SIGNAL PROCESSING

DEPARTMENT OF ELECTRICAL ENGINEERING

Digital Signal Processing EQ2300/ 2E1340

Final Examination 2011–12–20, 14.00–19.00
Sample Solutions

1.

- As the filter $H_1(z)$ has to move past the donwsampling by a factor 2 in the alternative implementation, the nobel identities (see Section 5 of complementary reading) will give us $H_{01}(z)$ as

$$H_{01}(z) = H_0(z)H_1(z^2) = (-\tfrac{1}{8} + \tfrac{1}{4}z^{-1} + \tfrac{3}{4}z^{-2} + \tfrac{1}{4}z^{-3} - \tfrac{1}{8}z^{-4})(\tfrac{1}{2} - z^{-2} + \tfrac{1}{2}z^{-4})$$
$$= -\tfrac{1}{16} + \tfrac{1}{8}z^{-1} + \tfrac{1}{2}z^{-2} - \tfrac{1}{8}z^{-3} - \tfrac{7}{8}z^{-4} - \tfrac{1}{8}z^{-5} + \tfrac{1}{2}z^{-6} + \tfrac{1}{8}z^{-7} - \tfrac{1}{16}z^{-8}$$

- In the direct implementation of the first system $H_0(z)$ (with 5 filter taps) has to be applied 4 times per output sample in $y_{01}(n)$ and $H_1(z)$ (with 3 filter taps) has to be applied 2 times per output sample in $y_{01}(n)$. The total number of multiplications per output sample in $y_{01}(n)$ is thus $4 \times 5 + 2 \times 3 = 26$. In the second implementation the filter H_{01} (with 9 taps) have to be applied 4 times per output sample in y_{01} and the number of multiplications become $4 = 36$. The first implementation is thus most efficient.
- The polyphase implementation is given as in Figure 2.15 in the complementary reading with polyphase filters $p_l(n) = h_{01}(nD + l)$, or more explicitly

$$p_0(n) = \{\underset{\uparrow}{-\tfrac{1}{16}}, -\tfrac{7}{8}, -\tfrac{1}{16}\},\ p_1(n) = \{\underset{\uparrow}{\tfrac{1}{8}}, -\tfrac{1}{8}\},\ p_2(n) = \{\underset{\uparrow}{\tfrac{1}{2}}, \tfrac{1}{2}\},\ \text{and},\ p_3(n) = \{\underset{\uparrow}{-\tfrac{1}{8}}, \tfrac{1}{8}\}.$$

Each of these filters have to be applied once for each output sample, yielding a total number of multiplications per output sample of $3 + 2 + 2 + 2 = 9$ which is significantly lower than the 36 multiplications required in the direct implementation, and also lower than the number of multiplications required in the first implementation.

2. (a) The equalizer

$$G(z) = \frac{1}{H(z)} = \frac{1}{1 + 0.5z^{-1}}$$

If causal, we consider the ROC: $|z| > 2$, then

$$g(n) = (-\frac{1}{2})^n u[n]$$

(b) The filter length $L = 20$

i. $X = N + L - 1 = 219$.

ii. The number of multiplications for a given block length M is

$$c(M) = \frac{M\log_2^M + M}{M - 20 + 1}, \quad L \le M \le X, M = 2^p.$$

$c(2^5) = 14.77$; $c(2^6) = 9.96$; $c(2^7) = 9.39$;

The optimal FFT block length is $M = 2^7 = 128$.

(c) We have to find a good combination of methods (Periodogram, modified periodogram, Bartlett, Welch, Black-Tukey) and windows (Rectangular, Bartlett, Hanning, Hamming, Blackman)

- Sidelode level ≤ -20dB $\Rightarrow$ Bartlett, Hanning, Hamming or Blackman windowing. In addition, notice that Periodogram and Bartlett's methods are excluded since they use Rectangular window by default (unless you state otherwise and clearly state what you do).
- Consistency $\Rightarrow$Welch's or Black-Tukey method. (Here modified periodogram is excluded)
- variance $\leq \frac{1}{2}P^2(e^{-jw}) \Rightarrow \begin{cases} \text{Welch's :} & \text{At least } K = 2 \text{ segments} \\ \text{Black-Tukey:} & \text{the window length } M \leq \frac{3}{4}N \end{cases}$
- Resolution $\Delta\nu \leq 0.013 \Rightarrow \begin{cases} \text{Welch's :} & \text{At most 2 segments, only Bartlett window} \\ \text{Black-Tukey:} & \text{all windows work for } M = \frac{3}{4}N \end{cases}$

Conclution: There are two alternatives for estimation. (1) Welch's methods with Bartlett window. (2) Black-Tukey method with different windows (except rectangular).

