SIGNAL PROCESSING

DEPARTMENT OF ELECTRICAL ENGINEERING

Digital Signal Processing EQ2300/ 2E1340

Final Examination 2011–06–01, 14.00–19.00
Sample Solutions

1. (a) Both Bartletts and Welchs methods trade resolution for variance, and with a segment length of $L = 32$ as opposed to the full data length of $N = 512$ the resultion is much lower. Thus we can see that B and D correspond of Bartlett and Welch. To differentiate between them, we note that the rectangular window of Bartletts method introduce sidelobes at -13dB. We see these in the graph for method B so we know that this is Bartlett and therefore that D is Welch.

The periodogram also has a rectanglar window so we should expect to see sidelobes (or spectral leakage) at -13dB. These can be seen in Method C so this is the peridodogram and A is the modified periodogram. A common error is to mistake this spectral leakage as loss of resolution, but note that the resolution is measured at -3dB and that C is actually more narrow at this level (although it is a bit hard to see).

(b) There will be $K = N/L = 512/32 = 16$ segments in Bartletts method. For Welchs method with 50% overlap there should be roughly twice as many, i.e., 32. A careful calculation taking into account edge effects give the number $K = 2N/L - 1 = 31$.

(c) The best plot to estimate the noise variance is D because the variance of the estimator (which is not the same thing as the noise variance) is smallest and the noise variance is not hidden by sidelobes. In plot D we can read the value -30dB off the noise floor.

2. The covariance matrix of $x(n)$, in general, is stated as,

$$R_x = \begin{bmatrix} r_{xx}(0) & r_{xx}^{\star}(1) \\ r_{xx}(1) & r_{xx}(0) \end{bmatrix},$$

where considering the given values for the auto-correlation function result in

$$R_x = \begin{bmatrix} \beta & 1-j \\ 1+j & \beta \end{bmatrix}.$$

Note that we should have a 2×2 covariance matrix since we only have one sinusoid. Applying the Pisarenko method, we obtain the noise power which is equal to the minimum eigenvalue of R_x, i.e.,

$$\det(R - \lambda I) = 0 \Rightarrow \lambda_1 = \beta - \sqrt{2} = \lambda_{min}, \lambda_2 = \beta + \sqrt{2}$$

since $\lambda_{min} = \sigma_w^2 = 1$, β is equal to $1 + \sqrt{2}$.

To find the noise eigenvector, v_{min}, we should solve the following equation

$$R_x \underline{v}_{min} = \lambda_{min} \underline{v}_{min}$$

where $\underline{v}_{min} = [v_{min}(0) \quad v_{min}(1)]$. This is the eigenvector which spans noise subspace.

$$v_{min} = \begin{bmatrix} \frac{-1+j}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Therefore, the eigenfilter is expressed as

$$\begin{aligned} V_{min}(z) &= \sum_{k=0}^{1} v_{min} z^{-k} \\ &= \frac{-1+j}{2} + \frac{1}{\sqrt{2}} z^{-1} \end{aligned} \quad ,$$

where its zero denotes (represent) the frequency, f_0,

$$z = \frac{1+j}{\sqrt{2}} = |r|e^{j\omega_0} \Rightarrow |r| = 1, \omega_0 = \frac{\pi}{4}$$

which results in $f_0 = \frac{\omega_0}{2\pi} = \frac{1}{8}$.

To determine the signal amplitude, we will find the signal eigenvector as

$$R_x \underline{v}_s = \lambda_2 \underline{v}_s \quad \underset{\lambda_2 = 1+2\sqrt{2}}{\Rightarrow} \quad \underline{v}_s = \begin{bmatrix} \frac{1-j}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Finally, the signal amplitude is attained by solving the following equation

$$\begin{aligned} |V_s(e^{j\omega_0})|^2 |A|^2 &= \lambda_2 - \sigma_w^2 \\ |\sum_{n=0}^{1} v_s(n) e^{-jn\frac{\pi}{4}}|^2 |A|^2 &= \lambda_2 - \sigma_w^2 \\ |(1-j)|^2 |A|^2 &= 1 + 2\sqrt{2} - 1 = 1 \quad \Rightarrow \quad |A| = 2^{\frac{1}{4}} = 1.1892. \end{aligned}$$

3. (a) From the difference equation

$$d(n) = \tfrac{1}{4} d(n-2) + v(n)$$

we can obtain the autocorrelation function of $d(n)$ as

$$r_d(k) = \begin{cases} \frac{16}{15} \left(\frac{1}{2}\right)^{|k|}, & k \text{ is even} \\ 0, & k \text{ is odd} \end{cases}$$

