SIGNAL PROCESSING

SCHOOL OF ELECTRICAL ENGINEERING

Digital Signal Processing EQ2300 / 2E1340

Final Examination 2014–01–15, 08.00–13.00 Examples of Solutions

1. a) The output $d[n]$ of the filter $h[n]$ will be a sum of two components, one $d_1[n]$ due to $x_1[n] = \cos(2\pi\nu_1 n)$ and one $d_2[n]$ due to $x_2[n] = \cos(2\pi\nu_2 n)$. For the first component, we have

$$d_1[n] = \sum_{k=-\infty}^{+\infty} h[k]x_1[n-k] = \frac{1}{2}\left[\sum_{k=-\infty}^{+\infty} h[k]e^{-j2\pi\nu_1(n-k)} + \sum_{k=-\infty}^{+\infty} h[k]e^{j2\pi\nu_1(n-k)}\right],$$

and factoring out the constant terms in the summations yields

$$d_1(n) = \frac{1}{2}e^{-j2\pi\nu_1 n}H(\nu)\,|_{\nu=-\nu_1} + \frac{1}{2}e^{j2\pi\nu_1 n}H(\nu)\,|_{\nu=\nu_1},$$

where $H(\nu) := 1 - \sqrt{2}e^{-j2\pi\nu} + e^{-j4\pi\nu}$ is the frequency response of the system $h[n]$. Plugging $\nu_1 = 1/16$ and similarly $-\nu_1$ in $H(\nu)$ yields $H(\nu_1) \approx 0.4 - j0.166$ and $H(-\nu_1) \approx 0.4 + j0.166$. Now following the same steps for the second component, we get that

$$d_2[n] = \frac{1}{2}e^{-j2\pi\nu_2 n}H(\nu_2) + \frac{1}{2}e^{j2\pi\nu_2 n}H(-\nu_2) = 0,$$

since $H(\nu_2) = H(-\nu_2) = 0$. In other words, the filter has a spectral null at $\nu_2 = 1/8$, thereby completely attenuating $x_2[n]$.

b) Downsampling will expand each periodic repetition of the spectrum of $d[n]$ along the f-axis by a factor of **2** so that the magnitude of the spectrum of $z(n)$ becomes

$$|Z(\nu)| = \frac{1}{4}\left|H\left(\frac{1}{16}\right)\right|\left[\delta(\nu - \frac{1}{8}) + \delta(\nu + \frac{1}{8})\right] \text{ in } \nu \in \left[-\frac{1}{2}, \frac{1}{2}\right)$$

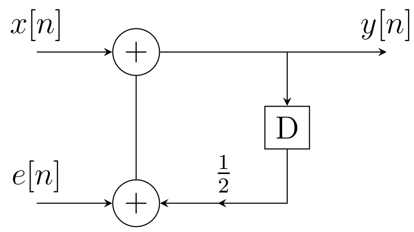
(here $\delta(\cdot)$ denotes the impulse function). Upsampling will compress $Z(\nu)$ by a factor of **2** along the f-axis, introducing two frequency components in the band $\left[0, \frac{1}{2}\right)$ at $\{\frac{1}{16}, \frac{7}{16}\}$ and similarly $\{-\frac{1}{16}, -\frac{7}{16}\}$ in the band $\left[-\frac{1}{2}, 0\right)$. Specifically, the magnitude of the spectrum $Y(\nu)$ will be

$$|Y(\nu)| = \frac{1}{4}\left|H\left(\frac{1}{16}\right)\right|\left[\delta(\nu \pm \frac{1}{16}) + \delta(\nu \pm \frac{7}{16})\right].$$

2. The autocorrelation function of the harmonic process (we know this from the material we covered in class) is given by $r_x[k] = A^2/2\cos(2\pi\nu_0 k)$, therefore the power spectral density (PSD) is given by

$$P_x(\nu) = \frac{A^2}{4}\Big[\delta(\nu-\nu_0)+\delta(\nu+\nu_0)\Big].$$

To calculate the SQNR, we use the well-known additive noise model after the multiplications which happen in the filter, i.e.,



where $e[n]$ is i.i.d. noise uniformly distributed in $\left[-\frac{\Delta}{2},+\frac{\Delta}{2}\right]$, where $\Delta = 2^{-b}$ (zero-mean with variance $\sigma_e^2 = \Delta^2/12$). The system in the figure is an AR(1) system and has frequency response

$$H(\nu) = \frac{1}{1-0.5e^{-j2\pi\nu}}.$$

First we calculate the PSD of the output due to the signal $x[n]$ only,

$$P_{y_x}(\nu) = P_x(\nu)|H(\nu)|^2 = \frac{A^2}{4}\left|H(\frac{1}{3})\right|^2\left[\delta(\nu-\frac{1}{3})+\delta(\nu+\frac{1}{3})\right]$$

which leads to an output signal power (one can also see this straight away from the sinus-in sinus-out rule)

$$\sigma_{y_x}^2 = \int_{-1/2}^{1/2} P_{y_x}(\nu)d\nu = \frac{A^2}{2}\left|H(\frac{1}{3})\right|^2.$$

