INFORMATION SCIENCE AND ENGINEERING

SCHOOL OF ELECTRICAL ENGINEERING

Digital Signal Processing EQ2300 / 2E1340

Final Examination 2017–01–08, 08.00–13.00 Examples of Solutions

1. a) The two main peaks are at the verge of being separable. This means that the frequency resolution is $\Delta\nu = 0.12 - 0.1 = 0.02$. With the formula for the resolution of a modified periodogram, using a Hamming window, we get

$$\Delta\nu \approx \frac{1.30}{L} \quad \Rightarrow \quad L \approx \frac{1.30}{0.02} = 65\,.$$

The actual value of L used in generating the plot is $L = 64$, which conveniently is a power of 2.

b) Assuming that $L = 64$, and using the fact that we have 50% overlap, we get

$$K = \frac{N}{L} + \frac{N-L}{L} = 31\,,$$

or

$$K \approx 0.5\frac{N}{L} = 32$$

in case we do not wish to be exact about the edge effects for the overlapping blocks.

c) There are many ways to deal with this. A bit better separation would be obtained if we chose a window with a narrower main lobe, such as, e.g., a Bartlett window. This would affect the spectral leakage negatively, but is an ok solution. Another strategy would be to increase L to, say, $L = 128$ while keeping the same window. This would affect the variance negatively, which may also be ok.

d) In this case, we would need to reduce spectral lekage as this is what covers the weak sinusoid, not the noise. We could thus choose a window with better suppression of the slide lobes, such as a Blackman window. The weak sinusoid is $20\log_{10} 0.005 \approx -46$ dB below the strong ones, so the -58 dB sidelobe level of the Blackman window would be sufficient.

e) We could use Blackman-Tukey's spectrum estimator which has better variability, i.e., lower variance at the same frequency resolution. If we chose the window in Blackman-Tukey's estimator as $w_{\mathrm{BT}}[n] = w[n] * w[-n]$ where $w[n]$ is the length $L = 64$ Hamming window and normalized appropriately, we would get $W_{\mathrm{BT}}(\nu) \propto |W(\nu)|^2$, and very similar Bias properties (and thus resolution) with Blackman-Tukey's method as with Welch's method.

2. a) The standard (biased) ACF-estimator is given by

$$r_x[k] = \begin{cases} \frac{1}{N}\sum_{n=0}^{N-1} x[n]x[n+|k|] & |k| < N \\ 0 & |k| \geq N \end{cases}$$

where in this case $N = 3$. The explicit calculation would then yield

$$r_x[0] = \frac{1}{3}\left(2^2 + 1^2 + 1^2\right) = \frac{6}{3} = 2$$

$$r_x[1] = \frac{1}{3}\Big(2 \times 1 + 1 \times 1\Big) = \frac{3}{3} = 1$$
$$r_x[2] = \frac{1}{3}\Big(2 \times 1\Big) = \frac{2}{3}\,.$$

b) We find the AR-coefficient by solving the Yule walker questions, which in this case are given by

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -\begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \frac{-1}{2^2 - 1^2} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} = -\frac{1}{9} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

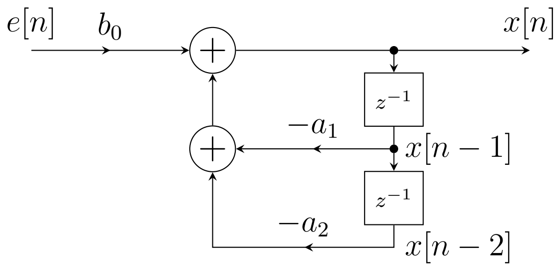
and

$$b_0^2 = r_x[0] - \sum_{k=1}^{2} a_k r_x[k] = 2 - \frac{4}{9} \times 1 - \frac{1}{9} \times \frac{2}{3} = \frac{40}{27}$$

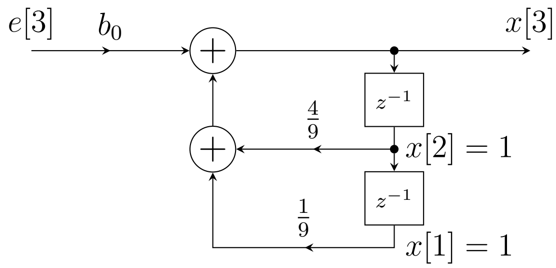
or

$$b_0 = \frac{2\sqrt{30}}{9}\,.$$

c) We can equivalently implement the AR-system using the following schematic.



This makes a bit easier to see the influence of prior samples $x[n]$ on the next. Particularizing to the values we have for a_1 and a_2, as well as $x[2]$ and $x[1]$, we obtain the following schematic.



The minimum mean square error (MMSE) estimate of $x[3]$ can now be obtained from the MMSE estimate of $e[3]$, which is 0, and thus $\hat{x}[3] = \frac{4}{9} \times 1 + \frac{1}{9} \times 1 = \frac{5}{9}$ is a reasonable estimate of $x[3]$. Given the connection between AR modeling and linear MMSE (LMMSE) estimation, we would get the same estimate of $x[3]$ if we had derived the LMMSE estimate of $x[3]$, and used the estimates for the ACF that we calculated in part a), but the given explanation is arguably more intuitive.

