SOLUTIONS

E 95 **Digital Signalbehandling**, 2E1340

Final Examination 2004–12–14, 14.00–19.00

1. You could either recognize the algorithm directly as a polyphase implementation or think about the ordinary decimation structure and notice that the procedure is equivalent to using the anti-alias filter

$$h(n) = \frac{1}{2}(\delta(n) + \delta(n+1))$$

before decimation.

a) A block diagram of this polyphase implementation is given in Figure 1. The polyphase filters are given by $p_k(n) = h(2n+k) \quad \Rightarrow$

$$p_0(n) = h(2n) = \frac{1}{2}\delta(n),$$

$$p_1(n) = h(2n+1) = \frac{1}{2}\delta(n+1).$$

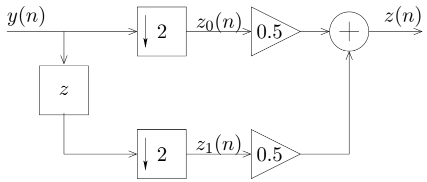

Figure 1: Polyphase implementation

b) The frequency response of the filter is $H(f) = \frac{1}{2}(1+e^{j2\pi f})$. The signal before decimation has DTFT $X(f) = H(f)Y(f)$. After decimation, the DTFT is given by

$$Z(f) = \frac{1}{2}\sum_{k=0}^{1} X\left(\frac{f-k}{2}\right) = \frac{1}{4}\left((1+e^{j\pi f})Y\left(\frac{f}{2}\right) + (1-e^{j\pi f})Y\left(\frac{f-1}{2}\right)\right).$$

c) It is easy to see that $h(n)$ is a low-pass filter, for instance by looking at the square of the magnitude of $H(f)$:

$$|H(f)|^2 = \frac{1}{4}(1+e^{j2\pi f})(1+e^{-j2\pi f}) = \frac{1}{2}(1+\frac{1}{2}(e^{j2\pi f}+e^{-2\pi f})) = \frac{1}{2}(1+\cos(2\pi f)),$$

and we get

$$|H(f)|^2 = \begin{cases} 1, & f=0 \\ 1/2, & f=1/4 \\ 0, & f=1/2. \end{cases}$$

This looks like a low-pass filter with 3dB cut-off at $f = 1/4$, which is what we would like for an anti-aliasing filter designed for decimation by 2. If the input signal has much of its power located in high frequencies, then the proposed algorithm works to reduce the aliasing. Note, however, that the filter is not very sharp. This means that not all aliasing is removed, and also that some of the low frequencies are attenuated. In other words, if the input signal has low-pass character, we might be better off without the filter.

2. The quantization after the multiplication is approximated by a noise source $e(n)$ with power $\sigma_{ee}^2 = 2^{-2(B-1)}/12$. The transfer function from the noise source to the output is $H(z) = 1/(1-az^{-1})$, so the spectral density of the noise $q(n)$ at the output is

$$P_{qq}(f) = P_{ee}(f)|H(f)|^2 = \frac{2^{-2(B-1)}}{12}\left|\frac{1}{1-ae^{-j2\pi f}}\right|^2 = \frac{2^{-2(B-1)}}{12(1-2a\cos(2\pi f)+a^2)}$$

The input sinusoidal signal will only add Dirac pulses at $f = \pm f_0$ to the spectral density $P_{yy}(f)$ of the total output. So, at $f = 0$, we have

$$1.186\cdot 10^{-5} \approx P_{yy}(0) = P_{qq}(0) = \frac{2^{-14}}{12(1-a)^2}$$

and at $f = 1/2$ we get

$$2.812\cdot 10^{-6} \approx P_{yy}(\frac{1}{2}) = P_{qq}(\frac{1}{2}) = \frac{2^{-14}}{12(1+a)^2}$$

Both these equations are approximately solved by $a = 0.345$.

3. a) The recommended method is to use the periodogram together with zero padding (or some model based technique). Actually, this is equivalent to maximum likelihood since we only have a single frequency, so it gives the lowest variance of the estimated frequency (this is somewhat surprising, since the variance of the power spectrum density is worse than many other methods). The use of an averaging method, such as Bartlett's, is not recommended, since it does not give a better localization of the frequency of the sinusoid (possibly with the exception of extremely low signal to noise levels, where the main peak of the spectrum cannot even be dected in the periodogram). Note that resolution itself is not an issue, since there is only a single frequency to find.

b) Again, the periodogram is recommended if you want to stick to non-parametric techniques, since it provides the best resolution. Otherwise, a parametric technique like non-linear least squares, AR-modeling or some subspace method is preferable, since it can give provide good accuracy and "super-resolution" even when the number of samples is small. Again, averaging or windowing is discouraged, because it will decrease the resolution which might jeopardize the ability to localize both peaks.

c) In this case we would recommend some non-parametric method with averaging to reduce the variance and windowing to reduce the leakage. For example the Welch or Blackman-Tukey method is recommended, whereas the periodogram clearly is not recommended.

