SIGNAL PROCESSING

DEPARTMENT OF ELECTRICAL ENGINEERING

E 105 **Digital Signalbehandling** EQ2300/ 2E1340

Final Examination 2009–06–04, 14.00–19.00
Sample Solutions

1. The quantization caused by the multiplication by -0.9 is, as usual, modelled as additive white noise. Downsampling of the white noise results in white noise of the same variance. Hence the total noise at the output is

$$\sigma_q^2 = 2\sigma_e^2 \sum_{n=0}^{\infty} (0.5)^{2n}$$

where σ_e^2 is the quantization noise given by $\sigma_e^2 = \frac{2^{-2B}}{12}$. Then, we have,

$$\sigma_q^2 = 2\frac{2^{-2B}}{9}$$

. Equating the resulting SQNR expression to the given numerical value, we have,

$$30 = 10 \log_{10} \frac{0.1}{2\frac{2^{-2B}}{9}}$$

Further simplification yields a lower (integer) bound on B as 6

2. (a) The pole of $\frac{1}{H(z)}$ is 2, which is outside the unit circle. The noise variance is unbounded as the impulse response of $\frac{1}{H(z)}$ grows as 2^n.

(b) The pole is now at 1/2 which is within the unit circle. Noise variance will be $\sum_k \left(\frac{1}{2}\right)^{2k}$ which is finite.

(c) The idea is called *Time reverse filtering*. You can time reverse into $y(-n)$ ($Y(z^{-1})$) and filter it with the time reversed filter $h(-n)$ ($\frac{1}{H(z^{-1})}$). The output ($\frac{Y(z^{-1})}{H(z^{-1})}$) is again flipped. The infeasibility arises during the flipping of $y(n)$ and the output since they can be infinite in length.

(d) When $x(n)$ has length N, $v(n)$ has length $N+1$. Since the noise samples are uncorrelated, samples of $w(n), n \geq N+1$ do not yield any useful information. Hence the signal $y(n)$ can be considered as a $N+1$ length sequence. Being of finite length, $y(n)$ can be flipped after collecting all the $N+1$ samples. Further, we are interested only in the first N elements of the sequence corresponding to $\frac{Y(z^{-1})}{H(z^{-1})}$ (as $x(n)$ was of length N). As with $y(n)$, the output can be flipped after collecting all the N elements. This incurs an additional delay of N. One can also use matrix inversion to obtain the outputs with a delay of $N+1$.

3. (a) We need to satisfy the following condition for resolving the two components

$$\frac{\triangle f}{f_s} \geq \frac{0.89}{M} \tag{1}$$

Similarly, to satisfy the second requirement, we need,

$$K \approx 100 \tag{2}$$

Furthermore, we also have that $KM = Tf_s$. Combining these equations, we have,

$$\begin{aligned} M &\geq \frac{0.89 f_s}{\triangle f} \\ K &\approx 100 \\ KM &\geq \frac{89 f_s}{\triangle f} \\ T f_s &\geq \frac{89 f_s}{\triangle f} \end{aligned} \tag{3}$$

which in independent of f_s. The only condition required is that f_s should be above the Nyquist sampling frequency. If equation 3 is not satisfied, the requirements of the problem are not met. The result is somewhat counter-intuitive in the sense that increasing the data samples does to seem to aid the Bartlett method.

(b) In the true spectrum of the signal, the noise level will appear as a horizontal line. If we use a spectral estimation method with a high variance, the values of the spectrum will vary significantly from experiment to experiment and from frequency to frequency, so it will be very difficult to recognize the straight line and read its value. To conclude, we want a method with as low variance as possible, i.e. the Bartlett method.

(c) Here, we want to distinguish the two peaks even if they are very closely located in frequency, i.e. we want as high resolution as possible. Therefore, the Periodogram method is preferable.

