SIGNAL PROCESSING

SCHOOL OF ELECTRICAL ENGINEERING

Digital Signal Processing EQ2300 / 2E1340

Final Examination 2015–01–14, 08.00–13.00 Examples of Solutions

1. a) **iv**. The magnitude of b_0^2 is the mean squared modeling error, and constitutes a basis for model selection criteria like AIC or MDL. See also the answer for b).

b) **ii**. The power of the error of the MMSE optimal FIR one step ahead predictor is the same as the b_0^2 coefficient in the AR model, and

$$b_0^2 = r_x[0] + \sum_{k=1}^{N} a_k r_x[k]$$

where $h[k] = -a_{k+1}$ follows from the connection between the Wiener-Hopf and Yule-Walker equations.

c) The second circuit is nothing more than the polyphase implementation of the former circuit. This can be seen from previous knowledge of QMF banks (e.g., from the video lectures) or just by studying the part leading up to $v_0[m]$, which is the standard polyphase implementation of $H(z)$ followed by the downsampling. Thus, $p_0[n] = h[2n]$ and $p_1[n] = h[2n+1]$.

d) The formula for the number of complex valued multiplication can be written as

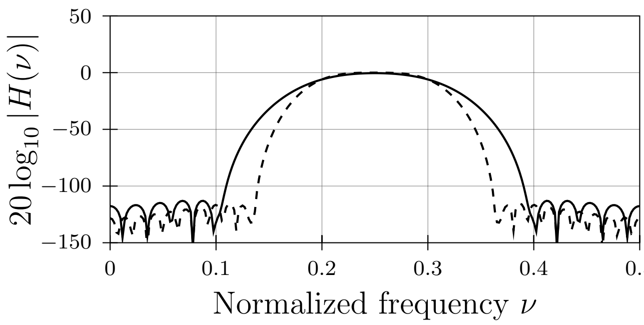
$$C = \frac{2 \times \frac{N}{2}\log_2 N + N}{N - M + 1}$$

which makes explicit the factor $\frac{N}{2}\log_2 N$, which is the complexity of the FFT as well as of the the inverse FFT. Replacing this with N^2, which is the complexity of the direct DFT or IDFT computation, yields

$$C = \frac{2N^2 + N}{N - M + 1} = \frac{N(2N+1)}{N - M + 1},$$

which is higher than the direct computation of the linear convolution.

2. a) The magnitude response of the filter will be given by



which corresponds to

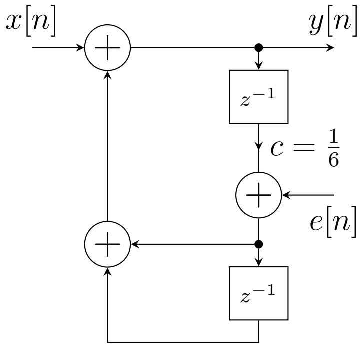
$$h[n] = h_{\mathrm{I}}[n]w[n] \quad \Leftrightarrow \quad H(\nu) = H_{\mathrm{I}}(\nu) * W(\nu) = \int_0^1 H_{\mathrm{I}}(\nu - \xi)W(\xi)d\xi\,.$$

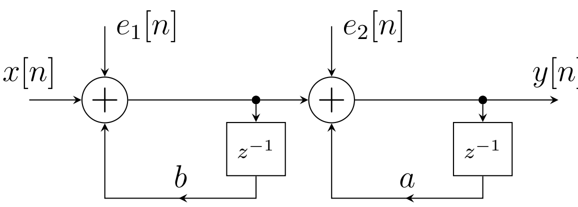
If the filter was made longer the width of the main lobe of the window would become narrower and the transition from pass band to stop band would be more rapid, as shown by the dashed curve illustrating a length 61 FIR filter.

b) The designed filter will be a linear phase FIR filter with a group delay of $\tau = (41-1)/2 = 20$. As ν_0 is in the middle of the passband the amplitude would not be seriously affected, and the output would be $y[n] \approx \sin(2\pi\nu_0(n-20)+\phi_0)$.

c) Using the complexity per sample formula for overlap add from Problem 1, we get that C_N (the complexity when using a length N FFT satisfies $C_{64} \approx 18.8$, $C_{128} \approx 11.6$, $C_{256} \approx 10.7$ and $C_{512} \approx 10.8$. It would thus make sense to pick $N = 256$ as this yields the lowest complexity. Note also that we only consider N on the form $N = 2^p$ as this is required for the basic radix-2 FFT algorithm.

