SOLUTIONS

E 81 **Digital Signalbehandling**, 2E1340

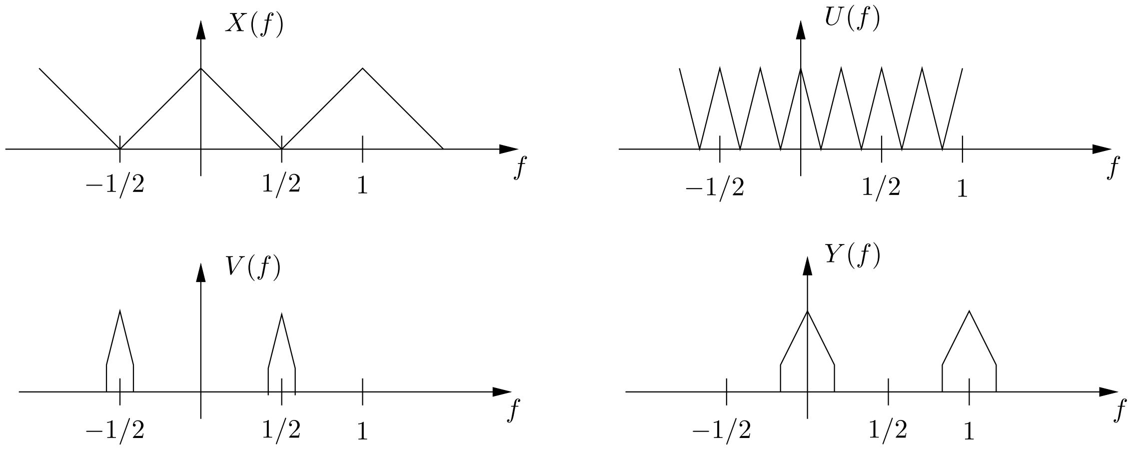
Final Examination 2000–04–26, 0900–1300

1. a) $y_2(n)$, $y_3(n)$ and $y_4(n)$ are stationary.
The interpolated signal $y_1(n)$ is not stationary, since it has zeros inserted at fixed positions, however combining interpolation with a random time shift as in the signal $y_2(n)$ gives a stationary result. Decimation does not destroy stationarity.

b)

$$X(k) = \frac{1}{N} Y((-k)_{\text{mod}(N)}) = \{\underset{\uparrow}{0.2},\ 1,\ 0.8 - 0.2i,\ 0.6,\ 0.4 + 0.2i\}$$

2. a) Introduce the notation $u(n)$ and $v(n)$ for the signal before and after the filter, respectively. Then if, for example, the input signal has a triangle shaped spectrum, the signal spectra are given by the following figure.



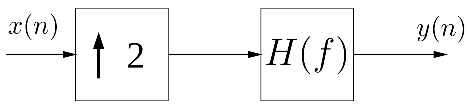
This follows from

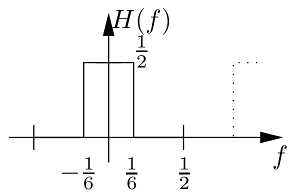
$$U(f) = X(4f)$$
$$V(f) = U(f)H(f)$$
$$Y(f) = \frac{1}{2} \sum_{k=0}^{1} V\left(\frac{f-k}{2}\right)$$

In general,

$$Y(f) = \begin{cases} \frac{1}{2} X(2f) & |f| \le \frac{1}{6} \\ 0 & \frac{1}{6} < |f| \le \frac{1}{2} \end{cases}$$

b) The system can be implemented by an interpolation with a factor 2 followed by a filter:





3. a) The spectrum of the signal has four peaks (per period) at frequencies $\pm f_1$ and $\pm f_2$. Assume that the amplitude of the two sinusoids is A. The signal in each segment is multiplied with a rectangular window $w(n) = 1, n = 0, 1, \ldots, (N-1)$, so the spectrum is convolved with

$$W(f) = \mathcal{F}\{w(n)\} = \frac{\sin(\pi f N)}{\sin(\pi f)} e^{-j\pi f(N-1)} \approx N \frac{\sin(\pi f N)}{\pi f N} e^{-j\pi f(N-1)}$$

if f is small. When the resulting spectrum $Y(f) = A(W(f-f_1) + W(f+f_1) + W(f-f_2) + W(f+f_2))$ is calculated using DFT, $\max[|Y(k)|] \approx AN$ and only one k value for each frequency peak will make $|k - f_i N| < 0.5$, such that $AW(k/N - f_i) > \gamma$, all other Y(k) will be set to zero. Thus, only four $Y(k)$ values have to be transmitted, which gives:

- Number of bits/segment $= 4(\log_2(N) + 16)$
- Number of bits/sample $= \frac{4\log_2(N)+64}{N}$

b) A larger segment size N gives better compression. However, if the flute player changes tone during a segment, the result will be wrong, so each segment must be significantly shorter than the shortest tone of the melody.

c) Assume that we add $M - N$ zeros to each segment. Calculate the FFT of length M instead of N and use $\log_2(M)$ bits for each index.

Each frequency peak will be more accurately described, both since we use a finer grid of frequency values and since we will use several $Y(k)$ values to describe each peak ($\approx M/N$ non-zero $Y(k)$ values for each peak if the same γ is used). The disadvantage is that more bits have to be transmitted for each segment, since we transmit more values and use more bits for each value. Thus, zero-padding can be used to determine the trade-off between accuracy and compression.

