SIGNAL PROCESSING

SCHOOL OF ELECTRICAL ENGINEERING

Digital Signal Processing EQ2300 / 2E1340

Final Examination 2012–12–13, 14.00–19.00 Examples of Solutions

1. a) Upsampling by 6 gives $\{\underset{\uparrow}{1}, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 3, 0, 0, 0, 0, 0, 4, 0, 0, 0, 0, 0\}$ which after downsampling by 4 gives $y(m) = \{\underset{\uparrow}{1}, 0, 0, 3, 0, 0\}$.

Answer: i).

b) As can be seen from the phase resonse, the filter is linear phase. A stable, causal IIR system cannot be linear phase, so i) is excluded. All Type 3 linear phase FIR systems have a zero both at $f = 0$ and $f = 1/2$, which excludes iii). All Type 2 linear phase FIR systems have a zero at $f = 1/2$, which agrees with the plotted magnitude response. As an alternative, you could check the value of the phase response at $f = 0$, which according to Table 4.1 in the book is 0 for Type 2 and $\pi/2$ for Type 3.

Answer: ii)

c) The Matlab command produces the periodogram estimate. An M point DFT evaluates the TDFT at the frequency points $f = k/M$, so the peak $k_0 = 88$ corresponds to a peak at $f_0 = 88/M$.

Answer: iv)

d) Two different features of the periodogram can be used to solve the problem. The first zero of $W_R(f)$ (using the notations of Tsakonas&Bengtsson "Some Notes on Non-Parametric Spectral Estimation") is at $f = 1/N$, so the main lobe extends from $f = -1/N$ to $f = 1/N$. From the figure, we see that the main lobe extends $8/M$ to the left and right around $f = f_0$, so we can conclude that $1/N = 8/M$, i.e., $N = M/8 = 32$. Alternatively, we can exploit that the height of the peak for a sinusoidal signal with amplitude A is $NA^2/4$, which gives $4^2N/4 = 128$, again resulting in $N = 32$.

Answer: ii)

2. The result of the multiplication by c has to be rounded off, resulting in an additive quantization noise $e(n)$ added after the multiplication. The standard approximations state that $e(n)$ is white, uncorrelated with $x(n)$ and has power $\sigma_e^2 = 2^{-2B}/12$, where in this example $B = 7$ (number of bits **excluding** the sign bit). Since the system is linear, superposition applies meaning that the output signal is the sum of the contribution from $x(n)$ and $e(n)$. Denote the output stemming from $x(n)$ by $y_x(n)$ and the output stemming from $e(n)$ by $y_e(n)$, so that $y(n) = y_x(n)+y_e(n)$. Since $x(n)$ and $e(n)$ are uncorrelated, also $y_x(n)$ and $y_e(n)$ are uncorrelated and consequently, the total power spectral density of the output is $P_y(f) = P_{y_x}(f) + P_{y_e}(f)$.

Contribution from $x(n)$: The transfer function from $x(n)$ to $y(n)$ is given by

$$H(z) = \frac{1 + cz^{-1}}{1 - cz^{-1}}$$

so

$$P_{y_x}(f) = P_x(f)\,|H(f)|^2 = 0.1\frac{|1 + ce^{-j2\pi f}|^2}{|1 - ce^{-j2\pi f}|^2} = 0.1\frac{1 + c^2 + 2c\cos(2\pi f)}{1 + c^2 - 2c\cos(2\pi f)}$$