3. The quantization noise of the two circuits is modeled as shown below





,

where $e[n], e_1[n]$ and $e_2[n]$ are white uncorrelated uniform processes with variance $\sigma_e^2 = \frac{2^{-2B}}{12}$.

a) The transfer function implemented by the first system is easily calculated in the z-transform domain

$$Y(z) = X(z) + cz^{-1}\left(Y(z) + z^{-1}Y(z)\right) \Rightarrow Y(z)\left(1 - cz^{-1} - cz^{-2}\right) = X(z)$$

$$H(z) = \frac{1}{1 - cz^{-1} - cz^{-2}}\,.$$

However, the noise input and the signal input do not happen at the same point of the circuit. From the noise input to the output the transfer function can be analogously calculated, and it results in

$$H_e(z) = \frac{1 + z^{-1}}{1 - cz^{-1} - cz^{-2}} = \frac{1}{D(z)}(1 + z^{-1})\,,$$

where $D(z) \triangleq 1 - cz^{-1} - cz^{-2}$. This was to be expected because two different versions of $e[n]$ get to the output point, the one that goes directly towards the sum, and the one that gets delayed first. Splitting the second order denominator into two first order denominators (for $c = \frac{1}{6}$) yields

$$\frac{1}{D(z)} = \frac{1}{1 - 1/6z^{-1} - 1/6z^{-2}} = \frac{3/5}{1 - 1/2z^{-1}} + \frac{2/5}{1 + 1/3z^{-1}}\,,$$

and by taking the inverse z-transform of $H_e(z)$ to find $h_e[n]$ (the impulse response from the quantization noise to the output) we obtain

$$h_e[n] = \frac{3}{5}\left[\left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{2}\right)^{n-1} u[n-1]\right] + \frac{2}{5}\left[\left(-\frac{1}{3}\right)^n u[n] + \left(-\frac{1}{3}\right)^{n-1} u[n-1]\right]$$

which can be simplified as $h_e[0] = 1$ and

$$h_e[n] = \frac{9}{10}\left(\frac{1}{2}\right)^{n-1} + \frac{4}{15}\left(-\frac{1}{3}\right)^{n-1} \quad \text{for } n \geq 1.$$

Finally, we can obtain

$$\sum_{n=0}^{\infty} h_e^2[n] = h_e^2[0] + \sum_{n=1}^{\infty} h_e^2[n] = 1 + \left(\frac{9}{10}\right)^2 \frac{1}{1-1/4} + \left(\frac{4}{15}\right)^2 \frac{1}{1-1/9} + 2\frac{9}{10}\frac{4}{15}\frac{1}{1+1/6}$$

which simplifies to

$$\sum_{n=0}^{\infty} h_e^2[n] = \frac{18}{7} \approx 2.57\,.$$

The noise power at the output is thus given by

$$\sigma_a^2 = \frac{2^{-2B}}{12}\frac{18}{7}\,.$$

b) The second circuit implements the transfer function resulting from cascading two first order AR blocks, i.e.

$$H(z) = \frac{1}{1-bz^{-1}}\frac{1}{1-az^{-1}}\,.$$

Therefore, for both circuits to implement the same function the constants have to be given by the poles in the second order system. Thus, $\{a, b\} = \{1/2, -1/3\}$, this is, either $(a, b) = (1/2, -1/3)$ or $(a, b) = (-1/3, 1/2)$. The impulse response from $e_1[n]$ to $y[n]$ is the same as the one from $x[n]$ to $y[n]$, i.e. the inverse z-transform of $H(z) = \frac{1}{D(z)}$ [see computations for part a)]. Therefore, its expression is

$$h_{e_1}[n] = \frac{3}{5}\left(\frac{1}{2}\right)^n u[n] + \frac{2}{5}\left(-\frac{1}{3}\right)^n u[n]$$

regardless of how we choose the values of a and b. However the impulse response from $e_2[n]$ to $y[n]$ is given by $h_{e_2}[n] = a^n u[n]$ and we have

$$\sum_{n=0}^{\infty} h_{e_2}^2[n] = \frac{1}{1-a^2} = \left\{a = -\frac{1}{3}\right\} = \frac{1}{1-1/9}$$

where the last equality is obtain by assigning $a = -1/3$. If we had chosen $a = 1/2$ we would have gotten a larger value. Continuing with $h_{e_1}[n]$ we obtain

$$\sum_{n=0}^{\infty} h_{e_1}^2[n] = \cdots = \frac{15}{14}$$

by a similar computation as in part a). We get an overall noise power at the output which is

$$\sigma_b^2 = \frac{2^{-2B}}{12}\left(\sum_{n=0}^{\infty} h_{e_1}^2[n] + \sum_{n=0}^{\infty} h_{e_2}^2[n]\right) = \frac{2^{-2B}}{12} \times \frac{123}{56}$$

where

$$\frac{123}{56} \approx 2.19\,.$$

c) The circuit in b) yields a smaller overall fixed point quantization noise power at the output, giving $\sigma_b^2 \approx 2.19\sigma_e^2$ instead of the worse, larger value obtained in the circuit in a) ($\sigma_a^2 \approx 2.57\sigma_e^2$).

