SOLUTIONS

E 90 **Digital Signalbehandling**, 2E1340

Final Examination 2003–04–30, 08.00–13.00

1. a) Since the filter is an FIR filter, it is stable for all choices of α.

b) The inverse of a lattice filter is stable iff all reflection coefficients are within the unit circle (compare to the Schur-Cohn stability test). Thus, the inverse of $g(n)$ is stable iff $|\beta| < 1$.

c) The normalized frequency of the peak is $f = 200/1024$. The downsampled signal corresponds to a sampling frequency of 16 000/3Hz, so the main frequency could be $F = 200/1024 \cdot 16\,000/3 \approx 1042$Hz. However, because of aliasing at the downsampling, it could also be $F = 1042 + 16\,000/3 = 6375$Hz or $F = 16\,000/3 - 1042 \approx 4292$Hz. (Here we assumed that the original sampling was performed using an anti-aliasing filter, otherwise there are infinitely many possible solutions.)

2. Quantization errors occur at the output of all the 5 multiplications and at the initial D/A conversion. Using the standard approximations, we get 6 additional noise sources as shown in Figure 1.

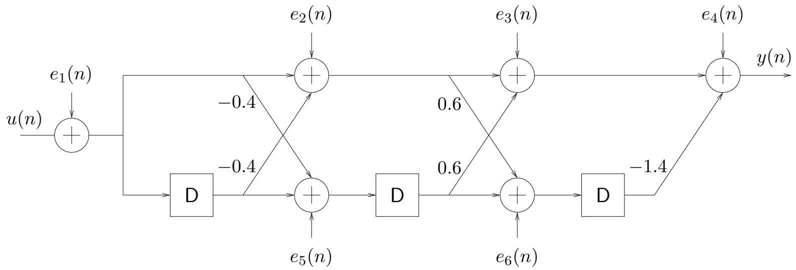

Figure 1: Equivalent model with quantization noise sources

Let $h_k(n)$ denote the impulse response from noise source $e_k(n)$ to the output. Here

$h_1(n) = \{1, -1.48, 1.496, -1.4\}$ (Use the Levinson-Durbin recursion)
$h_2(n) = \{1, -0.84\}$
$h_3(n) = h_4(n) = \{1\}$
$h_5(n) = \{0, 0.6, -1.4\}$
$h_6(n) = \{0, -1.4\}$

Then the total noise power at the output is

$$\sigma_q^2 = \sigma_e^2 \sum_k \sum_n |h_k(n)|^2 = 15.374016\sigma_e^2 ,$$

where σ_e^2 is the power of each quantization noise source. Normally during the course, we assume that the fixed point representation is used to represent numbers in the range $[-1, 1]$.

However, here we must use a scaling (at least conceptually) so that it represent numbers in the range $[-30, 30]$ in order to avoid overflow. This means that $\sigma_e^2 = 30^2 \cdot 2^{-2\cdot 9}/12$ and the total quantization noise power at the output is $15.37\sigma_e^2 \approx 4.4 \cdot 10^{-3}$

3. From Figure 2 it is easy to see that the output $y(n)$ corresponds to the **circular** convolution of length M of $d(n)$ and $h(n)$ as long as $K \geq L$. Applying the DFT of length M to the signals, this results in

$$Y(k) = H(k)D(k) ,$$

where $H(k)$ is the M-point DFT of $h(k)$ (zero-padded to length M).

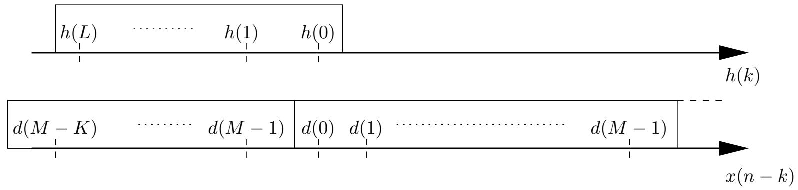

Figure 2: Illustration of the cyclic prefix

Alternative solution with sums and indices in case you don't like figures:

If $K \geq L$, (we also need $M \geq L$) we get

$$\begin{aligned} y(0) = r(K+1) &= \sum_{k=0}^{L} h(k)x(K+1-k) = h(0)d(0) + \sum_{k=1}^{L} h(k)d(M-k) \\ &= \sum_{k=0}^{L} h(k)x((-k)_M) \\ y(1) = r(K+2) &= \sum_{k=0}^{L} h(k)x(K+2-k) = h(0)d(1) + h(1)d(0) + \sum_{k=2}^{L} h(k)d(1+M-k) \\ &= \sum_{k=0}^{L} h(k)d((1-k)_M) \end{aligned}$$

