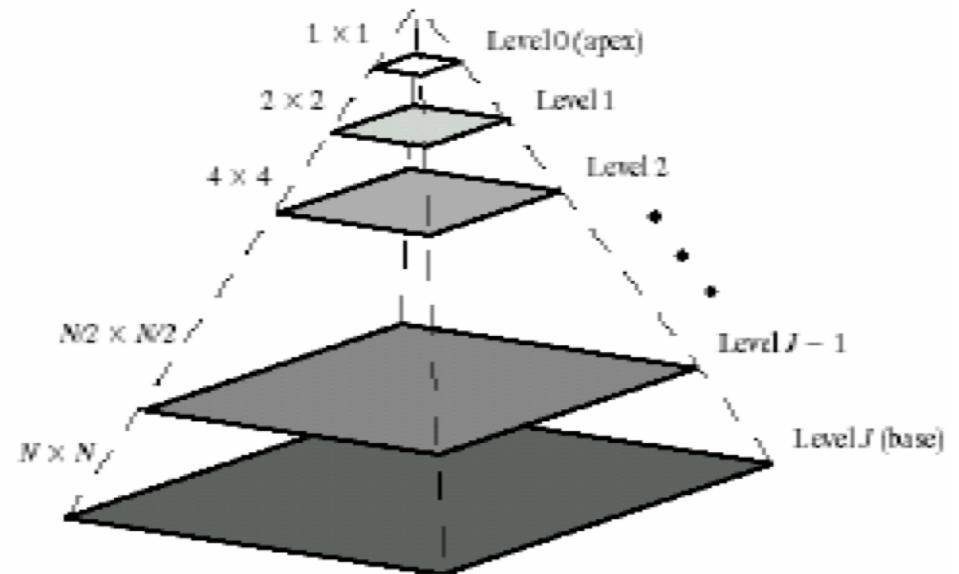
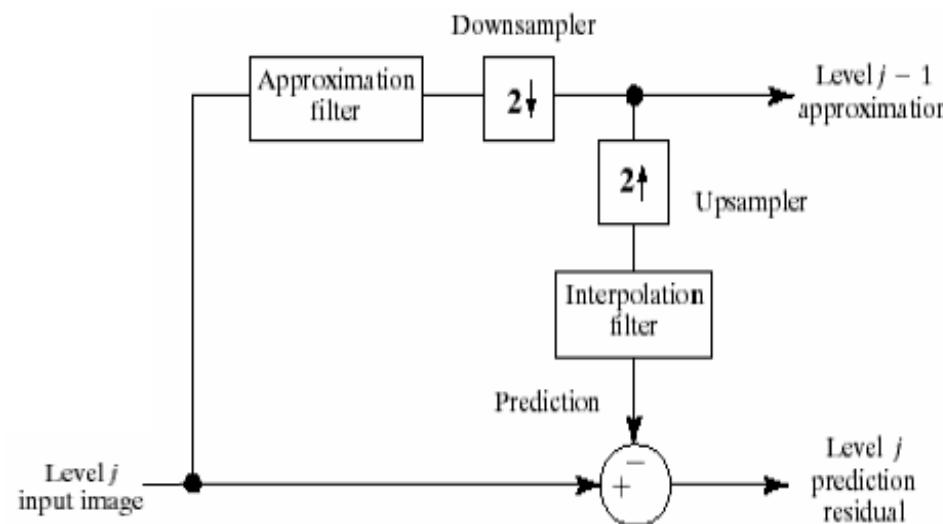


Multiresolution Processing

- Gaussian and Laplacian pyramid
- Discrete Wavelet Transform (DWT)
 - Two-channel filterbank with perfect reconstruction
 - Lifting implementation
 - Conjugate quadrature filters
- Wavelet theory
 - Wavelet basis
 - Scaling function and wavelet function

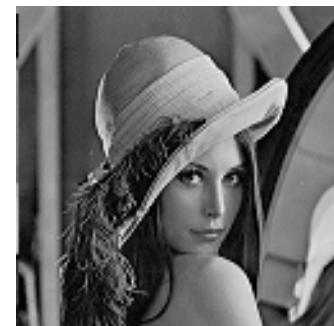


Image Pyramids



[Burt, Adelson, 1983]

Image Pyramid Example



Gaussian pyramid



Laplacian pyramid



Overcomplete Representation

Number of samples to be encoded =

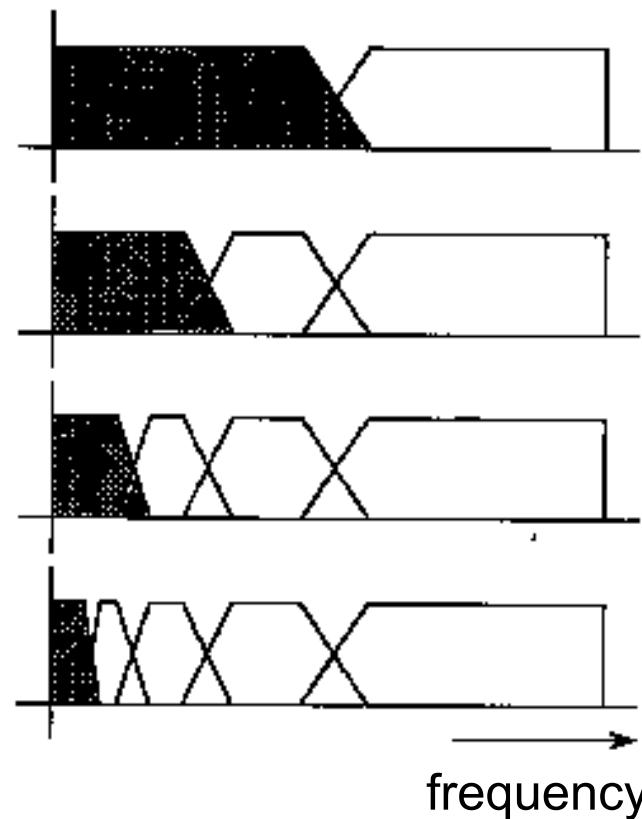
$$\left(1 + \frac{1}{N} + \frac{1}{N^2} + \dots\right) = \frac{N}{N-1} \times \text{number of original image samples}$$

