SOLUTIONS

E 96 **Digital Signalbehandling**, 2E1340

Final Examination 2005–03–31, 08.00–13.00

1. The parameters give $\beta = 40 \cdot 10^6\text{Hz}/10 \cdot 10^{-3}\text{s} = 4 \cdot 10^9[\text{s}^{-2}]$.

a) The number of samples during one sweep of the chirp signal is $N = (T - \Delta_{\max})F_s = 19600$. Since no zero-padding is used, the number of frequency samples will also be N, so the step between two DFT values will be $1/N$ in normalized frequency corresponding to $F_s/N = 102.04$Hz. Since $\hat{d} = c/(2\beta)\hat{\nu}$, this corresponds to a distance of $c/(2\beta) \cdot 102.04 \approx 3.83$m. So, the distance can be determined with an accuracy of $\pm 3.83/2 = \pm 1.91$m if there is no noise in the measurements. Use zero-padding to get even better accuracy.

b) The resolution of a periodogram is about $0.9/N$. Similar calculations as in a) show that this corresponds to a resolution in distance that is about 3.4m, so it should not be any problems to resolve two targets that are separated by 10m.

c) Within each period of the chirp signal, i.e. for each set of $N = 19600$ samples in this case, we have a normal frequency estimation problem. As long as there is only a single target, there is only a single frequency to look for and the periodogram is equivalent to maximum likelihood which is the optimal solution. If there are several targets, a model based method can provide better resolution and lower variance than the periodogram.

Another method to reduce the variance, in the case of noisy data, is to use measurements from several periods of the chirp signal. This, of course, will only work if the target does not move too quickly. The easiest method is to average the periodograms obtained over several measurement periods. Since there is a gap between each set of samples, it is difficult to use several periods of data to increase the resolution.

2. a) Let $L = 8000$ denote the number of signal samples and $M = 20$ be the length of the filter. A direct implementation will then require M multiplications for each output value, i.e. $M(L + M - 1) = 160\,380$ multiplications in total. As an alternative we consider overlap-add or overlap-save implementations using an FFT length of N. Then, the calculation will be divided into segments, where each segment results in $N - M + 1$ output values and requires $2N/4 \log_2(N) + N$ complex valued multiplications (only $N/4 \log_2(N)$ multiplications are needed for each of the FFTs and IFFTs if we exploit that the input of the FFT and the output of the IFFT are real valued). The total number of segments needed is $\lceil (L + M - 1)/(N - M + 1) \rceil$ (where $\lceil a \rceil$ denotes a rounded up to the next integer). This means that a total of

$$C(N) = \lceil (L + M - 1)/(N - M + 1) \rceil (2N \log_2(N) + 4N)$$

real valued multiplications are needed. Trying different values of $N = 2^p$ shows that the lowest complexity $C(N)$ is $C(128) = 170\,496$ which is higher than the direct implementation.

To conclude, since the filter is so short, the direct implementation using $160\,380$ multiplications is the best here. However, with a longer filter, an FFT based solution would have been better.

b) Sampling of the signal $x(t)$ yields the digital signal $x(n)$ for $n = 1, 2, ..., N$. As we know, the periodogram is proportional to the squared magnitude of the DFT of the windowed signal for large N. The number of real valued multiplications needed for calculating the FFT is

$$C_{FFT}(N) = 4\frac{N \log_2(N)}{4}$$

(again exploiting that the input signal is real valued). Calculating the squared magnitude of the FFT values requires an extra $2N$ real valued multiplications, hence the total number of multiplications needed for the periodogram is

$$C_{\text{periodogram}}(N) = N \log_2(N) + 2N$$

In particular, $C_{periodogram}(8192) = 122\,880$.

c) To be able to use the FFT from the filtering in the calculation of the Periodogram, the length of the FFT needs to be the same as the length of the periodogram. If the lengths are not equal you wont be able to combine the information from the segments due to the zero-padding used to get linear convolution. If the lengths are equal, you can multiply the FFT of the signal with the FFT of the filter and then plot the squared magnitude. (using a fixed filter the complexity of calculating the FFT of the filter can be neglected). The resulting complexity is that of the periodogram plus N complex valued multiplications, i.e.

$$C_{\text{periodogram+filter}}(N) = C_{\text{periodogram}}(N) + 4N = N \log_2(N) + 6N$$

With 8000 time samples and a filter length of 20, an FFT length of at least 8019 is needed, so 8192 is a good choice which gives $C_{\text{periodogram+filter}}(8192) = 155\,648$. Of course, this is a much better implementation of the full system than just taking the solutions from a) and b) and putting them together.

3. a) There is no risk of overflow since we can represent all values in the range $[-1, 1]$ and the maximum output from the filter is given by $2 \cdot 0.11 + 2 \cdot 0.18 + 0.21 = 0.79$. Round-off errors are modeled as uniformly distributed white noise with variance $\sigma_e^2 = \frac{\Delta^2}{12}$, where $\Delta = 2^{-(b-1)}$ is the step size of the quantizer. The A/D-converter acts as a noise source with variance $\sigma_{12}^2 = \frac{2^{-22}}{12}$ at the input. This noise will pass through the entire filter and give the contribution $\sigma_{12}^2 \sum_{n=0}^{4} h^2(n)$ to the noise power at the output. Each filter tap gives additional round-off noise with variance $\sigma_{16}^2 = \frac{2^{-30}}{12}$ directly at the output.

