EQ2300, Digital Signal Processing

Collection of Problems

Stockholm

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Contents

Chapter 1

Introduction, Review

1.1 Determine the convolution between $x(n) = \{\overset{\downarrow}{1}, 1\}$ and $y(n) = \{\overset{\downarrow}{1}, 1, 1\}$.

See Solution.

1.2 Let E be the energy and P the power of the signal $x(n)$. Which statement is wrong (C is an arbitrary constant $0 < C < \infty$)?

A	B	C	D
$E = C \Rightarrow P = 0$	$P = C \Rightarrow E = \infty$	$E = \infty \Rightarrow P = \infty$	$P = \infty \Rightarrow E = \infty$

See Solution.

1.3 In the design of either analog or digital filters, we often approximate a specified magnitude characteristic, without particular regard to the phase. For example, standard design techniques for lowpass and bandpass filters are derived from consideration of the magnitude characteristics only.

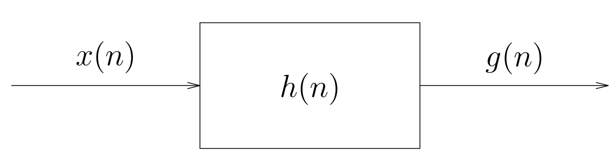
In many filtering problems, one would ideally like the phase characteristics to be zero or linear. For causal filters it is impossible to have zero phase. However, for many digital filtering applications, it is not necessary that the impulse response of the filter be zero for $n < 0$ if the processing is not to be carried out in real time.

One technique commonly used in digital filtering when the data to be filtered are of finite duration and stored, for example, on a disc or magnetic tape, is to process the data forward and then backward through the same filter.

Let $h(n)$ be the impulse response of a causal filter with an arbitrary phase characteristic, Assume that $h(n)$ is real and denote its Fourier transform by $H(e^{j\omega})$. Let $x(n)$ be the data that we want to filter. The filtering operation is performed as follows:

(a) *Method A:*

i.



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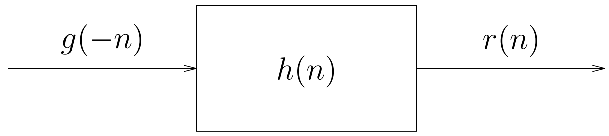


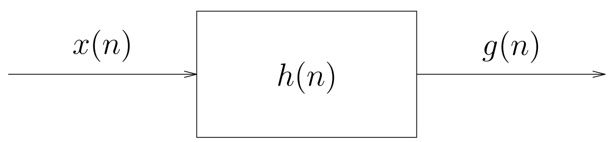
iii. $y(n) = r(-n)$

i. Determine the overall impulse response $h_1(n)$ that relates $x(n)$ and $y(n)$, and show that it has a zero-phase characteristic.

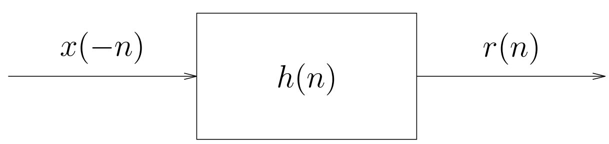
ii. Determine $|H_1(e^{j\omega})|$ and express it in terms of $|H(e^{j\omega})|$ and $\arg\{H(e^{j\omega})\}$.

(b) *Method B:*
Process $x(n)$ through the filter $h(n)$ to get $g(n)$. Also, process $x(n)$ backward through $h(n)$ to get $r(n)$. The output $y(n)$ is then taken as the sum $g(n)$ plus $r(-n)$

i.



ii.



iii. $y(n) = g(n) + r(-n)$

This composite set of operations can be represented by a filter, with input $x(n)$, output $y(n)$, and impulse response $h_2(n)$.

i. Show that the composite filter $h_2(n)$ has a zero-phase characteristic.

ii. Determine $|H_2(e^{j\omega})|$ and express it in terms of $|H(e^{j\omega})|$ and $\arg\{H(e^{j\omega})\}$.

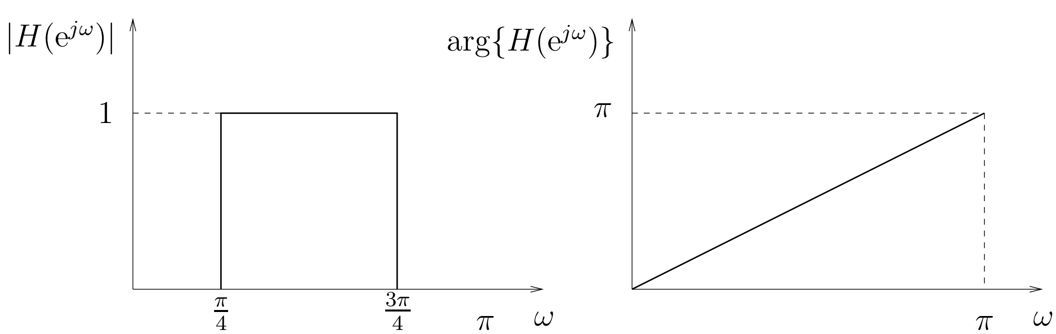


Figure 1.1:

(c) Suppose that we are given a sequence of finite duration, on which we would like to perform a bandpass, zero-phase operation. Furthermore, assume that we are given the bandpass filter $h(n)$, with frequency response as specified in Figure 1.1, which has the magnitude characteristic that we desire but linear phase. To achieve zero phase, we could use either method (A) or (B). Determine and sketch $|H_1(e^{j\omega})|$ and $|H_2(e^{j\omega})|$. From these results which method would you use to achieve the desired bandpass filtering operation? Explain why. More generally, if $h(n)$ has the desired magnitude but a nonlinear phase characteristic which method is preferable to achieve a zero-phase characteristic?

See Solution.

1.4 Show that any signal can be decomposed into an even and an odd component. Is the decomposition unique? Illustrate your arguments using the signal

$$x(n) = \{2, 3, \underset{\uparrow}{4}, 5, 6\}$$

See Solution.

1.5 Show that the energy (power) of a real-valued energy (power) signal is equal to the sum of the energies (powers) of its even and odd components. See Solution.

1.6 Let H_1 be the causal system described by the difference equation

$$y[n] = \frac{7}{12}y[n-1] - \frac{1}{12}y[n-2] + x[n-1] - \frac{1}{2}x[n-2]$$

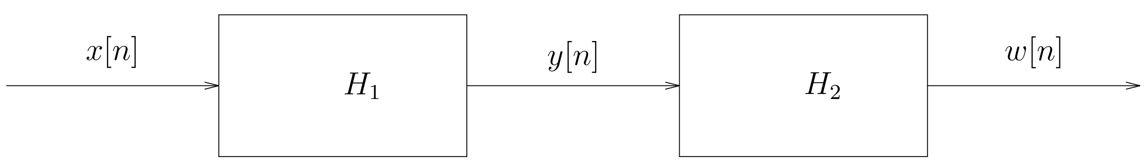


Figure 1.2: Overall system for Exercise 1.6. H_1 is characterized by the given difference equation, H_2 is analyzed for the different choices of $w[n]$ given in the headlines.

(a) Determine the system H_2 in Fig. 1.2 so that $w[n] = x[n]$. Is the inverse system H_2 causal?

(b) Determine the system H_2 in Fig. 1.2 so that $w[n] = x[n-1]$. Is the inverse system H_2 causal? Explain.

(c) Determine the difference equations for the system H_2 in parts (a) and (b).

See Solution.

1.7 Let $y[k] = \sin(\omega T k)$ with $k \in \mathbb{Z}$, determine a so that $y[k]$ satisfies the difference equation

$$y[k] - a y[k-1] + y[k-2] = 0.$$

See Solution.

Chapter 2

Covariance Sequence, Z-transform

[2.1] Determine the deterministic correlation function $r_{x,y}(\ell)$ between $x(n) = \{\overset{\downarrow}{1}, 1\}$ and $y(n) = \{\overset{\downarrow}{1}, 1, 1\}$. ($r_{x,y}(\ell) = \sum_k x(k)y(k-\ell)$)
See Solution.

[2.2] Determine the circular convolution between $x(n) = \{\overset{\downarrow}{1}, 1, 0, 0\}$ and $y(n) = \{\overset{\downarrow}{1}, 1, 1, 1\}$ (for $N = 4$). See Solution.

[2.3] Qrt likes to calculate the *linear* convolution of $x(n) = \{1, 1\}$ and $y(n) = \{1, 1, 1\}$ using the fast Fourier transform (FFT). What is the required minimum number of data points N in the FFT calculation? See Solution.

[2.4] Determine the (two-sided) z-transform of $x(n) = (1/2)^n$ for $n \geq 0$ and $x(n) = 0$ for $n < 0$.
($X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$) See Solution.

[2.5] Determine the one-sided z-transform of $x(n) = (1/2)^n$ for $n \geq 0$ and $x(n) = (1/2)^{-n}$ for $n < 0$.
($X^+(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$) See Solution.

[2.6] Determine the radius of convergence (ROC) for the (two-sided) z-transform of $x(n) = (1/2)^n$ for $n \geq 0$ and $x(n) = 0$ for $n < 0$. See Solution.

[2.7] (a) Show that for any constant $r \in \mathbb{C}$, and any $M, N \in \mathbb{Z}$, we have

$$\sum_{n=M}^{N} r^n = \begin{cases} \frac{r^M - r^{N+1}}{1-r} & \text{if } r \neq 1 \\ N - M + 1 & \text{if } r = 1 \end{cases}.$$

(b) Show that if $|r| < 1$, then

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

(c) Show that if $r \in \mathbb{R}$ with $|r| < 1$, then

$$\sum_{n=0}^{\infty} n r^{n-1} = \frac{1}{(1-r)^2}.$$

Solution not available.

[2.8] The signal $x(n)$ is defined as

$$x(n) = \begin{cases} \left(\frac{1}{4}\right)^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Determine the deterministic autocorrelation sequence $r_{xx}(k)$.
$(r_{xx}(k) = \sum_n x(n)x(n-k))$.
See Solution.

[2.9] Determine all possible impulse responses $h[n]$ that would yield the transfer function

$$H(z) = \frac{5z^{-1}}{(1-2z^{-1})(3-z^{-1})}.$$

Specify their corresponding ROCs and their properties in terms of stability and causality.
See Solution.

[2.10] For which values of the constant a is the transfer function $H(z)$ below stable?

$$H(z) = \frac{z^3 + z^2 + az + \frac{1}{2}}{z^3 + az^2 + z + \frac{1}{2}}$$

See Solution.

[2.11] a) The transfer function of a filter is

$$H(z) = \frac{1}{z+4}$$

and is valid for $|z| < 4$. Is this filter

i) causal?
ii) stable?

b) Find a stable impulse response for a system with the transfer function

$$H(z) = \frac{1}{(z-4)(z-0.1)}$$

Also, calculate the region of convergence (ROC) where the expression is valid.

See Solution.

Chapter 3

Discrete Fourier Transform, Fast Fourier Transform

[3.1] The first 3 points of a four point FFT ($f = 2\pi k/4$, $k = 0, 1, 2, 3$) of a real-valued sequence are $\{10,\ -2+2i,\ -2\}$. The remaining point is...

A	B	C	D	E	F	G	H	I	J	
10	-10	$+2-2i$	$-2-2i$	$-2+2i$	$+2+2i$	2	$2i$	-2	$-2i$	See Solution.

[3.2] Consider the sequence $\{1, -1, 1, 0, -1, 1, -1\}$. The Fourier transform of the sequence has a zero for $f = 1/6$. Verify that this is the case by

(a) taking the Fourier transform and letting $f = 1/6$.

(b) computing the spectrum by taking the Fourier transform of the correlation sequence and letting $f = 1/6$.

(c) taking the Z-transform and factoring the polynomial to see if $f = 1/6$ is a zero.

See Solution.

[3.3] Let $X[k]$ be the DFT of the real sequence $x[n]$, $n = 0, \ldots, N-1$. Also, let $Y[k]$ be the DFT of the sequence $y[n]$, which is the reversed signal, i.e., $y[n] = x[N-1-n]$. Express $X[k]$ in terms of $Y[k]$.

See Solution.

[3.4] Analog data to be spectrum-analyzed are sampled at 10 kHz and the DFT of 1024 samples computed. Determine the frequency spacing between spectral samples.

See Solution.

[3.5] Let $x[n]$ and $y[n]$ be two sequences with

$$\begin{aligned} x[n] &= 0 \qquad \text{for } n < 0 \text{ and } n \geq 8 \\ y[n] &= 0 \qquad \text{for } n < 0 \text{ and } n \geq 20 \end{aligned}$$

A 20-point DFT is performed on $x[n]$ and $y[n]$. The two DFT's are multiplied and an inverse DFT is performed resulting in a new sequence $r[n]$.

(a) Which elements of $r[n]$ coincide with the linear convolution of $x[n]$ and $y[n]$?

(b) How should the procedure be changed so that all elements of $r[n]$ correspond to elements of the linear convolution of $x[n]$ and $y[n]$?

See Solution.

[3.6] We wish to filter a long sequence through a FIR filter of length 128. We consider three options:

(a) Direct filtering

(b) Overlap-save method with a 256-point FFT

(c) Overlap-save method with a 512-point FFT

Recall that an FFT requires $\frac{N}{2}\log_2(N)$ complex multiplications. Determine the number of multiplications required by each method and establish and explain which method is most efficient.

See Solution.

[3.7] We wish to compute the DFTs $X(k)$ and $Y(k)$ of the two real sequences $x(n)$ and $y(n)$. We have software code that performs the DFT on complex sequences. We use the existing code on the complex sequence $z(n) = x(n) + jy(n)$ to obtain $Z(k)$. Determine how $X(k)$ and $Y(k)$ may be obtained from $Z(k)$.

See Solution.

[3.8] The infinite sequence $x(n) = (\frac{1}{2})^n$, $n \geq 0$, has a Fourier transform $X(f)$. The finite sequence $y(n)$ has eight terms, $y(0), \ldots, y(7)$. The 8-point DFT of $y(n)$ is denoted $Y(k)$. Now, it turns out that $Y(k) = X(k/8)$, $k = 0, \ldots, 7$. Determine the finite sequence $y(n)$. See Solution.

[3.9] Given the sequences

$$x_1[n] = \{\underset{\uparrow}{1},\ 2,\ 3,\ 1\} \qquad x_2[n] = \{\underset{\uparrow}{4},\ 3,\ 2,\ 2\}$$

determine

(a) Their linear convolution.

(b) Their circular convolution, using direct computation in the time-domain.

(c) Their circular convolution, using the 4-point DFT and IDFT.

See Solution.

[3.10] When computing a 9-point DFT, the sequence could be padded with seven zeros and a 16-point FFT algorithm used. However, to avoid this padding and extra computation, three 3-point DFTs may be used. The left hand side of Figure 3.2 is composed of three identical "butterflies". Determine the coefficients, a_{ij}, in this "butterfly", see Figure 3.1.

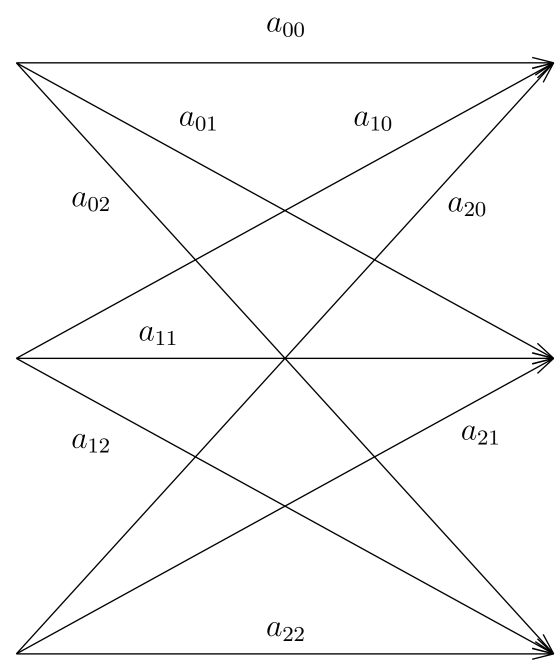
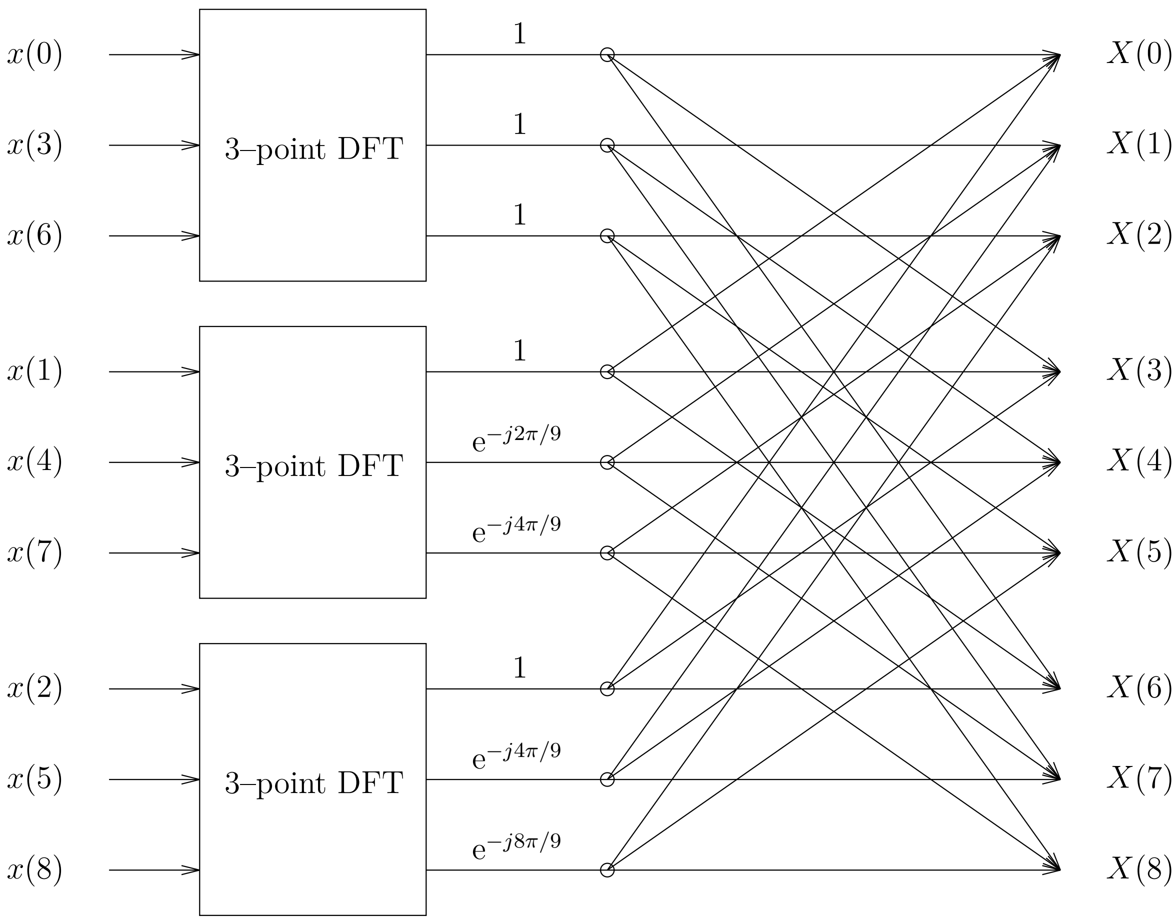


Figure 3.1: “Butterfly”

See Solution.



$x(0)$
$x(3)$
$x(6)$
$x(1)$
$x(4)$
$x(7)$
$x(2)$
$x(5)$
$x(8)$
3–point DFT
3–point DFT
3–point DFT
1
1
1
1
$e^{-j2\pi/9}$
$e^{-j4\pi/9}$
1
$e^{-j4\pi/9}$
$e^{-j8\pi/9}$
$X(0)$
$X(1)$
$X(2)$
$X(3)$
$X(4)$
$X(5)$
$X(6)$
$X(7)$
$X(8)$

Figure 3.2:

[3.11] In many applications, we want to analyze data using a sliding window. Assume that at time n, we have calculated an N-point DFT, $X^{(n)}(k)$, $k = 0, \ldots, N-1$, from measured data $x^{(n)} = \{x(n), \ldots, x(n+N-1)\}$

a) Show that, given the DFT of x^n, you can calculate $X^{(n+1)}(k)$, $k = 0, \ldots, N-1$, i.e. the DFT of $x^{(n+1)} = \{x(n+1), \ldots, x(n+N)\}$, using the equation

$$X^{(n+1)}(k) = [X^{(n)}(k) - x(n) + x(n+N)]e^{j\frac{2\pi k}{N}}$$

b) First calculate the DFT of data sequence $\{1, 2, 0, 1\}$ and then use the algorithm in a) to calculate the DFT of $\{2, 0, 1, 2\}$.

See Solution.

[3.12] Let $x(t)$ be a continuous, real valued and periodic signal (period P) which is band-limited to the interval $[-M/P,\ M/P]$ Hz. Assume that M is an integer, so that the Fourier series expansion of $x(t)$ is

$$x(t) = \sum_{k=-M}^{M} c_k e^{j2\pi kt/P}.$$

Form the sequence $x_1(n)$ by sampling $x(t)$ with the sampling rate $T_1 = \frac{P}{2M}$, i.e.

$$x_1(n) = x(nT_1), \quad n = 0, 1, ..., 2M-1.$$

Let $X_1(k)$ be the DFT of one period of $x_1(n)$ beginning at $n = 0$

$$X_1(k) = \sum_{n=0}^{2M-1} x_1(n)e^{-j2\pi nk/2M}$$

Based on $x_1(n)$, we want to determine the sequence $x_2(n)$ corresponding to sampling $x(t)$ with *double* the sampling rate:

$$x_2(n) = x(n\frac{T_1}{2}), \quad n = 0, 1, ..., 4M-1.$$

Let $X_2(k)$ be the DFT of one period of $x_2(n)$. Show how to calculate $X_2(k)$ directly from $X_1(k)$.

See Solution.

[3.13] Consider the following periodic signal:

$$x(n) = \{\ldots,\ 1,\ 0,\ 1,\ 2,\ \underset{\uparrow}{3},\ 2,\ 1,\ 0,\ 1, \ldots\}$$

(a) Sketch the signal $x(n)$ and its magnitude and phase in the frequency domain.

(b) Using the results above, verify Parseval's identity by computing the power in the time and frequency domains.

Solution not available.

[3.14] a) Determine the circular convolution between the sequence $x = \{0, 0, 1, 0, 1, 0, 0\}$ and $y = \{1, 2, 3, 4, 5, 6, 7\}$.

b) Define the sequences

$$y = \{0, 0, 1, 0, 1, 0, 0\}$$

and

$$x = \{1, 2, 3, 4, 5, 6, 7\}$$

We form the new sequence

$$z = \mathrm{IDFT}[\mathrm{DFT}[x] \cdot \mathrm{DFT}[y]]$$

Calculate z.

See Solution.

[3.15] A DSP is used to collect and process signals. The DSP contains a circuit that can perform N-point FFTs. Often we need to calculate FFTs of longer sequences to get sufficient frequency resolution. How can this be achieved in an efficient way? Comment on the general case and on the corresponding gains in frequency resolution.
(Hint: consider sequences of length $2N$ to begin with.)

See Solution.

[3.16] The sequence $y(n)$ is calculated according to

$$y(n) = \mathrm{IDFT}_{50}[X_1 \cdot X_2]$$

where $X_1 = \mathrm{DFT}_{50}[x_1]$ and $X_2 = \mathrm{DFT}_{50}[x_2]$. The sequences $x_1(n)$ and $x_2(n)$ are defined as $x_1(n) \neq 0$ within the interval $10 \leq n \leq 29$ and 0 outside $x_2(n) \neq 0$ within the interval $0 \leq n \leq 49$ and 0 outside

a) For what values of n can we be certain that $y(n)$ is identical to the linear convolution of $x_1(n)$ and $x_2(n)$?

b) Describe mathematically a method to calculate the linear convolution using 32-point FFTs and IFFTs!

See Solution.

[3.17] Let $x(t)$ be a continuous, real valued and periodic signal (period P) that is band limited over the interval $[-M/P,\ M/P]$ Hz. Assume that M is an integer, so that the Fourier series expansion of $x(t)$ is

$$x(t) = \sum_{k=-M}^{M} c_k e^{j2\pi kt/P}.$$

Form the sequence $y(n)$ by sampling $x(t)$ with the sampling rate $T_1 = \frac{P}{2M}$, i.e.:

$$y(n) = x(nT_1), \quad n = 0, 1, ..., 2M - 1.$$

Let $Y(k)$ be the DFT of a period of $y(n)$ starting at $n = 0$

$$Y(k) = \sum_{n=0}^{2M-1} y(n) e^{-j2\pi nk/2M}$$

Determine $Y(k)$ as a function of c_k.

See Solution.

[3.18] Given the two sequences $x(n)$ and $h(n)$, defined by

$$x(n) = \{\underset{\uparrow}{1}, 2, 0, -1, 0, -1, 1, 2, 1, 1, 0, -1\} \text{ and } h(n) = \{\underset{\uparrow}{1}, -1, 1\}$$

Determine the linear convolution of these two sequences using the, so called

a) Overlap-add method and

b) Overlap-save method

respectively, using **three** data blocks from the long sequence.

See Solution.

Chapter 4

Decimation & Interpolation, Multirate

[4.1] Consider a $\uparrow 3$ interpolator with a 18-tap FIR low-pass filter. An alternative implementation is in a polyphase structure with q_1 filters where each filter has q_2 taps. Determine q_1 and q_2! See Solution.

[4.2] $h[n] = \{h_0, h_1, \ldots, h_N\}$ where $N = M \cdot K$, is the impulse response of a low-pass filter with transfer function magnitude

$$|H(\omega)|^2 \approx \begin{cases} 1 & \text{for } |\omega| \leq \Omega \text{ with } \Omega < \frac{\pi}{M} \\ 0 & \text{otherwise.} \end{cases}$$

Determine the pass-band of $\bar{h}[n] = \{h_0, h_M, \cdots, h_{MK}\}$.

See Solution.

[4.3] Consider the signal $x(n) = a^n$, $n \geq 0$, $|a| < 1$.

(a) Determine the Fourier transform (spectrum) $X(\omega)$ of $x(n)$.

(b) The signal $x(n)$ is applied to a decimator which reduces the rate by a factor of 2. Determine the Fourier transform of the output.

(c) Show that the transformed output is simply the Fourier transform of $x(2n)$.

See Solution.

[4.4] If someone had asked you to upsample a signal by a factor 2, before you took this course on Digital Signal Processing, you would probably have used linear interpolation i.e between every two samples, you would have inserted a new sample which is the average of the two surrounding old samples.

How can this simple strategy be described within the framework for interpolation given in this course? Assume the standard interpolation structure with an interpolation component followed by a filter.

a) Determine the impulse response of the filter needed in the linear interpolation upsampling circuit.

b) Determine the corresponding frequency response.

c) Compare to the ideal interpolation filter in terms of causality and spectrum properties.

d) Determine a polyphase implementation of the linear interpolation circuit.

e) Determine the spectrum of the output $y(n)$ when the spectrum of the input $x(n)$ is

$$X(\omega) = \begin{cases} 1 & 0 \leq |\omega| \leq 0.2\pi \\ 0 & \text{otherwise} \end{cases}$$

f) Determine the spectrum of $y(n)$ when the spectrum of $x(n)$ is

$$X(\omega) = \begin{cases} 1 & 0.7\pi \leq |\omega| \leq 0.9\pi \\ 0 & \text{otherwise} \end{cases}$$

Hint: The linear interpolation upsampling circuit should realize the following function:

$$y(n) = \begin{cases} x(n/2) & n \text{ even} \\ \frac{1}{2}\left[x\left(\frac{n-1}{2}\right) + x\left(\frac{n+1}{2}\right)\right] & n \text{ odd} \end{cases}$$

See Solution.

[4.5] Consider an arbitrary digital filter $h[n]$ with transfer function

$$H(z) = \sum_{n=-\infty}^{+\infty} h[n] z^{-n} .$$

(a) Prove the known result

$$H(z) = \mathrm{P}_0\left(z^2\right) + z^{-1}\mathrm{P}_1\left(z^2\right) ,$$

where $p_0[n] = h[2n]$ and $p_1[n] = h[2n+1]$ are the polyphase decomposition components with decimation 2.

(b) Particularly, let

$$H(z) = \frac{a + z^{-1}}{1 + az^{-1}} .$$

Write down expressions for the two components of the polyphase decomposition assuming $H(z)$ is a causal filter, i.e. report $p_0[n]$, $\mathrm{P}_0(z)$, $p_1[n]$, and $\mathrm{P}_1(z)$. Notice that $\forall a \in \mathbb{R}$, $H(z)$ is all-pass. Are the polyphase components all-pass as well?

See Solution.

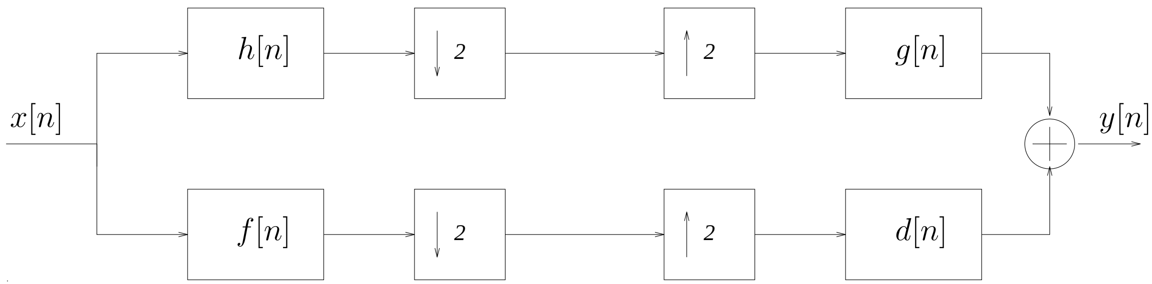


Figure 4.1: Filter bank under consideration in Exercise 4.6.

[4.6] A signal can be analyzed by dividing its spectrum into narrow frequency bands and examining each band separately. The signal in each frequency band is decimated to keep the same total amount of data, but it is still possible to restore the signal without distortion. If the implementable filters $h[n]$, $g[n]$, $f[n]$ and $d[n]$ are determined appropriately, the aliasing from the decimation will be eliminated completely.

(a) Consider the filter bank in Figure 4.1 and determine requirements on the filter transfer functions $H(\nu)$, $G(\nu)$, $F(\nu)$ and $D(\nu)$, so that the output signal $y[n]$ is identical to the input signal $x[n]$ except for a delay.

(b) Assume that the filters $h[n]$ and $f[n]$ are so-called "Quadrature Mirror Filters", i.e. $F(\nu) = H\left(\nu - \frac{1}{2}\right)$ and that $H(\nu) = G(\nu)$. Determine $D(\nu)$, and the relationships between the impulse responses $h[n]$, $f[n]$ and $d[n]$. Also, determine the requirements on $H(\nu)$ for a distortion-free transmission.

See Solution.

[4.7] Prove the following expressions for an interpolator of order I.

a) The impulse response $h(n)$ can be expressed as

$$h(n) = \sum_{k=0}^{I-1} \tilde{p}_k(n-k)$$

where

$$\tilde{p}_k(n) = \begin{cases} p_k(n/I), & n = 0, \pm I, \pm 2I, \dots \\ 0, & \text{otherwise} \end{cases}$$

b) $H(z)$ may be expressed as

$$H(z) = \sum_{k=0}^{I-1} z^{-k}\tilde{P}_k(z) = \sum_{k=0}^{I-1} z^{-k} P_k(z^I)$$

where z corresponds to the data rate at $h(n)$.

c) Now let z correspond to the data rate at $p(n)$, then

$$P_k(z) = \frac{1}{I} \sum_{n=-\infty}^{\infty} \sum_{l=0}^{I-1} h(n) e^{\frac{j2\pi l(n-k)}{I}} z^{-(n-k)/I} =$$

$$\Big/ z = e^{j\omega I} \Big/ = \frac{1}{I} \sum_{l=0}^{I-1} H\left(\omega - \frac{2\pi l}{I}\right) e^{jk(\omega - 2\pi l/I)} .$$

Note that ω corresponds to the data rate at $h(n)$, not at $p(n)$.

