SIGNAL PROCESSING

SCHOOL OF ELECTRICAL ENGINEERING

Digital Signal Processing EQ2300 / 2E1340

Final Examination 2016–01–16, 8:00–13:00 Reference Solutions

1. a) Using the graphic methodology for the circular convolution we obtain

$$\begin{array}{cccccccc}
\mathbf{3} & \mathbf{0} & \mathbf{2} & \mathbf{0} & \mathbf{1} & \mathbf{0} & & \\
3 & 0 & 0 & 0 & 1 & 2 & \rightarrow & \mathbf{10} \\
2 & 3 & 0 & 0 & 0 & 1 & \rightarrow & \mathbf{6} \\
1 & 2 & 3 & 0 & 0 & 0 & \rightarrow & \mathbf{9} \\
0 & 1 & 2 & 3 & 0 & 0 & \rightarrow & \mathbf{4} \\
0 & 0 & 1 & 2 & 3 & 0 & \rightarrow & \mathbf{5} \\
0 & 0 & 0 & 1 & 2 & 3 & \rightarrow & \mathbf{2}\,.
\end{array}$$

Thus, $\{3,\ 0,\ 2,\ 0,\ 1,\ 0\}$ ⑥ $\{3,\ 2,\ 1,\ 0,\ 0,\ 0\} = \{10, 6, 9, 4, 5, 2\}$.

b) When a signal is real, its FFT is hermitian in the sense that

$$X[N-k] = \sum_{n=0}^{N} x[n] \exp\left(-j\frac{2\pi}{N}(N-k)n\right) = \sum_{n=0}^{N} x[n] \exp\left(j\frac{2\pi}{N}kn\right) = X^*[k]\,.$$

Thus, we have that $X[4] = X^*[2] = -j$ and $X[5] = X^*[1] = 1$, i.e.

$$\mathcal{F}_6\{x[n]\} = \{2,\ 1,\ j,\ 0,\ -j,\ 1\}\,.$$

c) $y[m] = x[3m] = \sin(2\pi\, 3\, 0.4m + 0.3) = \sin(2\pi\, 1.2m + 0.3) = \sin(2\pi\, 0.2m + 0.3)$. Therefore, $\nu_y = 0.2$ and $\phi_y = \phi_x = 0.3$.

d) Observe that $G(z) = H(-z)$ implies that the DFTs of $g[n]$ and $h[n]$ are related through $G(\nu) = H(-z)|_{z=\exp j2\pi\nu} = H\left(\nu + \frac{1}{2}\right)$. Therefore

$$g[n] = \exp\left(-j2\pi\frac{1}{2}n\right) h[n] = (-1)^n h[n]$$

and $G(\nu)$ is a low pass filter. To gain some intuition into this, note that for $\nu = 0$, $G(0) = H\left(0 + \frac{1}{2}\right) = H\left(\frac{1}{2}\right)$.

e) The most likely true model order is 7. The coefficient b_0^2 can be interpreted as the mean square fitting error of the AR model. Therefore, we see that from order 7 on, the mean square error does not decrease, meaning that all the extra coefficients are being assigned very small values and the fitting of the models is still being done with the first 7.

2. a) We recognize from the expression of the estimate that we are using the periodogram as an estimator for the spectral density of $x[n]$. We know that, when using the periodogram, the resolution will be that corresponding to a rectangular window of length N. Measuring the resolution in terms of the 3 dB bandwidth yields that

$$\Delta\nu_{3\text{dB}} = \frac{0.89}{N} = 8.9 \times 10^{-4}\,.$$

b) The spectral resolution with respect to the original signal $x(t)$, i.e. in the time-continuous domain, is related by the sampling frequency to the spectral resolution in the time-discrete domain. Indeed, the 3 dB bandwidth with respect to the time-continuous signal is

$$\Delta f_{3\text{dB}} = f_s \Delta\nu_{3\text{dB}} = 10^3 \text{ Hz } \times 8.9 \times 10^{-4} = 0.89 \text{ Hz } .$$

