SIGNAL PROCESSING

SCHOOL OF ELECTRICAL ENGINEERING

Digital Signal Processing EQ2300 / 2E1340

Final Examination 2016–03–16, 14.00–19.00 Examples of Solutions

1. a) i) $X[0]$ is always the sum of $x[n]$, i.e.,

$$X[0] = \sum_{n=0}^{N-1} x[n] \underbrace{e^{-j2\pi 0n/N}}_{e^0=1} = \sum_{n=0}^{N-1} x[n] = 22$$

ii) $X[N/2]$ is always the alternating sum of $x[n]$, i.e.,

$$X[5] = \sum_{n=0}^{N-1} x[n] \underbrace{e^{-j2\pi 5n/N}}_{e^{-j\pi n}=(-1)^n} = \sum_{n=0}^{N-1} x[n](-1)^n = -2$$

iii) This is a scaled version of the inverse DTFT at $n = 0$, i.e.,

$$\sum_{k=0}^{9} X[k] = \sum_{k=0}^{N-1} \underbrace{\sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}}_{X[k]} = \sum_{n=0}^{N-1} x[n] \underbrace{\sum_{k=0}^{N-1} e^{-j2\pi kn/N}}_{N\delta[n]} = Nx[0] = 10{\times}2 = 20$$

iv) This is a scaled version of the inverse DTFT at $n = 4$, i.e.,

$$\sum_{k=0}^{9} X[k]e^{j4\pi k/5} = \sum_{k=0}^{N-1} X[k]e^{-j2\pi k4/N} = Nx[4] = 10 \times 3 = 30$$

v) Parseval's relation imply that the DTFT is energy preserving up to a scaling constant, i.e.,

$$\sum_{k=0}^{9} \left|X[k]\right|^2 = N\frac{1}{N}\sum_{k=0}^{N-1} \left|X[k]\right|^2 \overset{\text{Parseval}}{=} N \sum_{n=0}^{N-1} \left|x[n]\right|^2 = 10 \times 80 = 800$$

b) Note that what the fixed point implementation does is to change the filter coefficients (and add quantization noise), but the change is deterministic in the sense that, say, 0.3 is always rounded off to the same number, regardless of where in the filter it appears so symmetries are preserved.

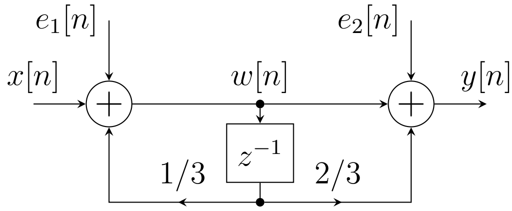
i) False. An FIR filter is always stable, no matter what the coefficients are.

ii) True. The frequency response is a function of the coefficients. Compare with $h[n] = a\delta[n]$ where $H(\nu) = a$ so $H(\nu)$ changes if a does.

iii) False. As noted above, the fixed point implementation does not alter symmetries so the filter will still be Type I and the linear phase property remains.

c) The periodogram suffers from spectral lekage and poor variance. Introducing a windows as, e.g., in the modified or windowed periodogram, can reduce spectral lekage (sidelobes), and introducing block averaging (as in Bartlett's or Welch's methods) will give a better variance of the spectrum estimate.

2. We will introduce the helper signal $w[n]$ as shown below, where we have also added the quantization noise for part a).



The impulse responce from $x[n]$ to $w[n]$ is that of a first order AR circuit, given by

$$h_1[n] = \left(\tfrac{1}{3}\right)^n u[n]$$

and the impulse responce from $w[n]$ to $y[n]$ is that of a first order MA circuit, given by

$$h_2[n] = \delta[n] + \tfrac{2}{3}\delta[n-1]\,.$$

The full impulse response from $x[n]$ to $y[n]$ is

$$h[n] = h_1[n] * h_2[n] = \left(\tfrac{1}{3}\right)^n u[n] + \tfrac{2}{3}\left(\tfrac{1}{3}\right)^{n-1} u[n-1] = \delta[n] + \left(\tfrac{1}{3}\right)^{n-1} u[n-1]\,.$$

