SIGNAL PROCESSING

SCHOOL OF ELECTRICAL ENGINEERING

Digital Signal Processing   EQ2300 / 2E1340

Final Examination 2017–01–11, 08.00–13.00 Examples of Solutions

1. a) In general, if $x[n] = \delta[n-m]$ we have $X[k] = e^{-j2\pi\frac{mk}{N}}$ which gives:

i) $x_{\mathrm{A}}[n] = \delta[n]$ and $\mathcal{F}_8\{x_{\mathrm{A}}[n]\} = e^{-j2\pi\frac{0k}{N}} = e^0 = 1 = X_{\mathrm{III}}[k]$,

ii) $x_{\mathrm{B}}[n] = \delta[n-1]$ and $\mathcal{F}_8\{x_{\mathrm{A}}[n]\} = e^{-j2\pi\frac{k}{8}} = \cos\big(2\pi\frac{k}{8}\big) - \sin\big(2\pi\frac{k}{8}\big) = X_{\mathrm{IV}}[k]$,

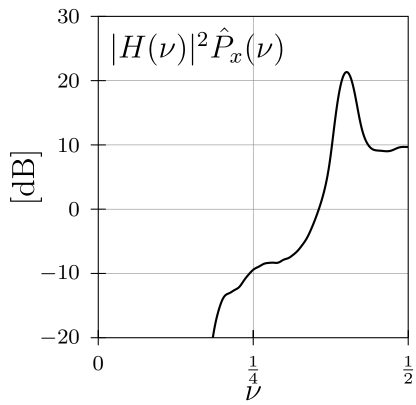
iii) $x_{\mathrm{C}}[n] = \delta[n-4]$ and $\mathcal{F}_8\{x_{\mathrm{D}}[n]\} = e^{-j2\pi\frac{4k}{8}} = e^{-j\pi k} = (-1)^k = X_{\mathrm{I}}[k]$, and

iv) $x_{\mathrm{D}}[n] = \delta[n-7]$ and $\mathcal{F}_8\{x_{\mathrm{D}}[n]\} = e^{-j2\pi\frac{7k}{8}} = e^{j2\pi\frac{k}{8}} = \cos\big(2\pi\frac{k}{8}\big) + \sin\big(2\pi\frac{k}{8}\big) = X_{\mathrm{II}}[k]$.

b) iii) $H(z) = 1 + \frac{1}{2}z^{-1}$ does not have linear phase, because it is neither symmetric or anti-symmetric.

c) The Hamming window had a wider main lobe and lower sidelobes. This implies that that the designed filter will have a less rapid transition from the pass-band to the stop-band, but better suppression of frequencies in the stop-band.

d) The filter $H(\nu)$ effectively removes all low frequencies, resulting in



which according to

$$P_y(\nu) = \frac{1}{2}\sum_{k=0}^{1} P_x\left(\frac{\nu-k}{D}\right)\left|H\left(\frac{\nu-k}{D}\right)\right|^2,$$

and since $|H(\nu)|^2\hat{P}_x(\nu)$ almost exclusively cover the upper part of the frequency range, gets flipped and stretched and becomes spectrum A.

2. a) If h has length $M = 15$, we must at least have $N \geq M = 15$ so that the N-point zero padded DFT of $h[n]$ is valid. However, $N = 15$ is not the most efficient choice. The complexity per signal sample, is given by

$$C_N = \frac{N \log_2 2N}{N - M + 1}$$

which for $M = 15$ and N of the form $N = 2^p$ has a minimum for $N = 64$ in which case $C_N = 8.96$ as tabulated in the summary notes. For this value of N, the number of samples read from the input for each block is $L = N - M + 1 = 50$. Consequently, we set `N = 64` and `L = 50` to get the least computationally complex implementation.

b) If we read 50 samples form the input for each block, and have 1000 samples to process, we need to repeat the procedure for $1000/50 = 20$ blocks. This implies that $B = 20$ or `B = 20`. The complexity per block is $N \log_2 2N$ so the total complexity becomes

$$C = BN \log_2 2N = 20 \times 64 \times \log_2 128 = 8960\,.$$

This figure should be increase by another $\frac{N}{2} \log_2 N = 192$ if we also wish to account for the complexity of transforming $h[n]$, but this part is marginal.

