EQ2300, Digital Signal Processing

Tutorials

Stockholm

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Contents

Tutorial 1

Prerequisites Review

1.1 (a) Show that for any constant $r \in \mathbb{C}$, and any $M, N \in \mathbb{Z}$, we have

$$\sum_{n=M}^{N} r^n = \begin{cases} \frac{r^M - r^{N+1}}{1-r} & \text{if } \ r \neq 1 \\ N - M + 1 & \text{if } \ r = 1 \end{cases} .$$

(b) Show that if $|r| < 1$, then

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} .$$

(c) Show that if $r \in \mathbb{R}$ with $|r| < 1$, then

$$\sum_{n=0}^{\infty} n r^{n-1} = \frac{1}{(1-r)^2} .$$

See Solution.

1.2 Let H_1 be the causal system described by the difference equation

$$y[n] = \frac{7}{12} y[n-1] - \frac{1}{12} y[n-2] + x[n-1] - \frac{1}{2} x[n-2]$$

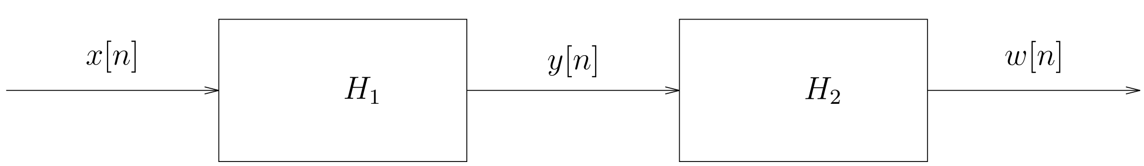

Figure 1.1: Overall system for Exercise 1.2. H_1 is characterized by the given difference equation, H_2 is analyzed for the different choices of $w[n]$ given in the headlines.

(a) Determine the system H_2 in Fig. 1.1 so that $w[n] = x[n]$. Is the inverse system H_2 causal?

(b) Determine the system H_2 in Fig. 1.1 so that $w[n] = x[n-1]$. Is the inverse system H_2 causal? Explain.

(c) Determine the difference equations for the system H_2 in parts (a) and (b).

See Solution.

1.3 Determine all possible impulse responses $h[n]$ that would yield the transfer function

$$H(z) = \frac{5z^{-1}}{(1-2z^{-1})(3-z^{-1})}\,.$$

Specify their corresponding ROCs and their properties in terms of stability and causality.

See Solution.

1.4 Let $y[k] = \sin(\omega T k)$ with $k \in \mathbb{Z}$, determine a so that $y[k]$ satisfies the difference equation

$$y[k] - ay[k-1] + y[k-2] = 0.$$

See Solution.

Tutorial 2

The DFT and the FFT

2.1 a) Show that the DTFT of

$$x[n] = \begin{cases} 1 & \text{if } 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

is given by

$$X(\nu) = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi\nu n} = e^{-j\pi\nu(N-1)}\frac{\sin(\pi\nu N)}{\sin(\pi\nu)} \,,$$

when $\nu \notin \mathbb{Z}$. You may be helped by the identity

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j} \,.$$

b) What happens when $\nu \in \mathbb{Z}$ (when ν takes on an integer value)?

c) Let

$$X_N[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi\frac{kn}{N}}$$

be the N-point DFT of $x[n]$. What is $X_N[k]$ and how does it relate to $X(\nu)$?

See Solution.

2.2 a) Let for some $0 < \nu_0 < \frac{1}{2}$ the DTFT $X(\nu)$ of $x[n]$ be given by

$$X(\nu) = \begin{cases} 1 & \text{if } |\nu| \leq \nu_0 \\ 0 & \text{otherwise} \end{cases}$$

when $\nu \in [-\frac{1}{2}, \frac{1}{2}]$. Show that

$$x[n] = \begin{cases} 2\nu_0 & \text{if } n = 0 \\ \frac{\sin(2\pi\nu_0 n)}{\pi n} & \text{otherwise} \end{cases} \,.$$

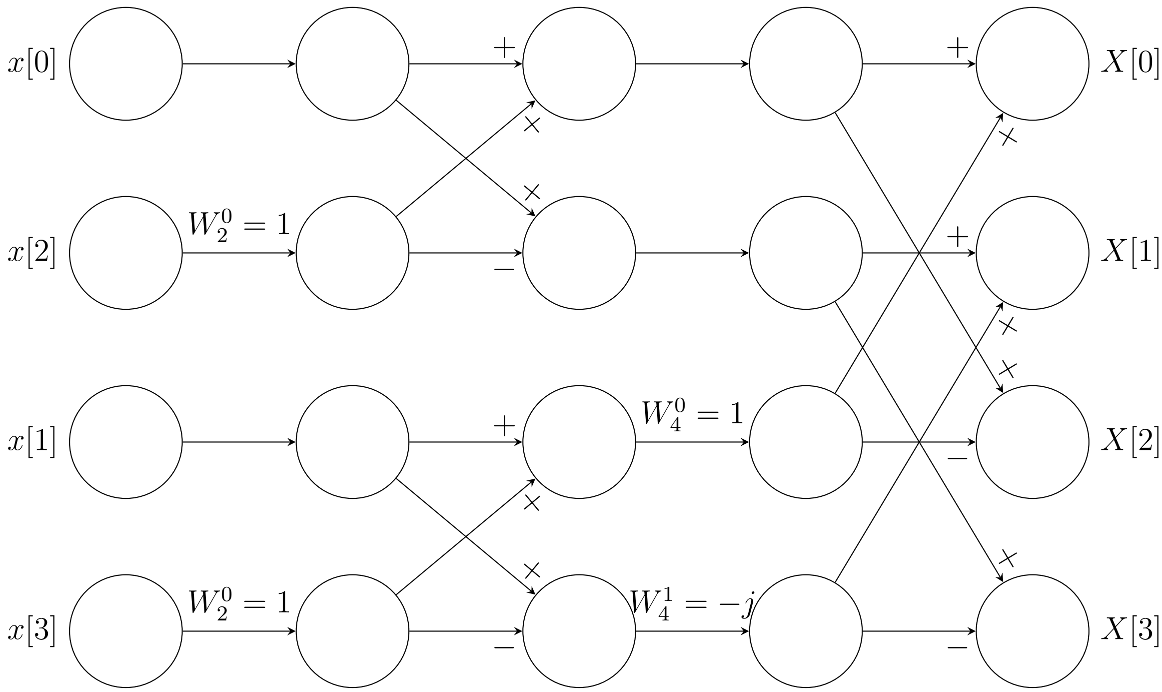

Figure 2.1: FFT Algorithm. Figure used for Exercise 2.3.

b) If

$$Y(\nu) = \begin{cases} c & \text{if } |\nu| \leq \nu_0 \\ 0 & \text{otherwise} \end{cases}$$

is the DTFT of $y[n]$, then what should c be so that $y[0] = 1$?

See Solution.

2.3 a) Let $x[n] = \{\underset{\uparrow}{1}, 2, -2, -1\}$ and evaluate the 4-point DFT $X[k]$ of $x[n]$ by direct computation of

$$X[k] = \sum_{n=0}^{3} x[n] e^{-j2\pi\frac{kn}{4}} \quad \text{for} \quad k = 0, \ldots, 3\,.$$

b) Compute $X[n]$ using the 4-point FFT algorithm by filling in the circles of Fig. 2.1 from left to right.

c) Count, in your manual computations and in Fig. 2.1, the number of complex-valued multiplications needed to get the result. Contrast and verify against the theoretical results presented in class.

d) Write a simple function in MATLAB or Octave that computes the DFT with 4 samples. Once you have verified it works, compare it to the built-in function **fft** using the **tic** and **toc** commands.

See Solution.

2.4 Let $X[k]$ be the DFT of the real sequence $x[n]$, $n = 0, \ldots, N-1$. Also, let $Y[k]$ be the DFT of the sequence $y[n]$, which is the reversed signal, i.e., $y[n] = x[N-1-n]$. Express $X[k]$ in terms of $Y[k]$.

See Solution.

Tutorial 3

Filtering with the FFT

3.1 Given the sequences

$$x_1[n] = \{1,\ \underset{\uparrow}{2},\ 3,\ 1\} \qquad x_2[n] = \{4,\ \underset{\uparrow}{3},\ 2,\ 2\}$$

determine

(a) Their linear convolution.

(b) Their circular convolution, using direct computation in the time-domain.

(c) Their circular convolution, using the 4-point DFT and IDFT.

See Solution.

3.2 Let $x[n]$ and $y[n]$ be two sequences with

$$\begin{array}{ll} x[n] = 0 & \text{for } n < 0 \text{ and } n \geq 8 \\ y[n] = 0 & \text{for } n < 0 \text{ and } n \geq 20 \end{array}$$

A 20-point DFT is performed on $x[n]$ and $y[n]$. The two DFT's are multiplied and an inverse DFT is performed resulting in a new sequence $r[n]$.

(a) Which elements of $r[n]$ coincide with the linear convolution of $x[n]$ and $y[n]$?

(b) How should the procedure be changed so that all elements of $r[n]$ correspond to elements of the linear convolution of $x[n]$ and $y[n]$?

See Solution.

3.3 We wish to filter a long sequence through a FIR filter of length 128. We consider three options:

(a) Direct filtering

(b) Overlap-save method with a 256-point FFT

(c) Overlap-save method with a 512-point FFT

Recall that an FFT requires $\frac{N}{2}\log_2(N)$ complex multiplications. Determine the number of multiplications required by each method and establish and explain which method is most efficient.

See Solution.

Tutorial 4

FIR Filters and FIR Approximations

4.1 Consider the following windows, which are always zero outside the specified interval, and where you can assume that $M \geq 3$ and odd if convenient.

- Rectangular window
$$w_R[m] = 1, \; |m| \leq M.$$
- Barlett window
$$w_B[m] = 1 - \frac{|m|}{M} \quad |m| \leq M.$$
- Raised Cosine window, known by the Hann or Hanning window ($\alpha = \beta = 0.5$) and the Hamming window ($\alpha = 0.54$ and $\beta = 0.46$)
$$w_H[m] = \alpha + \beta \cos\left(\frac{2\pi m}{2M+1}\right) \quad |m| \leq M.$$

(a) Determine the DTFT $W(\nu)$ of each of the above windows and sketch each of these functions of ν.

(b) Show that the smallest positive frequency ν where the transfer function is zero is

- $1/(2M+1)$ for the rectangular window,
- $1/M$ for the Bartlett window,
- and $2/(2M+1)$ for the raised cosine window.

See Solution.

4.2 A linear phase length $M = 41$ FIR filter, approximating an ideal bandpass filter $H_I(\nu)$, where

$$|H_{\mathrm{I}}(\nu)| = \begin{cases} 1 & 0.2 \leq |\nu| \leq 0.3 \\ 0 & \text{otherwise} \end{cases} \quad \text{for } |\nu| \leq 0.5,$$

is designed using the window method with a Chebyshev window $w[n] \in \mathbb{R}$. The magnitude of the DTFT of the window, $W(\nu) = \mathcal{F}\{w[n]\}$, is shown in Fig. 4.1.

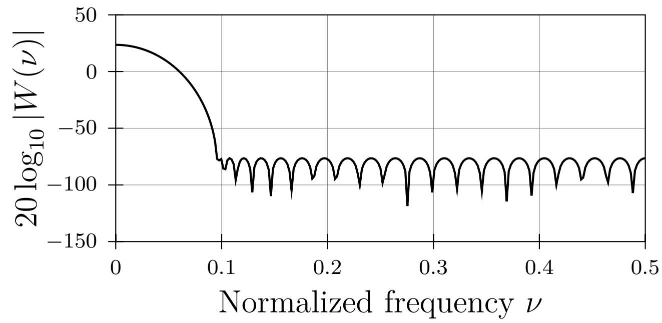

Figure 4.1: Spectrum of the window function $w[n]$ used for Exercise 4.2.

(a) Sketch the expected magnitude of the frequency response of the designed filter and illustrate how this response would change if we were to increase the length of the FIR filter. The sketch does not have to be perfect but should show that you understand how the frequency response of the designed filter depends on the ideal target response and the window's properties, and to the extent possible you should appropriately label important points on both axes of the sketch.

(b) Assuming that the filter is implemented causally, i.e., that $h[n] = 0$ for $n < 0$ and $n \geq M = 41$, and $x[n] = \sin(2\pi\nu_0 n + \phi_0)$ where $\nu_0 = 0.25$ is filtered by the designed filter. What would (approximately) the filtered signal be?

(c) Assuming that the designed FIR filter is to be implemented using overlap add for a long input signal, propose a good FFT length N for the implementation and motivate your choice. Specify what the number of complex valued multiplications per signal sample will be for the implementation.

See Solution.

Tutorial 5

Quantization and Finite-Precision

5.1 Uniform quantization is the effect of a nonlinear function, $Q(x)$, acting on a sequence $x[n]$, as seen in Fig. 5.1. In order to ease analysis, this may be modeled as an additive noise term $e[n]$ perturbing the signal, as seen in Equation (5.1).

$$y[n] = Q\left[x[n]\right] = x[n] + e[n] \tag{5.1}$$

Note, then, $y[n]$ as the output of the quantizer. Let $x[n]$ be a white sequence with zero-mean and variance σ_x^2. If the quantization level, Δ, is small relative to σ_x^2, we can model $e[n]$ as white, uniformly distributed between $-\frac{\Delta}{2}$ and $\frac{\Delta}{2}$, and uncorrelated with the input signal $x[n]$. Under this assumptions,

(a) determine the mean, variance, and correlation sequence for $e[n]$.

(b) determine the SNR in $y[n]$, i.e. $\mathrm{SNR}_{y[n]} = \sigma_x^2/\sigma_e^2$.

(c) if the signal $y[n]$ is filtered through a system with impulse response

$$h[n] = \begin{cases} 0 & \text{for } n < 0 \\ \frac{1}{2}\left(a^n + (-a)^n\right) & \text{for } n \geq 0 \end{cases},$$

compute the variance of the noise and the SNR at the output of the filter.

See Solution.

5.2 In Fig. 5.2 we see two different realizations of a first-order system. The realizations are implemented in binary fix-point arithmetic, 2-complement representation, and with round-off errors occurring in the multiplications. Compare the two realizations in terms of round-off noise variance at the output.

See Solution.

5.3 We want to implement the transfer function

$$H(z) = \frac{(1 + .5z^{-1})(1 + .25z^{-1})}{(1 - .5z^{-1})(1 - .25z^{-1})}$$

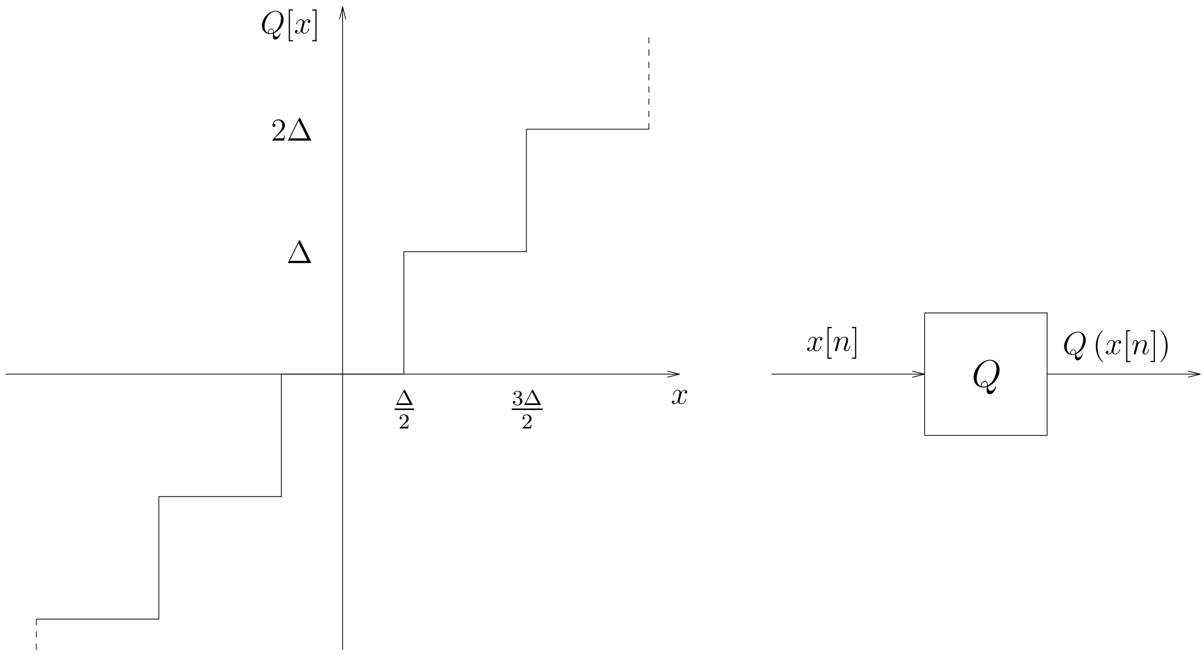

Figure 5.1: Quantization and its statistical model, illustrating Exercise 5.1.

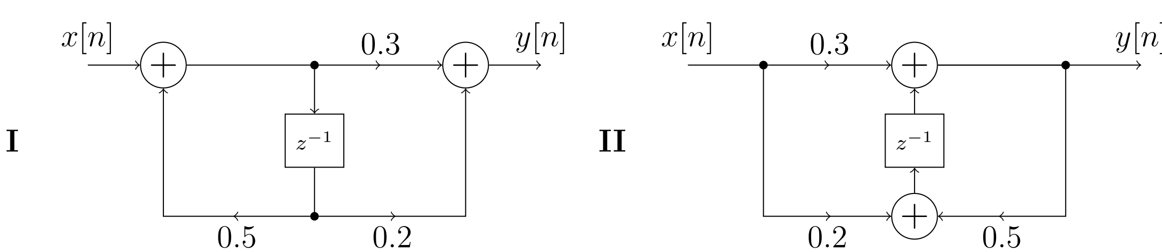

Figure 5.2: Two realizations of the first-order system discussed in Exercise 5.2.

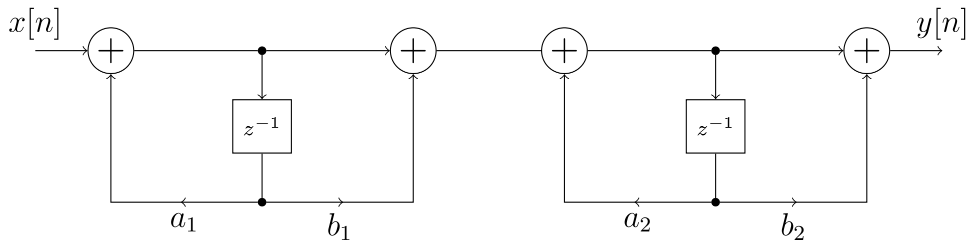

Figure 5.3: Cascade implementation of the second-order transfer function in Exercise 5.3.

as two first order filters in cascade form, see Fig. 5.3. How should a_1, a_2, b_1 and b_2 be determined in order to minimize the quantization noise (caused by the multipliers) at the output?

See Solution.

Tutorial 6

Fixed-Point Filter Implementation

6.1 The performance and stability of an IIR filter depends on the pole locations, so it is important to know how quantization of the filter coefficients affects the pole locations $\{p_i\}_{i=1}^N$. Consider the transfer function

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum\limits_{k=0}^{M} b_k z^{-k}}{1 + \sum\limits_{k=1}^{N} \alpha_k z^{-k}}.$$

The denominator polynomial, $A(z)$, can be factorized as

$$A(z) = 1 + \sum_{k=1}^{N} \alpha_k z^{-k} = \prod_{i=1}^{N} \left(1 - p_i z^{-1}\right).$$

We wish to know $\frac{\partial p_i}{\partial \alpha_k}$, which, for small deviations, will tell us that a δ_{α_k} change in coefficient α_k yields a $\frac{\partial p_i}{\partial \alpha_k}\delta_{\alpha_k}$ change in the pole location. In other words, $\frac{\partial p_i}{\partial \alpha_k}$ is the *sensitivity* of the pole location to quantization of α_k.

