SOLUTIONS

E 88 **Digital Signalbehandling**, 2E1340

Final Examination 2002–08–23, 14.00–18.00

1. We can categorize spectral estimation methods into non-parametric (Fourier based) and parametric (model based) methods. In the parametric methods the signal is assumed to emanate from a specific model with certain parameters to be determined. The non-parametric methods use fewer assumptions on the signal and just look for periodicities in the signal. As long as the assumptions on the signal model is relevant, the parametric methods can give more accurate estimates than the non-parametric methods. On the other hand, the non-parametric methods are more robust if the signal model is completely unknown. Another general problem of the parametric methods is to determine the model order.

Non-parametric:

Periodogram: Coincides with Maximum Likelihood if the signal is a single cisoid. In general, it provides better resolution than many other non-parametric methods but has a high variance.

Bartlett: Uses averaging to decrease the variance at the expense of lower resolution.

Welch: Similar to Bartlett, but the variance is slightly lower since the averaging is done over overlapping segments of data and the data is windowed. The windowing could also reduce the leakage.

Blackman-Tukey: Windowing the estimated autocorrelation sequence is another method to get lower variance and leakage but lower resolution than the periodogram.

Parametric:

AR-based methods: Assumes that the true spectrum is given by an AR-model (not true for a line spectrum). Good for signals with reasonably smooth spectra.
Many different methods have been suggested to estimate the AR parameters: LS, Burg, Yule-Walker, to mention a few.

Subspace methods: If the true signal is a sum of sinusoids (line spectrum), the data covariance matrix is low-rank. This fact is exploited in the, so-called, subspace methods.

Pisarenko: Simple subspace method with low accuracy since only a single eigenvector is used.

MUSIC: Uses more eigenvectors to provide better accuracy.

Maximum Likelihood: Based on a specific signal model, it is always possible to derive a maximum likelihood method. It can be shown that ML provides the best possible estimation accuracy (almost always), but the computational complexity is often very high compared to other methods.

2. a) Using the standard approximation, we model the quantization as additive white noise with power $\sigma_q^2 = \frac{2^{-2b}}{12}$. The two quantization noise sources (corresponding to the two multiplications), both enter the circuit at the addition in the lower branch. The impulse response from the noise sources to the output is $h(n) = \beta^n u(n)$, so the resulting noise power at the output is

$$\sigma_{y,q}^2 = 2\sigma_q^2 \sum_{n=0}^{\infty} |h(n)|^2 = \frac{2^{-2b}}{6} \frac{1}{1-\beta^2}$$

b) $\beta = 2/3$ gives $\sigma_{y,q}^2 = 0.3 \cdot 2^{-2b}$, see Figure 1.

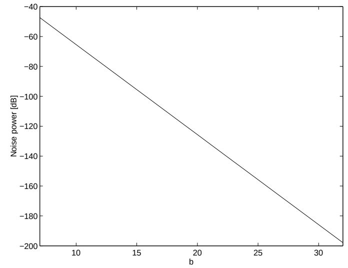

Figure 1: Quantization noise power as a function of b.

3. Note that $p_0(n) = h(2n)$ is simply a decimated version of $h(n)$, thus

$$P_0(f) = \frac{1}{2}\left(H(\frac{f}{2}) + H(\frac{f-1}{2})\right) = \frac{1}{4}(f - (f-1)) = \frac{1}{4},\ 0 \le f < 1$$

which means that $P_0(f) = \frac{1}{4}$ for all f.

Likewise, $p_1(n) = h(2n+1)$ is a decimated version of $g(n) = h(n+1)$ (note that $G(f) = e^{j2\pi f}H(f)$). This gives

$$P_1(f) = \frac{1}{2}\left(e^{j\pi f}H(\frac{f}{2}) + e^{j\pi(f-1)}H(\frac{f-1}{2})\right) = \frac{1}{4}e^{j\pi f}(f-(1-f))) = \frac{e^{j\pi f}(2f-1)}{4},\ 0 \le f < 1$$

For $-1/2 < f < 0$, we have $P_1(f) = P_1(f+1) = -\frac{e^{j\pi f}(2f+1)}{4}$. To summarize,

$$P_0(f) = \frac{1}{4}, \qquad P_1(f) = \begin{cases} \frac{e^{j\pi f}(2f-1)}{4} & 0 \le f < 1/2 \\ -\frac{e^{j\pi f}(2f+1)}{4} & -1/2 < f < 0 \end{cases}$$

4. Since the fundamental frequency is 20Hz, $T = 1/20$, so the Fourier series yields $x(t) = \sum_{m=-10}^{10} c_m e^{j2\pi \cdot 20mt}$. Denoting the sampling frequency by T_s, the FFT of the sampled signal is, $X(k) = \sum_{n=0}^{N-1} x(nT_s)e^{-j2\pi\frac{nk}{N}} = \sum_{m=-10}^{10} c_m \sum_{n=0}^{N-1} e^{-j2\pi\frac{n}{N}(20mNT_s-k)} = \sum_{m=-10}^{10} c_m N\delta((k-20mNT_s))_N$.

a) For the fundamental frequency ($m = 1$, $k = 5$), $5 = 20NT_s \Longrightarrow T_s = \frac{1}{4N} = \frac{1}{2048}$ seconds.

b) Since $X(k) = \sum_{m=-10}^{10} c_m N\delta((k-5m))_N$, we get $c_m = \frac{1}{N}X((5m))_N = \frac{1}{512}X((5m))_{512}$.

5. a) The data model is $\mathbf{x} = \mathbf{H}\boldsymbol{\theta}$ where

$$\mathbf{H} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The LS solution of this problem is

$$\hat{\boldsymbol{\theta}} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{x} = \begin{bmatrix} x(0) \\ x(1) \end{bmatrix}$$

b) The constraint of this problem is $\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$=0.

c) Setting the derivative to zero gives $\mathbf{H}^T\mathbf{H}\boldsymbol{\theta} + \mathbf{A}^T\boldsymbol{\lambda}/2 = \mathbf{H}^T\mathbf{x}$. In order to determine $\boldsymbol{\lambda}$, we also need the original constraint (i.e. setting the derivative of the Lagrangian with respect to $\boldsymbol{\lambda}$ to zero). The resulting system of equations is

$$\begin{bmatrix} \mathbf{H}^T\mathbf{H} & \mathbf{A}^T \\ \mathbf{A} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\theta} \\ \boldsymbol{\lambda}/2 \end{bmatrix} = \begin{bmatrix} \mathbf{H}^T\mathbf{X} \\ \mathbf{b} \end{bmatrix}$$

For this problem, we get

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \\ \lambda/2 \end{bmatrix} = \begin{bmatrix} x(0) \\ x(1) \\ 0 \end{bmatrix}$$

Adding the two first lines gives $\hat{\theta}_1 + \hat{\theta}_2 = x(0) + x(1)$ which combined with the third line gives

$$\hat{\theta}_1 = \hat{\theta}_2 = \frac{1}{2}[x(0) + x(1)]$$

d) The data model of this constrained LS problem can be reformulated as $\mathbf{x} = \mathbf{H}_T\boldsymbol{\theta}_T$ where $\theta_T = \theta_1 = \theta_2$. The resulting LS estimate of θ_T is $\hat{\theta}_T = \frac{1}{2}[x(0) + x(1)]$, which is the same answer as above.