KTH, SIGNAL PROCESSING LAB

SCHOOL OF ELECTRICAL ENGINEERING

Digital Signal Processing EQ2300/ 2E1340

Final Examination 2010–12–16, 8.00–13.00

Literature:

- Hayes: *Statistical Digital Signal Processing and Modeling*
  or
  Proakis, Manolakis: *Digital Signal Processing*
- Bengtsson: *Complementary Reading in Digital Signal Processing*
- Begtsson and Jaldén: *Summary slides*
- *Beta – Mathematics Handbook*
- *Collection of Formulas in Signal Processing, KTH*
- Unprogrammed pocket calculator.
- A dictionary.

Notice:

- Answer in English or Swedish.
- At most one problem should be treated per page.
- Answers without motivation/justification carry no rewards.
- Write your name and *personnummer* on each page.
- Write the number of solution pages on the cover page.

The exam consists of five problems with a maximum of 10 points each. For a passing grade, 24 points are normally required.

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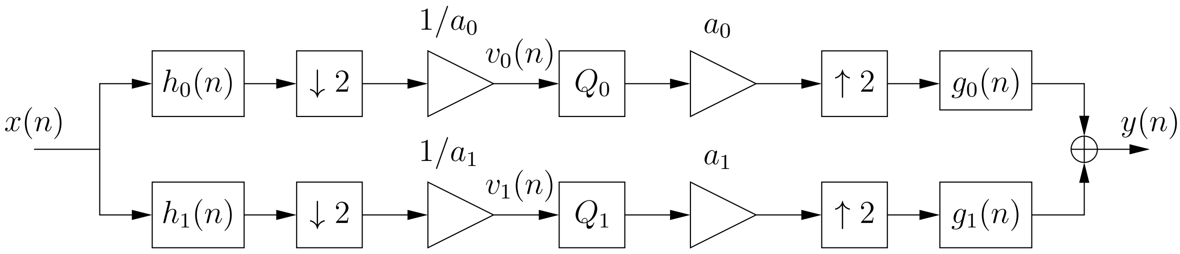
Results: Will be reported within three working weeks on "My pages".

Solutions: Will be available on the course homepage after the exam.

Good luck!

1.



The reconstructing filter bank depicted above is the same as used in the second project, i.e., the impulse responses of the filters are given by

$$h_0(n) = \{\underset{\uparrow}{-\tfrac{1}{8}}, \tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{4}, -\tfrac{1}{8}\} \qquad h_1(n) = \{\underset{\uparrow}{\tfrac{1}{2}}, -1, \tfrac{1}{2}\}$$

and

$$g_0(n) = \{\underset{\uparrow}{\tfrac{1}{2}}, 1, \tfrac{1}{2}\} \qquad g_1(n) = \{\underset{\uparrow}{\tfrac{1}{8}}, \tfrac{1}{4}, -\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{1}{8}\}\,.$$

Without quantization, this yields perfect reconstruction with a 3 sample delay according to $y(n) = x(n-l)$ for $l = 3$. However, we are interested in the effects that quantization has on the above system, and will seek to develop a formula that could allow us to optimally allocate bits to the two quantizers in order to minimize the quantization errors.

To this end, assume that the quantizers Q_0 and Q_1 implement $B_0 + 1$ and $B_1 + 1$ bit uniform quantization of the range $(-1, 1)$. Further, the signal amplification factors a_0 and a_1 are chosen in such a way that the inputs to the quantizers, $v_0(n)$ and $v_1(n)$, are always within the range $(-1, 1)$ so that no overflow occurs.

(a) Using the stochastic approximation of round-off errors, compute a formula for approximating the power of the round-off error present in the output signal $y(n)$, i.e, obtain an approximation of

$$P_Q = E\{(y(n) - x(n-l))^2\}\,.$$

Express your answer as a function of a_0, a_1, B_0 and B_1. You may for assume that the up-sampling introduces a random delay such that the resulting signals after up-sampling are wide sense stationary. (8p)

(b) Discuss what assumptions your approximation is based on. (2p)

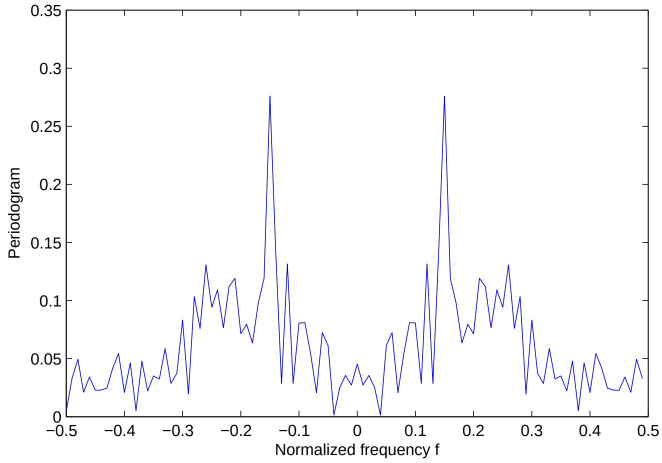
2. We are given N samples from a wide sense stationary (WSS) process $x(n)$, $n = 0, 1, ..., N-1$. The autocorrelation of $x(n)$ can be estimated via

$$\hat{r}_x(k) = \frac{1}{N} \sum_{n=0}^{N-1} x(n)x(n-k)$$

for $k = 0, \ldots, N-1$. For $k = -(N-1), \ldots, 1$ we have that $\hat{r}_x(k) = \hat{r}_x(-k)$, and we let $\hat{r}_x(k) = 0$ for $|k| \geq N$. The periodogram of $x(n)$, given by

$$\hat{P}_{\text{per}}(f) = \frac{1}{N} \left| \sum_{k=-\infty}^{\infty} \hat{r}_x(k) e^{-j2\pi f} \right|^2$$

is plotted below.