(a) Using the chain rule, compute the partial derivative $\frac{\partial p_i}{\partial \alpha_k}$.

(b) What is the take-home message of your analysis?

(c) How can we reduce this sensitivity to IIR filter coefficient quantization?

See Solution.

6.2 A fourth order, symmetric FIR filter with impulse response $h_{\text{direct}}[n] = [b_0, b_1, b_2, b_1, b_0] = [\frac{1}{3}, \frac{2}{3}, 1, \frac{2}{3}, \frac{1}{3}]$ is to be implemented.

(a) The fourth order filter $h_{\text{direct}}[n]$ can be implemented as a cascade $h_{\text{cascade}}[n] = h_{\text{direct}}[n]$ of two second order symmetric filters $h_1[n] = [b_{10}, b_{11}, b_{10}]$ and $h_2[n] = [b_{20}, b_{21}, b_{20}]$. Determine $h_1[n]$ and $h_2[n]$ such that $h_1[n] = h_2[n]$.
Hint: No math required. A plot of $h_{\text{direct}}[n]$ may help.

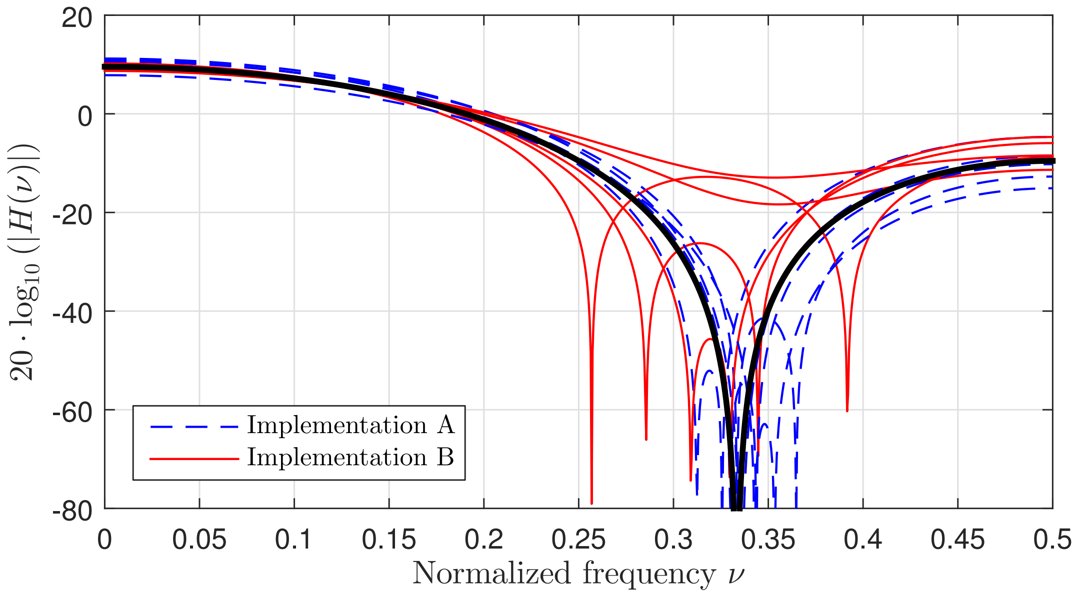

Figure 6.1: Impulse response under coefficient disturbances for different implementations.

(b) Determine the zeros of $H_1(z)$. Thus, at which frequency ν will a sinusoid $x[n] = A \cdot \sin(2\pi\nu n)$ be completely filtered out?

(c) Determine all zeros of $H_{\text{direct}}(z)$.

(d) For a M-th order filter $h[n] = [b_0, \ldots, b_M]$, the sensitivity of the zeros to a small change in the filter parameter b_k is given as

$$\frac{\partial z_i}{\partial b_k} = \frac{z_i^{M-k}}{b_0 \prod_{l \neq i}(z_i - z_l)}.$$

Determine the sensitivity of the zero locations of the forth order filter $H_{\text{direct}}(z)$ to small changes in b_k. What do you conclude?

(e) Is the sensitivity of any of the zeros of the second order filter $H_1(z)$ to a small change in b_{1k} finite?

(f) In Fig. 6.1, we show the impulse response of the ideal filter $h_{\text{direct}}[n] = h_{\text{cascade}}[n] = h_1[n] * h_2[n]$ (thick solid line). Furthermore, we show the impulse responses of the filters when their coefficients are disturbed by small changes: $\tilde{h}_{\text{direct}}[n] = h_{\text{direct}}[n] + e[n]$, and $\tilde{h}_{\text{cascade}}[n] = (h_1[n] + e_1[n]) * (h_2[n] + e_2[n])$ for several instances of the random disturbances $e[n]$, $e_1[n]$, $e_2[n]$, which are white and uniformly distributed between -0.1 and 0.1. Given your previous findings, argue which implementation ($\tilde{h}_{\text{direct}}[n]$ or $\tilde{h}_{\text{cascade}}[n]$) corresponds to the dashed lines (A) and which implementation corresponds to the thin solid lines (B).

See Solution.

Tutorial 7

Nonparametric Spectral Estimation

The Periodogram and the Modified Periodogram

7.1 We obtain $N = 10000$ samples of data $\{x[0], \ldots, x[N-1]\}$ at a sampling frequency $F_s = 1000\,\mathrm{Hz}$. The signal is a sinusoid in a signal-independent additive noise, i.e.

$$x[n] = A \sin(\omega_0 n) + e[n]\,,$$

and we know that the frequency of the sinusoid F_0 is smaller than $F_s/2$, i.e., that the sampling fulfilled the Nyquist criterion. To estimate the frequency F_0 and the amplitude A of the sinusoid we calculate the periodogram $P(\omega)$ of the signal in the interval $[0, \pi]$, i.e.,

$$P(\omega) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x[n] e^{-j\omega n}\right|^2 = \frac{1}{N}|X(\omega)|^2$$

where $X(\omega)$ is the DTFT of the sampled sequence $x[n]$. The plot of the obtained $P(\omega)$ is shown in Figure 7.1.

a) Estimate the frequency F_0 (in Hz).

b) Estimate the amplitude A of the sinusoid.

c) Estimate the noise power σ^2. For this last section, take into account that the actual implementation of the periodogram in Figure 7.1 was done using an N-point DFT, and thus, the average value stated in the figure corresponds to

$$\frac{1}{N}\sum_{k=0}^{N-1} P(\omega_k)$$

with $\omega_k = 2\pi \frac{k}{N}$.

See Solution.

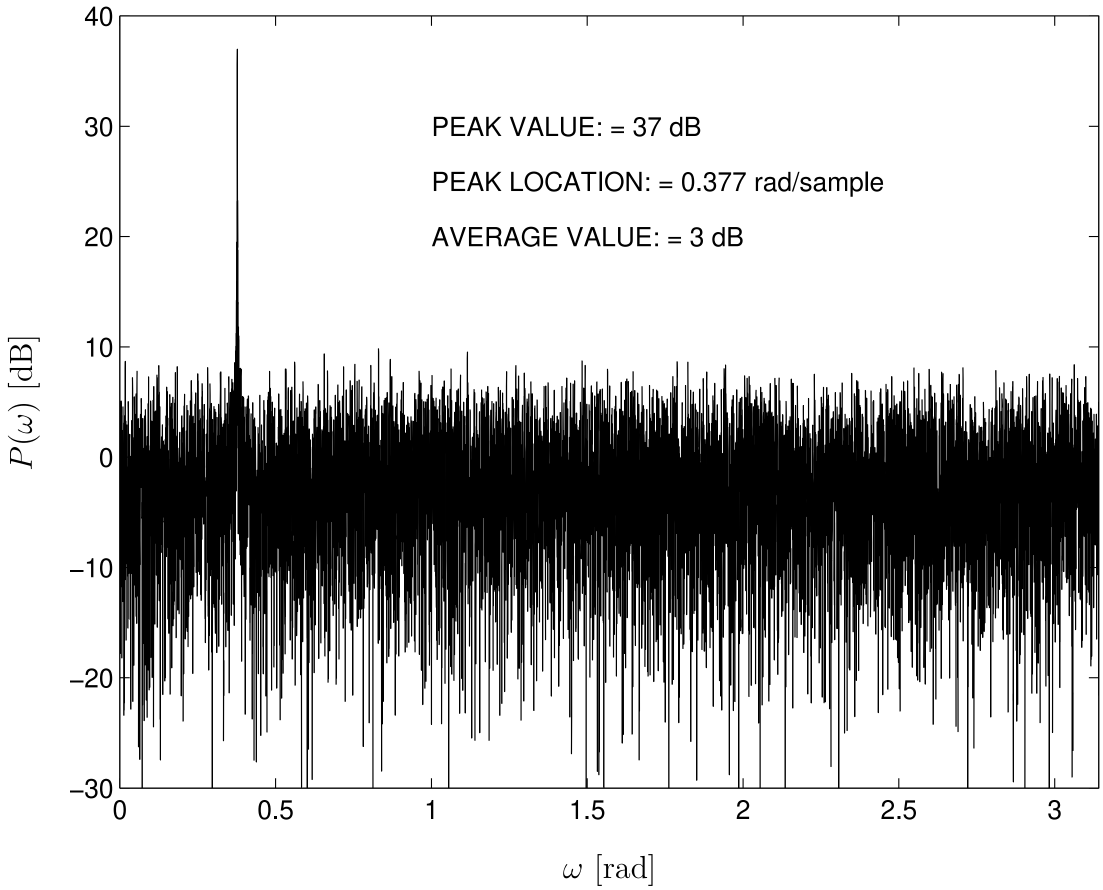

Figure 7.1: Periodogram obtained by processing the received signal $x[n]$ in Exercise 7.1.
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7.2 We wish to analyze the frequency content of a signal $x(t)$. The signal is anti-alias filtered, sampled, windowed, and the DFT is computed for a large number of points. Four different windows are applied and the absolute value of the DFT is plotted. Note that this DFT will actually be proportional to the modified periodogram of the sampled signal $x[n]$. If the original spectral content of the signal $x(t)$ is the one displayed in Fig. 7.2, combine each window with its resulting modified periodogram in Fig. 7.3.

See Solution.

7.3 In this exercise we will prove that the periodogram is an unbiased estimatior of the PSD for white noise. Let $y[n]$ be zero-mean white noise with variance σ^2 and let

$$Y(\nu) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} y[n] e^{-j2\pi\nu n}$$

denote its normalized DTFT. Then,

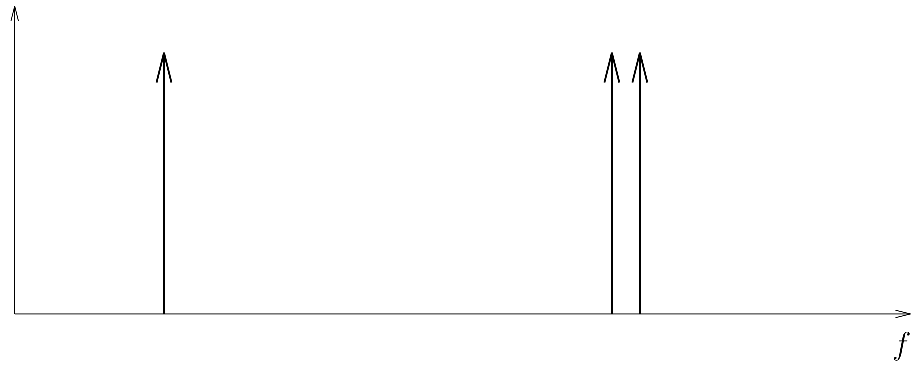


Figure 7.2: True spectrum of the signal $x(t)$ in Exercise 7.2.

(a) derive the correlation of the normalized DFT at different frequencies, i.e. compute $\mathrm{E}\{Y(\nu_1)Y^*(\nu_2)\}$.

(b) use the result of the previous calculation to conclude that the periodogram $\hat{P}_{yy}(\nu) = |Y(\nu)|^2$ is an unbiased estimator of the PSD of $y[n]$.

See Solution.

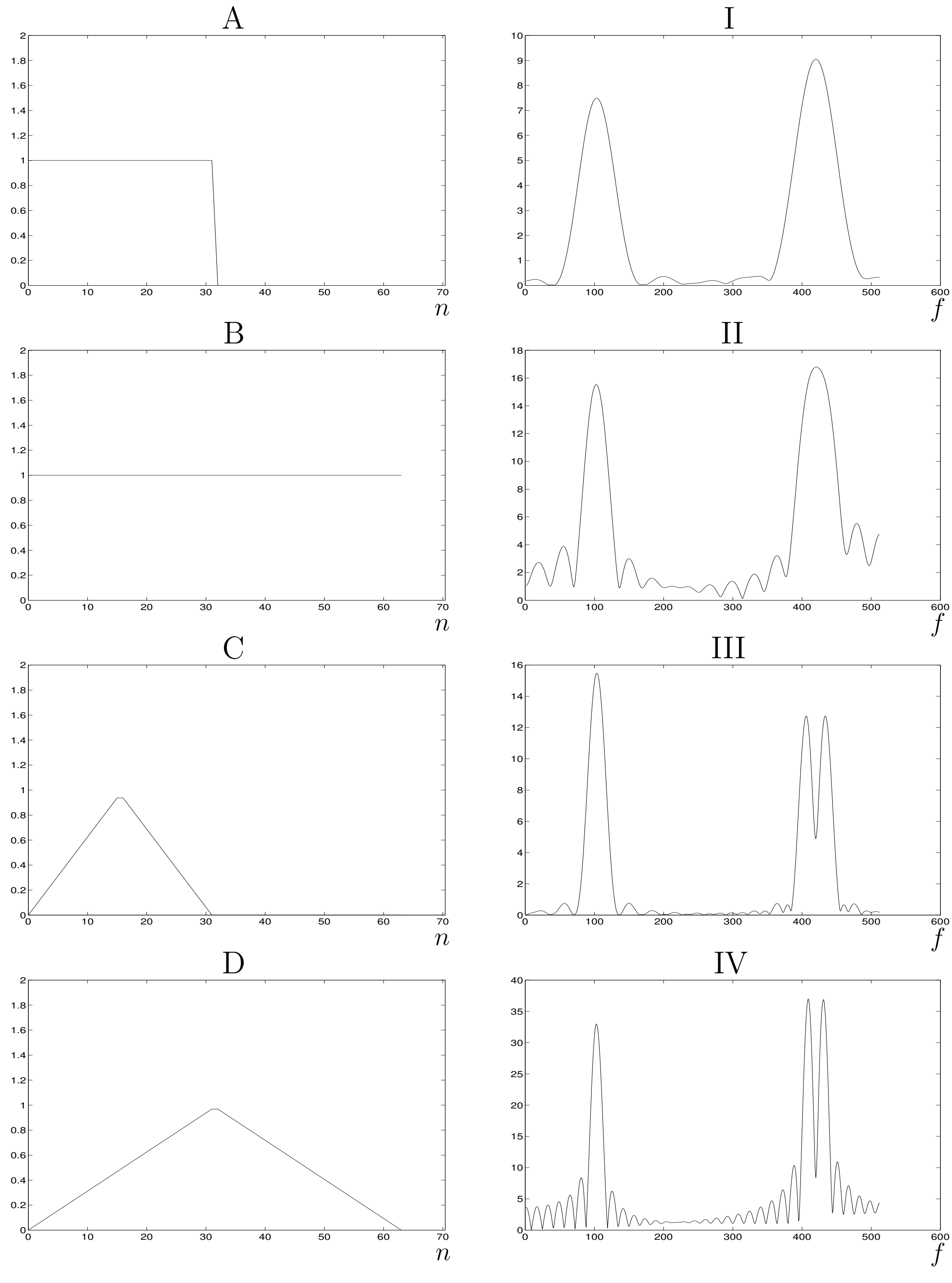


Figure 7.3: Windows and modified periodograms to be paired in Exercise 7.2.

Tutorial 8

Nonparametric Spectral Estimation

Barlett, Welch and Blackman-Tukey

8.1 We wish to estimate the spectrum of a signal using Welch's method. The signal is anti-aliased filtered and sampled at 12 kHz. We form K data segments, each containing M samples and a triangular window is applied on each segment.

(a) Assume that the signal contains two frequency components separated by 200 Hz. How large must M be chosen to ensure sufficient frequency resolution?

(b) How many segments K should we average over to ensure that the variance in the estimate is smaller than a 5% of the square of the correct spectral density?

See Solution.

8.2 The power spectral density of a real-valued sinusoid in white zero-mean Gaussian noise is estimated using a number of nonparametric methods. There are $N = 512$ samples available for the estimation and the estimates are shown in Fig. 8.1. The four different methods used are (not in any particular order):

- The Periodogram. All $N = 512$ samples are used.
- The Modified Periodogram. All $N = 512$ samples and a Hamming window is used.
- Bartlett's method. The $N = 512$ samples are divided into segments of length $L = 32$.
- Welch's method. A Hamming window is used and the $N = 512$ samples are divided into segments of length $L = 32$ with 50% overlap.

(a) Specify which plot corresponds to which method, i.e., specify the estimators labeled as Method A, Method B, Method C and Method D. Remember to fully motivate your answers.

(b) How many segments can be used in Bartlett's method and in Welch's method respectively?

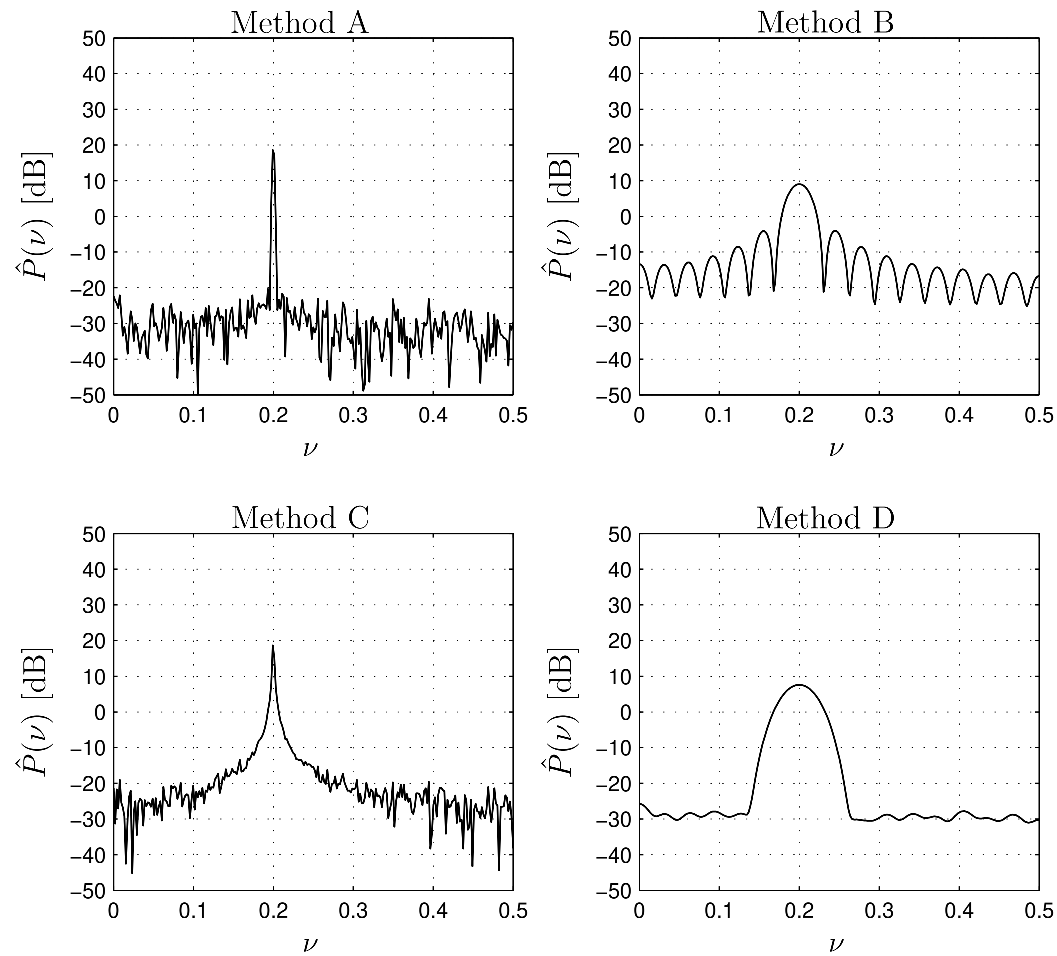


Figure 8.1: Different non-parametric spectral estimates.