a) Beginning under the assumption of infinite precision, we have that

$$\begin{aligned}
\sigma_y^2 =& \mathrm{E}\{y^2[n]\} \overset{(*)}{=} \sigma_x^2 \sum_{m=\infty}^{\infty} h^2[n] = \sigma_x^2 \left(1 + \sum_{m=1}^{\infty} \left(\frac{1}{3}\right)^{2(m-1)}\right) \\
=& \sigma_x^2 \left(1 + \sum_{m=0}^{\infty} \left(\frac{1}{9}\right)^{m}\right) = \sigma_x^2 \left(1 + \frac{1}{1-1/9}\right) = \frac{17}{8}\sigma_x^2 = \frac{17}{80}\,,
\end{aligned}$$

where the last step used that $\sigma_x^2 = 0.1 = 1/10$. The computation above uses the fact that $x[n]$ is white in order for $(*)$ to hold. Considering the quantization noise sources, the first source $e_1[n]$ reaches the output the same way that $x[n]$ does, so its contribution to the overall quantization noise will be

$$\frac{17}{8}\sigma_{e_1}^2 = \frac{17}{8}\frac{2^{-2B}}{12}\,.$$

The second noise source $e_2[n]$ reaches the output directly, so its contribution will just be

$$\sigma_{e_2}^2 = \frac{2^{-2B}}{12}$$

and the total power of the quantization noise at the output will thus be

$$\sigma_e^2 \triangleq \sigma_{e_1}^2 + \sigma_{e_2}^2 = \left(\frac{17}{8} + 1\right)\frac{2^{-2B}}{12} = \frac{25}{8}\frac{2^{-2B}}{12}\,.$$

The signal to noise ratio (SNR) becomes

$$\mathrm{SNR} = \frac{\sigma_y^2}{\sigma_e^2} = \frac{17/80}{25/8}\frac{12}{2^{-2B}} = \frac{204}{250}2^{2B} \geq \underbrace{10^{10/10}}_{10\,\mathrm{dB}} = 10$$

which implies that B has to be chosen such that

$$2^{2B} \geq \frac{250 \times 10}{204} \quad \Leftrightarrow \quad B \geq \frac{1}{2}\log_2 \frac{250 \times 10}{204} \approx 1.8\,.$$

As B needs to be integer values, this implies that $B = 2$ bits for the magnitude is sufficient.

b) The maximum values of the internal signals are attained immediately after the summations. In particular, we have

$$
\begin{aligned}
\left|w[n]\right| =& \left|h_1[n] * x[n]\right| = \left|\sum_{m=-\infty}^{\infty} x[n-m]h_1[m]\right| \leq \sum_{m=-\infty}^{\infty} \left|x[n-m]\right|\left|h_1[m]\right| \\
\leq& \gamma \sum_{m=-\infty}^{\infty} \left|h_1[m]\right| = \gamma \sum_{m=-\infty}^{\infty} h_1[m] = \gamma \sum_{m=0}^{\infty} \left(\frac{1}{3}\right)^m = \frac{\gamma}{1-1/3} = \frac{3}{2}\gamma
\end{aligned}
$$

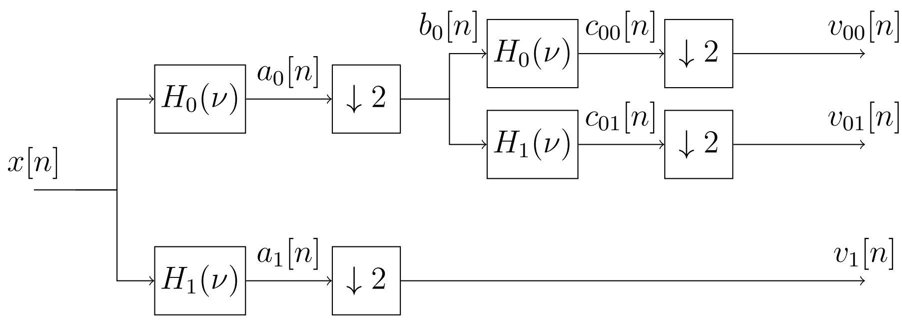
and

$$
\begin{aligned}
\left|y[n]\right| \leq& \gamma \sum_{m=-\infty}^{\infty} \left|h[m]\right| = \gamma \left(1 + \sum_{m=1}^{\infty} \left(\frac{1}{3}\right)^{m-1}\right) \\
=& \gamma \left(1 + \sum_{m=0}^{\infty} \left(\frac{1}{3}\right)^{m}\right) = \gamma \left(1 + \frac{1}{1-1/3}\right) = \frac{5}{2}\gamma\,.
\end{aligned}
$$

