SOLUTIONS

E 100 **Digital Signalbehandling**, 2E1340

Final Examination 2006–12–19, 08.00–13.00

1. (a) Total effective noise at the input of the lattice is $e(n) = e_1(n) + e_2(n) - k_2 e_3(n)$ so the variance is

$$\sigma_e^2 = \sigma^2 + \sigma^2 + k_2^2\sigma^2 = (2 + k_2^2)\frac{2^{-2(b-1)}}{12}$$

The transfer function of the lattice filter

$$H(z) = \frac{1}{1 + (k_1k_2 + k_3)z^{-1} + k_2z^{-2}} = \frac{1}{(1 - 5/9z^{-1})(1 - 3/5z^{-1})} = \frac{27/2}{1 - 3/5z^{-1}} - \frac{25/2}{1 - 5/9z^{-1}}$$

and

$$h(n) = \left[\frac{27}{2}(\frac{3}{5})^n - \frac{25}{2}(\frac{5}{9})^n\right]u(n)$$

so the total round-off noise power at the output of the lattice is

$$P_Q = \sigma_e^2 \sum_{n=-\infty}^{\infty} |h(n)|^2 \approx 4.52\sigma_e^2 = 0.0031$$

(b) This lattice is stable, which follows directly from the pole locations found above. Using the Schur-Cohn Stability test (the reflection coefficients are $[-0.8667,\ 0.3333]$) is overkill here.

2. The general conditions for perfect reconstruction with delay L are

$$G_0(z)H_0(-z) + G_1(z)H_1(-z) = 0 \quad (1)$$
$$G_0(z)H_0(z) + G_1(z)H_1(z) = 2z^{-L} \quad (2)$$

Inserting the given filters in (1) gives

$$a + (b - a)z^{-1} - bz^{-2} + c + (c + d)z^{-1} + dz^{-2} = 0$$

i.e.

$$\begin{cases} a + c = 0 \\ b - a + c + d = 0 \\ d - b = 0 \end{cases}$$

which gives $a = b = -c = d$. Inserting these into (2) gives

$$2z^{-L} = a + (a + a)z^{-1} + az^{-2} - a + (a + a)z^{-1} - az^{-2} = 4az^{-1}$$

so the only solution is $L = 1$ and $a = b = -c = d = 1/2$.

3. a) The only common spectral estimate that we can derive from $|X(k)|$ is the periodogram, $\hat{P}^{\text{Periodogram}}(f) = |X(k)|^2/N$, so $\hat{P}^{\text{Periodogram}}(f) \le P_{\max}(f)$ corresponds to $|X(k)| \le \sqrt{NP_{\max}}$. The cutoff frequency $f = 0.2$ corresponds to $k = 0.2 \cdot N \approx 410$. To summarize, we should verify if

$$|X(k)| \le \begin{cases} \sqrt{2048 \cdot 1} \approx 45 & 0 \le k \le 410 \\ \sqrt{2048 \cdot 0.1} \approx 14 & 410 < k \le 1024 \end{cases}$$

This is clearly not the case in Figure 4. However, since the periodogram has a large variance, it may very well happen that the true spectrum is below the mask. Therefore, a reasonable answer is "*The figure does not provide sufficient information.*".

b) Here we have even less information. Since $|\mathrm{Re}[X(k)]| \le |X(k)|$, we cannot say much, even if $\mathrm{Re}[X(k)]^2/N$ happens to be below $P_{\max}(f)$. Again, the most reasonable answer is "*The figure does not provide sufficient information.*".

c) If we assume that the signal is Gaussian distributed, we know that the variance of the Bartlett spectral estimate is about $P^2(f)/L$, and here, the number of segments is $L = 16$, so the standard deviation is 1/4 of the true value. Since all but the highest peaks (which we hope are above and not below the true spectrum) are clearly below 80% of the mask, we can claim "*Yes, the power spectral density of the signal is below the mask, with high probability.*"

4. The MSE is given by

$$J = E\left[(s(n) - \hat{s}(n))^2\right] = E\left[(s(n) - ay(n) - by(n-1))^2\right]$$

To minimize J we should find a and b such that $\frac{\partial J}{\partial a} = 0$ and $\frac{\partial J}{\partial b} = 0$.