(c) Argue which plot is best for estimating the noise variance and give an estimated numerical value for the noise variance.

See Solution.

8.3 The spectral density of a time-discrete signal $x[n]$ is estimated using the Welch method. The window used is

$$w[n] = \cos^2\left(\frac{n\pi}{N}\right)$$

where N is the number of samples in the block. The signal $x[n]$ is defined for $-\frac{N}{2} < n \leq \frac{N}{2}$. Show that the N-point DFT for $x[n]w[n]$ may be computed from the DFT of $x[n]$ without explicitly forming $x[n]w[n]$.

See Solution.

Tutorial 9

Parametric Spectral Estimation

9.1 A linear predictor is used to construct a linear-phase filter to adaptively cancel a sinusoidal signal $s[n] = A\cos(2\pi fn + \phi)$, where ϕ is an unknown phase term that we model as $\phi \sim \mathcal{U}[0, 2\pi)$, f is a known deterministic frequency and A a fixed unknown amplitude. The linear predictor has the structure specified in Fig. 9.1, where the blocks marked as D are simple registers, which delay the signal 1 sample.

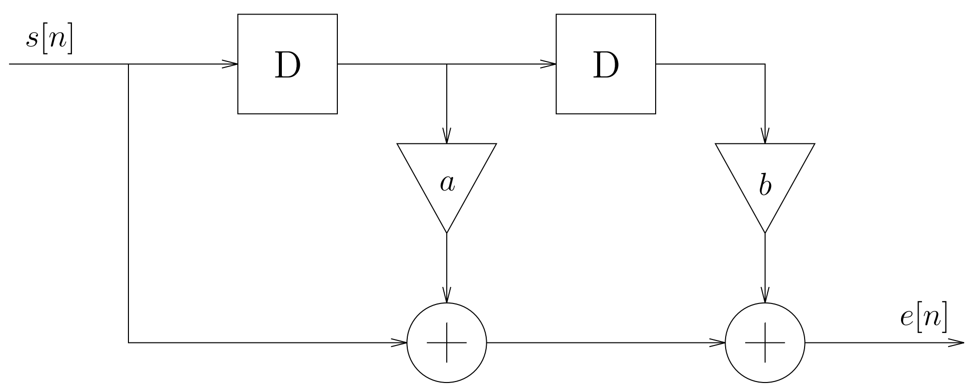


Figure 9.1: Figure representing the linear predictor under design in Exercise 9.1.

Determine the coefficients a and b in terms of the properties of $s[n]$ in order to cancel the signal. Additionally, express the frequency of $s(n)$ in terms of the filter coefficients.

See Solution.

9.2 Suppose that we would like to estimate the power spectrum of an AR(2) process

$$x[n] + a_1 x[n-1] + a_2 x[n-2] = b_{x,0} w[n]$$

where $w[n]$ is unit variance white noise. However, our measurements of $x[n]$ are noisy, and what we observe is the process

$$y[n] = x[n] + v[n]$$

where the measurement noise, $v[n]$, is uncorrelated with $x[n]$. It is known that $v[n]$ is a first-order moving average process,

$$v[n] = b_0 q[n] + b_1 q[n-1]$$

where $q[n]$ is white noise. Based on measurements of $v[n]$, the power spectrum of $v[n]$ is estimated to be

$$\hat{P}_v(\omega) = 3 + 2\cos(\omega).$$

From $y[n]$ we estimate the following values of the autocorrelation sequence $r_y[k]$,

$$\hat{r}_y[0] = 5 \quad ; \quad \hat{r}_y[1] = 2 \quad ; \quad \hat{r}_y[2] = 0 \quad ; \quad \hat{r}_y[3] = -1 \quad ; \quad \hat{r}_y[4] = 0.5\,.$$

Using all of the given information, estimate the power spectrum of $x[n]$.

See Solution.

9.3 A common method to estimate the parameters of an AR-process is to first estimate the covariance function and then solve Yule-Walker equations to find the parameters. An unbiased estimator of the autocorrelation function is

$$\hat{r}_y[k] = \frac{1}{N-|k|} \sum_{n=0}^{N-|k|-1} y[n]y[n+|k|] \tag{9.1}$$

(a) Using (9.1), determine the estimate of the AR parameters b_0^2, a_1, a_2 from the following data sequence generated by a stationary AR(2) process: $y[0] = 2$, $y[1] = 1$, $y[2] = 2$. Is the estimate of the autocorrelation function given by (9.1) reasonable? Where are the poles of the filter included in the AR(2) model for this process? Explain the implications of these results and evaluate the resulting AR(2) model.

(b) A biased estimator of the autocorrelation function with considerably better variance properties is

$$\hat{r}_y[k] = \frac{1}{N} \sum_{n=0}^{N-|k|-1} y[n]y[n+|k|]\,. \tag{9.2}$$

Redo 9.3a with this new estimate and yield new conclusions.

See Solution.

Tutorial 10

Multirate Signal Processing

Decimation and Interpolation

10.1 In the system of Fig. 10.1, $X(f)$ (the Fourier transform of a continuous-time signal $x(t)$) and $H(\nu)$ (the transfer function of a discrete-time filter $h[n]$) are shown.

(a) Determine the largest possible value of B such that no aliasing appears at the decimation.

A	B	C	D	E
$B = 1/4$	$B = 3/4$	$B = 1/2$	$B = 1$	something else

(b) From all the options shown in Fig. 10.2, which one is the transform of the output signal $y(t)$ for the specific choice of B made in the previous question?

(c) At what frequency should the D/C converter be operated so that $y(t)$ simply becomes a low-pass filtered version of $x(t)$?

See Solution.

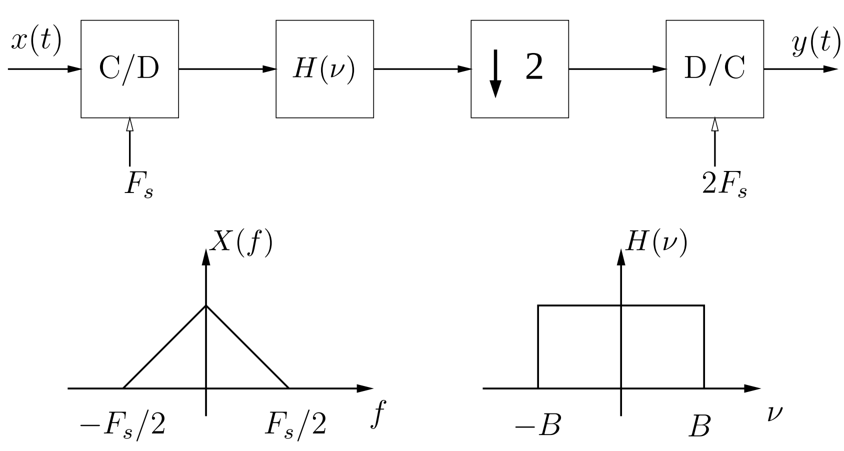


Figure 10.1: System discussed in Exercise 10.1a.

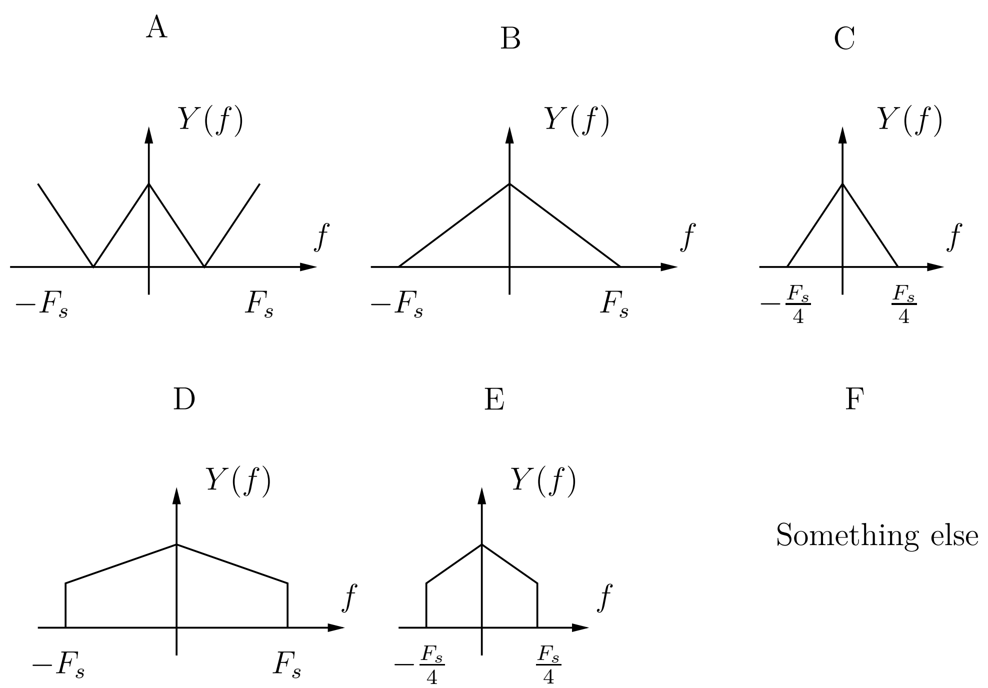


Figure 10.2: Possible spectrum of the continuous-time signal $x(t)$ discussed in Exercise 10.1b.

10.2 $h[n] = \{h_0, h_1, \ldots, h_N\}$ where $N = M \cdot K$, is the impulse response of a low-pass filter with transfer function magnitude

$$|H(\omega)|^2 \approx \begin{cases} 1 & \text{for } |\omega| \leq \Omega \text{ with } \Omega < \frac{\pi}{M} \\ 0 & \text{otherwise.} \end{cases}$$

Determine the pass-band of $\bar{h}[n] = \{h_0, h_M, \cdots, h_{MK}\}$.

See Solution.

10.3 Suppose that you obtained a sequence $s[n]$ by filtering a speech signal $s_c(t)$ with a continuous-time low-pass filter with a cutoff frequency of 5 kHz and then sampling the resulting output at a 10 kHz rate, as shown in Fig. 10.3.

Unfortunately, you destroyed the speech signal $s_c(t)$ once the sequence $s[n]$ was stored. Later, you find that what you should have done is followed the process shown in Fig. 10.4 instead.

Develop a method to obtain $s_1[n]$ from $s[n]$ using discrete-time signal processing techniques. Your method may require a very large amount of computation, but should not require a C/D or D/C converter. If your method uses a discrete-time filter, you should specify the frequency response of the filter.

See Solution.

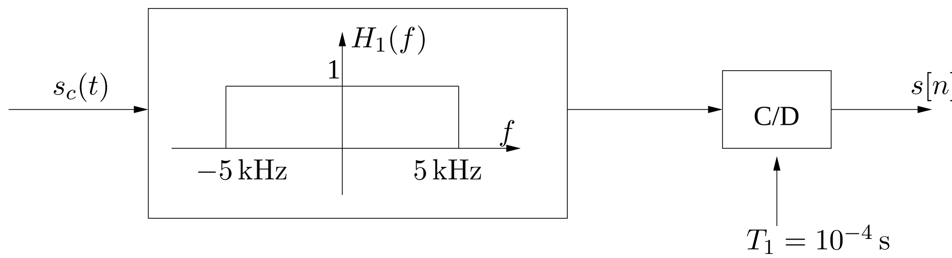


Figure 10.3: Original sampling of the speech signal studied in Exercise 10.3.

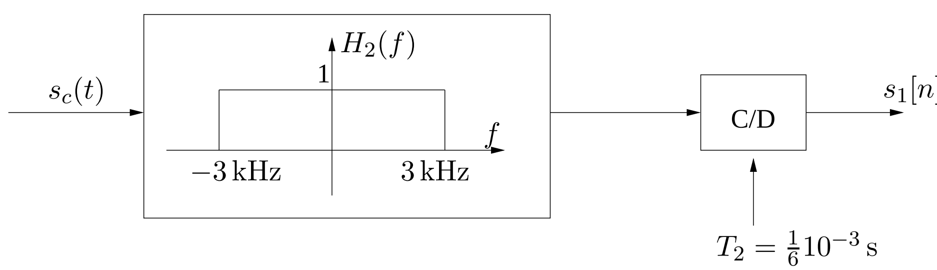


Figure 10.4: Desired sampling of the speech signal. Exercise 10.3 asks to recover this hypothetical signal $s_1[n]$ from the $s[n]$ obtained according to Fig. 10.3.

Tutorial 11

Multirate Signal Processing

Filter Banks

11.1 If $x[n] = \{x[0], x[1], \ldots, x[N]\}$, with $N = MK$, is a band-pass sequence with pass-band centered around ω_0, then $\bar{x}[n] = \{x[0], x[K], \cdots, x[MK]\}$ has center frequency $\bar{\omega}_0$ (consider that all quantities below are well inside $[0, \pi)$), where

$$\begin{array}{cccc} \text{A} & \text{B} & \text{C} & \text{D} \\ \bar{\omega}_0 = \omega_0 & \bar{\omega}_0 = K\omega_0 & \bar{\omega}_0 = M\omega_0 & \bar{\omega}_0 = N\omega_0 \end{array}$$

$$\begin{array}{ccc} \text{E} & \text{F} & \text{G} \\ \bar{\omega}_0 = \omega_0/K & \bar{\omega}_0 = \omega_0/M & \bar{\omega}_0 = \omega_0/N \end{array}.$$

See Solution.

11.2 A signal can be analyzed by dividing its spectrum into narrow frequency bands and examining each band separately. The signal in each frequency band is decimated to keep the same total amount of data, but it is still possible to restore the signal without distortion. If the implementable filters $h[n]$, $g[n]$, $f[n]$ and $d[n]$ are determined appropriately, the aliasing from the decimation will be eliminated completely.

(a) Consider the filter bank in Figure 11.1 and determine requirements on the filter transfer functions $H(\nu)$, $G(\nu)$, $F(\nu)$ and $D(\nu)$, so that the output signal $y[n]$ is identical to the input signal $x[n]$ except for a delay.

(b) Assume that the filters $h[n]$ and $f[n]$ are so-called "Quadrature Mirror Filters", i.e. $F(\nu) = H\left(\nu - \frac{1}{2}\right)$ and that $H(\nu) = G(\nu)$. Determine $D(\nu)$, and the relationships between the impulse responses $h[n]$, $f[n]$ and $d[n]$. Also, determine the requirements on $H(\nu)$ for a distortion-free transmission.

See Solution.

11.3 Consider an arbitrary digital filter $h[n]$ with transfer function

$$H(z) = \sum_{n=-\infty}^{+\infty} h[n]z^{-n}.$$

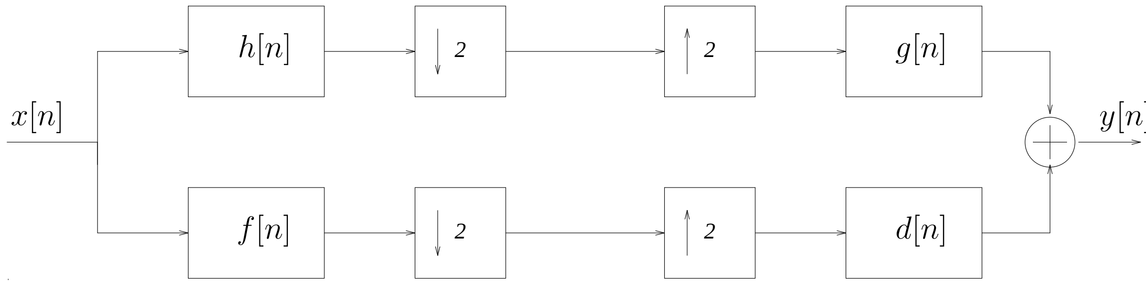


Figure 11.1: Filter bank under consideration in Exercise 11.2 and Exercise 11.4.

(a) Prove the known result

$$H(z) = \mathrm{P}_0\left(z^2\right) + z^{-1}\mathrm{P}_1\left(z^2\right) ,$$

where $p_0[n] = h[2n]$ and $p_1[n] = h[2n+1]$ are the polyphase decomposition components with decimation 2.

(b) Particularly, let

$$H(z) = \frac{a + z^{-1}}{1 + az^{-1}} .$$

Write down expressions for the two components of the polyphase decomposition assuming $H(z)$ is a causal filter, i.e. report $p_0[n]$, $\mathrm{P}_0(z)$, $p_1[n]$, and $\mathrm{P}_1(z)$. Notice that $\forall a \in \mathbb{R}$, $H(z)$ is all-pass. Are the polyphase components all-pass as well?

See Solution.

11.4 A multi-rate reconstructing filter bank is implemented as shown in Fig. 11.1. The filter $h[n]$ has an impulse response given by

$$h[n] = \{\underset{\uparrow}{1}, \tfrac{1}{2}\}$$

Obtain *causal* 2-tap FIR filters $f[n]$, $g[n]$, and $d[n]$ such that perfect reconstruction is achieved, i.e., $y[n] = x[n-L]$ for some delay $L \in \mathbb{N}$, and specify the value of L. Design the filter $f[n]$ such that $F(\nu = 0) = 0$, i.e., $F(\nu)$ has a high-pass characteristic with zero gain for constant signals.

See Solution.

Tutorial 12

Advanced Topics and Recent Exam Problems

12.1 In this problem, we will consider the "double integration" system for quantization with noise shaping shown in Fig. 12.1.

In this system, $H_1(z) = \dfrac{1}{1-z^{-1}}$ and $H_2(z) = \dfrac{z^{-1}}{1-z^{-1}}$ and the frequency response of the decimation filter is

$$H_3(\omega) = \begin{cases} 1, & |\omega| < \pi/M \\ 0, & \pi/M \leq |\omega| \leq \pi \end{cases}.$$

The noise source $e[n]$, which statistically models a quantizer, is assumed to be a zero-mean white-noise signal that is uniformly distributed and has power $\sigma_e^2 = \Delta^2/12$.

(a) Determine an equation for $Y(z)$ in terms of $X(z)$ and $E(z)$. From the Z-transform relation, show that $y[n]$ can be expressed in the form $y(n) = x[n-1] + f[n]$, where $f[n]$ is the output due to the noise source $e[n]$. What is the time-domain relation between $f[n]$ and $e[n]$?

(b) Now use that $e[n]$ is the white-noise signal previously described. Use the result from the previous section to show that the power spectrum of the noise $f[n]$ is

$$P_{ff}(\omega) = 16\sigma_e^2 \sin^4(\omega/2)\,.$$

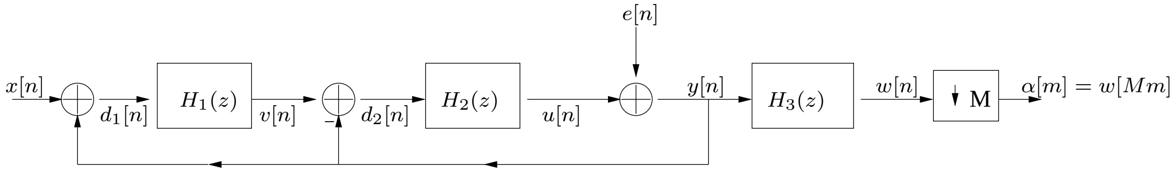


Figure 12.1: Noise shaping example.

What is the total noise power, σ_f^2, in the signal $y[n]$? Sketch and compare the power spectra $P_{ee}(\omega)$ and $P_{ff}(\omega)$ for $0 \leq \omega \leq \pi$.

