^{1/2} |H(\nu)|^2 \sigma_e^2 d\nu = \sigma_e^2 \int_{-1/2}^{1/2} |H(\nu)|^2 d\nu.$$

The variance of each of the noise sources depends on the number of bits. It is

$$\sigma_i^2 = \frac{\Delta^2}{12} = \frac{2^{-2b}}{12}$$

Similarly

$$E[y(n)^2] = r_y(0) = \sigma_x^2 \int_{-1/2}^{1/2} |H(\nu)|^2 d\nu.$$

Thus the SQNR becomes

$$\frac{\sigma_x^2}{\sigma_e^2} = \frac{12\sigma_x^2}{2^{-2b}L}.$$

(c) If the filter is intended as a whitening filter then the output that is due to the actual signal is white, $P_{yy} = P_{xx}|H(f)|^2 = \sigma_y^2$ and the SQNR must then be

$$\frac{12\sigma_y^2}{2^{-2b}L\int_{-1/2}^{1/2}|H(\nu)|^2 d\nu} = \frac{12\sigma_y^2}{2^{-2b}L\int_{-1/2}^{1/2}\frac{\sigma_y^2}{P_{xx}(\nu)}d\nu} = \frac{12}{2^{-2b}L\int_{-1/2}^{1/2}\frac{1}{P_{xx}(\nu)}d\nu} \quad (2)$$

(d) Clearly the SQNR can be very low if the filter has poles close to the unit circle (since the power spectral density on the input process then is almost zero at certain frequencies).

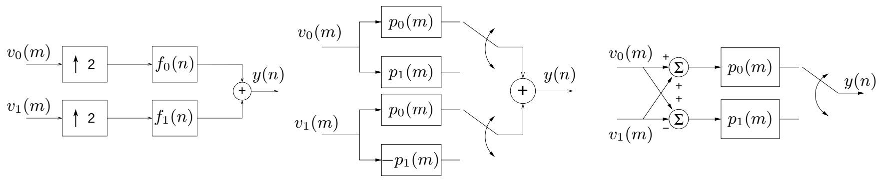

Figure 2: Synthesis filter bank. Direct implementation (left), polyphase implementation (middle), optimized version (right).

4. (a) The relationship $F_1(f) = F_0(f-1/2)$ corresponds to $f_1(n) = e^{-j\pi n} f_0(n) = (-1)^n f_0(n)$. Therefore, if

$$p_0(m) = f_0(2m)$$
$$p_1(m) = f_0(2m+1)$$

is a polyphase decomposition of $f_0(n)$, then the polyphase decomposition of $f_1(n)$ is given by $p_0(m)$ and $-p_1(m)$. Figure 2 illustrates how the direct implementation can be converted into a polyphase implementation (middle figure), exploiting the symmetry. Since the filters are linear, this is equivalent to the requested system in the right figure.

(b) In the direct implementation, the outputs of both the filters $f_0(n)$ and $f_1(n)$ are used to obtain each sample of $y(n)$, i.e. $2L$ multiplications are needed for each output value. In the polyphase implementation, the even numbered output samples are obtained from the filter $p_0(n)$ (using $\lceil L/2 \rceil$ multiplications) and the odd numbered output samples from $p_1(n)$ (using $\lfloor L/2 \rfloor$ multiplications). Therefore, only $L/2$ multiplications are needed for each output sample. i.e. the polyphase implementation is a factor 4 faster than the direct implementation, if we only count multiplications.

5. a) After sampling at F_s sampling frequency, the 50Hz frequency source will be of the form $w(n) = A\cos(2\pi f_{50} n + \phi)$, where $f_{50} = 50/F_s$. From basic courses in signals and systems, we know that the filter $H(f)$ will only affect the amplitude and phase of the sinusoid, so

$$y(n) = s(n) + |H(f_{50})| A\cos(2\pi f_{50} n + \phi + \angle[H(f_{50})])$$

The same reasoning shows that

$$\begin{aligned} x(n) =& y(n) - |G(f_{50})|A\cos(2\pi f_{50} n + \phi + \angle[G(f_{50})]) \\ =& s(n) + |H(f_{50})|A\cos(2\pi f_{50} n + \phi + \angle[H(f_{50})]) - |G(f_{50})|A\cos(2\pi f_{50} n + \phi + \angle[G(f_{50})]) \end{aligned}$$

If $|H(f_{50})| = |G(f_{50})|$ and $\angle[H(f_{50})] = \angle[G(f_{50})]$, then the sinusoid has been canceled completely and $x(n) = s(n)$. This solution will minimize the power of $x(n)$, since $s(n)$ is uncorrelated with the sinusoid (to be precise, we also have to make the assumption that s(n) does not contain any pure sinusoid with the same frequency, which is equivalent to that the the spectral density of $s(n)$ does not have any Dirac pulse at f_{50}). To see this, notice that Power$[x(n)]$ = Power$[s(n)]$+Power[sinusoid] which is minimized if and only if the power of the sinusoid is zero.
The two conditions $|H(f_{50})| = |G(f_{50})|$ and $\angle[H(f_{50})] = \angle[G(f_{50})]$ are only true if

$$H(f_{50}) = G(f_{50}) = g_0 + g_1 e^{j2\pi f_{50}} = g_0 + g_1\cos(2\pi f_{50}) + jg_1\sin(2\pi f_{50})$$

which is solved by

$$\begin{cases} g_1 = \frac{\mathrm{Im}[H(f_{50})}{\sin(2\pi f_{50})} \\ g_0 = \mathrm{Re}[H(f_{50})] - g_1 \cos(2\pi f_{50}) = \mathrm{Re}[H(f_{50})] - g_1 \cot(2\pi f_{50}) \end{cases}$$

b) If we view the output of G_f as an estimate of $y(n)$ and view $x(n)$ as the resulting estimation error, then we can use the standard theory for MMSE estimation, which gives the so-called normal equations

$$\begin{bmatrix} g_0 \\ g_1 \end{bmatrix} = \mathbf{R}_{ww}^{-1}\mathbf{r}_{wy} = \begin{bmatrix} r_{ww}(0) & r_{ww}(1) \\ r_{ww}(1) & r_{ww}(0) \end{bmatrix}^{-1} \begin{bmatrix} r_{yw}(0) \\ r_{yw}(1) \end{bmatrix}$$

where you can insert estimated correlation coefficients, obtained by

$$\hat{r}_{yw}(k) = \frac{1}{N}\sum_{n=k}^{N-1} y(n)w(n-k), \quad k = 0, 1$$

and similarly for $\hat{r}_{ww}(k)$.