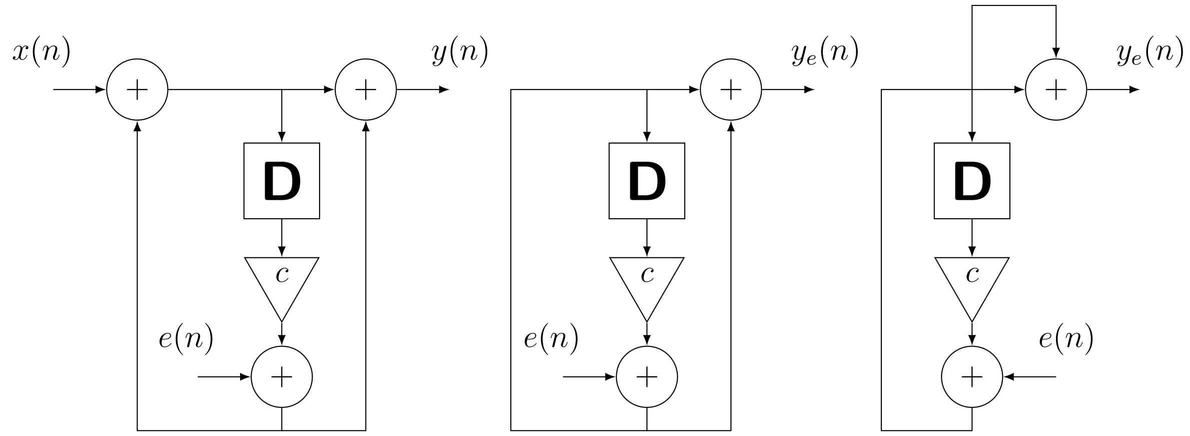

Figure 1: Approximate model for the quantization noise (left). Equivalent system considering only $e(n)$ as the input (middle and right).

Contributions from $e(n)$: The middle graph in Fig. 1 shows what the system looks like when only $e(n)$ is the input. Noting that the same system can be drawn as in the right hand graph, if follows that the transfer function from $e(n)$ to $y_e(n)$ is

$$G(z) = \frac{2}{1 - cz^{-1}}$$

Consequently,

$$P_{y_e}(f) = P_e(f)\,|G(f)|^2 = \sigma_e^2 \frac{4}{|1 - ce^{-j2\pi f}|^2} = \frac{2^{-14}/3}{1 + c^2 - 2c\cos(2\pi f)}$$

The total power spectral density at the output is therefore

$$P_y(f) = P_{y_x}(f) + P_{y_e}(f) = \frac{2^{-14}/3 + 0.1(1 + c^2) + 0.2c\cos(2\pi f)}{1 + c^2 - 2c\cos(2\pi f)}$$

3. The DTFT of $x(n)$ is given by

$$X(f) = \sum_{m=-\infty}^{\infty} \frac{A_1}{2}\big(\delta(f - m - 1/16) + \delta(f - m + 1/16)\big) + \frac{A_2}{2}\big(\delta(f - m - 1/8) + \delta(f - m + 1/8)\big)$$

After downsampling by $D = 6$, we obtain

$$W(f) = \frac{1}{6}\sum_{k=0}^{5} X\left(\frac{f-k}{6}\right)$$

The peaks of $X(f)$ at $f = \pm 1/16$ will end up as peaks of $W(f)$ at $f/6 = \pm 1/16$, i.e. at $f = \pm 3/8$, plus periodic repetitions. The peaks of $X(f)$ at $f = \pm 1/8$ will end up as peaks of $W(f)$ at $f/6 = \pm 1/8$, i.e. at $f = \pm 3/4$, plus periodic repetitions. The only of the latter that end up in the interval $f \leq 1/2$ are at $f = \pm 1/4$. What happens to the amplitudes? Consider first what happens in the time domain. Clearly, downsampling a single sinusoidal signal will result in a sinusoidal signal with the same amplitude, so we can directly conclude that the amplitudes of the two sinusoids will remain A_1 and A_2, respectively. To see this mathematically, consider for example the term $A_1/2\delta(f - 1/16)$ in $X(f)$. The corresponding term with $k = 0$ in $W(f)$ will be

$$\frac{1}{6}\frac{A_1}{2}\delta\left(\frac{f}{6} - \frac{1}{16}\right) = \frac{1}{6}\frac{A_1}{2}\delta\left(\frac{f - 6/16}{6}\right) = \frac{A_1}{2}\delta\left(f - \frac{3}{8}\right)$$