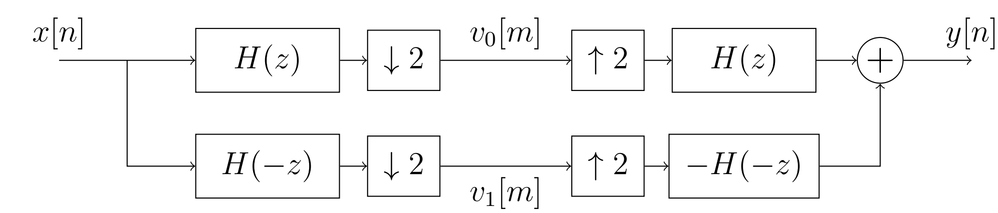
(c) Now assume that $X(\omega) = 0$ for $\pi/M < \omega \leq \pi$. Argue that the output of $H_3(z)$ is $w[n] = x[n-1] + g[n]$. Describe in your own words what $g[n]$ is.

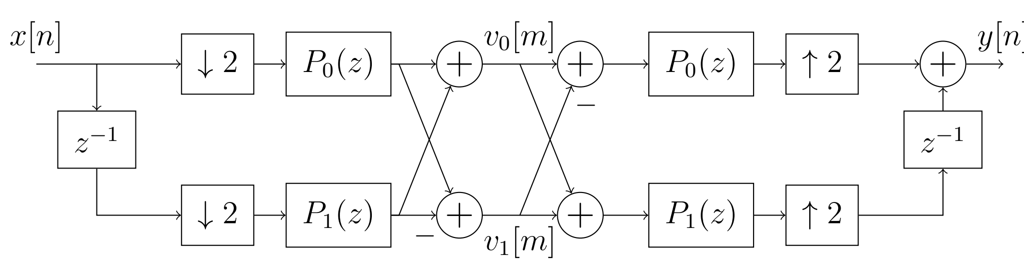
(d) Determine an expression for the noise power σ_g^2 at the output of the decimation filter. Assume that $\pi/M << \pi$, i.e., that M is large, so that you can use the approximation $\sin x \approx x$ to simplify the evaluation of the integral.

(e) After the decimator, the output is $\alpha[m] = w[Mm] = x[Mm-1] + q[m]$, where $q[m] = g[Mm]$. Now suppose that $x[n] = x_c(nT)$ (i.e., $x[n]$ was obtained by sampling a continuous-time signal $x_c(t)$). What condition must be satisfied by $X_c(\Omega)$, the Fourier transform of $x_c(t)$, so that $x[n-1]$ will pass through the filter unchanged? Express the *signal component* at the output $\alpha[n]$ in terms of $x_c(t)$. What is the total noise power σ_q^2 at the output? Give an expression for the power spectrum of the noise at the output, and, on the same set of axes, sketch the power spectra $P_{ee}(\omega)$ and $P_{qq}(\omega)$ for $0 \leq \omega \leq \pi$.

See Solution.

12.2 A quadrature mirror filter (QMF) bank with perfect reconstruction is implemented in two different ways as shown below. Both implementations yield the same $v_0[m]$, $v_1[m]$, and $y[n]$ given the same $x[n]$. Give expressions for $p_0[n]$ and $p_1[n]$ in terms of the base filter $h[n]$, where $P_0(z)$, $P_1(z)$, and $H(z)$ are the respective transfer functions of the filters. (3p)





See Solution.

Solutions

Prerequisites Review

[1.1] (a) For $r = 1$, we have that

$$\sum_{n=M}^{N} r^n = \sum_{n=M}^{N} 1 = N - M + 1\,.$$

Otherwise,

$$(1-r)\sum_{n=M}^{N} r^n = \sum_{n=M}^{N} r^n - \sum_{k=M+1}^{N+1} r^k = r^M - r^{N+1} \Rightarrow \sum_{n=M}^{N} r^n = \frac{r^M - r^{N+1}}{1-r}\,.$$

(b) For any value $r \in \mathbb{C}$ with $|r| < 1$ we know that

$$\lim_{N\to+\infty}\left[r^N\right] = 0\,.$$

Thus, for $|r| < 1$,

$$\lim_{N\to+\infty}\left[\sum_{n=0}^{N} r^n\right] = \lim_{N\to+\infty}\left[\frac{r^0 - r^{N+1}}{1-r}\right] = \frac{1}{1-r}\,.$$

(c) Define the function $f : (-1,1) \to \mathbb{R}$ such that

$$f(r) = r^n\,.$$

Then, linearity of the derivative yields

$$\sum_{n=0}^{\infty} n r^{n-1} = \sum_{n=0}^{\infty} \frac{\partial}{\partial r}\{f(r)\} = \frac{\partial}{\partial r}\left\{\sum_{n=0}^{\infty} f(r)\right\} = \frac{\partial}{\partial r}\left\{\frac{1}{1-r}\right\} = \frac{1}{(1-r)^2}\,.$$

See Exercise.

[1.2] (a) Determine H_2 knowing $w[n] = x[n]$. Then, $X(z) = W(z)$.
But, applying the Convolution theorem for the cascading of linear time invariant (LTI) systems, we also have $W(z) = H_1(z)H_2(z)X(z)$ and consequently, $H_1(z)H_2(z) = 1 \Rightarrow H_2(z) = \dfrac{1}{H_1(z)}$.

Let's find $H_1(z) = \dfrac{Y(z)}{X(z)}$, then. First, transform the Finite Differences Equation:

$$Y(z) = \frac{7}{12}Y(z)z^{-1} - \frac{1}{12}Y(z)z^{-2} + X(z)z^{-1} - \frac{1}{2}X(z)z^{-2}$$
$$\Updownarrow$$
$$Y(z)\left(1 - \frac{7}{12}z^{-1} + \frac{1}{12}z^{-2}\right) = X(z)\left(z^{-1} - \frac{1}{2}X(z)z^{-2}\right)$$
$$\Updownarrow$$
$$\frac{Y(z)}{X(z)} = \frac{z^{-1} - \frac{1}{2}z^{-2}}{1 - \frac{7}{12}z^{-1} + \frac{1}{12}z^{-2}}.$$

Thus, $H_1(z) = \dfrac{\left(z - \frac{1}{2}\right)}{\left(z - \frac{1}{3}\right)\left(z - \frac{1}{4}\right)}$ and therefore $H_2(z) = \dfrac{\left(z - \frac{1}{3}\right)\left(z - \frac{1}{4}\right)}{\left(z - \frac{1}{2}\right)}$. The poles of $H_2(z)$ are $z_1 = 1/2$ and $z_2 = +\infty$ (because $\lim_{z\to+\infty} (|H_2(z)|) = +\infty$). Therefore, the system can not be causal (its ROC can not include the pole $z_2 = +\infty$) and thus, the ROC of the system is the set $\{z \in \mathbb{C} \mid |z| < 1/2\}$.

(b) Determine H_2 knowing $w[n] = x[n-1]$.
In this case we have $W(z) = X(z)z^{-1}$. Following the steps above we get

$$H_2(z) = \frac{z^{-1}}{H_1(z)} = \frac{1}{zH_1(z)} = \frac{\left(z - \frac{1}{3}\right)\left(z - \frac{1}{4}\right)}{z\left(z - \frac{1}{2}\right)}.$$

The poles of $H_2(z)$ are, now, $z_1 = 0$ and $z_2 = 1/2$. If we pick the ROC of the system (now possible) as the set $\{z \in \mathbb{C} \mid |z| > 1/2\}$, then H_2 is causal and stable.

Explanatory note:

Time domain	Z-domain
A delay of one sample has allowed us to find $w[n]$ with only samples from the past or present ($m < n$).	A pole at 0 has replaced the pole at $+\infty$, allowing for a ROC with the "causal" shape.

(c) In a) we have

$$H_2(z) = \frac{W(z)}{Y(z)} = \frac{\left(z - \frac{1}{3}\right)\left(z - \frac{1}{4}\right)}{\left(z - \frac{1}{2}\right)}$$
$$\Updownarrow$$
$$W(z)\left(z - \frac{1}{2}\right) = \left(z - \frac{1}{3}\right)\left(z - \frac{1}{4}\right)Y(z)$$

$$\Updownarrow$$

$$zW(z) - \frac{W(z)}{2} = z^2 Y(z) - \frac{7}{12} zY(z) + \frac{Y(z)}{12}$$

Now, by the inverse Z transform, we get

$$w[n] - \frac{1}{2} w[n-1] = y[n+1] - \frac{7}{12} y[n] + \frac{1}{12} y[n-1]\,.$$

In b) we have

$$H_2(z) = \frac{W(z)}{Y(z)} = \frac{\left(z - \frac{1}{3}\right)\left(z - \frac{1}{4}\right)}{z\left(z - \frac{1}{2}\right)}$$

$$\Updownarrow$$

$$H_2(z) = \frac{\frac{1}{12} z^{-2} - \frac{7}{12} z^{-1} + 1}{1 - \frac{1}{2} z^{-1}}$$

$$\Updownarrow$$

$$W(z)\left(1 - \frac{1}{2} z^{-1}\right) = Y(z)\left(\frac{1}{12} z^{-2} - \frac{7}{12} z^{-1} + 1\right).$$

Now, by the inverse Z transform, we get

$$w[n] - \frac{1}{2} w[n-1] = \frac{1}{12} y[n-2] - \frac{7}{12} y[n-1] + y[n].$$

See Exercise.

[1.3] The technique known as partial fraction expansion sets

$$H(z) = \frac{Az^{-1} + B}{1 - \frac{1}{3} z^{-1}} + \frac{Cz^{-1} + D}{1 - 2z^{-1}}$$

to obtain $A = C = 0$ and $D = -A = 1$. This is done because the resulting summands are easier to inverse Z-transform. Then, we obtain,

$$H(z) = \frac{5z^{-1}}{(1 - 2z^{-1})(3 - z^{-1})} = \underbrace{\frac{1}{1 - 2z^{-1}}}_{H_1(z)} + \underbrace{\frac{-1}{1 - \frac{1}{3} z^{-1}}}_{H_2(z)}.$$

$H_1(z)$:

$$h_{1,1}[n] = 2^n u[n], \quad \mathrm{ROC}_{1,1} = \{|z| > 2\},$$
$$h_{1,2}[n] = -2^n u[-n-1], \quad \mathrm{ROC}_{1,2} = \{|z| < 2\}.$$

$H_2(z)$:

$$h_{2,1}[n] = -\left(\frac{1}{3}\right)^n u[n], \quad \mathrm{ROC}_{2,1} = \{|z| > \frac{1}{3}\},$$

$$h_{2,2}[n] = \left(\frac{1}{3}\right)^n u[-n-1], \quad \mathrm{ROC}_{2,2} = \{|z| < \frac{1}{3}\}.$$

$h[n]$ is given by

$$h[n] = h_{1,i}[n] + h_{2,j}[n], \quad \mathrm{ROC} = \mathrm{ROC}_{1,\mathrm{i}} \cap \mathrm{ROC}_{2,\mathrm{j}}.$$

Combinations with non-empty ROC are

$$h[n] = [2^n - (1/3)^n]u(n), \quad \mathrm{ROC} = \{|z| > 2\},$$

which yields a causal but BIBO unstable system,

$$h[n] = -2^n u(-n-1) - (1/3)^n u(n), \quad \mathrm{ROC} = \{1/3 < |z| < 2\},$$

which yields a noncausal (not causal nor anti-causal) and BIBO stable system, and

$$h[n] = [(1/3)^n - 2^n]u(-n-1), \quad \mathrm{ROC} = \{|z| < 1/3\},$$

which yields an anti-causal BIBO unstable system.

See Exercise.

[1.4] If $\omega T = \pi q$, with $q \in \mathbb{Z}$, a can take any value because $\forall k \in \mathbb{Z}, y[k] = 0$.

Otherwise, consider the Z transform of the difference equation,

$$Y(z)\left(1 - az^{-1} + z^{-2}\right) = 0\,,$$

which allows us to think on the problem as designing a filter with impulse response

$$H(z) = 1 - az^{-1} + z^{-2}$$

that cancels the signal $Y(z)$.

An equivalent equation, then, is

$$z^2 - az + 1 = 0\,, \forall z \in \mathbb{C} \mid Y(z) \neq 0$$

which results in

$$z = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - 1} = \frac{a}{2} \pm j\sqrt{1 - \frac{a^2}{4}}\,.$$

Because our signal has a component only at frequency ω, we want, in particular, that $H(z)|_{z=e^{\pm j\omega}} = 0$. Thus,

$$z = e^{\pm j\omega T} = \cos(\omega T) \pm j\sin(\omega T) = \frac{a}{2} \pm j\sqrt{1 - \frac{a^2}{4}}\,,$$

which is fulfilled for $a = 2\cos(\omega T)$.

See Exercise.

The DFT and the FFT

[2.1] a) Recall that

$$x[n] = \begin{cases} 1 & \text{if } 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}.$$

Given this definition, we can use the provided tips to derive:

$$\begin{aligned} X(\nu) &= \sum_{n=-\infty}^{+\infty} x[n] e^{-j2\pi\nu n} && \{\text{by definition of } x[n]\} \\ &= \sum_{n=0}^{N-1} e^{-j2\pi\nu n} && \left\{\text{when } e^{-j2\pi\nu n} \neq 1 \Leftrightarrow \nu \notin \mathbb{Z}\right\} \\ &= \frac{1-e^{-j2\pi\nu N}}{1-e^{-j2\pi\nu}} = \frac{e^{-j\pi\nu N}}{e^{-j\pi\nu}} \frac{e^{j\pi\nu N} - e^{-j\pi\nu N}}{e^{j\pi\nu} - e^{-j\pi\nu}} && \left\{\text{using } \sin(x) = \frac{e^{jx}-e^{-jx}}{2j}\right\} \\ &= e^{-j\pi\nu(N-1)} \frac{\sin(\pi\nu N)}{\sin(\pi\nu)}. \end{aligned}$$

b) When $\nu \in \mathbb{Z}$, the exponential summands $e^{-j2\pi\nu n} = \cos(2\pi\nu n) - j\sin(2\pi\nu n) = 1$, and thus,

$$X(\nu) = \sum_{n=0}^{N-1} 1 = N, \forall \nu \in \mathbb{Z}.$$

c) The required DFT is defined for $k = 0, \ldots, N-1$ as

$$\begin{aligned} X_N[k] &= \sum_{n=0}^{N-1} x[n] e^{-j2\pi\frac{kn}{N}} && \{\text{using a) and b)}\} \\ &= X(\nu)\big|_{\nu=\frac{k}{N}} = \begin{cases} N & \text{if } k=0 \\ e^{-j\pi\frac{(N-1)k}{N}} \frac{\sin(\pi k)}{\sin(\frac{\pi k}{N})} = 0 & \text{if } k \neq 0 \end{cases}. \end{aligned}$$

This result may seem counter-intuitive, because we know that $x[n]$ actually varies in time. However, note that the N-point DFT only takes into account the behavior of the signal from sample 0 to sample $N-1$. Thus, as far as the N-point DFT is concerned, $x[n]$ is just a constant signal. Figure 2 shows both the time and frequency domain representations of $x[n]$ when $N=5$, as well as the 5- and 7-point DFTs of $x[n]$.

See Exercise.

[2.2] a) Recall that

$$X(\nu) = \begin{cases} 1 & \text{if } |\nu| \leq \nu_0 \\ 0 & \text{otherwise}, \end{cases}$$

when $\nu \in [-\frac{1}{2}, \frac{1}{2}]$. Then

$$\begin{aligned}
x[n] &= \int_0^1 X(\nu) e^{j2\pi\nu n} d\nu && \{\text{by periodicity of } X(\nu)\} \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} X(\nu) e^{j2\pi\nu n} d\nu && \{\text{by definition of } X(\nu)\} \\
&= \int_{-\nu_0}^{\nu_0} e^{j2\pi\nu n} d\nu && \{\text{by Euler's formula}\} \\
&= \int_{-\nu_0}^{\nu_0} (\cos(2\pi\nu n) + j\sin(2\pi\nu n))\, d\nu && \{\text{by (anti-)symmetry of (sin) cos}\} \\
&= 2\int_0^{\nu_0} \cos(2\pi\nu n) d\nu && \{\text{if } n \neq 0\} \\
&= \frac{1}{\pi n}\int_0^{\nu_0} 2\pi n \cos(2\pi\nu n) d\nu \\
&= \frac{[\sin(2\pi\nu n)]_0^{\nu_0}}{\pi n} = \frac{\sin(2\pi\nu_0 n)}{\pi n},
\end{aligned}$$

and

$$x[0] = 2\int_0^{\nu_0} \cos(2\pi\nu 0) d\nu = 2\int_0^{\nu_0} 1 d\nu = 2\nu_0\,.$$

b) It follows from the definition of $Y(\nu)$ that $Y(\nu) = cX(\nu)$. This imples $y[n] = cx[n]$, and then $y[0] = cx[0] = c2\nu_0 = 1 \Rightarrow c = \frac{1}{2\nu_0}$. Note here the explicit relation between the area of the Fourier transform $Y(\nu)$ and the first sample of $y[n]$, i.e. $y[0] = \int_0^1 Y(\nu)d\nu$. Observe that, in knowledge of this relation, the answer was obvious, since $Y(\nu)$ is a rectangle of width $2\nu_0$ and height c.

See Exercise.

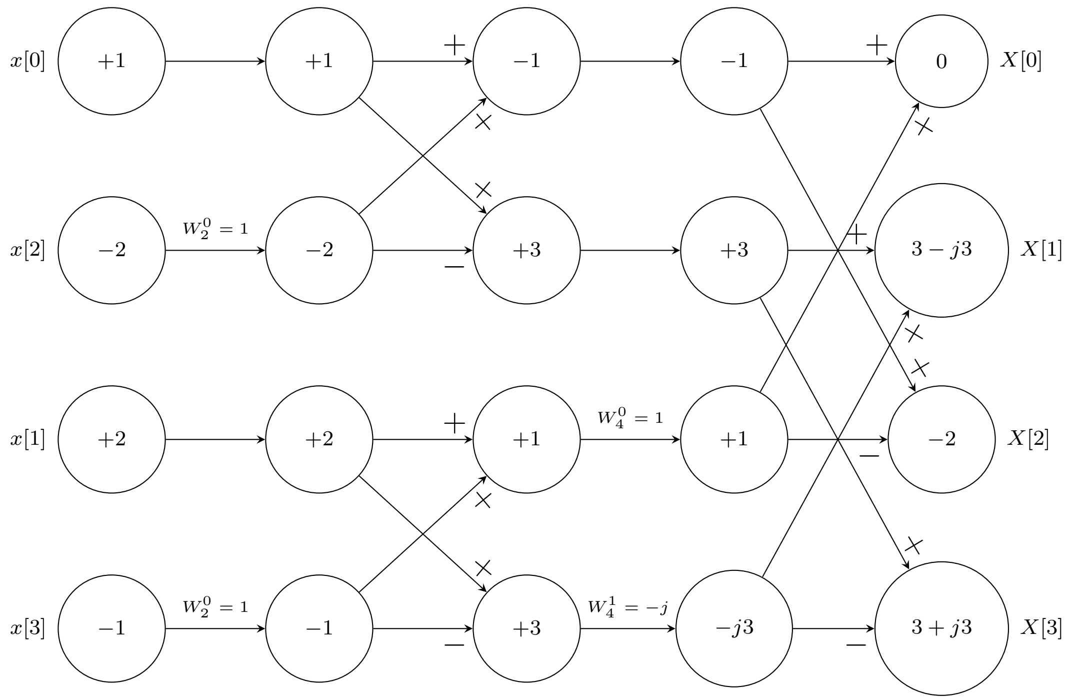
[2.3] a) Recall that $x[n] = \{\underset{\uparrow}{1}, 2, -2, -1\}$. Then,

$$X[k] = \sum_{n=0}^{3} x[n] e^{-j2\pi\frac{kn}{4}} \quad \text{for} \quad k = 0, \ldots, 3\,,$$

and the following table illustrates the steps to compute each $X[k]$.