Solution not available.

[4.8] Consider a signal $x(n)$ with the Fourier transform $X(\omega)$, where

$$X(\omega) = 0 \quad \text{for} \quad \omega_m < |\omega| < \pi$$

a) Show that the signal $x(n)$ can be recovered from $x(nD)$ where $D = 2, 3, 4, ...$ as long as the sampling frequency $\omega_s = 2\pi/D \geq 2\omega_m$.

b) Show that $x(n)$ can be reconstructed by

$$x(n) = \sum_{k=-\infty}^{\infty} x(kD) h(n - kD)$$

and determine $h(n)$.

See Solution.

[4.9] a) A signal $x(n)$ goes through a kind of combined filter and decimator, resulting in the signal $y(n)$, described by

$$y(n) = \sum_{k=0}^{N} h(k) x(2n - Lk) \tag{4.1}$$

The filter $h(n)$ is order N and L is an integer ≥ 1. We assume that $x(n) = 0$ when $n < 0$.

Express the Z transform $Y(z)$ in $H(z)$ and $X(z)$.

(Hint: If $y(n) = u(2n)$ then $Y(z) = 1/2\left[U(z^{1/2}) + U(-(z^{1/2}))\right]$.)

b) Let $L = 4$, $N = 1$ and

$$h(n) = \begin{cases} 1 & n = 0, 1 \\ 0 & \text{otherwise} \end{cases} \qquad f(n) = \begin{cases} 1 & n = 0, 1, 2, 3 \\ 0 & \text{otherwise} \end{cases} .$$

Show that $f(n)$ passes unchanged through the device (4.1), that is

$$f(n) = \sum_{k=0}^{N} h(k) f(2n - Lk)$$

Perform the calculations in the time domain.

c) Verify the calculations from exercise a) by inserting $F(z)$ in the expression for $Y(z)$, showing that the relation in exercise b) holds also in the frequency domain. (Hint: $F(z) = \frac{1-z^{-4}}{1-z^{-1}}$.)

See Solution.

Chapter 5

Finite Word Length Effects

[5.1] Uniform quantization is the effect of a nonlinear function, $Q(x)$, acting on a sequence $x[n]$, as seen in Fig. 5.1. In order to ease analysis, this may be modeled as an additive noise term $e[n]$ perturbing the signal, as seen in Equation (5.1).

$$y[n] = Q\left[x[n]\right] = x[n] + e[n] \tag{5.1}$$

Note, then, $y[n]$ as the output of the quantizer. Let $x[n]$ be a white sequence with zero-mean and variance σ_x^2. If the quantization level, Δ, is small relative to σ_x^2, we can model $e[n]$ as white, uniformly distributed between $-\frac{\Delta}{2}$ and $\frac{\Delta}{2}$, and uncorrelated with the input signal $x[n]$. Under this assumptions,

(a) determine the mean, variance, and correlation sequence for $e[n]$.

(b) determine the SNR in $y[n]$, i.e. $\text{SNR}_{y[n]} = \sigma_x^2/\sigma_e^2$.

(c) if the signal $y[n]$ is filtered through a system with impulse response

$$h[n] = \begin{cases} 0 & \text{for } n < 0 \\ \frac{1}{2}\left(a^n + (-a)^n\right) & \text{for } n \geq 0 \end{cases},$$

compute the variance of the noise and the SNR at the output of the filter.

See Solution.

[5.2] Consider the second-order system in Figure 5.2. The coefficients a and b are represented exactly and the products $ay(n-1)$ and $by(n-2)$ are not truncated. Two-complement addition is used with f according to Figure 5.3.

(a) Determine the stability region for the case with no overflow.

(b) If $x(n) = 0$, what are the conditions for no overflow?

Solution not available.

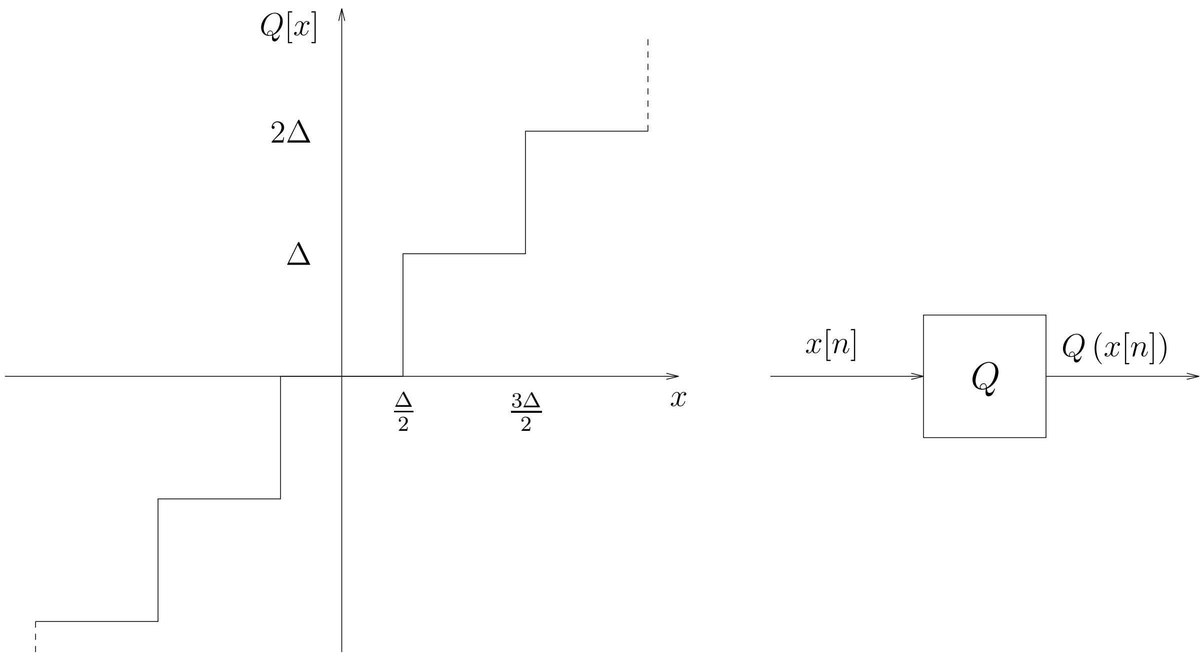


Figure 5.1: Quantization and its statistical model, illustrating Exercise 5.1.

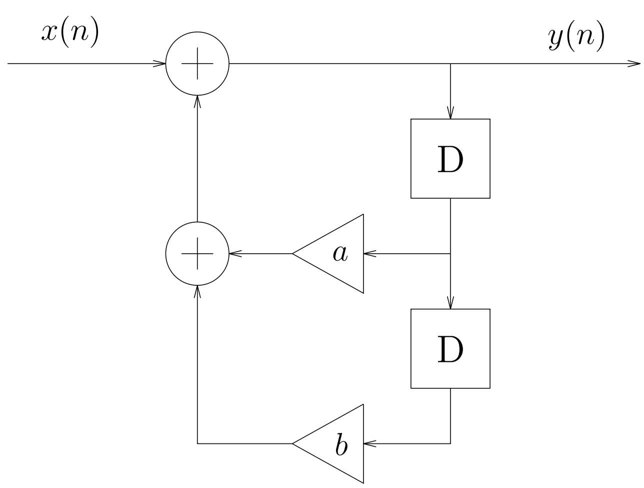


Figure 5.2:

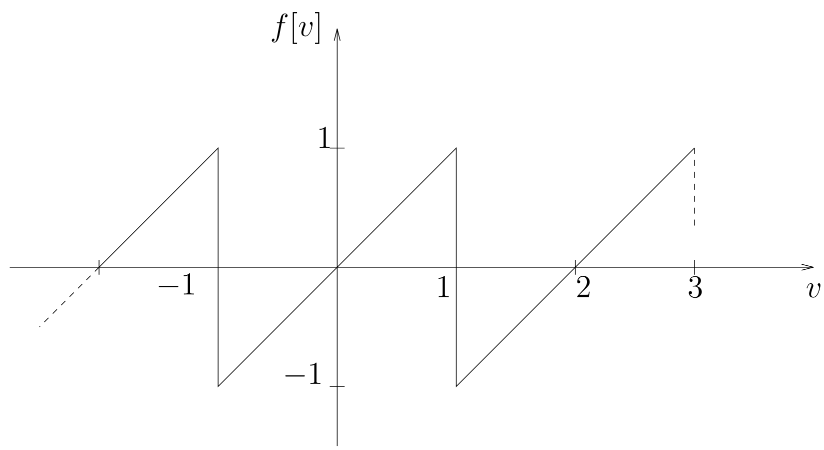


Figure 5.3:

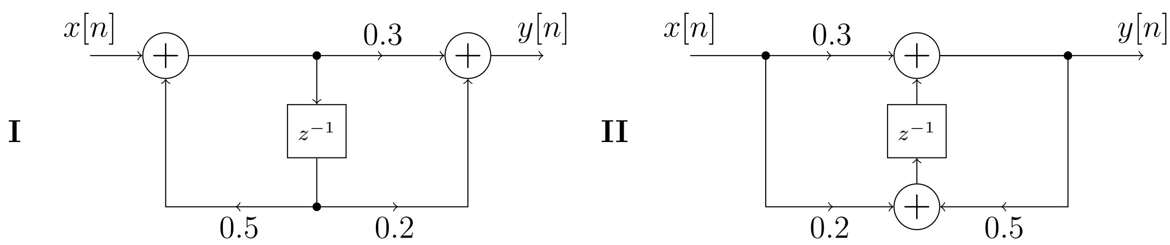


Figure 5.4: Two realizations of the first-order system discussed in Exercise 5.3.

[5.3] In Fig. 5.4 we see two different realizations of a first-order system. The realizations are implemented in binary fix-point arithmetic, 2-complement representation, and with round-off errors occurring in the multiplications. Compare the two realizations in terms of round-off noise variance at the output.

See Solution.

[5.4] A digital filter is implemented with fix-point arithmetic where the data is represented with b bits excluding the sign bit. Multiplication results in a $2b$ product which is rounded to b bits. The additions do not cause overflow.

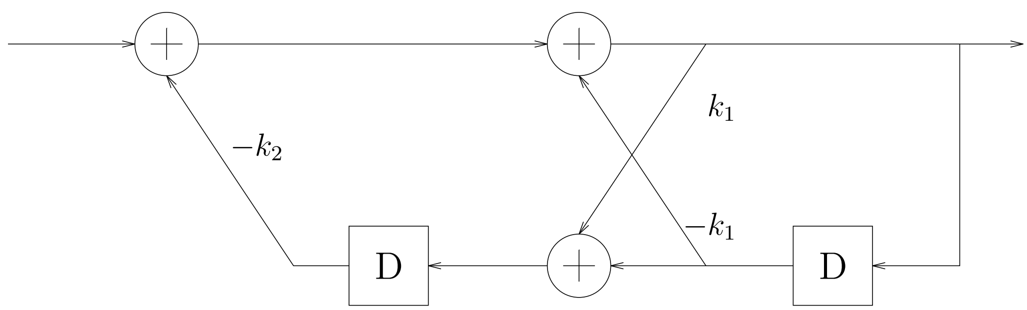


Figure 5.5:

Determine the round-off noise variance on the output of the second-order lattice filter in Figure 5.5 when $k_1 = -\frac{10}{11}$ and $k_2 = \frac{3}{8}$.

See Solution.

[5.5] We want to implement the transfer function

$$H(z) = \frac{(1 + .5z^{-1})(1 + .25z^{-1})}{(1 - .5z^{-1})(1 - .25z^{-1})}$$

as two first order filters in cascade form, see Fig. 5.6. How should a_1, a_2, b_1 and b_2 be determined in order to minimize the quantization noise (caused by the multipliers) at the output?

See Solution.

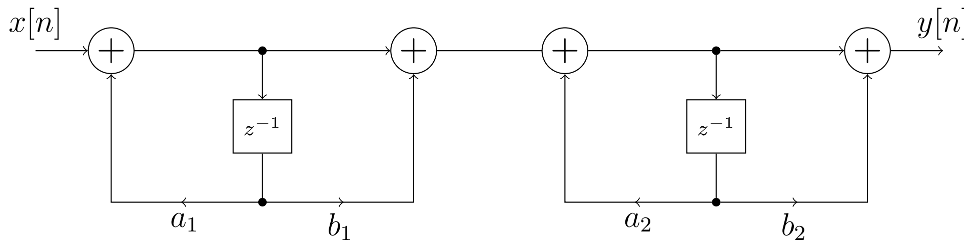


Figure 5.6: Cascade implementation of the second-order transfer function in Exercise 5.5.

[5.6] Determine for which values of a that the filter in Figure 5.7 is stable without overflow. The input signal, $x(n)$, is limited to the interval $[0, 1]$ and the fixed-point representation can handle numbers in the interval $[-1, 2[$.

a) Assume that the initial state is arbitrary.

b) Assume that the initial state is 0.

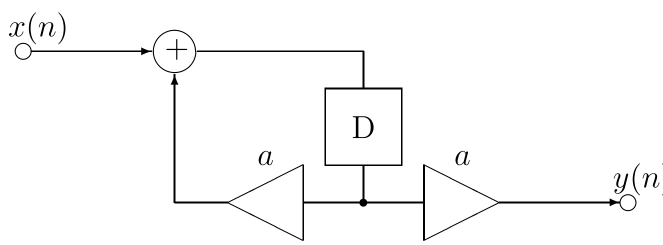


Figure 5.7:

See Solution.

[5.7] Figure 5.8 shows two filters implemented using binary fixed-point arithmetic with round-off.

i) Which realization gives less round-off noise (caused by round off errors from the multipliers) at the output when $b_0 = 2$, $b_1 = 1/3$, $a_1 = 1/2$, $k = 3$?

ii) Which realization gives less round-off noise at the output when $b_0 = 2$, $b_1 = 2$, $a_1 = 2$, $k = 1/2$?

iii) Determine the value of a_1, as a function of b_0, b_1, k, that gives the smallest round-off noise of the first filter?

See Solution.

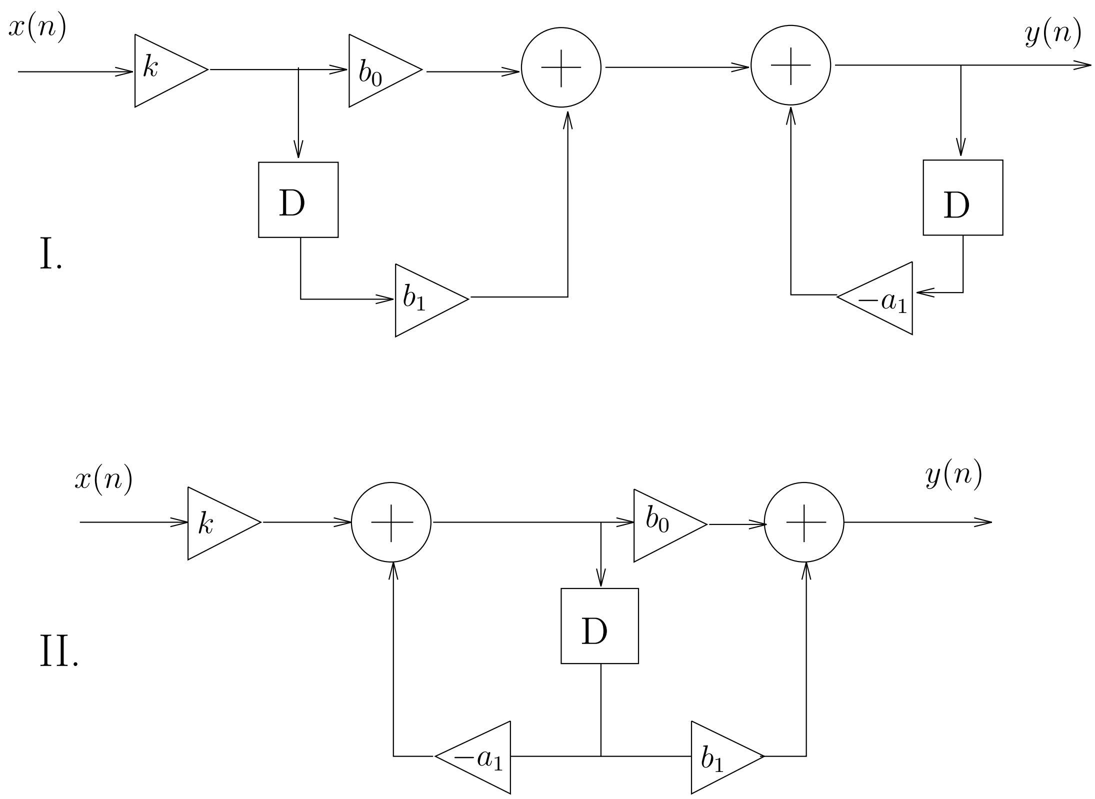


Figure 5.8:

[5.8] In figure 5.9, the input signal is sampled at a sampling rate of 8 kHz. The samples are represented using $b + 1$ bits, of which one bit is a sign bit. The signal is filtered according to the figure, using round-off at the multiplier.

a) What is the noise at the output caused by A/D conversion and round off?

b) Assume that the input signal is a sinusoid with a frequency of 2 kHz, and with an amplitude set to 50 % of maximum range of the A/D converter. Determine b so that the output SNR exceeds 50 dB. Use the result from a).

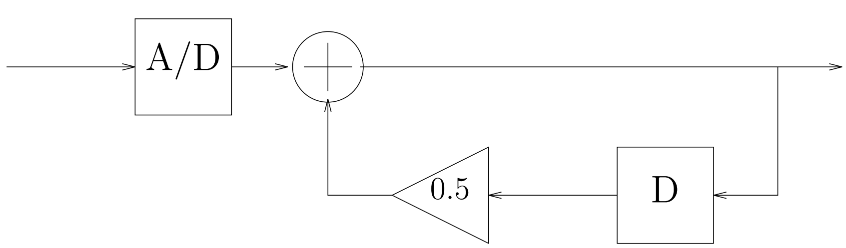


Figure 5.9:

See Solution.

[5.9] Figure 5.10 shows a filter implemented as a cascade of first order sections. Binary fixed-point arithmetic with round-off is used.

a) You can choose in which order to implement these two first order sections. Which order will you choose to get the smallest round-off noise (caused by round-off errors from the multipliers) at the output.

b) Let σ_e^2 be the variance of the round-off noise. Find a closed form expression for the output noise variance caused by round-off errors in the multipliers.

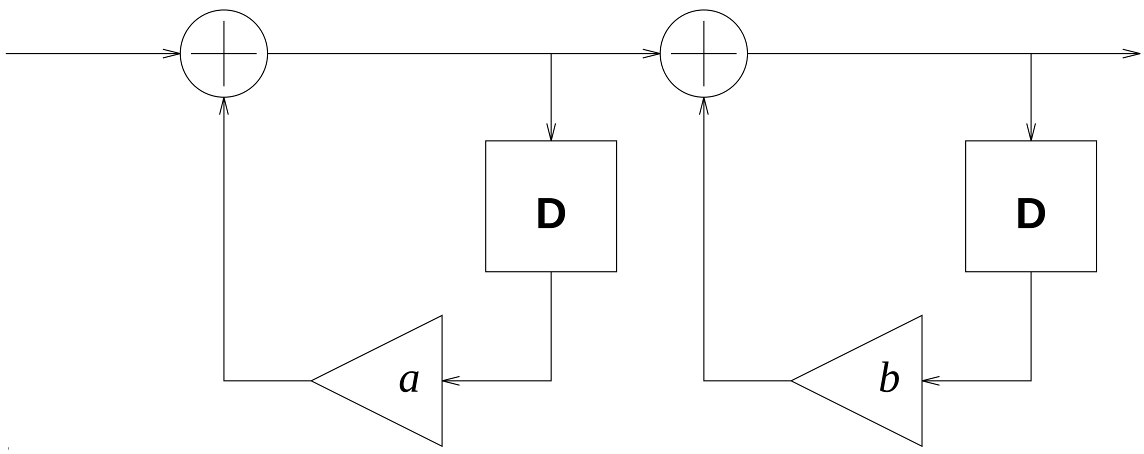


Figure 5.10:

See Solution.

Chapter 6

Non-Parametric Spectral Estimation

[6.1] We wish to analyze the frequency content of a signal $x(t)$. The signal is anti-alias filtered, sampled, windowed, and the DFT is computed for a large number of points. Four different windows are applied and the absolute value of the DFT is plotted. Note that this DFT will actually be proportional to the modified periodogram of the sampled signal $x[n]$. If the original spectral content of the signal $x(t)$ is the one displayed in Fig. 6.1, combine each window with its resulting modified periodogram in Fig. 6.2.

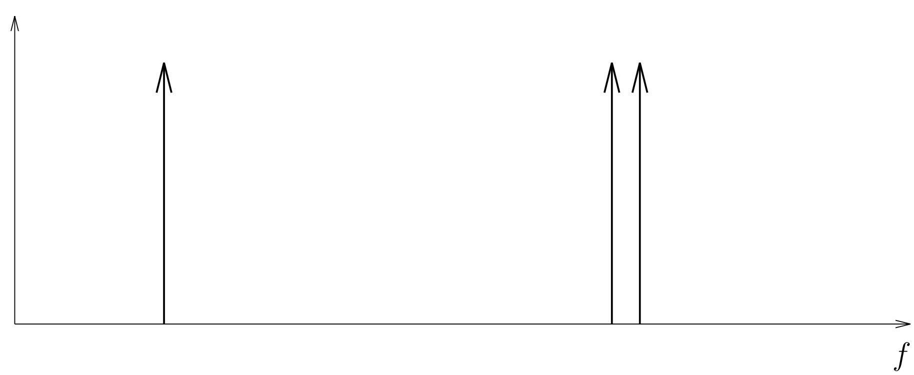


Figure 6.1: True spectrum of the signal $x(t)$ in Exercise 6.1.

See Solution.

[6.2] The Bartlett method is used to estimate the power spectrum of a signal $x(n)$, i.e., we form K data segments, each containing M samples. We know that the power spectrum consists of a single peak with a 3-dB bandwidth of 0.01 cycle per sample, but we do not know the location of the peak.

(a) Assuming that N is large, determine the value of $M = N/K$ so that the spectral window is narrower than the peak.

(b) Explain why it is not advantageous to increase M beyond the value obtained above.

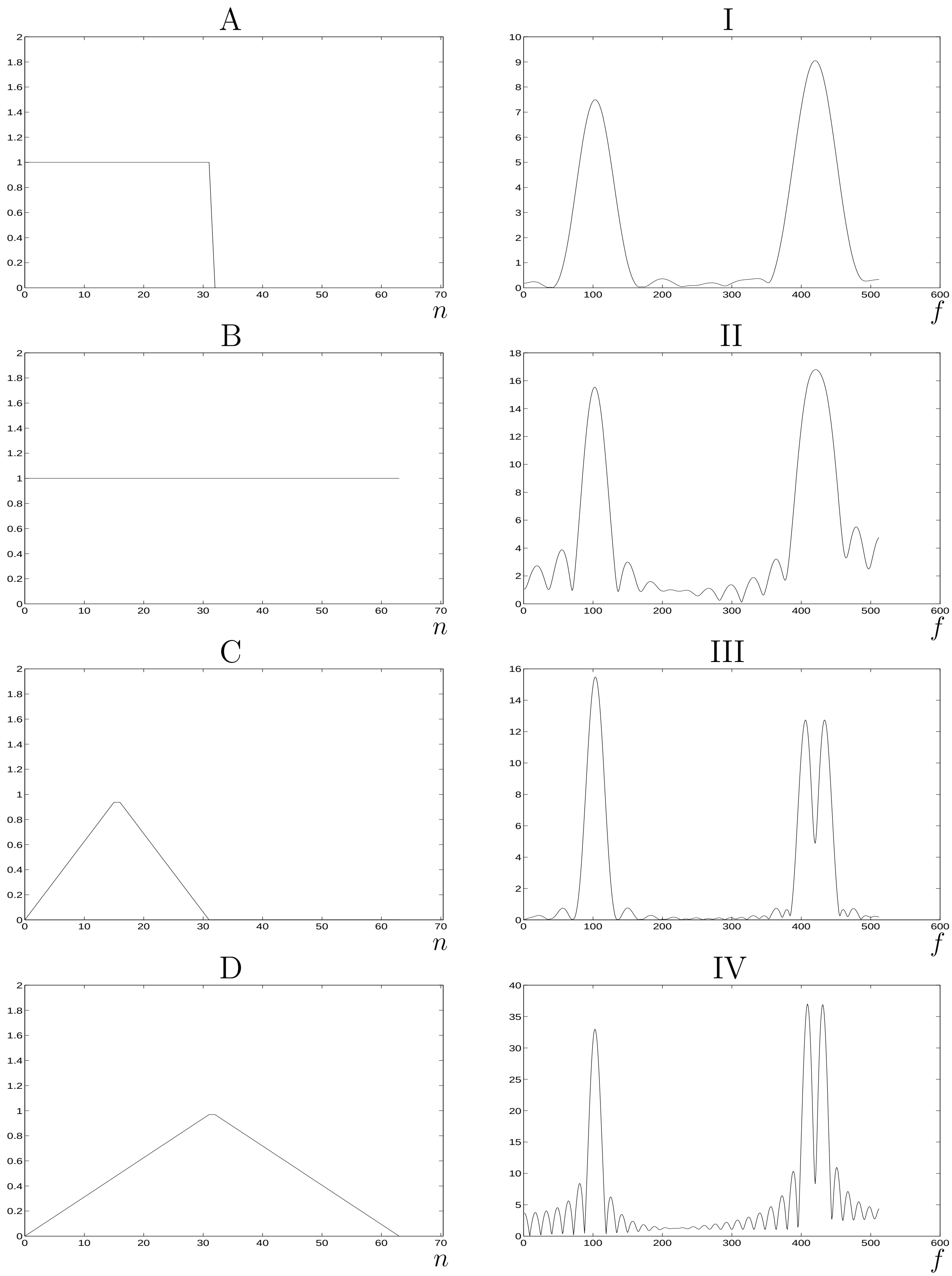


Figure 6.2: Windows and modified periodograms to be paired in Exercise 6.1.

See Solution.

[6.3] We wish to estimate the spectrum of a signal using Welch's method. The signal is anti-aliased filtered and sampled at 12 kHz. We form K data segments, each containing M samples and a triangular window is applied on each segment.

(a) Assume that the signal contains two frequency components separated by 200 Hz. How large must M be chosen to ensure sufficient frequency resolution?

(b) How many segments K should we average over to ensure that the variance in the estimate is smaller than a 5% of the square of the correct spectral density?

See Solution.

[6.4] The spectral density of a time-discrete signal $x[n]$ is estimated using the Welch method. The window used is

$$w[n] = \cos^2\left(\frac{n\pi}{N}\right)$$

where N is the number of samples in the block. The signal $x[n]$ is defined for $-\frac{N}{2} < n \leq \frac{N}{2}$. Show that the N-point DFT for $x[n]w[n]$ may be computed from the DFT of $x[n]$ without explicitly forming $x[n]w[n]$.

See Solution.

[6.5] In the Welch method, a record of length N is segmented into K sequences of length M.

$$x^{(k)}[n] = x[n + kM - M] \text{ for } 0 \leq n \leq M-1, \text{ and } 1 \leq k \leq K,$$

and these segments are windowed before computing K modified periodograms

$$J_M^{(k)}(\omega) = \frac{1}{MU}\left|\sum_{n=0}^{M-1} x^{(k)}[n]w[n]\mathrm{e}^{-j\omega n}\right|^2 \text{ with } 1 \leq k \leq K,$$

where

$$U = \frac{1}{M}\sum_{n=0}^{M-1} w^2[n]\,.$$

The spectrum estimate is then defined as

$$B_{xx}^w(\omega) = \frac{1}{K}\sum_{k=1}^{K} J_M^{(k)}(\omega)\,.$$

Show that

$$\begin{aligned}\mathrm{E}[B_{xx}^w(\omega)] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{xx}(\theta)W(\omega-\theta)d\theta \\ &= \frac{1}{2\pi}(P_{xx}(\theta) * W(\theta))(\omega)\,,\end{aligned}$$

where

$$W(\omega) = \frac{1}{MU}\left|\sum_{n=0}^{M-1} w[n]e^{-j\omega n}\right|^2 .$$

Hint: Make use of the fact that if $x(n)$ is zero-mean, then

$$r_{xx}[m] = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{xx}(\omega)e^{j\omega m}d\omega .$$

Solution not available.

[6.6] When windowing is applied to the autocorrelation lags in the Blackman-Tukey spectral estimation method, the spectrum is smoothed which helps to reduce the variance. The ratio of the variance of the Blackman-Tukey estimate to the variance of the periodogram is approximately

$$R = \frac{\mathrm{var}[P_{xx}^W(f)]}{\mathrm{var}[P_{xx}(f)]} = \frac{1}{N}\sum_{m=-M}^{M} w^2(m) = \frac{1}{N}\int_{-\pi}^{\pi} W^2(e^{j2\pi f})df .$$

where N is the record length and $2M+1$ is the total window length. Thus, by adjusting the shape and length of the window, the variance of $P_{xx}^W(f)$ can be reduced over that of the periodogram.

Another measure of the amount of smoothing caused by windowing is the width of the main lobe which causes a reduction in resolution. In this case we define this bandwidth as the symmetric interval between the first negative and positive frequencies at which $W(f) = 0$.

Consider the following windows:

Rectangular

$$w_R(m) = 1 \quad |m| \le M$$

Bartlett

$$w_B(m) = 1 - \frac{|m|}{M} \quad |m| \le M$$

Raised Cosine

$$w_H(m) = \alpha + \beta\cos\left(\frac{2\pi m}{2M+1}\right) \quad |m| \le M$$

Note: If $\alpha = \beta = 0.5$ this is the Hann window (often called Hanning window) and if $\alpha = 0.54$ and $\beta = 0.46$ this is the Hamming window.

All the windows are zero outside the specified interval.

(a) Determine the Fourier transform of each of the above windows and sketch each of these functions of f.

(b) For each of these windows, show that the entries in the following table are correct. Assume that $M >> 1$.

Window Name	Approximate Width of Main Lobe	Approximate Variance Ratio (R)
Rectangular	$1/M$	$2M/N$
Bartlett	$2/M$	$2M/(3N)$
Raised Cosine	$2/M$	$2M(\alpha^2 + \beta^2/2)/N$

See Solution.

[6.7] When using the Welch method (without overlapping segments) for spectrum estimation, we can reduce the variance by increasing the number of segments, K. Use the theoretical expression for the variance of the estimate and calculate the relative reduction in variance when going from K to K+1 segments. Sketch the function for $1 \leq K \leq 10$. What is the asymptotic (in K) performance?

See Solution.

[6.8] We obtain $N = 10000$ samples of data $\{x[0], \ldots, x[N-1]\}$ at a sampling frequency $F_s = 1000\,\text{Hz}$. The signal is a sinusoid in a signal-independent additive noise, i.e.

$$x[n] = A \sin(\omega_0 n) + e[n]\,,$$

and we know that the frequency of the sinusoid F_0 is smaller than $F_s/2$, i.e., that the sampling fulfilled the Nyquist criterion. To estimate the frequency F_0 and the amplitude A of the sinusoid we calculate the periodogram $P(\omega)$ of the signal in the interval $[0, \pi]$, i.e.,

$$P(\omega) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x[n]e^{-j\omega n}\right|^2 = \frac{1}{N}|X(\omega)|^2$$

where $X(\omega)$ is the DTFT of the sampled sequence $x[n]$. The plot of the obtained $P(\omega)$ is shown in Figure 6.3.

a) Estimate the frequency F_0 (in Hz).

b) Estimate the amplitude A of the sinusoid.

c) Estimate the noise power σ^2. For this last section, take into account that the actual implementation of the periodogram in Figure 6.3 was done using an N-point DFT, and thus, the average value stated in the figure corresponds to

$$\frac{1}{N}\sum_{k=0}^{N-1} P(\omega_k)$$

with $\omega_k = 2\pi\frac{k}{N}$.

See Solution.

