SIGNAL PROCESSING

DEPARTMENT OF ELECTRICAL ENGINEERING

Digital Signal Processing EQ2300/ 2E1340

Final Examination 2009–09–15, 14.00–19.00
Sample Solutions

1. (a) The ratio between powers of sinusiod 1 and 3 (and 2 and 3) is $A_1^2/A_3^2 = 100 =$ 20dB. We thus need a window with sidelobe subpression of at least 20 dB (or 23 dB if you view the addition of sinusoid 1 and 2). The rectangular window can thus not be used (and would normally not be used in Welch's method anyway). Let's pick the Hanning window which gives sidelobes 12 dB below the weakest sinusoid. Sinusoid 1 and 2 are close so we need sufficiently good resolution to keep them appart. We have in particular (note that $\Delta\omega$ refers to angular frequency, and we should remove the 2π in Table 1)

$$\Delta f = 0.1 \geq \frac{1.44}{L}$$

where L is the length of the window. We thus need at least $L \geq 144$. We should probably pick it a little bit higher to get good separation between sinusoid 1 and 2. Note here that many choices are correct, the main issue is how the solution is motivated.

(b) As we have 3 sinusoids, the size of $\mathbf{R}_x$ should be 3+1=4 to get a noise subspace of dimension one, which is required by Pizarenkos method.

2. (a) The multiplications with a_1 and a_2 introduce quantization noise with power

$$\sigma_q^2 = \frac{2^{-2B}}{12}\,.$$

Note that the noise from a_1 pass through the whole filter, while the noise from the a_2 only pass through the second stage of the filter.
As

$$H(z) = \frac{1}{(1-az^{-1})(1+2az^{-1})} = \frac{1}{3(1-az^{-1})} + \frac{2}{3(1+2az^{-1})}$$

we get an impulse response of

$$h(n) = \frac{1}{3}a^{-n} + \frac{2}{3}(-2a)^{-n}$$

for $n \geq 0$, $h(0) = 0$ for $n < 0$. Further,

$$|h(n)|^2 = \left(\frac{1}{3}a^{-n} + \frac{2}{3}(-2a)^{-n}\right)^2 = \frac{17}{9}(a^2)^{-n} + \frac{4}{9}(-2a^2)^{-n}\,,$$

for $n \geq 0$ which implies

$$\sum_{k=0}^{\infty} |h(n)|^2 = \frac{17}{9}\frac{1}{1-a^2} + \frac{4}{9}\frac{1}{1+2a^2} = \frac{7+10a^2}{3(1-a^2)(1+2a^2)}$$

and

$$\sigma_1^2 = \sigma_q^2 \sum_{k=0}^{\infty} |h(n)|^2 = \frac{2^{-2B}}{12} \frac{7+10a^2}{3(1-a^2)(1+2a^2)}.$$

This expression does not depend on the order of a and $-2a$. The noise generated by multiplication by a_2 is (where $h_2(n)$ is the impulse response of the second stage)

$$\sigma_2^2 = \sigma_q^2 \sum_{k=0}^{\infty} |h_2(n)|^2 = \sigma_q^2 \sum_{k=0}^{\infty} (a_2^2)^n = \sigma_q^2 \frac{1}{1-a_2^2}.$$

Comparing $a_2 = -2a$ with $a_2 = a$ we see that $a_2 = a$ will lead to lower noise at the output from the second circuit. The total noise will then be

$$\sigma_1^2 + \sigma_2^2 = \frac{2^{-2B}}{12} \left(\frac{7+10a^2}{3(1-a^2)(1+2a^2)} + \frac{1}{1-a^2} \right) = \frac{2^{-2B}}{12} \frac{10+16a^2}{3(1-a^2)(1+2a^2)}.$$

3. (a) The optimal filter coefficients that would minimize the MSE are obtained from the Wiener-Hopf equation

$$W_{opt} = R_x^{-1} r_{zx}$$

computing

$$\begin{aligned} r_x(k) &= E[x(n)x(n-k)] \\ &= E[(v(n)+u(n))(v(n-k)+u(n-k))] \\ &= E[v(n)v(n-k)] + E[u(n)u(n-k)] \\ &= \delta(k) + \sigma_u^2\delta(k) = (1+\sigma_u^2)\delta(k) \end{aligned}$$

and computing r_{zx}

$$\begin{aligned} r_{zx}(k) &= E[z(n)x(n-k)] \\ &= E[(d(n)+g(n))(v(n-k)+u(n-k))] \\ &= E[g(n)v(n-k)] \end{aligned}$$

since the noise sources are independent of each other and the desired signal.

