SOLUTIONS

E 86 **Digital Signalbehandling**, 2E1340

Final Examination 2001–12–21, 14.00–18.00

1. a) A. (Since we know that the signal is causal, the transform is identical to the single-sided Z-transform.)

b) C. (See Table 5.2 in the book.)

c) A. (Half segment length $\Rightarrow$ half resolution)

d) B. (The residual error of the LS solution should be the norm of the noise vector.)

2. First, calculate the quantization noise at the output. The only round-off occurs at the multiplication and is modeled as additional white noise with variance $\sigma_e^2 = \frac{2^{-2b}}{12}$. The transfer function from the noise source to the output is $H(z) = \frac{1}{1-0.7z^{-1}}$, which gives the impulse response $h(n) = 0.7^n$. Thus, the quantization noise power at the output is

$$P_{\text{noise}} = \sigma_e^2 \sum_{n=0}^{\infty} h^2(n) = \frac{2^{-2b}}{12(1-0.7^2)} = \frac{2^{-2b}}{6.12}$$

Next, the signal amplitude should be maximized to give maximum SNR. Since a linear filter only changes the amplitude and phase of a sinusoidal input, we know that the output (without quantization) is of the form $y(n) = \alpha\cos(2\pi f_0 n + \phi)$, where

$$\alpha = A\left|H(e^{j2\pi f_0})\right| = \frac{A}{\sqrt{1.49 - 1.4\cos(2\pi f_0)}}$$

As long as we limit A such that the maximum amplitude of both the input and the output 1, it is easy to see that there will be no overflow. Thus, the maximum amplitude of the output signal is

$$\alpha_{\max} = \min\{\frac{1}{\sqrt{1.49 - 1.4\cos(2\pi f_0)}}, 1\} = \begin{cases} 1 & |f_0| \le \frac{\arccos(0.35)}{2\pi} \\ \frac{1}{\sqrt{1.49-1.4\cos(2\pi f_0)}} & \frac{\arccos(0.35)}{2\pi} < |f_0| \le 0.5 \end{cases}$$

For a given output amplitude α, the average signal power on the output is

$$P_{\text{signal}} = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\alpha^2\cos^2(2\pi f_0 n + \phi) = \alpha^2 \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\frac{1+\cos(4\pi f_0 n + 2\phi)}{2} = \frac{\alpha^2}{2}$$

The resulting maximum SNR is

$$\text{SNR}_{\max} = \frac{P_{\text{signal,max}}}{P_{\text{noise}}} = \begin{cases} 3.06\cdot 2^{2b} & |f_0| \le 0.19 \\ \frac{3.06\cdot 2^{2b}}{1.49-1.4\cos(2\pi f_0)} & 0.19 < |f_0| \le 0.5 \end{cases}$$

A more conservative constraint on $A = \max|x(n)|$ which guarantees that no overflow appears, no matter what input function is used, can be obtained as explained on page 588 of the book:

$$A_{\max} = \frac{1}{\sum|h(n)|} = 0.3$$

3. a)

$$\begin{aligned} V(f) =& G_1(f)F(2f)\frac{1}{2}(H_1(f)Y(f) + H_1(f-\frac{1}{2})Y(f-\frac{1}{2})) \\ &+ G_0(f)F(2f)\frac{1}{2}(H_0(f)Y(f) + H_0(f-\frac{1}{2})Y(f-\frac{1}{2})) \\ =& \frac{1}{2}F(2f)((G_1(f)H_1(f) + G_0(f)H_0(f))Y(f) \\ &+ (G_1(f)H_1(f-\frac{1}{2}) + G_0(f)H_0(f-\frac{1}{2}))Y(f-\frac{1}{2})) \\ =& e^{-j2\pi fL}F(2f)Y(f) \end{aligned}$$

Thus, the whole filter bank corresponds to a filter with transfer function $e^{-j2\pi fL}F(2f)$.

b) The factor $e^{-j2\pi fL}$ corresponds to a delay by L and

$$\mathcal{F}^{-1}[F(2f)] = \tilde{f}(n) = \begin{cases} f(n/2) & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Therefore,

$$v(n) = \sum_k \tilde{f}(k)y(n-L-k) = \Big/ k = 2m \Big/ = \sum_m f(m)y(n-L-2m)$$

c) If $F(f)$ corresponds to a Δ-step delay, then $v(n) = y(n-L-2\Delta)$.

4. a) According to the Levinson-Durbin algorithm, the estimated prediction error is given by the following recursion:

$$\hat{\sigma}_0^2 = \hat{\sigma}_y^2$$
$$\hat{\sigma}_p^2 = \hat{\sigma}_{p-1}^2(1-|K_p|^2),\ p = 1, 2, \ldots$$

Thus, we have

p	1	2	3	4	5
$\hat{\sigma}_p^2$	0.19	0.0684	0.0438	0.0420	0.0382
AIC(p)	−1.58	−2.52	−2.89	−2.85	−2.86

AIC(p) is minimum for $p = 3$.

b) Using the Levinson-Durbin iteration to go from reflection coefficients to filter coefficients, we get

Order, m	$a_m(0)$	$a_m(1)$	$a_m(2)$	$a_m(3)$
1	1	0.9		
2	1	0.18	-0.8	
3	1	0.66	-0.908	-0.6

Thus, the estimated AR-model is $y(n) + 0.66y(n-1) - 0.908y(n-2) - 0.6y(n-3) = e(n)$, where $e(n)$ is white noise with variance $\sigma_e^2 = \hat{\sigma}_3^2 \approx 0.0438$.

5. a) $k_1 = \frac{F_1}{F_s}N = 213$ and $k_2 = (1-\frac{F_1}{F_s})N = 300$ due to aliasing as $F_2 > \frac{F_s}{2}$. Peaks corresponding to negative frequencies: $k_3 = 1024 - 213 = 811$ and $k_4 = 1024 - 300 = 724$. Since the frequency separation between the peaks is $\gg 1/100$, there are no resolution problems.

b) The anti-alias filter included in the decimation will remove the sinusoid with the higher frequency. Thus the resulting frequency of the reconstructed signal is 208 Hz.

c) The eigenvector corresponds to the coefficients in the $A(z)$ polynomial, i.e. the solution is found by setting $A(z) = 1 - 1.854z^{-1} + 9.000z^{-2} = 0$. The normalized frequencies are then found as $\hat{f} = \arg(\frac{z}{2\pi}) = \pm 0.2$ and the $F = 0.2\cdot 1000$ Hz $= 200$ Hz. In practice, you should be skeptical to the frequency estimate if the roots are this far from the unit circle.