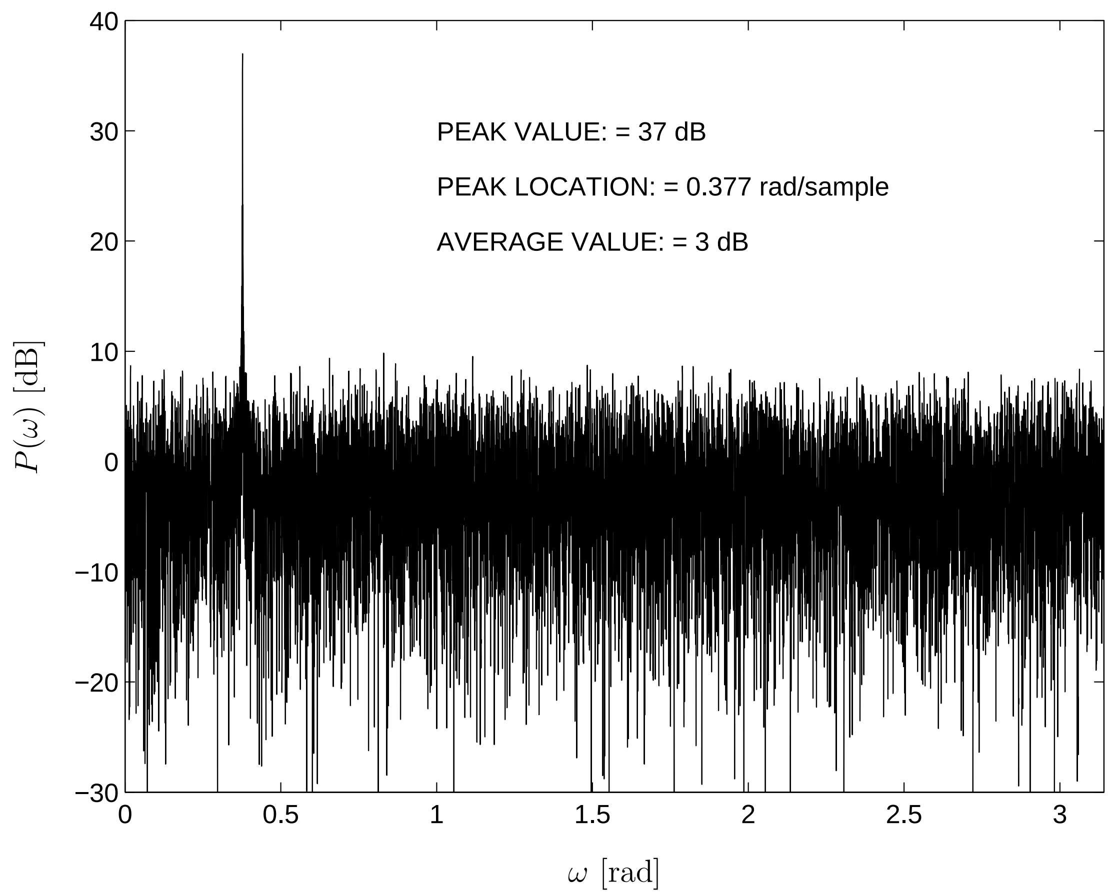


Figure 6.3: Periodogram obtained by processing the received signal $x[n]$ in Exercise 6.8.
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Chapter 7

Model Based Spectral Estimation

[7.1] Determine the least-squares FIR inverse of length 3 to the system with impulse response

$$h(n) = \begin{cases} 2 & n = 0 \\ 1 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Also, determine the minimum squared error e_{min}. See Solution.

[7.2] Consider the signal

$$x(n) = \begin{cases} a^n, & n \geq 0 \\ 0, & \text{otherwise} \end{cases} \qquad -1 < a < 1$$

(a) Show that the optimum first-order one-step predictor, defined by

$$e(n) = x(n) - \hat{x}(n) = x(n) - a_1 x(n-1)$$

is specified by $a_1 = a$ and the minimum squared prediction error is given by $\sum_{n=0}^{\infty} e^2(n) = 1$.

(b) Determine the second-order predictor

$$e(n) = x(n) - a_1 x(n-1) - a_2 x(n-2)$$

and the corresponding minimum squared prediction error.

See Solution.

[7.3] A linear predictor is used to construct a linear-phase filter to adaptively cancel a sinusoidal signal $s[n] = A\cos(2\pi fn + \phi)$, where ϕ is an unknown phase term that we model as $\phi \sim \mathcal{U}[0, 2\pi)$, f is a known deterministic frequency and A a fixed unknown amplitude. The linear predictor has the structure specified in Fig. 7.1, where the blocks marked as D are simple registers, which delay the signal 1 sample.

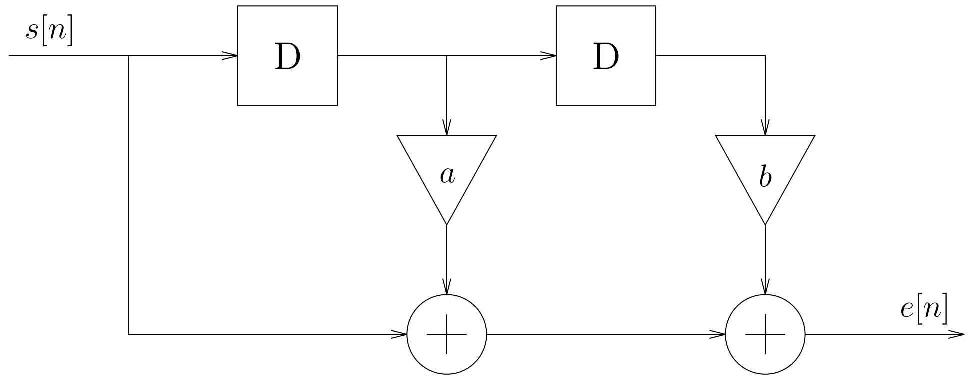


Figure 7.1: Figure representing the linear predictor under design in Exercise 7.3.

Determine the coefficients a and b in terms of the properties of $s[n]$ in order to cancel the signal. Additionally, express the frequency of $s(n)$ in terms of the filter coefficients.

See Solution.

[7.4] The signal $x(t)$ is sampled at $x(nT)$ and has a covariance function $r_x(\tau)$. The discrete time signal is linearly interpolated between the sample values, i.e., $x(nT + T/2)$ is a linear combination of $x(nT)$ and $x(nT + T)$. The linear combination should minimize the mean squared error between the interpolated value and the true value. Determine the linear combination and compare against straight line interpolation.

See Solution.

[7.5] Show that the mean residual power (i.e. the expected value of the inverse filter output power) is given by

$$\mathrm{E}\{e^2(n)\} = r(0) + \sum_{i=1}^{P} a_i r(i)$$

where $r(i)$ is the covariance function of the input sequence, a_i are the transversal filter coefficients, and P is the order of the filter. Solution not available.

[7.6] Using the observations $y(n-1), \ldots, y(n-P)$ we wish to form an estimate, $\hat{y}(n+d)$, of $y(n+d)$. $\hat{y}(n+d)$ should be a linear combination of the observed samples.

$$\hat{y}(n+d) = \sum_{k=1}^{P} a_k y(n-k)$$

The estimate should minimize the mean squared error. Determine a system of equations from which a_k may be computed in terms of $r_y(m)$. See Solution.

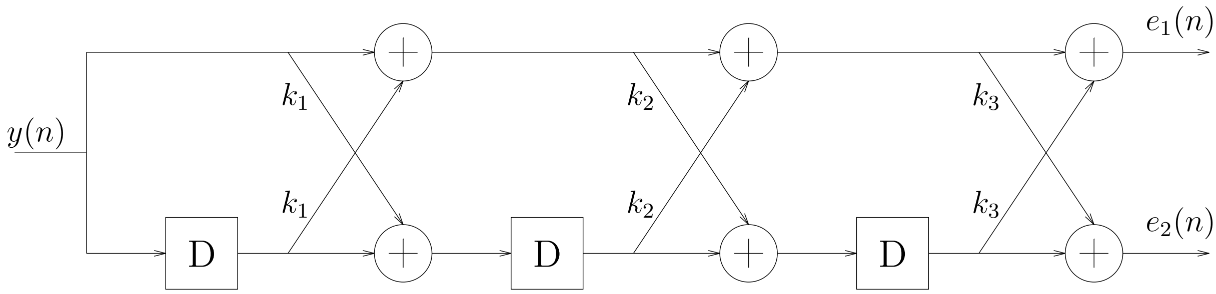


Figure 7.2:

[7.7] In EEG-research a digital lattice filter is sometimes used to detect epileptic signals.

Usually, up to ten layers are used but here we have three layers.

(a) Show that the transfer functions $E_1(z)/Y(z)$ and $E_2(z)/Y(z)$ may also be realized as transversal filters. Determine the transversal filter coefficients, a_i.

(b) Compute the transversal filter coefficients, a_i, from the reflection coefficients, k_i, using the relationship between a_i and k_i in Levinson-Durbin's algorithm.

(c) What must hold for the reflection coefficients for the filter to be stable?

See Solution.

[7.8] Show that for a first-order lattice filter

$$E^{(1)} = E^{(0)}(1 - K_1^2)$$

where $E^{(i)}$ is the expected value of the i^{th}-order residual power, $\mathrm{E}\{e_i^2(n)\}$. Compare against the Levinson-Durbin equations. See Solution.

[7.9] The time continuous process $x(t)$ has a covariance function

$$r_x(\tau) = \frac{1}{2}\mathrm{e}^{-a_1|\tau|} + \frac{1}{2}\mathrm{e}^{-a_2|\tau|}$$

The process is sampled with an interval T_s. The resulting time discrete process is approximated by a third order AR process. Use the Yule-Walker equations to determine the coefficients of the AR process. We have

$$\mathrm{e}^{a_1 T_s} = 2 \qquad \mathrm{e}^{a_2 T_s} = 3$$

See Solution.

[7.10] Determine the parameters $\{k_m\}$ of the lattice filter (often termed reflection coefficients) corresponding to the FIR filter described by the system function

$$H(z) = 1 + 2z^{-1} + \frac{1}{3}z^{-2}$$

See Solution.

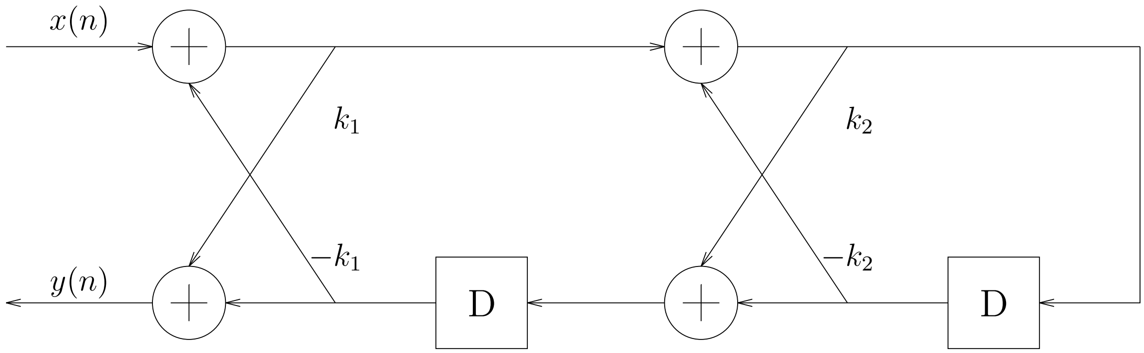


Figure 7.3:

[7.11] Consider the filter in Figure 7.3.

(a) Determine the transfer function

$$H(z) = \frac{Y(z)}{X(z)}$$

(b) For which values of k_1 and k_2 is the filter stable?

See Solution.

[7.12] (a) Sketch the lattice realization for the resonator

$$H(z) = \frac{1}{1 - (2r\cos\omega_0)z^{-1} + r^2 z^{-2}}$$

(b) What happens if $r = 1$?

See Solution.

[7.13] A common method to estimate the parameters of an AR-process is to first estimate the covariance function and then solve Yule-Walker equations to find the parameters. An unbiased estimator of the autocorrelation function is

$$\hat{r}_y[k] = \frac{1}{N - |k|} \sum_{n=0}^{N-|k|-1} y[n]y[n + |k|] \tag{7.1}$$

(a) Using (7.1), determine the estimate of the AR parameters b_0^2, a_1, a_2 from the following data sequence generated by a stationary AR(2) process: $y[0] = 2$, $y[1] = 1$, $y[2] = 2$. Is the estimate of the autocorrelation function given by (7.1) reasonable? Where are the poles of the filter included in the AR(2) model for this process? Explain the implications of these results and evaluate the resulting AR(2) model.

(b) A biased estimator of the autocorrelation function with considerably better variance properties is

$$\hat{r}_y[k] = \frac{1}{N} \sum_{n=0}^{N-|k|-1} y[n]y[n+|k|]\,. \tag{7.2}$$

Redo 7.13a with this new estimate and yield new conclusions.

See Solution.

[7.14] We want to estimate the parameters of an AR(2) model from measured data y

$$y(n) + a_1 y(n-1) + a_2 y(n-2) = v(n) \tag{7.3}$$

where $v(n)$ is white noise. Assume that in reality y is described by the (unknown) model

$$y(n) = b_0 e(n) + b_1 e(n-1)$$

where $e(n)$ is white noise with zero mean and variance 1.

a) Assume that the least squares method is used to estimate the parameters in (7.3). What will the estimates $\hat{a}_1$ and $\hat{a}_2$ converge to if the true parameters are $b_0 = 1$ and $b_1 = 0.3$

b) What is the impulse response for $n = 0, 1, 2$ of the estimated AR-model when the number of measured data goes to infinity. Compare with the true signal impulse response.

See Solution.

[7.15] Let the following AR-model be given

$$y(n) - 0.3y(n-1) + 0.02y(n-2) = e(n).$$

What are the corresponding reflection coefficients?

See Solution.

[7.16] a) Determine the d-step-ahead predictor $\hat{y}(n+d) = ay(n)$ that minimizes the mean squared error

$$E[(y(n+d) - \hat{y}(n+d))^2]$$

for an arbitrary signal $y(k)$ with a given covariance function $r_y(k)$.

b) Do the same but with $\hat{y}(n+d) = \sum_{k=0}^{m} a_k y(n-k)$.

c) What is $\hat{y}(n+d)$ and the variance of the estimation error, when $m = 2$ and $y(k)$ is white noise with variance 1 filtered by $H(z) = 1 + \frac{1}{2}z^{-1}$?

See Solution.

[7.17] Consider the signal

$$y(k) = gv(k) + w(k), \quad k = 0, 1, 2, \ldots, M-1.$$

where $v(k)$ is a known signal, g is a stochastic variable with $E\{g\} = 0$, $E\{g^2\} = \lambda$, $w(k)$ is white noise with variance 1 and $w(k)$ is independent of g. Determine the coefficients in a linear estimator of g

$$\hat{g} = \sum_{k=0}^{M-1} h(k)y(k)$$

such that the mean square error

$$E\{(g - \hat{g})^2\}$$

is minimized.

See Solution.

[7.18] When reading bits stored on a magnetic floppy disk, the read head registers the following signal (approximately); $ae^{-(t-\tau)^2}$ corresponds to a "1" and $-ae^{-(t-\tau)^2}$ to a "0". Normally, there is some uncertainty on the exact position of the pulse. When the information is stored densely, the information bits will overlap. The measured time discrete signal can be modeled as

$$x(n) = \sum_{k=1}^{r} a_k e^{-(n-\tau_k)^2} + v(n) \qquad n = 0,\ 1, \ldots, N-1$$

where $v(n)$ is additive white noise. The sign of a_k decides whether it is a "1" or a "0".

a) Assume that the τ_k are known. What is the least squares estimate ($\min \sum_{n=0}^{N-1} v^2(n)$) of a_k? Matrix expressions are acceptable.

b) Assume that the τ_k are unknown. Show that the least squares estimate of $\tau_1, \ldots, \tau_r$ can be found from the following maximization problem

$$\max_{\tau_k} \bar{x}^T P(\tau_1, \ldots, \tau_r)\bar{x}, \qquad P(\tau_1, \ldots, \tau_r) = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T$$

where $\bar{x} = \begin{bmatrix} x(0), & x(1), & \ldots, & x(N-1) \end{bmatrix}$ and the ijth element of $\mathbf{X}$ is given by

$$\{\mathbf{X}\}_{ij} = e^{-(i-1-\tau_j)^2}$$

c) Let $N = 2$, $r = 2$, $\tau_1 = 0$, $\tau_2 = 1$ and consider the estimate in part a). What are the conditions on $x(0)$ and $x(1)$, to find an estimate of two consecutive "1" ($\hat{a}_1 > 0$ and $\hat{a}_2 > 0$)?

See Solution.

[7.19] Given a system with impulse response $h(n) = \mathrm{e}^{-an} \cos \omega n, \;\; n \geq 0$, where ω and a are unknown constants.

a) Determine an ARMA-model corresponding to the system $h(n)$.

b) We know that ARMA-processes give a set of nonlinear equations when the coefficients are to be estimated. Hence we want to approximate the model in a) with an AR(2) model. Determine the coefficients of the optimal process and the corresponding least squares error. You do not have to solve the set of equations.

See Solution.

[7.20] We want to estimate an Nth order AR-process from the sequence $x(n)$, $n = 0, 1, ..., N+1$. Consider the Yule-Walker equations where the covariance sequence, $r(k)$, is estimated by

$$\hat{r}(k) = \frac{1}{N+2} \sum_{n=0}^{N-|k|+1} x(n)x(n+|k|)$$

show that this is equivalent to solving the following least squares problem

$$\min \sum_{n=1}^{N+1} |x(n) - \varphi^T(n)\theta|^2$$

where $\varphi(n) = [-x(n-1), \ldots, -x(n-N)]^T$, $\theta = [a_1, \ldots, a_N]^T$ and $x(n) = 0$, $n < 0$ and $n > N+1$.

See Solution.

[7.21] We consider a number of complex sinusoids in noise,

$$x[n] = \sum_{k=1}^{d} a_k e^{j\omega_k n} + w[n], \text{ for } n = 0, 1, \ldots, N-1$$

where a_k, $k = 1, \ldots, d < N$ are complex numbers describing the amplitude and phase of the sinusoids. The frequencies ω_k, $k = 1, \ldots, d < N$ are known. The noise, $w[n]$, is complex valued and white with zero mean and variance σ_w^2.

(a) Formulate the least squares estimate of a_k.

(b) What happens to the estimate when $N \to \infty$? Is the estimator unbiased ?

(c) Determine the covariance matrix for the estimate and interpret the result.

See Solution.

Chapter 8

Adaptive Filters

(This chapter is mainly kept here for historical reasons.)

[8.1] An adaptive filter has is described by figure 8.1. The input $y(n)$ is a real val-

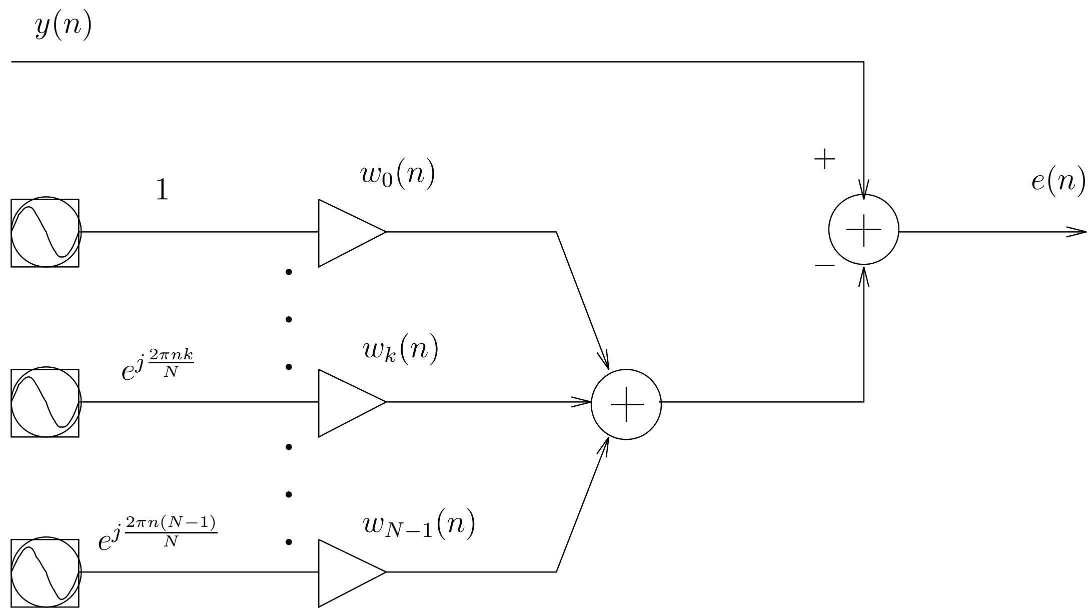


Figure 8.1:

ued periodic signal. The signal sources generate the complex signals $e^{j2\pi\frac{nk}{N}}$. The coefficients $w_k(n)$ are complex valued and $e(n)$ is the complex valued error signal.

The coefficients $w_k(n)$ are updated for each time instance n with a step in the direction of the negative gradient to minimize the error $|e(n)|^2$.

a) After some time, the gadget has converged. What do the w_k represent? Justify your answer!

b) Specify a formula for updating the coefficients in the negative gradient direction according to

$$w_k(n+1) = w_k(n) \quad + \quad ???$$

Note that most of the entities are complex valued.
Hint: Divide w_k into its real and imaginary part.

See Solution.

[8.2] Consider the adaptive FIR filter

$$y(n) = \sum_{k=0}^{M-1} \theta_k^n x(n-k)$$

$$\theta(n+1) = w\theta(n) + \mu\phi(n)e(n)$$

where

$$\theta(n) = [\theta_0^n \dots \theta_{M-1}^n]^T$$
$$\phi(n) = [x(n) \dots x(n-M+1)]^T$$

μ is the step size, $0 < w < 1$ and $e(n)$ is the error signal. The filter is used as a one-step-ahead predictor and the error signal is the difference between the true value and the predicted value.

The signal to be predicted is given by $u(n) = au(n-1) + v(n)$ where $v(n)$ is stationary white noise. Determine the values of μ so that the filter will converge? What will the filter converge to? Consider the average behavior, $E\{\theta(n)\} = \bar{\theta}(n)$, and let $M = 1$.

See Solution.

[8.3] Show that the LMS algorithm converges for an AR(1) model

$$y(n) + a_1 y(n-1) = e(n)$$

if and only if the step size is within $0 < \mu < 2/r_y(0)$. Solution not available.

[8.4] Consider the cost function

$$V(\theta) = \theta^2 + 40\theta + 28$$

where θ is an unknown parameter (scalar). Suppose that we search for the minimum of V by using the steepest descent algorithm

$$\theta(n+1) = \theta(n) - \frac{1}{2}\mu g(n)$$

where $g(n)$ is the gradient.

a) Determine the optimal value of θ and the minimum cost.

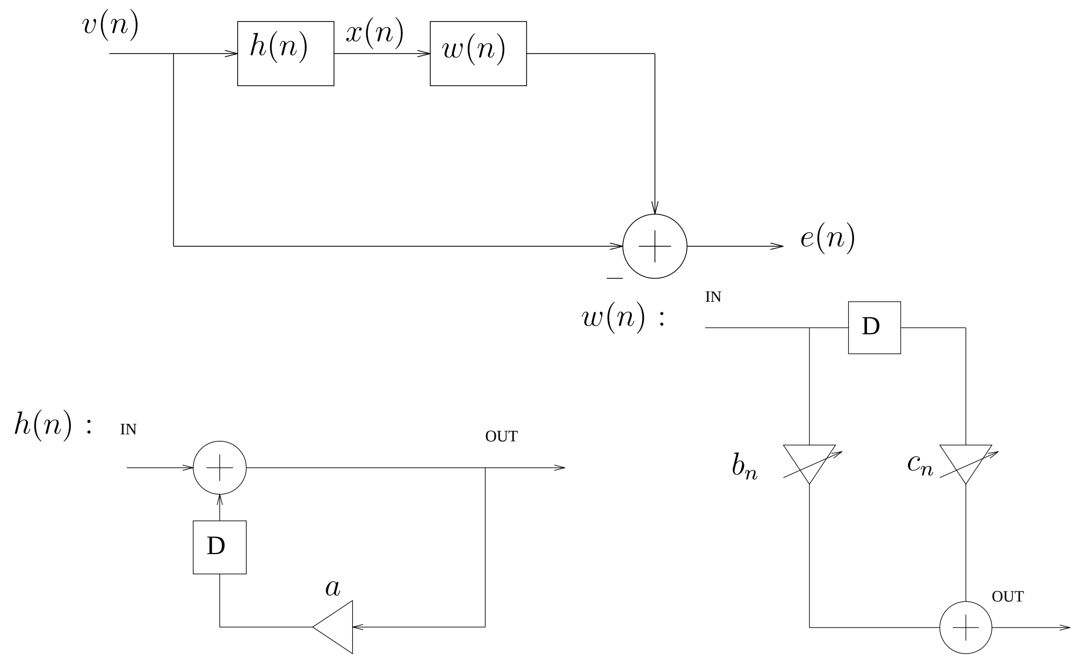


Figure 8.2:

b) Determine the range of values of μ that provides a stable algorithm.

c) Plot the expression for V as a function of n for a value of μ in the stable range and some initial value.

Solution not available.

[8.5] We wish to determine the unknown parameter a in the filter $h(n)$ using the adaptive filter $w(n)$, see figure 8.2. It is known that $\mid a \mid< 0.5$. The adaptive filter is designed to minimize $J = E\{e^2(n)\}$.
$v(n)$ is white noise with $E\{v(n)v(n+k)\} = \delta(k)\sigma_v^2$.

a) Algorithm 1:

$$w_{n+1} = w_n - \mu \bigtriangledown J$$

where $w_n = [b_n \;\; c_n]^T$ are the filter coefficients at time n, μ is the step size and $\bigtriangledown J$ is the gradient for J . Determine $\bigtriangledown J$!

b) Why cannot the algorithm in a) be implemented?

c) Algorithm 2: The filter coefficients are adapted using the LMS algorithm of order 2. What is the largest possible step size μ so that the filter will converge?

d) What should b_n and c_n converge to if the step size goes to zero?

See Solution.

Chapter 9

Subspace Based Techniques

[9.1] We model a number of signal samples, $x(0)$, $x(1)$, ..., $x(N-1)$ as two complex sinusoids,

$$\hat{x}(n) = s_1 e^{j(w_1 n + \phi_1)} + s_2 e^{j(w_2 n + \phi_2)}, \qquad n = 0, 1, \ldots, N-1$$

$$x(n) = \hat{x}(n) + e(n)$$

where $e(n)$ is a noise term. This can be expressed in matrix form

$$\bar{x} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} = \mathbf{X}\bar{s} + \bar{e}$$

$$\bar{s} = \begin{bmatrix} s_1 e^{j\phi_1} \\ s_2 e^{j\phi_2} \end{bmatrix}$$

where $\bar{e}$ is defined in similarly to $\bar{x}$.

a) Determine $\mathbf{X}$.

b) We want to determine s_i and ϕ_i, $i = 1, 2$ by minimizing $|\bar{e}|^2$. Show that

$$|\bar{e}|^2 = (\bar{s} - \mathbf{X}^\dagger \bar{x})^* \mathbf{X}^* \mathbf{X} (\bar{s} - \mathbf{X}^\dagger \bar{x}) + \bar{x}^* P \bar{x}$$

where

$$P = \mathbf{I} - \mathbf{X}\mathbf{X}^\dagger, \qquad \mathbf{X}^\dagger = (\mathbf{X}^* \mathbf{X})^{-1} \mathbf{X}^*$$

$\{\cdot\}^*$ means complex conjugate transpose and $\mathbf{I}$ is the identity matrix. Find the estimate $\bar{s}$ that minimizes the expressions above?

c) Assume that the noise $e(n)$ is white with variance σ_e^2. Show how to estimate σ_e^2.

d) Determine s_i and ϕ_i, $i = 1, 2$ when $\omega_2 - \omega_1 = 2\pi k/N$ for some integer $k \neq 0$ and $|k| < N$. What happens when $e(n) = 0$? What happens if $\omega_1 = \omega_2$?

See Solution.

[9.2] We will use the Pisarenko method to estimate a complex sinusoid in white noise. Let

$$x(n) = ae^{j(2\pi fn+\phi)} + v(n), \quad n = 0, 1, \ldots, N-1$$

where a and f are unknowns, ϕ is uniformly distributed over $[0 \quad 2\pi]$ and $v(n)$ is white noise with zero mean. The noise is complex valued with unknown variance $\mathrm{E}\{v(n)v^*(n)\} = \sigma^2$ and $\mathrm{E}\{v^2(n)\} = 0$.

The Pisarenko method works as usual but for complex valued signals, the covariance matrix is defined as

$$R = \begin{bmatrix} r_{xx}(0) & r_{xx}^*(1) & \cdots & r_{xx}^*(m) \\ r_{xx}(1) & r_{xx}(0) & \cdots & r_{xx}^*(m-1) \\ \vdots & & \ddots & \vdots \\ r_{xx}(m) & r_{xx}(m-1) & \cdots & r_{xx}(0) \end{bmatrix}$$

where $r_{xx}(k) = \mathrm{E}\{(x(n) - \mathrm{E}\{x(n)\})(x(n-k) - \mathrm{E}\{x(n-k)\})^*\}$.

(a) Write down the steps in the Pisarenko method to estimate σ^2 and f from data $x(n)$, $n = 0, 1, \ldots, N-1$.

(b) Determine the autocovariance function, $r_{xx}(k)$ for $x(n)$. You may need to know that $\mathrm{E}\{\cos\phi\} = \mathrm{E}\{\sin\phi\} = 0$.

(c) Let $m = 1$. Show that the Pisarenko method gives the correct value of σ^2 and f.

(d) Let $m = 2$ and $f = .5$. Determine the eigenvalues of R. What happens when trying to apply the Pisarenko method? Explain.

(e) Can you find a simple modification of the Pisarenko method that works in the above case?

See Solution.

[9.3] The autocorrelation sequence for a process $y(n)$ is given by

$$r(k) = \sigma^2 \cos(2\pi f_0 k)$$

We want to find the periodicities (and their frequencies) in $y(n)$ using two different methods.

a) Adapt an AR model to the process. Determine the suitable order, justify your choice. Determine the frequency by the position of the poles.

b) Use the Pisarenko method to estimate the frequency.
Hint: try $\lambda_{min} = 0$.

See Solution.

[9.4] Let

$$x(n) = a\lambda^n + e(n), \qquad n = 0, 1, \ldots, N-1$$

where a and λ are two unknown complex valued numbers and $e(n)$ is complex valued white noise with zero mean and variance 1. We can express this in matrix form

$$\bar{x} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} = \mathbf{X}a + \bar{e}$$

where $\bar{e}$ is defined similarly to $\bar{x}$.

a) Determine λ and a by minimizing $|\bar{e}|^2$. Show that

$$\hat{a} = \mathbf{X}^\dagger \bar{x}, \qquad \mathbf{X}^\dagger = (\mathbf{X}^*\mathbf{X})^{-1}\mathbf{X}^*$$

where $\{\cdot\}^*$ denotes complex conjugate transpose.

b) Show that $\hat{\lambda}$ could be obtained by minimizing

$$\bar{x}^* P(\lambda)\bar{x}, \qquad P(\lambda) = \mathbf{I} - \mathbf{X}\mathbf{X}^\dagger$$

where $\mathbf{I}$ is the identity matrix.

c) Assume now that $x(n) = a\lambda_1^n + e(n)$. What will $\hat{\lambda}$ be when $E\{\bar{x}^* P(\lambda)\bar{x}\}$ is minimized?

It maybe useful to know that $\mathbf{y}^*\mathbf{X}\mathbf{X}^\dagger\mathbf{y}$ is maximized when $\mathbf{X} = \mathbf{y}$.

See Solution.

[9.5] Measurements of a stationary signal has given the following estimates of the autocorrelation of the process, $r_y(0) = 3.1321, r_y(1) = 1.8271, r_y(2) = -0.2262, r_y(3) = -1.9563, r_y(4) = -2.2103$. We know that the signal consists of 2 different sinusoids plus the white Gaussian noise.

Use the Pisarenko method to determine the frequencies of the two sinusoids as well as the noise power.

