EQ2330 Image and Video Processing
Tutorial #3: Frequency Domain Filtering

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Recap from Tutorial #2

Image Bluring & Sharpening

- Usually a signal/image has the majority of its energy in the low- and mid-frequency range of the amplitude spectrum.
- At higher frequencies, the information of interest is often blurred by noise.
- Thus, a filter that reduced high-frequency components can reduce the visible effects of noise.
- Blurring V.S. Sharpening
  - Low-pass filter: Reduces high freq. components/Spatial blurring/Denoising.
  - High-pass filter: Addresses high freq. components/Sharpening/Increase noise.

Recap from Tutorial #2

Image Denoising Examples

- Average filtering: $\frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, or $\frac{1}{5}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, or $\frac{1}{16}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$
- Wiener filtering (Minimize mean square error filtering).

Image Sharpening Examples:

- Unsharp masking:
  - Subtract blurred image from original image, to obtain the "high frequency/edges/sharpened" parts (called unsharp mask)
  - Add this mask to original image.
- Laplacian operator:
  - Subtract Laplacian operator from original image.
  - Add the residual to the original image. Then the image can be sharpened.

Recap from Tutorial #2

DFT & Convolution Theorem

- Discrete Fourier Transform:
  - $F(u) = \sum_{x=0}^{M} f(x)e^{-j2\pi ux/M}, u \in [0, M-1]$.
  - $f(x) = \sum_{x=0}^{m} F(u)e^{j2\pi ux/M}, x \in [0, M-1]$.
- Convolution Theorem:
  - $\mathcal{F}\{f(t)h(t)\} = F(u) \star H(u)$.
  - $\mathcal{F}\{f(t) \star h(t)\} = F(u)H(u)$.
- DC Component:
  - $F(0) = \sum_{x=0}^{M} f(x) = M\frac{1}{M}\sum_{x=0}^{M} f(x) = M\bar{f}(x)$.
  - If a filter has DC component $H(0) = 0$, due to convolutional theorem, the output image has DC component $G(0) = H(0)F(0) = 0 \times F(0) = 0$. And $G(0) = M\bar{g}(x)$, thus $\bar{g}(x) = 0$.

This Tutorial

- Discrete Fourier Transform
- Convolution theorem.
- Low-pass filter reduces noise, and high-pass filter sharpen images.
- Zero padding to avoid wrap-around error.

Exercise # 3: Problem 4.3

Hint: Firstly find $\mathcal{F}^{-1}\{H(u,v)\}$, and then find $\mathcal{F}^{-1}\{H(u,v)F(u,v)\}$ using convolution theorem.

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Ans:

- The inverse Fourier transform of $H(u,v) = A$ is $h(x,y) = A\delta(x,y)$.
- Then from convolution theorem, we obtain that

$$\begin{aligned}\mathcal{F}^{-1}\{H(u,v)F(u,v)\} &= h(x,y) \star f(x,y)\\ &= A\delta(x,y) \star f(x,y)\\ &= Af(x,y).\end{aligned}$$

Exercise #3: Problem 4.11

Prove convolution theorem of continuous case, i.e.,
$\mathcal{F}\{f(x) \star h(x)\} = H(u)F(u)$ and $\mathcal{F}\{f(x)h(x)\} = H(u) \star F(u)$.

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Ans:

$$
\begin{aligned}
\mathcal{F}\{f(x) \star h(x)\} &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\alpha)h(x-\alpha)d\alpha \right] e^{-j2\pi ux} dx \\
&= \int_{-\infty}^{\infty} f(\alpha) \left[\int_{-\infty}^{\infty} h(x-\alpha)e^{-j2\pi ux} dx \right] d\alpha \\
&= \int_{-\infty}^{\infty} f(\alpha) \left[\int_{-\infty}^{\infty} h(x-\alpha)e^{-j2\pi u(x-\alpha)} dx \right] e^{-j2\pi u\alpha} d\alpha \\
&= \int_{-\infty}^{\infty} f(\alpha)H(u)e^{-j2\pi u\alpha} d\alpha \\
&= H(u) \int_{-\infty}^{\infty} f(\alpha)e^{-j2\pi u\alpha} d\alpha \\
&= H(u)F(u).
\end{aligned}
$$

Exercise #3: Problem 4.11 (cont.)