↑
subsampling factor

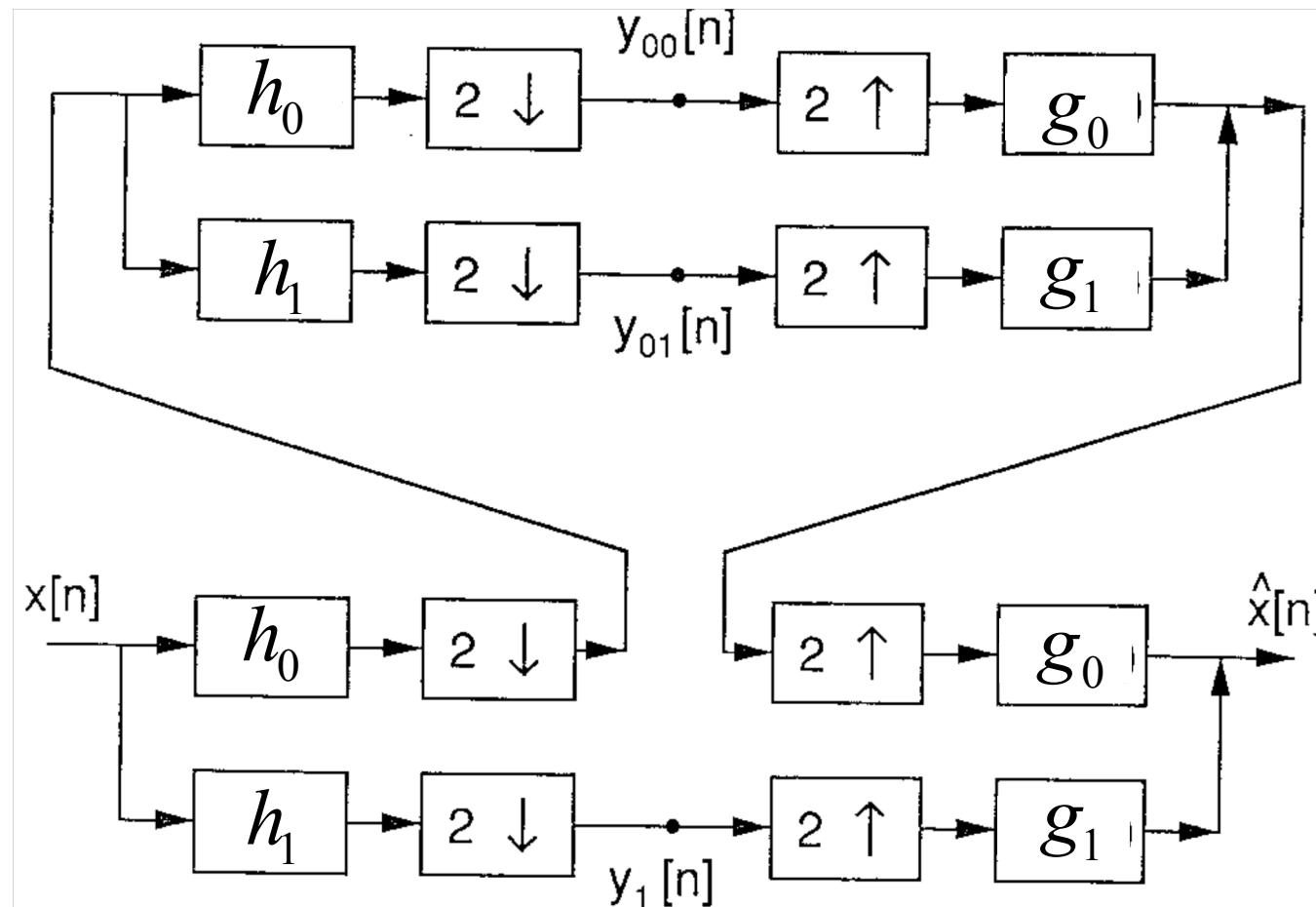


1-D Discrete Wavelet Transform (DWT)

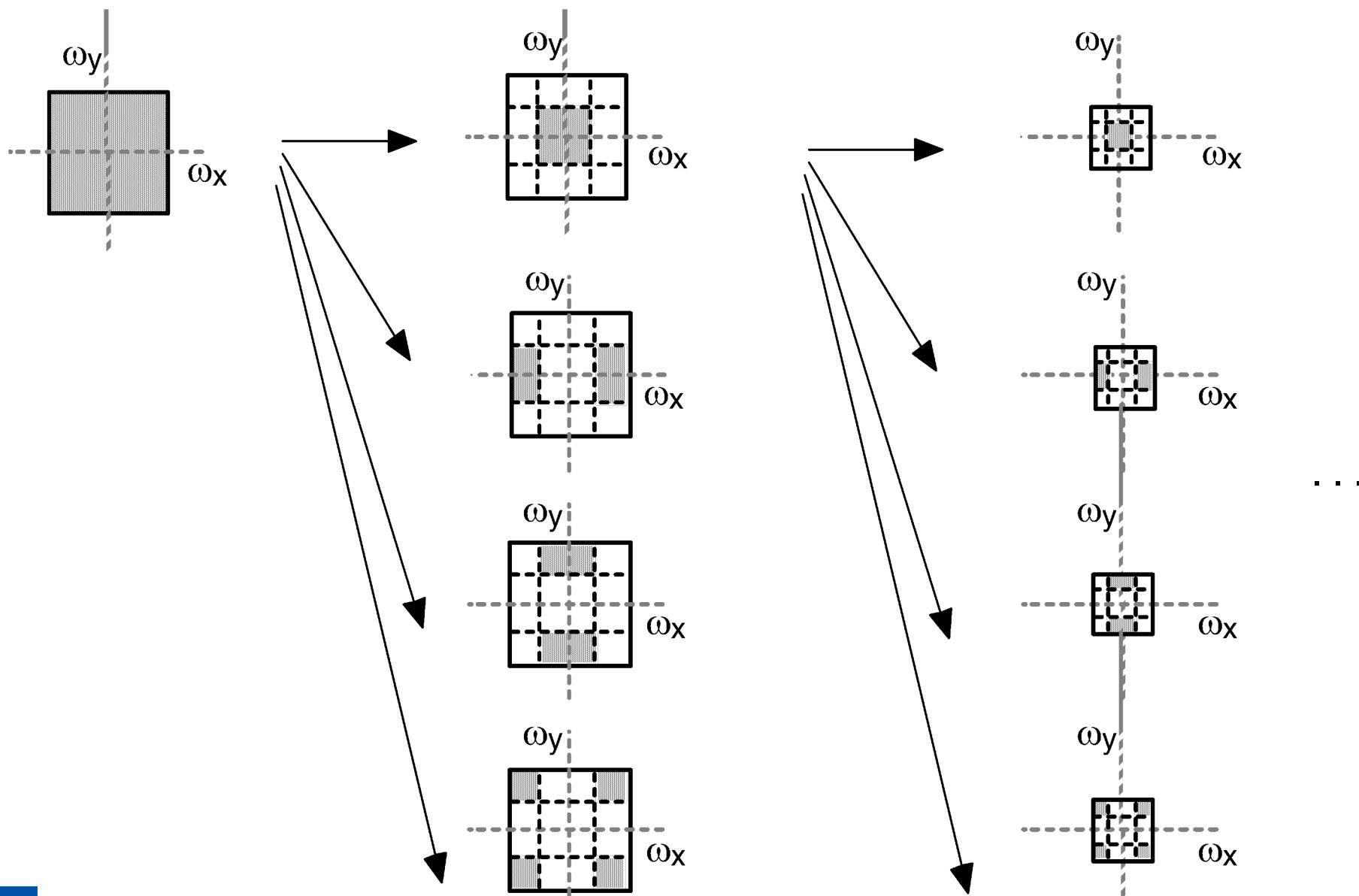
- Recursive application of a two-band filter bank to the lowpass band of the previous stage yields octave band splitting:



Cascaded Analysis / Synthesis Filterbanks



2-D Discrete Wavelet Transform



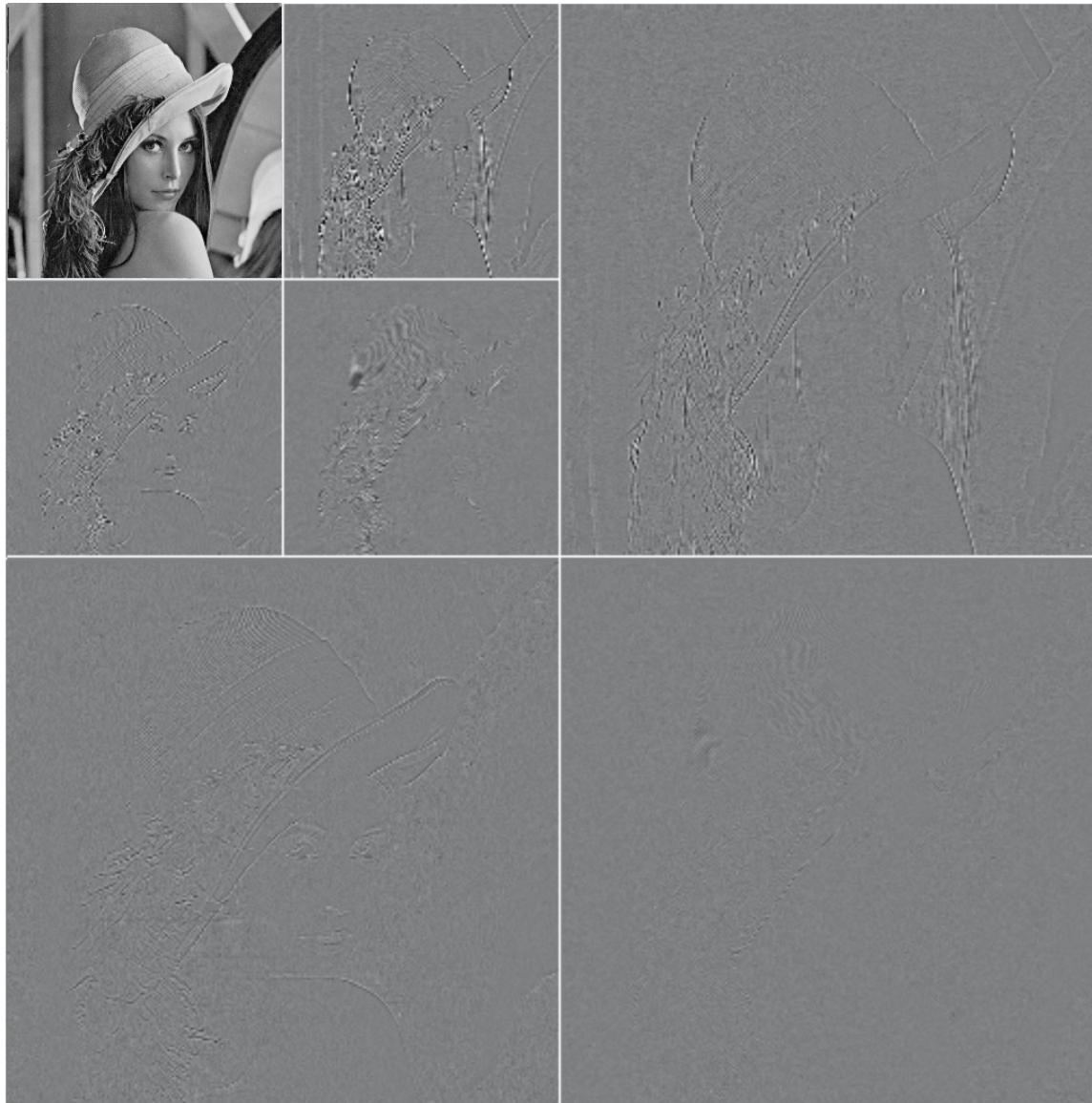
2-D Discrete Wavelet Transform Example



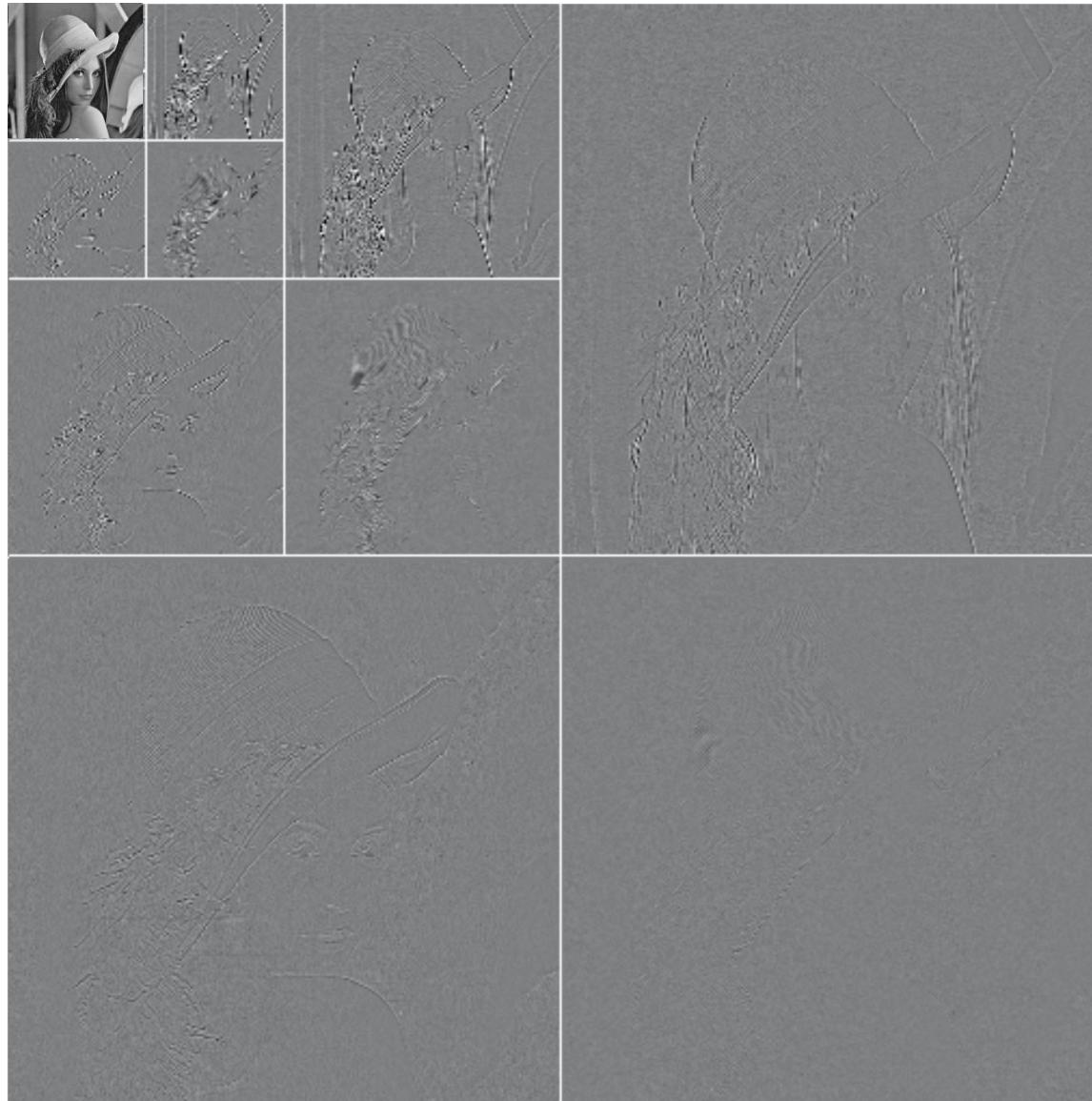
2-D Discrete Wavelet Transform Example



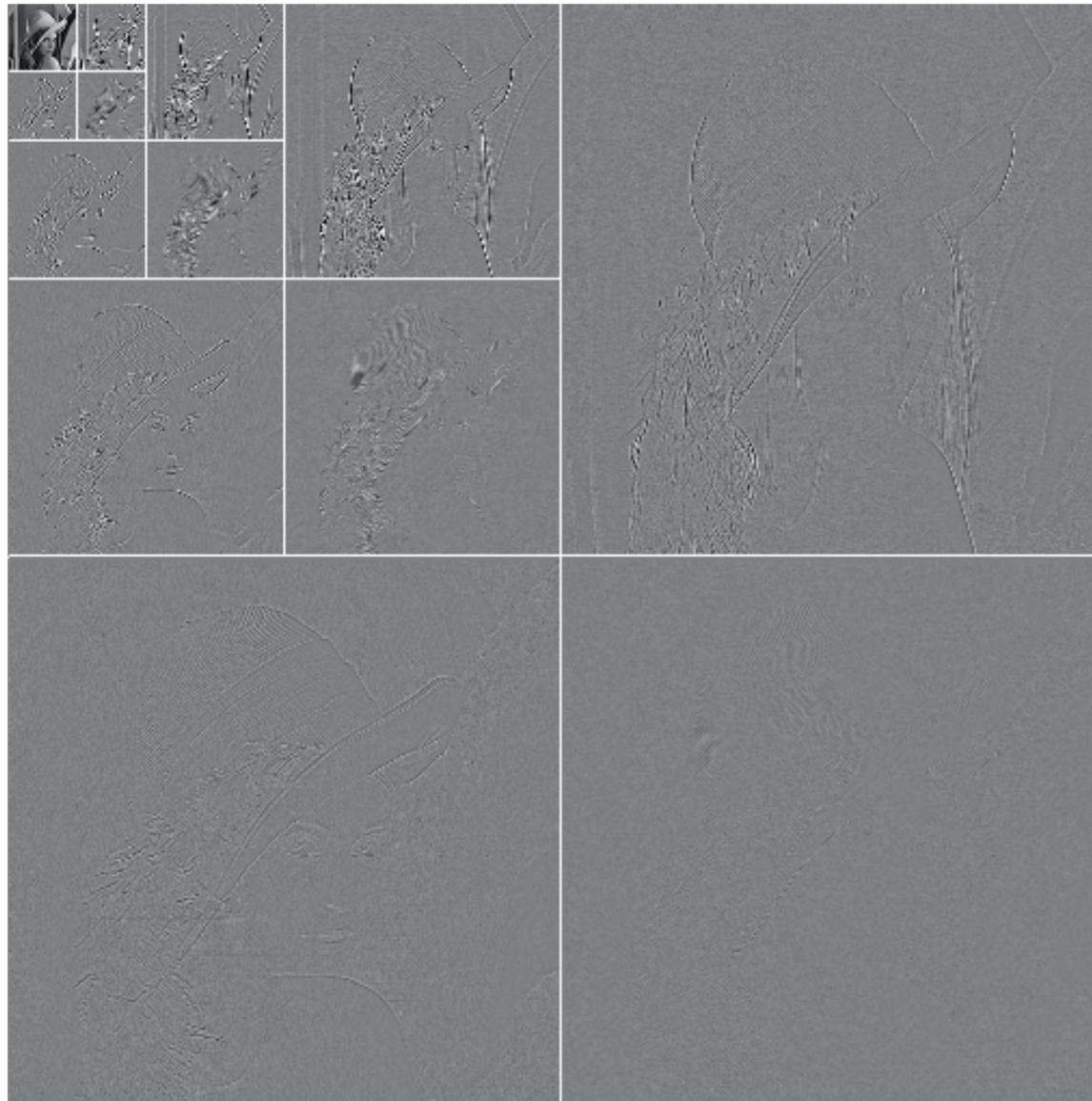
2-D Discrete Wavelet Transform Example



2-D Discrete Wavelet Transform Example



2-D Discrete Wavelet Transform Example

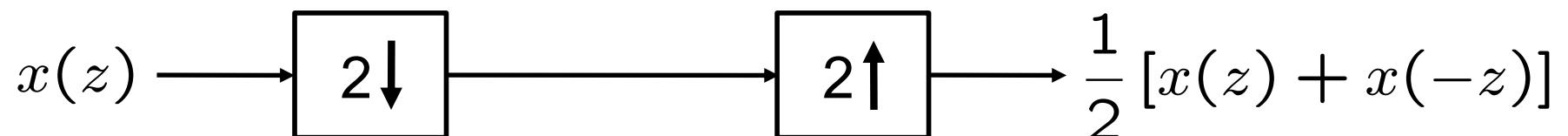


Review: Z-Transform and Subsampling

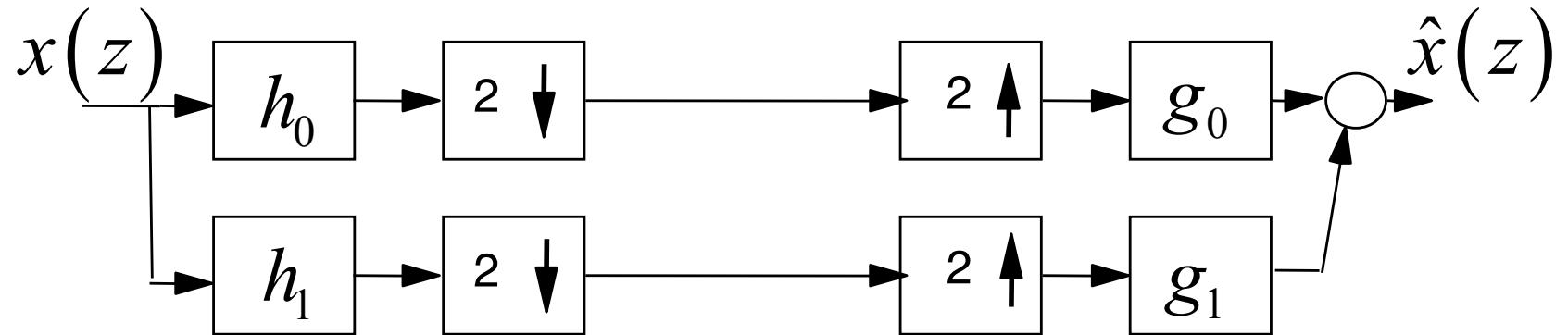