Hint: Some of the following relationship may be useful.

$$\begin{pmatrix} 3.1321 & 1.8271 & -0.2262 & -1.9563 & -2.2103 \\ 1.8271 & 3.1321 & 1.8271 & -0.2262 & -1.9563 \\ -0.2262 & 1.8271 & 3.1321 & 1.8271 & -0.2262 \\ -1.9563 & -0.2262 & 1.8271 & 3.1321 & 1.8271 \\ -2.2103 & -1.9563 & -0.2262 & 1.8271 & 3.1321 \end{pmatrix} \approx TT^T$$

where

$$T = \begin{pmatrix} 0.1104 & -0.2862 & -0.6601 & 0.1140 & -1.6091 \\ -0.3083 & 0.3709 & -0.1533 & -1.1551 & -1.2416 \\ 0.4307 & 0.0000 & 0.1189 & -1.7124 & 0.0000 \\ -0.3083 & -0.3709 & -0.1533 & -1.1551 & 1.2416 \\ 0.1104 & 0.2862 & -0.6601 & 0.1140 & 1.6091 \end{pmatrix}$$

$$T^TT \approx \begin{pmatrix} 0.4000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.4390 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.9325 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 5.6272 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 8.2616 \end{pmatrix}$$

The following table lists the roots r_1, r_2, r_3, r_4 of the equation $a_4z^4+a_3z^3+a_2z^2+a_1z+a_0$ for five different sets of coefficients.

	Eq. 1	Eq. 2	Eq. 3	Eq. 4	Eq. 5
a_4	0.1104	−0.2862	−0.6601	0.1140	−1.6091
a_3	−0.3083	0.3709	−0.1533	−1.1551	−1.2416
a_2	0.4307	0.0000	0.1189	−1.7124	0.0000
a_1	−0.3083	−0.3709	−0.1533	−1.1551	1.2416
a_0	0.1104	0.2862	−0.6601	0.1140	1.6091
r_1	$0.5878 + 0.8090j$	$0.6480 + 0.7616j$	$0.6825 + 0.7309j$	11.5092	$-0.3858 + 0.9226j$
r_2	$0.5878 - 0.8090j$	$0.6480 - 0.7616j$	$0.6825 - 0.7309j$	$-0.7337 + 0.6795j$	$-0.3858 - 0.9226j$
r_3	$0.8090 + 0.5878j$	−1.0000	$-0.7986 + 0.6019j$	$-0.7337 - 0.6795j$	1.0000
r_4	$0.8090 - 0.5878j$	1.0000	$-0.7986 - 0.6019j$	0.0869	−1.0000

See Solution.

Chapter 10

Review of Matrix Theory

The purpose of this section is to review some results from matrix theory which are required for the courses in Signal Processing. Some simple exercises are included at the end. If you have any problems with these, review your linear algebra.

10.1 Matrices and Vectors

An $m \times n$ matrix A has m rows and n columns with matrix elements a_{ij}, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Sometimes one writes

$$A = \{a_{ij}\}$$

A matrix with the same number of rows and columns is called *square*. When $n = 1$, A is called a column vector of dimension m. When $m = 1$, A is called a row vector of dimension n. The elements of an $n-$dimensional vector are denoted a_i, $i = 1, 2, \ldots, n$.

10.2 Addition, Subtraction, and Multiplication

Two matrices with the same dimensions (same number of rows and columns) may be added or subtracted.

$$C = A + B, \quad \text{i.e.} \quad c_{ij} = a_{ij} + b_{ij}$$

A matrix A multiplied by a scalar k, multiplies each entry by k, i.e.,

$$C = kA, \quad c_{ij} = ka_{ij}$$

Two matrices A and B with dimensions $m \times p$ and $p \times n$ respectively, may be multiplied.

$$C = AB, \quad c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

Note that C is $m \times n$ and that even if $m = n$ matrix multiplication is *not* commutative, i.e., $AB \neq BA$ in general. However, matrix multiplication is associative,

$$A(BC) = (AB)C$$

The identity matrix of order n is defined as

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

For matrices with proper dimensions we have

$$AI = A \qquad IB = B$$

A matrix with all the elements equal to zero is called the zero matrix.

10.3 Matrix transpose

Let A be an $m \times n$ matrix. The matrix transpose of A is denoted A^T (sometimes A') and is an $n \times m$ matrix defined by

$$B = A^T \qquad b_{ji} = a_{ij}$$

Note that

$$(AB)^T = B^T A^T$$

For matrices with complex elements it is convenient to define the Hermitian transpose (or complex conjugate transpose) denoted A^* (sometimes A^H). It is defined by

$$B = A^* \quad b_{ji} = \text{conj}(a_{ij})$$

where $\text{conj}(\cdot)$ denotes complex conjugate ($\text{conj}(a + ib) = a - ib$). Of course, for real matrices $A^* = A^T$.

10.4 Symmetric Matrix

If

$$A = A^T$$

the matrix A is called *symmetric*. If

$$A = A^*$$

the matrix A is called *Hermitian*. Note that symmetric and Hermitian matrices are square.

10.5 Trace

Let A be an $n \times n$ matrix. The trace of the matrix is the sum of the diagonal elements, i.e.,

$$\mathrm{Tr}\{A\} = \sum_{k=1}^{n} a_{kk}$$

10.6 Determinant

Let A be an $n \times n$ (square) matrix. The determinant of A is denoted $|A|$ or $\det(A)$ and is defined by

$$|A| = \sum_{j=1}^{n} a_{ij}\gamma_{ij} \quad \text{for any} \quad i = 1, 2, \ldots, n$$

where γ_{ij} denotes the *cofactor* corresponding to a_{ij} and is defined by

$$\gamma_{ij} = (-1)^{i+j}|M_{ij}|$$

where M_{ij} is the $(n-1)\times(n-1)$ matrix obtained by deleting the i^{th} row and j^{th} column of A. The definition above is often referred to as *Laplace's expansion.* To compute $|M_{ij}|$ the expansion above is applied recursively until the cofactors are scalars in which case $\det(a) = a$.

Note that for two square matrices, A and B, of the same dimensions

$$|AB| = |A||B|$$

Remark: Do not worry if you do not understand the definition of the determinant immediately. Make sure that you know how to compute the determinant of a 3×3 matrix though.

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

10.7 Invertability

A square matrix A is *invertible* or *nonsingular* if and only if $|A| \neq 0$. The matrix is singular if $|A| = 0$. The inverse of A is denoted A^{-1} and has the property that

$$AA^{-1} = A^{-1}A = I$$

Note that if $|A| \neq 0$ and $|B| \neq 0$

$$(AB)^{-1} = B^{-1}A^{-1}$$

The inverse of a matrix A may be computed as

$$A^{-1} = \frac{1}{|A|}\tilde{A}$$

where $\tilde{A}^T = \{\gamma_{ij}\}$. $\tilde{A}$ is termed the *adjugate* matrix of A.

10.8 Rank of Matrices

The rank of an $n \times m$ matrix A is equal to the number of linearly independent columns (or, equivalently, rows) of A. Apparently, the rank of A is always $\leq \min(n, m)$. If the rank of A equals $\min(n, m)$ then A is said to be full rank. If $n = m$ then A is full rank if and only if A is non-singular.

10.9 Orthogonal Vectors and Matrices

Let x and y be two $n \times 1$ vectors. These two vectors are said to be *orthogonal* if

$$x^* y = \sum_{i=1}^{n} x_i^* y_i = 0$$

It the vectors are real-valued, this of course reduces to $x^T y = 0$.

If a matrix satisfies

$$A^* A = I$$

it is called *orthogonal* (or sometimes *orthonormal* or *unitary*). A real, orthogonal matrix satisfies $A^T A = I$ of course.

10.10 Positive Definite Matrices

Let A be a Hermitian matrix ($A = A^*$). The matrix A is *positive semidefinite* if the quadratic form (quadratic in the entries of x)

$$x^* A x \geq 0$$

for all vectors x. If the equality holds only when $x = 0$, A is *positive definite*.

When referring to positive (semi)definite matrices we implicitly assume Hermitian matrices.

10.11 Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix. A non-zero vector p for which

$$Ap = \lambda p$$

is said to be an *eigenvector* of the matrix A with an associated *eigenvalue* λ (λ is a scalar which may be complex). Note that if p is an eigenvector so is kp for any non-zero scalar k. The polynomial

$$|\lambda I - A|$$

is termed the *characteristic polynomial*. The zeros of the characteristic polynomial are the eigenvalues of A. The matrix A has n eigenvalues, λ_i, $i = 1, 2, \ldots, n$, since the characteristic polynomial is of degree n. If the eigenvalues are distinct, the eigenvectors p_i, $i = 1, 2, \ldots, n$, are unique.

Also, it is true that the determinant of A is equal to the product of the eigenvalues

$$|A| = \prod_{k=1}^{n} \lambda_k$$

Thus, if A is singular it has at least one eigenvalue equal to zero.

Note also that the trace of A is equal to the sum of the eigenvalues

$$\mathrm{Tr}\{A\} = \sum_{k=1}^{n} \lambda_k$$

For any Hermitian matrix A, we can find a decomposition

$$A = \sum_{i=1}^{n} \lambda_i e_i e_i^*$$

where λ_i are the eigenvalues and e_i are the corresponding eigenvectors of A. This is referred to as the *eigendecomposition* of the matrix A. The eigenvectors are orthogonal and normalized to unit length, $||e_i||^2 = 1$, i.e.,

$$e_i^* e_j = ||e_i||^2 \delta_{ij} = \begin{cases} 1 & \text{when} \quad i = j \\ 0 & \text{when} \quad i \neq j \end{cases}$$

These vectors are then termed *ortho-normal.* This implies that

$$AE = E\Lambda \qquad E = [e_1, e_2, \ldots, e_n]$$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

In other words, $E^* E = I$.

A symmetric or Hermitian matrix has real eigenvalues.

A positive definite matrix has positive eigenvalues.

10.12 Vector Norm

The norm of a vector is a measure of the length of the vector. The two-norm of a vector x is defined as

$$||x|| = \left(\sum_{k=1}^{n} |x_k|^2 \right)^{1/2}$$

10.13 Frobenius Norm of a Matrix

The Frobenius norm of an $m \times n$ matrix A is defined as

$$||A||_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}$$

Note that

$$||A||_F^2 = \mathrm{Tr}\{A^* A\}$$

10.14 Differentiation

Let $f(x)$ be a real valued scalar function of a real valued n-dimensional vector x. The gradient of f is defined as the column vector

$$\frac{df(x)}{dx} = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}.$$

The matrix with second order partial derivatives is called the Hessian matrix:

$$H = \frac{d^2 f(x)}{dx\, dx^T} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & & \\ \vdots & & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}$$

Note that H is an $n \times n$ symmetric matrix.

10.15 Matrix Problems

[10.1] Multiply the two matrices below

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

[10.2] Compute

$$\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

[10.3] Compute

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1}$$

[10.4] Show that matrix multiplication is associative.

[10.5] Show that

$$\mathrm{Tr}\{A+B\} = \mathrm{Tr}\{B+A\} = \mathrm{Tr}\{A\} + \mathrm{Tr}\{B\}$$

[10.6] Let A and B^T be $m \times n$ matrices. Show that

$$\mathrm{Tr}\{AB\} = \mathrm{Tr}\{B^T A^T\} = \mathrm{Tr}\{BA\}$$

[10.7] Let $|A| \neq 0$ and $|B| \neq 0$, show that

$$(AB)^{-1} = B^{-1}A^{-1}$$

[10.8] Show that

$$||A||_F^2 = \mathrm{Tr}\{A^* A\}$$

[10.9] Show that A and A^T have the same rank.

[10.10] What is the rank of $\begin{bmatrix} 1 & 2 & 4 & 1 \\ 2 & 4 & 8 & 2 \\ 3 & 6 & 2 & 0 \end{bmatrix}$?

[10.11] Let A be a real, square matrix. Show that

$$x^T A x$$

depends only on the symmetric part of A, i.e., $1/2(A + A^T)$.

[10.12] For a Hermitian matrix we have

$$AE = E\Lambda$$

with the definitions above. Show that $E^* = E^{-1}$.

[10.13] Show that for a Hermitian matrix

$$\mathrm{Tr}\{A\} = \sum_{k=1}^{n} \lambda_k$$

Remark: The above is true for any square matrix.

[10.14] Show that for a Hermitian matrix

$$|A| = \prod_{k=1}^{n} \lambda_k$$

Remark: The above is true for any square matrix.

[10.15] Let A be an $n \times n$ matrix. Show that

$$|kA| = k^n |A|$$

for a scalar k.

[10.16] Let $A = E\Lambda E^*$ with the definitions above. Find the eigendecomposition of A^{-1}.

[10.17] Show that A^*A is positive semidefinite for any matrix A.

[10.18] Show that all eigenvalues of a positive definite matrix are positive. (The inverse is also true, a Hermitian matrix is positive semidefinite if and only if all eigenvalues are positive.)

[10.19] What can you say about the eigenvalues of a positive semidefinite matrix?

[10.20] Let x and a be n-dimensional real vectors and let A be a real symmetric $n \times n$ matrix. Show that

$$\frac{d(x^T a)}{dx} = a$$
$$\frac{d(x^T Ax)}{dx} = 2Ax$$
$$\frac{d^2(x^T Ax)}{dx\, dx^T} = 2A$$

[10.21] Consider the following minimization problem

$$\min_x \|Ax - b\|^2 = \min_x (Ax - b)^T (Ax - b)$$

where A is $n \times m$, x is $m \times 1$, and b is $n \times 1$. The matrix A is assumed to be full rank.

a) What can you say about the problem when $n = m$, $n < m$, and $n > m$?

Let $n > m$. Solve the minimization problem by

b) Differentiation.
c) Completing the squares.
d) Orthogonality condition.
e) What would you do in the complex case?

See Solution.

Chapter 11

Selected Short Problems

[11.1] A 625Hz tone (a real-valued signal) is sampled with sampling frequency 1000Hz. Determine the normalized frequency f of the sampled signal, ($f \in (0, \frac{1}{2})$).

A	B	C	D	E	F	G
1/8	1/4	3/8	1/2	5/8	3/4	7/8

[11.2] If $x(n) = \{x(0), x(1), \ldots, x(N)\}$ where $N = M \cdot K$ is a sequence with pass-band $|X(\omega)|^2 \approx 1$ for $|\omega| \leq \Omega \ll \pi$, then $\bar{x}(n) = \{x(0), x(M), \cdots, x(MK)\}$ has the pass band in ...

A	B	C	D	E
$\lvert\omega\rvert \leq K\Omega$	$\lvert\omega\rvert \leq M\Omega$	$\lvert\omega\rvert \leq \Omega$	$\lvert\omega\rvert \leq \Omega/K$	$\lvert\omega\rvert \leq \Omega/M$

[11.3] Consider the periodogram as an estimator of the power spectral density. Which statement is wrong? The periodogram is ...... estimator of the power spectral density.

A	B	C	D
a biased	a unbiased	an asymptotically unbiased	not a consistent

[11.4] Osquar wants to filter his measured data $x(n)$ first by a filter with impulse response $h_1(n)$ and then by a filter with impulse response $h_2(n)$. By mistake, he confuses the data $x(n)$ with the impulse response $h_2(n)$, i.e. he filters his "measured data" $h_2(n)$ first by $h_1(n)$ and then by the "filter" with impulse response $x(n)$. How does the differ the actual output signal $y(n)$ from the desired signal $y_0(n)$?

A	B	C	D
$y(n) = y_0(n)$	$y(n) = -y_0(n)$	$y(n) = y_0(-n)$	$y(n) = -y_0(-n)$

[11.5] Osqulda wants to filter her measured data $x(n)$ by the asymptoticly stable filter $H_1(z) = \frac{1}{1+az^{-1}}$. Osquar claims that she could just as well filter the data first through the filter $H_1(z) = \frac{1}{1+bz^{-1}}$ and then by the filter $H_2(z) = \frac{1+bz^{-1}}{1+az^{-1}}$. Ignoring possible transient effects, the two procedures are equivalent for three of the cases below. In which case is Osquar wrong?

A	B	C	D
$\lvert b\rvert < 1$	$\lvert b\rvert > 1$	$\lvert b\rvert < \lvert a\rvert$	$a = b$

See Solution.

[11.6] A 1400 Hz tone (a real-valued signal) is sampled with the sampling frequency $F_s = 2000$ Hz. Determine the relative angular frequency ω rad of the sampled signal.

A	B	C	D	E	F	G	H
0.15	0.30	0.35	0.70	0.94	1.88	2.20	4.40

[11.7] The first five values of the 8-point DFT $\{X(k)\}$ of a real-valued sequence $x(n)$ are given by $\{2,\ 1-j2.4,\ 0,\ 1-j0.48,\ 4\}$. Determine the remaining 3 values $X(5), X(6), X(7)$

	A	B	C	D
$X(5)$:	$1+j2.4$	$1+j0.48$	0	$1+j0.48$
$X(6)$:	0	0	$1+j2.4$	0
$X(7)$:	$1+j0.48$	$1-j2.4$	2	$1+j2.4$

[11.8] If $x(n) = \{x(0), x(1), \ldots, x(N)\}$ where $N = MK$ is a band-pass sequence with pass-band centered around Ω_0, then $\bar{x}(n) = \{x(0), x(K), \cdots, x(MK)\}$ has center frequency $\bar{\Omega}$ (the ideal case is considered so all quantities below are well inside $(0, \pi)$), where

A	B	C	D
$\bar{\Omega} = \Omega_0$	$\bar{\Omega} = K\Omega_0$	$\bar{\Omega} = M\Omega_0$	$\bar{\Omega} = N\Omega_0$

E	F	G
$\bar{\Omega} = \Omega_0/K$	$\bar{\Omega} = \Omega_0/M$	$\bar{\Omega} = \Omega_0/N$

.

[11.9] Consider the Pisarenko harmonic decomposition method for frequency estimation applied to a data sequence consisting of a noisy sine-wave with unknown frequency F Hz ($F < F_s/2$) sampled at $F_s = 1000$ Hz. An estimate of the autocorrelation matrix is calculated from the given sequence, and calculation of the eigenvector associated with the minimum eigenvalue gives the vector $(1\ 0\ 1)^T$. Determine the frequency F!

A	B	C
$F = 100$ Hz	$F = 150$ Hz	$F = 200$ Hz
D	**E**	**F**
$F = 250$ Hz	$F = 300$ Hz	$F = 350$ Hz

.

[11.10] An *allpass filter* ($|H(\omega)|^2 = 1$) has the following transfer function

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{a_0 + a_1 z^{-1} + a_2 z^{-2}}$$

If the input signal $x(n)$ to the filter is white Gaussian noise with variance $\sigma_x = 1$, What is the variance of the output signal $y(n)$?

D	E	F
1	$\frac{b_0^2+b_1^2+b_2^2}{a_0^2+a_1^2+a_2^2}$	$b_0^2+b_1^2+b_2^2+a_0^2+a_1^2+a_2^2$

D	E	F
$\frac{b_0+b_1+b_2}{a_0+a_1+a_2}$	$\frac{(b_0+b_1+b_2)^2}{(a_0+a_1+a_2)^2}$	π .

See Solution.

[11.11] (from your laboration) Assume that we sample with 10kHz. We send a sinusoid at 8kHz through the program `dsp-adda` which samples and reconstructs (assume ideal reconstruction matched to the sampling frequency) the output signal. The anti-aliasing filter has a cut off frequency above 10 kHz. Which tone (in KHz) will be recovered in the output signal?

A	B	C	D	E	F	G	H	I	J
1	2	3	4	5	6	7	8	9	10

[11.12] Consider an $\uparrow 3$ interpolator with an 18-tap FIR low-pass filter. An alternative implementation is in a polyphase structure with q_1 filters where each filter has q_2 taps. Determine q_1 and q_2!

A	B	C	D	E	F	G
$q_1 = 3$	$q_1 = 6$	$q_1 = 18$	$q_1 = 3$	$q_1 = 6$	$q_1 = 18$	$q_1 = 6$
$q_2 = 18$	$q_2 = 6$	$q_2 = 3$	$q_2 = 6$	$q_2 = 18$	$q_2 = 1$	$q_2 = 3$

[11.13] If $x(n) = \{x(0), x(1), \ldots, x(N)\}$ where $N = M \cdot K$ is a band-pass sequence with pass-band centered around Ω_0, then $\bar{x}(n) = \{x(0), x(M), \cdots, x(MK)\}$ has center frequency $\bar{\Omega}$ (the ideal case is considered so all quantities below are well inside $(0, \pi)$), where

A	B	C	D
$\bar{\Omega} = \Omega_0$	$\bar{\Omega} = K\Omega_0$	$\bar{\Omega} = M\Omega_0$	$\bar{\Omega} = N\Omega_0$

E	F	G
$\bar{\Omega} = \Omega_0/K$	$\bar{\Omega} = \Omega_0/M$	$\bar{\Omega} = \Omega_0/N$

[11.14] Consider the Pisarenko harmonic decomposition method for frequency estimation applied to a data sequence consisting of a noisy sine-wave with unknown frequency F Hz ($F < F_s/2$) sampled at $F_s = 1000$ Hz. An estimate of the autocorrelation matrix is calculated from the given sequence, and calculation of the eigenvector associated with the minimum eigenvalue gives the vector with roots $z = e^{\pm i\pi/5}$. Determine the frequency F!

A	B	C
$F = 100$ Hz	$F = 150$ Hz	$F = 200$ Hz

D	E	F
$F = 250$ Hz	$F = 300$ Hz	$F = 350$ Hz

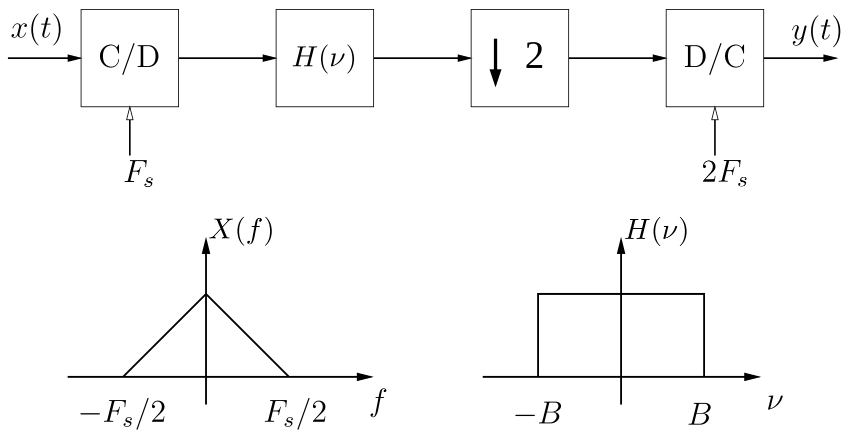


Figure 11.1: System discussed in Exercise 11.15a.

[11.15] The sequence $\{x(0) \cdots x(N-1)\}$ has been obtained by sampling a continuous time signal $x(t)$ with sampling time $T_s = 0.1$ s. The obtained sequence $\{x(0) \cdots x(N-1)\}$ is filtered by a time discrete anti-aliasing filter after which it is downsampled by a factor 4. How should we select the cut-off frequency (in Hz) of the anti-aliasing filter? (assume an ideal filter)

A	B	C	D	E	F
0.0125 Hz	0.125 Hz	1.25 Hz	12.5 Hz	125 Hz	1250 Hz

See Solution. In the system of Fig. 11.1, $X(f)$ (the Fourier transform of a continuous-time signal $x(t)$) and $H(\nu)$ (the transfer function of a discrete-time filter $h[n]$) are shown.

(a) Determine the largest possible value of B such that no aliasing appears at the decimation.

A	B	C	D	E
$B = 1/4$	$B = 3/4$	$B = 1/2$	$B = 1$	something else

(b) From all the options shown in Fig. 11.2, which one is the transform of the output signal $y(t)$ for the specific choice of B made in the previous question?

(c) At what frequency should the D/C converter be operated so that $y(t)$ simply becomes a low-pass filtered version of $x(t)$?

See Solution.

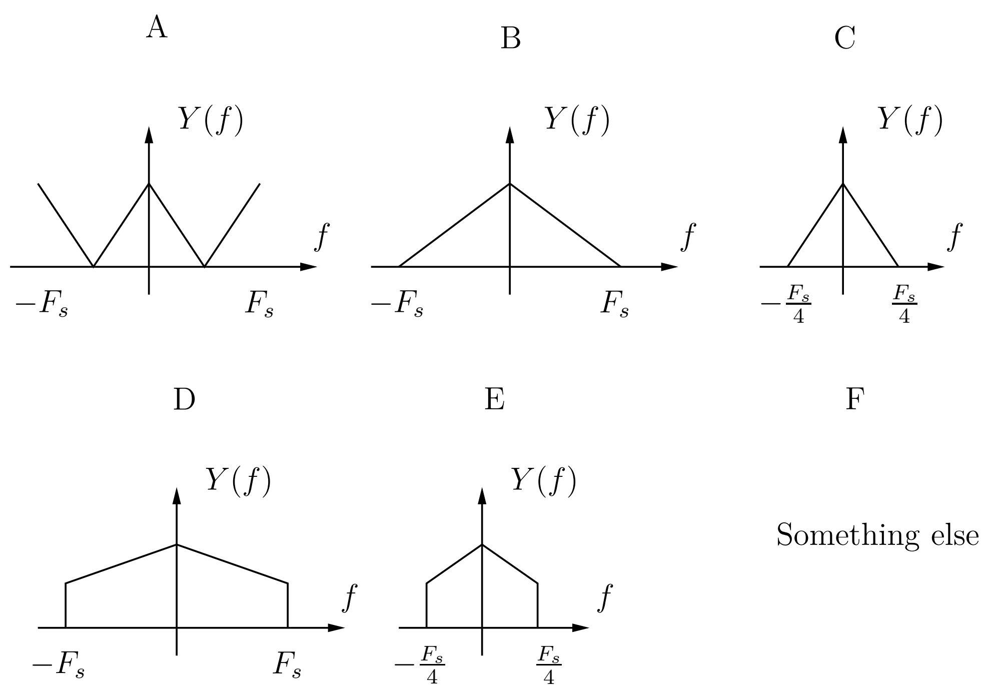


Figure 11.2: Possible spectrum of the continuous-time signal $x(t)$ discussed in Exercise 11.15b.

Hints

Hint for [1.2]: The power is the energy per time unit.

Hint for [1.3]: If $H(e^{j\omega}) = \mathcal{F}\{h(n)\}$, what is $\mathcal{F}\{h(-n)\}$?

Hint for [1.4]: If $x(n) = x_{\text{even}}(n) + x_{\text{odd}}(n)$, what is $x(-n)$?

Hint for [1.6]: Work with the transfer functions $H_1(z)$, $H_2(z)$. What is the total transfer function from $x(n)$ to $w(n)$? How can you see from a transfer function if the corresponding filter is causal?

Hint for [1.7]: One approach is to view $y(k) - ay(k-1) + y(k-2)$ as a linearly filtered version of $y(k)$. Where should the zeros of this filter be placed?

Another approach is to use the theory of characteristic equations for difference equations.

If you want to use the brute force approach to directly insert $y(k) = \sin(\omega kT)$ and solve for a, a hint is to consider the equivalent equation $y(k+1) - ay(k) + y(k-1) = 0$, which is easier to handle due to the symmetry.

Hint for [2.9]: If you know the poles of $X(z)$, how can you determine the boundaries between the different possible ROC:s?

Hint for [2.10]: Schur-Cohn!

Hint for [3.1]: DFTs of real valued signals have certain symmetry properties.

Hint for [3.7]: Exploit the symmetry properties for DFTs of real valued signals. What are the corresponding properties for purely imaginary signals?

Hint for [3.8]: Write $n = 8m + l$ and express the DTFT as a double sum over m and l.

Hint for [5.5]: Note that there is no need to calculate all the quantization noise contributions at the output, since some of the terms will be the same for both implementations.

Hint for [5.6]: There is no simple solution, just try to systematically go through the worst cases in every step.

Hint for [5.7]: If you get strange results in (ii), try to figure out what is wrong about the question.

Hint for [5.8]: What is the output signal of the filter if you for a moment ignore the round-off errors? What is the power of such a signal?

Hint for [5.9]: For a), see the hint of [5.11.5].

Hint for [6.1]: One method to calculate the DTFT of a triangular window is to view the triangle as the convolution of two rectangles. What happens in the frequency domain if you double the length of a window in the time domain?

Hint for [6.4]: Euler's formula.

Hint for [6.8]: What is the theoretical spectral density of $x(n)$? Why can we not use that formula directly to read of the amplitude of the sinusoid (the noise power, on the other hand, can be estimated directly from the periodogram)?

For the amplitude of the sinusoid, you have to analyze what the periodogram produces for a pure sinusoidal signal.

Hint for [7.1]: The problem is vaguely formulated. Try to think of different possible interpretations of the question.

One possible interpretation would be that we first should form the inverse of $h(n)$ and then find an FIR filter of length 3 that provides the best least-squares fit to this inverse filter. This would actually lead to a very simple solution. However, the answer we propose, uses another interpretation of the problem.

Hint for [7.2]: First try to figure out the answer based on your intuition, before doing any calculations.

Hint for [7.11]: In the course literature, you can find general results for this kind of lattice structure with feedback, but we do not go through it in this course. You can just as well use your general knowledge on linear systems to find the transfer function.

Hint for [7.12]: This question is not really relevant for the course.

Hint for [7.14]: First determine the autocovariance sequence for $y(n)$ in terms of b_0 and b_1.

Hint for [7.21]: Define vectors $\mathbf{x}$ and $\mathbf{a}$ containing all the N samples of $x(n)$ and the d coefficients a_k, respectively. Try to formulate the data model in the form $\mathbf{x} = \mathbf{\Omega a} + \mathbf{w}$ where the matrix $\mathbf{\Omega}$ is a function of the known frequencies.

For c), first figure out what the covariance matrix of the noise vector $\mathbf{w}$ is.

Hint for [9.5]: Information about the necessary eigenvalues and eigenvectors is hidden in the provided information about the matrix T. Another hint is to think of how the eigenvectors are used in the Pisarenko method and to compare the numbers in the table with the elements of T.

Hint for [10.13]: Use the previous question, together with properties of the trace operator from one of the earlier questions.

Hint for [10.14]: Same hint as in the previous question.

Answers and Selected Solutions

[1.1] $\{\overset{\downarrow}{1}, 2, 2, 1\}$ See Exercise.

[1.2] The wrong statement is: $E = \infty \Rightarrow P = \infty$. See Exercise.

[1.3] (a) $|H_1(\mathrm{e}^{j\omega})| = |H(\mathrm{e}^{j\omega})|^2$ and $\arg\{H_1(\mathrm{e}^{j\omega})\} = 0$.

(b) $|H_2(\mathrm{e}^{j\omega})| = 2\mathrm{Re}H(\mathrm{e}^{j\omega})$ and $\arg\{H_2(\mathrm{e}^{j\omega})\} = 0$.

(c) Method A is superior because of its amplitude response.

See Exercise.

[1.4]

$$\begin{aligned} x(n)_o &= \{-2, -1, 0, 1, 2\} \\ x(n)_e &= \{4, 4, 4, 4, 4\} \end{aligned}$$

See Exercise.

[1.5]

$$\begin{aligned} E &= \sum_{n=-\infty}^{\infty} x^2(n) = \sum_{n=-\infty}^{\infty} x_e^2(n) + \sum_{n=-\infty}^{\infty} x_o^2(n) + 2\sum_{n=-\infty}^{\infty} x_e(n)x_o(n) \\ &= E_e + E_o + \sum_{n=-\infty}^{\infty} x_e(n)x_o(n) - \sum_{n=-\infty}^{\infty} x_e(n)x_o(n) = E_e + E_o \end{aligned}$$

See Exercise.

[1.6] (a) Determine H_2 knowing $w[n] = x[n]$. Then, $X(z) = W(z)$.

But, applying the Convolution theorem for the cascading of linear time invariant (LTI) systems, we also have $W(z) = H_1(z)H_2(z)X(z)$ and consequently, $H_1(z)H_2(z) = 1 \Rightarrow H_2(z) = \dfrac{1}{H_1(z)}$.