c) The output will have length $1000 + 15 - 1 = 1014$, and computing each value required 15 multiplications, so the total complexity becomes

$$C = 1014 \times 15 = 15210\,.$$

The appoximation $C \approx 1000 \times 15 = 15000$ is also ok in this case. Either way, the complexity is larger than that of the overlap-add method.

d) To store the entire output sequence, we need to have $M \geq 1014$. This will also take care of the issue with the linear versus circular convolution. The smallest value of the form $M = 2^p$ that satisfies this is $M = 2^10 = 1024$. Since the implementation requires 3 M-point FFTs (2 FFTs and an inverse FFT) as well as M multiplications in the frequency domain, the total complexity becomes

$$C = \frac{3M \log_2 M}{2} + M = 16384\,.$$

This is worse than the overlap-add method, and even worse than the direct implementation of the convolution in the time domain.

3. a) From the summary notes, course book, and video lectures, we have for Bartlett's method that

$$\mathrm{E}\{\hat{P}_x^{\mathrm{B}}(\nu)\} = P_x(\nu) \circledast \frac{1}{L} |W_{\mathrm{R}}(\nu)|^2$$

where $W_{\mathrm{R}}(\nu$ is the discrete time Fourier transform of the rectangular window, given by $w_{\mathrm{R}}[n] = 1$ when $0 \leq n < L$, and $w_{\mathrm{R}}[n] = 0$ otherwise. We als have for Blackman-Tukey's method that

$$\mathrm{E}\{\hat{P}_x^{\mathrm{BT}}(\nu)\} \approx P_x(\nu) \circledast W^{2M+1}(\nu)$$

where $W^{2M+1}(\nu)$ is the length $2M + 1$ window (with maximum lag M) used in this method. Note here that $W^{2M+1}(\nu)$ is the discrete time Fourier transform of the window given in the hint, and that for this problem $M = L$.

Note that by the above, the requested result will follow if we show that $\frac{1}{L}|W_{\mathrm{R}}(\nu)|^2 = W^{2M+1}(\nu)$, which in the time domain is equivalent to

$$\frac{1}{L} w_{\mathrm{R}}[-n] * w_{\mathrm{R}}[n] = w[n] \,.$$

To this end, for $n \geq 0$ and $n \leq L$ we have

$$w_{\mathrm{R}}[-n] * w_{\mathrm{R}}[n] = \sum_{m=-\infty}^{\infty} w[m]w[m-n] = \sum_{m=n}^{L-1} 1 = L - n = L - |n| \,.$$

For $n \leq 0$ and $n \geq -L$ we have

$$w_{\mathrm{R}}[-n] * w_{\mathrm{R}}[n] = \sum_{m=-\infty}^{\infty} w[m]w[m-n] = \sum_{m=0}^{L-1+n} 1 = L + n = L - |n| \,.$$

When $|n| \geq L$ we have $w_{\mathrm{R}}[-n] * w_{\mathrm{R}}[n] = 0$ because the two parts do not overlap. Thus,

$$\frac{1}{L} w_{\mathrm{R}}[-n] * w_{\mathrm{R}}[n] = w[n] = \frac{L - |n|}{L} = \frac{M - |n|}{M} = w[n]$$

when $|n| < M$, and $\frac{1}{L} w_{\mathrm{R}}[-n] * w_{\mathrm{R}}[n] = w[n] = 0$ when $|n| \geq M$.

This established that $\mathrm{E}\{\hat{P}_x^{\mathrm{B}}(\nu)\} \approx \mathrm{E}\{\hat{P}_x^{\mathrm{BT}}(\nu)\}$. The reason that we do not have equality in this expression is that the formula for the mean of Blackman-Tukey's method neglects a convolution with the squared Fourier transform of a length N window that appears due to the biased estimation of the ACF $r_x[k]$, as is also the case for the basic periodogram estimate. This effect is however usually negligible in comparison with the convolution with $W^{2M+1}(\nu)$, and we can for the problem at hand also visually see in the figure that the approximation is quite good.

