SIGNAL PROCESSING

DEPARTMENT OF ELECTRICAL ENGINEERING

Digital Signal Processing EQ2300/ 2E1340

Final Examination 2010–06–03, 14.00–19.00
Sample Solutions

1. Let $H_1(z) = a + bz^{-1}$. Following the suggestion in the complementary reading, we take $F_1(z) = -H_0(-z) = -1 + \frac{1}{2}z^{-1}$ and $F_0(z) = H_1(-z) = a - bz^{-1}$ in which case we are guaranteed that

$$F_0(z)H_0(-z) + F_1(z)H_1(-z) = 0$$

Further, we have

$$\begin{aligned} F_0(z)H_0(z) + F_1(z)H_1(z) &= (a - bz^{-1})(1 + \tfrac{1}{2}z^{-1}) + (-1 + \tfrac{1}{2}z^{-1})(a + bz^{-1}) \\ &= (a - 2b)z^{-1} = 2z^{-1} \end{aligned}$$

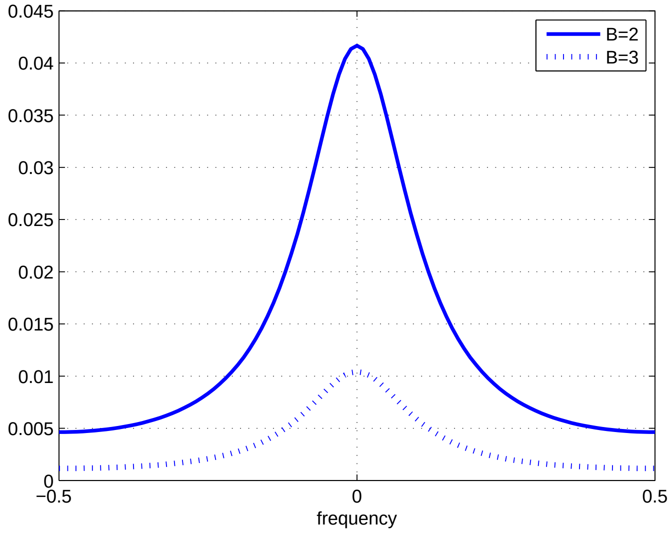
where the last equality is true if $a - 2b = 2$. Thus, we get perfect reconstruction with delay $L = 1$ whenever $a - 2b = 2$. Additionally, to get $H_1(f = 0) = H_1(z = 1) = 0$ requires that $a + b = 0$. Solving the two equation yield $a = \frac{2}{3}$ and $b = -\frac{2}{3}$ which completely specify the filters.

2. a) The variance of the quantization noise from each of the multipliers is $\sigma_q^2 = \frac{2^{-2B}}{12}$. This quantization noise is sequence of i.i.d. variables, therefore the decimation does not change the variance of the first quantization noise source. By adding the two quantization noise sources, we can consider one input quantization noise source with variance $2\sigma_q^2$, which passes through the second stage of the filter,i.e., $H_2(z) = \frac{1}{1-0.5z^{-1}}$ or $h_2(n) = (0.5)^n u(n)$. The output quantization noise variance is given by $\sigma_y^2 = 2\sigma_q^2 \sum_{-\infty}^{\infty} h_2^2(n) = \frac{2^{-2B+1}}{9}$.

b) The spectral density of the quantization noise at the output is given by

$$\begin{aligned} R_{yy}(f) = R_{qq}(f)|H(f)|^2 &= 2\sigma_q^2 \frac{1}{(1 - 0.5e^{-j2\pi f})(1 - 0.5e^{j2\pi f})} \\ &= \frac{2^{-2B}}{6} \frac{1}{(1 - \cos(2\pi f) + 0.25)} \end{aligned} \tag{1}$$

The spectrum is shown in the figure below for $B = 2$ and $B = 3$.