Let's find $H_1(z) = \dfrac{Y(z)}{X(z)}$, then. First, transform the Finite Differences Equation:

$$Y(z) = \frac{7}{12}Y(z)z^{-1} - \frac{1}{12}Y(z)z^{-2} + X(z)z^{-1} - \frac{1}{2}X(z)z^{-2}$$

$$\Updownarrow$$

$$Y(z)\left(1-\frac{7}{12}z^{-1}+\frac{1}{12}z^{-2}\right)=X(z)\left(z^{-1}-\frac{1}{2}X(z)z^{-2}\right)$$

$$\Updownarrow$$

$$\frac{Y(z)}{X(z)}=\frac{z^{-1}-\frac{1}{2}z^{-2}}{1-\frac{7}{12}z^{-1}+\frac{1}{12}z^{-2}}.$$

Thus, $H_1(z)=\frac{\left(z-\frac{1}{2}\right)}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{4}\right)}$ and therefore $H_2(z)=\frac{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{4}\right)}{\left(z-\frac{1}{2}\right)}$. The poles of $H_2(z)$ are $z_1=1/2$ and $z_2=+\infty$ (because $\lim_{z\to+\infty}(|H_2(z)|)=+\infty$). Therefore, the system can not be causal (its ROC can not include the pole $z_2=+\infty$) and thus, the ROC of the system is the set $\{z\in\mathbb{C}\mid |z|<1/2\}$.

(b) Determine H_2 knowing $w[n]=x[n-1]$.
In this case we have $W(z)=X(z)z^{-1}$. Following the steps above we get

$$H_2(z)=\frac{z^{-1}}{H_1(z)}=\frac{1}{zH_1(z)}=\frac{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{4}\right)}{z\left(z-\frac{1}{2}\right)}.$$

The poles of $H_2(z)$ are, now, $z_1=0$ and $z_2=1/2$. If we pick the ROC of the system (now possible) as the set $\{z\in\mathbb{C}\mid |z|>1/2\}$, then H_2 is causal and stable.

Explanatory note:

Time domain	Z-domain
A delay of one sample has allowed us to find $w[n]$ with only samples from the past or present ($m<n$).	A pole at 0 has replaced the pole at $+\infty$, allowing for a ROC with the "causal" shape.

(c) In a) we have

$$H_2(z)=\frac{W(z)}{Y(z)}=\frac{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{4}\right)}{\left(z-\frac{1}{2}\right)}$$

$$\Updownarrow$$

$$W(z)\left(z-\frac{1}{2}\right)=\left(z-\frac{1}{3}\right)\left(z-\frac{1}{4}\right)Y(z)$$

$$\Updownarrow$$

$$zW(z)-\frac{W(z)}{2}=z^2Y(z)-\frac{7}{12}zY(z)+\frac{Y(z)}{12}$$

Now, by the inverse Z transform, we get

$$w[n]-\frac{1}{2}w[n-1]=y[n+1]-\frac{7}{12}y[n]+\frac{1}{12}y[n-1]\,.$$

In b) we have

$$H_2(z) = \frac{W(z)}{Y(z)} = \frac{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{4}\right)}{z\left(z-\frac{1}{2}\right)}$$

$$\updownarrow$$

$$H_2(z) = \frac{\frac{1}{12}z^{-2} - \frac{7}{12}z^{-1} + 1}{1-\frac{1}{2}z^{-1}}$$

$$\updownarrow$$

$$W(z)\left(1-\frac{1}{2}z^{-1}\right) = Y(z)\left(\frac{1}{12}z^{-2} - \frac{7}{12}z^{-1} + 1\right).$$

Now, by the inverse Z transform, we get

$$w[n] - \frac{1}{2}w[n-1] = \frac{1}{12}y[n-2] - \frac{7}{12}y[n-1] + y[n].$$

See Exercise.

[1.7] If $\omega T = \pi q$, with $q \in \mathbb{Z}$, a can take any value because $\forall k \in \mathbb{Z}, y[k] = 0$.

Otherwise, consider the Z transform of the difference equation,

$$Y(z)\left(1 - az^{-1} + z^{-2}\right) = 0\,,$$

which allows us to think on the problem as designing a filter with impulse response

$$H(z) = 1 - az^{-1} + z^{-2}$$

that cancels the signal $Y(z)$.

An equivalent equation, then, is

$$z^2 - az + 1 = 0\,, \forall z \in \mathbb{C} \mid Y(z) \neq 0$$

which results in

$$z = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - 1} = \frac{a}{2} \pm j\sqrt{1 - \frac{a^2}{4}}\,.$$

Because our signal has a component only at frequency ω, we want, in particular, that $H(z)\big|_{z=e^{\pm j\omega}} = 0$. Thus,

$$z = e^{\pm j\omega T} = \cos(\omega T) \pm j\sin(\omega T) = \frac{a}{2} \pm j\sqrt{1-\frac{a^2}{4}}\,,$$

which is fulfilled for $a = 2\cos(\omega T)$.

See Exercise.

[2.1] $\{1, 2, \overset{\downarrow}{2}, 1\}$ See Exercise.

[2.2] $\{\overset{\downarrow}{2}, 2, 2, 2\}$ See Exercise.

[2.3] $N = 4$. See Exercise.

[2.4] $\frac{1}{1-0.5z^{-1}}$ See Exercise.

[2.5] $\frac{1}{1-0.5z^{-1}}$ See Exercise.

[2.6] $|z| > 0.5$. See Exercise.

[2.8] If $k \geq 0$,

$$r_{xx}(k) = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \left(\frac{1}{4}\right)^{n+k} = \ldots \frac{16}{15}\left(\frac{1}{4}\right)^k$$

Similar calculations for $k < 0$ (or can you find a smarter solution?) show that $r_{xx}(k) = \frac{16}{15}\left(\frac{1}{4}\right)^{|k|}$
See Exercise.

[2.9] The technique known as partial fraction expansion sets

$$H(z) = \frac{Az^{-1} + B}{1 - \frac{1}{3}z^{-1}} + \frac{Cz^{-1} + D}{1 - 2z^{-1}}$$

to obtain $A = C = 0$ and $D = -A = 1$. This is done because the resulting summands are easier to inverse Z-transform. Then, we obtain,

$$H(z) = \frac{5z^{-1}}{(1 - 2z^{-1})(3 - z^{-1})} = \underbrace{\frac{1}{1 - 2z^{-1}}}_{H_1(z)} + \underbrace{\frac{-1}{1 - \frac{1}{3}z^{-1}}}_{H_2(z)}.$$

$H_1(z)$:

$$h_{1,1}[n] = 2^n u[n], \quad \mathrm{ROC}_{1,1} = \{|z| > 2\},$$
$$h_{1,2}[n] = -2^n u[-n-1], \quad \mathrm{ROC}_{1,2} = \{|z| < 2\}.$$

$H_2(z)$:

$$h_{2,1}[n] = -\left(\frac{1}{3}\right)^n u[n], \quad \mathrm{ROC}_{2,1} = \{|z| > \frac{1}{3}\},$$
$$h_{2,2}[n] = \left(\frac{1}{3}\right)^n u[-n-1], \quad \mathrm{ROC}_{2,2} = \{|z| < \frac{1}{3}\}.$$

$h[n]$ is given by

$$h[n] = h_{1,i}[n] + h_{2,j}[n], \quad \mathrm{ROC} = \mathrm{ROC}_{1,\mathrm{i}} \cap \mathrm{ROC}_{2,\mathrm{j}}.$$

Combinations with non-empty ROC are

$$h[n] = [2^n - (1/3)^n]u(n), \quad \mathrm{ROC} = \{|z| > 2\},$$

which yields a causal but BIBO unstable system,

$$h[n] = -2^n u(-n-1) - (1/3)^n u(n), \quad \text{ROC} = \{1/3 < |z| < 2\},$$

which yields a noncausal (not causal nor anti-causal) and BIBO stable system, and

$$h[n] = [(1/3)^n - 2^n]u(-n-1), \quad \text{ROC} = \{|z| < 1/3\},$$

which yields an anti-causal BIBO unstable system.

See Exercise.

[2.10] Use the Schur-Cohn stability test. $H(z) = B(z)/A(z)$, where $A(z) = 1 + az^{-1} + z^{-2} + 1/2z^{-3}$, i.e. $a_3(n) = \{1, a, 1, 1/2\}$. Step-down recursion gives

$$\begin{aligned}
K_3 &= a_3(3) = 1/2 \\
a_2(1) &= \frac{1}{1-K_3^2}(a_3(1) - K_3 a_3(2)) = \frac{4a-2}{3} \\
K_2 = a_2(2) &= \frac{1}{1-K_3^2}(a_3(2) - K_3 a_3(1)) = \frac{4-2a}{3} \\
K_1 = a_1(1) &= \frac{1}{1-K_2^2}(a_2(1) - K_2 a_2(1)) = \frac{4a-2}{7-2a}
\end{aligned}$$

$|K_3| < 1$ holds always, $|K_2| < 1$ holds for $1/2 < a < 7/2$ and $|K_1| < 1$ holds for $-5/2 < a < 3/2$. This means that the filter is stable if and only if $1/2 < a < 3/2$.

See Exercise.

[2.11] a) i) No

ii) Yes

b)

$$\begin{aligned}
\mathbf{H}(z) &= \frac{1}{(z-4)(z-0.1)} = \frac{10}{39}\left[\frac{1}{z-4} - \frac{1}{z-0.1}\right] \\
&= \frac{10}{39}\left[\frac{-1/4}{1-z/4} - \frac{z^{-1}}{1-0.1z^{-1}}\right] \\
&= -\frac{10}{156}\sum_{k=0}^{\infty}(z/4)^k - \frac{10}{39}z^{-1}\sum_{k=0}^{\infty}(0.1z^{-1})^k \\
&= -\frac{10}{156}\sum_{k=-\infty}^{0} 4^k z^{-k} - \frac{100}{39}\sum_{k=1}^{\infty} 0.1^k z^{-k}.
\end{aligned}$$

But $\mathbf{H}(z) = \sum_{k=-\infty}^{\infty} h(k) z^{-k}$, therefore

$$h(k) = \begin{cases} -\frac{10}{156}4^k & k \le 0 \\ -\frac{100}{39}0.1^k & k > 0 \end{cases}$$

Region of Convergence: $0.1 < |z| < 4$.

See Exercise.

[3.1] D See Exercise.

[3.2] (a) $X(1/6) = 0$

(b) $P(e^{j2\pi f}) = \mathcal{F}\{r_x(k)\} = 0$ for $f = 1/6$.

(c) A zero at $f = 1/6$ corresponds to a factor $1 - z^{-1} + z^{-2}$.

$$X(z) = 1 - z^{-1} + z^{-2} - z^{-4} + z^{-5} - z^{-6} = (1 - z^{-1} + z^{-2})(1 - z^{-4})$$

See Exercise.

[3.3] For $k = 0, 1, \ldots, N-1$,

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi nk}{N}} && [n = N-1-l] \\ &= \sum_{l=0}^{N-1} x[N-1-l] e^{-j\frac{2\pi(N-1-l)k}{N}} = \sum_{l=0}^{N-1} y[l] e^{j\frac{2\pi lk}{N}} e^{j\frac{2\pi k}{N}} \\ &= e^{j\frac{2\pi k}{N}} \sum_{l=0}^{N-1} y[l] e^{-j\frac{2\pi l(-k)}{N}}, \end{aligned}$$

$Y[k]$ is defined only for $k = 0, 1, \ldots, N-1$. However, because the DTFT $Y(\nu)$ is periodic with period 1, evaluating the expression of $Y[k]$ on a k outside this range yields $Y[(-k)\mathrm{mod}(N)]$. Therefore, we conclude that, for $k = 0, 1, \ldots, N-1$, the DFTs of $y[n]$ and $x[n]$ are related through

$$X[k] = e^{j\frac{2\pi k}{N}} Y[(-k)\mathrm{mod}(N)] = \begin{cases} Y[0] & \text{for } k = 0 \\ e^{j\frac{2\pi k}{N}} Y[N-k] & \text{for } k = 1, ..., N-1 \end{cases}.$$

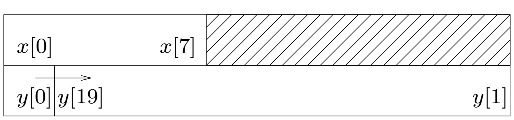
See Exercise.

[3.4]

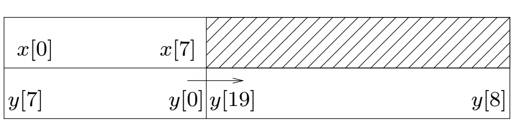
$$\frac{10 \text{ kHz}}{1024} = 9.76 \text{ Hz/sample}$$

See Exercise.

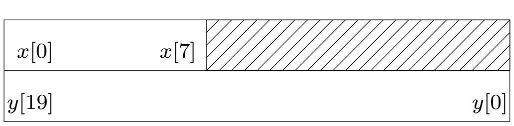
[3.5] The figure below uses the graphical strategy to compute the convolution as a reference to provide a solution.



$r[0]$ does not correspond to the linear convolution



$r[7]$ corresponds to the linear convolution



$r(19)$ still corresponds to the linear convolution

(a) The elements $r[n]$ for $n = 0, \dots, 6$ will not correspond to the linear convolution, since they involve products that do not appear in the linear convolution, e.g. $y[19] * x[1]$ as a contribution to r[0]. The elements $r[n]$ for $n = 7, \dots, 19$ will be precisely those found in the linear convolution, since they include all the same terms.

(b) From the figure, it is clear that the error is caused by the 7 last values of $y[n]$. This issue can be resolved by increasing the length of the sequences and the length of the DFTs to 27 by padding them with zeros.

See Exercise.

[3.6] (a) Direct filtering, 128 real multiplications/sample, one per filter coefficient.

(b) Overlap-save method with 256-point FFT,

$$4\,\frac{2 \cdot 8 \cdot \frac{256}{2} + 256}{129} \approx 71 \text{ real multiplications/sample}\,.$$

(c) Overlap-save method with 512-point FFT,

$$4\,\frac{2 \cdot 9 \cdot \frac{512}{2} + 512}{385} \approx 53 \text{ real multiplications/sample}\,.$$

See Exercise.

[3.7]

$$X(k) = \frac{1}{2}(Z(k) + Z^*(N-k))$$
$$Y(k) = \frac{j}{2}(-Z(k) + Z^*(N-k))$$

See Exercise.

[3.8]

$$y(n) = \left(\frac{1}{2}\right)^n \frac{256}{255}$$

See Exercise.

[3.9] Using the suggested methods, we get that

$$x_3[n] = (x_1 * x_2)[n] = \{4, 11, 20, 19, 13, 8, 2\},$$

and $x_4[n] = (x_1 \, ④ \, x_2)[n] = \{17,\ 19,\ 22,\ 19\}$.

See Exercise.

[3.10] $a_{00} = a_{10} = a_{01} = a_{20} = a_{02} = 1$, $a_{11} = a_{22} = e^{-j2\pi/3}$, $a_{21} = a_{12} = e^{j2\pi/3}$ See Exercise.

[3.11] a)

$$\begin{aligned}
X^{(n+1)}(k) =& \sum_{m=0}^{N-1} x(n+1+m)e^{-j\frac{2\pi km}{N}} \\
=& \sum_{m=-1}^{N-1} x(n+1+m)e^{-j\frac{2\pi km}{N}} - x(n)e^{j\frac{2\pi k}{N}} = \{l = m+1\} \\
=& \sum_{l=0}^{N} x(n+l)e^{-j\frac{2\pi kl}{N}} e^{j\frac{2\pi k}{N}} - x(n)e^{j\frac{2\pi k}{N}} \\
=& (X^{(n)}(k) - x(n) + x(n+N))e^{j\frac{2\pi k}{N}}
\end{aligned}$$

b)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1-j \\ -2 \\ 1+j \end{bmatrix}$$

Update:

$$X^{(1)}(k) = [X^{(0)}(k) - 1 + 2]e^{j2k\pi/4} = \{5, 1+2j, 1, 1-2j\}$$

See Exercise.

[3.12] $x_1(n) = x(nT)$, $n = 0, 1, \ldots, 2M-1$. The Nyquist criterion is fulfilled which means that $x_2(n)$ can be found through interpolation.

$$\begin{aligned}
X_2(k) &= \sum_{n=0}^{4M-1} x(\frac{nP}{4M})e^{-j\frac{2\pi nk}{4M}} = \sum_{n=0}^{4M-1}\sum_{l=-M}^{M} c_l e^{j\frac{2\pi nl}{4M}} e^{-j\frac{2\pi nk}{4M}} \\
&= \sum_{l=-M}^{M} c_l \sum_{n=0}^{4M-1} e^{j\frac{2\pi n}{4M}(l-k)} = 4M \sum_{r=-\infty}^{\infty}\sum_{l=-M}^{M} c_l \delta(l-k+r4M) \\
&= \begin{cases} 4Mc_k & 0 \le k \le M \\ 0 & M < k < 3M \\ 4Mc_{k-4M} & 3M \le k \le 4M-1 \end{cases}
\end{aligned}$$

$$\begin{aligned}
X_1(k) &= \sum_{n=0}^{2M-1} x(\frac{nP}{2M})e^{-j\frac{2\pi nk}{2M}} = \sum_{l=-M}^{M} c_l \sum_{n=0}^{2M-1} e^{j\frac{2\pi n}{2M}(l-k)} \\
&= 2M \sum_{r=-\infty}^{\infty}\sum_{l=-M}^{M} c_l \delta(l-k+r2M) \\
&= \begin{cases} 2Mc_k & 0 \le k < M \\ 2M(c_M + c_{-M}) & k = M \\ 2Mc_{k-2M} & M < k \le 2M-1 \end{cases}
\end{aligned}$$

This gives

$$X_2(k) = \begin{cases} 2X_1(k) & 0 \le k < M \\ X_1(k) & k = M \\ 0 & M < k < 3M \\ 2X_1(k-2M) & 3M \le k \le 4M-1 \end{cases}$$

See Exercise.

[3.14] a)

$$z(n) = \sum_k y(k)x((n-k))_N$$

$z(0) = 4+6 = 10$, $z(1) = 7+5 = 12$, $z(2) = 1+6 = 7$, $z(3) = 2+7 = 9$, $z(4) = 3+1 = 4$, $z(5) = 4+2 = 6$, $z(6) = 5+3 = 8$

b) $z = \{10, 12, 7, 9, 4, 6, 8\}$

See Exercise.

[3.15] Consider a sequence with length $2N$.

$$\begin{aligned} \mathrm{DFT}_{2N}(x(n)) = X(k) = \sum_{n=0}^{2N-1} x(n)e^{\frac{-j2\pi nk}{2N}} = \\ = \sum_{n=0}^{N-1} x(2n)e^{\frac{-j2\pi 2nk}{N}} + \sum_{n=0}^{N-1} x(2n+1)e^{\frac{-j2\pi(2n+1)k}{N}} = \\ = \sum_{n=0}^{N-1} x(2n)e^{\frac{-j2\pi nk}{N/2}} + e^{\frac{-j2\pi nk}{N}} \sum_{n=0}^{N-1} x(2n+1)e^{\frac{-j2\pi nk}{N/2}} = \\ = \mathrm{DFT}_N(x(2n)) + e^{\frac{-j2\pi nk}{N}}\mathrm{DFT}_N(x(2n+1)) \end{aligned}$$

That is one step of the "butterfly" implementation to solve a $2N$-point FFT by two N-point FFTs for the odd and even samples respectively. In the same way, we can handle sequences with length $2^k N$ by performing k steps of the "butterfly" outside the available FFT routine.

The frequency resolution is inversely proportional to the length of the sequence.

See Exercise.

[3.16] a) $y(29)$ through $y(49)$ are free from aliasing and are therefore identical with the linear convolution.

b) Consider the "overlap-save" method (see P&M). x_1 can be seen as an impulse response with $M = 20$. $N = 32 = M-1+L$ gives $L = 13$. Add $M-1$ zeros at the beginning of x_2 and generate a sequence with length N. Generate another sequence beginning with $x_1(10)$. Multiply the FFTs and then do IFFT, remove the first $M-1$ points and adjust the time index. Continue the above process with segments of x_2, each with length L.

See Exercise.

[3.17]

$$Y(k) = \sum_{n=0}^{2M-1} x(\frac{nP}{2M})e^{-j\frac{2\pi nk}{2M}} = \sum_{l=-M}^{M} c_l \sum_{n=0}^{2M-1} e^{j\frac{2\pi n}{2M}(l-k)} =$$

$$= 2M \sum_{l=-M}^{M} c_l \delta((l-k)_{\text{mod }(2M)}) = \begin{cases} 2Mc_k & 0 \le k < M \\ 2M(c_M + c_{-M}) & k = M \\ 2Mc_{k-2M} & M < k \le 2M-1 \end{cases}$$

See Exercise.

[3.18] a) Overlap-add

We have $N_x = 12, M = 3$ which gives $N_y = 12 + 3 - 1 = 14$. To get 3 data blocks, we choose $L = 4$ which means a circular convolution with 6 points.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$x(n)$	1	2	0	-1	0	-1	1	2	1	1	0	-1		
$h_0(n)$	1	-1	1	0	0	0								
$x_1(n)$	1	2	0	-1	0	0								
$x_2(n)$					0	-1	1	2	0	0				
$x_3(n)$									1	1	0	-1	0	0
$y_1(n)$	1	1	-1	1	1	-1								
$y_2(n)$					0	-1	2	0	-1	2				
$y_3(n)$									1	0	0	0	1	-1
$y(n)$	1	1	-1	1	1	-2	2	0	0	2	0	0	1	-1

This gives $N_y = 14$ which is sufficient for the desired linear convolution.

b) Overlap-save

In order to handle $N_y = 14$ with 'overlap-save', the number of convolution points must be at least 7. Let us use 7.

n			0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$x(n)$			1	2	0	-1	0	-1	1	2	1	1	0	-1			
$h_0(n)$			1	-1	1	0	0	0	0								
$x_1(n)$	0	0	1	2	0	-1	0										
$x_2(n)$						-1	0	-1	1	2	1	1					
$x_3(n)$											1	1	0	-1	0	0	0
$y_1(n)$	~~-1~~	~~0~~	1	1	-1	1	1										
$y_2(n)$						~~-1~~	~~2~~	-2	2	0	0	2					
$y_3(n)$											~~1~~	~~0~~	0	0	1	-1	0
$y(n)$	~~-1~~	~~0~~	1	1	-1	1	1	-2	2	0	0	2	0	0	1	-1	0

We get $N_y = 15$ which, again, is a sufficient number of output values to cover the full linear convolution.

See Exercise.

[4.1] $q_1 = 3$ and $q_2 = 6$. See Exercise.

[4.2] We first express the filter in the normalized frequency $\nu = \frac{\omega}{2\pi}$:

$$|H_{\text{norm}}(\nu)|^2 \approx \begin{cases} 1 & \text{for } |\nu| \leq \frac{\Omega}{2\pi} \text{ with } \Omega < \frac{\pi}{M} \\ 0 & \text{otherwise.} \end{cases}$$

The impulse response is downsampled by a factor M. This expands the spectrum by a factor of M. Because $\frac{\Omega}{2\pi} < \frac{1}{2M}$, we can disregard folding (aliasing) terms, and $\bar{h}[n]$ has its pass-band in $|\nu| \leq M\frac{\Omega}{2\pi}$ or $|\omega| \leq M\Omega$.

See Exercise.

[4.3] (a)

$$X(\omega) = \frac{1}{1 - ae^{-j\omega}}$$

(b)

$$X(\omega) = \frac{1}{1 - a^2e^{-j\omega}}$$

(c)

$$\mathcal{F}\{x(2n)\} = \frac{1}{1 - a^2e^{-j\omega}}$$

See Exercise.

[4.4]

a) The formulation in the hint, gives the polyphase solution directly with:

$$\begin{aligned} p_0(n) &= \delta(n) \\ p_1(n) &= \frac{1}{2}\left[\delta(n+1) + \delta(n)\right] \end{aligned}$$

The corresponding filter in the direct implementation is

$$h(n) = \{\ldots, p_0(-1), p_1(-1), \underset{\uparrow}{p_0(0)}, p_1(0), p_0(1), \ldots\} = \{\frac{1}{2}, \underset{\uparrow}{1}, \frac{1}{2}\}$$

b) $H(\omega) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} = 1 + \cos(\omega)$

c) Both filters will be non-causal low-pass filters. The ideal filter will preserve the spectral shape of the input while the linear interpolation filter will change it.

d) See a)

e)

$$Y(\omega) = \begin{cases} 1+\cos\omega & 0 \le |\omega| \le 0.1\pi \\ 1+\cos\omega & 0.9\pi \le |\omega| \le \pi \\ 0 & \text{otherwise} \end{cases}$$

f)

$$Y(\omega) = \begin{cases} 1+\cos\omega & 0.35\pi \le |\omega| \le 0.45\pi \\ 1+\cos\omega & 0.55\pi \le |\omega| \le 0.65\pi \\ 0 & \text{otherwise} \end{cases}$$

See Exercise.

[4.5] (a) Splitting the infinite sum in two, one containing even ns and one containing odd ns, yields

$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{+\infty} h[n]z^{-n} \\ &= \sum_{m=-\infty}^{+\infty} h[2m]z^{-2m} + \sum_{p=-\infty}^{+\infty} h[2p+1]z^{-(2p+1)} \\ &= \sum_{m=-\infty}^{+\infty} h[2m]\left(z^2\right)^{-m} + z^{-1}\sum_{p=-\infty}^{+\infty} h[2p+1]\left(z^2\right)^{-p} . \end{aligned}$$

Taking into account that

$$\mathrm{P}_0\left(z\right) = \sum_{n=-\infty}^{+\infty} p_0[n]z^{-n} = \sum_{m=-\infty}^{+\infty} h[2m]z^{-m} ,$$

and

$$\mathrm{P}_1\left(z\right) = \sum_{n=-\infty}^{+\infty} p_1[n]z^{-n} = \sum_{p=-\infty}^{+\infty} h[2p+1]z^{-p} ,$$

it is clear that

$$H(z) = \mathrm{P}_0\left(z^2\right) + z^{-1}\mathrm{P}_1\left(z^2\right) .$$

(b) The proposed transfer function $H(z)$ is all-pass $\forall a \in \mathbb{R}$, because

$$|H(z)|^2 = \frac{a+z^{-1}}{1+az^{-1}}\frac{a+z^{-*}}{1+az^{-*}} = \frac{a^2+2a\,\mathrm{Re}\left\{z^{-1}\right\}+|z|^{-2}}{1+2a\,\mathrm{Re}\left\{z^{-1}\right\}+a^2|z|^{-2}}$$

and, $\forall z \in \mathbb{C} \mid z = e^{j2\pi\nu}$ for some $\nu \in [0,1)$, $|H(z)|^2\big|_{z=e^{j2\pi\nu}} = 1$.
Using the inverse Z-transform, and taking into account that the filter is causal, we obtain the impulse response

$$h[n] = \frac{1}{a}\delta[n] + (a - \frac{1}{a})(-a)^n u[n] ,$$

where $u[n]$ is the discrete step function and $\delta[n]$ the Kronecker delta. Hence,

$$p_0[n] = h[2n] = \frac{1}{a}\delta[n] + (a - \frac{1}{a})(-a)^{2n} \text{ for } n \geq 0,$$

$$\begin{aligned} p_1[n] = h[2n+1] &= \frac{1}{a}\delta[2n+1] + (a - \frac{1}{a})(-a)^{2n+1}, \text{ for } n \geq 0 \\ &= (1-a^2)(-a)^{2n}, \text{ for } n \geq 0\,. \end{aligned}$$

Using now the Z-tranform, we obtain the expressions for the two components of the polyphase decomposition,

$$P_0(z) = a\frac{1-z^{-1}}{1-a^2z^{-1}},$$

$$P_1(z) = \frac{1-a^2}{1-a^2z^{-1}}.$$

In this case, both polyphase components can not be all-pass for the same parameter a. This can be proved by showing that, for this particular filter,

$$|H\nu)|^2 = |\mathrm{P}_0(\nu)|^2 + |\mathrm{P}_1|^2\,.$$

For further insight on this situation, the following MATLAB / Octave code helps visualizing the situation. Note that this frequency separation of the polyphase components is not a general property, but only a result for this case.

```
nu = -0.5:0.0001:0.5;
as = 0:0.2:1;
for a = as
  p0=a*(1-exp(-1i*4*pi*nu))./(1-a^2*exp(-1i*4*pi*nu));
  p1=(1-a^2)./(1-a^2*exp(-1i*4*pi*nu));
  plot(nu,abs(p0.^2),'r--',nu,abs(p1.^2),'b-.')
  hold on
end
legend('0','1')
```

See Exercise.

[4.6] (a) Consider the annotations in Fig. 11.3. Then, using the basic relations learned in the theory, we can derive the spectrum at each annotated spot as follows. For $a_0[n]$ and $a_1[n]$, the convolution property states

$$A_0(\nu) = X(\nu)H(\nu),\ A_1(\nu) = X(\nu)F(\nu)\,,$$

where $X(\nu)$ is the DTFT of $x[n]$, and $H(\nu)$ and $F(\nu)$ are the transfer functions for the filters $h[n]$ and $f[n]$, respectively. For $b_0[m]$ and $b_1[m]$, i.e., the signals after downsampling, the full spectrum is given by

$$B_0(\nu) = \frac{1}{2}\sum_{k=0}^{1} X\left(\frac{\nu-k}{2}\right) H\left(\frac{\nu-k}{2}\right), B_1(\nu) = \frac{1}{2}\sum_{k=0}^{1} X\left(\frac{\nu-k}{2}\right) F\left(\frac{\nu-k}{2}\right),$$

which contains the aliasing terms that could later hinder total reconstruction. Then, applying the relation for upsampling given in the theory, we obtain, for $c_0[n]$ and $c_1[n]$,

$$C_0(\nu) = \frac{1}{2}\sum_{k=0}^{1} X\left(\nu-\frac{k}{2}\right) H\left(\nu-\frac{k}{2}\right), C_1(\nu) = \frac{1}{2}\sum_{k=0}^{1} X\left(\nu-\frac{k}{2}\right) F\left(\nu-\frac{k}{2}\right).$$

Finally, the filters $g[n]$ and $d[n]$ (with transfer functions $G(\nu)$ and $D(\nu)$, respectively), have their effect, that, thanks to the convolution theorem, can be written as

$$\begin{aligned} E_0(\nu) &= G(\nu)\left[\frac{1}{2}\sum_{k=0}^{1} X\left(\nu-\frac{k}{2}\right) H\left(\nu-\frac{k}{2}\right)\right] \\ E_1(\nu) &= D(\nu)\left[\frac{1}{2}\sum_{k=0}^{1} X\left(\nu-\frac{k}{2}\right) F\left(\nu-\frac{k}{2}\right)\right] \end{aligned}.$$

Summarizing, then, we get the output $y[n]$ with spectrum

$$\begin{aligned} Y(\nu) &= E_0(\nu) + E_1(\nu) \\ &= \frac{1}{2}G(\nu)\left[X(\nu)H(\nu) + X\left(\nu-\frac{1}{2}\right)H\left(\nu-\frac{1}{2}\right)\right] \\ &+ \frac{1}{2}D(\nu)\left[X(\nu)F(\nu) + X\left(\nu-\frac{1}{2}\right)F\left(\nu-\frac{1}{2}\right)\right] \\ &= \frac{1}{2}X(\nu)\left[G(\nu)H(\nu) + D(\nu)F(\nu)\right] \\ &+ \frac{1}{2}X\left(\nu-\frac{1}{2}\right)\left[G(\nu)H\left(\nu-\frac{1}{2}\right) + D(\nu)F\left(\nu-\frac{1}{2}\right)\right]. \end{aligned}$$

We want $Y(\nu) = X(\nu)e^{-j2\pi M\nu}$ corresponding to a delay of M samples, which, directly from the previous expression, gives

$$\begin{aligned} G(\nu)H(\nu) + D(\nu)F(\nu) &= 2e^{-j2\pi\nu M} \\ G(\nu)H\left(\nu-\frac{1}{2}\right) + D(\nu)F\left(\nu-\frac{1}{2}\right) &= 0\,. \end{aligned}$$

(b) We have three pieces of information to work on

$$H^2(\nu) + D(\nu)F(\nu) = 2e^{-j2\pi\nu M} \tag{11.1}$$

$$H(\nu)H\left(\nu-\frac{1}{2}\right) + D(\nu)F\left(\nu-\frac{1}{2}\right) = 0 \tag{11.2}$$

$$F(\nu) = H\left(\nu-\frac{1}{2}\right), \tag{11.3}$$

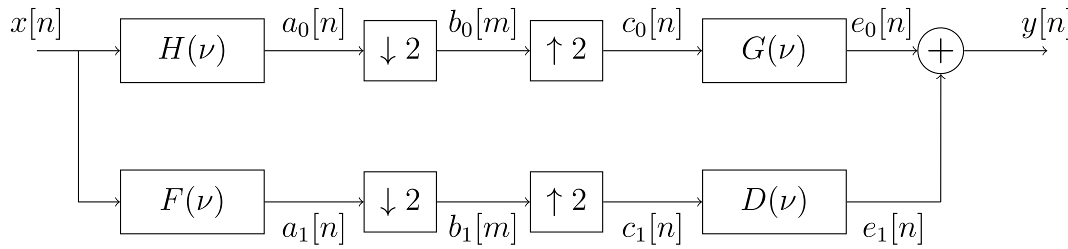


Figure 11.3: Annotated filter bank for solving Exercise 4.6. Mainly the same as Fig. 4.1, but annotated with reference signals to do the analysis.

where only the conclusions from the previous question and the new assumptions in the headlines have been used.