b) At frequencies where the noise dominates, we have for Bartlett's method that

$$\mathrm{Var}\{\hat{P}_x(\nu)\} \approx \frac{1}{K} P_x^2(\nu) \,.$$

For all other frequencies than $\nu = \nu_0$, we have $P_x(\nu) = \sigma^2 \approx 0.1$. Thus,

$$\mathrm{Var}\{\hat{P}_x(\nu)\} \approx \frac{1}{K} P_x^2(\nu) = \frac{\sigma^4}{16} \approx \frac{0.01}{16} = 6.25 \times 10^{-4} \,.$$

c) Similar to the previous part, we have for Blackman-Tukey's method that

$$\mathrm{Var}\{\hat{P}_x(\nu)\} \approx \left[\frac{1}{N} \sum_{k=-M}^{M} w^2[k] \right] P_x^2(\nu) \,,$$

where

$$\left[\frac{1}{N} \sum_{k=-M}^{M} w^2[k] \right] = \frac{2M}{3N}$$

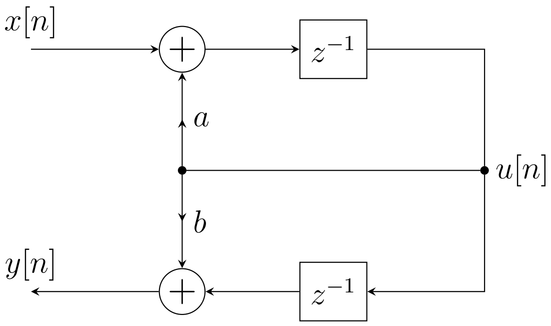
for the triangular window. Thus,

$$\mathrm{Var}\{\hat{P}_x(\nu)\} \approx \frac{2M}{3N} \sigma^4 = \frac{2 \times 64 \times 0.01}{3 \times 1024} \approx 4.17 \times 10^{-4} \,.$$

d) As the two estimators have approximately the same mean, and as Blackman-Tukey's estimator has better variance, we can argue that Blackman-Tukey's method is the one with better performance.

4. Problem with quantization noise

a) We begin by introducing the additional help variable $n[n]$ according to



Taking the z-transform of the signals involved, we get

$$U(z) = z^{-1}(X(z) + aU(z))$$

which implies that

$$U(z) = \frac{z^{-1}}{1 + az^{-1}} X(z)\,.$$

Next,

$$Y(z) = bU(z) + z^{-1}U(z) = \frac{bz^{-1} + z^{-2}}{1 + az^{-1}} X(z)$$

which finally implies that

$$H(z) = \frac{bz^{-1} + z^{-2}}{1 + az^{-1}}\,.$$

b) We get $h[n]$ by taking the inverse z-transform, according to

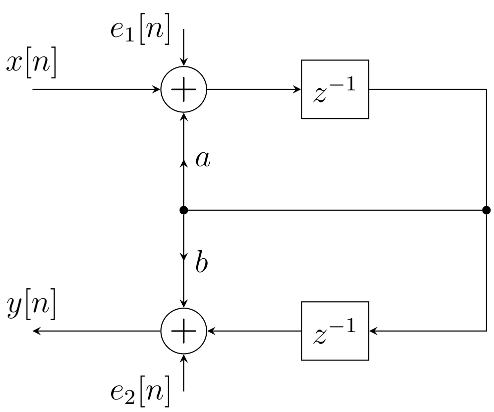
$$H(z) = bz^{-1}\frac{1}{1 + az^{-1}} + z^{-2}\frac{1}{1 + az^{-1}} \quad \Rightarrow \quad h[n] = ba^{n-1}u[n-1] + a^{n-2}u[n-2]$$

where $u[n]$ is the unit step function. The impulse response can also be written as

$$h[n] = b\delta[n-1] + (ba + 1)a^{n-2}u[n-2]\,,$$

which will be convenient later.

c) We get the equivalent noise model, by adding additive zero mean white noise sources with power $2^{-2B}/12$ after each multiplication as shown below.