4. a) The code implements Welch's spectrum estimator with a block length $L = 128$, a Hamming window, and 50% overlap ($D = 64 = L/2$).

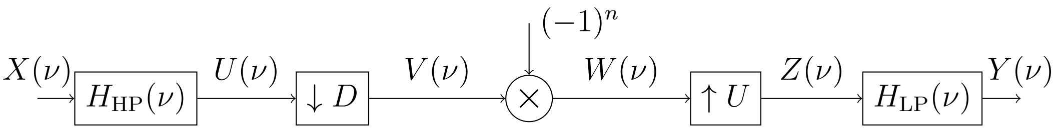
b) The spectrum estimate is computed for $\nu = \nu_k = k/512$ for $k = 0, \ldots, 511$.

c) In order to improve the resolution by a factor of two (reduce $\Delta\nu_{3\text{ dB}}$ from 0.01 to 0.005) one needs to double the block length L, i.e., set $L = 256$. To maintain the 50% overlap one needs to set $D = 128$. The rest of the code can be as it was.

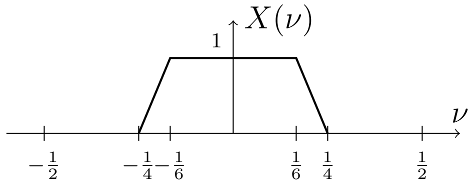
This change would roughly halve the number of blocks, which would increase the variance by a factor of 2. Being a bit more precise we can see that we would decrease the number of blocks from $K = 31$ to $K = 15$, so the variance would more precisely increase by a factor closer to $31/15 \approx 2$.

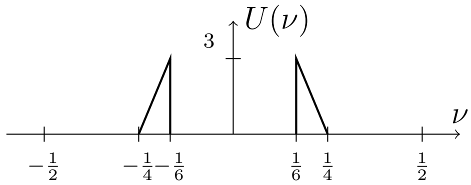
d) It is the window that mainly determines the spectral leakage so we would need to change the window to a window with lower side lobes. The Blackman window would be a candidate.

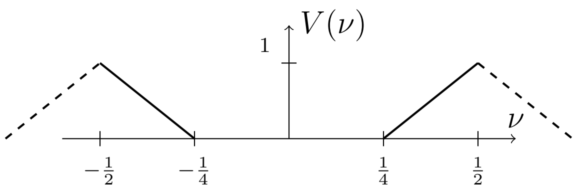
e) The spectrum appears to have 3 distinct peaks in the range $\nu \in [0, 1/2]$, or 6 in the range $\nu \in [0, 1]$. In order to model this we need a 6th order AR model.

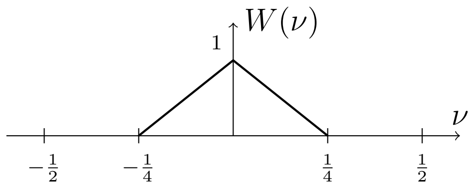


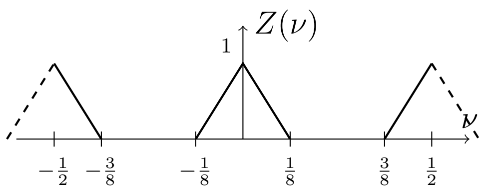
5. The circuit above will do the trick, with $D = 3$, $U = 2$, $\nu_{\mathrm{HP}} = \frac{1}{6}$, $c_{\mathrm{HP}} = 3$, $\nu_{\mathrm{LP}} = \frac{1}{8}$, and $c_{\mathrm{LP}} = 1$. This yields the following intermediate signals.

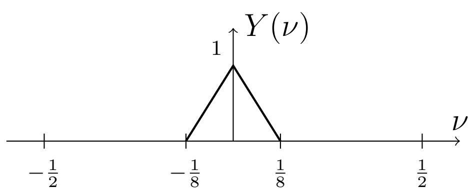












Note that there are many choices of c_{HP} and c_{LP}. The proposed solution will work as long as $c_{\mathrm{HP}} c_{\mathrm{LP}} = 3$, albeit with different intermediate signal. We can also choose ν_{LP} anywhere between $\frac{1}{8}$ and $\frac{3}{8}$ to remove the last high frequency components in $Z(\nu)$.