From (11.3) and the periodicity of the discrete spectrum we can extract that

$$F\left(\nu - \frac{1}{2}\right) = H(\nu)\,.$$

Analysis of (11.2) under these new conditions reveals that, if we assume $\exists \nu \in [0,1) \mid H(\nu) \neq 0$,

$$H(\nu)\left(H\left(\nu - \frac{1}{2}\right) + D(\nu)\right) = 0 \Rightarrow D(\nu) = -H\left(\nu - \frac{1}{2}\right)\,.$$

Therefore, we obtain that

$$\begin{array}{rcl} F(\nu) & = & H\left(\nu - \frac{1}{2}\right) \\ D(\nu) & = & -H\left(\nu - \frac{1}{2}\right) \end{array}\,.$$

By inverse DTFT, we obtian that the impulse responses are related through

$$f[n] = e^{j\pi n} h[n] = \left\{ \begin{array}{cc} h[n] & \text{for even } n \\ -h(n) & \text{for odd } n \end{array} \right.,$$

and similarly

$$d[n] = -e^{j\pi n} h(n) = \left\{ \begin{array}{cc} -h[n] & \text{for even } n \\ h(n) & \text{for odd } n \end{array} \right..$$

Finally, using these new results and (11.1), we obtain

$$H^2(\nu) + H^2\left(\nu - \frac{1}{2}\right) = 2e^{-j2\pi\nu M}\,.$$

See Exercise.

[4.8] $y(n) = x(nD)$ gives

$$Y(\omega) = \frac{1}{D}\sum_{k=0}^{D-1} X(\frac{\omega - 2\pi k}{D}) = \frac{1}{D}X(\frac{\omega}{D})$$

Since $X(\omega) = 0$, $\omega_m \leq \pi/D < |\omega| < \pi$.

a) Since the Fourier transform is one to one, the signal $x(n)$ can be recovered from $y(n)$.

b) We use interpolation with $z(nD) = y(n)$ and zero otherwise.

$$Z(\omega) = Y(\omega D)$$

Low pass filter with $H(\omega) = D$, $|\omega| < \pi/D$ and zero otherwise. We get

$$Z(\omega) = DY(\omega D) = X(\omega) \quad \text{för} \quad 0 < |\omega| < \pi/D$$

and zero otherwise.

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)e^{j\omega n}d\omega = \frac{D}{2\pi}\int_{-\pi/D}^{\pi/D} e^{j\omega n}d\omega = \frac{D\sin \pi n/D}{\pi n}, \quad n \neq 0$$

$h(0) = 1$

See Exercise.

[4.9] a) Let $u(n) = \sum_{k=0}^{N} h(k)x(n - Lk)$. Then,

$$U(z) = \sum_{k=0}^{N} h(k)\mathcal{Z}\left\{x(n - Lk)\right\} = \sum_{k=0}^{N} h(k)z^{-Lk}X(z) =$$

$$= X(z)\sum_{k=0}^{N} h(k)(z^L)^{-k} = X(z)H(z^L)$$

$y(n) = u(2n)$ gives

$$Y(z) = \frac{1}{2}\left(X(z^{1/2})H(z^{L/2}) + X(-(z^{1/2}))H(-(z^{L/2}))\right)$$

b) R.H.S =f(2n)+f(2n - 4) = { 1, 1, 0, 0} + { 0, 0, 1, 1 } = f(n)= L.H.S

c)

$$H(z) = 1 + z^{-1}$$

$$U(z) = F(z)H(z^L) = \frac{1 - z^{-4}}{1 - z^{-1}}(1 + z^{-4}) = \frac{1 - z^{-8}}{1 - z^{-1}}$$

$$Y(z) = \frac{1}{2}\left(U(z^{1/2}) + U(-(z^{1/2}))\right) = \frac{1 - z^{-4}}{2}\left(\frac{1}{1 - z^{-1/2}} + \frac{1}{1 + z^{-1/2}}\right) =$$

$$= \frac{1 - z^{-4}}{1 - z^{-1}} = F(z)$$

See Exercise.

[5.1] (a) By assumption, we have that

$$f_{E[n]}(e[n]) = \begin{cases} \frac{1}{\Delta} & \text{for } e[n] \in \left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right] \\ 0 & \text{otherwise} \end{cases} .$$

Then, we have that

$$m_e = \mathrm{E}\left\{e[n]\right\} = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \frac{1}{\Delta} e[n] de[n] = 0 ,$$

$$\sigma_e^2 = \mathrm{E}\left\{e^2[n]\right\} = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \frac{1}{\Delta} e^2[n] de[n] = \frac{1}{\Delta}\left[\frac{e^3[n]}{3}\right]_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} = \frac{\Delta^2}{12} .$$

Additionally, $e[n]$ is also white by assumption, and thus,

$$r_e[k] = \mathrm{E}\left\{e[n]e[n-k]\right\} = \frac{\Delta^2}{12}\delta[k] .$$

(b)

$$\mathrm{SNR}_{y[n]} = \frac{\sigma_x^2}{\sigma_e^2} = 12\frac{\sigma_x^2}{\Delta^2} .$$

(c) Because both $x[n]$ is white, we have that

$$\begin{aligned} \mathrm{E}\left\{((h*x)[n])^2\right\} &= \mathrm{E}\left\{\left(\sum_{m=0}^{\infty} h[m]x[n-m]\right)^2\right\} \\ &= \mathrm{E}\left\{\left(\sum_{m=0}^{\infty} h[m]x[n-m]\right)\left(\sum_{k=0}^{\infty} h[k]x[n-k]\right)\right\} \\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} h[m]h[k]\,\mathrm{E}\left\{x[n-m]x[n-k]\right\} \\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} h[m]h[k]\,\mathrm{E}\left\{x^2[n]\right\}\delta[k-m] \\ &= \sum_{m=0}^{+\infty} h^2[m]\,\mathrm{E}\left\{x^2[n]\right\} , \end{aligned}$$

where we used the autocorrelation of the white input signal

$$\mathrm{E}\left\{x[n-m]x[n-k]\right\} = \mathrm{E}\left\{x[n]x[n-k+m]\right\} = r_x[k-m] = \sigma_x^2\delta[k-m] .$$

Similarly, because $e[n]$ is white,

$$\mathrm{E}\left\{((h*e)[n])^2\right\} = \mathrm{E}\left\{e^2[n]\right\}\sum_{n=0}^{+\infty} h^2[n] .$$

Therefore,

$$\mathrm{SNR}_{\text{out}} = \frac{\mathrm{E}\left\{((h * x)[n])^2\right\}}{\mathrm{E}\left\{((h * e)[n])^2\right\}} = \mathrm{SNR}_{y[n]} = 12\frac{\sigma_x^2}{\Delta^2}\,.$$

Finally, the variance of the noise at the output of the filter will be

$$\mathrm{E}\left\{((h * e)[n])^2\right\} = \mathrm{E}\left\{e^2[n]\right\}\sum_{n=0}^{+\infty} h^2[n] = \frac{\Delta^2}{12}\sum_{n=0}^{+\infty} h^2[n]$$

with

$$\begin{aligned}
\sum_{n=0}^{+\infty} h^2[n] &= \frac{1}{4}\sum_{n=0}^{+\infty}(a^n + (-a)^n)^2 \\
&= \frac{1}{4}\sum_{m=0}^{+\infty}\left(a^{2m} + (-a)^{2m}\right)^2 + \frac{1}{4}\sum_{k=0}^{+\infty}\left(a^{2k+1} + (-a)^{2k+1}\right)^2 \\
&= \frac{1}{4}\sum_{m=0}^{+\infty} 4(a^4)^m = \frac{1}{1-a^4}\,.
\end{aligned}$$

See Exercise.

[5.3] Let us first analyze the performance with respect to the quantization noise of the first realization of the system, i.e. the one shown in Fig. 5.4.I. Let us model the quantization noise as a white, uniform, zero-mean, additive noise at the output of each multiplier. These noises are also assumed independent of each other. Note $e_1[n]$ the noise after the 0.5 multiplier, $e_2[n]$ the noise after the 0.3 multiplier, and $e_3[n]$ the noise after the 0.2 multiplier. Note σ_e^2 the power of each of these noises, which is the same, since this power depends only on the considered quantization. Note $\sigma_{y_{e_i}}^2$ the contributions of each of the noises to the noise at the output, such that the total noise power at the output can be written as $\sigma_{y_e}^2 = \sum_{i=1}^{3}\sigma_{y_{e_i}}^2$. It is clear that both $e_2[n]$ and $e_3[n]$ make their way directly into the output. $e_1[n]$, on the other hand, goes through the whole system and then reaches the output. Thus, $\sigma_{y_{e_2}}^2 = \sigma_{y_{e_3}}^2 = \sigma_e^2$, but

$$\sigma_{y_{e_1}}^2 = \mathrm{E}\left\{((e_1 * h)[n])^2\right\}\,.$$

Through the transfer function of the system and the inverse Z-transform, it is easy to obtain that $h[n] = 0.3\,2^{-n}u[n] + 0.2\,2^{-n+1}u[n-1]$. Thus, because $e_1[n]$ is white,

$$\begin{aligned}
\sigma_{y_{e_1}}^2 &= \sigma_e^2\sum_{n=-\infty}^{+\infty} h^2[n] = \sigma_e^2\left[0.3^2 + \sum_{n=1}^{+\infty}\left(0.3\,2^{-n} + 0.2\,2^{-n+1}\right)^2\right] \\
&= \sigma_e^2\left[0.3^2 + \sum_{n=1}^{+\infty}\left(\frac{3}{10}\frac{1}{2^n} + \frac{2}{10}\frac{2}{2^n}\right)^2\right] = \sigma_e^2\left[0.3^2 + \left(\frac{7}{10}\right)^2\sum_{n=1}^{+\infty}\frac{1}{4^n}\right] \\
&= \sigma_e^2\left[0.3^2 + \frac{49}{100}\frac{\frac{1}{4}}{1-\frac{1}{4}}\right] = \sigma_e^2\left[0.3^2 + \frac{49}{300}\right] \approx 0.2533\sigma_e^2\,.
\end{aligned}$$

Therefore, for the realization of the system presented in Fig. 5.4.I, the total quantization noise at the output is

$$\sigma_{\mathrm{I}}^2 = \sigma_{y_e}^2 = \sum_{i=1}^{3} \sigma_{y_{e_i}}^2 = (1 + 1 + 0.2533)\sigma_e^2 = 2.2533\sigma_e^2 \,.$$

Now, then, let us evaluate the quantization noise power at the output of the second realization of the system, i.e. the one shown in Fig. 5.4.II. We will use the same naming convention, i.e. we will note $e_1[n]$ the noise after the 0.5 multiplier, $e_2[n]$ the noise after the 0.3 multiplier, and $e_3[n]$ the noise after the 0.2 multiplier. An anaysis of the system deriving its transfer function and inverse Z-transforming will reveal that the system has exactly the same impulse response $h[n]$. However, the entry points of the quantization noises are essentially different. In this case, $e_2[n]$ only has to go through the AR part of the circuit, or, in other terms,

$$Y_{e_2}(z) = E_2(z) + 0.5z^{-1}Y_{e_2}(z) \Rightarrow H_2(z) = \frac{Y_{e_2}(z)}{E_2(z)} = \frac{1}{1 - 0.5z^{-1}} \,.$$

Therefore, the impulse response of the system that $e_2[n]$ goes through before reaching the output is $h_2[n] = 2^{-n}u[n]$ (obtained by inverse Z-transform). Because the input point of the noises $e_1[n]$ and $e_3[n]$ is only one delay away from the entry point for $e_2[n]$, it is not surprising that

$$H_1(z) = H_3(z) = z^{-1}H_2(z) = \frac{z^{-1}}{1 - 0.5z^{-1}}$$

which yields $h_1[n] = h_3[n] = h_2[n-1] = 2^{-n+1}u[n-1]$.

For all three quantization noises, we have that, because they are white,

$$\sigma_{y_{e_i}}^2 = \mathrm{E}\left\{((e_i * h)[n])^2\right\} = \sigma_e^2 \sum_{n=-\infty}^{+\infty} h_i^2[n] \,.$$

In fact, because their impulse responses are the same except for a one-sample delay, and we are summing the whole infinite impulse response,

$$\sum_{n=-\infty}^{+\infty} h_1^2[n] = \sum_{n=-\infty}^{+\infty} h_2^2[n] = \sum_{n=-\infty}^{+\infty} h_3^2[n] \,,$$

and

$$\sum_{n=-\infty}^{+\infty} h_2^2[n] = \sum_{n=0}^{+\infty} \frac{1}{2^{2n}} = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3} \,.$$

Therefore, for the realization of the system presented in Fig. 5.4.II, the total quantization noise at the output is

$$\sigma_{\mathrm{II}}^2 = \sigma_{y_e}^2 = \sum_{i=1}^{3} \sigma_{y_{e_i}}^2 = \left(\frac{4}{3} + \frac{4}{3} + \frac{4}{3}\right)\sigma_e^2 = 4\sigma_e^2 > \sigma_{\mathrm{I}}^2 \,.$$

See Exercise.

[5.4] Quantization noise with variance $\sigma_e^2 = 2^{-2b}/12$ enters the filter in three places. At the input of the filter, the total quantization noise is $\epsilon(n) = e_1(n) + e_2(n) - k_2 e_3(n-1)$, with variance $\sigma_\epsilon^2 = (2 + k_2^2)\sigma_e^2$. The transfer function of the filter is

$$H(z) = \frac{1}{1 + k_1(1+k_2)z^{-1} + k_2 z^{-2}} = \ldots = \frac{3}{1 - \frac{3}{4}z^{-1}} - \frac{2}{1 - \frac{1}{2}z^{-1}}.$$

$$h(n) = [3(3/4)^n - 2(1/2)^n]u(n)$$

$$\sigma_y^2 = \sum_{n=-\infty}^{\infty} h^2(n) \cdot \sigma_\epsilon^2 = \frac{704}{105}\sigma_\epsilon^2 = \frac{704}{105}(2 + \frac{9}{64})\frac{2^{-2b}}{12} \approx 1.2 \cdot 2^{-2b}$$

See Exercise.

[5.5] Let us refer to the uniform, white, additive and independent quantization noises at the output of each multiplier as $e_i[n], i = 1, 2, 3, 4$, respectively, from left to right in Fig. 5.6. Both $e_1[n]$ and $e_4[n]$ will not be affected by the choice of parameters, because the former will always travel through the whole transfer function, while the latter will always go directly to the output. $e_2[n]$ and $e_3[n]$, on the other hand, will both travel through the second stage of the system before getting to the output. Therefore, we will have to select a_2 and b_2 in order to make the second stage as resistant to quantization noise at its input as possible.

Analyzing the overall transfer function results in identifying that the possible choices of the parameters that will yield the desired system are

a_1	b_1	a_2	b_2
0.25	0.25	0.5	0.5
0.5	0.25	0.25	0.5
0.25	0.5	0.5	0.25
0.5	0.5	0.25	0.25

The transfer function of the second stage is

$$H_2(z) = \frac{1 + b_2 z^{-1}}{1 - a_2 z^{-1}} = \frac{z}{z - a_2} + b_2 z^{-1}\frac{z}{z - a_2}.$$

Because we $e_2[n]$ and $e_3[n]$, the power of their contribution to the output will be

$$\sigma_{y_{e_2}}^2 = \sigma_{y_{e_3}}^2 = \sigma_e^2 \sum_{n=-\infty}^{\infty} h_2^2[n],$$

where σ_e^2 is a constant value dependent only on the specific quantization used.

From the expression of the Z-transform above, it is immediate to extract that $h_2[n] = a_2^n u[n] + b_2 a_2^{n-1} u[n-1]$. Thus,

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} h_2^2[n] &= 1 + \sum_{n=1}^{+\infty} \left(a_2^n + b_2 a_2^{n-1}\right)^2 \\
&= 1 + \sum_{n=1}^{+\infty} \left[(a_2 + b_2) a_2^{n-1}\right]^2 \\
&= 1 + (a_2 + b_2)^2 \sum_{n=1}^{+\infty} a_2^{2n-2} = 1 + (a_2 + b_2)^2 \sum_{q=0}^{+\infty} a_2^{2q} \\
&= 1 + \frac{(a_2 + b_2)^2}{1 - a_2^2}.
\end{aligned}$$

Then, the noise components at the output from $e_2[n]$ and $e_3[n]$ will be scaled by the following factors, for the following combinations of parameters

a_1	b_1	a_2	b_2	$\sum_{n=-\infty}^{\infty} h_2^2[n]$
0.25	0.25	0.5	0.5	2.33
0.5	0.25	0.25	0.5	1.60
0.25	0.5	0.5	0.25	1.75
0.5	0.5	0.25	0.25	1.27

Therefore, in terms of quantization noise, the optimal parameters that implement the desired transfer function are $a_1 = 0.5$, $b_1 = 0.5$, $a_2 = 0.25$ and $b_2 = 0.25$.

See Exercise.

[5.6] Introduce a new signal variable $z(n)$ after the addition, requirement: $-1 \le z(n) < 2$

a) Assume $-1 \le z(k) < 2$.

$a \ge 0$**:** $z(k+1) = x(k+1) + az(k) \le 1 + az(k)$. $z(k+1)$ is surely < 2 if $\mathbf{a < 1/2}$.

$a < 0$**:** $z(k+1) \ge az(k)$. $z(k+1)$ is surely ≥ -1 if $\mathbf{a \ge -1/2}$. Then, also $z(k+1) \le 1 + az(k) \le 1/2 < 2$.
Answer: $\underline{-\frac{1}{2} \le a < \frac{1}{2}}$

b)

$z(n)_{max} = \frac{1}{1-a}$ when $x(n) = 1\ \forall n$ and $a > 0$
$z(n)_{max} < 2 \Longrightarrow a < \frac{1}{2}$
$z(n)_{min} = \frac{a}{1-a^2}$ when $\begin{cases} x(n) = 1 & n-k = \text{odd} \\ x(n) = 0 & n-k = \text{even} \end{cases}$ and $a < 0$
$z(n)_{min} \ge -1 \Longrightarrow a \ge \frac{1-\sqrt{5}}{2}$

$-1 \le y(n) < 2$ is always fulfilled when the above requirements are fulfilled
Answer: $\underline{\frac{1-\sqrt{5}}{2} \le a < \frac{1}{2}}$
See Exercise.

[5.7] The noise from k is filtered by the same transfer function in both cases and thus gives the same contribution.
Case I: The noise from b_0, b_1 and a_1 is filtered by $H_1(z) = 1/(1 + a_1 z^{-1})$.
Case II: The noise from a_1 is filtered by $H(z) = (b_0 + b_1 z^{-1})/(1 + a_1 z^{-1})$ and the noise from b_0 and b_1 goes directly to the output.
Therefore, we need to determine which of the expressions below gives the smallest round-off noise.

$$\sigma_I = 3 \sum_{n=0}^{\infty} h_1^2(n) \qquad \sigma_{II} = \sum_{n=0}^{\infty} h^2(n) + 2$$

$$H_1(z) = 1/(1 + a_1 z^{-1}) = 1 - a_1 z^{-1} + a_1^2 z^{-2} + \ldots \implies h_1(n) = (-a_1)^n$$

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}} = (b_0 + b_1 z^{-1})(1 - a_1 z^{-1} + a_1^2 z^{-2} + \ldots)$$

$$\implies h(0) = b_0,\ h(n) = (-a_1)^{n-1}(b_1 - b_0 a_1), \quad n > 0$$

$$\sigma_I = \frac{3}{1 - a_1^2} \qquad \sigma_{II} = 2 + b_0^2 + \frac{(b_1 - b_0 a_1)^2}{1 - a_1^2}$$

i) $\sigma_I = 4$, $\sigma_{II} = 6 + 16/27$, $\sigma_I < \sigma_{II}$.

ii) Unstable filter!

$$\sigma_I < \sigma_{II} \implies \frac{3}{1 - a_1^2} < 2 + b_0^2 + \frac{(b_1 - b_0 a_1)^2}{1 - a_1^2}$$

Stable filter requires $1 - a_1^2 > 0$. This gives

$$a_1^2 + b_0 b_1 a_1 + c < 0, \quad c = \frac{1 - b_0^2 - b_1^2}{2}$$

If $b_0^2 b_1^2/4 - c < 0$ then I will never be better than II. Otherwise,

$$-\frac{b_0 b_1}{2} - \sqrt{\frac{b_0^2 b_1^2}{4} - c} < a_1 < -\frac{b_0 b_1}{2} + \sqrt{\frac{b_0^2 b_1^2}{4} - c}.$$

See Exercise.

[5.8] a) The filter $H(z) = 1/(1 - 0.5z^{-1}) \implies h(n)0.5^n u(n)$. We get

$$\sum_{n=0}^{\infty} h^2(n) = \frac{4}{3}$$

Assume that the amplitude is scaled with max value 1. There are two noise contributors, one from A/D converter and one from round-off at the multiplication, both with variance

$$\sigma_e^2 = \frac{\Delta^2}{12} = \frac{2^{-2b}}{12}$$

Noise at the output

$$\sigma_{\text{ut}}^2 = 2\sigma_e^2 \sum_{n=0}^{\infty} h^2(n) = 2\frac{2^{-2b}}{12}\frac{4}{3} = \frac{2}{9}2^{-2b}$$

b) The input amplitude is 0.5. The normalized frequency is $2/8 = 1/4$. The amplitude amplification is

$$|H(j\omega)|_{\omega=2\pi/4} = \left|\frac{1}{1-0.5e^{-j2\pi/4}}\right| = \frac{2}{\sqrt{5}}$$

The output power is

$$\sigma_{\text{sig}}^2 = \left(\frac{1}{\sqrt{2}}\frac{1}{2}\frac{2}{\sqrt{5}}\right)^2 = \frac{1}{10}$$

$$\text{SNR} = \sigma_{\text{sig}}^2/\sigma_{\text{ut}}^2 = \frac{9}{20}2^{2b} > 10^5 \quad \Longrightarrow \quad b \geq 9.$$

See Exercise.

[5.9] The impulse response for the first section is $h_1(n) = a^n, n = 0, 1, \ldots$ and the impulse response for the second section is $h_2(n) = b^n, n = 0, 1, \ldots$. The total impulse response is $h(n) = h_1(n) * h_2(n)$. We assume this is a stable filter. The noise variance at the output is

$$\sigma_{ut}^2 = \sigma_e^2[\sum_{n=0}^{\infty} h^2(n) + \sum_{n=0}^{\infty} h_i^2(n)]$$

where $i = 1$ or $i = 2$.

a) Correspondingly, we should choose the smaller one of

$$\sum_{n=0}^{\infty} h_1^2(n) = \frac{1}{1-a^2} \quad \text{and} \quad \sum_{n=0}^{\infty} h_2^2(n) = \frac{1}{1-b^2}$$

as the last section. If $|a| < |b|$ choose h_1 as the last section and if $|a| > |b|$ choose h_2 as the last section.

b) Determine

$$\begin{aligned} h(n) &= h_1(n) * h_2(n) = \sum_{k=0}^{n} a^k b^{n-k} = \sum_{k=0}^{\infty} a^k b^{n-k} - \sum_{k=n+1}^{\infty} a^k b^{n-k} \\ &= \begin{cases} \frac{b^{n+1}}{b-a} - \frac{a^{n+1}}{b-a} & a \neq b \\ \sum_{k=0}^{n} a^n = (n+1)a^n & a = b \end{cases} \end{aligned}$$

We get $(a \neq b)$

$$\begin{aligned}\sum_{n=0}^{\infty} h^2(n) &= \sum_{n=0}^{\infty}\left(\frac{b^{2n}b^2}{(b-a)^2} - 2\frac{a^n b^n ab}{(b-a)^2} + \frac{a^{2n}a^2}{(b-a)^2}\right) \\ &= \frac{1}{(b-a)^2}\left(\frac{b^2}{1-b^2} - 2\frac{ab}{1-ab} + \frac{a^2}{1-a^2}\right)\end{aligned}$$

We get

$$\sigma_{ut}^2 = \sigma_e^2\left[\frac{1}{(b-a)^2}\left(\frac{b^2}{1-b^2} - 2\frac{ab}{1-ab} + \frac{a^2}{1-a^2}\right) + \frac{1}{1-p^2}\right]$$

where $p = a$ or $p = b$ depending on the order.

See Exercise.

[6.1] Very basic properties of the time-frequency domains relationship have to be recalled,

- a wider (time) window will provoke a narrower (frequency) main lobe,
- a thinner (time) window will provoke a wider (frequency) main lobe.

Additionally, a basic characteritzation of the rectangular and triangular windows will say that,

- a rectangular window has larger (power) side lobes,
- a triangular window has smaller (power) side lobes.

Using these properties, we have that: A $\leftrightarrow$ II, B $\leftrightarrow$ IV, C $\leftrightarrow$ I, and D $\leftrightarrow$ III.

See Exercise.

[6.2] (a) Δf is 0.01. Therefore

$$\frac{0.9}{M} < 0.01 \quad \Longrightarrow \quad M > 90$$

(b) Increasing M will not give us a better estimate of the location of the peak, but it will give us a better estimate of its width.

See Exercise.

[6.3] (a) In this problem, the triangular or Bartlett window is used in Welch's method. Therefore, if we define *sufficient* resolution as the $3\,dB$ bandwidth being thinner than the frequency difference ΔF we want to distinguish, we have,

$$\frac{1.28}{M} < \frac{\Delta F}{F_s} \Rightarrow M > 1.28\frac{F_s}{\Delta F} = 1.28\frac{12000}{200} \approx 77.$$

(b) For Welch's method with a Bartlett window, the variance is, approximately,

$$\mathrm{Var}[\hat{P}_W(w)] \approx \begin{cases} \frac{1}{K} P_x^2(w) & \text{with no overlap} \\ \frac{9}{8K} P_x^2(w) & \text{with 50\% overlap} \end{cases},$$

where $P_x(w)$ is the power spectrum density. For example, let us assume there's no overlap,

$$0.05\, P_x^2(w) \geq \frac{1}{K} P_x^2(w).$$

Therefore, assuming the non-overlapping case, $K \geq 20$.

See Exercise.

[6.4] Consider first that $w[n] = \cos^2\left(\frac{n\pi}{N}\right)$, and, therefore,

$$\begin{aligned} w[n] &= \left(\frac{e^{j2\pi\frac{n}{2N}} + e^{-j2\pi\frac{n}{2N}}}{2}\right)^2 \\ &= \frac{1}{2} + \frac{e^{j2\pi\frac{1}{N}n} + e^{-j2\pi\frac{1}{N}n}}{4} \end{aligned}$$

$$\begin{aligned} \mathrm{DFT}_N\{w[n]\} = W[k] &= \frac{1}{2} N\delta[k \bmod N] \\ &+ \frac{1}{4} N\delta[(k-1) \bmod N] + \frac{1}{4} N\delta[(k+1) \bmod N]. \end{aligned}$$

This implies that

$$\begin{aligned} \mathrm{DFT}\{x[n]w[n]\} &= \frac{1}{N} X[k] \circledN W[k] \\ &= \frac{1}{2} X[k] + \frac{1}{4} X[k-1] + \frac{1}{4} X[k+1]. \end{aligned}$$

See Exercise.

[6.6] (a)

Rectangular

$$\begin{aligned} W_R(f) &= \sum_{m=-M}^{M} e^{-j2\pi f m} = e^{j2\pi f M} \frac{1 - e^{-j2\pi f(2M+1)}}{1 - e^{-j2\pi f}} \\ &= \frac{e^{j\pi f(2M+1)} - e^{-j\pi f(2M+1)}}{e^{j\pi f} - e^{-j\pi f}} = \frac{\sin(\pi f(2M+1))}{\sin(\pi f)} \end{aligned}$$

For $f = 0$, $W_R(0) = 2M + 1$.

Bartlett This triangular window can be seen as the convolution of two rectangular windows of length M. The highest value of this correlation will be M if the rectangular windows have height 1, so the result should be divided by M to get the correct scaling. The convolution corresponds to a multiplication in the frequency domain, so

$$W_B(f) = \frac{1}{M} \frac{\sin^2(\pi f(M)))}{\sin^2(\pi f)}$$

Raised cosine Since $\cos(x) = (e^{jx} + e^{-jx})/2$, we get

$$\begin{aligned} W_H(f) =& \alpha \frac{e^{j\pi f(2M+1)} - e^{-j\pi f(2M+1)}}{e^{j\pi f} - e^{-j\pi f}} + \frac{\beta}{2} \frac{e^{j\pi(f(2M+1)-1)} - e^{-j\pi(f(2M+1)-1)}}{e^{j\pi(f-1/(2M+1))} - e^{-j\pi(f-1/(2M+1))}} \\ &+ \frac{\beta}{2} \frac{e^{j\pi(f(2M+1)+1)} - e^{-j\pi(f(2M+1)+1)}}{e^{j\pi(f+\frac{1}{2M+1}} - e^{-j\pi(f+\frac{1}{2M+1})}} \\ =& \alpha \frac{\sin(\pi f(2M+1))}{\sin(\pi f)} + \frac{\beta}{2} \frac{\sin(\pi(f(2M+1)-1))}{\sin(\pi(f-\frac{1}{2M+1}))} \\ &+ \frac{\beta}{2} \frac{\sin(\pi(f(2M+1)+1))}{\sin(\pi(f+\frac{1}{2M+1}))} \end{aligned}$$

(b)

Rectangular $W_R(f)$ is zero when $f = k/(2M+1)$ for all integers k except for $k = 0$. This, gives the main lobe width $1/(2M+1) - (-1/(2M+1)) = 2/(2M+1)$. The variance ratio is

$$R = \frac{1}{N} \sum_{m=-M}^{M} w^2(m) = \frac{2M+1}{N}$$

Bartlett The main lobe width is $2/M$ since the zero crossings are at $f = k/M$.

$$\begin{aligned} R &= \frac{1}{N} \left(1 + 2 (\sum_{m=1}^{M} 1 - 2\frac{m}{M} + \frac{m^2}{M^2}) \right) \\ &= \frac{1}{N} \left(1 + 2M - 4\frac{M(M+1)}{2M} + 2\frac{M(M+1)(2M+1)}{6M^2} \right) \\ &= \frac{1}{N} \left(\frac{(M+1)(2M+1)}{3M} - 1 \right) \approx \frac{2M}{3N} \end{aligned}$$

Raised cosine $W_H(f)$ is zero for $f = k/(2M+1)$ except for $k = 0$ where the first term is non-zero, $k = 1$, where the second term is non-zero and $k = -1$, where the third term is non-zero. Thus, the main lobe width is $4/(2M+1)$.

$$\begin{aligned} R &= \frac{1}{N} \sum_{m=-M}^{M} \alpha^2 + 2\alpha\beta \cos\left(\frac{2\pi m}{2M+1}\right) + \frac{\beta^2}{2} \left(\cos\left(2\frac{2\pi m}{2M+1}\right) + 1 \right) \\ &= \frac{(2M+1)(\alpha^2 + \beta^2/2)}{N} \end{aligned}$$

since the sums are over one and two full periods of the cos function, respectively.