3. a) The time for processing the block is just the time between concequitive `Start` and `Stop` times, i.e., 0.2s.

b) The time between successive calls to the code can be obtained as the time between one `Start` time and the next, i.e., 0.5s.

c) We know from the theory of overlap save that each block will process $N - M + 1$ (new) samples. As the samples come in at 1 kHz, we know that we in 0.5s (the time between concequitive calls to the code) will collect 500 samples, thus, $N - M + 1 = 500$, and $M = N - 500 + 1 = 1024 - 500 + 1 = 525$.

d) The total number of complex values computations are

$$C = 2 \times \frac{N}{2} \log_2 N + N = 1024 \times 11 = 11264\,.$$

As this number of multiplications are done in 0.2s, we get

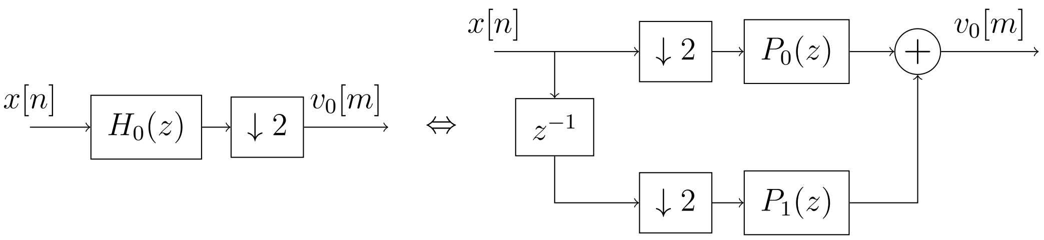
$$\frac{0.2\text{s}}{11264} \approx 1.8 \times 10^{-5}\text{s} = 18\mu\text{s}$$

per multiplication. If we do not have a calculator, and make the approximations $N \approx 10$ and $11 \approx 10$ (i.e., we assume that the FFTs dominate the complexity), we get $C = 10^4$ and $20\mu s$ per complex valued multiplication, which is close to the more exact answer.

e) With $M = 25$, the number of (new) samples per block for overlap save would be $L = N - M + 1 = 1000$. As the code was called every 0.5s when only 500 samples were collected per block, the code will now be called every 1.0s. However, since N remains unchanged, the processing time per block would remain at 0.2s. Thus, the representative output would be as follows.

```
Start: 1489.210 s
Done: 1489.410 s
Start: 1490.210 s
Done: 1489.410 s
Start: 1491.210 s
Done: 1491.410 s
```

4. a) If we particularize the schematic to the computation of only $v_0[m]$, we get the following polyphase equivalent implementation with $P_0(z) = \frac{1}{\sqrt{2}}$ and $P_1(z) = \frac{1}{\sqrt{2}}$.



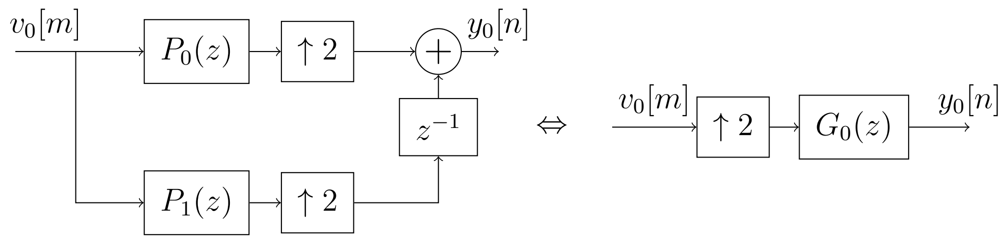
We can now use the standard expression for polyphase filters, i.e.,

$$H_0(z) = \sum_{k=0}^{1} z^{-k} P_k(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} z^{-1};$$

Doing the same for $v_1[m]$ yields after some manipulations an equivalent implementation, with the only difference that $P_1(z) = -\frac{1}{\sqrt{2}}$. Thus,

$$H_1(z) = \sum_{k=0}^{1} z^{-k} P_k(z) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} z^{-1} = H_0(-z)\,.$$

Considering the contribution of $v_0[m]$ to the output, in order to get $G_0(z)$, will yield the following polyphase decomposition, again with $P_0(z) = \frac{1}{\sqrt{2}}$ and $P_1(z) = \frac{1}{\sqrt{2}}$.



