EQ2330 Image and Video Processing
Tutorial #4: Image Restoration

Linghui Zhou
linghui@kth.se

Change schedule of Tutorial on 13rd, 20th, 27th Nov

Recap

Cross Correlation

- Cross correlation $\phi_{fg}(m,n) = \mathbb{E}\{f(x+m, y+n)g^*(x,y)\}$.
- Special case: Autocorrelation $\phi_{ff}(m,n) = \mathbb{E}\{f(x+m, y+n)f^*(x,y)\}$.

Cross Spectrum

- Cross Spectral Density: Fourier transform of cross correlation

$$\Phi_{fg} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \phi_{fg}(m,n) e^{-jw_x m - jw_y n}$$

- Special case: Power Spectral Density

$$\Phi_{ff} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \phi_{ff}(m,n) e^{-jw_x m - jw_y n}$$

Recap

Properties of Cross Spectra Density

- If $g(x, y) = f(x, y) * h(x, y)$, then
  - $\Phi_{gg}(w_x, w_y) = |H(w_x, w_y)|^2 \Phi_{ff}(w_x, w_y)$.
  - $\Phi_{sg}(w_x, w_y) = \Phi_{sf}(w_x, w_y) H^*(w_x, w_y)$.
- If $f(x, y)$ and $n(x, y)$ are uncorrelated, and $\mathbb{E}(n(x, y)) = 0$, then
  - $\phi_{fg}(m, n) = \mathbb{E}\{f(x+m, y+n) n^*(x, y)\} = \mathbb{E}\{f(x+m, y+n)\}\mathbb{E}\{n^*(x, y)\} = 0$
  - $\Phi_{fg}(m, n) = 0$
  - Message: If two signals are uncorrelated and one of them has zero mean, then their cross spectra density is 0.

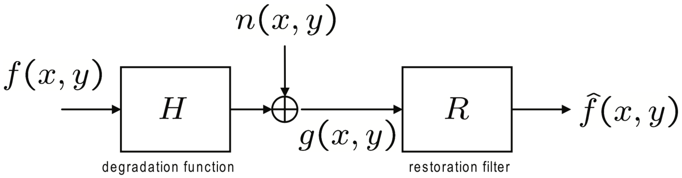
Recap

White Gaussian Noise $n(x, y) \sim \mathcal{N}(0, \sigma^2)$

- The values at any pair of time instant (or spatial location in the scenario of images) are i.i.d. (identically and independently distributed). And hence uncorrelated.
- Auto correlation function: $\phi_{nn}(m, n) = \mathbb{E}\{n(x+m, y+n)n^*(x, y)\}$.
  - When $(m, n) \neq (0, 0)$, $\phi_{nn}(m, n) = \mathbb{E}\{n(x+m, y+n)\}\mathbb{E}\{n^*(x, y)\} = 0$.
  - When $(m, n) = (0, 0)$, $\phi_{nn}(m, n) = \mathbb{E}\{n(x, y)n^*(x, y)\} = \sigma^2$.
  - $\phi_{nn}(m, n) = \sigma^2\delta(m, n)$.
- Power Spectral Density: $\Phi_{nn}(w_x, w_y) = \phi_{nn}(0, 0)e^{-jw_x 0 - jw_y 0} = \sigma^2$.

Recap

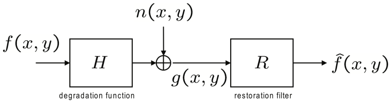
Restoration filter



- Time domain: $g(x, y) = f(x, y) * h(x, y) + n(x, y)$.
- Frequency domain: $G(u, v) = F(u, v)H(u, v) + N(u, v)$.
- Restoration filter options: Inverse filter, Wiener filter, Constrained Least Square filter, etc.

Recap

Restoration filter



- Inverse filter: $R_I(u,v) = \frac{1}{H(u,v)}$.
- Wiener filter: $R_W = \frac{H^*(u,v)}{|H(u,v)|^2 + K}$, where $K = \frac{|N(u,v)|^2}{|F(u,v)|^2}$.
- Constrained Least Square filter: $R_C(u,v) = \frac{H^*(u,v)}{|H(u,v)|^2 + \gamma |P(u,v)|^2}$, where $P(u,v) = \mathcal{F}\left\{\begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}\right\}$ and γ is a parameter that can be adjusted (according to the visual perception).