Term	Expression	Expansion	Simplification	Result
$X[0]$	$\sum_{n=0}^{3} x[n]$	$1+2-2-1$	$1+2-2-1$	0
$X[1]$	$\sum_{n=0}^{3} x[n]e^{-j2\pi\frac{n}{4}}$	$1+2e^{-j\frac{\pi}{2}}-2e^{-j\pi}-e^{-j\frac{3\pi}{2}}$	$1-j2+2-j$	$3-j3$
$X[2]$	$\sum_{n=0}^{3} x[n]e^{-j2\pi\frac{2n}{4}}$	$1+2e^{-j\pi}-2e^{-j2\pi}-e^{-j3\pi}$	$1-2-2+1$	-2
$X[3]$	$\sum_{n=0}^{3} x[n]e^{-j2\pi\frac{3n}{4}}$	$1+2e^{-j\frac{3\pi}{2}}-2e^{-j3\pi}-e^{-j\frac{9\pi}{2}}$	$1+j2+2+j$	$3+j3$

b) Filling Fig. 2.1 from left to right leads to the following result.



We therefore conclude that both the direct computation in a) and the procedure in Fig. 2.1 lead to the same result.

c) For the direct computation of the 4-samples DFT, 16 complex-valued multiplications are needed, i.e. those of the samples $x[n], n = 0, \ldots, 3$ with 4 different complex unit-circle values. This agrees with the formula presented in class, $N^2 = 4^2 = 16$. For the computation using Fig. 2.1, only those links wich have multiplying values are counted, i.e. 4 links. This agrees with the theory, because $\frac{N\log_2(N)}{2} = \frac{4\log_2(4)}{2} = \frac{8}{2} = 4$.

d) The following function works in both Octave and MATLAB. It does a very basic check of the input data and proceeds to compute the 4-samples DFT by direct computation. The code is vectorized to improve performance.

```
function [ ret ] = DFT_4( x )
 % Checking dimensions, we want a column vector
  % Get dimensions
  check = size(x);
  % If it is not a column vector, make it one
  if check(2) ~= 1
   % Equivalently, x = x(:)
   x = reshape( x, prod(check), 1 );
  end
  % Truncate to 4 samples
   x = x( 1:4 );
 % Compute DFT 4
  % Create indices matrix
   k = [0;1;2;3]; n=k';
   args = k * n;
```

```
    % Generate Fourier matrix
     W = exp(-1i*2*pi*args/4);
    % Get DFT
     ret = W*x;
end
```

We can compare the two implementations in terms of how long do they take to compute the 4-samples DFT by using the code in the following listing.

```
Nit = 1000;
tdirect = zeros( 1, Nit);
tradix2 = zeros( 1, Nit );
for ii = 1:Nit
    a = rand( 4, 1 );
    tic; DFT_4( a ); tdirect(ii) = toc;
    tic; fft( a, 4 ); tradix2(ii) = toc;
end
mean(tdirect./tradix2)
```

This suggested, with MATLAB 2014b, that **fft** is 8 times faster than our implementation. Note, however, that this is not a comparison between the algorithms, but between the implementations. Even if the radix-2 algorithm is faster than direct computation, it does not account for the difference. In fact, the function **fft** from MATLAB or Octave is implemented in C/C++, which makes it much faster than anything we can implement.

See Exercise.

[2.4] For $k = 0, 1, \ldots, N-1$,

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi nk}{N}} & [n = N-1-l] \\ &= \sum_{l=0}^{N-1} x[N-1-l] e^{-j\frac{2\pi(N-1-l)k}{N}} = \sum_{l=0}^{N-1} y[l] e^{j\frac{2\pi lk}{N}} e^{j\frac{2\pi k}{N}} \\ &= e^{j\frac{2\pi k}{N}} \sum_{l=0}^{N-1} y[l] e^{-j\frac{2\pi l(-k)}{N}}, \end{aligned}$$

$Y[k]$ is defined only for $k = 0, 1, \ldots, N-1$. However, because the DTFT $Y(\nu)$ is periodic with period 1, evaluating the expression of $Y[k]$ on a k outside this range yields $Y[(-k)\mathrm{mod}(N)]$. Therefore, we conclude that, for $k = 0, 1, \ldots, N-1$, the DFTs of $y[n]$ and $x[n]$ are related through

$$X[k] = e^{j\frac{2\pi k}{N}} Y[(-k)\mathrm{mod}(N)] = \begin{cases} Y[0] & \text{for } k = 0 \\ e^{j\frac{2\pi k}{N}} Y[N-k] & \text{for } k = 1, ..., N-1 \end{cases}.$$

See Exercise.

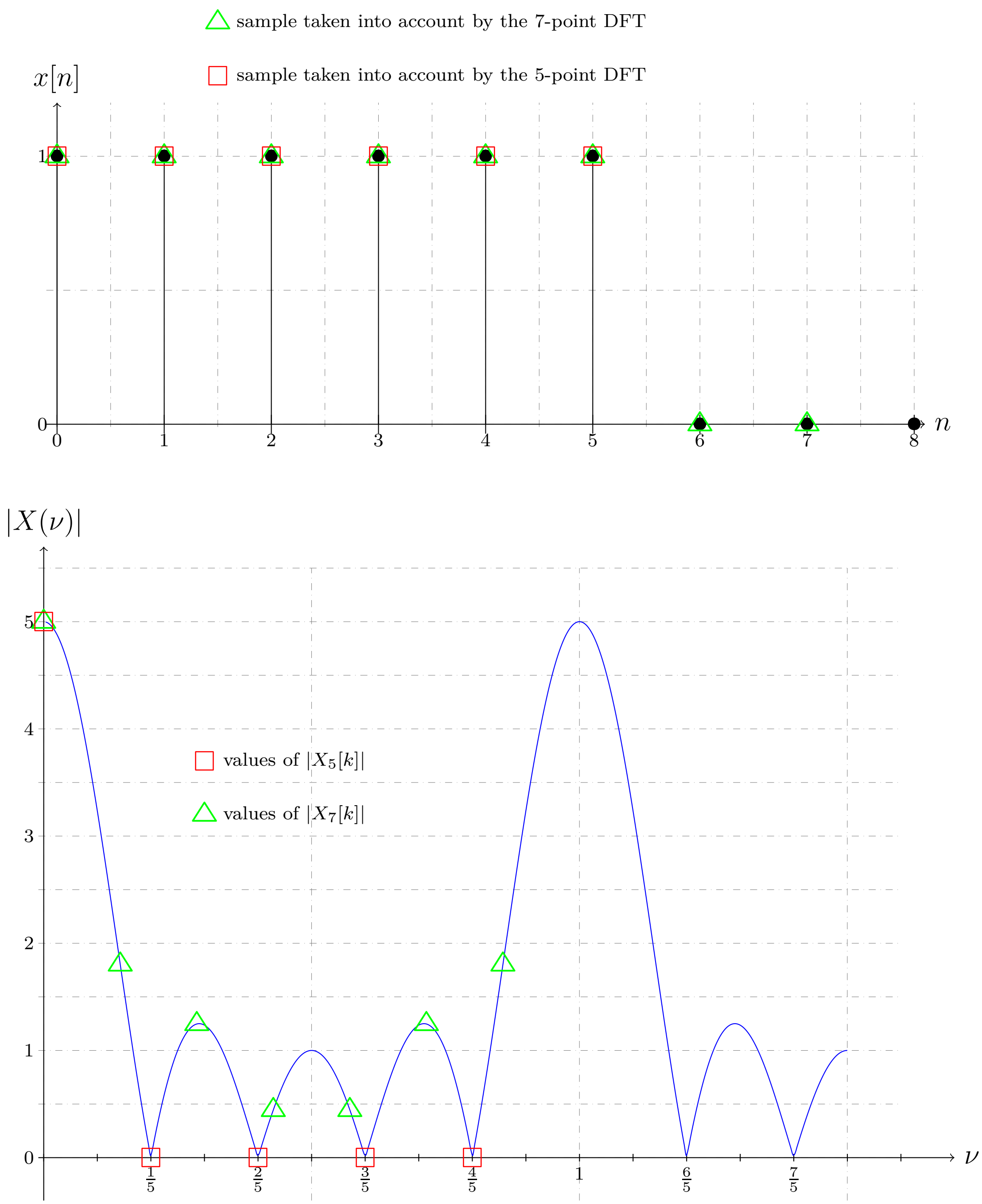


Figure 2: The upper part of the Figure shows the signal $x[n]$ defined in Exercise 2.1 in Chapter 2 when $N = 5$. Note that the 5-point DFT only takes into account the non-varying part of $x[n]$. The lower part of the Figure shows the DTFT $X(\nu)$ of $x[n]$, as well as the 5- and 7-point DFTs. Note that the 5-point DFT represents $x[n]$ as a 0 frequency signal.

Filtering with the FFT

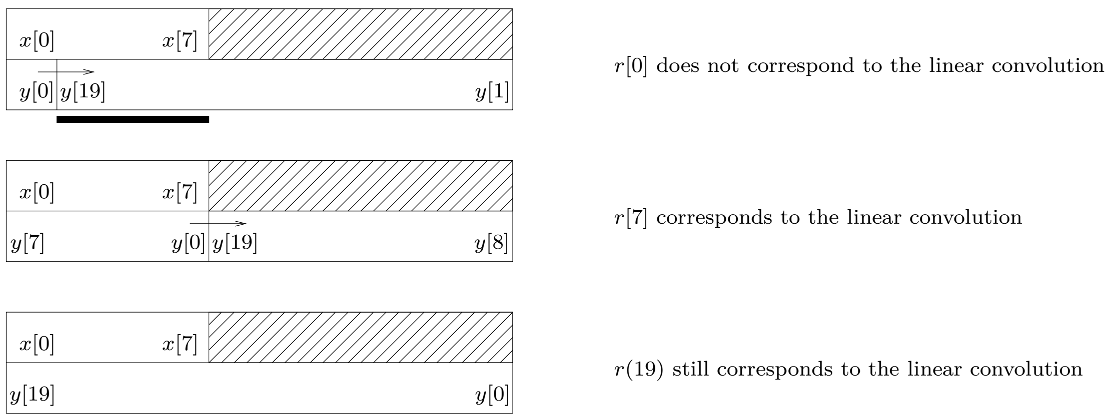
[3.1] Using the suggested methods, we get that

$$x_3[n] = (x_1 * x_2)[n] = \{4, 11, 20, 19, 13, 8, 2\},$$

and $x_4[n] = (x_1 \, \textcircled{4} \, x_2)[n] = \{17,\ 19,\ 22,\ 19\}$.

See Exercise.

[3.2] The figure below uses the graphical strategy to compute the convolution as a reference to provide a solution.



(a) The elements $r[n]$ for $n = 0, \ldots, 6$ will not correspond to the linear convolution, since they involve products that do not appear in the linear convolution, e.g. $y[19] * x[1]$ as a contribution to r[0]. The elements $r[n]$ for $n = 7, \ldots, 19$ will be precisely those found in the linear convolution, since they include all the same terms.

(b) From the figure, it is clear that the error is caused by the 7 last values of $y[n]$. This issue can be resolved by increasing the length of the sequences and the length of the DFTs to 27 by padding them with zeros.

See Exercise.

[3.3] (a) Direct filtering, 128 real multiplications/sample, one per filter coefficient.

(b) Overlap-save method with 256-point FFT,

$$4\,\frac{2 \cdot 8 \cdot \frac{256}{2} + 256}{129} \approx 71 \text{ real multiplications/sample}\,.$$

(c) Overlap-save method with 512-point FFT,

$$4\,\frac{2 \cdot 9 \cdot \frac{512}{2} + 512}{385} \approx 53 \text{ real multiplications/sample}\,.$$

See Exercise.

FIR Filters and FIR Approximations

[4.1] (a)

- Rectangular window
By direct definition and using Euler's identity and the geometric sum, we obtain

$$\begin{aligned} W_R(\nu) &= \sum_{m=-\infty}^{\infty} w_r[m]e^{-j2\pi\nu m} = \sum_{m=-M}^{M} e^{-j2\pi\nu m} = e^{j2\pi\nu M}\sum_{n=0}^{2M} e^{-j2\pi\nu m} \\ &= e^{j2\pi\nu M}\frac{1-e^{-j2\pi\nu(2M+1)}}{1-e^{-j2\pi\nu}} = \frac{e^{j\pi\nu 2M}}{e^{-j\pi\nu}}\frac{1-e^{-j2\pi\nu(2M+1)}}{e^{j\pi\nu}-e^{-j\pi\nu}} \\ &= e^{j\pi\nu(2M+1)}\frac{1-e^{-j2\pi\nu(2M+1)}}{e^{j\pi\nu}-e^{-j\pi\nu}} = \frac{e^{j\pi\nu(2M+1)}-e^{-j\pi\nu(2M+1)}}{e^{j\pi\nu}-e^{-j\pi\nu}} \\ &= \frac{\sin(\pi\nu(2M+1))}{\sin(\pi\nu)}, \end{aligned}$$

for $\nu \neq 0$. For $\nu = 0$, $e^{-j2\pi\nu m} = 1$ and $W_R(0) = 2M+1$. Note here that the term $2M+1$ corresponds to the length of the window.

- Barlett window
The Barlett or triangular window can be seen as the convolution of two rectangular windows. In particular, if $M \geq 3$ and is odd, we can build the triangular window with parameter M as the convolution of two rectangular windows from the previous point with parameter $M' = (M-1)/2$ with a scaling factor, i.e.

$$w_{B,M}[m] = \frac{1}{M}(w_{R,M'} * w_{R,M'})[m]\,. \tag{1}$$

This can be proved by using the definition of convolution, and is clear for small examples using the graphical convolution, e.g. for $M = 3$ and $M' = 1$. The length of the rectangular windows with parameter M' is $2M'+1 = M$. Therefore, the highest of their convolution will be M, so the division by M is the correct scaling to match w_B. By the convolution theorem, convolution in temporal domain corresponds to multiplication in the frequency domain, so

$$W_B(\nu) = \frac{1}{M}\frac{\sin^2(\pi\nu M))}{\sin^2(\pi\nu)}\,.$$

This transform is valid even when our assumptions do not hold, e.g. $M < 3$ or is even, but its proof is much longer.

- Raised cosine window

Since $\cos(x) = (e^{jx} + e^{-jx})/2$, we get

$$\begin{aligned}W_H(\nu) =&\alpha\frac{e^{j\pi\nu(2M+1)} - e^{-j\pi\nu(2M+1)}}{e^{j\pi\nu} - e^{-j\pi\nu}} + \frac{\beta}{2}\frac{e^{j\pi(\nu(2M+1)-1)} - e^{-j\pi(\nu(2M+1)-1)}}{e^{j\pi(\nu-1/(2M+1))} - e^{-j\pi(\nu-1/(2M+1))}}\\ &+ \frac{\beta}{2}\frac{e^{j\pi(\nu(2M+1)+1)} - e^{-j\pi(\nu(2M+1)+1)}}{e^{j\pi(\nu+\frac{1}{2M+1}} - e^{-j\pi(\nu+\frac{1}{2M+1})}}\\ =&\alpha\frac{\sin(\pi\nu(2M+1))}{\sin(\pi\nu)} + \frac{\beta}{2}\frac{\sin(\pi(\nu(2M+1)-1))}{\sin(\pi(\nu-\frac{1}{2M+1}))}\\ &+ \frac{\beta}{2}\frac{\sin(\pi(\nu(2M+1)+1))}{\sin(\pi(\nu+\frac{1}{2M+1}))}\end{aligned} \quad (2)$$

(b)
- Rectangular window
  $W_R(\nu)$ is zero when $\nu = k/(2M+1)$ for all integers k except for $k = 0$. This makes the first positive ν with $W_R(\nu) = 0$ be $\nu = 1/(2M+1)$.
- Barlett window
  The first zero with positive ν is $\nu = 1/M$.
- Raised cosine window
  $W_H(\nu)$ is zero for $\nu = k/(2M+1)$ except for three cases: $k = 0$ where the first term in (2) is non-zero; $k = 1$, where the second term in (2)is non-zero; and $k = -1$, where the third term in (2) is non-zero. Thus, the first positive ν that makes it zero is $\nu = 2/(2M+1)$.

See Exercise.

[4.2] a) The magnitude response of the filter will be the one shown in Fig. 3, which corresponds to

$$h[n] = h_{\mathrm{I}}[n]w[n] \quad \Leftrightarrow \quad H(\nu) = H_{\mathrm{I}}(\nu) * W(\nu) = \int_0^1 H_{\mathrm{I}}(\nu-\xi)W(\xi)d\xi\,.$$

If the filter was made longer the width of the main lobe of the window would become narrower and the transition from pass-band to stop-band would be more rapid.

b) The designed filter will be a linear phase FIR filter with a group delay of $\tau = (41-1)/2 = 20$. As ν_0 is in the middle of the passband the amplitude would not be seriously affected, and the output would be $y[n] \approx \sin(2\pi\nu_0(n-20)+\phi_0)$.

c) Using the complexity per sample formula for overlap-add, we get that C_N (the complexity when using a length N FFT satisfies $C_{64} \approx 18.8$, $C_{128} \approx 11.6$, $C_{256} \approx 10.7$ and $C_{512} \approx 10.8$. It would thus make sense to pick $N = 256$ as this yields the lowest complexity. Note also that we only consider N on the form $N = 2^p$ as this is required for the basic radix-2 FFT algorithm.

See Exercise.

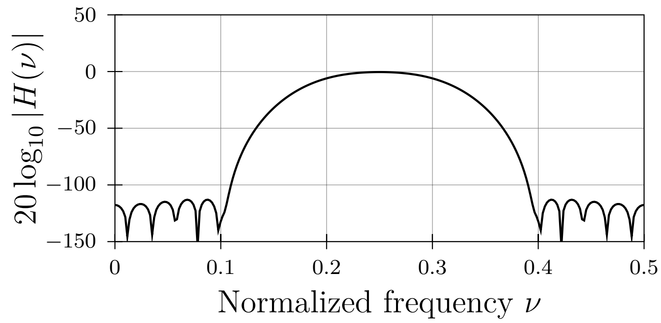


Figure 3: Solution to the first question in Exercise 4.2. Magnitude response of the designed filter.

Quantization and Finite-Precision

[5.1] (a) By assumption, we have that

$$f_{E[n]}(e[n]) = \begin{cases} \frac{1}{\Delta} & \text{for } e[n] \in \left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right] \\ 0 & \text{otherwise} \end{cases} .$$

Then, we have that

$$m_e = \mathrm{E}\left\{e[n]\right\} = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \frac{1}{\Delta} e[n] de[n] = 0\,,$$

$$\sigma_e^2 = \mathrm{E}\left\{e^2[n]\right\} = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \frac{1}{\Delta} e^2[n] de[n] = \frac{1}{\Delta}\left[\frac{e^3[n]}{3}\right]_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} = \frac{\Delta^2}{12}\,.$$

Additionally, $e[n]$ is also white by assumption, and thus,

$$r_e[k] = \mathrm{E}\left\{e[n]e[n-k]\right\} = \frac{\Delta^2}{12}\delta[k]\,.$$

(b)

$$\mathrm{SNR}_{y[n]} = \frac{\sigma_x^2}{\sigma_e^2} = 12\frac{\sigma_x^2}{\Delta^2}\,.$$

(c) Because both $x[n]$ is white, we have that

$$\begin{aligned}
\mathrm{E}\left\{((h*x)[n])^2\right\} &= \mathrm{E}\left\{\left(\sum_{m=0}^{\infty} h[m]x[n-m]\right)^2\right\} \\
&= \mathrm{E}\left\{\left(\sum_{m=0}^{\infty} h[m]x[n-m]\right)\left(\sum_{k=0}^{\infty} h[k]x[n-k]\right)\right\} \\
&= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} h[m]h[k]\,\mathrm{E}\left\{x[n-m]x[n-k]\right\} \\
&= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} h[m]h[k]\,\mathrm{E}\left\{x^2[n]\right\}\delta[k-m] \\
&= \sum_{m=0}^{+\infty} h^2[m]\,\mathrm{E}\left\{x^2[n]\right\},
\end{aligned}$$

where we used the autocorrelation of the white input signal

$$\mathrm{E}\left\{x[n-m]x[n-k]\right\} = \mathrm{E}\left\{x[n]x[n-k+m]\right\} = r_x[k-m] = \sigma_x^2\delta[k-m]\,.$$

Similarly, because $e[n]$ is white,

$$\mathrm{E}\left\{((h*e)[n])^2\right\} = \mathrm{E}\left\{e^2[n]\right\}\sum_{n=0}^{+\infty} h^2[n]\,.$$

Therefore,

$$\mathrm{SNR}_{\text{out}} = \frac{\mathrm{E}\left\{((h*x)[n])^2\right\}}{\mathrm{E}\left\{((h*e)[n])^2\right\}} = \mathrm{SNR}_{y[n]} = 12\frac{\sigma_x^2}{\Delta^2}\,.$$

Finally, the variance of the noise at the output of the filter will be

$$\mathrm{E}\left\{((h*e)[n])^2\right\} = \mathrm{E}\left\{e^2[n]\right\}\sum_{n=0}^{+\infty} h^2[n] = \frac{\Delta^2}{12}\sum_{n=0}^{+\infty} h^2[n]$$

with

$$\begin{aligned}
\sum_{n=0}^{+\infty} h^2[n] &= \frac{1}{4}\sum_{n=0}^{+\infty}\left(a^n + (-a)^n\right)^2 \\
&= \frac{1}{4}\sum_{m=0}^{+\infty}\left(a^{2m} + (-a)^{2m}\right)^2 + \frac{1}{4}\sum_{k=0}^{+\infty}\left(a^{2k+1} + (-a)^{2k+1}\right)^2 \\
&= \frac{1}{4}\sum_{m=0}^{+\infty} 4(a^4)^m = \frac{1}{1-a^4}\,.
\end{aligned}$$

See Exercise.