See Exercise.

[6.7] See P& M (12.2.21): $\mathrm{var}[P(f)] \approx \frac{1}{K}\Gamma^2(f)$ i.e. $\mathrm{var}[P]_K \sim \frac{1}{K}$ and $\mathrm{var}[P]_{K+1} \sim \frac{1}{K+1}$

$$F(K) = \frac{\Delta \mathrm{var}[P]_K}{\mathrm{var}[P]_{K+1}} = \frac{\mathrm{var}[P]_K - \mathrm{var}[P]_{K+1}}{\mathrm{var}[P]_{K+1}} = \frac{\frac{1}{K} - \frac{1}{K+1}}{\frac{1}{K+1}} = \frac{1}{K}$$

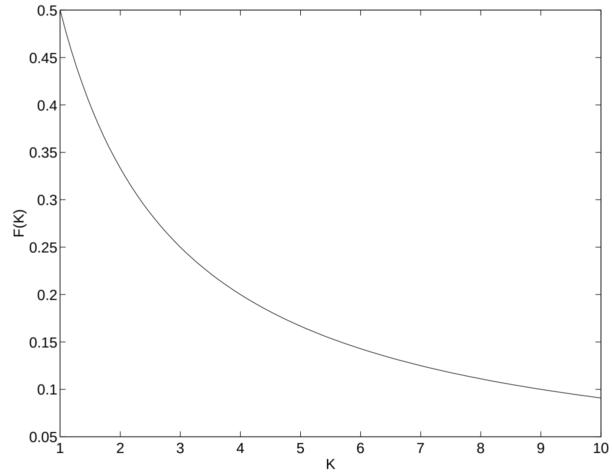


Figure 11.4: The requested curve in problem [5.11.7].

$\lim_{K\to\infty} F(K) = 0$

See Exercise.

[6.8] This exercise is more challenging than it appears. We have to use our theoretical knowledge in this applied setting to estimate the given quantities. Consider first that the model on the sampled signal is $x[n] = s[n] + e[n]$ where $s[n] = A\sin(\omega_0 n)$, where $\omega_0 = 2\pi F_0/F_s$, and $e[n]$ is the noise.

From the plot, it can be concluded that the noise is white, which was not stated in the headlines. We can deduce this because we know that the spectrum of $s[n]$ has a single peak in the range $\omega \in [0, \pi]$, and thus, the remaining spectrum in Figure 6.3, must correspond to the noise. A flat power spectral density implies a delta correlation function, and thus, the noise is white.

(a) The peak in the periodogram, as we mentioned, comes from the signal term $s[n]$. In particular, we have a peak at $\omega = \omega_0$, which in Figure 6.3 can recognise as the peak is at $\omega_0 = 0.377\,\text{rad/sample}$, which gives an estimate $\hat{F}_0 = \frac{\omega_0}{2\pi} F_s = 60\,\text{Hz}$.

(b) At the location of the peak, i.e. $\omega = \omega_0$, all the energy of $s[n]$ is concentrated, so the contribution from $e[n]$ can be neglected. To see this, recall that when observing the logarithmic transformation of a large number, small variations are compressed, so they can be disregarded. Therefore,

$$
\begin{aligned}
P(\omega_0) \approx P_s(\omega_0) &= \frac{1}{N}\left|\sum_{n=0}^{N-1} A\sin(\omega_0 n)\, e^{-j\omega_0 n}\right|^2 \\
&= \frac{A^2}{N}\left|\frac{1}{2j}\sum_{n=0}^{N-1}\left(e^{j\omega_0 n} - e^{-j\omega_0 n}\right) e^{-j\omega_0 n}\right|^2 \\
&= \frac{A^2}{4N}\left|\sum_{n=0}^{N-1}\left(1 - e^{-j2\omega_0 n}\right)\right|^2 = \frac{A^2}{4N}\left|N - \sum_{n=0}^{N-1} e^{-j2\omega_0 n}\right|^2 \\
&= \frac{A^2}{4N}\left|N - \frac{e^{-j2\omega_0 N} - 1}{e^{-j2\omega_0} - 1}\right|^2 \approx \frac{A^2 N}{4}\,.
\end{aligned}
$$

Here, we used the definition of the periodogram of a signa, Euler's identity for $\sin(x)$, the geometric sum formula, and that, because $\omega_0 = 0.377\,\text{rad/sample}$ and $N = 10000$,

$$
\left|\frac{e^{-j2\omega_0 N} - 1}{e^{-j2\omega_0} - 1}\right| \approx |0.23 + 0.07j| \approx 0.24 << N\,.
$$

Then, we have that $10^{3.7} = P(\omega_0) = A^2 N/4$, which yields

$$
\hat{A} = \sqrt{\frac{4 \cdot 10^{3.7}}{N}} = 1.4159\,.
$$

Note, as a curiosity, that the true value used for generating these data was $A = \sqrt{2} = 1.4142$.

(c) In order to know the noise power, we need to somehow link it to the remaining quantity provided by Figure 6.3. We are given the average value of the periodogram, i.e.

$$
\frac{1}{N}\sum_{k=0}^{N-1} P(\omega_k)\,.
$$

To relate this average to our unknown variable σ^2, we can consider the contribution of the noise $e[n]$ to the periodogram, i.e.,

$$
P_e(\omega) = \frac{1}{N}\left|\sum_{n=0}^{N-1} e[n] e^{-j\omega n}\right|^2\,.
$$

However, this quantity is random, so there is no direct way to compare it to our measurement. However, considering its expected value $\mathrm{E}[P_e(\omega)]$ instead will provide the link between our model and the measurement, and yield an estimate for σ^2. Indeed,

$$\begin{aligned}
\mathrm{E}[P_e(\omega)] &= \frac{1}{N}\,\mathrm{E}\left[\left|\sum_{n=0}^{N-1} e[n]e^{-j\omega n}\right|^2\right] \\
&= \frac{1}{N}\,\mathrm{E}\left[\left(\sum_{n=0}^{N-1} e[n]e^{-j\omega n}\right)\left(\sum_{m=0}^{N-1} e[m]e^{-j\omega m}\right)^*\right] \\
&= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} \mathrm{E}\left\{e[n]e[m]\right\} e^{-j\omega(n-m)} = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} \sigma^2\delta[n-m]e^{-j\omega(n-m)} \\
&= \frac{1}{N}\sum_{n=0}^{N-1}\sigma^2 = \sigma^2\,.
\end{aligned}$$

Here, we expand the squared absolute value as the product of a complex by its conjugate, we use the linearity of the expectation so that only random quantities remain inside it, and then, we use the fact that $e[n]$ is a white noise process to finalize the result.

Because our sample size is large, the division of the space $\omega \in [0, 2\pi]$ is very fine, and thus, we can assume that there are $k_1, k_2 \in \{0, 1, \ldots, N-1\}$ such that $\omega_{k_1} = \omega_0$ and $\omega_{k_2} = 2\pi - \omega_0$. Then, we have that

$$\begin{aligned}
\frac{1}{N}\sum_{k=0}^{N-1} \mathrm{E}\left[P(\omega_k)\right] &= \frac{1}{N}\left(\sum_{k=0}^{N-1} \mathrm{E}\left[P_e(\omega_k)\right] + \sum_{q=0}^{N-1} P_s(\omega_q)\right) \\
&= \frac{1}{N}\left(\sum_{k=0}^{N-1} \sigma^2 + 2\frac{A^2N}{4}\right) = \sigma^2 + \frac{A^2}{2}\,.
\end{aligned}$$

Here, we have first used the independence between the signal and the noise, then used the previous result for $P_s(\omega_0)$ and $P_s(2\pi - \omega_0)$, as well as the knowledge that $P_s(\omega) = 0$ for all $\omega \neq \omega_0, 2\pi - \omega_0$ in $[0, 2\pi]$, and finally incorporated the result for the expected value of $P_e(\omega)$.

This final expression, then, yields the estimate $\hat{\sigma^2} = 10^{0.3} - \frac{A^2}{2} \approx 1$.

See Exercise.

[7.1] The question is vaguely formulated and can be interpreted in many different ways. However, here we choose to find the FIR filter $g(n)$ that minimizes the average power of the error $\epsilon(n)$ in Figure 11.5 when the input signal $w(n)$ is white noise with variance one, (in other words, all frequencies are considered equally important when we calculate the error).

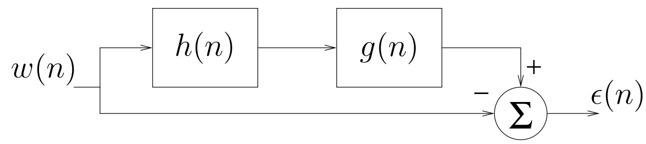


Figure 11.5:

We have $\epsilon(n) = \underbrace{(h(n) * g(n) - 1)}_{f(n)} * w(n)$. Writing $f(n)$ in vector form gives

$$\underbrace{\begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(2) \end{bmatrix}}_{\mathbf{f}} = \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} g(0) \\ g(1) \\ g(2) \end{bmatrix}}_{\mathbf{g}} - \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{b}}$$

The average error power is

$$\begin{aligned} e = r_{\epsilon\epsilon}(0) &= \sum_n |f(n)|^2 = \mathbf{f}^T\mathbf{f} = (\mathbf{A}\mathbf{g} - \mathbf{b})^T(\mathbf{A}\mathbf{g} - \mathbf{b}) \\ &= \mathbf{g}^T\mathbf{A}^T\mathbf{A}\mathbf{g} - \mathbf{b}^T\mathbf{A}\mathbf{g} - \mathbf{g}^T\mathbf{A}^T\mathbf{b} + \mathbf{b}^T\mathbf{b} \\ &= \left(\mathbf{g} - (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}\right)^T \mathbf{A}^T\mathbf{A} \left(\mathbf{g} - (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}\right) + \mathbf{b}^T\mathbf{b} - \mathbf{b}^T\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b} \\ &\geq \mathbf{b}^T\mathbf{b} - \mathbf{b}^T\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b} \end{aligned}$$

with equality iff

$$\begin{bmatrix} g(0) \\ g(1) \\ g(2) \end{bmatrix} = \mathbf{g}_{\min} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b} = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 5 & 2 \\ 0 & 2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{85} \begin{bmatrix} 42 \\ -20 \\ 8 \end{bmatrix}$$

and the resulting minimum squared error is

$$e_{\min} = \mathbf{b}^T\mathbf{b} - \mathbf{b}^T\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b} = \mathbf{b}^T\mathbf{b} - \mathbf{b}^T\mathbf{A}\mathbf{g}_{\min} = \frac{1}{85}$$

See Exercise.

[7.2] (a)

$$a_1 = \frac{r_x(1)}{r_x(0)} = \frac{\frac{a}{1-a^2}}{\frac{1}{1-a^2}} = a$$

(b)

$$a_1 = a, \quad a_2 = 0, \quad e_{min} = 1$$

See Exercise.

[7.3] Because the filter in Fig. 7.1 is an MA(2) system, a way to approach this problem is to model the given signal as an AR(2) process and then choose the parameters of the MA(2) system to cancel this model. In a general case, this would not yield a perfect cancellation, but only the best linear cancellation possible. However, in this case, the signal $s[n]$ is completely cancelled because it is an AR(2) process.

Therefore, let's fit an AR(2) model to the proposed signal. The autocorrelation function is

$$r_{ss}[k] = \mathrm{E}[s[n]s[n-k]] = \frac{A^2}{2}\,\mathrm{E}[\cos(2\pi f(2n+k)+2\phi)+\cos(2\pi fk)] = \frac{A^2}{2}\cos(2\pi k)\,.$$

Thus, the Yule-Walker equations give

$$\begin{bmatrix} r_{ss}(0) & r_{ss}(1) \\ r_{ss}(1) & r_{ss}(0) \end{bmatrix} \begin{bmatrix} a \\ ab \end{bmatrix} = \begin{bmatrix} -r_{ss}(1) \\ -r_{ss}(2) \end{bmatrix},$$

i.e.,

$$\begin{aligned} \begin{bmatrix} a \\ b \end{bmatrix} &= \left(\frac{A^2}{2}\begin{bmatrix} 1 & \cos(2\pi f) \\ \cos(2\pi f) & 1 \end{bmatrix}\right)^{-1} \frac{A^2}{2}\begin{bmatrix} -\cos(2\pi f) \\ -\cos(4\pi f) \end{bmatrix} \\ &= \frac{1}{1-\cos^2(2\pi f)}\begin{bmatrix} 1 & -\cos(2\pi f) \\ -\cos(2\pi f) & 1 \end{bmatrix}\begin{bmatrix} -\cos(2\pi f) \\ 1-2\cos^2(2\pi f) \end{bmatrix} = \begin{bmatrix} -2\cos(2\pi f) \\ 1 \end{bmatrix}. \end{aligned}$$

The frequency can be found through $f = \frac{1}{2\pi}\arccos(-\frac{a}{2})$.

See Exercise.

[7.4] We want to find $\hat{x}(nT+T/2) = a_1 x(nT) + a_2 x(nT+T)$,
such that $\epsilon = \mathrm{E}\{e^2(nT+T/2)\}$ is minimized.

$$\begin{aligned} \mathrm{E}\{e^2\} &= \mathrm{E}\{(x(nT+T/2)-\hat{x}(nT+T/2))^2\} = \ldots = \\ &= r_x(0)+a_1^2 r_x(0)+a_2^2 r_x(0)+2a_1a_2r_x(T)-2a_1r_x(T/2)-2a_2r_x(T/2) \end{aligned}$$

Differentiate and set to zero, $\frac{\partial\epsilon}{\partial a_i} = 0, \quad \forall i \quad \Rightarrow$

$$\begin{cases} \frac{\partial\epsilon}{\partial a_1} = 2a_1r_x(0)+2a_2r_x(T)-2r_x(T/2) = 0 \\ \frac{\partial\epsilon}{\partial a_2} = 2a_2r_x(0)+2a_1r_x(T)-2r_x(T/2) = 0 \end{cases}$$

Solving for a_1, a_2 gives

$$a_1 = a_2 = \frac{r_x(T/2)}{r_x(0)+r_x(T)}$$

and

$$\hat{x}(nT+\frac{T}{2}) = \frac{r_x(T/2)}{r_x(0)+r_x(T)}\left(x(nT)+x(nT+T)\right).$$

This result should be compared to linear interpolation, where $a_1 = a_2 = 1/2$. See Exercise.

[7.6] Yule-Walker equations. See Exercise.

[7.7] (a)

$$H_1(z) = 1 + (k_1 + k_1k_2 + k_2k_3)z^{-1} + (k_2 + k_1k_3 + k_1k_2k_3)z^{-2} + k_3z^{-3}$$

$$H_2(z) = k_3 + (k_2 + k_1k_3 + k_1k_2k_3)z^{-1} + (k_1 + k_1k_2 + k_2k_3)z^{-2} + z^{-3}$$

$H_2(z)$ is the reverse polynomial of $H_1(z)$, i.e., $H_2(z) = z^{-3}H_1(z^{-1})$, and vice versa.

(b)

$$a_1 = k_1 + k_1k_2 + k_2k_3 \quad a_2 = k_2 + k_1k_3 + k_1k_2k_3 \quad a_3 = k_3$$

(c) All the poles are in the origin for a transversal filter!

See Exercise.

[7.8] A general derivation of the fact $E^{(n)} = E^{(n-1)}(1 - |K_n|^2)$ can be found in the derivation of the Levinson-Durbin algorithm in the text book. However, for a first-order filter as in Figure 11.6, it is easy to do a brute force derivation. Assume

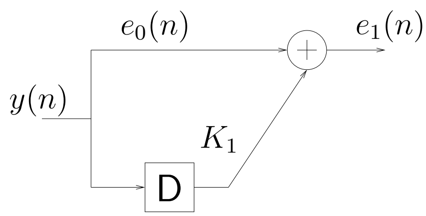


Figure 11.6: First-order lattice filter

that the input signal $y(n)$ has the autocorrelation function $r_{yy}(k)$. We see directly that $E^{(0)} = r_{yy}(0)$, the power of $y(n)$. From Yule-Walker (or the Levinson-Durbin recursion), $K_1 = a_1(1) = -r_{yy}(1)/r_{yy}(0)$, which gives $e_1(n) = \underbrace{\{1,\ K_1\}}_{h_1(n)} * y(n)$ with power

$$\begin{aligned} E^{(1)} = r_{e_1e_1}(0) &= \sum_k \sum_l h_1(l)h_1(m)r_{yy}(0 - l + m) \\ &= r_{yy}(0) - \frac{r_{yy}(1)}{r_{yy}(0)}r_{yy}(1) - \frac{r_{yy}(1)}{r_{yy}(0)}r_{yy}(1) + \left(\frac{r_{yy}(1)}{r_{yy}(0)}\right)^2 r_{yy}(0) \\ &= r_{yy}(0)\left(1 - \left(\frac{r_{yy}(1)}{r_{yy}(0)}\right)^2\right) = E^{(0)}(1 - K_1^2) \end{aligned}$$

See Exercise.

[7.9]

$$a_1 = -0.4131 \quad a_2 = -0.0069 \quad a_3 = -0.0035$$

See Exercise.

[7.10]

$$k_1 = \frac{3}{2} \quad k_2 = \frac{1}{3}$$

See Exercise.

[7.11] (a)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{k_1 + k_2(1+k_1)z^{-1} + z^{-2}}{1 + k_2(1+k_1)z^{-1} + k_1 z^{-2}}$$

(b) Shur-Cohns test gives $|k_1| < 1$ and $|k_2| < 1$.

See Exercise.

[7.12] (a)

$$k_2 = r^2 \quad k_1 = -\frac{2r\cos\omega_0}{1+r^2}$$

(b) The second reflection coefficient becomes 1 and the system becomes an oscillator.

See Exercise.

[7.13] (a) Using (7.1) results in $\hat{r}_y[0] = 3$, $\hat{r}_y[1] = 2$, $\hat{r}_y[2] = 4 > \hat{r}_y[0]$, which is not reasonable for an autocorrelation function.

The Yule-Walker equations, in this context, give:

$$\begin{bmatrix} \hat{r}_y[0] & \hat{r}_y[1] \\ \hat{r}_y[1] & \hat{r}_y[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -\hat{r}_y[1] \\ -\hat{r}_y[2] \end{bmatrix} \Leftrightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2/5 \\ -8/5 \end{bmatrix}.$$

The poles of the resulting filter $\frac{1}{A(z)}$ are in $z_1 = -1/5 - \sqrt{1/25 + 8/5} \approx -1.48$ and $z_2 = -1/5 + \sqrt{1/25 + 8/5} = 1.08$, which makes the filter unstable (because it has to be causal). This results from a severe model mismatch, because an AR(2) process is expected to decay, while our samples did not. Note that an unstable filter in an AR model would signify a signal that can grow indefinitely. Finally, the best evidence of model mismatch is that the first YW equation gives us that

$$b_0^2 = \sum_{k=1}^{3} a_k \hat{r}_y[k] = -2.6\,.$$

(b) Using (7.2) results in $\hat{r}_y[0] = 3$, $\hat{r}_y[1] = 4/3$, $\hat{r}_y[2] = 4/3$, which is a proper estimator of the autocorrelation function, because most of the energy is concentrated in $\hat{r}_y[0]]$. The Yule-Walker equations give, in this context, give:

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -4/13 \\ -4/13 \end{bmatrix}.$$

The poles of the resulting filter $\frac{1}{A(z)}$ are in $z_1 = 2/13 + \sqrt{4/169 + 4/13} = 0.73$ and $z_2 = 2/13 - \sqrt{4/169 + 4/13} = -0.42$, which makes the filter stable (because it has to be causal). Finally, the parameter b_0^2 yields a consistent value too,

$$b_0^2 = \sum_{k=1}^{3} a_k \hat{r}_y[k] = 2.18\,.$$

This problem exemplifies why (7.2) is much more frequently used than (7.1), at least for low sample sizes. (7.2) has improved statistical performance because it weights down those values for which the biased estimator has a larger variance. This tecnique is known in statistics as *shrinking* an unbiased estimator to decrease its MSE.

See Exercise.

[7.14] a) The Yule-Walker equations give:

$$\begin{bmatrix} r_y(0) & r_y(1) \\ r_y(1) & r_y(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -r_y(1) \\ -r_y(2) \end{bmatrix}$$

$r_y(0) = E\{y^2(n)\} = E\{(b_0 e(n) + b_1 e(n-1))^2\} = b_0^2 + b_1^2 = 1.09$
$r_y(1) = E\{y(n)y(n-1)\} = E\{(b_0 e(n) + b_1 e(n-1))(b_0 e(n-1) + b_1 e(n-2))\}$
$= b_1 b_0 = 0.3$
$r_y(2) = E\{y(n)y(n-2)\} = 0.$
The coefficients converge to

$$a_1 \approx -0.2978, \qquad a_2 \approx 0.0820.$$

b) The impulse response for the estimated model:

$$\hat{h}(0) = 1, \quad \hat{h}(1) = -a_1, \quad \hat{h}(2) = -a_2 + a_1^2.$$

The impulse response for the true model:

$$h(0) = 1, \quad h(1) = 0.3, \quad h(2) = 0.$$

See Exercise.

[7.15]

$$\begin{aligned}
1:\ & a_1^2 && = -0.3; a_2^2 = 0.02 \\
2:\ & m && = 2 \\
3:\ & \rho_2 && = a_2^2 = 0.02 \\
4:\ & a_1^1 && = (a_1^2 - \rho_2 a_1^2)/(1-\rho_2^2) \\
& && = ((-0.3) + 0.02 \cdot 0.3)/(1-(0.02)^2) \\
& && = -0.294 \\
5:\ & m && = 1 \\
3:\ & \rho_1 && = -0.294
\end{aligned}$$

See Exercise.

[7.16] a) $\hat{y}(n+d) = ay(n)$
Minimize $E[(y(n+d) - \hat{y}(n+d))^2] = E[y^2(n+d) - 2ay(n+d)y(n) + a^2y^2(n)]$
$= r_y(0) - 2ar_y(d) + a^2r_y(0)$. Minimized when $a = r_y(d)/r_y(0)$.

b) Minimize $V = E[(y(n+d) - \sum_{k=0}^{m} a_k y(n-k))^2] = E[(y(n+d) - \phi^T(n)\theta)^2]$, where

$$\phi^T(n) = [y(n)\ y(n-1)\ \ldots\ y(n-m)]$$
$$\theta^T = [a_0\ a_1\ \ldots\ a_m]$$

$V = E[y^2(n+d) - 2\theta^T\phi(n)y(n+d) + \theta^T\phi(n)\phi^T(n)\theta] = r_y(0) - 2\theta^T\mathbf{r} + \theta^T\mathbf{R}\theta$

where

$$\mathbf{r}^T = [r_y(d)\ \ldots\ r_y(m+d)]$$
$$\mathbf{R} = \begin{bmatrix} r_y(0) & \ldots & r_y(m) \\ \vdots & \ddots & \vdots \\ r_y(m) & \ldots & r_y(0) \end{bmatrix}$$

We have

$$V = r_y(0) + (\theta - \mathbf{R}^{-1}\mathbf{r})^T\mathbf{R}(\theta - \mathbf{R}^{-1}\mathbf{r}) - \mathbf{r}^T\mathbf{R}^{-1}\mathbf{r}$$

V is minimized when $\theta = \mathbf{R}^{-1}\mathbf{r}$, since $\mathbf{R}$ is positive definite

c) Let the noise variance be σ^2. Then $r_y(0) = \sigma^2 5/4$, $r_y(1) = \sigma^2/2$, $r_y(2) = 0$. We get for $d = 1$

$$\begin{bmatrix} 5 & 2 & 0 \\ 2 & 5 & 2 \\ 0 & 2 & 5 \end{bmatrix} \theta = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \theta = \begin{bmatrix} 0.49 \\ -0.24 \\ 0.09 \end{bmatrix}$$

For $d \geq 2$, $\theta = 0$.

See Exercise.

[7.17] Let $\hat{g} = \mathbf{h}^T\mathbf{y}$ where $\mathbf{h}^T = [h(0), \ldots, h(M-1)]$, $\mathbf{y}^T = [y(0), \ldots, y(M-1)]$ and correspondingly for $\mathbf{v}$ and $\mathbf{w}$.

$$\begin{aligned} E\{(g - \hat{g})^2\} &= E\{(g - \mathbf{h}^T\mathbf{y})^2\} = E\{g^2 - 2\mathbf{h}^T\mathbf{y}g + \mathbf{h}^T\mathbf{y}\mathbf{y}^T\mathbf{h}\} \\ &= \lambda + E\{-2\mathbf{h}^T(g\mathbf{v} + \mathbf{w})g + \mathbf{h}^T(g\mathbf{v} + \mathbf{w})(g\mathbf{v} + \mathbf{w})^T\mathbf{h}\} \\ &= \lambda - 2\lambda\mathbf{h}^T\mathbf{v} + \lambda\mathbf{h}^T\mathbf{v}\mathbf{v}^T\mathbf{h} + \mathbf{h}^T\mathbf{h} \end{aligned}$$

where we use the fact that g is independent of $\mathbf{w}$ and that $E\{\mathbf{w}\mathbf{w}^T\} = \mathbf{I}$. Note the similarities to the derivation of the Yule-Walker equations if we write

$$E\{(g - \hat{g})^2\} = \lambda - 2\mathbf{h}^T\mathbf{r} + \mathbf{h}^T\mathbf{R}\mathbf{h}$$

where $\mathbf{R} = \lambda\mathbf{v}\mathbf{v}^T + \mathbf{I}$ and $\mathbf{r} = \lambda\mathbf{v}$. Then, we know that the solution is given by the set of equations $\mathbf{R}\mathbf{h} = \mathbf{r}$. Therefore

$$\mathbf{h} = \mathbf{R}^{-1}\mathbf{r} = (\lambda\mathbf{v}\mathbf{v}^T + \mathbf{I})^{-1}\lambda\mathbf{v}$$

See Exercise.

[7.18] In matrix form

$$\bar{x} = \mathbf{X}a + \bar{v}$$

where $a = [a_1, \ldots, a_r]^T$.

a)

$$\begin{aligned} |\bar{v}|^2 &= \bar{v}^T\bar{v} = (\bar{x} - \mathbf{X}a)^T(\bar{x} - \mathbf{X}a) \\ &= a^T\mathbf{X}^T\mathbf{X}a - a^T\mathbf{X}^T\bar{x} - \bar{x}^T\mathbf{X}a + \bar{x}^T\bar{x} \\ &= a^T\mathbf{X}^T\mathbf{X}a - a^T\mathbf{X}^T\bar{x} - \bar{x}^T\mathbf{X}a + \bar{x}^T\mathbf{X}\mathbf{X}^\dagger\bar{x} + \bar{x}^T\bar{x} - \bar{x}^T\mathbf{X}\mathbf{X}^\dagger\bar{x} \\ &= (a - \mathbf{X}^\dagger\bar{x})^T\mathbf{X}^T\mathbf{X}(a - \mathbf{X}^\dagger\bar{x}) + \bar{x}^T(\mathbf{I} - P)\bar{x} \end{aligned}$$

The estimate that minimizes $|\bar{v}|^2$ is given by $\hat{a} = \mathbf{X}^\dagger\bar{x}$, since $\mathbf{X}^T\mathbf{X}$ is positive definite and $\bar{x}^T(\mathbf{I} - P)\bar{x}$ is independent of a.

b) Substitute the result from a) in the least squares criterion,

$$\begin{aligned} |\bar{v}|^2 &= \bar{v}^T\bar{v} = (\bar{x} - \mathbf{X}a)^T(\bar{x} - \mathbf{X}a) \\ &= (\bar{x} - \mathbf{X}\mathbf{X}^\dagger\bar{x})^T(\bar{x} - \mathbf{X}\mathbf{X}^\dagger\bar{x}) = \bar{x}^T\bar{x} - \bar{x}^TP\bar{x} \end{aligned}$$

which is the same as maximizing

$$\bar{x}^TP\bar{x}$$

c)

$$e^{-1}x(1) < x(0) < e\, x(1)$$

See Exercise.

[7.19] a)

$$\begin{aligned} H(z) &= \sum_{n=0}^{\infty} \mathrm{e}^{-an}\cos\omega n z^{-n} \qquad \{\text{euler}\} \\ &= \frac{1}{2}\left(\frac{1}{1-\mathrm{e}^{-a+i\omega}z^{-1}} + \frac{1}{1-\mathrm{e}^{-a-i\omega}z^{-1}}\right) \\ &= \frac{1-\mathrm{e}^{-a}\cos\omega z^{-1}}{1-2\mathrm{e}^{-a}\cos\omega z^{-1} + \mathrm{e}^{-2a}z^{-2}} \end{aligned}$$

i.e. $\quad y(n) - 2\mathrm{e}^{-a}\cos\omega y(n-1) + \mathrm{e}^{-2a}y(n-2) = e(n) - \mathrm{e}^{-a}\cos\omega e(n-1).$

b)

$$\hat{y}(n) = c_1y(n-1) + c_2y(n-2)$$

The Yule-Walker equations give $\begin{bmatrix} r_y(0) & r_y(1) \\ r_y(1) & r_y(0) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} r_y(1) \\ r_y(2) \end{bmatrix}$

$r_y(k)$ is taken from (a). Let $\alpha = \mathrm{e}^{-a}$, $\beta = \cos\omega$, then:

$$\begin{cases} r_y(0) - 2\alpha\beta r_y(1) + \alpha^2 r_y(2) = r_{ey}(0) - \alpha\beta r_{ey}(-1) \\ r_y(1) - 2\alpha\beta r_y(0) + \alpha^2 r_y(1) = -\alpha\beta r_{ey}(0) \\ r_y(2) - 2\alpha\beta r_y(1) + \alpha^2 r_y(0) = 0 \\ r_{ey}(0) = \sigma^2 \\ r_{ey}(-1) - 2\alpha\beta r_{ey}(0) = -\alpha\beta\sigma^2 \end{cases}.$$

This gives: $$\begin{cases} r_y(0) - 2\alpha\beta r_y(1) + \alpha^2 r_y(2) & = (1-\alpha^2\beta^2)\sigma^2 \\ 2\alpha\beta r_y(0) - (1+\alpha^2) r_y(1) & = -\alpha\beta\sigma^2 \\ \alpha^2 r_y(0) - 2\alpha\beta r_y(1) + r_y(2) & = 0 \end{cases}$$

The least squares error: $\mathrm{E}\{\varepsilon^2\} = r_y(0) - [c_1 \; c_2]^T [r_y(1) \; r_y(2)]$

See Exercise.

[7.20] The least squares method gives the following system of equations

$$\frac{1}{N+2}\sum_{n=1}^{N+1} \varphi(n)\varphi^T(n)\theta = \frac{1}{N+2}\sum_{n=1}^{N+1}\varphi(n)x(n)$$

Look at row i of the system of equations

$$\frac{1}{N+2}\sum_{n=1}^{N+1} x(n-i)[x(n-1),\ldots,x(n-N)]\theta = \frac{1}{N+2}\sum_{n=1}^{N+1} -x(n-i)x(n)$$

Consider

$$\frac{1}{N+2}\sum_{n=1}^{N+1} x(n-i)x(n-k) = ... = \frac{1}{N+2}\sum_{m=1}^{N-|i-k|+1} x(m)x(m+|i-k|) = \hat{r}(i-k)$$

since $x(n) = 0$, $n < 0$ and $n > N+1$. This gives

$$[\hat{r}(i-1), \hat{r}(i-2), \ldots, \hat{r}(i-N)]\theta = -\hat{r}(i)$$

which is equal to one row from the Yule-Walker equations.

See Exercise.