d) In the previous part, we see that $e_1[n]$ reach the output after passing though the entire filter (it enters the circuit at the same place as $x[n]$), while $e_2[n]$ reaches the output directly. To get the power of $e_1[n]$ at the output, we need to multiply the power of the input with

$$\sum_{n=-\infty}^{\infty} |h[n]|^2 .$$

To this end, we have

$$\sum_{n=-\infty}^{\infty} |h[n]|^2 = b^2 + (ba+1)^2 \sum_{n=2}^{\infty} a^{2(n-2)} = b^2 + \frac{(ba+1)^2}{1-a^2} = \frac{b^2+2ab+1}{1-a^2},$$

where we have used the second expression for $h[n]$ from part b). he power of $e_2[n]$ at the output will be the same as it is at the insertion point. Thus, the total power of teh noise at the output of the circuit will be given by

$$\sigma_e^2 = \frac{2^{-2B}}{12}\left(1 + \frac{b^2+2ab+1}{1-a^2}\right) = \frac{2^{-2B}}{12}\frac{2+b^2-a^2+2ab}{1-a^2}.$$

5. a) Note first that we have

$$p_0[n] = h[2n] = \frac{\sin(\pi n)}{\pi n} = 0$$

when $n \neq 0$, as $\sin(\pi n) = 0$ for all integer n, and $p_0[0] = h[0] = 1$. Consequently, $p_0[n] = \delta[n]$. Note next that $y[2n] = p_0[n] * x[n] = x[n]$ for any n which was to be proven. Similarly, $y[2n+1] = p_1[n] * x[n]$ although this was not explicitly asked about in the questions.

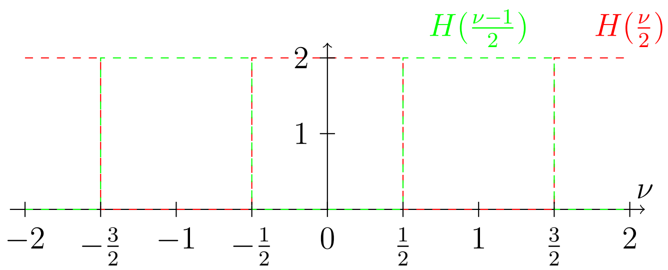
b) As $p_0[n] = h[2n]$ is just a down-sampled version of $h[n]$, we have from the summery notes that

$$P_0(\nu) = \frac{1}{2}\sum_{k=0}^{1} H\left(\frac{\nu-k}{2}\right) = \frac{1}{2}H\left(\frac{\nu}{2}\right) + \frac{1}{2}H\left(\frac{\nu-1}{2}\right).$$

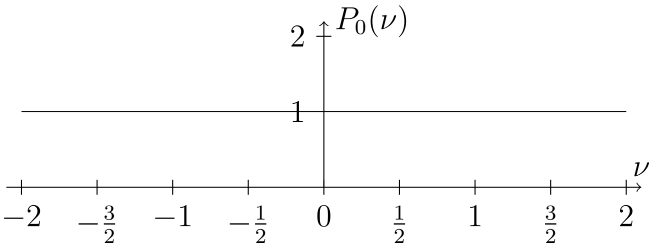
When $0 < \nu| < \frac{1}{2}$ we have that $\frac{|\nu|}{2} < \frac{1}{4}$ and $\frac{1}{4} < \frac{|\nu-1|}{2} < \frac{1}{2}$, which implies that

$$H\left(\frac{\nu}{2}\right) = 2 \quad \text{and} \quad H\left(\frac{\nu-1}{2}\right) = 0$$

in the ideal case. This implies that $P_0(\nu) = 1$ for $0 < \nu < \frac{1}{4}$. An equivalent argument gives $P_0(\nu) = 1$ for $\frac{1}{4} < \nu < \frac{1}{2}$, and the rest follows by symmetry and periodicity. Graphically, we have



and thus



Note here that we do not have to worry about magnitude and phase, as all terms involved are real valued.

c) Let $g[n] = h[n+1]$. Then $p_1[n] = g[2n]$and

$$P_1(\nu) = \frac{1}{2} G\left(\frac{\nu}{2}\right) + \frac{1}{2} G\left(\frac{\nu - 1}{2}\right)$$