The autocorrelation function of $x(n)$ is

$$r_x(k) = r_d(k) + r_w(k),$$

and the cross-correlation

$$r_{dx}(k) = \mathbb{E}\{d(n+1)x(n-k)\} = r_d(k+1)$$

The Wiener-Hopf equations for FIR Wiener filter are

$$R_x w = R_{dx}$$

which for the first order FIR linear predictor becomes

$$\begin{bmatrix} r_x(0) & 0 \\ 0 & r_x(0) \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \end{bmatrix} = \begin{bmatrix} r_d(1) \\ r_d(2) \end{bmatrix} .$$

We get $w(0) = 0$, $w(1) = 4/31$, and a mean square error

$$\varepsilon = r_d(0) - w(1) r_d(2) = \frac{32}{31} \approx 1.03$$

(b) Since $r_d(1) = r_d(3) = 0$ and the additive noise $v(n)$, is white, then $x(n), x(n-2), x(n-4), \ldots$ is of no use estimating $d(n+1)$. Therefore, a better estimator that use only two samples is the following

$$W(z) = w(1)z^{-1} + w(3)z^{-3}$$

$$Solving the Wiener-Hopf equations \begin{bmatrix} r_x(2) & r_x(0) \\ r_x(0) & r_x(2) \end{bmatrix} \begin{bmatrix} w(1) \\ w(3) \end{bmatrix} = \begin{bmatrix} r_d(2) \\ r_d(4) \end{bmatrix}$$

gives us $w(1) = 8/47, w(3) = 1/63$. The mean square error

$$\varepsilon = r_d(0) - w(1)r_d(2) - w(3)r_d(4) = \frac{45313}{44416} \approx 1.02$$

which is smaller than value in part (a).

4. (a) Given the dynamic range of the ADC, ± 2, and the operating dynamic range of the ADC, the attenuator should limit the amplitude of the input signal to be within this range. Since the auto-correlation is given to be $r_{xx}(\tau) = 5\delta(\tau)$, the variance is obtained by evaluating the auto-correlation at $\tau = 0 \Longrightarrow \sigma_{xx}^2 = r_{xx}(0) = 5$. The signal is uniformly distributed in $[-A_x, A_x]$ and so the variance will be $\sigma_{xx}^2 = (A_x - (-A_x))^2/12 = A_x^2/3 \Longrightarrow A_x = \sqrt{15}$. The signal range clearly exceeds the ADC's dynamic range, therefore, the attenuator should be set at $A = \frac{2}{\sqrt{15}}$.

The quantization noise variance at the ADC is given by $\sigma_{ADC}^2 = \frac{\Delta^2}{12}$. Where Δ is the step size for the given dynamic range of the ADC. And so $\Delta = (V_{max} - V_{min})/2^b = (2-(-2))/2^6 = 1/2^4$. Therefore, $\sigma_{ADC}^2 = \frac{2^{-8}}{12}$.

$$\text{SQNR} = 10\log_{10}\left(\frac{\sigma_{xQ}^2}{\sigma_{ADC}^2}\right) = 10\log_{10}\left(\frac{A^2\sigma_{xx}^2}{\sigma_{ADC}^2}\right) = 10\log_{10}\left(\frac{1.34}{2^8 \cdot 12}\right) = 36.12\text{dB}. \tag{1}$$

(b) Let σ_o^2 be the filtered quantization noise,

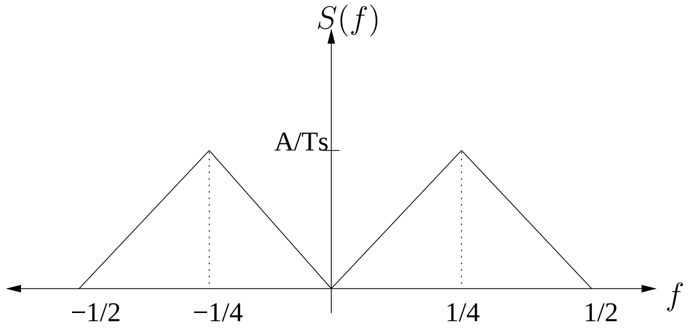
$$\sigma_o^2 = \sigma_{e_{adc}}\sum_{n=0}^{\infty} h^2(n) + \sigma_{e_{filt}}\sum_{n=0}^{\infty} h^2(n) = (\sigma_{e_{adc}} + \sigma_{e_{filt}})\sum_{n=0}^{\infty} h^2(n) \tag{2}$$

And the quantization noise variance within the filter is

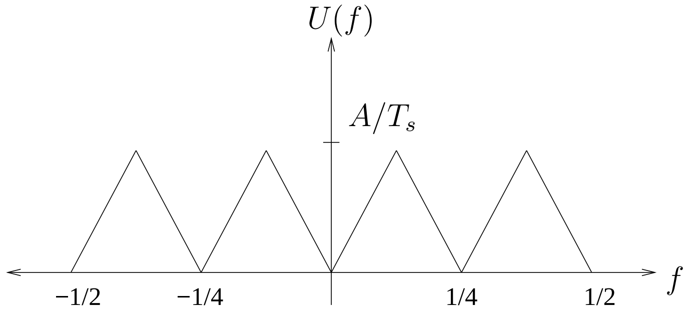
$$\sigma_{filt}^2 = \frac{1}{12}(\frac{1}{2^8})^2 \tag{3}$$