[7.21] In matrix notation

$$\bar{x} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} = \mathbf{\Omega a} + \bar{w}$$

where $\bar{w}$ is defined similarly to $\bar{x}$ and

$$\mathbf{\Omega} = \begin{bmatrix} 1 & \cdots & 1 \\ e^{j\omega_1} & \cdots & e^{j\omega_d} \\ \vdots & \ddots & \vdots \\ e^{j\omega_1(N-1)} & \cdots & e^{j\omega_d(N-1)} \end{bmatrix} \qquad \mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_d \end{bmatrix}$$

a) Minimize $|\bar{w}|^2$.

$$\begin{aligned} |\bar{w}|^2 &= \bar{w}^*\bar{w} = (\bar{x} - \mathbf{\Omega a})^*(\bar{x} - \mathbf{\Omega a}) \\ &= \mathbf{a}^*\mathbf{\Omega}^*\mathbf{\Omega a} - \mathbf{a}^*\mathbf{\Omega}^*\bar{x} - \bar{x}^*\mathbf{\Omega a} + \bar{x}^*\bar{x} \\ &= \mathbf{a}^*\mathbf{\Omega}^*\mathbf{\Omega a} - \mathbf{a}^*\mathbf{\Omega}^*\bar{x} - \bar{x}^*\mathbf{\Omega a} + \bar{x}^*\mathbf{\Omega\Omega}^\dagger\bar{x} + \bar{x}^*\bar{x} - \bar{x}^*\mathbf{\Omega\Omega}^\dagger\bar{x} \\ &= (\mathbf{a} - \mathbf{\Omega}^\dagger\bar{x})^*\mathbf{\Omega}^*\mathbf{\Omega}(\mathbf{a} - \mathbf{\Omega}^\dagger\bar{x}) + \bar{x}^*P\bar{x} \end{aligned}$$

where $P = \mathbf{I} - \mathbf{\Omega\Omega}^\dagger$. The estimate that minimizes $|\bar{w}|^2$ is given by $\hat{\mathbf{a}} = \mathbf{\Omega}^\dagger\bar{x}$ since $\mathbf{y}^*\mathbf{\Omega}^*\mathbf{\Omega y} = |\mathbf{\Omega y}|^2$ is positive and $\bar{x}^*P\bar{x}$ is independent of $\mathbf{a}$.

b)

$$\hat{\mathbf{a}} = \mathbf{\Omega}^\dagger\bar{x} = \mathbf{\Omega}^\dagger(\mathbf{\Omega a} + \bar{w}) = \mathbf{a} + \mathbf{\Omega}^\dagger\bar{w} \to \mathbf{a}$$

since $\mathbf{\Omega}^*\mathbf{\Omega} \to N\mathbf{I}$ and the noise is uncorrelated with $\mathbf{a}$. The estimate is unbiased since

$$\mathrm{E}\{\hat{\mathbf{a}}\} = \mathbf{a} + \mathbf{\Omega}^\dagger\,\mathrm{E}\{\bar{w}\} = \mathbf{a}$$

c)

$$\mathrm{E}\{(\hat{\mathbf{a}} - \mathbf{a})(\hat{\mathbf{a}} - \mathbf{a})^*\} = \mathbf{\Omega}^\dagger\,\mathrm{E}\{\bar{w}\bar{w}^*\}\mathbf{\Omega}^{\dagger *} = \sigma_w^2(\mathbf{\Omega}^*\mathbf{\Omega})^{-1}$$

The more the noise, the higher the variance. The variance goes to zero when $N \to \infty$ and the estimate errors are uncorrelated. If $\mathbf{\Omega}$ looses rank (the frequencies are too close), the variance will increase.

See Exercise.

[8.1] a)

$$e(n) = y(n) - \sum_{k=0}^{N-1} w_k(n)e^{j2\pi\frac{nk}{N}}$$

$$|e(n)|^2 = \left(y(n) - \sum_{k=0}^{N-1} w_k(n)e^{j2\pi\frac{nk}{N}}\right)\left(y(n) - \sum_{k=0}^{N-1} w_k(n)e^{j2\pi\frac{nk}{N}}\right)^*$$

$|e(n)|^2$ is minimized when $y(n) = \sum_{k=0}^{N-1} w_k e^{j2\pi\frac{nk}{N}}$. Compare with DFT, IDFT.

$$w_k = \frac{Y(k)}{N} = \frac{1}{N}\sum_{n=0}^{N-1} y(n)e^{-j2\pi\frac{nk}{N}}.$$

b) Split w_k into the real part, ($\mathrm{Re}\{w_k\} = \bar{w}_k$) and the imaginary part, ($\mathrm{Im}\{w_k\} = \tilde{w}_k$).

$$\bar{w}_k(n+1) = \bar{w}_k(n) - \mu\frac{\partial|e(n)|^2}{\partial\bar{w}_k(n)}, \qquad \tilde{w}_k(n+1) = \tilde{w}_k(n) - \mu\frac{\partial|e(n)|^2}{\partial\tilde{w}_k(n)}.$$

This can be combined into

$$\bar{w}_k(n+1) + j\tilde{w}_k(n+1) = \bar{w}_k(n) + j\tilde{w}_k(n) - \mu\left(\frac{\partial|e(n)|^2}{\partial\bar{w}_k(n)} + j\frac{\partial|e(n)|^2}{\partial\tilde{w}_k(n)}\right)$$

Define
$$\frac{\partial f(w_k(n))}{\partial w_k(n)} = \frac{\partial f(w_k(n))}{\partial \bar{w}_k(n)} + j\frac{\partial f(w_k(n))}{\partial \tilde{w}_k(n)}.$$

Note that
$$\frac{\partial w_k(n)}{\partial w_k(n)} = \frac{\partial w_k(n)}{\partial \bar{w}_k(n)} + j\frac{\partial w_k(n)}{\partial \tilde{w}_k(n)} = 1 + j^2 = 0$$

and
$$\frac{\partial w_k^*(n)}{\partial w_k(n)} = 1 - j^2 = 2.$$

This gives
$$\frac{\partial |e(n)|^2}{\partial w_k(n)} = e(n)\frac{\partial}{\partial w_k(n)}\left(y^*(n) - \sum_{l=0}^{N-1} w_l^*(n)e^{-j2\pi\frac{nl}{N}}\right) = -2e(n)e^{-j2\pi\frac{nk}{N}}.$$

Update in the negative gradient direction with the step size μ:
$$w_k(n+1) = w_k(n) + 2\mu e(n)e^{-j2\pi\frac{nk}{N}}.$$

See Exercise.

[8.2]
$$\theta(n+1) = w\theta(n) + \mu\phi(n)e(n) = w\theta(n) + \mu u(n)[u(n+1) - u(n)\theta(n)]$$

Consider the average behavior, $E\{\theta(n)\} = \bar{\theta}(n)$,
$$\bar{\theta}(n+1) = w\bar{\theta}(n) + \mu r_1 - \mu r_0\bar{\theta}(n)$$

$r_1 = ar_0$ gives
$$\bar{\theta}(n+1) = (w - \mu r_0)^{n+1}\bar{\theta}(0) + \sum_{k=0}^{n}(w - \mu r_0)^k a r_0 \mu$$

If $|w - \mu r_0| < 1 \implies (w-1)/r_0 < \mu < (w+1)/r_0$, the filter converges to
$$\bar{\theta}(n+1) \longrightarrow \frac{ar_0\mu}{1 - w + \mu r_0}$$

See Exercise.

[8.5]
$$\begin{aligned} r_x(k) &= \sum_l \sum_m h(l)h(m)r_v(k-l+m) \qquad \{r_v(k) = \sigma_v^2\delta(k)\} \\ &= \sum_l \sum_m h(l)h(m)\delta(k-l+m) \end{aligned}$$

$$H(z) = \frac{1}{1 - az^{-1}} \Rightarrow h(n) = a^n u(n)$$

$$r_x(0) = \sigma_v^2 \sum_m a^{2n} = \sigma_v^2 \frac{1}{1-a^2}$$

$$r_x(1) = \{l = m+1\} = \sigma_v^2 \sum_m a^{m+1} a^m = \sigma_v^2 \frac{a}{1-a^2}$$

$$y(n) = x(n)b_n + x(n-1)c_n$$

$$e(n) = y(n) - v(n) = x(n)b_n + x(n-1)c_n - v(n)$$

$$\begin{aligned} e^2(n) = {} & x^2(n)b_n^2 + x^2(n-1)c_n^2 + v^2(n) \\ & + 2x(n)x(n-1)b_n c_n - 2c_n x(n-1)v(n) - 2b_n v(n)x(n) \end{aligned}$$

$$J = E\{e^2(n)\} = r_x(0)b_n^2 + r_x(0)c_n^2 + r_v(0) + 2r_x(1)b_n c_n - 0 - 2b_n r_v(0)$$

$$\frac{dJ}{db_n} = 2r_x(0)b_n + 2r_x(1)c_n - 2r_v(0)$$

$$\frac{dJ}{dc_n} = 2r_x(0)c_n + 2r_x(1)b_n$$

$$\bigtriangledown J = \begin{bmatrix} 2\sigma_v^2 \frac{1}{1-a^2} b_n + 2\sigma_v^2 \frac{a}{1-a^2} c_n - 2\sigma_v^2 \\ 2\sigma_v^2 \frac{1}{1-a^2} c_n + 2\sigma_v^2 \frac{a}{1-a^2} b_n \end{bmatrix}$$

b) We do not know a

c)

$$0 < \mu < \frac{2}{\lambda_{max}}$$

$$\lambda_{max} < M r_x(0) = M\sigma_v^2 \frac{1}{1-a^2}$$

$$|a| < 0.5$$

$$2\sigma_v^2 \frac{1}{1-\frac{1}{4}} = \sigma_v^2 \frac{2 \cdot 4}{3} = \sigma_v^2 \frac{8}{3}$$

$$\mu_{max} = \frac{2}{\frac{8}{3}\sigma_v^2} = \frac{3}{4\sigma_v^2}$$

d) $b_n \to 1$ and $c_n \to -a$, can be shown, for example, from $\bigtriangledown J = 0$.

See Exercise.

[9.1] a)

$$\mathbf{X} = \begin{bmatrix} 1 & 1 \\ e^{j\omega_1} & e^{j\omega_2} \\ e^{j2\omega_1} & e^{j2\omega_2} \\ \vdots & \vdots \\ e^{j(N-1)\omega_1} & e^{j(N-1)\omega_2} \end{bmatrix}$$

b)

$$
\begin{aligned}
(\bar{s} - \mathbf{X}^\dagger \bar{x})^* \mathbf{X}^* \mathbf{X} (\bar{s} - \mathbf{X}^\dagger \bar{x}) + \bar{x}^* P \bar{x} &= \bar{s}^* \mathbf{X}^* \mathbf{X} \bar{s} - \bar{s}^* \mathbf{X}^* \bar{x} - \bar{x}^* \mathbf{X} \bar{s} + \bar{x}^* \mathbf{X} \mathbf{X}^\dagger \bar{x} \\
&\quad + \bar{x}^* \bar{x} - \bar{x}^* \mathbf{X} \mathbf{X}^\dagger \bar{x} \\
&= \bar{s}^* \mathbf{X}^* \mathbf{X} \bar{s} - \bar{s}^* \mathbf{X}^* \bar{x} - \bar{x}^* \mathbf{X} \bar{s} + \bar{x}^* \bar{x} \\
&= (\bar{x} - \mathbf{X} \bar{s})^* (\bar{x} - \mathbf{X} \bar{s}) = \bar{e}^* \bar{e} = \|\bar{e}\|^2
\end{aligned}
$$

The estimate that minimizes $\|\bar{e}\|^2$ is given by $\hat{\bar{s}} = \mathbf{X}^\dagger \bar{x}$, since $\mathbf{X}^* \mathbf{X}$ is positive definite and $\bar{x}^* P \bar{x}$ is independent of $\bar{s}$.

c) $\hat{e}(n) = x(n) - \hat{x}(n)$, so

$$
\hat{\sigma}_e^2 = \frac{1}{N} \hat{\bar{e}}^* \hat{\bar{e}} = \min \frac{1}{N} \|\bar{e}\|^2 = \frac{\bar{x}^* P \bar{x}}{N}
$$

d)

$$
\begin{aligned}
\mathbf{X}^* \mathbf{X} &= \begin{bmatrix} N & \sum_{n=0}^{N-1} e^{jn(\omega_2 - \omega_1)} \\ \sum_{n=0}^{N-1} e^{jn(\omega_1 - \omega_2)} & N \end{bmatrix} \\
&= \begin{bmatrix} N & e^{j(\omega_2 - \omega_1)(N-1)/2} \frac{\sin((\omega_2 - \omega_1)N/2)}{\sin((\omega_2 - \omega_1)/2)} \\ e^{-j(\omega_2 - \omega_1)(N-1)/2} \frac{\sin((\omega_2 - \omega_1)N/2)}{\sin((\omega_2 - \omega_1)/2)} & N \end{bmatrix}
\end{aligned}
$$

$\omega_2 - \omega_1 = 2\pi k/N \Rightarrow \sin((\omega_2 - \omega_1)N/2) = 0$ and $\sin((\omega_2 - \omega_1)/2) \neq 0$. Thus

$$
(\mathbf{X}^* \mathbf{X})^{-1} = \frac{1}{N} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

We get

$$
\hat{\bar{s}} = \mathbf{X}^\dagger \bar{x} = \frac{1}{N} \mathbf{X}^* \bar{x} = \frac{1}{N} \begin{bmatrix} \sum_{n=0}^{N-1} x(n) e^{-jn\omega_1} \\ \sum_{n=0}^{N-1} x(n) e^{-jn\omega_2} \end{bmatrix}
$$

s_i and ϕ_i are obtained as the magnitude and phase for each expression, respectively. If $e(n) = 0$

$$
\hat{\bar{s}} = \mathbf{X}^\dagger \bar{x} = \mathbf{X}^\dagger \hat{x} = \mathbf{X}^\dagger \mathbf{X} \bar{s} = \bar{s}
$$

We get the exact answer, note that this does not require $\omega_2 - \omega_1 = 2\pi k/N$. If $\omega_2 = \omega_1$, this method does not work since $\mathbf{X}^* \mathbf{X}$ then is singular (not invertible). This means that we should not attempt to model one exponential signal as two.

See Exercise.

[9.2] (a) i. Assume there are d complex sinusoids.

ii. Estimate $r_{xx}(k)$, $k = 0, 1, \ldots, d$.

iii. Form R with the estimated values.

iv. Calculate the smallest eigenvalue and the corresponding eigenvector, $[1 \;\; a_1 \;\; \ldots \;\; a_d]^T$, of R.

v. The smallest eigenvalue of R is used as the estimate of σ^2.

vi. Calculate the roots, z_i, of $\sum_{k=0}^{d} z^k a_{d-k}$.

vii. The arguments of z_i are the estimates of f_i, $\hat{f}_i = \arg(z_i)/2\pi$.

(b)

$$\mathrm{E}\{x(n)\} = \mathrm{E}\{ae^{j(2\pi fn+\phi)}\} + \mathrm{E}\{v(n)\} = ae^{j2\pi f}\,\mathrm{E}\{e^{j\phi}\} = 0$$
$$r_{xx}(k) = \mathrm{E}\{x(n)x^*(n-k)\} = |a|^2 e^{j2\pi fk} + \sigma^2\delta(k)$$

(c) Let $|a|^2 = p$. The characteristic function is $(\sigma^2 - \lambda)(\sigma^2 + 2p - \lambda) = 0$. The smallest $= \sigma^2$! The corresponding eigenvector is given by

$$\begin{bmatrix} p & pe^{-j2\pi f} \\ pe^{j2\pi f} & p \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \end{bmatrix} = 0$$

$a_1 = -e^{j2\pi f}$. The roots of $z - e^{j2\pi f} = 0$ are $z_1 = e^{j2\pi f}$ and $\hat{f} = 2\pi f/2\pi = f$!

(d)

$$R = \begin{bmatrix} p+\sigma^2 & -p & p \\ -p & p+\sigma^2 & -p \\ p & -p & p+\sigma^2 \end{bmatrix}$$

The eigenvalues are $\lambda_{1,2} = \sigma^2, \quad \lambda_3 = 3p + \sigma^2$. There are two smallest eigenvalues since we have one sinusoid (R is rank one). Therefore there are many eigenvectors with eigenvalue σ^2. More precisely, all vectors in the two dimensional space given by $1 - a_1 + a_2 = 0$ are eigenvectors of R and have eigenvalue σ^2. Every polynomial formed from such a vector has at least one root that gives the correct frequency.

(e) Take two vectors, e.g. $[1 \;\; 1 \;\; 0]^T$ and $[1 \;\; 0 \;\; -1]^T$ that have eigenvalue σ^2. Generate two polynomials and determine their roots which, in this case, are ± 1 and $0, \; -1$. The true root will be common to both polynomials, i.e. $-1 = e^{j2\pi f} \Rightarrow f = .5$.

See Exercise.

[9.3] a) Use the Levinson-Durbin algorithm.

$$\begin{aligned}
&\mathbf{m = 0} \qquad e_0 = r_y(0) = \sigma^2 \\
&\mathbf{m = 1} \qquad a_1^{(1)} = k_1 = -\frac{r_y(1)}{r_y(0)} = -\cos(2\pi f_0) \\
&\qquad\qquad\; e_1 = (1 - k_1^2)e_0 = \sigma^2 \sin^2(2\pi f_0) \\
&\mathbf{m = 2} \qquad a_2^{(2)} = k_2 = -\frac{cos(4\pi f_0) - \cos^2(2\pi f_0)}{1 - \cos^2(2\pi f_0)} = 1 \\
&\qquad\qquad\; a_1^{(2)} = a_1^{(1)} + k_2 a_1^{(1)} = -2\cos(2\pi f_0) \\
&\qquad\qquad\; e_2 = (1 - k_2^2)e_1 = 0
\end{aligned}$$

Determine the poles, $z^2 - 2\cos(2\pi f_0)z + 1 = 0$.

$$z_{1,2} = \cos(2\pi f_0) \pm j\sin(2\pi f_0) = e^{\pm j2\pi f_0}$$

Frequencies $\pm f_0$.

b) Determine the smallest eigenvalue and the corresponding eigenvector.

$$\sigma^2 \det \begin{bmatrix} 1-\lambda & \cos(2\pi f_0) & \cos(4\pi f_0) \\ \cos(2\pi f_0) & 1-\lambda & \cos(2\pi f_0) \\ \cos(4\pi f_0) & \cos(2\pi f_0) & 1-\lambda \end{bmatrix} = 0$$

This gives $\lambda_{min} = 0$ and the corresponding eigenvector is

$$e_{min}^T = [1 \quad -2\cos(2\pi f_0) \quad 1]$$

Similar to the above, this gives

$$z_{1,2} = \cos(2\pi f_0) \pm j\sin(2\pi f_0) = e^{\pm j2\pi f_0}$$

Frequencies $\pm f_0$.

See Exercise.

[9.4] a)

$$\begin{aligned} |\bar{e}|^2 &= \bar{e}^*\bar{e} = (\bar{x} - \mathbf{X}a)^*(\bar{x} - \mathbf{X}a) \\ &= a^*\mathbf{X}^*\mathbf{X}a - a^*\mathbf{X}^*\bar{x} - \bar{x}^*\mathbf{X}a + \bar{x}^*\bar{x} \\ &= a^*\mathbf{X}^*\mathbf{X}a - a^*\mathbf{X}^*\bar{x} - \bar{x}^*\mathbf{X}a + \bar{x}^*\mathbf{X}\mathbf{X}^\dagger\bar{x} + \bar{x}^*\bar{x} - \bar{x}^*\mathbf{X}\mathbf{X}^\dagger\bar{x} \\ &= (a - \mathbf{X}^\dagger\bar{x})^*\mathbf{X}^*\mathbf{X}(a - \mathbf{X}^\dagger\bar{x}) + \bar{x}^*P\bar{x} \end{aligned}$$

The estimate that minimizes $|\bar{e}|^2$ is given by $\hat{a} = \mathbf{X}^\dagger\bar{x}$ since $\mathbf{X}^*\mathbf{X}$ is positive definite and $\bar{x}^*P\bar{x}$ is independent of a.

b) Substitute the result from a) in the least squares criterion.

$$\begin{aligned} |\bar{e}|^2 &= \bar{e}^*\bar{e} = (\bar{x} - \mathbf{X}a)^*(\bar{x} - \mathbf{X}a) \\ &= (\bar{x} - \mathbf{X}\mathbf{X}^\dagger\bar{x})^*(\bar{x} - \mathbf{X}\mathbf{X}^\dagger\bar{x}) = \bar{x}^*P^*P\bar{x} \\ &= \bar{x}^*P\bar{x} \end{aligned}$$

c)

$$\begin{aligned} E\{\bar{x}^*P(\lambda)\bar{x}\} &= E\{(\mathbf{X}_1a + \bar{e})^*P(\lambda)(\mathbf{X}_1a + \bar{e})\} \\ &= a^*\mathbf{X}_1^*P(\lambda)\mathbf{X}_1a + E\{\bar{e}^*P(\lambda)\bar{e}\} \\ &= a^*\mathbf{X}_1^*P(\lambda)\mathbf{X}_1a + \mathrm{Tr}\{P(\lambda)E\{\bar{e}\bar{e}^*\}\} \\ &= a^*\mathbf{X}_1^*P(\lambda)\mathbf{X}_1a + \mathrm{Tr}\{P(\lambda)\} \\ &= a^*\mathbf{X}_1^*(\mathbf{I} - \mathbf{X}\mathbf{X}^\dagger)\mathbf{X}_1a + N - 1 \\ &= a^*\mathbf{X}_1^*\mathbf{X}_1a - a^*\mathbf{X}_1^*\mathbf{X}\mathbf{X}^\dagger\mathbf{X}_1a + N - 1 \end{aligned}$$

Only $-a^*\mathbf{X}_1^*\mathbf{X}\mathbf{X}^\dagger\mathbf{X}_1a$ depends on λ. This expression is minimized when $\lambda = \lambda_1$, i.e. $\hat{\lambda} = \lambda_1$.

See Exercise.

[9.5] Let t_1, t_2, t_3, t_4, t_5 be the columns of matrix T and introduce the notation R ($R = TT^T$) for the matrix with autocorrelation values. We see that Rt_1 is equivalent with the first column of $RT = TT^TT$, i.e. $0.4000t_1$ since T^TT is a diagonal matrix. Thus, t_1 is an eigenvector of R with eigenvalue 0.4000. Similarly, $t_2, \ldots, t_5$ are the eigenvectors of R with the corresponding eigenvalues, given by the respective diagonal element of T^TT.

The smallest eigenvalue is clearly 0.4000 and $b = t_1$ is the corresponding eigenvector. According to the Pisarenko method, we should find the roots of equation

$$b_1 + b_2z^{-1} + b_3z^{-2} + b_3z^{-3} + b_5z^{-4} = 0 \quad \text{i.e.} \quad b_1z^4 + b_2z^3 + b_3z^2 + b_4z^1 + b_5 = 0$$

(In the book, The first coefficient is one, but the roots of the equation do not change when all the coefficients are multiplied by a constant.)

The first column of the table gives the roots we are looking for, i.e. $r_{1,2} = 0.5878 \pm 0.8090j$, $r_{3,4} = 0.8090 \pm 0.5878j$. The frequency estimates are thus $f_1 = arg[r_1]/2\pi \approx 0.15$ and $f_2 = arg[r_3]/2\pi \approx 0.10$.

The estimate of the noise power is given by the smallest eigenvalue of R, i.e. $\sigma^2 \approx 0.40$.

See Exercise.

[10.1] $\begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$

[10.2] -2

[10.3] $\begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$

[10.5] Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \{a_{ij}\}$$

and

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} = \{b_{ij}\}.$$

Obviously,

$$A + B = B + A = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} + b_{n1} & a_{n2} + b_{n2} & \cdots & a_{nn} + b_{nn} \end{bmatrix} = \{a_{ij} + b_{ij}\}.$$

Hence

$$\mathrm{Tr}\{A+B\} = \mathrm{Tr}\{B+A\} = \sum_{i=1}^{n}(a_{ii}+b_{ii}) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = \mathrm{Tr}\{A\} + \mathrm{Tr}\{B\}$$

[10.10] 2

[10.12] Since A is Hermitian, there is an eigenvalue decomposition such that $AE = E\Lambda$, where $E = [e_1, e_2, \ldots, e_n]$ and

$$e_i^* e_j = \begin{cases} 1 & \text{when } \; i = j \\ 0 & \text{when } \; i \neq j \end{cases} .$$

Hence $E^*E = I$ (E is orthonormal), where I denotes identity matrix. Since I is full rank, E is full rank. Therefore, $E^* = IE^{-1} = E^{-1}$.

[10.16] $A^{-1} = E\Lambda^{-1}E^*$

[10.17] For any vector x, $x^*A^*Ax = (Ax)^*(Ax) = \|Ax\|^2 \geq 0$

[10.18] For any eigenvalue λ with eigenvector p, $Ap = \lambda p$. Thus, $p^*Ap = \lambda p^*p$ and $\lambda = \frac{p^*Ap}{p^*p} > 0$ if A is positive definite.

[10.19] All eigenvectors $\lambda_i \geq 0$.

[10.21] (a) When $n = m$ there is a unique solution $x = A^{-1}b$ such that the error $Ax - b$ is zero. When $n < m$ there are fewer equations than unknowns and hence infinitely many solutions. If $m > n$, there are more equations than unknowns and typically no solution making the error equal to zero. However, there is a unique solution to the minimization problem (see below).

(b) Using the differentiation rules of problem [10.10.20],

$$\frac{d}{dx} x^T A^T A x - b^T A x - x^T A^T b + b^T b = 2A^T A x - A^T b - A^T b = 0$$

iff $A^T A x = A^T b$. The matrix $(A^T A)$ is square and non-singular since A is full rank. Hence, the unique solution is $x = (A^T A)^{-1} A^T b$.

(c) The criterion can be rewritten as (verify!)

$$\begin{aligned} & x^T A^T A x - b^T A x - x^T A^T b + b^T b \\ & = \left[x - (A^T A)^{-1} A^T b\right]^T (A^T A) \left[x - (A^T A)^{-1} A^T b\right] - b^T A (A^T A)^{-1} A^T b + b^T b \end{aligned}$$

Apparently, the choice $x = (A^T A)^{-1} A^T b$ makes the criterion as small as possible ($= b^T b - b^T A (A^T A)^{-1} A^T b \geq 0$).

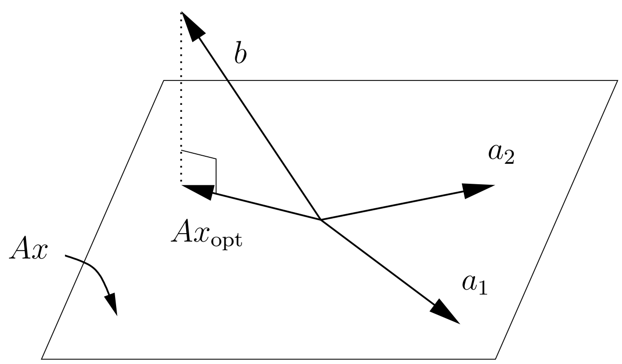


Figure 11.7: Illustration of the orthogonality condition

(d) The columns $a_1, \ldots, a_m$ of the matrix $A = [a_1, \ldots, a_m]$ can be seen as vectors in an n-dimensional space. Since $Ax = \sum a_k x_k$, all vectors of the form Ax are confined to the m-dimensional hyperplane spanned by the vectors $a_1, \ldots, a_m$. The vector b will typically be outside this hyperplane, but the point in the plane that is closest to b is clearly the orthogonal projection down to the plane, see Figure 11.7. In other words, the residual vector $Ax_{\text{opt}} - b$ should be orthogonal to all the columns of A. That is, $a_k^T(Ax_{\text{opt}} - b) = 0$ for $k = 1, \ldots, m$ or, equivalently,

$$A^T(Ax - b) = 0$$

This gives directly

$$(A^T A)x = A^T b.$$

(e) Completing the squares or using the orthogonality condition can be applied directly to the case of complex valued variables, just replace every $(\cdot)^T$ with $(\cdot)^*$. However, if we wish to use differentiation, it is more tricky since $\|Ax - b\|^2$ is not an analytic function of x so the ordinary derivative is not defined. It is possible to define a complex gradient, though, that gives the correct result, see Section 2.3.10 of Hayes book.

See Exercise.

[11.1] 625Hz corresponds to an angle of $-3\pi/4$. The folding around $-\pi$ results in an angle $3\pi/4$ that corresponds to the normalized frequency $f = 3/8$. Correct answer $\boxed{C}$.

[11.2] The sequence is downsampled by a factor M. The bandwidth of the resulting signal is $M\Omega$. Correct answer $\boxed{B}$.

[11.3] The periodogram is biased but asymptotically unbiased. Correct answer $\boxed{B}$.

[11.4]

$$\begin{aligned} y_0(n) =& (x(n) * h_1(n)) * h_2(n) = x(n) * (h_1(n) * h_2(n)) \\ =& (h_1(n) * h_2(n)) * x(n) = (h_2(n) * h_1(n)) * x(n) = y(n) \end{aligned}$$

Correct answer $\boxed{A}$.

[11.5] b has to be such that the filter is asymptotically stable, that is $|b| < 1$. Correct answer $\boxed{B}$.

See Exercise.

[11.6] 1400Hz corresponds to an angle of -0.6π. The folding around $-\pi$ results in an angle $0.6\pi = 1.885$ rad. Correct answer $\boxed{F}$.

[11.7] Correct answer $\boxed{D}$.

[11.8] The sequence is downsampled by a factor K. The new center frequency is $\bar{\Omega} = K\Omega_0$. Correct answer $\boxed{B}$.

[11.9] The roots of $z^2 + 1 = 0$ are $z = e^{\pm i\pi/2}$ so $\omega = \pi/2$. $F = F_s \cdot \omega/(2\pi) = 250$ Hz. Correct answer $\boxed{D}$.

[11.10] $\text{var}[y(n)] = \text{var}[x(n)] \cdot \sum_{n=-\infty}^{\infty} h(n)^2$ where $h(n)$ is the (infinite) impulse response of $H(z)$. Parsevals identity gives $\sum_{n=-\infty}^{\infty} h(n)^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(\omega)|^2 d\omega = 1$. Correct answer $\boxed{A}$.

See Exercise.

[11.11] 8000Hz corresponds to an angle of $-2\pi/5$. The folding results in the angle $2\pi/5$ so $F = (2\pi/5) \cdot (1/2\pi) \cdot 10 = 2$ kHz. Correct answer $\boxed{B}$.

[11.12] Correct answer $\boxed{D}$.

[11.13] The sequence is downsampled by a factor M. The new center frequency is $\bar{\Omega} = M\Omega_0$. Correct answer $\boxed{C}$.

[11.14] The roots are $z = e^{\pm i\pi/5}$ so $\omega = \pi/5$. $F = F_s \cdot \omega/(2\pi) = 100$ Hz. Correct answer $\boxed{A}$.

[11.15] $T_s = 0.1$ s gives sampling frequency $F_s = 10$ Hz. $\{x(n)\}$ has frequency components in $(0, 5)$ Hz. $x(n)$ should be filtered by an anti-aliasing filter with cut-off frequency $5/4 = 1.25$ Hz. Correct answer $\boxed{C}$.

See Exercise.

(a) Because the input signal is occupying all the possible spectrum (is being sampled at precisely the Nyquist rate), there is a need for the antialiasing filter $H(\nu)$. To see this, consider the signals $(x * h)[n]$ and $y[n]$ at each side of the downsampler. Because

$$Y(\nu) = \frac{1}{2}\sum_{k=0}^{1} H\left(\frac{\nu - k}{2}\right) X\left(\frac{\nu - k}{2}\right),$$

there is a clear possibility of aliasing if $H(\nu)$ does not cut the upper frequency components in $X(\nu)$. In particular, $B \leq \frac{1}{4}$ is necessary. Thus, the correct answer is $\boxed{\text{A}}$

(b) We know that the spectrum of $x[n]$ has been cut by the antialiasing filter at $B = \frac{1}{4}$. Thus, we know it will look like a triangle with chopped slopes. This leaves only options $\boxed{\text{D}}$ and $\boxed{\text{E}}$. However, we see that $\boxed{\text{E}}$ is exactly how the signal would look if it were converted to analog (at rate F_s) just after the filter. After downsampling, this shape will be expanded from the range $\left[-\frac{1}{4}, \frac{1}{4}\right]$ to the range $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Moreover, the D/A converter works at twice the frequency the A/D works, and thus, the resulting spectrum will have the same shape, but in the range $[-F_s, F_s]$. Therefore, the correct answer is $\boxed{\text{D}}$.

(c) If the D/C converter operates at rate $F_s/2$, the signal will look like in option $\boxed{\text{E}}$, which is the low-pass filtered version of the input signal.

See Exercise.