c) The radix-2 FFT algorithm can only work on signals of length 2^b with $b \in \mathbb{N} \setminus \{0\}$. In order to preserve all the spectral content, we need to choose the smallest b such that $2^b \geq N$ and zero-pad our signal with $2^b - N$ zeros at the end. In this case, the smallest such value is $b = 10$, for which $2^b = 1024$ and $2^b - N = 24$. Therefore, we have to zero-pad $x[n]$ with 24 zeros and then use the FFT_{1024} to compute the values of $\hat{P}_x(\nu)$ at discrete frequencies $\nu = \nu_k$ with $k = 0, \dots, 1023$.

d) The FFT_{1024} of the zero-padded signal is a discrete signal $\tilde{X}[k]$ that fulfills

$$\tilde{X}[k] = X(\nu)|_{\nu = \frac{k}{1024}} ,$$

where $X(\nu)$ is the DTFT of $x[n]$. Thus, the spacing between the discrete frequencies that correspond to any two consecutive ks is $\frac{1}{1024}$. Consequently, the spacing in Hz between the continuous frequencies that correspond to two consecutive samples of the FFT_{1024} is

$$\Delta f = \frac{1}{1024} \times 10^3 \text{ Hz } = \frac{1}{1.024} \text{ Hz } \approx 0.97656 \text{ Hz } .$$

e) The only approximations that we have for the variances of these spectral density estimators depend on the assumption that the process $x[n]$ is a "noisy" AR-type process, and that we have a relatively large number of samples. Because this is our only comparison ground for these methods, we will assume these conditions are fulfilled at all points during this section. The variance of the periodogram estimator is approximately

$$\text{Var}\left\{\hat{P}_x(\nu)\right\} \approx P_x^2(\nu) .$$

The Welch estimator with 50 % overlap and Barlett window $\hat{P}_x^W(\nu)$ has approximate variance

$$\text{Var}\left\{\hat{P}_x^W(\nu)\right\} \approx \frac{9}{8K} P_x^2(\nu) ,$$

where K is the number of blocks in which the signal $x[n]$ has been divided within the Welch estimator. Thus, the reduction factor on the variance from using the Welch estimator will always be

$$\frac{\text{Var}\left\{\hat{P}_x(\nu)\right\}}{\text{Var}\left\{\hat{P}_x^W(\nu)\right\}} = \frac{8K}{9} .$$

It is clear that higher Ks will yield higher reduction factors, and thus, we are interested in finding the higher K that can be used for the Welch estimation of $P_x(\nu)$ with 50 % overlap, Barlett window and $N = 1000$ while yielding a 3 dB bandwidth smaller than 10 Hz. The 3 dB bandwidth of the Welch estimator (in Hz) is

$$\Delta f_{3\text{dB}}^W = f_s \Delta\nu_{3\text{dB}}^W = 10^3 \text{ Hz } \times \frac{1.28}{L} \leq 10 \text{ Hz } .$$

This implies that $L \geq 128$. With a 50 % overlap, the delay between blocks is fixed to $D = \frac{L}{2}$. Moreover, the obvious condition that we can not use more samples than those that we have implies that

$$(K-1)D + L - 1 \leq N - 1 \Rightarrow KD - D + L \leq N \Rightarrow K \leq \frac{N-L}{D} + 1\,.$$

Note that, in general, we will not necessarily use all the N samples, but only the first $(K-1)D + L - 1$.

Therefore, in this specific case $K \leq \frac{N-L}{D} + 1 = \frac{2N}{L} - 1 \leq \frac{2000}{128} - 1 \approx 14.625$. Because $K \in \mathbb{Z}$, the maximum K we can get is 14, which yields a reduction factor of $\frac{8\times 14}{9} \approx 12.44$.