The number of multiplications required in the different described cases are

3. (a) $L_1 + L_2$

(b) $L_1 + L_2 - 1$

(c) $\frac{N_1 \log_2(2N_1)}{N_1 - L_1 + 1} + \frac{N_2 \log_2(2N_2)}{N_2 - L_2 + 1}$

(d) $\frac{N \log_2(2N)}{N - L + 1}$

(e) Since $L_1 = 5$ and $L_2 = 10$, then we find the number of required multiplications for

i. part (a): $L_1 + L_2 = 15$,

ii. part (b): $L_1 + L_2 - 1 = 14$,

iii. part (c): since the optimal FFT length is equal to $N_1 = 16$ for $L_1 = 5$, then the corresponding number of $mult./sample = 6.67$. Similarly, by $N_2 = 64$ the number of $mult./sample = 8.15$. Finally the overall number of $mult./sample = 14.82$,

iv. part (d): in this case $L = L_1 + L_2 - 1 = 14$ which results in $N = 64$, then the corresponding number of $mult./sample \approx 8.96$.

By comparing all the values obtained, the method (d) is the most efficient.

4. (a) Theory tells us that $\mathrm{E}\{\hat{P}_{xx}(f)\} = P_{xx}(f) * W_B(f)$. Thus, if the 3dB width of the main lobe of $W_B(f)$, which is

$$\Delta = \frac{0.89}{L},$$

is large in relation to the bandwidth of $P_{xx}(f)$, changing f_B will not significantly alter the appearance of $P_{xx}(f) * W_B(f)$. In other words, the resolution of the spectrum estimation must be sufficiently large in relation to what we try to see in the spectrum. Setting

$$\Delta = \frac{0.89}{L} \gg 10^{-3}$$

yields that $L \gg 890$ in order to get the resulting required.

(b) The spectrum of the dowsampled signal $y(n) = x(nD)$ satisfies

$$R_{yy}(f) = \frac{1}{D} R_{xx}(f/D)$$

as long as $R_{xx}(f) = 0$ for $0 \leq f \leq \frac{1}{2D}$ (otherwise aliasing would occur). Thus, if we should be able to represent all frequencies $f \leq 10^{-3}$ it should hold that $D \leq 500$. Thus, we could downsample the signal by up to a factor 500. Doing so would save much computation in the spectrum estimation if we are only interested in normalized frequencies up to 10^{-3}.

5. (a) The inverse of the given system function is

$$H^{-1}(Z) = 1 + \alpha Z^{-1} + 0.4Z^{-2} + 0.6Z^{-3}$$

To implement this system using a lattice filter structure, we must find the reflection coefficients that generate the polynomial $H^{-1}(z)$ or $\bar{H}(Z)$.
Now following the step-down recursion we proceed by

$$H_3^{-1}(Z) = 1 + \alpha Z^{-1} + 0.4Z^{-2} + 0.6Z^{-3}$$

we see that $\Gamma_3 = 0.6$

Next, generating the second-order system, $\bar{H}_2(Z)$

$$\begin{aligned}
H_2^{-1}(Z) &= \frac{1}{1-(\Gamma_3)^2}[H_3^{-1}(Z) - \Gamma_3 Z^{-3} H_3^{-1}(Z^{-1})] \\
&= \frac{1}{1-(0.6)^2}[1 + \alpha Z^{-1} + 0.4Z^{-2} + 0.6Z^{-3} - 0.6Z^{-3}(1 + \alpha Z + 0.4Z^2 + 0.6Z^3)] \\
&= 1 + \frac{\alpha - 0.24}{0.64} Z^{-1} + \frac{0.4 - 0.6\alpha}{0.64} Z^{-2}
\end{aligned}$$

and so the second reflection coefficient,Γ_2, is $\Gamma_2 = \frac{0.4-0.6\alpha}{0.64}$.

