Exercise #10. Image segmentation

Exam March 2010:

2. (a) The objective of finding an optimal threshold is to minimize the erroneous classification, i.e., a background pixel classified as foreground and a foreground pixel classified as background.

The error probability is:

$P_e(T) = P_{fg} \cdot \int_T^{\infty} f_{fg}(t)\,dt + P_{bg} \int_{-\infty}^{T} f_{bg}(t)\,dt,$

where we use T to denote the threshold, and we use the convention that low pixel values represent the foreground and high pixel values represent the background.

(b) To minimize $P_e(T)$, we differentiate $P_e(T)$ and set it to 0:

$$\frac{\partial P_e(T)}{\partial T} = 0 \Rightarrow P_{fg} \cdot f_{fg}(T) = P_{bg}\, f_{bg}(T) \quad ①$$

(c) Plug $f_{fg}(x)$, f_{bg} in ①, we obtain

$$P_{fg} \cdot \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{(T-\mu_1)^2}{2\sigma_1^2}\right) = P_{bg} \cdot \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left(-\frac{(T-\mu_2)^2}{2\sigma_2^2}\right).$$

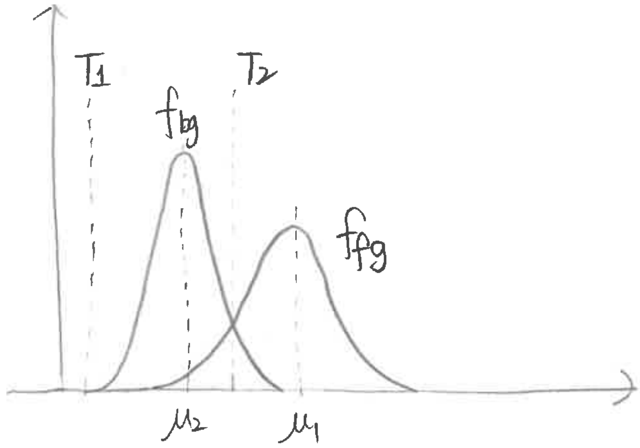
$$\exp\left[-\frac{(T-\mu_1)^2}{2\sigma_1^2} - \frac{(T-\mu_2)^2}{2\sigma_2^2}\right] = \frac{P_{bg}}{P_{fg}} \cdot \frac{\sigma_1}{\sigma_2}$$

$$\frac{\sigma_1^2(T-\mu_2)^2 - \sigma_2^2(T-\mu_1)^2}{2\sigma_1^2\sigma_2^2} = \ln\left(\frac{P_{bg}}{P_{fg}} \cdot \frac{\sigma_1}{\sigma_2}\right)$$

$$\underbrace{T^2(\sigma_1^2-\sigma_2^2)}_{a} + \underbrace{(2\sigma_2^2\mu_1 - 2\sigma_1^2\mu_2)}_{b}T + \underbrace{\left(\sigma_1^2\mu_2^2 - \sigma_2^2\mu_1^2 - 2\sigma_1^2\sigma_2^2\ln\left(\frac{P_{bg}}{P_{fg}} \cdot \frac{\sigma_1}{\sigma_2}\right)\right)}_{c} = 0$$

Therefore, we get a quadratic expression of form $aT^2+bT+c=0$.

(d)



Pixels with Gray-level between $[T_1, T_2]$ is classified as background.

(e). There is a single optimal threshold if $a=0$, i.e., $\sigma_1 = \sigma_2$.

The threshold is then

$$T = -\frac{c}{b} = \frac{1}{2}(\mu_1+\mu_2) - \frac{\sigma_1^2}{\mu_1-\mu_2} \cdot \ln\left(\frac{P_{bg}}{P_{fg}}\right).$$

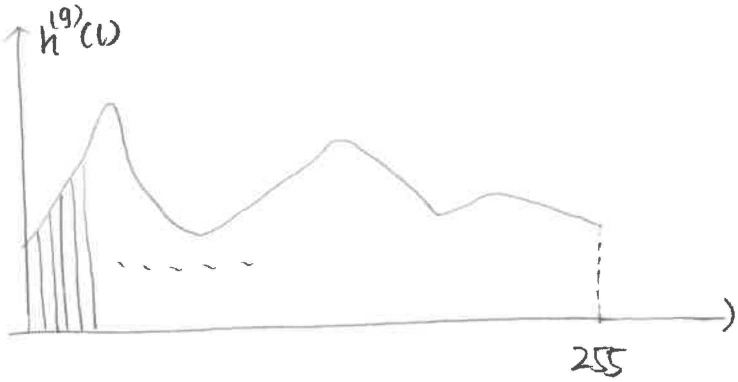
(f). $T = \frac{\mu_1+\mu_2}{2}$.

Exam May 2009.

(a) Histogram equalization transform:

$$g(x,y) = \frac{255}{N} \sum_{l=0}^{f(x,y)} h(l), \text{ where } N = \sum_{l=0}^{255} h(l). \quad ②$$

(b)



(c) Criterion: Minimizing probability of misclassification.

Mathematical expression: $T = \arg\min_T \; P_b \sum_{l=0}^{T} P_o(l) + P_p \sum_{l=T+1}^{255} P_p(l)$

(d) Neglect the effect of rounding operation and conside the case that $H(\cdot)$ is pixel-wise, non-decreasing, invertible mapping with

$$j=H(l) \Longleftrightarrow l=H^{-1}(j) \qquad \text{(invertible)}$$

$$l<l' \Longleftrightarrow H(l)<H(l') \qquad \text{(non-decreasing)}$$

for all gray-level values $l, l' \in L=\{l: h(l) \neq 0\}$.

(e) Define notations: $J=\{H(l): l \in L\}$

$L^T=\{l \in L: l \leq T\}$, $L_T=\{l \in L: l>T\}$

$J^T=\{j \in J: j \leq T\}$, $J_T=\{j \in T: j>T\}$.

The transform $H(\cdot)$ does not change the priors:

$$P_p^{(f)}=P_p^{(g)}=P_p, \quad P_o^{(f)}=P_o^{(g)}=P_o.$$

Due to ② in (a), we obtain

$$p_p^{(g)}(j)=\sum_{\{l \in L: H(l)=j\}} p_p^{(f)}(l)=p_p^{(f)}(H^{-1}(j))$$

$$p_o^{(g)}(j)=\sum_{\{l \in L: H(l)=j\}} p_o^{(f)}(l)=p_o^{(f)}(H^{-1}(j)).$$

The optimal thresholds are given by:

$$T^{(f)}=\underset{T}{\operatorname{argmin}}\ P_p \sum_{l \in L^T} p_p^{(f)}(l)+P_o \sum_{l \in L^T} p_o^{(f)}(l)$$

$$T^{(g)}=\underset{T}{\operatorname{argmin}}\ P_p \sum_{j \in J^T} p_p^{(g)}(l)+P_o \sum_{l \in J_T} p_o^{(g)}(j)$$

$$=\underset{T}{\operatorname{argmin}}\ P_p \sum_{j \in J^T} p_p^{(f)}(H^{-1}(j))+P_o \sum_{j \in J_T} p_o^{(f)}(H^{-1}(j))$$

$$=\underset{T}{\operatorname{argmin}}\ P_p \sum_{l \in L^{H^{-1}(T)}} p_p^{(f)}(l)+P_o \sum_{l \in L_{H^{-1}(T)}} p_o^{(f)}(l)$$

$$=H(T^{(f)})$$

This implies segmentation of f gives the same results as segmentation of g.