KTH, SIGNAL PROCESSING LAB

SCHOOL OF ELECTRICAL ENGINEERING

Digital Signal Processing EQ2300/ 2E1340

Final Examination 2009–12–15, 14.00–19.00

Literature:

- Hayes: *Statistical Digital Signal Processing and Modeling*
  or
  Proakis, Manolakis: *Digital Signal Processing*
- Bengtsson: *Complementary Reading in Digital Signal Processing*
- Begtsson and Jaldén: *Summary slides*
- *Beta – Mathematics Handbook*
- *Collection of Formulas in Signal Processing, KTH*
- Unprogrammed pocket calculator.

Notice:

- Answer in English or Swedish.
- At most one problem should be treated per page.
- Answers without motivation/justification carry no rewards.
- Write your name and *personnummer* on each page.
- Write the number of solution pages on the cover page.

The exam consists of five problems with a maximum of 10 points each. For a passing grade, 24 points are normally required.

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Results: Will be reported within three working weeks on “My pages”.

Solutions: Will be available on the course homepage after the exam.

Good luck!

Window	Sidelobe level (dB)	3 dB BW $(\Delta\omega)_{3\text{dB}}$
Rectangular	-13	$0.89(2\pi/N)$
Bartlett	-27	$1.28(2\pi/N)$
Hanning	-32	$1.44(2\pi/N)$
Hamming	-43	$1.30(2\pi/N)$
Blackman	-58	$1.68(2\pi/N)$

Table 1: Properties of a few commonly used windows of length N.

1. The signal $x(n)$ is a sum of three complex sinusodial signals, with amplitudes $A_1 = A_2 = 1$ and $A_3 = 0.1$, and corresponding normalized frequencies $f_1 = 0.2$, $f_2 = 0.21$ and $f_3 = 0.25$.

- Suppose that the spectrum of $x(n)$ is to be estimated from $N = 1000$ samples of $x(n)$ using Welch's method with 50% overlap. Choose an appropriate window length and window type so that the three sinusoids can be clearly seen in the spectrum. Remember to motivate your answer. (8p)

  For your convenience, some commonly used windows and their properties are listed in Table 1.

- If you instead used Pizarenkos method to estimate the frequencies from the covariance matrix of $x(n)$, $\mathbf{R}_x$, what would be the size of $\mathbf{R}_x$. (2p)

2. A signal $x(n)$ has to be filtered by the filter

$$H(z) = \frac{1}{(1 - az^{-1})(1 + 2az^{-1})}$$

where $0 \leq a < 0.5$, and we should design a circuit that does this. A suggested implementation is shown in Figure 1. Note that $(a_1, a_2) = (a, -2a)$ and $(a_1, a_2) = (-2a, a)$ will both lead to equivalent filters when working with real values, but may lead to different results when implemented in fixed-point. To study the fixed-point implementation, we assume that all (internal) signals are in the range $[-1, 1]$, that no overflow occur, that the circuit is implemented in fixed point with $B + 1$ bits (with one bit representing the sign), and that the additive noise model for fixed-point implementations apply.

(a) Compute the noise-power due to quantization at the output $y(n)$ for the two different implementations, $(a_1, a_2) = (a, -2a)$ and $(a_1, a_2) = (-2a, a)$, and state which implementation leads to lower noise-power. (10p)

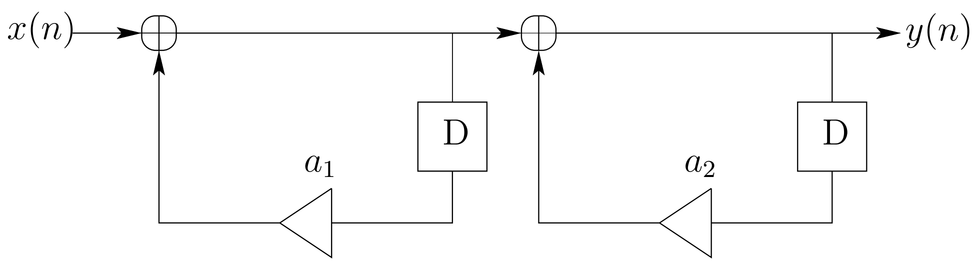

Figure 1: Suggested implementation of filter $H(z)$

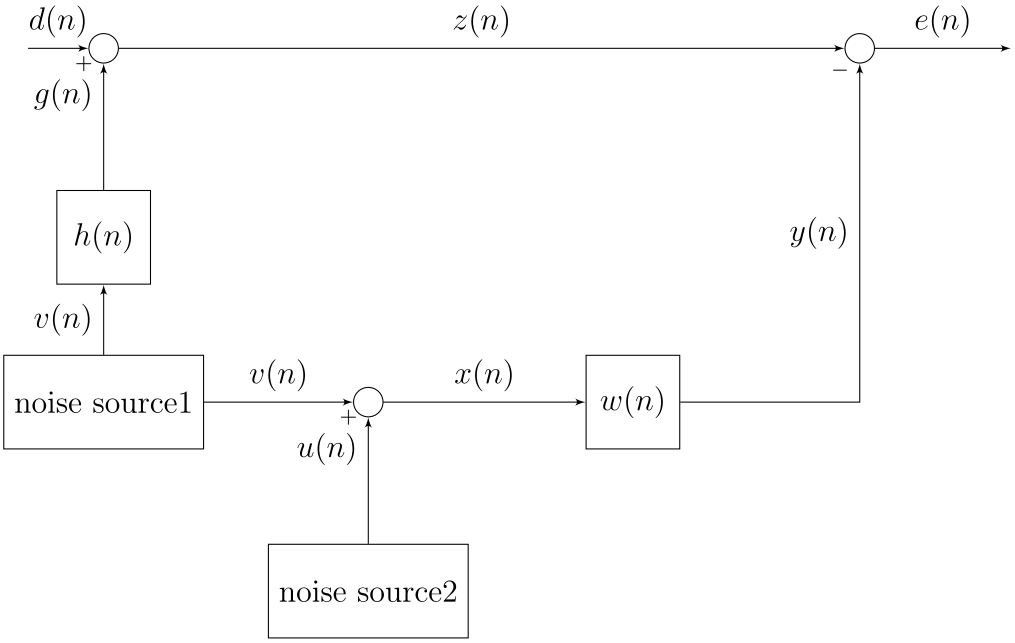

Figure 2: A system for noise subtraction.