Ans: (cont.)

$$
\begin{aligned}
\mathcal{F}^{-1}\{F(u) \star H(u)\} &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} F(\alpha) H(u-\alpha) d\alpha \right] e^{j2\pi ux} du \\
&= \int_{-\infty}^{\infty} F(\alpha) \left[\int_{-\infty}^{\infty} H(u-\alpha) e^{j2\pi ux} du \right] d\alpha \\
&= \int_{-\infty}^{\infty} F(\alpha) \left[\int_{-\infty}^{\infty} H(u-\alpha) e^{j2\pi x(u-\alpha)} du \right] e^{j2\pi x\alpha} d\alpha \\
&= \int_{-\infty}^{\infty} F(\alpha) h(x) e^{j2\pi x\alpha} d\alpha \\
&= h(x) \int_{-\infty}^{\infty} F(\alpha) e^{j2\pi x\alpha} d\alpha \\
&= h(x) f(x).
\end{aligned}
$$

Exercise #3: Problem 4.14

(i) Hint: $f(x - x_0, y - y_0) \Leftrightarrow F(u, v)e^{-j2\pi(\frac{ux_0}{M} + \frac{vy_0}{N})}$

Exercise #3: Problem 4.14

(i) Hint: $f(x - x_0, y - y_0) \Leftrightarrow F(u, v)e^{-j2\pi(\frac{ux_0}{M} + \frac{vy_0}{N})}$

Ans:

The average of four immediate neighbors of $f(x, y)$ is
$g(x, y) = \frac{1}{4}[f(x+1, y) + f(x-1, y) + f(x, y+1) + f(x, y-1)]$.
The Fourier transform of $g(x, y)$ is

$$\begin{aligned} G(u, v) &= \frac{1}{4}F(u, v)\left(e^{j2\pi\frac{u}{M}} + e^{-j2\pi\frac{u}{M}} + e^{j2\pi\frac{v}{N}} + e^{-j2\pi\frac{v}{N}}\right) \\ &= \frac{1}{2}F(u, v)\left(cos2\pi\frac{u}{M} + cos2\pi\frac{v}{N}\right). \end{aligned}$$

Therefore the equivalent filter is $H(u, v) = \frac{1}{2}\left(cos2\pi\frac{u}{M} + cos2\pi\frac{v}{N}\right)$.

Exercise #3: Problem 4.14

(ii) Hint: Consider one variable first, and then generalize to two variables.

Exercise #3: Problem 4.14

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Ans:

Given $H(u,v) = \frac{1}{2}\left(cos2\pi\frac{u}{M} + cos2\pi\frac{v}{N}\right)$.

- Consider u. When $u = 0$, $|\frac{1}{2}cos2\pi\frac{u}{M}| = \frac{1}{2}$. When $u = \frac{M}{4}$, $|2jsin\left(\frac{\pi u}{M}\right)e^{j\pi\frac{u}{M}}| = 0$.
- The above argument holds for v.
- The filter takes value 1 at the origin $(0,0)$ and decreases as the distance from the origin increases, which is the characteristic of a lowpass filter.

Exercise #3: Problem 4.15

(i) Hint: Consider first order derivatives in both x direction y direction.

Ans:
Consider the sum of first derivative in in both x direction y direction

$$g(x, y) = f(x+1, y) - f(x, y) + f(x, y+1) - f(x, y).$$

The corresponding Fourier transform is

$$G(u, v) = F(u, v) \left(e^{j2\pi \frac{u}{M}} - 1 + e^{j2\pi \frac{v}{N}} - 1 \right).$$

Exercise #1: Problem 4.15

Ans: (cont.)

