SIGNAL PROCESSING

SCHOOL OF ELECTRICAL ENGINEERING

Digital Signal Processing EQ2300 / 2E1340

Final Examination 2013–05–30, 08.00–13.00 Examples of Solutions

1. a) Let $x(n) = \{\underset{\uparrow}{3}, 2, 1\}$. Then,

$$w(n) = x(n) \circledcirc \{\underset{\uparrow}{3}, 0, -1\} = 3x(n) - x((n-2)_{\text{mod }3}) = 3\cdot\{\underset{\uparrow}{3}, 2, 1\} - \{\underset{\uparrow}{2}, 1, 3\} = \{\underset{\uparrow}{7}, 5, 0\}$$

Answer: iv) $w(n) = \{\underset{\uparrow}{7}, 5, 0\}$

b) Both curves look fairly similar and except for the frequency range around the main peaks, they stay fairly close to the red curve, which which indicates that the variance is reasonably low. The two main "peaks" (rather "blobs") indicate that $x(n)$ contains two sinusoids with frequencies $f \approx 0.12$ and $f \approx 0.21$. The main lobes are fairly wide and we also recognize typical side-lobes. The edge-to-edge width of the main lobe for the the Bartlett method with $K = 16$ would be $2K/N = 1/32 \approx 0.03$, which agrees well with the plotted curves.

All these facts strongly hint that the Bartlett method with $K = 16$ segments was used, since the estimates have a fairly low variance and also low (=poor) resolution (wide main lobes).

Answer: i)

c) Comparing with Figure 1, the estimates in Figure 2 clearly show larger variations, both between the two curves, from frequency value to frequency value and compared to the red curve. In other words, this estimate has much larger variance. On the other hand, the main lobes are much narrower so the estimates have much higher (=better) resolution. This indicates that the Periodogram method was used.

Answer: v)

2. The AR(1) model gives that the frequency response from $u(n)$ to $x(n)$ is $1/(1 - 0.2e^{-j\omega}) = 1/H(\omega)$, so the output due to $x(n)$, say $y_x(n)$ has power spectral density

$$P_{y_x}(\omega) = P_x(\omega)\,|H(\omega)|^2 = \sigma_u^2 \left|\frac{1}{H(\omega)}\right|^2 |H(\omega)|^2 = \sigma_u^2,$$

i.e. $y_x(n)$, is white noise with power $\sigma_u^2 = 1$ (in fact, $y_x(n) = e(n)$).

The quantization error is modeled as usual as an additive stochastic noise $e(n)$, assumed to be zero mean with variance $\sigma_e^2 = \frac{\Delta^2}{12}$.

Introduce the notation $a = 0.2$. The power spectral density of the output due to noise is

$$P_{y_e}(\omega) = \sigma_e^2 |H(\omega)|^2 = \sigma_e^2(1 + a^2 - ae^{j\omega} - ae^{-j\omega})$$

which yields the autocorrelation

$$r_{y_e}(k) = (1 + a^2)\sigma_e^2\delta(k) - a\sigma_e^2\delta(k+1) - a\sigma_e^2\delta(k-1)$$

and evaluating at zero yields the power of the noise at the output

$$\sigma_{y_e}^2 = r_{y_e}(0) = (1+a^2)\sigma_e^2 = (1+a^2)\frac{\Delta^2}{12}$$

(as an alternative, directly use the expression $\sigma_{y_e}^2 = \sigma_e^2 \sum |h(n)|^2$). Therefore the SQNR at the output will be

$$\text{SQNR} = \frac{12}{(1+a^2)\Delta^2} = \frac{12}{1.04\Delta^2}.$$

3. a) Since the system has an Mth order pole at zero, and zeros at $\pm 1/2$,

$$H(z) = a\frac{(z-1/2)(z+1/2)}{z^M} = \frac{a(z^2-1/4)}{z^M} = az^{-M}(z^2-1/4) = az^{-M+2}(1-1/4z^{-2})$$

Since $H(1) = 3/4 \Rightarrow a(1-1/4) = 3/4 \Rightarrow a = 1$. Thus,

$$H(z) = z^{-M+2}(1-1/4z^{-2})$$

It follows that $h(n) = \mathcal{Z}^{-1}\{z^{-M+2} - 1/4z^{-M}\} = \delta(n-(M-2)) - 1/4\delta(n-M)$.

b) The filter is BIBO stable since there are no poles outside the unit circle (you could alternatively check that the RoC contains the unit circle).

c) Because $M \geq 2$ is assumed, inspection of the impulse response reveals that the filter is causal.

d) If $M = 2K$ is an even number, we can write $H(z) = G(Z^2)$ if $G(z) = z^{-K+1}(1 - 1/4z^{-1})$. Then, the Noble identities show that $r(n) = y(n)$. The corresponding impulse response is $g(n) = \delta(n-(K-1)) - 1/4\delta(n-K)$. If M is odd, on the other hand, no $G(z)$ exists so that both outputs are identical. One way to prove this is to consider the polyphase implementation of the upper branch, see Figure 1. According to the polyphase theory, $p_0(n) = h(2n)$ and $p_1(n) = h(2n+1)$. Comparing Figure 1 to the lower branch of the original system, we see that $r(n) = y(n)$ iff $G(z) = P_0(z)$ and $P_1(z) = 0$. If M is odd, the resulting $p_1(n)$ is clearly non-zero, since both non-zero taps of $h(n)$ appear at odd numbered n, which proves that it is impossible to obtain $r(n) = y(n)$, no matter how $G(z)$ is chosen. (When M is even, $P_0(z) = G(z)$ and $P_1(z) = 0$ holds).
Answer: M has to be an even number and then $G(z) = z^{-K+1}(1-1/4z^{-1})$.