3. a) The frequency response $H_A(\nu)$ corresponds to the windowing method for filter design, and the frequency response $H_B(\nu)$ corresponds to the frequency sampling method for filter design. This can be seen because $H_B(\nu)$ periodically (with period $\frac{1}{M+1}$) fits the template of the ideal filter perfectly. This creates the most revealing behavior, which is the fluctuation of the filter response in the pass-band, which can not be explained by convolution with a monomodal (single maximum) window. Moreover, $H_A(\nu)$ has a shape consistent with a convolution with a Hanning window. Finally, the scale of these plots is linear, and thus, the ripples of the window in $H_A(\nu)$ are not and should not be visible.

b) For a filter of length $M+1$ (order M), the frequency sampling design method tells us to fix $H_1[k] = H_I(\nu)|_{\nu=\frac{k}{M+1}}$, where $H_I(\nu)$ is the response of the ideal filter. However, we only have information on the modulus of the ideal filter. This implies that we have to explicitly choose the phase. Luckily, for a Type I linear phase filter of Mth order (with M even) we have studied that we have to factor the samples we take of our ideal filter in such a way that

$$H_I\left(\frac{k}{M+1}\right) = A[k]\exp^{j2\pi\theta[k]} \text{ with } \theta[k] = -\frac{kM}{2(M+1)} \text{ for } k = 0, 1, \ldots, M\,,$$

and $A[k] = A[M-k+1] \in \mathbb{R}$ for $k = 1, 2, \ldots, \frac{M}{2}$. Using what we know of our ideal filter, i.e.

$$H_I(\nu) = \begin{cases} 1 & \text{for } \nu \in \left[\frac{1}{4}, \frac{3}{4}\right] \\ 0 & \text{otherwise} \end{cases} \text{ for } \nu \in [0, 1]$$

we obtain that $A[k] = \left|H_I\left(\frac{k}{M+1}\right)\right| = \left\{\underset{\uparrow}{0}, 0, 0, 1, 1, 1, \ldots\right\}$, in which we only specify the first $\frac{M}{2}$ samples because of the symmetry property stated above. With such a choice, we have studied that we will get an impulse response

$$\begin{aligned} h[n] &= \frac{1}{M+1}\left[A[0] + 2\sum_{k=1}^{\frac{M}{2}}(-1)^k A[k]\cos\left(\frac{\pi k(1+2n)}{M+1}\right)\right] \\ &= \frac{2}{11}\left[-\cos\left(\frac{\pi 3(1+2n)}{11}\right) + \cos\left(\frac{\pi 4(1+2n)}{11}\right) - \cos\left(\frac{\pi 5(1+2n)}{11}\right)\right], \end{aligned}$$

for $n = 0, \ldots, 10$. Note that the phase term $\theta[k] = -\frac{kM}{2(M+1)}$ is consistent with a group delay of $\frac{M}{2}$ expressed as a linear phase, and that we do not need to verify that the resulting impulse response fulfills the Type I conditions, because we have designed it forcing these conditions.

c) The windowing design method tells us to obtain the IIR impulse response of the ideal filter in mind and multiply it by the desired window function. Our ideal filter would have the modulus in the figure and a linear phase expressing its group delay of $\frac{M}{2}$ samples. Thus,

$$H_I(\nu) = \begin{cases} \exp\left(-j2\pi\frac{M}{2}\right) & \text{for } \nu \in \left[\frac{1}{4}, \frac{3}{4}\right] \\ 0 & \text{otherwise} \end{cases} \text{ for } \nu \in [0, 1] \ .$$