4. The assumption on $y(n)$ means that $r_y(1) = r_y(3) = r_y(5) = 0$.

(a) The Yule-Walker equations take the following form, when inserting the known zeros

$$\begin{bmatrix} r_y(0) & 0 & r_y(2) & 0 \\ 0 & r_y(0) & 0 & r_y(2) \\ r_y(2) & 0 & r_y(0) & 0 \\ 0 & r_y(2) & 0 & r_y(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ r_y(2) \\ 0 \\ r_y(4) \end{bmatrix}$$

Because of the zeros, this system of equations decouples into the following two system of equations

$$\begin{aligned} \begin{bmatrix} r_y(0) & r_y(2) \\ r_y(2) & r_y(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} r_y(0) & r_y(2) \\ r_y(2) & r_y(0) \end{bmatrix} \begin{bmatrix} a_2 \\ a_4 \end{bmatrix} &= \begin{bmatrix} r_y(2) \\ r_y(4) \end{bmatrix} \end{aligned}$$

which have the solution $a_1 = a_3 = 0$ and

$$\begin{bmatrix} a_2 \\ a_4 \end{bmatrix} = \frac{1}{r_y^2(0) - r_y^2(2)} \begin{bmatrix} r_y(0)r_y(2) - r_y(2)r_y(4) \\ r_y(0)r_y(4) - r_y^2(2) \end{bmatrix}$$

(b) This time, the Yule-Walker equations are

$$\begin{bmatrix} r_y(0) & 0 & r_y(2) & 0 \\ 0 & r_y(0) & 0 & r_y(2) \\ r_y(2) & 0 & r_y(0) & 0 \\ 0 & r_y(2) & 0 & r_y(0) \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} r_y(2) \\ 0 \\ r_y(4) \\ 0 \end{bmatrix}$$

which decouples into

$$\begin{bmatrix} r_y(0) & r_y(2) \\ r_y(2) & r_y(0) \end{bmatrix} \begin{bmatrix} b_1 \\ b_3 \end{bmatrix} = \begin{bmatrix} r_y(2) \\ r_y(4) \end{bmatrix}$$
$$\begin{bmatrix} r_y(0) & r_y(2) \\ r_y(2) & r_y(0) \end{bmatrix} \begin{bmatrix} b_2 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

with solutions $b_1 = a_2, b_3 = a_4, b_2 = b_4 = 0$.

(c) The most straightforward solution is to write out the MSE expression and find the optimality conditions. An alternative is to first use the above results to determine the MMSE estimator of $u(n)$ of the form

$$\widehat{u}_{unconstr}(n) = d_1 y(n-1) + d_2 y(n-2) + d_3 y(n-3) + d_4 y(n-4)$$

Since $u(n) = y(n) + y(n+1)$, and the estimators use the same observations as in a) and b), we must have $\widehat{u}_{unconstr}(n) = \widehat{y}_1(n) + \widehat{y}_2(n)$, i.e. $d_1 = d_2 = a_2$ and $d_3 = d_4 = a_4$, which happens to have exactly the form required for the requested $\widehat{u}(n)$. Consequently, $c_1 = a_2$ and $c_2 = a_4$.

5. (a) i. The impulse response of $F(z)$ is shifted $2k$ steps in time compared to $H(z)$. Since the polyphase components contain every second value of the impulse response, we get $Q_0(z) = \alpha z^{-k} P_0(z)$ and $Q_1(z) = \alpha z^{-k} P_1(z)$

ii. Similarly, $Q_1(z) = \alpha z^{-k} P_0(z)$ and $Q_0(z) = \alpha z^{-(k+1)} P_1(z)$

(b) It can be shown that $P_0(z) = \frac{1}{1-c^2 z^{-1}}, P_1(z) = \frac{c}{1-c^2 z^{-1}}, |z| > c$. Then, $Q_0(z) = \frac{a_0 + a_1 c z^{-1} + a_2 z^{-1}}{1-c^2 z^{-1}}, Q_1(z) = \frac{a_0 c + a_1 + a_2 c z^{-1}}{1-c^2 z^{-1}}, |z| > c$

(c) The direct implementation requires 3 multiplications (clearly seen when implemented as a difference equation). For the polyphase implementation, 3 multiplications per samples for each of the two filters are needed, but since these happen at half the rate, this corresponds to 3 multiplications per sample at the original rate. The result in the compendium that the number of multiplications reduces by a factor of two only holds for FIR filters, not for IIR filters in general.