3. (a) Considering the autocorrelation sequence of $x(n)$ as $r_x(k) = \alpha^{|k|}$ and using the Levinson-Durbin recursion, we have

$$a_0(0) = 1, \qquad \epsilon_0 = r_x(0) = 1,$$

for $j = 0$,

$$\begin{aligned} \gamma_0 = \quad r_x(1) = &\quad \alpha \\ \Gamma_1 = \quad -\frac{\gamma_0}{\epsilon_0} = &\quad -\alpha \\ a_1(1) = \quad \Gamma_1 = &\quad -\alpha \\ \epsilon_1 = \epsilon_0[1 - |\Gamma_1|^2] &= 1 - \alpha^2, \end{aligned}$$

for $j = 1$,

$$\gamma_1 = r_x(2) + a_1(1)r_x(1) = \alpha^2 - \alpha^2 = 0 \Rightarrow \Gamma_2 = 0 \quad (\epsilon_2 = \epsilon_1)$$

which implies that $x(n)$ is a first-order AR process. Then, $\mathbf{a_1} = [1 \;\; a_1(1)]^T = [1 \;\; -\alpha]^T$ and $|b_0|^2 = \epsilon_1 = 1 - \alpha^2$. The transfer function is obtained as

$$H(z) = \frac{\sqrt{1-\alpha^2}}{1 - \alpha z^{-1}}.$$

(b) The autocorrelation sequence of $y(n)$ is

$$r_y(k) = r_x(k) + \sigma_w^2\delta(k) = \alpha^{|k|} + \sigma_w^2\delta(k),$$

then, using the Yule-Walker equations,

$$\begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \end{bmatrix} \begin{bmatrix} a_2(1) \\ a_2(2) \end{bmatrix} = - \begin{bmatrix} r_x(1) \\ r_x(2) \end{bmatrix}$$

we get,

$$\begin{bmatrix} 1+\sigma_w^2 & \alpha \\ \alpha & 1+\sigma_w^2 \end{bmatrix} \begin{bmatrix} a_2(1) \\ a_2(2) \end{bmatrix} = - \begin{bmatrix} \alpha \\ \alpha^2 \end{bmatrix}.$$

Then, the AR parameters, $a_2(1)$ and $a_2(2)$, can be found as

$$a_2(1) = \frac{-[\alpha(1+\sigma_w^2) - \alpha^3]}{(1+\sigma_w^2)^2 - \alpha^2}$$
$$a_2(2) = \frac{-[-\alpha^2 + \alpha^2(1+\sigma_w^2)]}{(1+\sigma_w^2)^2 - \alpha^2}$$

(c) If $\sigma_w^2 \to 0$, then

$$a_2(1) \to -\alpha, \qquad a_2(2) \to 0$$

which shows if there is no noise, $a_2(2)$ tends to zero and we get the first-order AR process, consequently, the same AR parameter is obtained as that of part (a) for $a_1(1)$. In general, if the white noise is added, the poles of the transfer function move toward the origin.

4. a) The first step is to find the power spectrum of $d(n)$. This is relatively straightforward, we have that

$$P_d(e^{j\omega}) = \frac{\sigma_e^2}{|1-0.3e^{-j\omega}|^2} = \frac{1}{1-0.3^2}\frac{1-0.3^2}{1+0.3^2-0.6\cos\omega}$$

Now, the filter $h(n)$ has a frequency response given by $H(e^{j\omega}) = 1 - e^{-j\omega}$, so, the spectrum of the signal after the filtering is

$$\begin{aligned} P_f(e^{j\omega}) &= P_d(e^{j\omega})|1-e^{-j\omega}|^2 = P_d(e^{j\omega})\left(2 - e^{-j\omega} - e^{j\omega}\right) \\ &= 2P_d(e^{j\omega}) - e^{-j\omega}P_d(e^{j\omega}) - e^{j\omega}P_d(e^{j\omega}) \end{aligned}$$