Note that in both cases the absolute values can be dropped due to the positivity of the impulse response, and the inequalities can be satisfied with equality by letting $x[n] = \gamma > 0$ so they are known to be tight. The most restrictive condition will be that we for no overflow must have $\left|y[n]\right| \leq 1$ which implies that

$$
\gamma \leq \frac{2}{5}\,.
$$

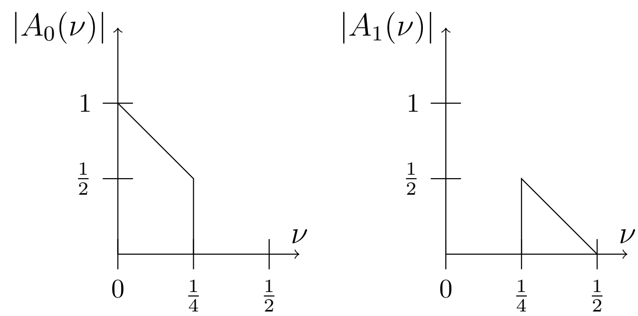
3. We introduce the following intermediate signals.

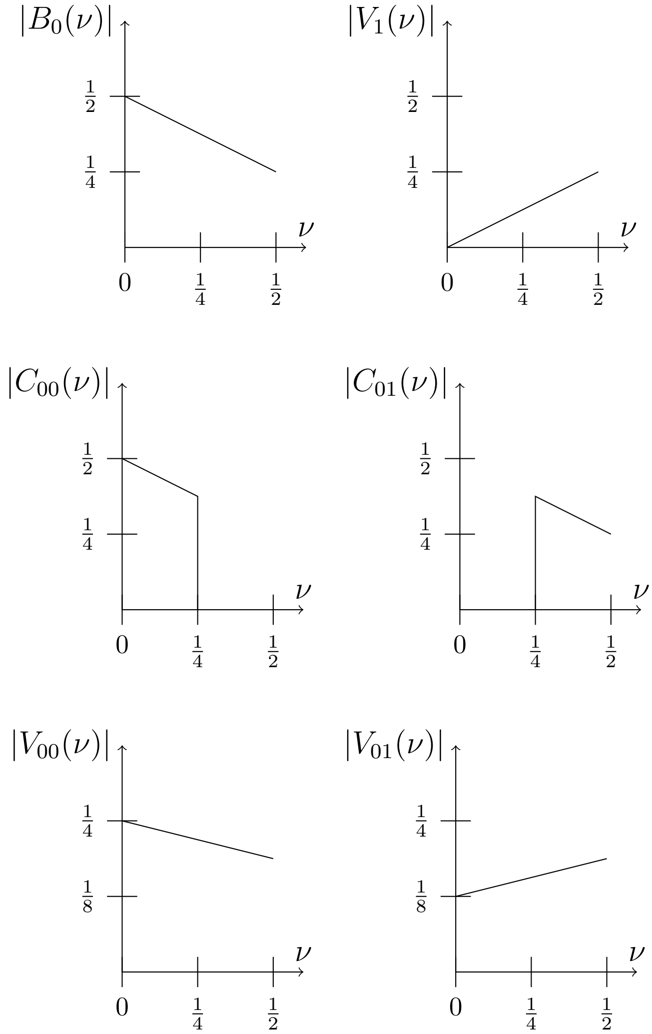


We will make repeated use of the formulas

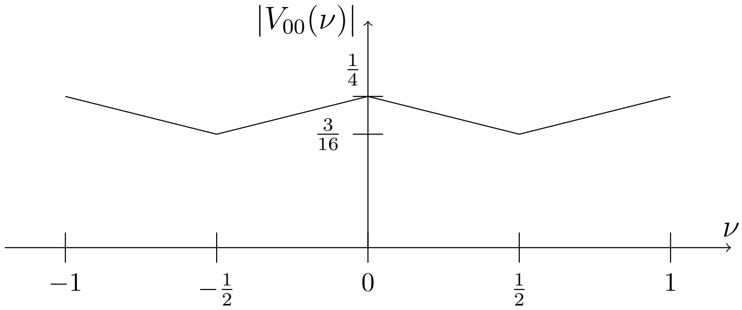
$$
Y(\nu) = H(\nu)X(\nu) \qquad \text{and} \qquad Y(\nu) = \frac{1}{2}\sum_{k=0}^{1} X\left(\frac{\nu - k}{D}\right)
$$