- Generalization of the discrete-time Fourier transform

$$x(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \quad z \in \mathcal{C} \quad r^- < |z| < r^+$$

- Fourier transform on unit circle: Substitute $z=e^{j\omega}$
- Downsampling and upsampling by factor 2



Two-Channel Filterbank



$$\hat{x}(z) = \frac{1}{2} [h_0(z)g_0(z) + h_1(z)g_1(z)]x(z)$$

$$+ \frac{1}{2} [h_0(-z)g_0(z) + h_1(-z)g_1(z)]x(-z)$$

Aliasing

- Aliasing cancellation if :

$$g_0(z) = h_1(-z)$$

$$-g_1(z) = h_0(-z)$$



Example: 2-Channel FB with Perfect Reconstruction

- Impulse responses, analysis filters:
Lowpass highpass

$$\left(\frac{-1}{4}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{-1}{4} \right) \quad \left(\frac{-1}{4}, \frac{1}{2}, \frac{-1}{4} \right)$$

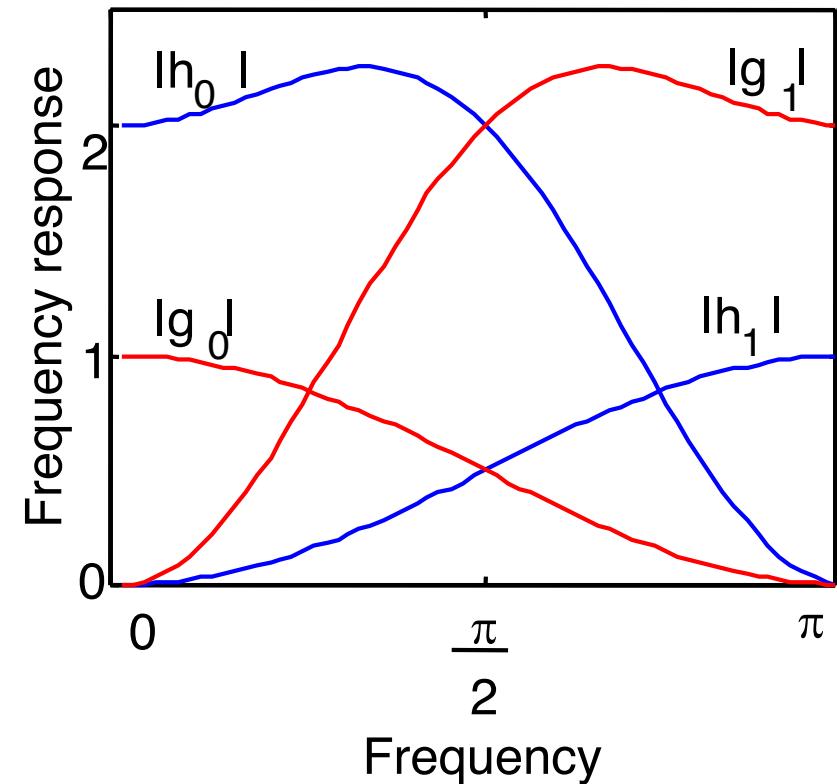
- Impulse responses, synthesis filters

Lowpass highpass

$$\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4} \right) \quad \left(\frac{1}{4}, \frac{1}{2}, \frac{-3}{2}, \frac{1}{2}, \frac{1}{4} \right)$$