Since $h(n) = 0.5^n u(n) \Longrightarrow \sum_{n=0}^{\infty} h^2(n) = \frac{1}{1-0.5^2}$ therefore, $\sigma_o^2 = \frac{4}{3}(\frac{1}{12}(\frac{1}{2^8})^2 + \frac{1}{12}(\frac{1}{2^4})^2) = 4.35 \cdot 10^{-4}$ and the output SNR $= 10\log_{10}\frac{(4/3)(1.34)}{4.35 \cdot 10^{-4}} = 36.1\text{dB}$

5. (a) Since $s_a(t)$ is sampled at the Niquist rate, the DTFT of the sampled speech signal $s(n)$ is as illustrated in next figure (of course there are infinite such replicas in the bands $[-k/2, k/2], k \in \mathbb{N}\backslash\{0,1\}$, so here we only draw what happens in $[-1/2, 1/2]$):



Then, upsampling by a factor of 2 scales the frequency axis of $S(f)$ by a factor of 2 as illustrated next:



(b) Starting from the difference equation and taking the Fourier transform on both sides yields

$$Y(f) = U(f) + \frac{1}{2}e^{-j2\pi f}U(f) + \frac{1}{2}e^{j2\pi f}U(f),$$

from which we get that the frequency response of the filter is

$$H(f) = \frac{Y(f)}{U(f)} = 1 + \cos(2\pi f).$$

The impulse response is therefore given by $h(n) = \delta(n) + \frac{1}{2}\delta(n-1) + \frac{1}{2}\delta(n+1)$. To see the effect of this filter on $u(n)$, note that due to upsampling, $u(n) = 0$ for n odd. Therefore, with $y(n) = u(n) + \frac{1}{2}[u(n-1) + u(n+1)]$ it follows that

$$y(n) = \begin{cases} u(n), & \text{for } n \text{ odd} \\ \frac{1}{2}[u(n-1) + u(n+1)], & \text{for } n \text{ even.} \end{cases}$$

Thus, the even-index values of $u(n)$ are unchanged, and the odd-index values are the average of the two neighboring values. The filter is therefore a linear interpolator, just like the one we learned in class.

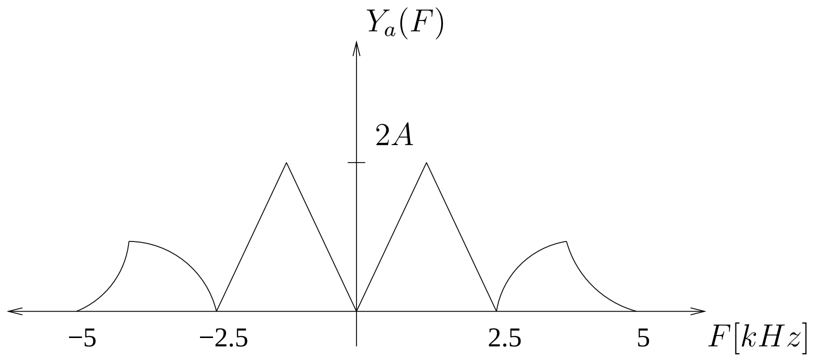
(c) The output of the D/C converter $y_a(t)$ has a Fourier transform $Y_a(F)$

$$Y_a(F) = \begin{cases} T_s\, Y(F\, T_s), & \text{for } |F| \leq F_s/2 \\ 0, & \text{otherwise.} \end{cases}$$

Since $Y(f) = H(f)S(2f) = H(f)S(2f)$, then

$$Y_a(F) = \begin{cases} T_s\, H(F\, T_s)S(2\ F\ T_s) = [1 + \cos(2\pi F/F_s)]\, S_a(2F), & \text{for } |F| \leq F_s/2 \\ 0, & \text{otherwise.} \end{cases}$$

This is illustrated graphically in the next figure:



Thus, $y_a(t)$ does not correspond to slowed-down speech due to the images of $S_a(F)$ that occur in the frequency range 2.5kHz − 5kHz and the non-ideal linear interpolator.

(d) Doubling the *sampling time* from T_s to $2T_s$ will result in aliasing in $S(f)$, and therefore can only make the approximation worse. On the other hand, doubling the *sampling rate* from F_s to $2F_s$ will eliminate the images of the $S_a(F)$ that occur in the range 2.5kHz − 5kHz, thus resulting in a better approximation.