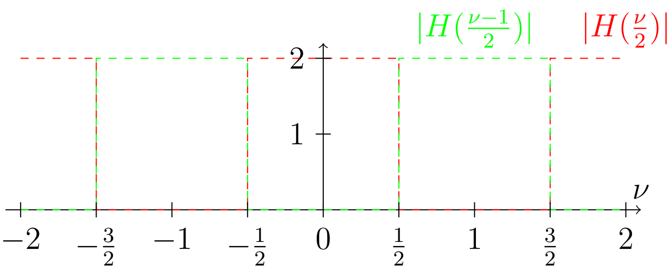
by the solution to part b). Further,

$$G(\nu) = e^{j2\pi\nu} H(\nu)$$

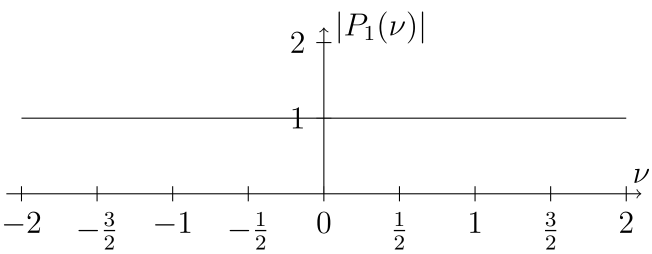
which implies that

$$P_1(\nu) = \frac{1}{2} e^{j2\pi\frac{\nu}{2}} H\left(\frac{\nu}{2}\right) + \frac{1}{2} e^{j2\pi\frac{\nu-1}{2}} H\left(\frac{\nu-1}{2}\right) = \frac{1}{2} e^{j\pi\nu} H\left(\frac{\nu}{2}\right) - \frac{1}{2} e^{j\pi\nu} H\left(\frac{\nu-1}{2}\right).$$

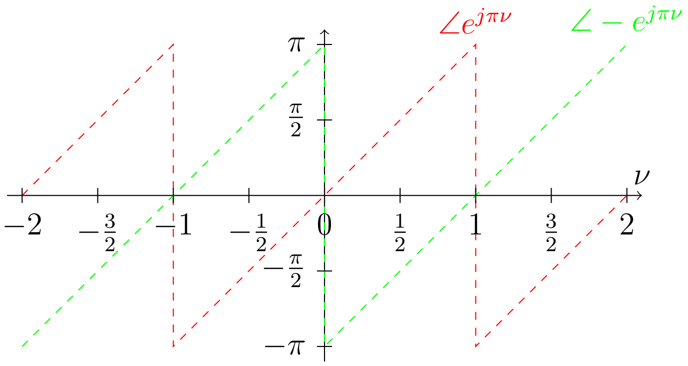
d) As $H\left(\frac{\nu}{2}\right)$ and $H\left(\frac{\nu-1}{2}\right)$ do not overlap in the sense that one terms is zero whenever the other terms is non-zero, we can treat the magnitude and phase of each term separately. For the magnitude, we can proceed as in the graphical solution to part b), and obtain



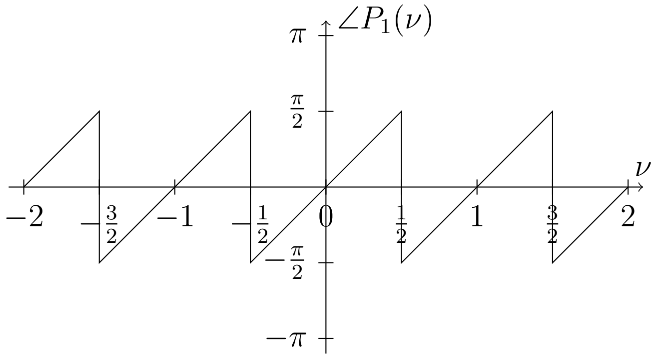
and thus



For the phase, it helps to plot



which is together with the intermediate magnitude plot implies that



Note now that for $|\nu| < \frac{1}{2}$, it follows that $P_1(\nu) = e^{j\pi\nu} = e^{-j2\pi\tau\nu}$ for $\tau = -\frac{1}{2}$, which looks like a time shift (time-advance) of half a sample. Note however that we cannot have $P_1(\nu) = e^{j\pi\nu}$ across all ν as $P_1(\nu)$ need to be periodic with period 1, so the interpretation of $p_1[n]$ implementing half a sample delay is somewhat flawed.