SOLUTIONS

E 101 **Digital Signalbehandling**, 2E1340

Final Examination 2007–06–04, 14.00–19.00

1. a) Downsampling by 2 gives $Y(f) = \frac{1}{2}(X(f/2) + X((f-1)/2)$, which together with Fig. 1 shows that $Y_1(f)$ is the correct result.

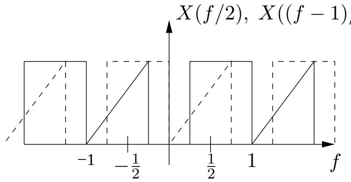

Figure 1: $X(2f)$ (solid lines) and $X((f-1)/2)$ (dashed lines).

b) The desired resolution in normalized frequency should be about 100Hz/48kHz=1/480. The different methods give roughly a resolution of $1/M$, where M is the length of each segment of data.

Method:	M	Approximative resolution $1/M$
i)	16384	1/16384
ii)	16384/128=128	1/128
iii)	16384/16=1024	1/1024

Clearly, alternative ii) gives too low resolution, whereas alternatives i) and iii) give sufficient resolution. However, the variance is 16 times lower in alternative iii) than for the periodogram, so iii) is the preferable solution.

c) The normalized frequency of the sampled signal will be $f = 440\text{Hz}/8\text{kHz} = 0.055$. The DFT of length 4096, gives 4096 values in the frequency domain, so the highest peak should occur at $4096f \approx 225$, i.e. alternative ii).

2. a) Model the quantization noise as additive white noise with power $\sigma_e^2 = \frac{2^{-30}}{12}$, which is added after each multiplication. The noise resulting from the multiplication by 1/3 goes through the filter $H_1(z) = 1 + \frac{1}{5}z^{-1}$ whereas the noise from the multiplication by 1/5 appears directly at the output. Therefore, the power spectral density of the total quantization noise at the output is

$$P_{q_yq_y}(z) = \sigma_e^2 H_1(z)H_1(z^{-1}) + \sigma_e^2 = \sigma_e^2(2.04 + 0.2z^{-1} + 0.2z)$$

corresponding to the autocorrelation sequence

$$r_{q_yq_y}(k) = \begin{cases} 2.04\sigma_e^2, & k = 0 \\ 0.2\sigma_e^2, & k = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

This gives $\mathrm{E}[q_y^2(n)] = r_{q_yq_y}(0) = 2.04\sigma_e^2 \approx 1.58 \cdot 10^{-10}$.

b) Linear interpolation gives

$$\begin{cases} w(2n) = y(n) \\ w(2n-1) = \frac{y(n)+y(n-1)}{2} \end{cases}$$

so that

$$\begin{cases} \mathrm{E}[q_w^2(2n)] = \mathrm{E}[q_y^2(n)] \approx 1.58 \cdot 10^{-10} \\ \mathrm{E}[q_w^2(2n-1)] = \mathrm{E}[(\frac{q_y(n)+q_y(n-1))^2}{4} + \sigma_e^2 = \frac{r_{q_yq_y}(0)+r_{q_yq_y}(1)}{2} + \sigma_e^2 \approx 1.65 \cdot 10^{-10} \end{cases}$$

The extra term σ_e^2 for odd numbered samples, corresponds to the quantization from the division by 2 in the interpolation. (Since a division by 2 corresponds to a bit shift, it is possible to find a somewhat more accurate expression for the quantization error, but the expression given above is a reasonable approximation).

3. The indirect form of the periodogram is given by

$$\hat{P}(f) = \sum_{n=-N+1}^{N-1} \hat{r}(n)e^{-j2\pi fn}$$

so if we want to use an FFT of length $M = 2N-1$, we should use

$$\begin{aligned} \hat{P}(k) = \hat{P}(f)\rfloor_{f=\frac{k}{2N-1}} &= \sum_{n=-N+1}^{N-1} \hat{r}(n)e^{-j2\pi kn/M} \\ &= \sum_{n=0}^{N-1} \hat{r}(n)e^{-j2\pi kn/M} + \sum_{n=N}^{M-1} \hat{r}(n-M)e^{-j2\pi kn/M} \end{aligned}$$

i.e. the proper periodogram is given by

$$\hat{P}(k) = \texttt{FFT}([\underset{\uparrow}{\hat{r}(0)}, \hat{r}(1)\ldots, \hat{r}(N-1), \hat{r}(-N+1), \ldots, \hat{r}(-1)])$$

a) What the student calculates is the DFT of a circularly rotated version of the autocorrelation data,

$$\hat{P}_1(k) = \sum_{n=0}^{M-1} \hat{r}((n-N+1) \mod M)e^{-j2\pi kn/M} = e^{-j2\pi k(N-1)/M}\hat{P}(k)$$

Since we know that the periodogram $\hat{P}(k)$ is always real valued and non-negative, is easy to recover the periodogram, using $\hat{P}(k) = |\hat{P}_1(k)|$.

b) Since $\hat{P}(k)$ is real valued, $\mathrm{Re}[\hat{P}_1(k)] = \cos(2\pi k\frac{N-1}{M})\hat{P}(k)$, we can recover the periodogram using $\hat{P}(k) = \mathrm{Re}[\hat{P}_1(k)]/\cos(2\pi k\frac{N-1}{M})$. Note that the denominator is non-zero since $M = 2N-1$ is odd.

c) Using the symmetry of $\hat{r}(k)$,

$$2\,\mathrm{Re}[\hat{P}_2(k)] - \hat{r}(0) = \sum_{n=-N+1}^{N-1} \hat{r}(n)e^{-j2\pi kn/N} = \hat{P}(f)\rfloor_{f=\frac{k}{N}}$$

so at least if we also have access to $\hat{r}(0)$, then it is possible to calculate the periodogram from `phat2`, but only in N frequency points.

d) As can be seen in the previous answer, it is easy to recover the periodogram if you have access to $\hat{r}(0)$. However, if $\hat{r}(0)$ is unavailable, then there is an unknown additive constant which makes it impossible to determine an absolute power level, but the plot will still be useful if you want to find the dominating frequencies or only get the relative power difference between different frequency bands.