We now want to model $x(n)$ as an AR(p) process for some reasonably chosen p.

(a) Suggest a model order p, and motivate your choice. (3p)

(b) Determine the parameters in your AR(p) model, for the choice of p suggested in part (a). (3p)

(c) Now, suppose that you decide to increase the model order to $p+1$. Explain how you would compute the parameters of the AR($p+1$) from the parameters of the AR(p) model using the Levinson-Durbin recursion. (4p)

3. Consider a non-causal filter of length $M = 5$ with an impulse response $h(n)$ given by

$$h(n) = \{1, 2, \underset{\uparrow}{3}, 2, 1\}\,.$$

This is (a scaled version of) the filter used for linear interpolation when increasing the sample rate by a factor of 3. However, in this problem we will mainly be concerned with the filter itself, and not up-sampling.

(a) Determine the discrete-time Fourier transform (DTFT) $H(f)$ of $h(n)$. (2p)

(b) Let $x(n)$

$$x(n) = 2\cos(\pi n/4) + \cos(\pi n)$$

be the input sequence to the filter and let

$$y(n) = h(n) * x(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

be the corresponding output sequence. Compute $y(n)$ and its DTFT $Y(f)$. Simplify your answers as much as possible. (4p)

(c) Consider an alternative input sequence $x^\star(n) = x(n) + \cos(2\pi f^\star n)$ and its corresponding output $y^\star(n) = h(n) * x^\star(n)$. Find an $f^\star \in (0, \frac{1}{2})$ that leaves the output of the filter unchanged, i.e., an $f^\star$ such that $y^\star(n) = y(n)$. (4p)

4. At many times we wish to low pass discrete-time signals to retain only frequencies in the lower half of the spectrum. The ideal low pass filter with normalized cut-off frequency $f = 1/4$, i.e., $H(f) = 1$ for $0 \leq |f| \leq 1/4$ and $H(f) = 0$ for $1/4 \leq |f| \leq 1/2$, has an impulse response given by

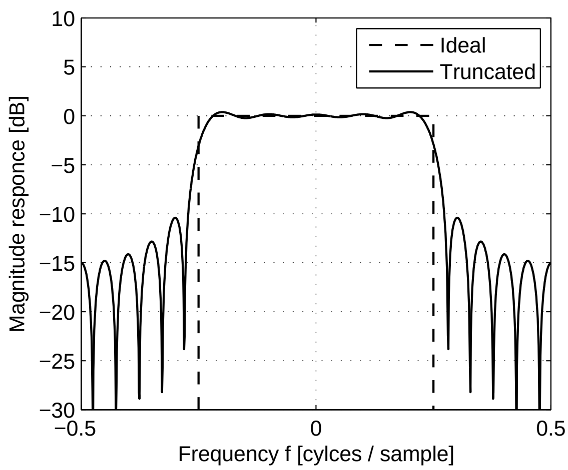
$$h(n) = \frac{\sin(\pi n/2)}{\pi n} = \frac{1}{2}\,\mathrm{sinc}(n/2)$$

and is thus not implementable in practise. A simple approach to get an implementable approximation of $h(n)$ is to truncate, i.e, to use $h(n) \approx h_M(n)$ where

$$h_M(n) = \begin{cases} h(n) & |n| \leq M \\ 0 & |n| > M \end{cases},$$

and we may be interested in the quality of this approximation.

To address this question, the frequency responses (i.e., the TDFT) of both the ideal filter $h(n)$ and that of $h_M(n)$ are shown below for $M = 10$.



We can see that truncation of the ideal filter leads to a widening of the pass-band and side-lobe leakage.

(a) Give an expression for the time-discrete fourier transform $H_M(f)$ of $h_M(n)$ for general values of M. You may give your answer in the form of an integral. (5p)

(b) Will the magnitude of the largest side-lobe be decreased if you increase M? Motivate your answer. (2p)

(c) Suggest a method to alter the truncated filter such that the side-lobes levels or the transfer function are reduced. Explain what other effects on the transfer function your alteration will have. (3p)

5. Assume that you are given a real-valued discrete-time signal $x(n)$ and that you wish to examine the presence of a periodic (cyclical) component at a specified (known) frequency f_0. We assume that the signal model is

$$s(n) = \alpha \cos(2\pi f_0 n) + \beta \sin(2\pi f_0 n), \quad n = 0, \cdots, N-1$$

and that we observe $s(n)$ embedded in additive noise $e(n)$, i.e., the observed signal is

$$x(n) = s(n) + e(n), \quad n = 0, \cdots, N-1$$

and then estimate the coefficients α and β that "fit" best the model to the data.

If $f_0 = k/N$, where k is an integer taking on any of the values $k = 1, 2, \cdots, N/2 - 1$, find the *least squares* estimate (LSE) of α and β. Simplify your expressions as much as possible. (10p)

Hint: The following trigonometric identities may be useful: $2\sin(\theta)\cos(\theta) = \sin(2\theta)$, and $2\cos^2(\theta) = 1 + \cos(2\theta)$. Also, note that

$$\sum_{n=0}^{N-1} \cos(\gamma n) = \sum_{n=0}^{N-1} \Re\left(e^{j\gamma n}\right) = \Re\left(\sum_{n=0}^{N-1} e^{j\gamma n}\right)$$

where $\Re(a)$ denotes the real valued part of $a \in \mathbb{C}$.