[5.2] Let us first analyze the performance with respect to the quantization noise of the first realization of the system, i.e. the one shown in Fig. 5.2.I. Let us model the quantization noise as a white, uniform, zero-mean, additive noise at the output of each multiplier. These noises are also assumed independent of each other. Note $e_1[n]$ the noise after the 0.5 multiplier, $e_2[n]$ the noise after the 0.3 multiplier, and $e_3[n]$ the noise after the 0.2 multiplier. Note σ_e^2 the power of each of these noises, which is the same, since this power depends only on the considered quantization. Note $\sigma_{y_{e_i}}^2$ the contributions of each of the noises to the noise at the output, such that the total noise power at the output can be written as $\sigma_{y_e}^2 = \sum_{i=1}^{3} \sigma_{y_{e_i}}^2$. It is clear that both $e_2[n]$ and $e_3[n]$ make their way directly into the output. $e_1[n]$, on the other hand, goes through the whole system and then reaches the output. Thus, $\sigma_{y_{e_2}}^2 = \sigma_{y_{e_3}}^2 = \sigma_e^2$, but

$$\sigma_{y_{e_1}}^2 = \mathrm{E}\left\{((e_1 * h)[n])^2\right\} .$$

Through the transfer function of the system and the inverse Z-transform, it is easy to obtain that $h[n] = 0.3\,2^{-n}u[n] + 0.2\,2^{-n+1}u[n-1]$. Thus, because $e_1[n]$ is white,

$$\begin{aligned}
\sigma_{y_{e_1}}^2 &= \sigma_e^2 \sum_{n=-\infty}^{+\infty} h^2[n] = \sigma_e^2 \left[0.3^2 + \sum_{n=1}^{+\infty} \left(0.3\,2^{-n} + 0.2\,2^{-n+1}\right)^2\right] \\
&= \sigma_e^2 \left[0.3^2 + \sum_{n=1}^{+\infty} \left(\frac{3}{10}\frac{1}{2^n} + \frac{2}{10}\frac{2}{2^n}\right)^2\right] = \sigma_e^2 \left[0.3^2 + \left(\frac{7}{10}\right)^2 \sum_{n=1}^{+\infty} \frac{1}{4^n}\right] \\
&= \sigma_e^2 \left[0.3^2 + \frac{49}{100}\frac{\frac{1}{4}}{1-\frac{1}{4}}\right] = \sigma_e^2 \left[0.3^2 + \frac{49}{300}\right] \approx 0.2533\sigma_e^2 .
\end{aligned}$$

Therefore, for the realization of the system presented in Fig. 5.2.I, the total quantization noise at the output is

$$\sigma_{\mathrm{I}}^2 = \sigma_{y_e}^2 = \sum_{i=1}^{3} \sigma_{y_{e_i}}^2 = (1 + 1 + 0.2533)\sigma_e^2 = 2.2533\sigma_e^2 .$$

Now, then, let us evaluate the quantization noise power at the output of the second realization of the system, i.e. the one shown in Fig. 5.2.II. We will use the same naming convention, i.e. we will note $e_1[n]$ the noise after the 0.5 multiplier, $e_2[n]$ the noise after the 0.3 multiplier, and $e_3[n]$ the noise after the 0.2 multiplier. An anaysis of the system deriving its transfer function and inverse Z-transforming will reveal that the system has exactly the same impulse response $h[n]$. However, the entry points of the quantization noises are essentially different. In this case, $e_2[n]$ only has to go through the AR part of the circuit, or, in other terms,

$$Y_{e_2}(z) = E_2(z) + 0.5z^{-1}Y_{e_2}(z) \Rightarrow H_2(z) = \frac{Y_{e_2}(z)}{E_2(z)} = \frac{1}{1-0.5z^{-1}} .$$

Therefore, the impulse response of the system that $e_2[n]$ goes through before reaching the output is $h_2[n] = 2^{-n}u[n]$ (obtained by inverse Z-transform). Because the input point of the noises $e_1[n]$ and $e_3[n]$ is only one delay away from the entry point for $e_2[n]$, it is not surprising that

$$H_1(z) = H_3(z) = z^{-1}H_2(z) = \frac{z^{-1}}{1-0.5z^{-1}}$$

which yields $h_1[n] = h_3[n] = h_2[n-1] = 2^{-n+1}u[n-1]$.

For all three quantization noises, we have that, because they are white,

$$\sigma^2_{y_{e_i}} = \mathrm{E}\left\{((e_i * h)[n])^2\right\} = \sigma_e^2 \sum_{n=-\infty}^{+\infty} h_i^2[n]\,.$$

In fact, because their impulse responses are the same except for a one-sample delay, and we are summing the whole infinite impulse response,

$$\sum_{n=-\infty}^{+\infty} h_1^2[n] = \sum_{n=-\infty}^{+\infty} h_2^2[n] = \sum_{n=-\infty}^{+\infty} h_3^2[n]\,,$$

and

$$\sum_{n=-\infty}^{+\infty} h_2^2[n] = \sum_{n=0}^{+\infty} \frac{1}{2^{2n}} = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}\,.$$

Therefore, for the realization of the system presented in Fig. 5.2.II, the total quantization noise at the output is

$$\sigma^2_{\mathrm{II}} = \sigma^2_{y_e} = \sum_{i=1}^{3} \sigma^2_{y_{e_i}} = \left(\frac{4}{3}+\frac{4}{3}+\frac{4}{3}\right)\sigma_e^2 = 4\sigma_e^2 > \sigma^2_{\mathrm{I}}\,.$$

See Exercise.

[5.3] Let us refer to the uniform, white, additive and independent quantization noises at the output of each multiplier as $e_i[n], i = 1, 2, 3, 4$, respectively, from left to right in Fig. 5.3. Both $e_1[n]$ and $e_4[n]$ will not be affected by the choice of parameters, because the former will always travel through the whole transfer function, while the latter will always go directly to the output. $e_2[n]$ and $e_3[n]$, on the other hand, will both travel through the second stage of the system before getting to the output. Therefore, we will have to select a_2 and b_2 in order to make the second stage as resistant to quantization noise at its input as possible.

Analyzing the overall transfer function results in identifying that the possible choices of the parameters that will yield the desired system are

a_1	b_1	a_2	b_2
0.25	0.25	0.5	0.5
0.5	0.25	0.25	0.5
0.25	0.5	0.5	0.25
0.5	0.5	0.25	0.25

The transfer function of the second stage is

$$H_2(z) = \frac{1 + b_2 z^{-1}}{1 - a_2 z^{-1}} = \frac{z}{z - a_2} + b_2 z^{-1} \frac{z}{z - a_2}\,.$$

Because we $e_2[n]$ and $e_3[n]$, the power of their contribution to the output will be

$$\sigma^2_{y_{e_2}} = \sigma^2_{y_{e_3}} = \sigma_e^2 \sum_{n=-\infty}^{\infty} h_2^2[n]\,,$$

where σ_e^2 is a constant value dependent only on the specific quantization used.

From the expression of the Z-transform above, it is immediate to extract that $h_2[n] = a_2^n u[n] + b_2 a_2^{n-1} u[n-1]$. Thus,

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} h_2^2[n] &= 1 + \sum_{n=1}^{+\infty} \left(a_2^n + b_2 a_2^{n-1}\right)^2 \\
&= 1 + \sum_{n=1}^{+\infty} \left[(a_2 + b_2) a_2^{n-1}\right]^2 \\
&= 1 + (a_2 + b_2)^2 \sum_{n=1}^{+\infty} a_2^{2n-2} = 1 + (a_2 + b_2)^2 \sum_{q=0}^{+\infty} a_2^{2q} \\
&= 1 + \frac{(a_2 + b_2)^2}{1 - a_2^2}\,.
\end{aligned}$$

Then, the noise components at the output from $e_2[n]$ and $e_3[n]$ will be scaled by the following factors, for the following combinations of parameters

a_1	b_1	a_2	b_2	$\sum_{n=-\infty}^{\infty} h_2^2[n]$
0.25	0.25	0.5	0.5	2.33
0.5	0.25	0.25	0.5	1.60
0.25	0.5	0.5	0.25	1.75
0.5	0.5	0.25	0.25	1.27

Therefore, in terms of quantization noise, the optimal parameters that implement the desired transfer function are $a_1 = 0.5$, $b_1 = 0.5$, $a_2 = 0.25$ and $b_2 = 0.25$.

See Exercise.

Fixed-Point Filter Implementation

[6.1] (a) We have that

$$\left.\frac{\partial A(z)}{\partial \alpha_k}\right|_{z=p_i} = \left.\frac{\partial A(z)}{\partial z} \frac{\partial z}{\partial \alpha_k}\right|_{z=p_i},$$

which yields

$$\frac{\partial p_i}{\partial \alpha_k} = \frac{\left.\frac{\partial A(z)}{\partial \alpha_k}\right|_{z=p_i}}{\left.\frac{\partial A(z)}{\partial z}\right|_{z=p_i}} .$$

All we need basically is to compute the partial derivatives in the numerator and the denominator of this expession.

The numerator expands to

$$\frac{\partial A(z)}{\partial \alpha_k} = \partial \left\{ 1 + \sum_{l=1, l\neq k}^{N} \alpha_l z^{-l} + \alpha_k z^{-k} \right\} \Big/ \partial \alpha_k = z^{-k} ,$$

which is easily evaluated to

$$\left.\frac{\partial A(z)}{\partial \alpha_k}\right|_{z=p_i} = p_i^{-k} .$$

The denominator expands to

$$\begin{aligned}\frac{\partial A(z)}{\partial z} &= \partial \left\{ \prod_{k=1}^{N} (1 - p_k z^{-1}) \right\} \Big/ \partial z \\ &= \sum_{k=1}^{N} p_k z^{-2} \prod_{j=1, j\neq k}^{N} (1 - p_j z^{-1}) ,\end{aligned}$$

which yields

$$\begin{aligned}\left.\frac{\partial A(z)}{\partial z}\right|_{z=p_i} &= \sum_{k=1}^{N} p_k p_i^{-2} \prod_{j=1, j\neq k}^{N} (1 - p_j p_i^{-1}) \\ &= p_i^{-1} \prod_{j=1, j\neq i}^{N} (1 - p_j p_i^{-1}) \\ &= p_i^{-1} p_i^{-N+1} \prod_{j=1, j\neq i}^{N} (p_i - p_j) = p_i^{-N} \prod_{j=1, j\neq i}^{N} (p_i - p_j)\end{aligned}$$

Hence, the sensitivity of the pole p_i with respect to the change in α_k is given by the formula

$$\frac{\partial p_i}{\partial \alpha_k} = \frac{p_i^{N-k}}{\prod_{j=1, j\neq i}^{N} (p_i - p_j)} \tag{3}$$

(b) The take-home message can be seen directly from (3). As the poles get closer together, the sensitivity increases greatly. So as the filter order increases and more poles get stuffed closer together inside the unit circle (assuming causality), the error introduced by coefficient quantization in the pole locations will grow.

(c) We can reduce the sensitivity by using cascade- or parallel-form implementations. The numerator and denominator polynomials can be factored off-line at very high precision and grouped into second-order sections, which are then quantized section by section. The sensitivity of the quantization is thus that of second-order, rather than N-th order, polynomials. This yields major improvements in the frequency response of the overall filter, and is almost always done in practice.

See Exercise.

[6.2] (a) We see that $h_{\text{direct}}[n]$ has triangular shape, which can be obtained as a convolution of two rectangular pulses. Thus, $h_{\text{direct}}[n] = h_1[n] * h_2[n]$ with $h_1[n] = h_2[n] = \frac{1}{\sqrt{3}}[1, 1, 1]$.

(b) We have $H_1(z) = \frac{1}{\sqrt{3}}\left(1 + z^{-1} + z^{-2}\right)$. The zeros are at

$$z_1 = \frac{-1}{2} + \frac{j\sqrt{3}}{2}, \qquad z_2 = \frac{-1}{2} - \frac{j\sqrt{3}}{2}.$$

With $z_{1/2} = e^{j2\pi\nu_{1/2}}$ we get $\nu_1 = \frac{1}{3}$ and $\nu_2 = -\frac{1}{3}$. Thus, a sinusoid signal $x(t) = A \cdot \sin\left(2\pi\frac{1}{3}\right)$ will be filtered out.

(c) We have $H_{\text{direct}}(z) = H_{\text{cascade}}(z) = H_1(z)H_2(z)$. Thus, $H_{\text{direct}}(z)$ has a double zero at

$$z_{1/3} = \frac{-1}{2} + \frac{j\sqrt{3}}{2}$$

and a double zero at

$$z_{2/4} = \frac{-1}{2} - \frac{j\sqrt{3}}{2}.$$

(d) All zeros of $H_{\text{direct}}(z)$ are double zeros. Therefore, the partial derivative

$$\frac{\partial z_i}{\partial b_k}$$

is infinite, i.e. it is not differentiable. This indicates that the filter is very sensitive to a small change in the coefficient. Note that the displacement of the zero will not be infinite, as this was only a first order analysis.

(e) The two zeros of $H_1(z)$ are far apart, and the derivative is therefore finite. The filter $H_1(z)$ is thus not extremely sensitive to a small perturbation in the coefficients.

(f) We expect that the zeros in the direct implementation will be heavily displaced (due to the infinite derivative), and the zeros in the cascade implementation will only be slightly displaced. Implementation A (dashed lines) therefore corresponds to the cascade implementation, B (thin solid lines) to the direct implementation.

See Exercise.

Nonparametric Spectral Estimation

The Periodogram and the Modified Periodogram

[7.1] This exercise is more challenging than it appears. We have to use our theoretical knowledge in this applied setting to estimate the given quantities. Consider first that the model on the sampled signal is $x[n] = s[n] + e[n]$ where $s[n] = A\sin(\omega_0 n)$, where $\omega_0 = 2\pi F_0/F_s$, and $e[n]$ is the noise.

From the plot, it can be concluded that the noise is white, which was not stated in the headlines. We can deduce this because we know that the spectrum of $s[n]$ has a single peak in the range $\omega \in [0, \pi]$, and thus, the remaining spectrum in Figure 7.1, must correspond to the noise. A flat power spectral density implies a delta correlation function, and thus, the noise is white.

(a) The peak in the periodogram, as we mentioned, comes from the signal term $s[n]$. In particular, we have a peak at $\omega = \omega_0$, which in Figure 7.1 can recognise as the peak is at $\omega_0 = 0.377\,\text{rad/sample}$, which gives an estimate $\hat{F}_0 = \frac{\omega_0}{2\pi}F_s = 60\,\text{Hz}$.

(b) At the location of the peak, i.e. $\omega = \omega_0$, all the energy of $s[n]$ is concentrated, so the contribution from $e[n]$ can be neglected. To see this, recall that when observing the logarithmic transformation of a large number, small variations are compressed, so they can be disregarded. Therefore,

$$
\begin{aligned}
P(\omega_0) \approx P_s(\omega_0) &= \frac{1}{N}\left|\sum_{n=0}^{N-1} A\sin(\omega_0 n)\, e^{-j\omega_0 n}\right|^2 \\
&= \frac{A^2}{N}\left|\frac{1}{2j}\sum_{n=0}^{N-1}\left(e^{j\omega_0 n} - e^{-j\omega_0 n}\right)e^{-j\omega_0 n}\right|^2 \\
&= \frac{A^2}{4N}\left|\sum_{n=0}^{N-1}\left(1 - e^{-j2\omega_0 n}\right)\right|^2 = \frac{A^2}{4N}\left|N - \sum_{n=0}^{N-1} e^{-j2\omega_0 n}\right|^2 \\
&= \frac{A^2}{4N}\left|N - \frac{e^{-j2\omega_0 N} - 1}{e^{-j2\omega_0} - 1}\right|^2 \approx \frac{A^2 N}{4}\,.
\end{aligned}
$$

Here, we used the definition of the periodogram of a signa, Euler's identity for $\sin(x)$, the geometric sum formula, and that, because $\omega_0 = 0.377\,\text{rad/sample}$ and $N = 10000$,

$$
\left|\frac{e^{-j2\omega_0 N} - 1}{e^{-j2\omega_0} - 1}\right| \approx |0.23 + 0.07j| \approx 0.24 << N\,.
$$

Then, we have that $10^{3.7} = P(\omega_0) = A^2N/4$, which yields

$$
\hat{A} = \sqrt{\frac{4\cdot 10^{3.7}}{N}} = 1.4159\,.
$$

Note, as a curiosity, that the true value used for generating these data was $A = \sqrt{2} = 1.4142$.

(c) In order to know the noise power, we need to somehow link it to the remaining quantity provided by Figure 7.1. We are given the average value of the periodogram, i.e.

$$\frac{1}{N}\sum_{k=0}^{N-1} P(\omega_k)\,.$$

To relate this average to our unknown variable σ^2, we can consider the contribution of the noise $e[n]$ to the periodogram, i.e.,

$$P_e(\omega) = \frac{1}{N}\left|\sum_{n=0}^{N-1} e[n]e^{-j\omega n}\right|^2 .$$

However, this quantity is random, so there is no direct way to compare it to our measurement. However, considering its expected value $\mathrm{E}[P_e(\omega)]$ instead will provide the link between our model and the measurement, and yield an estimate for σ^2. Indeed,

$$\begin{aligned}
\mathrm{E}[P_e(\omega)] &= \frac{1}{N}\,\mathrm{E}\left[\left|\sum_{n=0}^{N-1} e[n]e^{-j\omega n}\right|^2\right] \\
&= \frac{1}{N}\,\mathrm{E}\left[\left(\sum_{n=0}^{N-1} e[n]e^{-j\omega n}\right)\left(\sum_{m=0}^{N-1} e[m]e^{-j\omega m}\right)^*\right] \\
&= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} \mathrm{E}\left\{e[n]e[m]\right\} e^{-j\omega(n-m)} = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} \sigma^2\delta[n-m]e^{-j\omega(n-m)} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} \sigma^2 = \sigma^2 .
\end{aligned}$$

Here, we expand the squared absolute value as the product of a complex by its conjugate, we use the linearity of the expectation so that only random quantities remain inside it, and then, we use the fact that $e[n]$ is a white noise process to finalize the result.