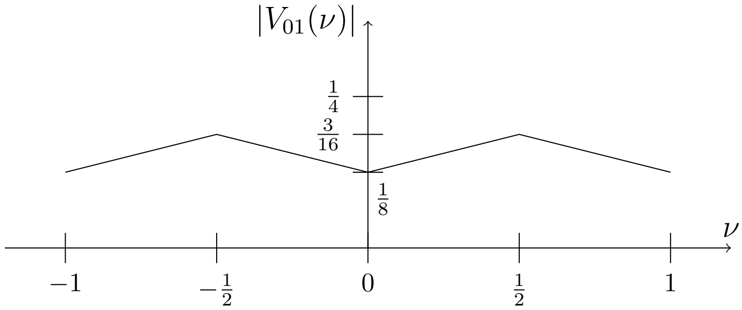
for filtering and downsampling respectively, albeit in graphically. For the range $\nu \in [0, 1/2]$, this yields

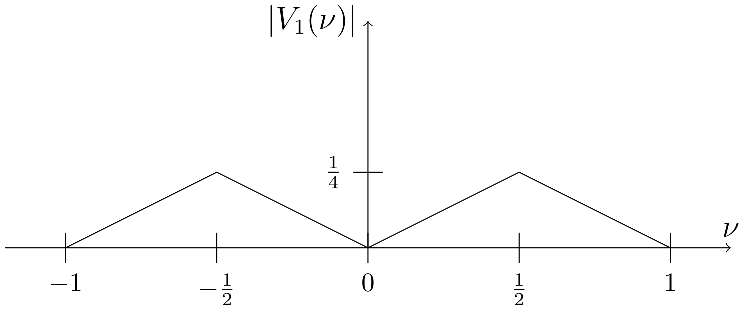




As all functions involved are real valued, their discrete Fourier transforms will be conjugate symmetric (the magnitude will be symmetric), and discrete Fourier transforms are always periodic with period 1. This allows us to extend them to the range $\nu \in [-1, 1]$ from the above, as follows.







4. a) The noise power can be read directly from the graph, where this is no sinus-signal, at -10 dB which corresponds to $\sigma^2 = 10^{-10/10} = 0.1$.

b) The value of L can be seen from the 3 dB resolution limit of the spectrum. We can see that the Peak at ν_1 can barley be told apart from the peak at ν_2, which implies that the resolution $\Delta\nu \approx 0.02$. Using the known formula for the resolution of a window of length L, when using the Bartlett window function, we get

$$\Delta\nu = \frac{1.28}{L} \approx 0.02 \quad \Leftrightarrow \quad L \approx 64$$

which in this case is the actual value used to generate the data. As we have 50% overlap it follows that $D = L/2 \approx 32$.

c) Using the formula for the bias of the estimator, we have that

$$\mathrm{E}\{\hat{P}_x^{\mathrm{W}}(\nu)\} = P_x(\nu) \circledast \frac{1}{LU}|W^{(L)}(\nu)|^2 \quad \text{where} \quad W^{(L)}(\nu) = \mathcal{F}\{w[n]\}\,.$$

The true spectrum is in this case given by

$$P_x(\nu) = \sum_{k=1}^{3} \frac{a^2}{4}\delta(\nu - \nu_k) + \sigma_x^2\,.$$

As the noise level is significantly below the signal level at the peaks in the spectra, this means that we expect the peaks to reach a level of

$$\frac{a^2}{4}\frac{1}{LU}|W^{(L)}(0)|^2\,,$$

at frequencies ν_1, ν_2 and ν_3. The value of $W^{(L)}(0)$ can (for even L) be obtained as

$$W^{(L)}(0) = \sum_{n=-\infty}^{\infty} w[n] = 2 \times \frac{2}{L-1} \sum_{n=0}^{(L-1)/2} n = \frac{L-1}{2}\,.$$

We also have that

$$LU = \sum_{n=0}^{L-1} w^2[n] = 2\frac{4}{(L-1)^2} \sum_{n=0}^{(L-1)/2} n^2 = \frac{L(L+1)}{3(L-1)}$$

so

$$\frac{1}{LU}|W^{(L)}(0)|^2 = \frac{3(L-1)^3}{4L(L+1)} \approx \frac{3L}{4}$$

where the last approximation holds for reasonably large L. Reading the peaks at around 11 dB, we obtain

$$\frac{a^2}{4}\frac{3L}{4} \approx 10^{11/10} = 10^{1.1}$$

which implies

$$a \approx \sqrt{\frac{16 \times 10^{1.1}}{3L}} \approx 1.024$$

for $L = 64$, which is not all that far away from the actual value of $a = 1$ that was used to generate the plot.