Therefore the equivalent filter function $H(u,v)$ in the frequency domain is

$$
\begin{aligned}
H(u,v) &= e^{j2\pi\frac{u}{M}} - 1 + e^{j2\pi\frac{v}{N}} - 1 \\
&= e^{j\pi\frac{u}{M}}\left(e^{j\pi\frac{u}{M}} - e^{-j\pi\frac{u}{M}}\right) + e^{j\pi\frac{v}{N}}\left(e^{j\pi\frac{v}{N}} - e^{-j\pi\frac{v}{N}}\right) \\
&= 2jsin\left(\frac{\pi u}{M}\right)e^{j\pi\frac{u}{M}} + 2jsin\left(\frac{\pi v}{N}\right)e^{j\pi\frac{v}{N}}.
\end{aligned}
$$

Exercise #1: Problem 4.15

(ii) Hint: Consider one variable first, and then generalize to two variables.

Exercise #1: Problem 4.15

(ii) Hint: Consider one variable first, and then generalize to two variables.

Ans:

Given $H(u,v) = 2jsin\left(\frac{\pi u}{M}\right) e^{j\pi\frac{u}{M}} + 2jsin\left(\frac{\pi v}{N}\right) e^{j\pi\frac{v}{N}}$.

- Consider u. When $u = 0$, $|2jsin\left(\frac{\pi u}{M}\right) e^{j\pi\frac{u}{M}}| = 0$. When $u = \frac{M}{2}$, $|2jsin\left(\frac{\pi u}{M}\right) e^{j\pi\frac{u}{M}}| = 2$.
- The above argument holds for v.
- The filter takes value 0 at the origin $(0,0)$ and increases as the distance from the origin increases, which is the characteristic of a highpass filter.

Exercise #3: Problem 4.4

Hint: Consider one variable first, and then generalize to two variables.

Exercise #3: Problem 4.4

Hint: Consider one variable first, and then generalize to two variables.

Ans:

We start with one variable and show find the inverse Fourier transform of $H(u) = e^{\frac{u^2}{2\sigma^2}}$ as follows

$$
\begin{aligned}
h(x) &= \int_{-\infty}^{\infty} e^{\frac{u^2}{2\sigma^2}} e^{j2\pi ux} du \\
&= \int_{-\infty}^{\infty} e^{\frac{(u-j2\pi\sigma^2 x)^2}{2\sigma^2}} e^{-\frac{(2\pi)^2\sigma^2 x^2}{2}} du \\
&= \left[\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{\frac{(u-j2\pi\sigma^2 x)^2}{2\sigma^2}} du \right] \sqrt{2\pi}\sigma e^{-\frac{(2\pi)^2\sigma^2 x^2}{2}} \\
&= \sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2 x^2}.
\end{aligned}
$$

Exercise #3: Problem 4.4

Ans: (cont.)

Now we consider two-dimension case as follows

$$
\begin{aligned}
h(x,y) &= A\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{u^2+v^2}{2\sigma^2}} e^{j2\pi(ux+vy)}\,dudv \\
&= A\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{\frac{u^2}{2\sigma^2}} e^{j2\pi ux}\,du\right] e^{\frac{v^2}{2\sigma^2}} e^{j2\pi vy}\,dv \\
&= A\int_{-\infty}^{\infty}\left[\sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2x^2}\right] e^{\frac{v^2}{2\sigma^2}} e^{j2\pi vy}\,dv \\
&= A\sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2x^2}\int_{-\infty}^{\infty} e^{\frac{v^2}{2\sigma^2}} e^{j2\pi vy}\,dv \\
&= A\sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2x^2}\sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2y^2} \\
&= A2\pi\sigma^2 e^{-2\pi^2\sigma^2(x^2+y^2)}.
\end{aligned}
$$

Exercise #3: Problem 4.5

Ans:

Given $H_{hp}(u,v) = 1 - H_{lp}(u,v)(u,v)$, we have that

$$\begin{aligned} h_{hp}(x,y) &= \delta(x,y) - h_{lp}(x,y) \\ &= \delta(x,y) - A2\pi\sigma^2 e^{-2\pi^2\sigma^2(x^2+y^2)}. \end{aligned}$$

Exercise #3: Problem 4.21

- The reason for padding is to establish a "buffer" between the periods that are implicit in DFT.
- Imaging the image in the left is duplicated infinitely many times to cover xy-plane. The result would be similar to a checkerboard, with each square being in the checkerboard being the image (and black extensions).
- If we do the same to the image on the right, the result is indistinguishable.
- Thus, either form of padding accomplish the same separation between images.