Because our sample size is large, the division of the space $\omega \in [0, 2\pi]$ is very fine, and thus, we can assume that there are $k_1, k_2 \in \{0, 1, \ldots, N-1\}$ such that $\omega_{k_1} = \omega_0$ and $\omega_{k_2} = 2\pi - \omega_0$. Then, we have that

$$\begin{aligned}
\frac{1}{N}\sum_{k=0}^{N-1} \mathrm{E}\left[P(\omega_k)\right] &= \frac{1}{N}\left(\sum_{k=0}^{N-1} \mathrm{E}\left[P_e(\omega_k)\right] + \sum_{q=0}^{N-1} P_s(\omega_q)\right) \\
&= \frac{1}{N}\left(\sum_{k=0}^{N-1} \sigma^2 + 2\frac{A^2 N}{4}\right) = \sigma^2 + \frac{A^2}{2} .
\end{aligned}$$

Here, we have first used the independence between the signal and the noise, then used the previous result for $P_s(\omega_0)$ and $P_s(2\pi - \omega_0)$, as well as the knowledge that $P_s(\omega) = 0$ for all $\omega \neq \omega_0, 2\pi - \omega_0$ in $[0, 2\pi]$, and finally incorporated the result for the expected value of $P_e(\omega)$.

This final expression, then, yields the estimate $\hat{\sigma^2} = 10^{0.3} - \frac{A^2}{2} \approx 1$.

See Exercise.

[7.2] Very basic properties of the time-frequency domains relationship have to be recalled,

- a wider (time) window will provoke a narrower (frequency) main lobe,
- a thinner (time) window will provoke a wider (frequency) main lobe.

Additionally, a basic characteritzation of the rectangular and triangular windows will say that,

- a rectangular window has larger (power) side lobes,
- a triangular window has smaller (power) side lobes.

Using these properties, we have that: A $\leftrightarrow$ II, B $\leftrightarrow$ IV, C $\leftrightarrow$ I, and D $\leftrightarrow$ III.

See Exercise.

[7.3] Recall that

$$Y(\nu) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} y[n] e^{-j2\pi\nu n} .$$

Then:

(a) Because $y[n]$ is white with correlation function $r_y[k] = \sigma^2 \delta[k]$,

$$\begin{aligned}
\mathrm{E}\left\{Y(\nu_1)Y^*(\nu_2)\right\} &= \frac{1}{N}\,\mathrm{E}\left\{\left(\sum_{n=0}^{N-1} y[n]e^{-j2\pi\nu_1 n}\right)\left(\sum_{m=0}^{N-1} y[m]e^{-j2\pi\nu_2 m}\right)^*\right\} \\
&= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} \mathrm{E}\left\{y[n]y^*[m]\right\} e^{-j2\pi\nu_1 n} e^{+j2\pi\nu_2 m} \\
&= \frac{\sigma^2}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} \delta[n-m] e^{-j2\pi\nu_1 n} e^{+j2\pi\nu_2 m} \\
&= \frac{\sigma^2}{N}\sum_{n=0}^{N-1} e^{-j2\pi(\nu_1-\nu_2)n} .
\end{aligned}$$

(b) We observe that $\hat{P}_{yy}(\nu) = |Y(\nu)|^2 = Y(\nu)Y^*(\nu)$, and, thus,

$$\begin{aligned}
\mathrm{Bias}\left(\hat{P}_{yy}(\nu)\right) &= \mathrm{E}\left\{Y(\nu)Y^*(\nu)\right\} - P_{yy}(\nu) \\
&= \frac{\sigma^2}{N}\sum_{n=0}^{N-1} 1 - \sigma^2 = 0 .
\end{aligned}$$

See Exercise.

Nonparametric Spectral Estimation

Barlett, Welch and Blackman-Tukey

[8.1] (a) In this problem, the triangular or Bartlett window is used in Welch's method. Therefore, if we define *sufficient* resolution as the $3\,dB$ bandwidth being thinner than the frequency difference ΔF we want to distinguish, we have,

$$\frac{1.28}{M} < \frac{\Delta F}{F_s} \Rightarrow M > 1.28\frac{F_s}{\Delta F} = 1.28\frac{12000}{200} \approx 77.$$

(b) For Welch's method with a Bartlett window, the variance is, approximately,

$$\mathrm{Var}[\hat{P}_W(w)] \approx \begin{cases} \frac{1}{K}P_x^2(w) & \text{with no overlap} \\ \frac{9}{8K}P_x^2(w) & \text{with 50\% overlap} \end{cases},$$

where $P_x(w)$ is the power spectrum density. For example, let us assume there's no overlap,

$$0.05\,P_x^2(w) \geq \frac{1}{K}P_x^2(w).$$

Therefore, assuming the non-overlapping case, $K \geq 20$.

See Exercise.

[8.2] (a) Each method is characterized by its fundamental properties. For methods A and C, we observe a very high resolution of the peak and high variance, and these methods therefore correspond to the periodogram and modified periodogram, but it is not directly clear which is which. At 3 dB attenuation, i.e. at $19\,\mathrm{dB} - 3\,\mathrm{dB} = 16\,\mathrm{dB}$ in the figure, the periodogram should provide a spectral resolution of $0.89/N$, whereas the modified periodogram with Hamming window has a resolution of $1.30/N$, where N=512 in both cases. In the figure, these differences (at the 3 dB attenuation level) are too small to measure. Note that, while it *looks* like the peak in Method C is wider than in A, the width must not be measured at the $-10\,\mathrm{dB}$ level, but at 16 dB. The periodogram is subject to high spectral leakage around the main peak, which corresponds to method C, where the spectral leakage can be seen from from $\nu = 0.1$ to $\nu = 0.3$. For the spectrum in method A, we observe a single, well-defined peak and only little spectral leakage. Thus, method A corresponds to the modified periodogram with a Hamming window. The Hamming window provides more than 41 dB attenuation of the side lobes, so the spectral leakage should be below $-22\,\mathrm{dB}$, which also matches with method A. Exhibiting much less variance and much worse resolution, but also clear spectral leakage, Barlett's method (Method B) is the least useful for any practical purpose in this case. Finally, Welch's method (Method D) shows the worse resolution by far, but also the best variance, thanks to the multiple windows with overlapping, which virtually increase the sample size.

(b) Barlett's method can, at most, obtain

$$K = \frac{512}{32} = 16$$

different segments from the data.

Welch's can get approximately twice as much, thanks to the 50% overlap. To be precise, for analyzing a sequence of total length N with Welch's method, where each block has length L, and overlaps with the previous block by D symbols (here, $D = L/2$), the number of segments is

$$K = 1 + \left\lfloor \frac{N-L}{L-D} \right\rfloor = 1 + \left\lfloor \frac{512-32}{32-16} \right\rfloor = 31. \tag{4}$$

This is because the first block uses L samples from the original sequence, and each following block takes $L - D$ new samples from the remaining sequence of length $N - L$.

(c) The best plot to estimate the noise variance is the one generated with Welch's method, i.e. Method D. The low variance makes measurement easy, and the spectral leakage is below the noise level, due to the use of a Hamming window. The peak, although very wide, can be clearly distinguished from the noise floor. The obtained measurement is a noise variance of $-30\,\text{dB}$.

See Exercise.

[8.3] Consider first that $w[n] = \cos^2\left(\frac{n\pi}{N}\right)$, and, therefore,

$$\begin{aligned} w[n] &= \left(\frac{e^{j2\pi\frac{n}{2N}} + e^{-j2\pi\frac{n}{2N}}}{2}\right)^2 \\ &= \frac{1}{2} + \frac{e^{j2\pi\frac{1}{N}n} + e^{-j2\pi\frac{1}{N}n}}{4} \end{aligned}$$

$$\begin{aligned} \text{DFT}_N\{w[n]\} = W[k] &= \frac{1}{2}N\delta[k \bmod N] \\ &+ \frac{1}{4}N\delta[(k-1) \bmod N] + \frac{1}{4}N\delta[(k+1) \bmod N]. \end{aligned}$$

This implies that

$$\begin{aligned} \text{DFT}\{x[n]w[n]\} &= \frac{1}{N}X[k] \, Ⓝ \, W[k] \\ &= \frac{1}{2}X[k] + \frac{1}{4}X[k-1] + \frac{1}{4}X[k+1]\,. \end{aligned}$$

See Exercise.

Parametric Spectral Estimation

[9.1] Because the filter in Fig. 9.1 is an MA(2) system, a way to approach this problem is to model the given signal as an AR(2) process and then choose the parameters of the MA(2) system to cancel this model. In a general case, this would not yield a perfect cancellation, but only the best linear cancellation possible. However, in this case, the signal $s[n]$ is completely cancelled because it is an AR(2) process.

Therefore, let's fit an AR(2) model to the proposed signal. The autocorrelation function is

$$r_{ss}[k] = \mathrm{E}[s[n]s[n-k]] = \frac{A^2}{2}\,\mathrm{E}[\cos(2\pi f(2n+k)+2\phi)+\cos(2\pi f k)] = \frac{A^2}{2}\cos(2\pi k)\,.$$

Thus, the Yule-Walker equations give

$$\begin{bmatrix} r_{ss}(0) & r_{ss}(1) \\ r_{ss}(1) & r_{ss}(0) \end{bmatrix} \begin{bmatrix} a \\ ab \end{bmatrix} = \begin{bmatrix} -r_{ss}(1) \\ -r_{ss}(2) \end{bmatrix}\,,$$

i.e.,

$$\begin{aligned}\begin{bmatrix} a \\ b \end{bmatrix} &= \left(\frac{A^2}{2}\begin{bmatrix} 1 & \cos(2\pi f) \\ \cos(2\pi f) & 1 \end{bmatrix}\right)^{-1} \frac{A^2}{2}\begin{bmatrix} -\cos(2\pi f) \\ -\cos(4\pi f) \end{bmatrix} \\ &= \frac{1}{1-\cos^2(2\pi f)}\begin{bmatrix} 1 & -\cos(2\pi f) \\ -\cos(2\pi f) & 1 \end{bmatrix}\begin{bmatrix} -\cos(2\pi f) \\ 1-2\cos^2(2\pi f) \end{bmatrix} = \begin{bmatrix} -2\cos(2\pi f) \\ 1 \end{bmatrix}.\end{aligned}$$

The frequency can be found through $f = \frac{1}{2\pi}\arccos(-\frac{a}{2})$.

See Exercise.

[9.2] We want to estimate the power espectrum of an $x[n]$, an AR(2) process, knowing:

$$y[n] = x[n] + v[n]. \tag{5}$$

$$v[n]:\ \text{first order moving average process uncorrelated with } x[n]. \tag{6}$$

$$v[n] = b_0 q[n] + b_1 q[n-1] \tag{7}$$

$$\hat{P}_v(\omega) = 3 + 2\cos(\omega). \tag{8}$$

$$\hat{\mathbf{r}}_y = \begin{bmatrix} 5 \\ 2 \\ 0 \\ -1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} \hat{r}_y[0] \\ \hat{r}_y[1] \\ \hat{r}_y[2] \\ \hat{r}_y[3] \\ \hat{r}_y[4] \end{bmatrix} \tag{9}$$

In order to solve this exercise we will follow the steps detailed below.

i) Extract the useful information.

(5) and (6) combined allow us to state that:

$$\begin{aligned} r_y[k] &= \mathrm{E}\left\{y[n]y[n-k]\right\} \underset{(5)}{=} \mathrm{E}\left\{(x[n]+v[n])(x[n-k]+v[n-k])\right\} \\ &= \mathrm{E}\left\{x[n]x[n-k]\right\} + \mathrm{E}\left\{x[n]v[n-k]\right\} + \mathrm{E}\left\{v[n]x[n-k]\right\} \\ &+ \mathrm{E}\left\{v[n]v[n-k]\right\} \\ &\underset{(6)}{=} r_x[k] + r_v[k]. \end{aligned}$$

Therefore,

$$r_x[k] = r_y[k] - r_v[k]. \tag{10}$$

(8) allows us to obtain $\hat{r}_v[k]$ from $\hat{P}_v(\omega)$ by inverse DTFT:

$$\hat{P}_v(\omega) = 3 + 2\cos(\omega) = 3 + 2\frac{e^{j\omega}+e^{-j\omega}}{2} = 3 + e^{j\omega} + e^{-j\omega}.$$

And now, by inverse DTFT:

$$\hat{r}_v[k] = 3\delta[k] + \delta[k-1] + \delta[k+1]. \tag{11}$$

ii) Compute $\hat{r}_x[k]$, for as many samples as we can.

We use (9), (10) and (11) to conclude

$$\hat{\mathbf{r}}_x = \begin{bmatrix} \hat{r}_x[0] \\ \hat{r}_x[1] \\ \hat{r}_x[2] \\ \hat{r}_x[3] \\ \hat{r}_x[4] \end{bmatrix} = \hat{\mathbf{r}}_y - \hat{\mathbf{r}}_v = \begin{bmatrix} 5 \\ 2 \\ 0 \\ -1 \\ 0.5 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \\ 0.5 \end{bmatrix}.$$

iii) Obtain an estimate of the parameters of the AR(2) process $x[n]$.

We could now use any spectral estimation technique we wanted, but since we know that $x[n]$ is an AR(2) process, we solve the Yule-Walker equations and find a_1 and a_2.

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 1/3 \end{bmatrix}.$$

Finally, we can estimate $\hat{b}_{x,0}^2 = \hat{r}_x[0] + \sum_{m=1}^{2} a_m \hat{r}_x[m] = 2 - \frac{2}{3}1 + \frac{1}{3}0 = 4/3$.

iv) Build the power spectrum estimate.

$$\hat{P}_x(\omega) = \frac{\hat{b}_{x,0}^2}{\left|1 + \sum_{m=1}^{2} a_m e^{-jm\omega}\right|^2} = \frac{4/3}{\left|1 - \frac{2}{3}e^{-j\omega} + \frac{1}{3}e^{-j2\omega}\right|^2}.$$

See Exercise.

[9.3] (a) Using (9.1) results in $\hat{r}_y[0] = 3$, $\hat{r}_y[1] = 2$, $\hat{r}_y[2] = 4 > \hat{r}_y[0]$, which is not reasonable for an autocorrelation function.

The Yule-Walker equations, in this context, give:

$$\begin{bmatrix} \hat{r}_y[0] & \hat{r}_y[1] \\ \hat{r}_y[1] & \hat{r}_y[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -\hat{r}_y[1] \\ -\hat{r}_y[2] \end{bmatrix} \Leftrightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2/5 \\ -8/5 \end{bmatrix}.$$

The poles of the resulting filter $\frac{1}{A(z)}$ are in $z_1 = -1/5 - \sqrt{1/25 + 8/5} \approx -1.48$ and $z_2 = -1/5 + \sqrt{1/25 + 8/5} = 1.08$, which makes the filter unstable (because it has to be causal). This results from a severe model mismatch, because an AR(2) process is expected to decay, while our samples did not. Note that an unstable filter in an AR model would signify a signal that can grow indefinitely. Finally, the best evidence of model mismatch is that the first YW equation gives us that

$$b_0^2 = \sum_{k=1}^{3} a_k \hat{r}_y[k] = -2.6\,.$$

(b) Using (9.2) results in $\hat{r}_y[0] = 3$, $\hat{r}_y[1] = 4/3$, $\hat{r}_y[2] = 4/3$, which is a proper estimator of the autocorrelation function, because most of the energy is concentrated in $\hat{r}_y[0]]$. The Yule-Walker equations give, in this context, give:

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -4/13 \\ -4/13 \end{bmatrix}.$$

The poles of the resulting filter $\frac{1}{A(z)}$ are in $z_1 = 2/13 + \sqrt{4/169 + 4/13} = 0.73$ and $z_2 = 2/13 - \sqrt{4/169 + 4/13} = -0.42$, which makes the filter stable (because it has to be causal). Finally, the parameter b_0^2 yields a consistent value too,

$$b_0^2 = \sum_{k=1}^{3} a_k \hat{r}_y[k] = 2.18\,.$$

This problem exemplifies why (9.2) is much more frequently used than (9.1), at least for low sample sizes. (9.2) has improved statistical performance because it weights down those values for which the biased estimator has a larger variance. This tecnique is known in statistics as *shrinking* an unbiased estimator to decrease its MSE.

See Exercise.

Multirate Signal Processing

Decimation and Interpolation

[10.1] (a) Because the input signal is occupying all the possible spectrum (is being sampled at precisely the Nyquist rate), there is a need for the antialiasing filter $H(\nu)$. To see this, consider the signals $(x * h)[n]$ and $y[n]$ at each side of the downsampler. Because

$$Y(\nu) = \frac{1}{2}\sum_{k=0}^{1} H\left(\frac{\nu-k}{2}\right) X\left(\frac{\nu-k}{2}\right),$$

there is a clear possibility of aliasing if $H(\nu)$ does not cut the upper frequency components in $X(\nu)$. In particular, $B \leq \frac{1}{4}$ is necessary. Thus, the correct answer is $\boxed{\text{A}}$

(b) We know that the spectrum of $x[n]$ has been cut by the antialiasing filter at $B = \frac{1}{4}$. Thus, we know it will look like a triangle with chopped slopes. This leaves only options $\boxed{\text{D}}$ and $\boxed{\text{E}}$. However, we see that $\boxed{\text{E}}$ is exactly how the signal would look if it were converted to analog (at rate F_s) just after the filter. After downsampling, this shape will be expanded from the range $\left[-\frac{1}{4}, \frac{1}{4}\right]$ to the range $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Moreover, the D/A converter works at twice the frequency the A/D works, and thus, the resulting spectrum will have the same shape, but in the range $[-F_s, F_s]$. Therefore, the correct answer is $\boxed{\text{D}}$.

(c) If the D/C converter operates at rate $F_s/2$, the signal will look like in option $\boxed{\text{E}}$, which is the low-pass filtered version of the input signal.

See Exercise.

[10.2] We first express the filter in the normalized frequency $\nu = \frac{\omega}{2\pi}$:

$$|H_{\text{norm}}(\nu)|^2 \approx \begin{cases} 1 & \text{for } |\nu| \leq \frac{\Omega}{2\pi} \text{ with } \Omega < \frac{\pi}{M} \\ 0 & \text{otherwise.} \end{cases}$$

The impulse response is downsampled by a factor M. This expands the spectrum by a factor of M. Because $\frac{\Omega}{2\pi} < \frac{1}{2M}$, we can disregard folding (aliasing) terms, and $\bar{h}[n]$ has its pass-band in $|\nu| \leq M\frac{\Omega}{2\pi}$ or $|\omega| \leq M\Omega$.

See Exercise.

[10.3] The signal that we have, i.e. $s[n]$, was sampled at a rate of $F_{s,1} = \frac{1}{10^{-4}\,\text{s}} = 10\,\text{kHz}$, while the signal that we want, i.e. $s_1[n]$, would have been sampled at a rate of $F_{s,2} = \frac{6}{10^{-3}\,\text{s}} = 6\,\text{kHz}$.

Recovering an alternate sampling of a continuous-time signal from a discrete-time sequence is not always possible. In particular, though, $F_{s,2} < F_{s,1}$ is a sufficient conditions if the signal $s(t)$ was sampled fulfilling Nyquist's criterion.

We want to perform a rate conversion of $\frac{6\,\text{kHz}}{10\,\text{kHz}} = \frac{3}{5}$. The simplest way to do that is first interpolating $s[n]$ by 3 and then decimating the result by 5. Fig. 4 shows how this can be done by using a single discrete-time filter $H(\nu)$ that combines the effect of the post-upsampling and the anti-aliasing ideal filters, i.e.

$$H(\nu) = \begin{cases} 3 & \text{for } |\nu| \leq \frac{1}{10} \\ 0 & \text{for } \frac{1}{10} < |\nu| \leq \frac{1}{2} \end{cases}.$$

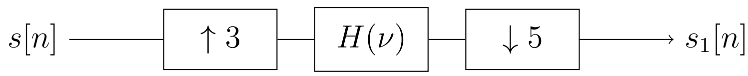


Figure 4: System that, with the proper filter $H(\nu)$, solves Exercise 10.3.

In this manner, the spectral information is preserved and $s_1[n]$ is obtained exactly as it had been sampled as specified in Fig. 10.4.

See Exercise.

Multirate Signal Processing

Filter Banks

[11.1] The sequence is downsampled by a factor K. The new center frequency is $\bar{\omega}_0 = K\omega_0$. The correct answer is $\boxed{B}$. See Exercise.