3. A classical application of Wiener filtering is noise subtraction. In this problem it is desirable to minimize the noise corrupting a sinusoidal signal. The setup for the application is shown in Figure 2 where $d(n) = A\sin(2\pi f n + \varphi)$ is the signal of interest. Here A is fixed and φ is uniformly distributed over $[0, 2\pi)$. Noise source 1 generates a white noise, $v(n)$, of zero mean and unit variance, which is filtered and added to the signal of interest, $d(n)$, to form $z(n) = d(n) + g(n)$. The impulse response $h(n)$ is given by

$$h(n) = \{\underset{\uparrow}{1}, 0.75\}$$

Assume that we can observe a reference signal $x(n)$ which is the sum of noise source 1 and noise source 2, where noise source 2 is white, zero mean, and of variance $\sigma_u^2 = 0.25$. Noise source 1 and 2 are mutually independent, and independent of $d(n)$. The aim is to use the information in $x(n)$ to remove as much of $g(n)$ as possible from $z(n)$, using the structure shown in Figure 2.

(a) Design a causal 2-tap FIR Wiener filter, $w(n)$, that minimize the variance of $e(n) = z(n) - y(n)$. Explain further why the same filter also minimize $E[|g(n) - y(n)|^2]$. (6p)

(b) Compute the resulting variance of $e(n)$, i.e., $E[|e(n)|^2]$. (2p)

(c) If the desired signal contained 2 sinusoids, i.e.,

$$d(n) = A_1 \sin(2\pi f_1 n + \phi_1) + A_2 \sin(2\pi f_2 n + \phi_2)\,,$$

how would this change your solutions in (a) and (b)? (2p)

4. Consider the finite-length sequence $x(n) = \{\underset{\uparrow}{1}, 0, 0, 0, 2\}$, and let $x_N(n)$ denote the sequence $x(n)$ zero-padded (at the end) to length N.

(a) Find the 8-point DFT of $x_8(n)$, that is $X_8(k) = \mathrm{DFT}_8\{x_8(n)\}$. (2p)

(b) Find $g_8(n)$ (a sequence of length 8) which has a DFT

$$G_8(k) = e^{j2k\frac{2\pi}{8}} X_8(k)$$

where $X_8(k)$ is the 8-point DFT of $x_8(n)$. (2p)

(c) Let $h(n) = \{\underset{\uparrow}{1}, 0, 1\}$ and $h_N(n)$ be $h(n)$ zero-padded (at the end) to length N. Let $x_N(n)$ be as above. Determine (compute)

i. $x_5(n) \circledS h_5(n)$ (2p)

ii. $x_8(n) ⑧ h_8(n)$ (2p)

where Ⓝ denotes N-point circular convolution.

(d) Determine the result of

$$\mathrm{IDFT}_8\{\mathrm{DFT}_8\{x_8(n)\} \cdot \mathrm{DFT}_8\{h_8(n)\}\}$$

where $\mathrm{DFT}_N\{\cdot\}$ and $\mathrm{IDFT}_N\{\cdot\}$ denotes the N-point DFT and its inverse, respectively. (2p)

5. The system in Figure 3 is used to downsample and upsample a stochastic signal $x(n) = A\cos(2\pi f_0 n + \varphi)$ where $A > 0$ and $0 < f_0 < \frac{1}{4}$ are deterministic and where φ is uniformly distributed over $[0, 2\pi)$. The interpolating filter $G(f)$ correspond to linear interpolation between samples, i.e., the impulse response of $G(f)$ is

$$g(n) = \{\tfrac{1}{2}, \underset{\uparrow}{1}, \tfrac{1}{2}\}.$$

As non-ideal interpolation is used there will be some reconstruction error $e(n) = y(n) - x(n)$ and we define the reconstruction signal-to-noise ratio (SNR) according to

$$\mathrm{SNR} = \frac{P_x}{P_e} = \frac{\mathrm{E}\big[|x(n)|^2\big]}{\mathrm{E}\big[|e(n)|^2\big]}.$$

(a) Give a closed form expression for SNR as a function of the normalized frequency f_0. Simplify your answer as much as possible. (8p)

(b) Show that SNR $\to \infty$ when $f_0 \to 0$, i.e., reconstruction becomes near perfect for low frequency inputs, and give an intuitive explanation for this result. (2p)

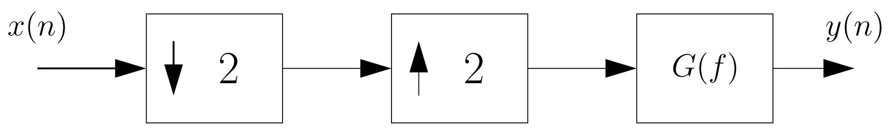

Figure 3: A decimating and interpolating circuit.