In general

$$\begin{aligned} y(n) = r(K+1+n) &= \sum_{k=0}^{L} h(k)x(K+1+n-k) \\ &= \sum_{k=0}^{n} h(k)d(n-k) + \sum_{k=n+1}^{L} h(k)d(M+n-k) = \sum_{k=0}^{L} h(k)d((n-k)_M) = h(n) Ⓜ d(n) \end{aligned}$$

Note that the cyclic prefix is a trick to make the linear convolution in the channel look like a circular convolution. This is in a way opposite to the ideas used in overlap-save/overlap-add, where the circular convolution is made to look like a linear convolution by adding zeros to the signal and the filter.

4. a) The DFT is defined: $X_k = \sum_{n=0}^{N-1} x_n e^{-j2\pi\frac{kn}{N}}$.
Let

$$\begin{aligned} N &= 2 \\ x_0 &= x(n-1) \\ x_1 &= x(n) \end{aligned}$$

We want X_0 and X_1 to match the DFT

$$\begin{aligned} X_0 &= x_0 + x_1 &= x(n-1) + x(n) \\ X_1 &= x_0 - x_1 &= x(n-1) - x(n) \end{aligned}$$

From the expressions above, $H_0(z)$ and $H_1(z)$ can be identified as

$$\begin{aligned} H_0(z) &= 1 + z^{-1} \\ H_1(z) &= -1 + z^{-1} \end{aligned}$$

Remark: The downsampling is the equivalent of block selection in the block-wise scheme, and does not influence the values of $X_{0,1}$.
For the reconstruction filters, let $v(n)$ be the output from $F_0(z)$ and $w(n)$ the output from $F_1(z)$. With $n = 2m$ we get in the time-domain

$$\begin{aligned} v(n) &= f_0(0)X_0(m) &+ f_0(1) \cdot 0 \\ v(n+1) &= f_0(0) \cdot 0 &+ f_0(1)X_0(m) \\ w(n) &= f_1(0)X_1(m) &+ f_1(1) \cdot 0 \\ w(n+1) &= f_1(0) \cdot 0 &+ f_1(1)X_1(m) \end{aligned}$$

$y = v + w \quad \Rightarrow$

$$\begin{aligned} y(n) &= f_0(0)X_0(m) + f_1(0)X_1(m) \\ y(n+1) &= f_0(1)X_0(m) + f_1(1)X_1(m) \end{aligned}$$

The IDFT is defined: $y_n = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j2\pi\frac{kn}{N}}$. With $y_0 = y(n)$ and $y_1 = y(n+1)$ the filter coefficients can be identified, and we get

$$\begin{aligned} F_0(z) &= \tfrac{1}{2}(1 + z^{-1}) \\ F_1(z) &= \tfrac{1}{2}(1 - z^{-1}) \end{aligned}$$

b) From Figure 4, we know that the conditions for perfect reconstruction must be fulfilled. Thus

$$F_0(z)H_0(z) + F_1(z)H_1(z) = 2z^{-L}$$

where L is the delay of the system. Using the expressions from a) gives

$$\frac{1}{2}(1 + z^{-1})(1 + z^{-1}) + \frac{1}{2}(1 - z^{-1})(-1 + z^{-1}) = 2z^{-1}$$

and

$$L = 1$$

In block-wise processing, at least one complete block of input samples must be available before any calculations can be performed. This indicates that we need a delay of $N - 1$ samples at the input. In the two-point case, the delay must be at least one sample, which is consistent with the delay of the filter bank.

5. a) Since multiplication in the time domain corresponds to a normalized circular convolution of the DFTs (see the collection of formulas),

$$\hat{P}_{xx}^M(k) = \hat{P}_{xx}^M(f)\rfloor_{f=\frac{k}{N}} = \frac{1}{N}\left|\mathrm{FFT}[w(n)x(n)]\right|^2$$
$$= \frac{1}{N}\left|\frac{1}{N}W(k) \circledN X(k)\right|^2 = \frac{1}{N^3}\left|\sum_{l=0}^{N-1} W(l)X((k-l)_N)\right|^2$$

b) Since $x(n)$ is not uniquely determined by $|X(k)|$, it is in general not possible to calculate $\hat{P}_{xx}^M(k)$ from the periodogram. In other words, the periodogram and the modified periodogram do not contain exactly the same information on the signal.