$$\begin{aligned} r_{zx}(k) &= E[(v(n)+0.75v(n-1))(v(n-k))] \\ &= \delta(k) + 0.75\delta(k-1) \end{aligned}$$

and so

$$R_x = \frac{1}{1+\sigma_u^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$r_{zx} = \begin{bmatrix} 1 \\ 0.75 \end{bmatrix}$$

therefore,

$$W_{opt} = (1+\sigma_u^2)^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.75 \end{bmatrix}$$

for σ_u^2=0.25.

$$W_{opt} = \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix}$$

The reason that the same filter minimize $E[|g(n)-y(n)|^2]$ is that $r_{zx}(k) = r_{gx}(k)$ which follows by the fact that $d(n)$ and $x(n)$ are independent.

(b) The corresponding minimum mean squared error is given by

$$\epsilon_{min} = r_z(0) - r_{zx}^T W_{opt} r_{zx} =$$

it is therefore necessary to compute the autocorrelation sequence of $z(n)$.

$$r_z(k) = E[(d(n) + g(n))(d(n-k) + g(n-k))]$$

$$r_z(k) = E[(d(n)d(n-k)] + E[g(n)g(n-k)] = r_d(k) + r_g(k)$$

since g(n) and d(n) are uncorrelated)

$$r_d(k) = E[(A\sin(2\pi fn + \phi))(A\sin(2\pi f(n-k) + \phi))] = 0.5A^2\cos(2\pi fk)$$

and

$$\begin{aligned} r_g(k) &= E[(v(n) + 0.75v(n-1))(v(n-k) + 0.75v(n-1-k))] \\ r_g(k) &= E[v(n)v(n-k)] + 0.75E[v(n-1)v(n-k)] + 0.75E[v(n)v(n-1-k)] \\ &\quad +0.75^2E[v(n-1)v(n-1-k)] \\ &= \delta(k) + 0.75\delta(k-1) + 0.75\delta(k+1) + 0.75^2\delta(k) \end{aligned}$$

and so,'

$$r_z(0) = r_d(0) + r_g(0) = 0.5A^2 + 1.56$$

therefore,

$$\epsilon_{min} = (0.5A^2 + 1.56) - r_{zx}^T W_{opt} = 0.5A^2 + 1.56 - 1.25$$

$$\epsilon_{min} = 0.5A^2 + 0.31$$

(c) The desired signal is corrupted by a noise which is uncorrelated to it. Since part of the reference input to the wiener filter is a noise signal generated from the same source as the corrupting noise in the desired signal,the best that the filter can do is to try to remove the signal component the is correlated to its reference signal.The optimal filter coefficients are therefore not dependent on the desired signal. The solution above would not change even if the desired signal is changed to two sinusoids (although the MSE in part b would).

4. (a) The DFT of $x_8(n)$ is seen to be

$$\begin{aligned} X_8(k) &= \sum_{n=0}^{7} x(n)e^{-j\frac{2\pi nk}{8}} \qquad\qquad k = 0, 1, ..., 7 \\ &= x(0) + x(4)e^{-j\pi k} \\ &= 1 + 2(-1)^k \end{aligned}$$

Hence, $X_8(k) = \{3, -1, 3, -1, 3, -1, 3, -1\}$.

(b) Based on DFT properties, we know that

$$x_8((n-l)_N) \Leftrightarrow e^{-j\frac{2\pi kl}{N}} \cdot X_8(k)$$

In this case, because $l = -2$, $x(n)$ is circularly shifted to the left by 2, and we have

$$G_8(k) = e^{j2k\frac{2\pi}{8}} X_8(k) \rightarrow g_8(n) = x_8((n+2)_8) = \{0, 0, 2, 0, 0, 0, 1, 0\}$$

(c) i. After zero padding, we obtain

$$x_5(n) = \{1, 0, 0, 0, 2\}$$

$$h_5(n) = \{1, 0, 1, 0, 0\}$$

Let

$$\begin{aligned} s_5(n) = x_5(n) ⑤ h_5(n) &= \sum_{k=0}^{4} x(k)h((n-k)_5) \\ &= x_5(0)h_5((n)_5) + x_5(4)h_5((n-4)_5) \end{aligned}$$