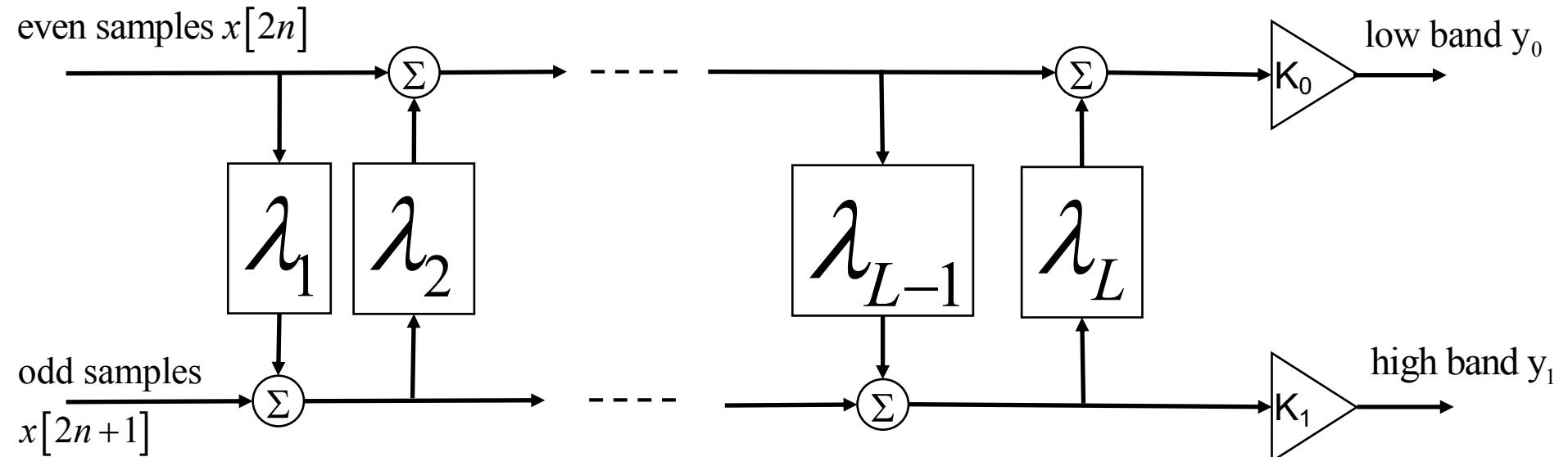
“Biorthogonal 5/3 filters”
“LeGall filters”

- Mandatory in JPEG2000
- Frequency responses:



Lifting

- Analysis filters

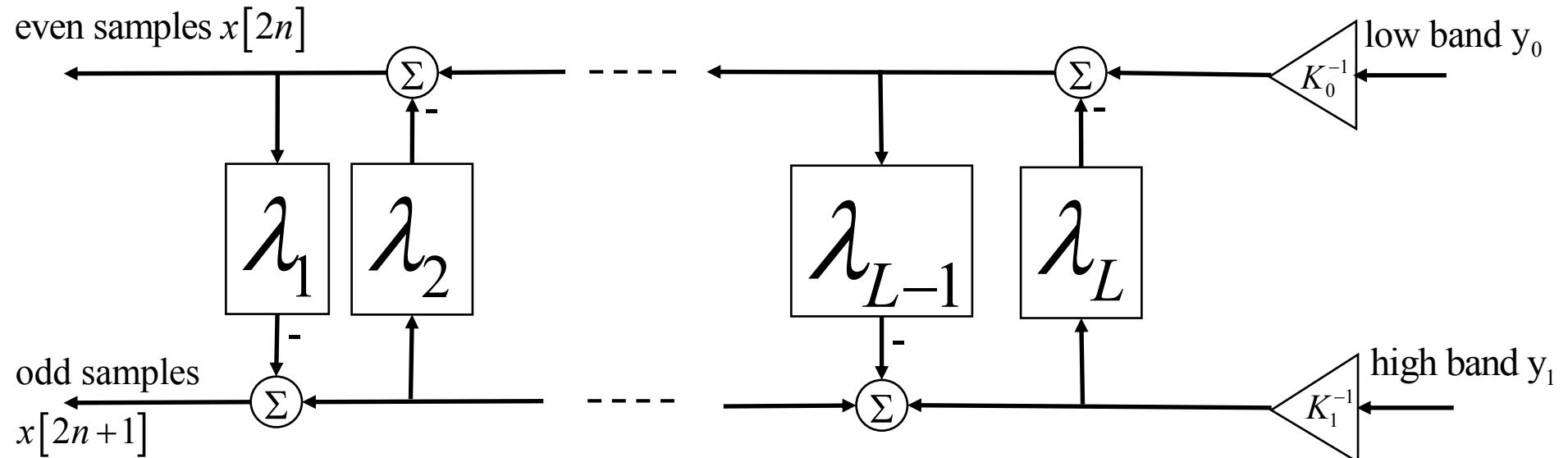


[Sweldens, 1996]

- L “lifting steps”
- First step can be interpreted as prediction of odd samples from the even samples