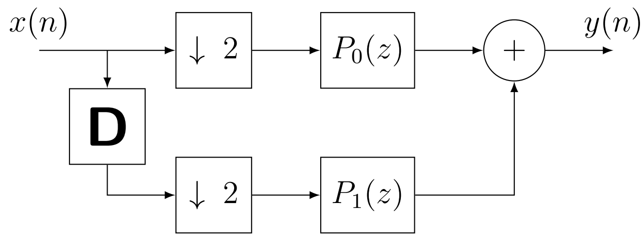

Figure 1: Polyphase implementation of the branch from $x(n)$ to $y(n)$.

4. a) One way to solve this problem is to use the decimation formula for TDFTs, $y_r(n) = x(2n) \longrightarrow Y_r(f) = \frac{1}{2}\Big(X(f) + X(f/2 - 1/2)\Big)$ and exploit the fact that $Y(k) = Y_r(f)\rfloor_{f=k/M}$.

An alternative is to start similarly to the derivation of the FFT algorithm,

$$\begin{aligned}
X(k) &= \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi nk}{N}} = \Big/ \begin{array}{l}\text{even: } n = 2m \\ \text{odd: } n = 2m+1\end{array} \Big/ \\
&= \sum_{m=0}^{N/2-1} \underbrace{x(2m)}_{=y_r(m)} e^{-j\frac{2\pi 2mk}{N}} + \sum_{m=0}^{N/2-1} \underbrace{x(2m+1)}_{y_i(m)} e^{-j\frac{2\pi(2m+1)k}{N}} \\
&= \sum_{m=0}^{M-1} y_r(m)e^{-j\frac{2\pi mk}{M}} + e^{-j\frac{2\pi k}{N}} \sum_{m=0}^{M-1} y_i(m)e^{-j\frac{2\pi mk}{M}} = Y_r(k) + e^{-j\frac{2\pi k}{N}} Y_i(k)
\end{aligned}$$

In order to get further, the trick is to replace k by $k + M$ in the previous formula and exploit that the DFT is periodic, which gives

$$X(k+M) = Y_r(k+M) + e^{-j\frac{2\pi k}{N}} e^{-j2\pi M/N} Y_i(k+M) = Y_r(k) - e^{-j\frac{2\pi k}{N}} Y_i(k)$$

Now, it is easy to solve for $Y_r(k)$ and $Y_i(k)$, resulting in

$$\begin{aligned}
Y_r(k) &= \frac{1}{2}\Big(X(k) + X(k+M)\Big) \\
Y_i(k) &= \frac{1}{2}\, e^{j\frac{2\pi k}{N}} \Big(X(k) - X(k+M)\Big)
\end{aligned}$$

The same result can also be derived by mimicking the proof of the corresponding DTFT result.

b) Combining the two results from a) and using the linearity of the DFT, we obtain

$$\begin{aligned}
Y(k) = Y_r(k) + jY_i(k) &= \frac{1}{2}\Big(X(k) + X(k+M)\Big) + \frac{j}{2}\, e^{j\frac{2\pi k}{N}} \Big(X(k) - X(k+M)\Big) \\
&= \frac{1}{2}\Big(X(k)(1 + je^{j\frac{2\pi k}{N}}) + X(k+M)(1 - je^{j\frac{2\pi k}{N}})\Big)
\end{aligned}$$

c) $k = 0$ gives $Y(0) = \frac{1}{2}\Big(X(0)(1+j) + X(M)(1-j)\Big) = 7/2 - j3/2$.

5. Let us start with the power of $y(n)$,

$$\sigma_y^2 = \sigma_w^2 = r_w(0) = \int_{-1/2}^{1/2} P_w(f)\, df = \int_{-1/4}^{1/4} 2 + \sin(2\pi f)\, df = \int_{-1/4}^{1/4} 2\, df = 1\ ,$$

since $\sin(2\pi f)$ is an odd function which integrated over a symmetric interval gives zero.

Next, the autocorrelation sequence for $v(n)$ is $r_v(k) = \sigma_v^2 \delta(k)$ since $v(n)$ is white noise. Therefore, the autocorrelation sequence for $x(n)$ is

$$r_x(k) = \begin{cases} r_y(0) - r_v(0) = \sigma_y^2 - \sigma_v^2 = 1/2 & \text{for } k = 0 \\ r_y(k) - r_v(0) = r_y(k) & \text{for } k \neq 0 \end{cases}$$

The AR(2) model for $x(n)$ is

$$x(n) + a_1 x(n-1) + a_2 x(n-2) = e(n),$$

where $e(n)$ is zero-mean white with variance σ_e^2. The Yule-Walker equations give

$$\begin{bmatrix} r_x(0) & \hat{r}_x(1) \\ \hat{r}_x(1) & r_x(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -\hat{r}_x(1) \\ -\hat{r}_x(2) \end{bmatrix},$$

i.e., with the numeric values inserted,

$$\begin{bmatrix} 1/2 & 1/4 \\ 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -1/4 \\ -1/6 \end{bmatrix},$$

from which one obtains the coefficients as $a_1 = -\frac{4}{9} \approx -0.44$ and $a_2 = -\frac{1}{9} \approx -0.11$. The variance of the driving noise $e(n)$ is estimated next as

$$\sigma_e^2 = r_x(0) + a_1\hat{r}_x(1) + a_2\hat{r}_x(2) = \frac{10}{27} \approx 0.37.$$

A parametric estimate for the power spectrum of $x(n)$ can therefore be formulated as

$$\hat{P}_x(f) = \frac{10/27}{\left|1 - \frac{4}{9}e^{-j2\pi f} - \frac{1}{9}e^{j4\pi f}\right|^2}.$$