KTH, SIGNAL PROCESSING

SCHOOL OF ELECTRICAL ENGINEERING

Digital Signal Processing EQ2300 / 2E1340

Final Examination 2016–03–16, 14:00-19:00

Literature:

- Jaldén: *Summary notes for EQ2300* (30 page printed material).
- *Beta – Mathematics Handbook*
- *Collection of Formulas in Signal Processing, KTH.*
- *One A4 of your own notes.* You may write on both sides, and it does not have to be hand written, but cannot contain full solutions to tutorial problems or previous exam problems.
- An unprogrammed pocket calculator.

Notice:

- Answer in English or Swedish.
- At most one problem should be treated per page.
- Answers without motivation/justification carry no rewards.
- Write your name and *personnummer* on each page.
- Write the number of solution pages on the cover page.

The exam consists of five problems with a maximum of 10 points each. For a passing grade, 24 points are normally required.

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Results: Will be reported within three working weeks on "My pages".

Solutions: Will be available on the course homepage after the exam.

Good luck!

1. A sequence of short questions to warm up...

a) Let $x[n]$ be the finite length sequence given by

$$x[n] = \{\underset{\uparrow}{2},\ 1,\ 1,\ 0,\ 3,\ 2,\ 0,\ 3,\ 4,\ 6\},$$

and let for $N = 10$

$$X[k] \triangleq \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

be the 10-point discrete Fourier transform (DFT) of $x[n]$. Determine

i) $X[0]$ (1p)

ii) $X[5]$ (1p)

iii) $\sum_{k=0}^{9} X[k]$ (1p)

iv) $\sum_{k=0}^{9} X[k] e^{j4\pi k/5}$ (1p)

v) $\sum_{k=0}^{9} \left|X[k]\right|^2$ (1p)

b) A Type I linear phase FIR filter filter is implemented as a convolution sum with fixed point arithmetics. What could happen? Answer with *true* or *false*, and a short motivation.

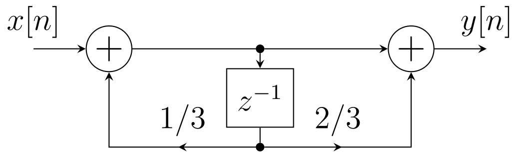
i) The filter can become unstable. (1p)

ii) The filter's frequency response can change from the ideal case. (1p)

iii) The linear phase property can be lost. (1p)

c) Give 2 good reasons for why one may not wish to use the periodogram for estimating the power spectrum of a stochastic process. (2p)

2. The circuit below is used to filter a signal $x[n]$ to yield $y[n]$.

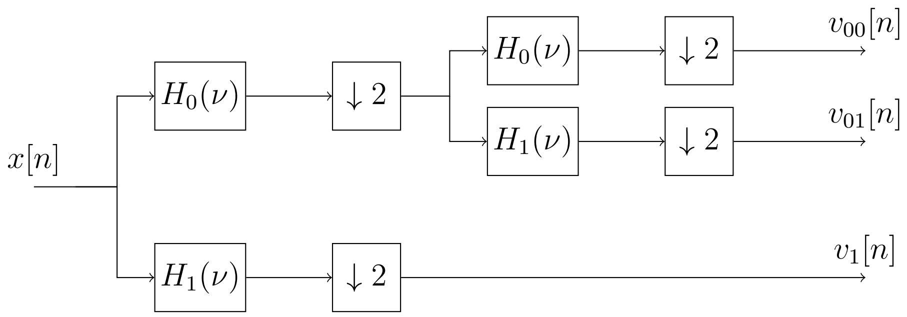


Assume that $x[n]$ can be modelled as a zero mean wide-sense stationary white noise process with power $\sigma_x^2 = \mathrm{E}\{x^2[n]\} = 0.1$. The goal is to implement the circuit with fixed point arithmetics using a $B+1$ bit signed magnitude representation of the range $[-1, 1]$.

a) Assuming that no overflow occurs, what is the minimum value of B, i.e., the number of bits used for the magnitude representation, that has to be used in order to give a signal to noise ratio at the output $y[n]$ of at least 10 dB. The signal to noise ratio is defined as the ratio between the power of $y[n]$, when computed with infinite precision, and the power of the quantization noise added by the $B+1$ bit fixed point implementation. (6p)

b) Assume that $\left|x[n]\right| \le \gamma$ for all n. What is then the largest value of γ for which we can guarantee that no overflow occurs in the circuit? (4p)