Problem 5.16 (Convolution and Impulse Function)

Given $h(x-\alpha, y-\beta) = e^{-\left[(x-\alpha)^2+(y-\beta)^2\right]}$ and $f(x,y) = \delta(x-a)$. What is $g(x,y)$?
Hint: $g(x,y) = f(x,y) \star h(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)h(x-\alpha, y-\beta)d\alpha d\beta$.

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Ans:

$$\begin{aligned}
g(x,y) = f(x,y) \star h(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)h(x-\alpha, y-\beta)d\alpha d\beta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(\alpha-a)e^{-\left[(x-\alpha)^2+(y-\beta)^2\right]}d\alpha d\beta \\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} \delta(\alpha-a)e^{-(x-\alpha)^2}d\alpha\right] e^{-(y-\beta)^2}d\beta \\
&= \int_{-\infty}^{\infty} e^{-(x-a)^2}e^{-(y-\beta)^2}d\beta = \sqrt{\pi}e^{-(x-a)^2}.
\end{aligned}$$

Problem 5.18 (Image Blurring by Uniform Acceleration)

Hint: Section 5.6.3 in Textbook.

- Suppose an image $f(x, y)$ undergoes motion $x_0(t)$ and $y_0(t)$ during time $[0, T]$ in x and y directions, respectively.
- Then the output image is $g(x, y) = \int_0^T f(x - x_0, y - y_0)dt$.
- The Fourier transform is $G(u, v) = F(u, v) \int_0^T e^{-j2\pi[ux_0(t)+vy_0(t)]}dt$.
- Define the blurring/degradation function $H(u, v) = \int_0^T e^{-j2\pi[ux_0(t)+vy_0(t)]}dt$.

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Ans:

$$
\begin{aligned}
H(u, v) &= \int_0^T e^{-j2\pi ux_0(t)}dt = \int_0^T e^{-j\pi uat^2}dt \\
&= \int_0^T cos(\pi uat^2) + jsin(\pi uat^2)dt \\
&= \frac{1}{\sqrt{\pi ua}}\left[\int_0^{\sqrt{\pi ua}T} cos(t^2)dt - j\int_0^{\sqrt{\pi ua}T} sin(t^2)dt\right]
\end{aligned}
$$

Problem 5.21 (Derivative and Fourier Transform)

$h(x,y) = \frac{x^2+y^2-2\sigma^2}{\sigma^4} e^{-\frac{x^2+y^2}{2\sigma^2}}$, show $H(u,v) = -8\pi^3\sigma^2(u^2+v^2)e^{-2\pi^2\sigma^2(u^2+v^2)}$.

Hint 1: Consider $s(x,y) = e^{-\frac{x^2+y^2}{2\sigma^2}}$.

Hint 2: $\mathcal{F}\{\frac{\partial^m s(x,y)}{\partial x^m} \frac{\partial^n s(x,y)}{\partial y^n}\} = (j2\pi u)^m (j2\pi v)^n S(u,v)$

Hint 3: $\mathcal{F}\{e^{-\frac{x^2+y^2}{2\sigma^2}}\} = 2\pi\sigma^2 e^{-2\pi^2\sigma^2(u^2+v^2)}$ (Exercise 3)

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Ans:

- $\frac{\partial s(x,y)}{\partial x} = -\frac{2x}{2\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$
- $\frac{\partial^2 s(x,y)}{\partial x^2} = -\frac{1}{\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} - \frac{x}{\sigma^2}(-\frac{2x}{2\sigma^2})e^{-\frac{x^2+y^2}{2\sigma^2}} = (-\frac{1}{\sigma^2} + \frac{x^2}{\sigma^4})e^{-\frac{x^2+y^2}{2\sigma^2}}$
- $\nabla^2 s(x,y) = \frac{\partial^2 s(x,y)}{\partial x^2} + \frac{\partial^2 s(x,y)}{\partial y^2} = (\frac{x^2+y^2-2\sigma^2}{\sigma^4})e^{-\frac{x^2+y^2}{2\sigma^2}} = h(x,y)$
- $\mathcal{F}\{h(x,y)\} = [(j2\pi u)^2 + (j2\pi v)^2]\mathcal{F}\{s(x,y)\} =$ $-8\pi^3\sigma^2(u^2+v^2)e^{-2\pi^2\sigma^2(u^2+v^2)}$

Problem 5.22 (Wiener Filter)

Given $H(u, v)$, find the expression for the Wiener Filter.