[11.2] (a) Consider the annotations in Fig. 5. Then, using the basic relations learned in the theory, we can derive the spectrum at each annotated spot as follows. For $a_0[n]$ and $a_1[n]$, the convolution property states

$$A_0(\nu) = X(\nu)H(\nu),\ A_1(\nu) = X(\nu)F(\nu)\,,$$

where $X(\nu)$ is the DTFT of $x[n]$, and $H(\nu)$ and $F(\nu)$ are the transfer functions for the filters $h[n]$ and $f[n]$, respectively. For $b_0[m]$ and $b_1[m]$, i.e., the signals after downsampling, the full spectrum is given by

$$B_0(\nu) = \frac{1}{2}\sum_{k=0}^{1} X\left(\frac{\nu-k}{2}\right) H\left(\frac{\nu-k}{2}\right), B_1(\nu) = \frac{1}{2}\sum_{k=0}^{1} X\left(\frac{\nu-k}{2}\right) F\left(\frac{\nu-k}{2}\right),$$

which contains the aliasing terms that could later hinder total reconstruction. Then, applying the relation for upsampling given in the theory, we obtain, for $c_0[n]$ and $c_1[n]$,

$$C_0(\nu) = \frac{1}{2}\sum_{k=0}^{1} X\left(\nu-\frac{k}{2}\right) H\left(\nu-\frac{k}{2}\right), C_1(\nu) = \frac{1}{2}\sum_{k=0}^{1} X\left(\nu-\frac{k}{2}\right) F\left(\nu-\frac{k}{2}\right).$$

Finally, the filters $g[n]$ and $d[n]$ (with transfer functions $G(\nu)$ and $D(\nu)$, respectively), have their effect, that, thanks to the convolution theorem, can be written as

$$\begin{aligned} E_0(\nu) &= G(\nu)\left[\frac{1}{2}\sum_{k=0}^{1} X\left(\nu-\frac{k}{2}\right) H\left(\nu-\frac{k}{2}\right)\right] \\ E_1(\nu) &= D(\nu)\left[\frac{1}{2}\sum_{k=0}^{1} X\left(\nu-\frac{k}{2}\right) F\left(\nu-\frac{k}{2}\right)\right] \end{aligned}.$$

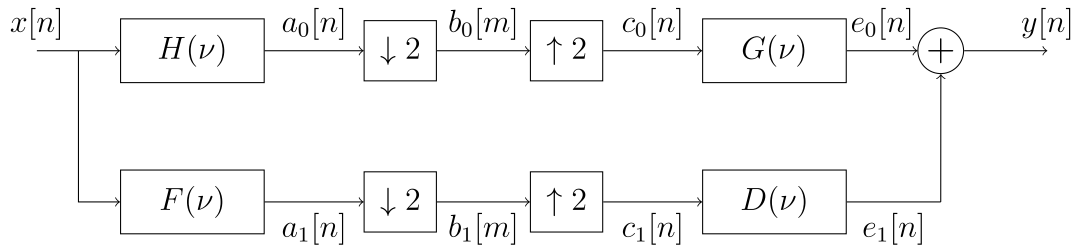


Figure 5: Annotated filter bank for solving Exercise 11.2. Mainly the same as Fig. 11.1, but annotated with reference signals to do the analysis.

Summarizing, then, we get the output $y[n]$ with spectrum

$$\begin{aligned}
Y(\nu) &= E_0(\nu) + E_1(\nu) \\
&= \frac{1}{2}G(\nu)\left[X(\nu)H(\nu) + X\left(\nu - \frac{1}{2}\right)H\left(\nu - \frac{1}{2}\right)\right] \\
&+ \frac{1}{2}D(\nu)\left[X(\nu)F(\nu) + X\left(\nu - \frac{1}{2}\right)F\left(\nu - \frac{1}{2}\right)\right] \\
&= \frac{1}{2}X(\nu)\left[G(\nu)H(\nu) + D(\nu)F(\nu)\right] \\
&+ \frac{1}{2}X\left(\nu - \frac{1}{2}\right)\left[G(\nu)H\left(\nu - \frac{1}{2}\right) + D(\nu)F\left(\nu - \frac{1}{2}\right)\right].
\end{aligned}$$

We want $Y(\nu) = X(\nu)e^{-j2\pi M\nu}$ corresponding to a delay of M samples, which, directly from the previous expression, gives

$$\begin{aligned}
G(\nu)H(\nu) + D(\nu)F(\nu) &= 2e^{-j2\pi\nu M} \\
G(\nu)H\left(\nu - \frac{1}{2}\right) + D(\nu)F\left(\nu - \frac{1}{2}\right) &= 0\,.
\end{aligned}$$

(b) We have three pieces of information to work on

$$H^2(\nu) + D(\nu)F(\nu) = 2e^{-j2\pi\nu M} \qquad (12)$$

$$H(\nu)H\left(\nu - \frac{1}{2}\right) + D(\nu)F\left(\nu - \frac{1}{2}\right) = 0 \qquad (13)$$

$$F(\nu) = H\left(\nu - \frac{1}{2}\right), \qquad (14)$$

where only the conclusions from the previous question and the new assumptions in the headlines have been used.

From (14) and the periodicity of the discrete spectrum we can extract that

$$F\left(\nu - \frac{1}{2}\right) = H(\nu)\,.$$

Analysis of (13) under these new conditions reveals that, if we assume $\exists \nu \in [0,1) \mid H(\nu) \neq 0$,

$$H(\nu)\left(H\left(\nu-\frac{1}{2}\right)+D(\nu)\right)=0 \Rightarrow D(\nu)=-H\left(\nu-\frac{1}{2}\right).$$

Therefore, we obtain that

$$\begin{array}{rcl} F(\nu) & = & H\left(\nu-\frac{1}{2}\right) \\ D(\nu) & = & -H\left(\nu-\frac{1}{2}\right) \end{array}.$$

By inverse DTFT, we obtian that the impulse responses are related through

$$f[n]=e^{j\pi n}h[n]=\begin{cases} h[n] & \text{for even } n \\ -h(n) & \text{for odd } n \end{cases},$$

and similarly

$$d[n]=-e^{j\pi n}h(n)=\begin{cases} -h[n] & \text{for even } n \\ h(n) & \text{for odd } n \end{cases}.$$

Finally, using these new results and (12), we obtain

$$H^2(\nu)+H^2\left(\nu-\frac{1}{2}\right)=2e^{-j2\pi\nu M}.$$

See Exercise.

[11.3] (a) Splitting the infinite sum in two, one containing even ns and one containing odd ns, yields

$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{+\infty} h[n]z^{-n} \\ &= \sum_{m=-\infty}^{+\infty} h[2m]z^{-2m} + \sum_{p=-\infty}^{+\infty} h[2p+1]z^{-(2p+1)} \\ &= \sum_{m=-\infty}^{+\infty} h[2m]\left(z^2\right)^{-m} + z^{-1}\sum_{p=-\infty}^{+\infty} h[2p+1]\left(z^2\right)^{-p}. \end{aligned}$$

Taking into account that

$$\mathrm{P}_0(z)=\sum_{n=-\infty}^{+\infty} p_0[n]z^{-n}=\sum_{m=-\infty}^{+\infty} h[2m]z^{-m},$$

and

$$\mathrm{P}_1(z)=\sum_{n=-\infty}^{+\infty} p_1[n]z^{-n}=\sum_{p=-\infty}^{+\infty} h[2p+1]z^{-p},$$

it is clear that

$$H(z)=\mathrm{P}_0\left(z^2\right)+z^{-1}\mathrm{P}_1\left(z^2\right).$$

(b) The proposed transfer function $H(z)$ is all-pass $\forall a \in \mathbb{R}$, because

$$|H(z)|^2 = \frac{a+z^{-1}}{1+az^{-1}}\frac{a+z^{-*}}{1+az^{-*}} = \frac{a^2+2a\,\mathrm{Re}\left\{z^{-1}\right\}+|z|^{-2}}{1+2a\,\mathrm{Re}\left\{z^{-1}\right\}+a^2|z|^{-2}}$$

and, $\forall z \in \mathbb{C} \mid z = e^{j2\pi\nu}$ for some $\nu \in [0,1)$, $|H(z)|^2\big|_{z=e^{j2\pi\nu}} = 1$.
Using the inverse Z-transform, and taking into account that the filter is causal, we obtain the impulse response

$$h[n] = \frac{1}{a}\delta[n] + (a - \frac{1}{a})(-a)^n u[n]\,,$$

where $u[n]$ is the discrete step function and $\delta[n]$ the Kronecker delta.
Hence,

$$p_0[n] = h[2n] = \frac{1}{a}\delta[n] + (a-\frac{1}{a})(-a)^{2n} \text{ for } n \geq 0,$$

$$\begin{aligned} p_1[n] = h[2n+1] &= \frac{1}{a}\delta[2n+1] + (a-\frac{1}{a})(-a)^{2n+1}, \text{ for } n \geq 0 \\ &= (1-a^2)(-a)^{2n}, \text{ for } n \geq 0\,. \end{aligned}$$

Using now the Z-tranform, we obtain the expressions for the two components of the polyphase decomposition,

$$P_0(z) = a\frac{1-z^{-1}}{1-a^2z^{-1}},$$

$$P_1(z) = \frac{1-a^2}{1-a^2z^{-1}}.$$

In this case, both polyphase components can not be all-pass for the same parameter a. This can be proved by showing that, for this particular filter,

$$|H\nu)|^2 = |\mathrm{P}_0(\nu)|^2 + |\mathrm{P}_1|^2\,.$$

For further insight on this situation, the following MATLAB / Octave code helps visualizing the situation. Note that this frequency separation of the polyphase components is not a general property, but only a result for this case.

```
nu = -0.5:0.0001:0.5;
as = 0:0.2:1;
for a = as
  p0=a*(1-exp(-1i*4*pi*nu))./(1-a^2*exp(-1i*4*pi*nu));
  p1=(1-a^2)./(1-a^2*exp(-1i*4*pi*nu));
  plot(nu,abs(p0.^2),'r--',nu,abs(p1.^2),'b-.')
  hold on
end
legend('0','1')
```

See Exercise.

[11.4] In the videolecture on filterbanks, it has been stated that a sufficient condition for perfect reconstruction is:

$$\begin{cases} D(z) = -H(-z) \\ G(z) = F(-z) \end{cases}$$

Obs: We consider it a good exercise trying to prove the sufficient condition stated above.

This means that if we get some filters that fulfill these conditions and the ones given in the problem, we will have a solution. However, note that there is no guarantee that we can do so, and not being able to find such filters would not imply that the problem has no solutions.

Let's, then, try it:

$$\begin{aligned}
&H(z) = 1 + \frac{1}{2}z^{-1} \\
&D(z) = -H(-z) = -1 + \frac{1}{2}z^{-1} \\
&G(z) = F(-z) = a - bz^{-1} \\
&F(z) = a + bz^{-1},\ F(z)\big|_{z=e^{j0}} = F(1) = a + b = 0 \Rightarrow a = -b.
\end{aligned}$$

Using the last equality we can rewrite

$$\begin{aligned}
&G(z) = F(-z) = a(1 + z^{-1}) \\
&F(z) = a(1 - z^{-1}).
\end{aligned}$$

To set the delay and determine a, we have to compute the term that will multiply $X(z)$ in the expression of $Y(z)$ and set it to z^{-L} for some $L \in \mathbb{N}$.

$$\begin{aligned}
\frac{1}{2}\left(G(z)H(z) + D(z)F(z)\right) &= \frac{1}{2}\left(\left(a(1+z^{-1})\right)\left(1 + \frac{1}{2}z^{-1}\right)\right. \\
&\left. + \left(-1 + \frac{1}{2}z^{-1}\right)\left(a(1 - z^{-1})\right)\right) \\
&= \frac{1}{2}\left(a + az^{-1} + \frac{a}{2}z^{-1} + \frac{a}{2}z^{-2}\right. \\
&\left. - a + \frac{a}{2}z^{-1} + az^{-1} - \frac{a}{2}z^{-2}\right) \\
&= \frac{3}{2}az^{-1} = z^{-1},\ \text{forcing } a = \frac{2}{3}.
\end{aligned}$$

Then, $a = \frac{2}{3}$, $b = -\frac{2}{3}$ and the delay is of one sample, i.e. $L = 1$. Concluding:

$$H(z) = 1 + \frac{1}{2}z^{-1}, G(z) = \frac{2}{3} + \frac{2}{3}z^{-1}$$
$$F(z) = \frac{2}{3} - \frac{2}{3}z^{-1}, D(z) = -1 + \frac{1}{2}z^{-1}$$

and

$$Y(z) = z^{-1}X(z).$$

See Exercise.

Advanced Topics and Recent Exam Problems

[12.1] (a) We have that

$$\begin{aligned} V(z) &= H_1(z)\left(X(z) - Y(z)\right) \\ U(z) &= H_2(z)\left(V(z) - Y(z)\right) \quad \text{and} \\ Y(z) &= U(z) + E(z) \\ &= \underbrace{\frac{H_1(z)H_2(z)}{1 + H_2(z)\left(1 + H_1(z)\right)}}_{H_{xy}(z)} X(z) + \underbrace{\frac{1}{1 + H_2(z)\left(1 + H_1(z)\right)} E(z)}_{H_{ey}(z)} \end{aligned}$$

Substituting $H_1(z) = \dfrac{1}{1 - z^{-1}}$ and $H_2(z) = \dfrac{z^{-1}}{1 - z^{-1}}$, we find $H_{xy}(z) = z^{-1}$ and $H_{ey}(z) = \left(1 - z^{-1}\right)^2$. Hence, the difference equation is $y[n] = x[n-1] + f[n]$, where $f[n] = e[n] - 2e[n-1] + e[n-2]$.

(b) We have that

$$\begin{aligned} P_{ff}(\omega) &= \sigma_e^2 |H_{ey}(\omega)|^2 = \sigma_e^2 |(1 - e^{-j\omega})^2|^2 \\ &= \sigma_e^2 (1 - e^{-j\omega})^2 (1 - e^{j\omega})^2 = \sigma_e^2 \left(2 - 2\cos\omega\right)^2 \\ &= 16\sigma_e^2 \sin^4(\omega/2). \end{aligned}$$

The total noise power σ_f^2 is the autocorrelation of $f[n]$ evaluated at 0:

$$\begin{aligned} \sigma_f^2 &= \mathrm{E}\left\{[e[n] - 2e[n-1] + e[n-2]]^2\right\} \\ &= \mathrm{E}\left\{e^2[n]\right\} + 4\,\mathrm{E}\left\{e^2[n-1]\right\} + \mathrm{E}\left\{e^2[n-2]\right\} \\ &= 6\sigma_e^2, \end{aligned}$$

where we have used linearity of expectations and the fact that, since $e[n]$ is white, $\mathrm{E}\left\{e[n]e[n-k]\right\} = 0$ for $k \neq 0$.

The sketch of the power spectra can be seen in Fig. 6.

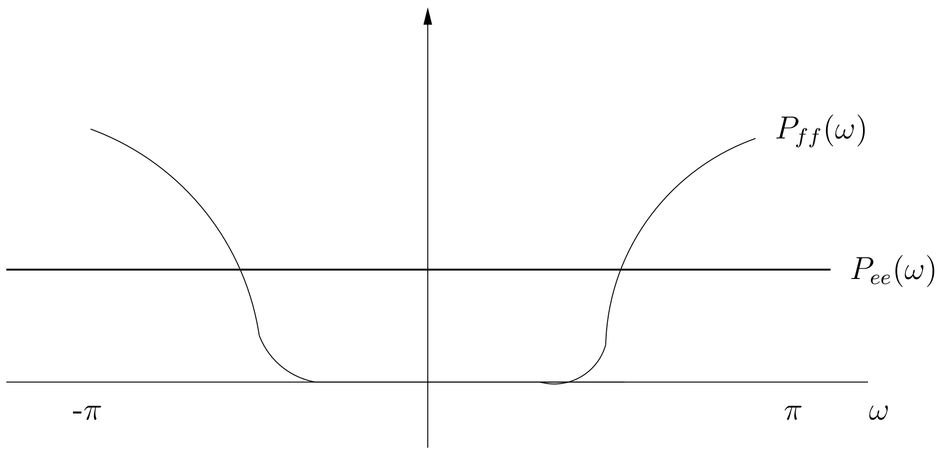


Figure 6: First sketch of power spectra required in Exercise 12.1.

(c) Since $X(\omega)$ is bandlimited, $x(n) * h_3[n] = x[n]$. Hence,

$$\begin{aligned} w[n] = y[n] * h_3[n] &= (x[n-1] + f[n]) * h_3[n] \\ &= x[n-1] + g[n], \end{aligned}$$

where $g[n]$ is the quantization noise in the region $|\omega| < \pi/M$.

(d) For a small x, $\sin x \approx x$. Therefore,

$$\begin{aligned} \sigma_g^2 &= \frac{1}{2\pi} \int_{-\pi/M}^{\pi/M} \sigma_e^2 \left(2\sin(\omega/2)\right)^4 d\omega \\ &\approx \frac{1}{2\pi} \int_{-\pi/M}^{\pi/M} \sigma_e^2 \left(2(\omega/2)\right)^4 d\omega = \frac{\sigma_e^2}{2\pi} \frac{\omega^5}{5} \Bigg|_{-\pi/M}^{\pi/M} \\ &= \frac{\sigma_e^2 \pi^4}{5M^5}. \end{aligned}$$

(e) $X_c(\Omega)$ must be sufficiently bandlimited, so that $X(\omega) = X_c(\Omega T)$ is zero for $|\omega| > \pi/M$. Hence, $X_c(\Omega) = 0$ for $|\Omega| > \pi/MT$. Assuming this is satisfied then $\alpha_x[n] = x[Mn-1] = x_c[MTn-T]$. Downsampling does not change the variance of the noise and hence $\sigma_q^2 = \sigma_g^2$. We have that

$$P_{qq}(\omega) = \frac{1}{M} P_{gg}(\omega/M) = \frac{16}{M} \sigma_e^2 \sin^4\left(\frac{\omega}{2M}\right),$$

and hence the resulting sketch of the spectra is the one shown in Fig. 7.

See Exercise.

[12.2] The second circuit is nothing more than the polyphase implementation of the former circuit. This can be seen from previous knowledge of QMF banks or

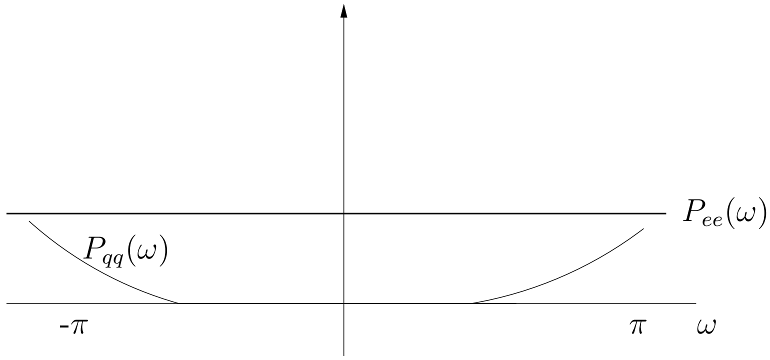


Figure 7: Final sketch of power spectra required in Exercise 12.1 for $M \approx 2.5$.

just by studying the part leading up to $v_0[m]$, which is the standard polyphase implementation of $H(z)$. A clue observation is that, given the known relation

$$H(z) = \mathrm{P}_0\left(z^2\right) + z^{-1}\mathrm{P}_1\left(z^2\right) ,$$

it follows directly that

$$\begin{aligned} G(z) = -H(-z) &= -\mathrm{P}_0\left((-z)^2\right) + z^{-1}\mathrm{P}_1\left((-z)^2\right) \\ &= -\mathrm{P}_0\left(z^2\right) + z^{-1}\mathrm{P}_1\left(-z^2\right) , \end{aligned}$$

which intrinsically relates the polyphase implementation of QMFs. Thus, $p_0[n] = h[2n]$ and $p_1[n] = h[2n+1]$.

See Exercise.