SOLUTIONS

E 83 **Digital Signalbehandling**, 2E1340

Final Examination 2000–12–20, 0900–1300

1. a) The DFT is real-valued if $x(n) = x^*((-n) \mod N)$, which holds for the signals $x_4(n)$ and $x_6(n)$.

b) The proper way to do resampling by a factor 5/3 is to first interpolate by 5 and then decimate by 3, i.e. `y=decimate(interp(x,5),3);`. In the frequency domain, this gives

$$Y(f) = \begin{cases} \frac{5}{3}X(\frac{5}{3}f) & |f| \le \frac{3}{10}\} \\ 0 & \text{otherwise} \end{cases}$$

which corresponds well to signal $y_1(n)$ in the figure.

If the decimation is performed first, resulting in an intermediate signal $u(n)$,

$$U(f) = \frac{1}{3}X\left(\frac{f}{3}\right), \quad |f| \le \frac{1}{2}$$

which after interpolation by 5 gives the result $y(n)$, where

$$Y(f) = \begin{cases} 5U(5f) & |f| \le \frac{1}{10} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{5}{3}X(\frac{5}{3}f) & |f| \le \frac{1}{10} \\ 0 & \text{otherwise} \end{cases}$$

Thus, `y=interp(decimate(x,3),5);` corresponds to signal $y_2(n)$ in the figures.

When the resampling is performed without filtering, `y=interp(decimate(x,3,1),5,1);`, the resulting signal contains aliasing and the spectrum is periodic with period $1/I = 1/5$ which is clearly visible in the figure for $y_3(n)$.

c) In parametric spectral estimation, the estimated spectrum is given by

$$P(f) = \hat{\Gamma}(f) = |\hat{H}(f)|^2\hat{\sigma}_e^2$$

For an AR model of order M, $\hat{H}(f) = \frac{1}{\hat{A}(f)}$ has M poles and can have at most M peaks in the spectrum. Thus, $P_4(f)$ must correspond to the AR(3) model (method (iii)) and $P_2(f)$ must correspond to the AR(10) model (method (iv)).

The Blackman-Tukey method gives a more smooth spectrum estimate compared to the periodogram, so $P_1(f)$ must correspond to method (ii) (Blackman-Tukey) and $P_3(f)$ must correspond to method (i) (Periodogram).

2. Describe the round-off quantization in each multiplication by an additional noise source, $e_m(n)$ with $\gamma_{e_m}(k) = \sigma^2\delta(k)$, see Figure 1.

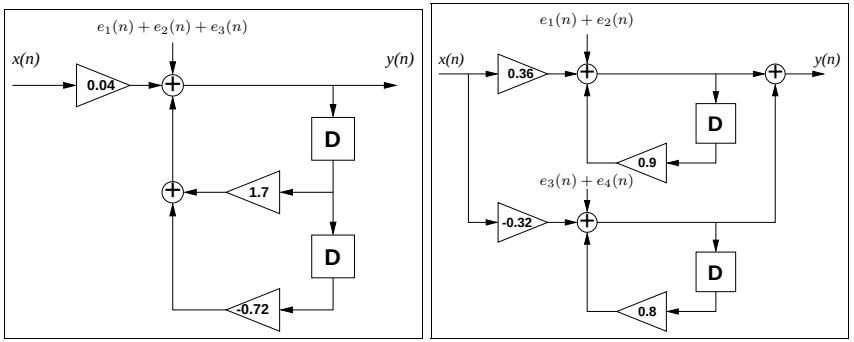

Figure 1:

For the implementation to the left, the transfer function from the noise sources to the output is

$$H_L(z) = \frac{1}{(1-0.9z^{-1})(1-0.8z^{-1})} = \frac{9}{(1-0.9z^{-1})} - \frac{8}{(1-0.8z^{-1})}$$

i.e., $h_L(n) = 9 \cdot 0.9^n - 8 \cdot 0.8^n$. The noise variance on the output is

$$3\sigma^2\sum_{n=0}^{\infty}|h_L(n)|^2 = 3\sigma^2\sum_{n=0}^{\infty}81 \cdot 0.81^n - 144 \cdot 0.72^n + 64 \cdot 0.64^n$$
$$= 3\sigma^2\left(\frac{81}{1-0.81} - \frac{144}{1-0.72} + \frac{64}{1-0.64}\right) \approx 269\sigma^2$$