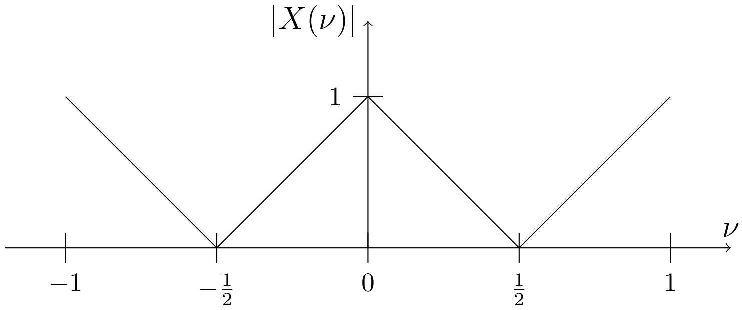
3. Consider the analysis part of a filter bank given by



where

$$H_0(\nu) \triangleq \begin{cases} 1 & |\nu| \leq \frac{1}{4} \\ 0 & \frac{1}{4} < |\nu| \leq \frac{1}{2} \end{cases} \qquad \text{and} \qquad H_1(\nu) \triangleq \begin{cases} 0 & |\nu| \leq \frac{1}{4} \\ 1 & \frac{1}{4} < |\nu| \leq \frac{1}{2} \end{cases}.$$

Let $x[n]$ have a discrete time Fourier transform given $X(\nu)$, with a magnitude given by the figure below.



Draw the magnitude of the discrete time Fourier transforms of $v_{00}[n]$, $v_{01}[n]$, and $v_1[n]$, i.e, $|V_{00}(\nu)|$, $|V_{01}(\nu)|$, and $|V_1(\nu)|$, as functions of ν over the range $\nu \in [-1, 1]$. Remember to label the all axes and add tick-marks on both axes where appropriate to illustrate relevant normalized frequencies ν and amplitudes (as done above). (10p)

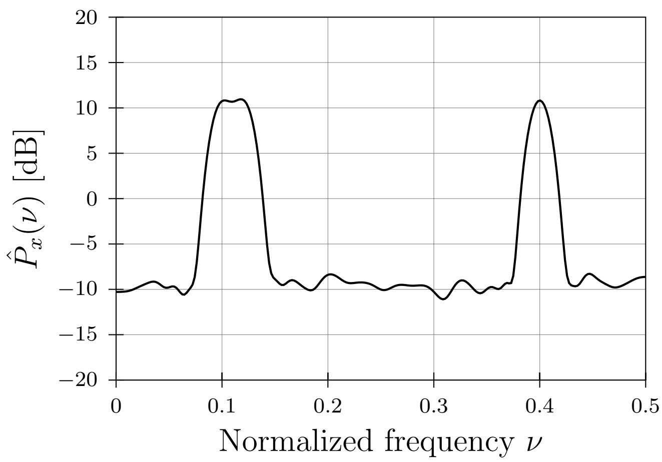
4. A discrete time signal $x[n]$ is given by

$$x[n] \triangleq \sum_{k=1}^{3} a \sin(2\pi\nu_k n + \phi_k) + v[n]$$

where $\nu_1 = 0.1$, $\nu_2 = 0.12$ and $\nu_3 = 0.4$, where $v[n]$ is zero mean white Gaussian noise of power σ^2, and where a is unknown. $N = 2048$ samples of $x[n]$ are stored in the vector `x`, and the spectrum of $x[n]$ is estimated using Welch's method with 50% overlap as in the following code, after replacing `???` with some chosen numbers.

```
N = 2048; R = 512; L = ???; D = ???;
K = floor((N-L)/D+1);
w = window('bartlett',L)'; U = 1/L*sum(abs(w).^2);
Ph = 0;
for k=0:K-1
    xk = x(k*D+1:k*D+L);
    Xk = fft(xk.*w,R);
    Ph = Ph + 1/(K*L*U)*abs(Xk).^2;
end;
```

The resulting spectrum estimate is shown below.



Note that you will, most likely, not be able to answer the questions below with exact values. It is important for evaluation that you carefully explain how you obtain your answers.