This yields

$$\begin{bmatrix} r_y(0) & r_y(1) \\ r_y(1) & r_y(0) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r_{sy}(0) \\ r_{sy}(1) \end{bmatrix}$$

Since $e(n)$ and $s(n)$ are uncorrelated, we have

$$\begin{aligned} P_y(z) &= H(z)H(z^{-1})P_s(z) + P_e(z) = \sigma_s^2(1 + z^{-1})(1 + z) + \sigma_e^2 \\ &= \sigma_s^2 z^{-1} + 2\sigma_s^2 + \sigma_e^2 + \sigma_s^2 z \end{aligned}$$

$$P_{sy}(z) = H(z)P_s(z) = \sigma_s^2(1 + z^{-1})$$

Thus, we have

$$\begin{bmatrix} 2\sigma_s^2 + \sigma_e^2 & \sigma_s^2 \\ \sigma_s^2 & 2\sigma_s^2 + \sigma_e^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sigma_s^2 \\ 0 \end{bmatrix}$$

Therefore $a = \frac{\sigma_s^2(2\sigma_s^2 + \sigma_e^2)}{(\sigma_s^2 + \sigma_e^2)(3\sigma_s^2 + \sigma_e^2)}$ and $b = \frac{-\sigma_s^4}{(\sigma_s^2 + \sigma_e^2)(3\sigma_s^2 + \sigma_e^2)}$.
When $\sigma_e^2 \ll \sigma_s^2$, $a = \frac{2}{3}$ and $b = \frac{-1}{3}$. They are independent of σ_s^2.

5. Since the frequency axis will be scaled by 4/3, the highest frequency that can be represented without aliasing will be $f_{\max} = 3/4$ and we will see below that it is possible to preserve all frequencies below $f_{\max} = 3/4$.

We can use the ordinary structure of rate conversion, as shown in Fig. 1, but use a filter of the form

$$H(f) = \begin{cases} C & 0 \le f < f_1 \\ 0 & \text{otherwise in } [0, 1[\end{cases}$$

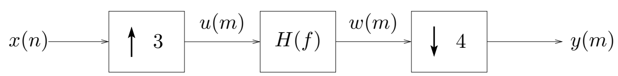

Figure 1: Implementation of rate conversion.

As can be seen in the following example, we can then use $f_1 = 1/4$, corresponding to $f_{\max} = 3/4$ and similarly to the ordinary rate conversion structure, it makes sense to use $C = 3$, corresponding to a scaling by $3/4$ of $X(f)$ (the extra factor $3/4$ that you see in the example, appears since the low pass filtering cuts the frequencies of the signal that have highest power in this specific example). Based on

$$\begin{aligned} U(f) &= X(3f) \\ W(f) &= H(f)U(f) \\ Y(f) &= \frac{1}{4}\sum_{k=0}^{3} W\left(\frac{f-k}{4}\right) \end{aligned}$$

we can easily draw Figure 2.

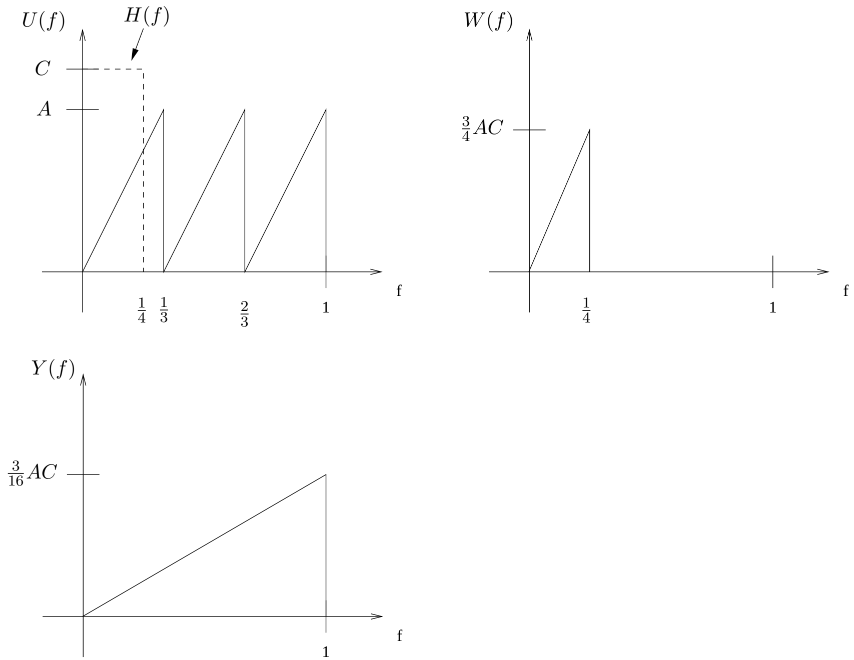

Figure 2: Fourier transforms of the signals.