For the implementation to the right, the transfer function from $e_1(n)$ and $e_2(n)$ to the output is

$$H_{R12}(z) = \frac{1}{(1-0.9z^{-1})}$$

and that from $e_3(n)$ and $e_4(n)$ to the output is

$$H_{R34}(z) = \frac{1}{(1-0.8z^{-1})}$$

corresponding to $h_{R1,2}(n)0.9^n$ and $h_{R3,4}(n)0.8^n$. The noise variance on the output is

$$2\sigma^2\sum_{n=0}^{\infty}|h_{R1,2}(n)|^2 + 2\sigma^2\sum_{n=0}^{\infty}|h_{R3,4}(n)|^2 = 2\sigma^2\left(\frac{1}{1-0.81} + \frac{1}{1-0.64}\right) \approx 16\sigma^2$$

Clearly, the parallel implementation to the right is significantly better than the direct implementation to the left.

3. Multiply the identity with $y(n-m)$ and take the expectation, resulting in

$$\begin{aligned}\mathrm{E}[y(n)y(n-m)] + a_1\,\mathrm{E}[y(n-1)y(n-m)] \\ = b_0\,\mathrm{E}[u(n)y(n-m)] + b_1\,\mathrm{E}[u(n-1)y(n-m)] + \mathrm{E}[e(n)y(n-m)]\end{aligned}$$

or

$$\gamma_{yy}(m) + a_1\gamma_{yy}(m-1) = b_0\gamma_{uy}(m) + b_1\gamma_{uy}(m-1) + \gamma_{ey}(m)$$

Since $y(n)$ only depends on current and older values of $e(n)$, $\gamma_{ey}(m) = 0$ for all $m > 0$. Multiplying the original equation by $e(n)$ and taking expecations, we see that $\gamma_{ey}(0) = \gamma_{ee}(0) = \sigma_e^2$. Since there are four parameters, we need four equations to be able to solve for the parameters. Using $m = 0, 1, 2, 3$, we get the following system of equations.

$$\begin{bmatrix}\gamma_{yy}(0) & \gamma_{yy}(1) & -\gamma_{uy}(0) & -\gamma_{uy}(-1) \\ \gamma_{yy}(1) & \gamma_{yy}(0) & -\gamma_{uy}(1) & -\gamma_{uy}(0) \\ \gamma_{yy}(2) & \gamma_{yy}(1) & -\gamma_{uy}(2) & -\gamma_{uy}(1) \\ \gamma_{yy}(3) & \gamma_{yy}(2) & -\gamma_{uy}(3) & -\gamma_{uy}(2)\end{bmatrix}\begin{bmatrix}1 \\ a_1 \\ b_0 \\ b_1\end{bmatrix} = \begin{bmatrix}\sigma_e^2 \\ 0 \\ 0 \\ 0\end{bmatrix}$$

By adding more equations, corresponding to higher values of m, we can get an overdetermined system of equations which can be solved in the least squares sence. Note, however, that that this will not necessarily provide better estimates since for estimated $\hat{\gamma}(m)$, the accuracy is often worse for higher values of m.

4. a) The "filter" uses the last $M = 3$ values of the input signal. According to the overlap-save method, the last $M-1 = 2$ values of each segment should be saved and appended at the beginning of the next segment (for the first segment, zeros are appended). In the output, the first $M-1 = 2$ values in each segment should be discarded. Each segment should contain $L = 5$ new data points.

Segment 1 $x_1(n) = \{\underbrace{0,0}_{\text{old}},\underbrace{1,2,3,-2,4}_{\text{new}}\}$. This gives (initial state zero):

$$p_1(n) = \{0^2, 0^2{+}0^2, 0^2{+}0^2{+}1^2, 0^2{+}1^2{+}2^2, 1^2{+}2^2{+}3^2, \dots\} = \{\underbrace{0,0,}_{\text{discard}}\ 1, 5, 14, 17, 29\}$$

Segment 2 $x_2(n) = \{\underbrace{-2,4,}_{\text{old}}\underbrace{-1,-3,2,0,2}_{\text{new}}\}$, which gives

$$p_2(n) = \{-2^2, -2^2+4^2, -2^2+4^2+-1^2, \dots\} = \{\underbrace{4,20,}_{\text{discard}}\ 21, 26, 14, 13, 8\}$$