This yields

$$G_0(z) = \sum_{k=0}^{1} z^{-k} P_k(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} z^{-1} = H_0(z)\,.$$

Repeating the same exercise for the influence of $v_1[m]$ on $y[n]$ yields $P_0(z) = -\frac{1}{\sqrt{2}}$ and $P_1(z) = \frac{1}{\sqrt{2}}$, which implies that

$$G_1(z) = \sum_{k=0}^{1} z^{-k} P_k(z) = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} z^{-1} = -H_0(-z)\,.$$

b) By part a), we have that $H_0(z) = H(z)$, $H_1(z) = H(-z)$, $G_0(z) = H(z)$, and $G_1(z) = -H(-z)$ for $H(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} z^{-1}$. This is the standard QMF construction which guarantees that there is no aliasing, i.e., that

$$G_0(z)H_0(-z) + G_1(z)H_1(-z) = 0\,.$$

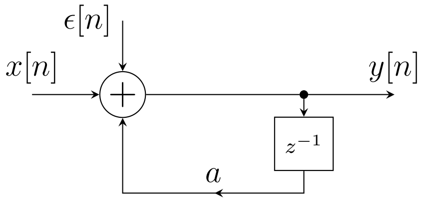
What remains to compute is

$$\begin{aligned}
&G_0(z)H_0(z) + G_1(z)H_1(z) \\
&= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} z^{-1}\right)\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} z^{-1}\right) + \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} z^{-1}\right)\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} z^{-1}\right) \\
&= \left(\frac{1}{2} + z^{-1} + \frac{1}{2} z^{-2}\right) + \left(-\frac{1}{2} + z^{-1} - \frac{1}{2} z^{-2}\right) = 2z^{-1}
\end{aligned}$$

which proves perfect reconstruction, with a delay $L = 1$.

Note: The original exam mistakenly defined L through $y[n - L] = x[n]$ in place of the common $y[n] = x[n - L]$. Thus, technically $L = -1$ would have been the right answer. Either answer, with sufficient explanation, will be accepted.

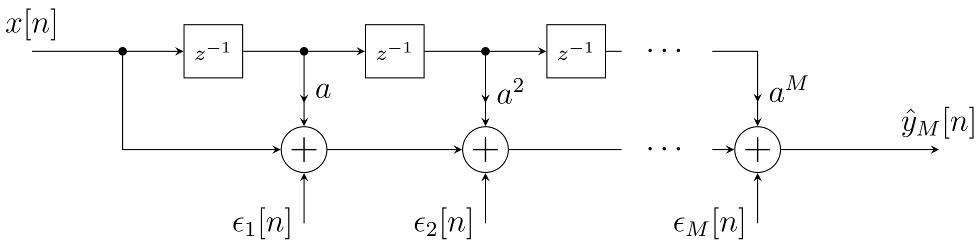
5. a) The additive quantization noise model is given by the following circuit, where $\epsilon[n]$ is the (zero mean white) quantization noise of power $\sigma_\epsilon^2 = \frac{2^{-2B}}{12}$.



As both the input sequence $x[n]$ and the quantization noise $\epsilon[n]$ are white sequences, and enter the circuit at the same place, they will be equally affected in terms of how much power remains in the output after passing the circuit. Thus,

$$\text{SQNR} = \frac{\sigma^2}{\frac{2^{-2B}}{12}} = 12\sigma^2 2^{2B}\,.$$

b) The additive quantization noise model is in this case given by the following circuit, where $\epsilon_1[n]$ to $\epsilon_M[n]$ are i.i.d. quantization noise sequences of power $\sigma_\epsilon^2 = \frac{2^{-2B}}{12}$.



As the power of independent sources add, the total power of the noise at the output will just be

$$M\frac{2^{-2B}}{12}\,.$$

However, in this case the noise is added directly to the output, so it will be affected differently than the input sequence $x[n]$. The power of output that is due to the input can however, as $x[n]$ is white, be computed as

$$\sigma^2 \sum_{m=1}^{M} \left|h[m]\right|^2 = \sigma^2 \sum_{m=1}^{M} (a^2)^m = \sigma^2 a^2 \frac{1-a^{2M}}{1-a^2}\,.$$

This yeilds

$$\text{SQNR} = \left(\sigma^2 a^2 \frac{1-a^{2M}}{1-a^2}\right) \Big/ \left(M\frac{2^{-2B}}{12}\right) = \frac{12\sigma^2 a^2 2^{2B}(1-a^{2M})}{M(1-a^2)}\,.$$

c) In case b), the noise is directly added to the output, so it will still be spectrally white based on the assumption that it is white at the insertion point. Thus,

$$P_{e,b}(\nu) = M\frac{2^{-2B}}{12}\,.$$

In case a), the quantization noise is added to the input, so it will be colored by the filter (circuit) at the output. Note that the frequency response of the circuit is

$$H(\nu) = \frac{1}{1-a^{-j2\pi\nu}}\,,$$

so

$$|H(\nu)|^2 = H(\nu)H^*(\nu) = \frac{1}{1-a^{-j2\pi\nu}}\frac{1}{1-a^{j2\pi\nu}} = \frac{1}{1+a^2-2a\cos(2\pi\nu)}\,.$$

This implies that

$$P_{e,a}(\nu) = \frac{2^{-2B}}{12(1+a^2-2a\cos(2\pi\nu))}\,.$$