You may be helped by knowing that Matlab command `w = window('bartlett',L)` creates a length L (triangular) window that corresponds to the function

$$w[n] = \frac{2}{L-1}\begin{cases} n & 0 \le n \le \frac{L-1}{2} \\ L-1-n & \frac{L-1}{2} \le n \le L-1 \\ 0 & n < 0 \text{ or } n > L-1 \end{cases} .$$

a) Obtain an estimate of the noise power σ^2. (2p)

b) Provide an informed estimate of how `L` and `D` were chosen, i.e., specify what you think their values were and why. (3p)

c) Estimate the common amplitude a of the three sinusoids in $x[n]$. (5p)

5. From the DSP course we know that the ideal discrete-time low-pass filter with cutoff frequency ν_c has frequency and impulse responses given by

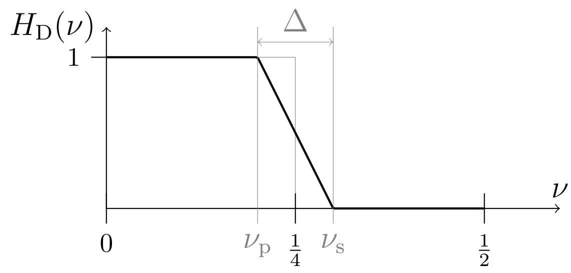
$$H_\mathrm{I}(\nu) = \begin{cases} 1 & |\nu| \le \nu_\mathrm{c} \\ 0 & \nu_\mathrm{c} < |\nu| \le \frac{1}{2} \end{cases} \qquad \text{and} \qquad h_\mathrm{I}(\nu) = \begin{cases} \frac{\sin(2\pi\nu_\mathrm{c} n)}{\pi n} & n \neq 0 \\ 2\nu_\mathrm{c} & n = 0 \end{cases}.$$

A filter with impulse response $h_\mathrm{I}[n]$ is not implementable, and truncating the filter leads to ripples in frequency domain (which is known as Gibb's phenomenon). One way to reduce the ripples is to multiply the impulse response with a window before truncating the impulse response. This leads to the windowed filter-design method. Another approach to reduce ripples is to change the desired response so that it transitions more smoothly from the pass-band to the stop-band.

To study a simple example of this, consider the desired frequency response given by

$$H_\mathrm{D}(\nu) = \begin{cases} 1 & |\nu| < \nu_\mathrm{p} \\ \frac{\nu_\mathrm{s} - \nu}{\nu_\mathrm{s} - \nu_\mathrm{p}} & \nu_\mathrm{p} \le |\nu| \le \nu_\mathrm{s} \\ 0 & \nu_\mathrm{s} < |\nu| \le \frac{1}{2} \end{cases}$$

where $0 \le \nu_\mathrm{p} < \nu_\mathrm{s} \le \frac{1}{2}$, where $\nu_\mathrm{c} = (\nu_\mathrm{p} + \nu_\mathrm{s})/2$, and where $\Delta = \nu_\mathrm{s} - \nu_\mathrm{p}$. The desired frequency response is illustrated below for the case where $\nu_\mathrm{p} = 0.2$ and $\nu_\mathrm{s} = 0.3$.



a) Show that the full impulse response corresponding to $H_\mathrm{D}(\nu)$ is given by (5p)

$$h_\mathrm{D}[n] = \begin{cases} \frac{\sin(\pi\Delta n)}{\pi\Delta n} \frac{\sin(2\pi\nu_\mathrm{c} n)}{\pi n} & n \neq 0 \\ 2\nu_\mathrm{c} & n = 0 \end{cases}.$$

b) For a given ν_c, let $\nu_\mathrm{p} = \nu_\mathrm{c} - \frac{\Delta}{2}$ and $\nu_\mathrm{p} = \nu_\mathrm{c} + \frac{\Delta}{2}$ where $\Delta > 0$, and let

$$h_\mathrm{I}^M[n] = \begin{cases} h_\mathrm{I}[n] & |n| \le M \\ 0 & |n| > M \end{cases} \qquad \text{and} \qquad h_\mathrm{D}^M[n] = \begin{cases} h_\mathrm{D}[n] & |n| \le M \\ 0 & |n| > M \end{cases}$$

be the respective truncated frequency response for some $M > 0$. Let $H_\mathrm{I}^M(\nu)$ and $H_\mathrm{D}^M(\nu)$ be the corresponding frequency responses, and define the mean square approximation errors in the frequency domain as

$$E_\mathrm{I}^M \triangleq \int_{-1/2}^{1/2} |H_\mathrm{I}^M(\nu) - H_\mathrm{I}(\nu)|^2 d\nu \qquad \text{and} \qquad E_\mathrm{D}^M \triangleq \int_{-1/2}^{1/2} |H_\mathrm{D}^M(\nu) - H_\mathrm{D}(\nu)|^2 d\nu\,.$$

Show that it always holds that $E_\mathrm{I}^M > E_\mathrm{D}^M$, i.e., $H_\mathrm{D}(\nu)$ is in a sense easier to approximate by an FIR filter than $H_\mathrm{I}(\nu)$. (5p)