Segment 3 $x_3(n) = \{\underbrace{0,2,}_{\text{old}}\underbrace{-1,3,-4,-2,1}_{\text{new}}\}$, which gives

$$p_3(n) = \{\underbrace{0,4,}_{\text{discard}}\ 5, 14, 26, 29, 21\}$$

The resulting output is $p(n) = \{\underset{\uparrow}{1}, 5, 14, 17, 29, 21, 26, 14, 13, 8, 5, 14, 26, 29, 21\}$.

b) In contrast to overlap-save, the overlap-add method only works for linear filters, in general (this is the answer we had in mind). However, for this particular non-linear system, the contributions from the different samples enter linearly into the output which means that overlap-add actually does work.

Note that if, for example, $p(n) = x^2(n) + (x(n-1) + x(n-2))^2$, then overlap-add will not give the correct result.

5. According to the text, $\gamma_k \approx Kr_k^{-\alpha}$. In order to write this in matrix form, define the following vectors:

$$\boldsymbol{\gamma} = \begin{bmatrix}\gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_{100}\end{bmatrix} \qquad \mathbf{a}(\alpha) = \begin{bmatrix}r_1^{-\alpha} \\ r_2^{-\alpha} \\ \vdots \\ r_{100}^{-\alpha}\end{bmatrix}$$

We want to find K and α such that $\boldsymbol{\gamma} \approx K\mathbf{a}(\alpha)$. Least squares approach: find the K and α that minimize $\|\boldsymbol{\gamma} - K\mathbf{a}(\alpha)\|^2$.

$$\begin{aligned}\|\boldsymbol{\gamma} - K\mathbf{a}(\alpha)\|^2 &= (\boldsymbol{\gamma} - K\mathbf{a}(\alpha))^T(\boldsymbol{\gamma} - K\mathbf{a}(\alpha)) = K^2\mathbf{a}(\alpha)^T\mathbf{a}(\alpha) - 2K\mathbf{a}(\alpha)^T\boldsymbol{\gamma} + \boldsymbol{\gamma}^T\boldsymbol{\gamma} \\ &= \mathbf{a}(\alpha)^T\mathbf{a}(\alpha)\left(K - \frac{\mathbf{a}(\alpha)^T\boldsymbol{\gamma}}{\mathbf{a}(\alpha)^T\mathbf{a}(\alpha)}\right)^2 + \boldsymbol{\gamma}^T\boldsymbol{\gamma} - \frac{(\mathbf{a}(\alpha)^T\boldsymbol{\gamma})^2}{\mathbf{a}(\alpha)^T\mathbf{a}(\alpha)} \geq \boldsymbol{\gamma}^T\boldsymbol{\gamma} - \frac{(\mathbf{a}(\alpha)^T\boldsymbol{\gamma})^2}{\mathbf{a}(\alpha)^T\mathbf{a}(\alpha)}\end{aligned}$$

with equality iff $K = \frac{\mathbf{a}(\alpha)^T\boldsymbol{\gamma}}{\mathbf{a}(\alpha)^T\mathbf{a}(\alpha)}$. When this optimal value of K is inserted into the LS criterion, we get the concentrated cost function

$$\tilde{V}(\alpha) = \boldsymbol{\gamma}^T\boldsymbol{\gamma} - \frac{(\mathbf{a}(\alpha)^T\boldsymbol{\gamma})^2}{\mathbf{a}(\alpha)^T\mathbf{a}(\alpha)}$$

which can be minimized numerically resulting in an optimal estimate $\hat{\alpha}$. Then, we can find the optimal estimate of K as $\hat{K} = \frac{\mathbf{a}(\hat{\alpha})^T\boldsymbol{\gamma}}{\mathbf{a}(\hat{\alpha})^T\mathbf{a}(\hat{\alpha})}$. This follows directly from

$$\|\boldsymbol{\gamma} - \hat{K}\mathbf{a}(\hat{\alpha})\|^2 = \tilde{V}(\hat{\alpha}) = \min_{\alpha}\tilde{V}(\alpha) = \min_{K,\alpha}\|\boldsymbol{\gamma} - K\mathbf{a}(\alpha)\|^2$$