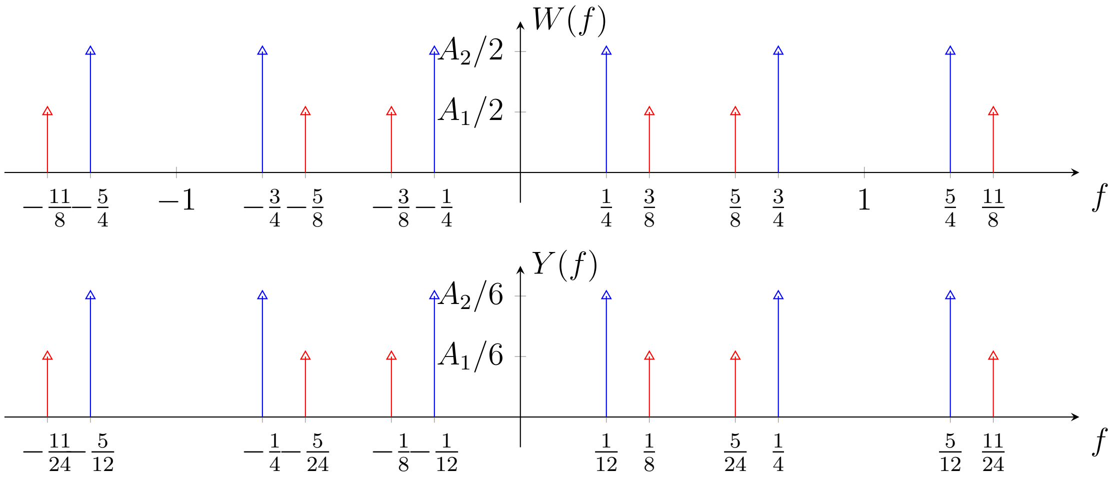

Figure 2: $W(f)$ and $Y(f)$

where the last equality follows from $\delta(f/C) = C\delta(f)$. To summarize,

$$W(f) = \frac{A_1}{2}\big(\delta(f-3/8)+\delta(f-m+3/8)\big)$$
$$+\frac{A_2}{2}\big(\delta(f-m-1/4)+\delta(f-m+1/4)\big) + \text{periodic repetitions},$$

see Fig. 2. An even simpler approach to reach the same result is to look at the time domain expressions,

$$w(k) = x(6k) = A_1 \cos\left(2\pi\frac{6}{16}k\right) + A_2 \cos\left(2\pi\frac{6}{8}k\right)$$

and note that $\cos\left(2\pi\frac{3}{4}k\right) = \cos\left(2\pi k - 2\pi\frac{1}{4}k\right) = \cos\left(2\pi\frac{1}{4}k\right)$.

After upsampling by $I = 3$ we obtain

$$Y(f) = W(3f)$$

giving the scaling of the frequency axis shown in Fig. 2. Note that we have to consider all peaks that end up in $|f| \le 1/2$ in $Y(f)$, i.e., all peaks of $W(f)$ in the interval $|f| \le 3/2$. The above mentioned relationship $\delta(f/C) = C\delta(f)$ means that the "height" of each Dirac pulse in $Y(f)$ should be 1/3 of the corresponding Dirac pulse in $W(f)$. In the time domain, the result displayed in Fig. 2 correspond to

$$y(n) = \frac{A_1}{3}\left(\cos\left(2\pi\frac{1}{8}k\right)+\cos\left(2\pi\frac{5}{24}k\right)+\cos\left(2\pi\frac{11}{24}k\right)\right)$$
$$+\frac{A_2}{3}\left(\cos\left(2\pi\frac{1}{12}k\right)+\cos\left(2\pi\frac{1}{4}k\right)+\cos\left(2\pi\frac{5}{12}k\right)\right)$$

The amplitude by 1/3 makes sense if you for example consider that $y(0) = w(0) = x(0)$ must hold.

Answer: See the lower plot of Fig. 2.

4. a) The transfer function from $e(n)$ to $x(n)$ is $H(z) = 1/(1 - 0.2z^{-1})$ so the spectral density of $x(n)$ is

$$P_x(f) = \sigma_e^2 |H(f)|^2 = \frac{1}{1 + \alpha^2 - 2\alpha \cos(2\pi f)}$$

where we used the notation $\alpha = 0.2$. Table lookup shows that the inverse DTFT is

$$r_{xx}(k) = \frac{1}{1 - \alpha^2} \alpha^{|k|} = \frac{1}{0.96} (0.2)^{|k|}$$

Since $e(n)$ and $w(n)$ are uncorrelated, also $x(n)$ and $w(n)$, so that

$$r_{yy}(k) = r_{xx}(k) + r_{ww}(k) = \frac{1}{0.96} (0.2)^{|k|} + 0.1\delta(k)$$

b) Use Yule-Walker to find the AR coefficients,

$$\begin{bmatrix} r_{yy}(0) & r_{yy}(1) \\ r_{yy}(1) & r_{yy}(0) \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \end{bmatrix} = - \begin{bmatrix} r_{yy}(1) \\ r_{yy}(2) \end{bmatrix}$$

which has the solution

$$\begin{aligned} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \end{bmatrix} &= -\frac{1}{r_{yy}^2(0) - r_{yy}^2(1)} \begin{bmatrix} r_{yy}(0) & -r_{yy}(1) \\ -r_{yy}(1) & r_{yy}(0) \end{bmatrix} \begin{bmatrix} r_{yy}(1) \\ r_{yy}(2) \end{bmatrix} \\ &= \frac{1}{r_{yy}^2(0) - r_{yy}^2(1)} \begin{bmatrix} r_{yy}(1)\,(r_{yy}(2) - r_{yy}(0)) \\ r_{yy}(0) r_{yy}(2) - r_{yy}^2(1) \end{bmatrix} \approx \begin{bmatrix} -0.182 \\ -0.003 \end{bmatrix} \end{aligned}$$