Next we calculate the power of the noise at the output, which is

$$\sigma_N^2 = \sigma_e^2\sum_{k=-\infty}^{+\infty} h^2[n],$$

and since $h[n] = 0.5^n u[n]$, we get that

$$\sigma_N^2 = \sigma_e^2\sum_{n=0}^{\infty}(\frac{1}{4})^n = \frac{4}{3}\sigma_e^2 = \frac{2^{-2b}}{9}.$$

This yields an SQNR of

$$\text{SQNR} = \frac{\sigma_{y_x}^2}{\sigma_N^2} = \frac{9A^2|H(\frac{1}{3})|^2}{2^{-2b+1}}.$$

3. a) One can prove this in a straightforward way by noting that

$$\hat{P}_{\mathrm{BT}} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{k=-M}^{M} w[k]\hat{r}_x[k]e^{-j2\pi k} d\nu = \sum_{k=-M}^{M} w[k]\hat{r}_x[k] \underbrace{\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi k} d\nu}_{\delta[k]} = w[0]r_x[0]\,.$$

Since $\hat{P}_{\mathrm{ML}} = r_x[0]$, it must hold that $w[0] = 1$ if $\hat{P}_{\mathrm{ML}} = \hat{P}_{\mathrm{BT}}$.

b) The bias formula for Blackman-Tukey states that

$$\mathrm{E}\{\hat{P}_{\mathrm{BT}}(\nu)\} \approx P_x(\nu) * W(\nu)$$

where $W(\nu)$ if the DTFT of the window used. Since the main lobe with of $W(\nu)$ is inversely proportional to the length f $w[n]$, the resolution is increased if we increase M, i.e., we are more likely to be able to resolve two nearby frequencies.

c) Increasing M makes the Blackman-Tukey estimate more like the Periodogram, and the variance increases with M.

4. a) We can implement a non-causal filter using a time-delayed causal counterpart, use the regular overlap save, and then shift the output sequence. Doing this, we get that

$$Z[k] = \mathcal{F}_N\{h[n-M]\} = \sum_{n=0}^{N-1} h[n-M]e^{-j2\pi nk/N}$$

where $N \geq 2M+1$, i.e., the block length of the DFT must be longer than the filters length. Compensating for the time shift gives $n_{\mathrm{shift}} = M$. As the length of the filter is $2M+1$, the first $2M$ samples of $y_{\mathrm{B}}[n]$ must be discarded, giving $n_1 = 2M$, and the bock overlap $n_{\mathrm{add}} = N - 2M$.

b) The complexity derivation follows the standard derivation but with $2M+1$ for the filters length, yielding

$$C = \frac{N\log_2 2N}{N-2M}\,.$$

5. a) The frequency response of the high-pass filter designed using the window method is given by

$$H(\nu) = H_{\text{ideal}}(\nu) * W(\nu)$$

where $W(\nu)$ is the DTFT of the window used. One has to ensure that the side lobe level of the window is sufficiently low, and that the drop from pass-band to stop-band is quick enough (determined by the width of the main lobe). The exact impulse response of the HP will be the mirror (around $\nu_c = 0.25$) of the LP responces shown in the figure, and one can see that the side lobes of the rectangular window are to high, and that the Chebychev window does not drop fast enough. The point $\nu_s = 0.2$ corresponds to the point $\nu = 0.3$ in the figure, and the Frequency response has there dropped only to about -30 dB, even though the slid lobe level eventually gives a suppression of over -100 dB. Consequently, only the *Hamming window* will satisfy the requirements.

b) The impulse response of a filter designed with the window method is

$$h[n] = w[n]h_{\text{ideal}}[n]\,.$$

In order to obtain a causal linear phase FIR HP filter we need to introduce a group delay of $\tau = M/2$, where $M = N - 1 = 40$ is the filter order, according to

$$H_{\text{ideal}}(\nu) = \begin{cases} 0 & 0 \le \nu \le \nu_c \\ e^{-j2\pi\tau\nu} & \nu_c < \nu \le 0.5 \end{cases}$$

where $\tau = 20$ and $\nu_c = 0.25$. Taking the inverse DTFT yields

$$h[n] = \delta[n-\tau] - \frac{\sin(2\pi\nu_c(n-\tau))}{\pi(n-\tau)} = \delta[n-20] - \frac{\sin(\pi(n-\tau)/2)}{\pi(n-20)}\,,$$

as one can also quickly see by noting that $H_{\text{HP}}(\nu) = 1 - H_{\text{LP}}(\nu)$. We obtain

$$h[n] = w[n]\left[\delta[n-20] - \frac{\sin(\pi(n-\tau)/2)}{\pi(n-20)}\right]$$

where $w[n]$ is the chosen window in the time domain.

c) Only the Chebychev window has good enough side lobe suppression, but it does not drop fast enough. We can fix this by increasing N. The steepness increases with N, while the side lobe level is largely unchanged, so using the Chebychev window with a larger N will work.