Using the Collection of Formulas (7.21, p.18) we have that the ACF of $f(n)$ is simply

$$r_f(k) = \frac{2}{1-0.3^2}0.3^{|k|} - \frac{1}{1-0.3^2}\left[0.3^{|k+1|} + 0.3^{|k-1|}\right]$$

Since $w(n)$ is uncorrelated with $f(n)$, the autocorrelation of $x(n)$, denoted as $r_x(k)$ will be

$$r_x(k) = r_f(k) + r_w(k) = r_f(k) + \sigma_w^2\delta(k)$$

b) We take the partial derivatives of J with respect to c, w_0 and w_1 and set them to zero. For c, we have

$$\frac{\partial J}{\partial c} = 0 \Leftrightarrow \mathbb{E}\{-\Big[c + w_0x(n) + w_1x(n-1) - d(n+1)\Big]\Big\} = 0$$

Since both $d(n)$ and $x(n)$ are zero-mean, we have that c should be equal to zero(verifying our intuition). We now set the partial derivatives of w_0 and w_1 equal to zero:

$$\frac{\partial J}{\partial w_0} = 0 \Leftrightarrow \mathbb{E}\{-\Big[c + w_0x(n) + w_1x(n-1) - d(n+1)\Big]x(n)\Big\} = 0$$

$$\frac{\partial J}{\partial w_1} = 0 \Leftrightarrow \mathbb{E}\{-\Big[c + w_0x(n) + w_1x(n-1) - d(n+1)\Big]x(n-1)\Big\} = 0,$$

and obtain the equations

$$
\begin{aligned}
&-r_{dx}(1) + w_0 r_x(0) + w_1 r_x(1) = 0 \\
&-r_{dx}(2) + w_0 r_x(1) + w_1 r_x(0) = 0
\end{aligned}
$$

The cross-correlation between $d(n)$ and $x(n)$ is

$$
\begin{aligned}
r_{dx}(k-1) &= \mathbb{E}\Big\{ d(n+1)x(n+k) \Big\} = \\
&= \mathbb{E}\Big\{ d(n+1)\Big[d(n+k) - d(n+k-1) + w(n+k) \Big] \Big\} \\
&= r_d(k-1) - r_d(k-2)
\end{aligned}
$$

Therefore, the solution of this two-by-two system can be found to be $w_0 = -0.63$ and $w_1 = -0.37$.

5. a) We replace Q with an additive noise sourse $e(n)$ and we follow the standard assumptions, namely, that $e(n)$ is zero-mean white (constant power spectrum) with power $\sigma_e^2 = \Delta^2/12$. This is depicted in Figure 1.

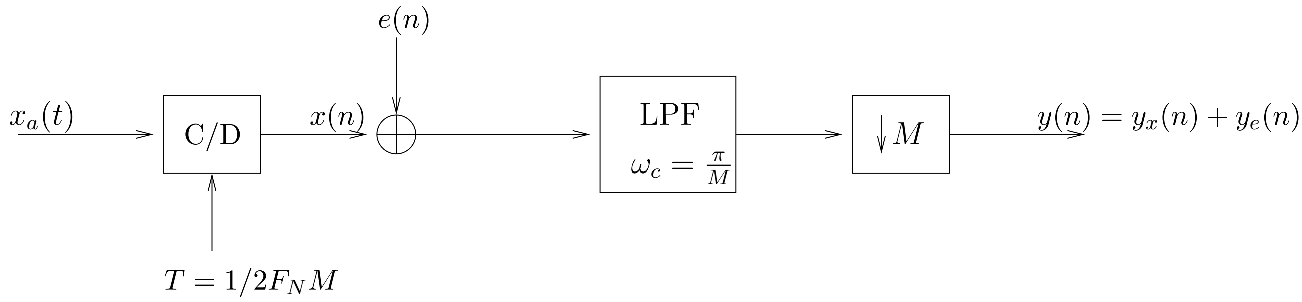

Figure 1:

The spectrum of $x(n)$ can be found easily from the spectrum of the analog signal. We have that

$$X(\omega) = \begin{cases} \frac{1}{T} X_a\left(\frac{\omega}{2\pi T}\right), & |\omega| \leq \pi/M \\ 0, & \text{otherwise} \end{cases}$$