The resulting AR(2) model is given by

$$\hat{y}(n) + \hat{a}_1 \hat{y}(n-1) + \hat{a}_2 \hat{y}(n-2) = v(n)$$

i.e., $\hat{y}(n) = g(n) * v(n)$, where $v(n)$ is white noise with power $\hat{\sigma}_v^2 = r_{yy}(0) + \hat{a}_1 r_{yy}(1) + \hat{a}_2 r_{yy}(2) \approx 1.18$ and the filter $g(n)$ has transfer function

$$G(z) = \frac{1}{1 + \hat{a}_1 z^{-1} + \hat{a}_2 z^{-2}}$$

c) The model based spectral estimate is given by the power spectral density of the estimated model:

$$\hat{P}_y(f) = \hat{\sigma}_v^2 |G(z)|^2 = \frac{\hat{\sigma}_v^2}{|1 + \hat{a}_1 e^{-j2\pi f} + \hat{a}_2 e^{-j4\pi f}|^2}$$

5. a) Since the length of $x(n)$ is $N = 50$, and the length of $h(n)$ is $K = 10$, the length of $y(n) = x(n) * h(n)$ is $N + K - 1 = 59$ samples (more precisely, $y(n)$ is in general non-zero for $0 \leq n \leq N + K - 2 = 58$.

b) Consider first the linear convolution sum. Since both $x(n)$ and $h(n)$ are zero for $n < 0$, we obtain

$$y(n) = \sum_{k=\infty}^{\infty} x(k)h(n-k) = \sum_{k=0}^{n} x(k)h(n-k)$$

Assume that $0 \leq n \leq N-1$ and split the sum defining the circular convolution into two parts, one where $(n-k)_{\text{mod } N} = n-k$ and one where $(n-k)_{\text{mod } N} = n-k+N$.

$$\begin{aligned} y_c(n) = x(n)\text{\textcircled{4}}h(n) &= \sum_{k=0}^{N-1} x(k)h((n-k)_{\text{mod } N}) \\ &= \sum_{k=0}^{n} x(k)h(n-k) + \sum_{k=n+1}^{N-1} x(k)h(n-k+N) \quad (1) \end{aligned}$$

The first sum coincides with the linear convolution $y(n)$. Compare the second sum with the linear convolution sum for $y(n+N)$,

$$y(n+N) = \sum_{k=0}^{n+N} x(k)h(n+N-k) = \sum_{k=n}^{N-1} x(k)h(n+N-k)$$

where the second inequality holds since $x(k) = 0$ when $k > N - 1$ and similarly if $k < n$, then $n - k + N > N \geq K$, so that $h(n - k + N) = 0$. This means that the non-zero terms of the second sum in (1) coincide with $y(n+N)$. This proves that $y_c(n) = y(n) + y(n + 50)$ holds when $0 \leq n \leq 49$.

It is recommended to visualize the above result graphically.

c) Using the above result and the given values, we have

$n = 0 \Rightarrow y(0) + y(50) = 10 \Rightarrow y(50) = 5$

$\vdots$

$n = 4 \Rightarrow y(4) + y(54) = 10 \Rightarrow y(54) = 5$

$n = 5 \Rightarrow y(5) + y(55) = 10 \Rightarrow y(55) = 10 - y(5) \Rightarrow y(5),\ y(55)$ cannot be determined

$\vdots$

$n = 8 \Rightarrow y(8) + y(58) = 10 \Rightarrow y(58) = 10 - y(8) \Rightarrow y(8),\ y(58)$ cannot be determined

$n = 9 \Rightarrow y(9) + y(59) = 10 \Rightarrow y(9) + 0 = 10 \Rightarrow y(9) = 10$

$\vdots$

$n = 49 \Rightarrow y(49) + y(99) = 10 \Rightarrow y(49) + 0 = 10 \Rightarrow y(49) = 10$

To summarize,

$$y(n) = \begin{cases} 5, & 0 \leq n \leq 4 \\ \text{unknown}, & 5 \leq n \leq 8 \\ 10, & 9 \leq n \leq 49 \\ 5, & 50 \leq n \leq 54 \\ \text{unknown}, & 55 \leq n \leq 58 \end{cases}$$