Therefore,

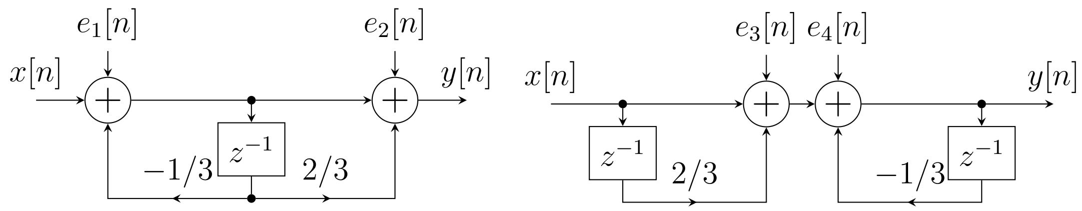
$$\begin{aligned} h_I[n] &= \int_0^1 H_I(\nu)\exp(j2\pi n\nu)d\nu = \int_{\frac{1}{4}}^{\frac{3}{4}} \exp\left(j2\pi\left(n-\frac{M}{2}\right)\nu\right)d\nu \\ &= \frac{1}{2j\,\pi\left(n-\frac{M}{2}\right)}\int_{\frac{1}{4}}^{\frac{3}{4}} j2\pi\left(n-\frac{M}{2}\right)\exp\left(j2\pi\left(n-\frac{M}{2}\right)\nu\right)d\nu \\ &= \frac{1}{2j\,\pi\left(n-\frac{M}{2}\right)}\left[\exp\left(j2\pi\left(n-\frac{M}{2}\right)\nu\right)\right]_{\frac{1}{4}}^{\frac{3}{4}} \\ &= \frac{\exp\left(j\pi\left(n-\frac{M}{2}\right)\right)}{\pi\left(n-\frac{M}{2}\right)}\frac{\exp\left(j\frac{\pi}{2}\left(n-\frac{M}{2}\right)\right)-\exp\left(-j\frac{\pi}{2}\left(n-\frac{M}{2}\right)\right)}{2j} \\ &= (-1)^{n-5}\frac{\sin\left(\frac{\pi}{2}(n-5)\right)}{\pi(n-5)} \end{aligned}$$

and

$$h_2[n] = (-1)^{n-5}\frac{\sin\left(\frac{\pi}{2}(n-5)\right)}{\pi(n-5)}w[n] \, .$$

Note that the symmetry of the ideal filter's impulse and the window provide the conditions for the resulting filter to be Type I.

4. a) Quantization noise gets inserted immediately after each multiplication as shown in the pictures below



b) In the case of implementation B both noise sources $e_3[n]$ and $e_4[n]$ passes though an AR1 circuit before reaching the output. With the choice of the feedback coefficient $-1/3$ this circuit has a pole at $z = -1/3$ and implements a high pass filter. The quantization noise in implementation B should thus appear high pass filtered, which is only consistent with spectrum $P_{e,2}(\nu)$. Consequently, the spectrum $P_{e,1}(\nu)$ corresponds to implementation A and the spectrum $P_{e,2}(\nu)$ corresponds to implementation B.

c) Starting with implementation A, the noise $e_1[n]$ passes through the entire system given by $H(z)$. Dividing $H(z)$ in partial fractions yields

$$H(z) = \frac{z+2/3}{z+1/3} = \frac{1+2/3z^{-1}}{1+1/3z^{-1}} = \frac{1}{1+1/3z^{-1}} + \frac{2/3z^{-1}}{1+1/3z^{-1}}$$

and the system impulse response is

$$h[n] = \left(-\frac{1}{3}\right)^n u[n] + \frac{2}{3}\left(-\frac{1}{3}\right)^{n-1} u[n-1]\,.$$

In particular, we have

$$h[0] = 1$$

and

$$h[n] = \left(-\frac{1}{3}\right)^n + \frac{2}{3}\left(-\frac{1}{3}\right)^{n-1} = \left(-\frac{1}{3}\right)^n - 2\left(-\frac{1}{3}\right)^n = -\left(-\frac{1}{3}\right)^n$$

when $n \geq 1$. Thus,

$$\sum_{n=0}^{\infty} |h[n]|^2 = 1 + \sum_{n=1}^{\infty}\left(\frac{1}{9}\right)^n = \sum_{n=0}^{\infty}\left(\frac{1}{9}\right)^n = \frac{1}{1-1/9} = \frac{9}{8}\,.$$