4. The filter implemented by the algorithm is $y(n) = x(n) + bx(n-1) + bx(n-2) + 0.5x(n-3) + ay(n-1) + ay(n-2) - 0.5y(n-3)$ so the transfer function is

$$H(z) = \frac{1 + bz^{-1} + bz^{-2} + 0.5z^{-3}}{1 - az^{-1} - az^{-2} + 0.5z^{-3}} = \frac{B(z)}{A(z)}$$

Only the parameter a will affect the poles, and thereby the stability, of the filter. We use the Schur-Cohn stability test to determine the values of a that give a stable filter. $A(z) = 1 - az^{-1} - az^{-2} + 0.5z^{-3}$ gives

$$a_3(0) = 1, \quad a_3(1) = -a, \quad a_3(2) = -a, \quad a_3(3) = \frac{1}{2}$$

The step-down recursion

$$\begin{aligned} \Gamma_m &= a_m(m) \\ a_{m-1}(k) &= \frac{a_m(k) - \Gamma_m a_m^*(m-k)}{1 - |\Gamma_m|^2}, \ k = 0, \ldots, m-1 \end{aligned}$$

gives

$$\begin{aligned} &\Gamma_3 = a_3(3) = \frac{1}{2}, \quad a_2(0) = 1, \quad a_2(1) = a_2(2) = \frac{-a - 1/2(-a)}{1 - 1/4} = \frac{-2a}{3} \\ &\Gamma_2 = a_2(2) = \frac{-2a}{3}, \quad a_1(0) = 1, \\ &a_1(1) = \frac{-2a/3 - (-2a/3)(-2a/3)}{1 - (2a/3)^2} = \frac{-2a/3(1 + 2a/3)}{(1 - 2a/3)(1 + 2a/3)} = \frac{-2a/3}{1 - 2a/3} \\ &\Gamma_1 = a_1(1) = \frac{-2a/3}{1 - 2a/3} \end{aligned}$$

The condition $|\Gamma_3| < 1$ is always fulfilled and $|\Gamma_2| < 1$ is fulfilled when $|a| < 3/2$. It remains to determine when $|\Gamma_1| < 1$. Since $|a| < 3/2$ hat to hold anyway, we see that $1 - 2a/3 > 0$ so $-1 < \Gamma_1 < 1$ is equivalent to

$$2a/3 - 1 < -2a/3 < 1 - 2a/3$$

The second inequality is trivially true, whereas the first one is only true when $a < 3/4$. To summarize, the filter is stable if and only if $-3/2 < a < 3/4$.

5. From the compendium, we know that the following conditions are necessary and sufficient to give perfect reconstruction with delay L,

$$\begin{aligned} F_0(z)H_0(-z) + F_1(z)H_1(-z) &= 0 \\ F_0(z)H_0(z) + F_1(z)H_1(z) &= 2z^{-L}\ . \end{aligned}$$

The same conditions expressed in terms of the DTFT are,

$$\begin{aligned} F_0(f)H_0(f - \frac{1}{2}) + F_1(f)H_1(f - \frac{1}{2}) &= 0 \\ F_0(f)H_0(f) + F_1(f)H_1(f) &= 2e^{-j2\pi fl}\ , \end{aligned}$$

From the figure, we see that $H_0(f - \frac{1}{2}) = H_0(f)$ and $H_1(f - \frac{1}{2}) = H_1(f)$, so if both conditions were true, we would get

$$0 = F_0(f)H_0(f - \frac{1}{2}) + F_1(f)H_1(f - \frac{1}{2}) = F_0(f)H_0(f) + F_1(f)H_1(f) = 2e^{-j2\pi fl}\ ,$$

which clearly is impossible. We can note that the same problem holds for all filters $H_k(z)$ with $H_k(z) = H_k(-z)$, for example all filters that can be expressed like $H_k(z) = B_k(z^2)/A_k(z^2)$ for some polynomials $B_k(z)$ and $A_k(z)$.

An alternative solution is to draw the spectrums of the signals $v_0(m)$ and $v_1(m)$ for some example input signal $x(n)$ and note that the aliasing introduced in $V_0(f)$ can never be recovered, even if we use information from $V_0(f)$.