As $h((-n)_N) = h(N-n)$ for $n = 0, \ldots, N-1$, we have

$$\begin{aligned} s_5(0) &= x_5(0)h_5(0) + x_5(4)h_5(1) = 1 \\ s_5(1) &= x_5(0)h_5(1) + x_5(4)h_5(2) = 2 \\ s_5(2) &= x_5(0)h_5(2) + x_5(4)h_5(3) = 1 \\ s_5(3) &= x_5(0)h_5(3) + x_5(4)h_5(4) = 0 \\ s_5(4) &= x_5(0)h_5(0) + x_5(4)h_5(0) = 2\,. \end{aligned}$$

ii. In this case, the zero-padded signals are

$$x_8(n) = \{1, 0, 0, 0, 2, 0, 0, 0\}$$

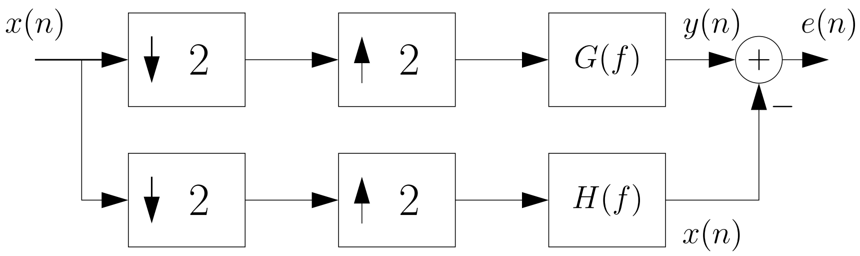
$$h_8(n) = \{1, 0, 1, 0, 0, 0, 0, 0\}$$

Proceeding in the previous task we get

$$w_8(n) = x_8(n) ⑧ h_8(n) = \{1, 0, 1, 0, 2, 0, 2, 0\}\,.$$

(d) As this computation is mathematically equivalent to circular convolution, the output is equal to $w_8(n)$ computed above.

5.



(a) Note that the error signal $e(n)$ can be obtained as the output of the above system where the ideal low-pass filter

$$H(f) = \begin{cases} 2 & |f| \le \frac{1}{4} \\ 0 & \frac{1}{4} < |f| \le \frac{1}{2} \end{cases}$$

leads to perfect reconstruction in the lower branch. By linearity, $e(n)$ may thus be viewed as the output after downsampling, upsamling and interpolation using a filter $G(f) - H(f)$. The power spectral density of the error is given by

$$R_{ee}(f) = \frac{1}{4}|G(f) - H(f)|^2 \Big(R_{xx}(f) + R_{xx}(f - \tfrac{1}{2})\Big)\,.$$

Using that $G(f) = 1 + \cos(2\pi f)$ and $R_{xx}(f) = \frac{A^2}{4}\delta(f - f_0)$ for $0 \le f \le \frac{1}{2}$ yields

$$R_{ee}(f) = \frac{1}{4}|1 - \cos(2\pi f)|^2 \tfrac{A^2}{4}\delta(f - f_0)$$

for $0 \le f \le \frac{1}{4}$ and

$$R_{ee}(f) = \frac{1}{4}|1 + \cos(2\pi f)|^2 \tfrac{A^2}{4}\delta(f - \frac{1}{2} + f_0)$$

for $\frac{1}{4} < f \le \frac{1}{2}$ which implies that

$$P_e = 2\int_0^{\frac{1}{2}} R_{ee}(f)df = \tfrac{A^2}{8}\Big[|1-\cos(2\pi f_0)|^2 + |1+\cos(2\pi(\tfrac{1}{2}-f_0))|^2\Big] = \tfrac{A^2}{4}|1-\cos(2\pi f_0)|^2 \,.$$

As $P_x = \frac{A^2}{2}$ we finally obtain

$$\mathrm{SNR} = \frac{2}{|1 - \cos(2\pi f_0)|^2} \,.$$

(b) It is seen from the above that $\mathrm{SNR} \to \infty$ when $f_0 \to 0$ as $\cos(2\pi f_0) = 1$ when $f_0 = 0$. This is intuitive since the lower the frequency f_0 is, the more constant (or slowly varying) $x(n)$ becomes, and it follows that $x(n)$ can be nearly perfectly reconstructed by linear interpolation in this case.