Therefore, the total noise at the output originating from $e_1[n]$ is

$$\sigma_{e_1}^2 = \frac{2^{-2B}}{12}\frac{9}{8}\,.$$

The noise from $e_2[n]$ directly reaches the output, and we get

$$\sigma_{e_2}^2 = \frac{2^{-2B}}{12}\,.$$

Summing up the total noise at the output yields

$$\sigma_e^2 = \sigma_{e_1}^2 + \sigma_{e_2}^2 = \frac{2^{-2B}}{12}\left(\frac{9}{8}+1\right) = \frac{2^{-2B}}{12}\frac{17}{8}$$

for the case of implementation A.
Continuing with implementation B, we see that both $e_3[n]$ and $e_4[n]$ pass though an AR1 filter with transfer function

$$G(z) = \frac{1}{1+1/3z^{-1}}\,,$$

or, equivalently, impulse response

$$g[n] = \left(-\frac{1}{3}\right)^n u[n]\,.$$

This yields a noise amplification of

$$\sum_{n=0}^{\infty} |g[n]|^2 = \sum_{n=0}^{\infty}\left(\frac{1}{9}\right)^n = \frac{1}{1-1/9} = \frac{9}{8}\,.$$

We thus get

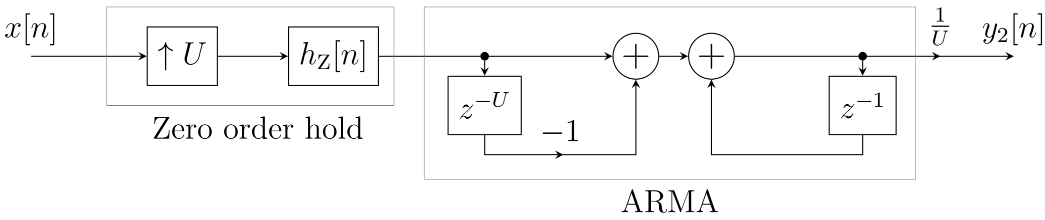
$$\sigma_{e_3}^2 = \sigma_{e_4}^2 = \frac{2^{-2B}}{12}\frac{9}{8}\,,$$

and

$$\sigma_e^2 = \sigma_{e_3}^2 + \sigma_{e_4}^2 = \frac{2^{-2B}}{12}\frac{18}{8} = \frac{2^{-2B}}{12}\frac{9}{4}\,.$$

In conclusion, implementation A has a very slight advantage in terms of quantization noise at the output.

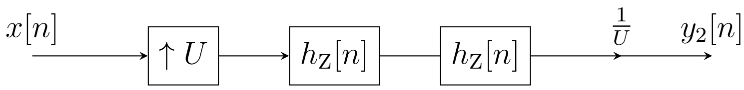
5. Using the Noble identities the system can be reformulated as



Now, the ARMA part above has a transfer function of

$$\frac{1 - z^{-U}}{1 - z^{-1}} = \sum_{n=0}^{U-1} z^{-n} ,$$

where the right hand part is equivalent to the z-transform for the zero order hold filter's impulse response. Thus, the system can be further simplified as



If we now look at the two consecutive applications of the zero order hold filter, we see that they are equivalent to a single filter with impulse response

$$h[n] \triangleq h_Z[n] * h_Z[n] = \sum_{m=-\infty}^{\infty} h_Z[m] h_Z[n-m] .$$

In order for the first term in the convolution sum to be non-zero, we require that $m \geq 0$ and $m \leq U-1$, and for the second term we require that $n-m \geq 0$ (or $m \leq n$) and $n - m \leq U - 1$ (or $m \geq n - (U-1)$). Thus, if $n < 0$ or $n > 2(U-1)$, we have $h[n] = 0$ as $h_Z[m]$ and $h_Z[n-m]$ do not overlap for any m. If $0 \leq n \leq U-1$, we have

$$h[n] = \sum_{m=0}^{n} 1 = n + 1 ,$$

and if $U \leq n \leq 2(U-1)$, we have

$$h[n] = \sum_{m=[n-(U-1)]}^{U-1} 1 = 2U - 1 - n .$$

Comparing this with $h_L[n]$, we see that $h[n] = U h_L(n-L)$ for $L = U-1$, which implies that after multiplication by $1/U$ at the end of the circuit, we have that $y_2[n] = y_1[n-L]$ where $L = U-1$. QED