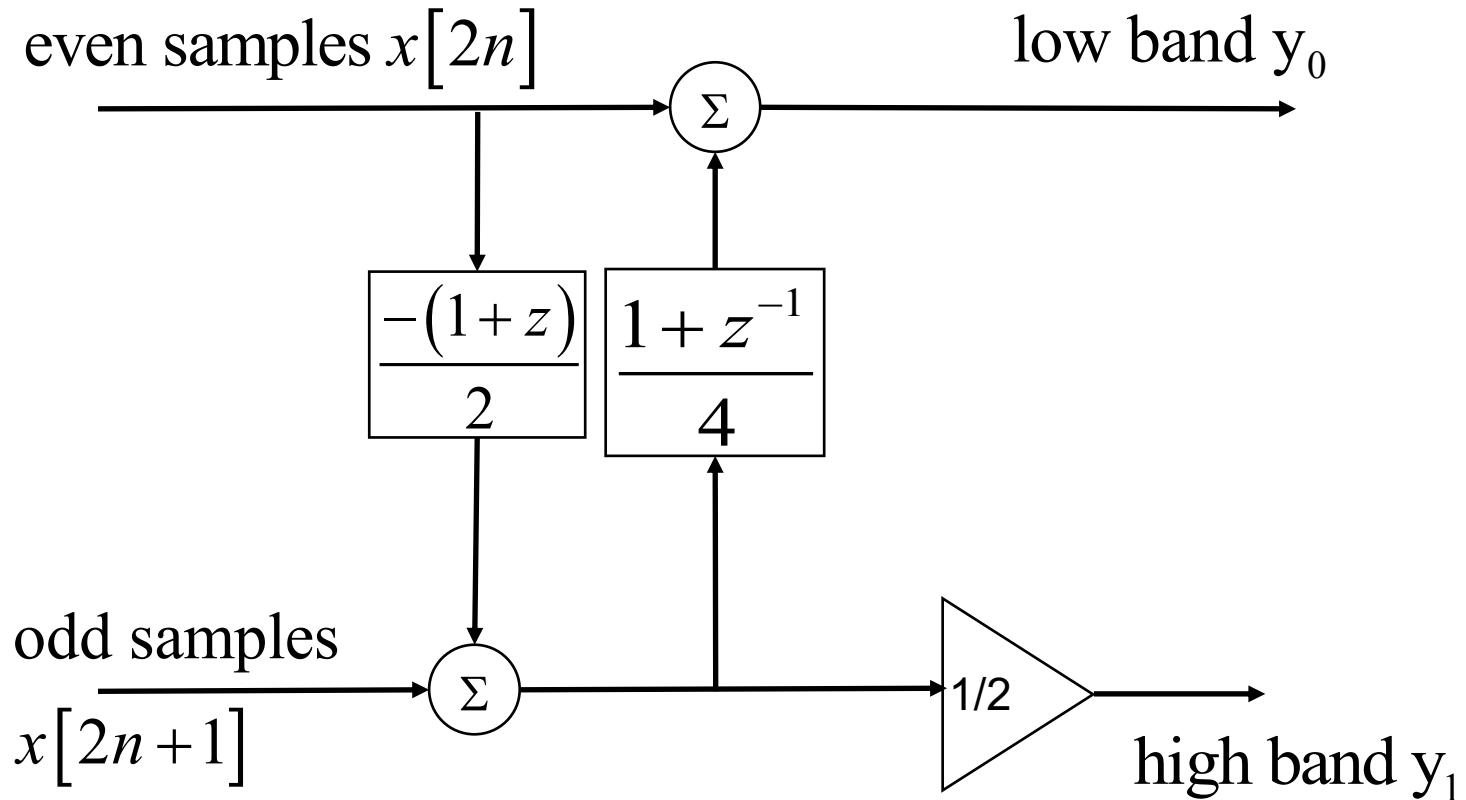
Lifting

- Synthesis filters



- Perfect reconstruction (biorthogonality) is directly built into lifting structure
- Powerful for both implementation and filter/wavelet design

Example: Lifting Implementation of 5/3 Filters



Verify by considering response to unit impulse in even and odd input channel.



Conjugate Quadrature Filters

- Achieve aliasing cancellation by

$$h_0(z) = g_0(z^{-1}) \equiv f(z)$$

$$h_1(z) = g_1(z^{-1}) = zf(-z^{-1})$$

Prototype filter

[Smith, Barnwell, 1986]

- Impulse responses

$$h_0[k] = g_0[-k] = f[k]$$

$$h_1[k] = g_1[-k] = (-1)^{k+1} f[-(k+1)]$$

- With perfect reconstruction: Orthonormal subband transform!
- Perfect reconstruction: Find power complementary prototype filter

$$|F(\omega)|^2 + |F(\omega \pm \pi)|^2 = 2$$



Wavelet Bases

- Consider Hilbert space $\mathcal{L}^2(\mathcal{R})$ of finite-energy functions
 $x = x(t)$
- Wavelet basis for $\mathcal{L}^2(\mathcal{R})$: Family of linearly independent functions

$$\psi_n^m(t) = \sqrt{2^{-m}} \psi(2^{-m}t - n)$$

“mother wavelet”

that span $\mathcal{L}^2(\mathcal{R})$. Hence, any signal $x \in \mathcal{L}^2(\mathcal{R})$ can be written as

$$x = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} y^{(m)}[n] \psi_n^{(m)}$$



Multiresolution Analysis

- Nested subspaces

$$\cdots \subset V^{(2)} \subset V^{(1)} \subset V^{(0)} \subset V^{(-1)} \subset V^{(-2)} \subset \cdots \subset \mathcal{L}^2(\mathcal{R})$$

Upward completeness $\bigcup_{m \in \mathbb{Z}} V^{(m)} = \mathcal{L}^2(\mathcal{R})$