5. a) Note that the expression

$$\frac{\sin(\pi\Delta n)}{\pi\Delta n}\frac{\sin(2\pi\nu_c n)}{\pi n}$$

is the product of two sinc-terms, one of which is the impulse response of the ideal low-pass filter. This means that the desired frequency response $H_D(\nu)$ should be possible to express as the convolution of two terms, one of which is the ideal frequency response. In particular, we have

$$\frac{\sin(\pi\Delta n)}{\pi\Delta n} = \frac{1}{\Delta}\frac{\sin(\pi\Delta n)}{\pi n} \quad \overset{\mathcal{F}}{\leftrightarrow} \quad \frac{1}{\Delta}\mathrm{rect}_\Delta(\nu)$$

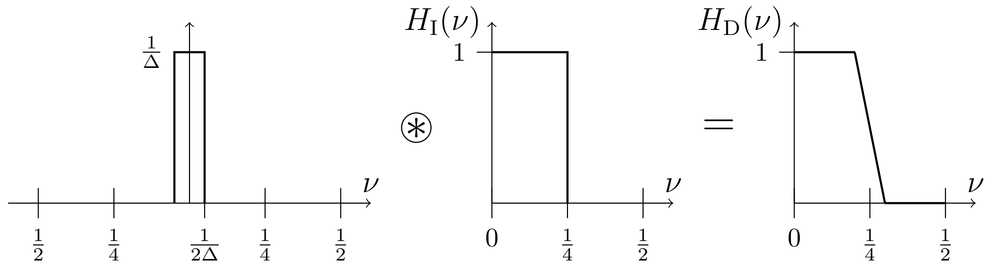
and

$$\frac{\sin(2\pi\nu_c n)}{\pi n} \quad \overset{\mathcal{F}}{\leftrightarrow} \quad \mathrm{rect}_{2\nu_c}(\nu) = H_I(\nu)\,,$$

so it should follow that

$$H_D(\nu) = \frac{1}{\Delta}\mathrm{rect}_\Delta(\nu) \circledast \mathrm{rect}_{2\nu_c}(\nu)\,.$$

This is however straightforwardly shown graphically, as illustrated below.



One interesting note that that the above argument actually shows that smoothing the frequency response as done in this problem is actually equivalent to windowing in the time domain. The only difference is really that the "window" in this case is not of finite length, so some explicit truncation is still needed.

b) Let

$$E_x^M(\nu) = H_x^M(\nu) - H_x(\nu)$$

where x is either I or D. By Parseval's relation we have that

$$E_x^M = \int_{-1/2}^{1/2} |E_x^M(\nu)|^2 d\nu = \sum_{n=-\infty}^{\infty} \left|e_x^M[n]\right|^2$$

where $e_x^M[n] = h_x^M[n] - h_x[n]$. As $h_x^M[n]$ is a truncated version of $h_x[n]$, where $h_x^M[n] = h_x[n]$ whenever $|n| \leq M$ and $h_x^M[n] = 0$ otherwise, it follows that $e_x^M[n] = 0$ when $|n| \leq M$ and $e_x^M[n] = h_x[n]$ otherwise. Thus,

$$E_x^M = 2\sum_{n>M} \left|h_x[n]\right|^2$$

where we have also used the fact that the impulse responses are symmetric. However, as

$$\left|\frac{\sin(\pi\Delta n)}{\pi\Delta n}\right| < 1$$

for all $n > 0$, it immediately follows that $\left|h_{\mathrm{D}}[n]\right|^2 < \left|h_{\mathrm{I}}[n]\right|^2$ for all $n > 0$, and that $E_{\mathrm{D}}^M < E_{\mathrm{I}}^M$. QED