4. For the two first order filters we have

$$H(z) = \frac{(1+0.4z^{-1})(1+0.2z^{-1})}{(1-0.2z^{-1})(1-0.4z^{-1})} = \frac{a+bz^{-1}}{1-0.2z^{-1}} + \frac{c+dz^{-1}}{1-0.4z^{-1}}$$

where $a + c = 1$. In order to minimize the number of multiplications, we choose $a = 0,\ c = 1$ or $a = 1,\ c = 0$. We do not have to consider the round-off noise directly on the output but only the round-off filtered through the first order filters (due to multiplication by 0.2 and 0.4. Choose the alternative that gives the smallest

$$\sum_{n=0}^{\infty} h_1^2(n) + \sum_{n=0}^{\infty} h_2^2(n)$$

where $h_1(n)$ and $h_2(n)$ are the impulse responses of the first order filters.

Alt I. Let $a = 0,\ c = 1$ which gives $b = -1.2,\ d = 2$. This gives

$$H_1(z) = \frac{-1.2z^{-1}}{1-0.2z^{-1}} \implies h_1(n) = -1.2 \cdot 0.2^{n-1} u(n-1)$$

$$H_2(z) = \frac{1+2z^{-1}}{1-0.4z^{-1}} \implies h_2(n) = 0.4^n u(n) + 2 \cdot 0.4^{n-1} u(n-1) = 1\delta(n) + 2.4 \cdot 0.4^{n-1} u(n-1)$$

Alt II. Let $a = 1,\ c = 0$ which gives $b = -1.4,\ d = 2.4$. This gives

$$H_1(z) = \frac{1-1.4z^{-1}}{1-0.2z^{-1}} \implies h_1(n) = 0.2^n u(n) - 1.4 \cdot 0.2^{n-1} u(n-1) = 1\delta(n) - 1.2 \cdot 0.2^{n-1} u(n-1)$$

$$H_2(z) = \frac{2.4z^{-1}}{1-0.4z^{-1}} \implies h_2(n) = 2.4 \cdot 0.4^{n-1} u(n-1)$$

We see that

$$\sum_{n=0}^{\infty} h_1^2(n) + \sum_{n=0}^{\infty} h_2^2(n) = 1 + 1.2^2 \frac{1}{1-0.2^2} + 2.4^2 \frac{1}{1-0.4^2} \approx 9.36$$

is the same in both cases. The choice does not matter.

5. In vector form we have

$$\boldsymbol{y} = \boldsymbol{p}a + \boldsymbol{n}$$

$\boldsymbol{y}^T = [y(0), \ldots, y(N-1)]$ and similarly for $\boldsymbol{p}$ and $\boldsymbol{n}$.

a)

$$\hat{a} = \arg\min_a |\boldsymbol{y} - \boldsymbol{p}a|^2 = \frac{\boldsymbol{p}^T\boldsymbol{y}}{\boldsymbol{p}^T\boldsymbol{p}} = \frac{1}{N}\boldsymbol{p}^T\boldsymbol{y}$$

b)

$$\mathrm{E}\{\hat{a}\} = \frac{1}{N}\mathrm{E}\{\boldsymbol{p}^T\boldsymbol{y}\} = \frac{1}{N}\mathrm{E}\{\boldsymbol{p}^T(\boldsymbol{p}a + \boldsymbol{n})\} = \frac{1}{N}\boldsymbol{p}^T\boldsymbol{p}a + \frac{1}{N}\boldsymbol{p}^T\mathrm{E}\{\boldsymbol{n})\} = a$$

$$\begin{aligned}
\mathrm{E}\{(\hat{a}-a)(\hat{a}-a) &= \mathrm{E}\{(\frac{1}{N}\boldsymbol{p}^T\boldsymbol{y} - a)(\frac{1}{N}\boldsymbol{p}^T\boldsymbol{y} - a)\} = \mathrm{E}\{\frac{1}{N^2}\boldsymbol{p}^T\boldsymbol{y}\boldsymbol{y}^T\boldsymbol{p}\} - a^2 \\
&= \frac{1}{N^2}\mathrm{E}\{\boldsymbol{p}^T(\boldsymbol{p}a + \boldsymbol{n})(\boldsymbol{p}a + \boldsymbol{n})^T\boldsymbol{p}\} - a^2 = \frac{1}{N^2}\left((\boldsymbol{p}^T\boldsymbol{p})^2a^2 + \boldsymbol{p}^T\mathrm{E}\{\boldsymbol{n}\boldsymbol{n}^T\}\boldsymbol{p}\right) - a^2 \\
&= a^2 + \frac{1}{N^2}\sigma^2\boldsymbol{p}^T\boldsymbol{p} - a^2 = \frac{\sigma^2}{N}
\end{aligned}$$

c) The error is given by $\hat{a} - a = \frac{1}{N}\boldsymbol{p}^T\boldsymbol{n}$. The error is a linear combination of Gaussian random variables and is therefore Gaussian. A mistake is made when the error exceeds a in the "wrong" direction and is independent of which bit is sent due to symmetry. The probability of making an error can thus be computed from

$$P_e = \int_a^\infty f_{\sigma_e^2}(x)dx$$

where $f_{\sigma_e^2}(x)$ is the probability density function for a zero-mean Gaussian random variable with variance $\sigma_e^2 = \sigma^2/N$.