Observe that the signal is bandlimited to π/M, therefore it passes through the LPF unaltered. Downsampling by a factor of M will expand $X(\omega)$ by a factor of M along the ω-axis and reduce the magnitude by a factor of M. Hence, the power of the signal component at the output of the filter is given by the integral

$$\sigma_{y_x}^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{M} X\left(\frac{\omega}{M}\right) d\omega$$

which can be calculated (note that this can be done also graphically for simplicity) to be

$$\sigma_{y_x}^2 = \frac{1}{2\pi}\frac{\pi}{MT} = \frac{1}{2MT}.$$

On the other hand, the quantization noise has a flat spectrum, so after filtering and downsampling by M, the power of the noise component will be

$$\sigma_{y_e}^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sigma_e^2}{M}\, d\omega = \frac{\sigma_e^2}{M}.$$

This means that the SQNR at the output $y(n)$ is $\text{SQNR} = \dfrac{\sigma_{y_x}^2}{\sigma_{y_e}^2} = \dfrac{6}{T\Delta^2}$.

b) As before, we model the quantization error as white noise and study the system shown in Figure 2.

It is clear that the signal $f(n)$ is the sum of two components: $f_x(n)$ due to the input $x(n)$ alone, and $f_e(n)$ due to the noise $e(n)$ alone. We denote the transfer function from $x(n)$ to $f(n)$ as H_{f_x} and the transfer function from $e(n)$ to $f(n)$ as H_{f_e}. These transfer functions can be calculated in a very straightforward manner and are

$$H_{f_x}(z) = 1 \quad \text{and} \quad H_{f_e}(z) = 1 - z^{-1}$$

Therefore, at the output $y(n)$, the power due to the input signal will be the same as before. For the noise $e(n)$, the power spectrum before decimation is

$$P_{ee}(e^{j\omega}) = \sigma_e^2|1 - e^{-j\omega}|^2 = 4\sigma_e^2 \sin^2\left(\frac{\omega}{2}\right),$$

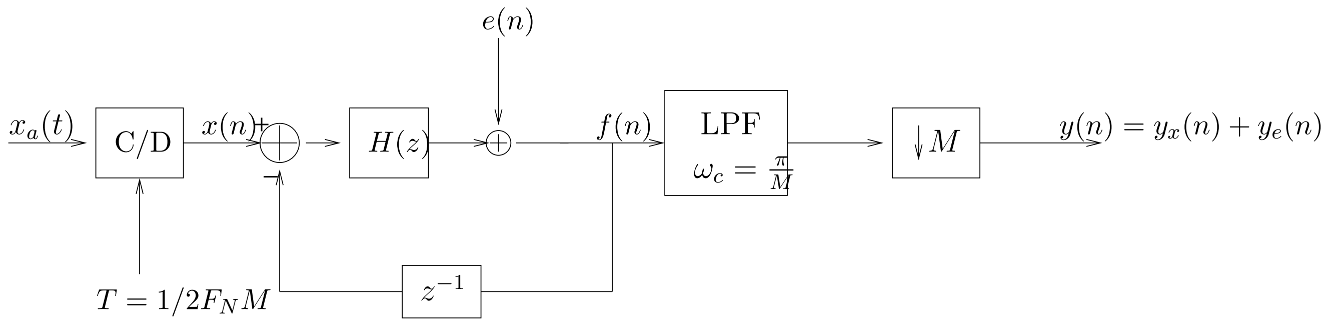

Figure 2:

hence, after filtering and downsampling,

$$P_{y_e}\left(e^{j\omega}\right) = \frac{4\sigma_e^2}{M}\sin^2\left(\frac{\omega}{2M}\right).$$

The power of the noise at the output is given by the integral

$$\sigma_{y_e}^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{4\sigma_e^2}{M}\sin^2\left(\frac{\omega}{2M}\right)d\omega,$$

which, using the simplification given in the hint, can be calculated easily to be

$$\sigma_{y_e}^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{4\sigma_e^2}{M}\left(\frac{\omega^2}{4M^2}\right)d\omega = \frac{\pi^2\Delta^2}{36M^3}.$$

Hence, the new SQNR$'$ is

$$\mathrm{SQNR}' = \left(\frac{1}{2MT}\right)\Big/\left(\frac{\pi^2\Delta^2}{36M^3}\right) = \frac{18}{\pi^2}\frac{M^2}{T\Delta^2},$$

and is much better than the first system when M is large.

c) By adding the loop we have managed to "shape" the noise power spectrum in such a way that most of the power will be outside the frequency π/M, and will be removed in decimation. This is the reason for the SQNR improvement.