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Ans:
The Wiener Filter is:

$$H_w(u, v) = \frac{H^*(u, v)}{|H(u, v)|^2 + K},$$

where K is the ratio of power spectra of the noise and the undergraded signal.

Problem 5.23 (Constrained Least Square Filter)

Given $H(u, v)$, find the expression for the Constrained Least Square Filter.

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Ans:

The Constrained Least Square Filter is:

$$H_c(u, v) = \frac{H^*(u, v)}{|H(u, v)|^2 + \gamma |P(u, v)|^2},$$

where $P(u, v) = \mathcal{F}\left\{ \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix} \right\}$ and γ is a parameter that can be adjusted.

Problem 5.24

Given the system model, show $|G(u,v)|^2 = |H(u,v)|^2|F(u,v)|^2 + |N(u,v)|^2$
Hint: $|G(u,v)|^2 = G(u,v)G^*(u,v)$.

Problem 5.24

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Hint: $|G(u,v)|^2 = G(u,v)G^*(u,v)$.

Ans:

Firstly, we have $G(u,v) = H(u,v)F(u,v) + N(u,v)$. Then we obtain

$$\begin{aligned}
|G(u,v)|^2 &= G(u,v)G^*(u,v) \\
&= [H(u,v)F(u,v) + N(u,v)][H(u,v)F(u,v) + N(u,v)]^* \\
&= [H(u,v)F(u,v) + N(u,v)][H^*(u,v)F^*(u,v) + N^*(u,v)] \\
&= H(u,v)F(u,v)H^*(u,v)F^*(u,v) + H(u,v)F(u,v)N^*(u,v) \\
&\qquad + N(u,v)H^*(u,v)F^*(u,v) + N(u,v)N^*(u,v) \\
&\overset{(a)}{=} |H(u,v)|^2|F(u,v)|^2 + |N(u,v)|^2.
\end{aligned}$$

where (a) holds since the image and noise are uncorrelated,

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- Given: Original camera, representation coins, image blurred due to out of focus.
- Objective: Restore the blurred image.

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Problem 5.27 (Image Restoration Example)

- Given: Original camera, representation coins, image blurred due to out of focus.
- Objective: Restore the blurred image.
- How? Hint: recall the formulation of Wiener filter, Constrained least square filter, etc.
- Idea: Estimate the degradation function, then apply filter operation.

Problem 5.27 (Image Restoration Example)

The principal steps are as follows:

1. Select coins as close as possible in size and content as the lost coins. Select a background that approximates the texture and brightness of the photos of the lost coins.

2. Set up the museum photographic camera in a geometry as close as possible to give images that resemble the images of the lost coins (this includes paying attention to illumination). Obtain a few test photos. To simplify experimentation, obtain a TV camera capable of giving images that resemble the test photos. This can be done by connecting the camera to an image processing system and generating digital images, which will be used in the experiment.

3. Obtain sets of images of each coin with different lens settings. The resulting images should approximate the aspect angle, size (in relation to the area occupied by the background), and blur of the photos of the lost coins.

4. The lens setting for each image in (3) is a model of the blurring process for the corresponding image of a lost coin. For each such setting, remove the coin and background and replace them with a small, bright dot on a uniform background, or other mechanism to approximate an impulse of light. Digitize the impulse. Its Fourier transform is the transfer function of the blurring process.

5. Digitize each (blurred) photo of a lost coin, and obtain its Fourier transform. At this point, we have $H(u, v)$ and $G(u, v)$ for each coin.

6. Obtain an approximation to $F(u, v)$ by using a Wiener filter. Equation (5.8-3) is particularly attractive because it gives an additional degree of freedom (K) for experimenting.