The total noise power at the output is then given by

$$\sigma_\epsilon^2 = \sigma_{12}^2 \sum_{n=0}^{4} h^2(n) + 5\sigma_{16}^2 = 0.1331\frac{2^{-22}}{12} + 5\frac{2^{-30}}{12} \approx 3 \cdot 10^{-9}.$$

b) The expression for the variance of the periodogram that is given in the text books is derived under the assumption that the input has a Gaussian distribution. In our case, however, all noise sources have uniform distributions! Since the Gaussian assumption does not hold, we can not expect to be able to predict the exact variance of the periodogram. However, the formula can still provide a good estimate.

4. a) Use the Noble identities to obtain

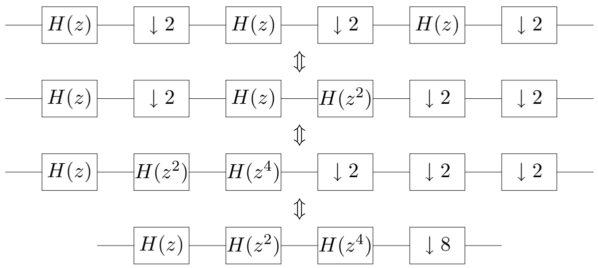

Figure 1: Using the Noble identities

$$G(z) = H(z^4)H(z^2)H(z) = (1 + 2z^{-4} + z^{-8})(1 + 2z^{-2} + z^{-4})(1 + 2z^{-1} + z^{-2}),$$

which gives

$$g(n) = \{\underset{\uparrow}{1}, 2, 3, 4, 5, 6, 7, 8, 7, 6, 5, 4, 3, 2, 1\}$$

and D=8.

b) If we assume that the pass-band of the first two stages is flat, we can get a rough estimate of the bandwidth of the filter by looking at the bandwidth of the final stage, i.e. the bandwidth of $H(z)$ divided by four.

Define the bandwidth as the frequency where $\frac{|H(f)|^2}{|H(0)|^2} = \frac{1}{2}$. This corresponds to the 3-dB bandwidth of the filter.

$$H(f) = 1 + 2e^{-j2\pi f} + e^{-j4\pi f}$$

$$\begin{aligned}|H(f)|^2 &= (1 + 2e^{-j2\pi f} + e^{-j4\pi f})(1 + 2e^{j2\pi f} + e^{j4\pi f}) \\ &= e^{-j2\pi f}(e^{j2\pi f} + 2 + e^{-j2\pi f})e^{j2\pi f}(e^{-j2\pi f} + 2 + e^{j2\pi f}) \\ &= (2 + 2\cos(2\pi f))^2\end{aligned}$$

$|H(0)|^2 = 16 \quad \Rightarrow$

$$\begin{aligned}(2 + 2\cos(2\pi f))^2 &= 8 \\ 2 + 2\cos(2\pi f) &= \pm\sqrt{8} = \pm 2\sqrt{2} \\ 1 + \cos(2\pi f) &= \sqrt{2} \\ \cos(2\pi f) &= \sqrt{2} - 1 \\ 2\pi f &= \arccos(\sqrt{2} - 1) \\ f &\approx 0.182\end{aligned}$$

A rough estimate of the bandwidth of $G(z)$ is then $f/4 \approx 0.0455$.

5. a) Consider the setup as follows: Each of the M radars will provide an estimate, $\hat{d}_i$ of the distance to the object.If the target is located at $\mathbf{e}_0 = [x_0, y_0]^T$, the actual distance between the target and RADAR i is $d_i = \|\mathbf{e}_0 - \mathbf{e}_i\|$.

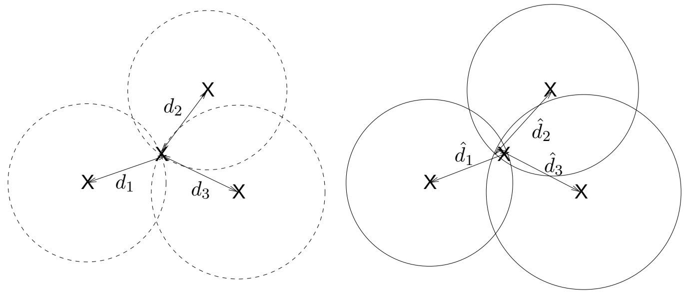

Figure 2: Exact localization using three RADARs (left), same situation with measurement errors (right).

One way to solve the problem is to use non-linear Least Squares (LS). The system of equations we want to solve is

$$\begin{cases}\|\mathbf{e}_0 - \mathbf{e}_1\| \approx \hat{d}_1 \\ \|\mathbf{e}_0 - \mathbf{e}_2\| \approx \hat{d}_2 \\ \vdots \quad \vdots \quad \vdots \\ \|\mathbf{e}_0 - \mathbf{e}_M\| \approx \hat{d}_M\end{cases}$$

The LS solution is given by the solution of

$$\min_{\mathbf{e}_0} \sum_{i=1}^{M} (\|\mathbf{e}_0 - \mathbf{e}_i\| - \hat{d}_i)^2$$

b) Geometrically, each $\hat{d}_i$ defines one circle. As long as there are no measurement errors, as in the left part of Fig. 2, all the circles will intersect exactly at the target. However, with measurement errors, as in the right part of the figure, the circles will in general not intersect in a single point and the LS solution finds the best approximation.

If there are only 2 RADARs, the two circles will intersect in exactly two points, where one of the points is an approximation of the location. In order to get a unique solution (at least without measurement errors), we need at least three RADARs. When there are estimation errors, the accuracy of the position obtained from the LS method will improve when we add more RADARs.

Extra possible solution: In most RADAR applications the radiation is not omnidirectional, and therefore we will have some idea of the main direction of the RADAR. Using this information and for some RADAR setups we will be able to obtain a unique estimate of the position of the target using only 2 RADARs. Still, it's better to have more.