Finally, the first order system would be,

$$\begin{aligned}
H_1^{-1}(Z) &= \frac{1}{1-(\Gamma_2)^2}[H_2^{-1}(Z) - \Gamma_2 Z^{-2} H_2^{-1}(Z^{-1})] \\
&= \frac{1}{1-(\frac{0.4-0.6\alpha}{0.64})^2}\left[1 + \left(\frac{\alpha - 0.24}{0.64}\right)Z^{-1} + \left(\frac{0.4-0.6\alpha}{0.64}\right)Z^{-2} - \right. \\
&\quad \left. \left(\frac{0.4-0.6\alpha}{0.64}\right)Z^{-2}\left\{1 + \left(\frac{\alpha-0.24}{0.64}\right)Z + \left(\frac{0.4-0.6\alpha}{0.64}\right)Z^2\right\}\right] \\
&= 1 + \left(\frac{\alpha - 0.24}{1.0401 - 0.6\alpha}\right)Z^{-1}
\end{aligned}$$

and therefore, $\Gamma_1 = \left(\frac{\alpha-0.24}{1.0401-0.6\alpha}\right)$. The structure of the lattice filter is as shown below.

(b) The system is stable when $|\Gamma_1|$, $|\Gamma_2|$ and $|\Gamma_3|$ are less than 1.
$|\Gamma_1| = 0.6 < 1$,
$|\Gamma_2| = |\frac{0.4-0.6\alpha}{0.64}| < 1$,

for $\alpha \leq \frac{0.4}{0.6}$
$=> \frac{0.4-0.6\alpha}{0.64} < 1$ and so $\alpha > -0.4$

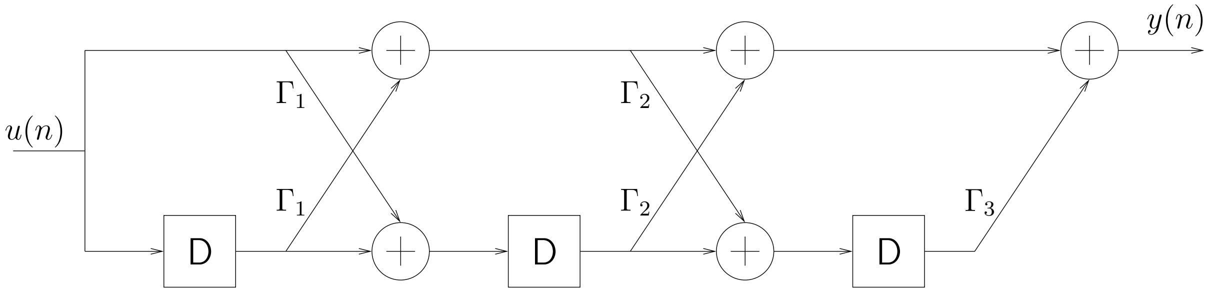


Figure 1: $H^{-1}(Z)$

for $\alpha > \frac{0.4}{0.6}$
$=> -\frac{(0.4-0.6\alpha)}{0.64} < 1$ and so $\alpha < 1.733$
Therefore, $|\Gamma_2| < 1$ for $-0.4 < \alpha < 1.733$

Finally, $|\Gamma_3| < 1 => \frac{|\alpha-0.24|}{|1.0401-0.6\alpha|} < 1$
for $0.24 \leq \alpha < 1.7335, |\Gamma_3| < 1$ when $\alpha - 0.24 < 1.0401 - 0.6\alpha$

$=> \alpha < 0.8$
for $\alpha < 0.24, |\Gamma_3| < 1$ when $-\alpha + 0.24 < 1.0401 - 0.6\alpha$
$=> -2 < \alpha < 0.24$
for $\alpha > 1.7335, |\Gamma_3| < 1$ when $\alpha - 0.24 < -1.0401 + 0.6\alpha$
$=> -2 > \alpha$ and $\alpha > 1.7335$ and so for this particular case $\alpha \in \{\}$

Therefore, $|\Gamma_3| < 1$ for $-2 < \alpha < 0.8$. And so the given system is stable for the set containing α , such that, $\alpha \in (-0.4, 1.733) \cap (-2, 0.8)$

i.e , the system is stable for $\alpha \in (-0.4, 0.8)$