4. a)

$$X(\omega) = \frac{1}{2}\delta(\omega - \omega_0)e^{j\phi} + \frac{1}{2}\delta(\omega + \omega_0)e^{-j\phi}$$

for $-\pi \leq \omega \leq \pi$, which gives (since $\delta(a\omega) = \delta(\omega)/a$)

$$\begin{aligned} W(\omega) = X(2\omega) =& \frac{1}{4}\delta\Big(\omega - \frac{\omega_0}{2}\Big)e^{j\phi} + \frac{1}{4}\delta\Big(\omega + \frac{\omega_0}{2}\Big)e^{-j\phi} \\ &+ \frac{1}{4}\delta\Big(\omega - \pi + \frac{\omega_0}{2}\Big)e^{-j\phi} + \frac{1}{4}\delta\Big(\omega + \pi - \frac{\omega_0}{2}\Big)e^{j\phi} \end{aligned}$$

or in the time domain

$$w(m) = \frac{1}{2}\cos\left(\frac{\omega_0}{2}m + \phi\right) + \frac{1}{2}\cos\left(\Big(\pi - \frac{\omega_0}{2}\Big)m - \phi\right)$$

so the frequencies are $\omega_0/2$ and $\pi - \omega_0/2$.

b) As is shown in the course literature, the ideal interpolation filter is given by

$$H(e^{j\omega}) = \begin{cases} 2 & \text{for } |\omega| < \pi/2 \\ 0 & \text{otherwise} \end{cases}$$

c) The filter for linear interpolation described in the compendium is non-causal. Delay it by one step to make it causal, which gives

$$H(z) = \frac{1}{2} + z^{-1} + \frac{1}{2}z^{-2}$$

d)

$$H(z) = \beta + \alpha z^{-1} + \beta z^{-2} = z^{-1}(\beta z + \alpha + \beta z^{-1})$$

gives

$$H(e^{j\omega}) = e^{-j\omega}(\alpha + 2\beta\cos(\omega)$$

The factor $e^{-j\omega}$ corresponds to a delay $L = 1$. To cancel the alias term at $\pi - \omega_0/2$ and get the desired amplitude at $\omega_0/2$, we need

$$\begin{cases} 2 = |H(e^{j\frac{\omega_0}{2}})| = \alpha + 2\beta\cos\left(\frac{\omega_0}{2}\right) \\ 0 = |H(e^{j\pi - j\frac{\omega_0}{2}})| = \alpha + 2\beta\cos\left(\pi - \frac{\omega_0}{2}\right) \end{cases}$$

which is solved by $\alpha = 1$ and $\beta = \frac{1}{2\cos\left(\frac{\omega_0}{2}\right)}$.

5. Let $\mathbf{w} = [w(0)\ \cdots\ w(M-1)]^T$, $\mathbf{y} = [y(0)\ \cdots\ y(M-1)]^T$, $\mathbf{x} = [x(0)\ \cdots\ x(M-1)]^T$ and $\mathbf{z} = [z(0)\ \cdots\ z(M-1)]^T$.

$$\begin{aligned} J = E\left[|h - \hat{h}|^2\right] =& E\left[|h - \mathbf{w}^T\mathbf{y}|^2\right] = E\left[|(1 - \mathbf{w}^T\mathbf{x})h - \mathbf{w}^T\mathbf{z}|^2\right] \\ =& (1 - \mathbf{w}^T\mathbf{x})(1 - \mathbf{x}^H\mathbf{w}^c)\sigma^2 + \mathbf{w}^T\mathbf{w}^c \\ =& \mathbf{w}^T(\sigma^2\mathbf{x}\mathbf{x}^H + \mathbf{I})\mathbf{w}^c - \mathbf{w}^T\mathbf{x}\sigma^2 - \mathbf{x}^H\mathbf{w}^c\sigma^2 + \sigma^2 \end{aligned}$$

By completing squares, we can rewrite J as

$$J = (\mathbf{w}^c - \hat{\mathbf{w}}^c)^H(\sigma^2\mathbf{x}\mathbf{x}^H + \mathbf{I})(\mathbf{w}^c - \hat{\mathbf{w}}^c) + \xi$$

where

$$\begin{aligned} \hat{\mathbf{w}} &= \left((\sigma^2\mathbf{x}\mathbf{x}^H + \mathbf{I})^{-1}\sigma^2\mathbf{x}\right)^c \\ \xi &= \sigma^2\left(1 - \mathbf{x}^H(\sigma^2\mathbf{x}\mathbf{x}^H + \mathbf{I})^{-1}\sigma^2\mathbf{x}\right) \end{aligned}$$

Since $\sigma^2\mathbf{x}\mathbf{x}^2 + \mathbf{I}$ is positive definite, the MSE is minimized by $\mathbf{w}_{opt} = \hat{\mathbf{w}}$ and the corresponding MSE is ξ.