Downward completeness $\bigcap_{m \in \mathbb{Z}} V^{(m)} = \{0\}$

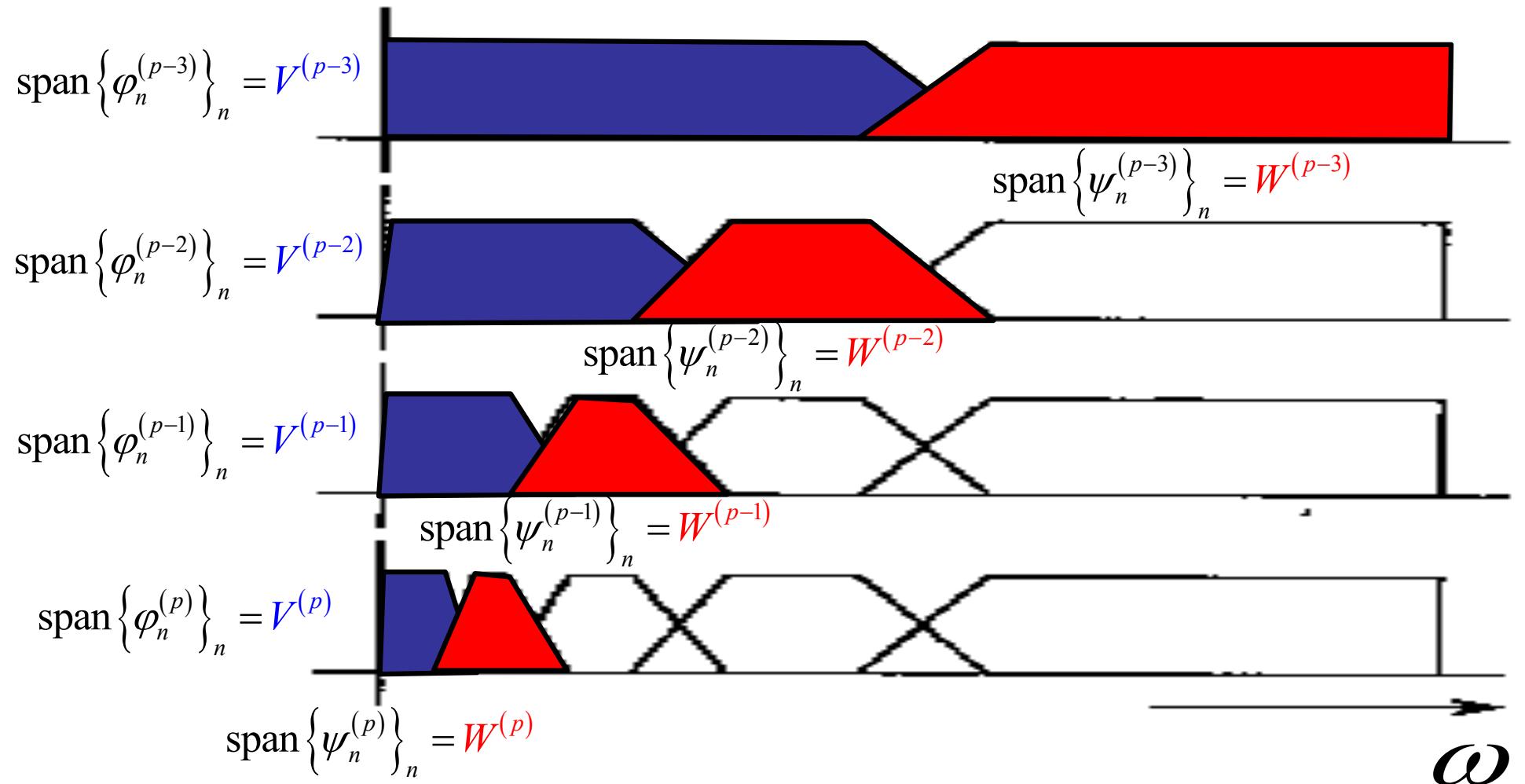
Self-similarity $x(t) \in V^{(0)} \quad \text{iff} \quad x(2^{-m}t) \in V^{(m)}$

Translation invariance $x(t) \in V^{(0)} \quad \text{iff} \quad x(t - n) \in V^{(0)} \quad \forall n \in \mathbb{Z}$

- There exists a **scaling function** $\varphi(t)$ with integer translates $\varphi_n(t) = \varphi(t - n)$ such that $\{\varphi_n\}_{n \in \mathbb{Z}}$ forms an orthonormal basis for $V^{(0)}$



Multiresolution Fourier Analysis



Relation to Subband Filters

Since $V^{(0)} \subset V^{(-1)}$, recursive definition of scaling function

$$\varphi(t) = \underbrace{\sum_{n=-\infty}^{\infty} g_0[n] \varphi_n^{(-1)}(t)}_{\text{linear combination of wavelets in } V^{(-1)}} = \sqrt{2} \sum_{n=-\infty}^{\infty} g_0[n] \varphi(2t-n)$$

Orthonormality

$$\begin{aligned}\delta[n] &= \langle \varphi_0^{(0)}, \varphi_n^{(0)} \rangle \\ &= 2 \int_{-\infty}^{\infty} \left(\sum_i g_0[i] \varphi(2t-i) \sum_j g_0[j] \varphi(2(t-n)-j) \right) dt \\ &= \sum_{i,j} g_0[i] g_0[j-2n] \langle \varphi_i^{(-1)}, \varphi_j^{(-1)} \rangle = \underbrace{\sum_i g_0[i] g_0[i-2n]}_{\text{unit norm and orthogonal to its 2-translates: corresponds to synthesis lowpass filter of orthonormal subband transform}}\end{aligned}$$



Wavelets from Scaling Functions

$W^{(p)}$ is orthonormal complement of $V^{(p)}$ in $V^{(p-1)}$

$$W^{(p)} \perp V^{(p)} \quad \text{and} \quad W^{(p)} \cup V^{(p)} = V^{(p-1)}$$

Orthonormal wavelet basis $\{\psi_n^{(0)}\}$ for $W^{(0)} \subset V^{(-1)}$

$$\psi(t) = \underbrace{\sum_{n=-\infty}^{\infty} g_1[n] \phi_n^{(-1)}(t)}_{\substack{\text{linear combination} \\ \text{of wavelets in } V^{(-1)}}} = \sqrt{2} \sum_{n=-\infty}^{\infty} g_1[n] \varphi(2t-n)$$

Using conjugate quadrature high-pass synthesis filter

$$g_1[n] = (-1)^{n+1} g_0[-(n-1)]$$

The mutually orthonormal functions, $\{\psi_n^{(0)}\}_{n \in \mathbb{Z}}$ and $\{\phi_n^{(0)}\}_{n \in \mathbb{Z}}$ together span $V^{(-1)}$.

Easy to extend to dilated versions of $\psi(t)$ to construct orthonormal wavelet basis

$$\left\{\psi_n^{(m)}\right\}_{n,m \in \mathbb{Z}} \text{ for } \mathcal{L}^2(\mathcal{R})$$



Calculating Wavelet Coefficients for a Continuous Signal

- Signal synthesis by discrete filter bank

Suppose continuous signal $x^{(0)}(t) = \sum_{n \in \mathbf{Z}} y_0^{(0)}[n] \varphi(t - n) = \sum_{n \in \mathbf{Z}} y_0^{(0)}[n] \varphi_n^{(0)} \in V^{(0)}$

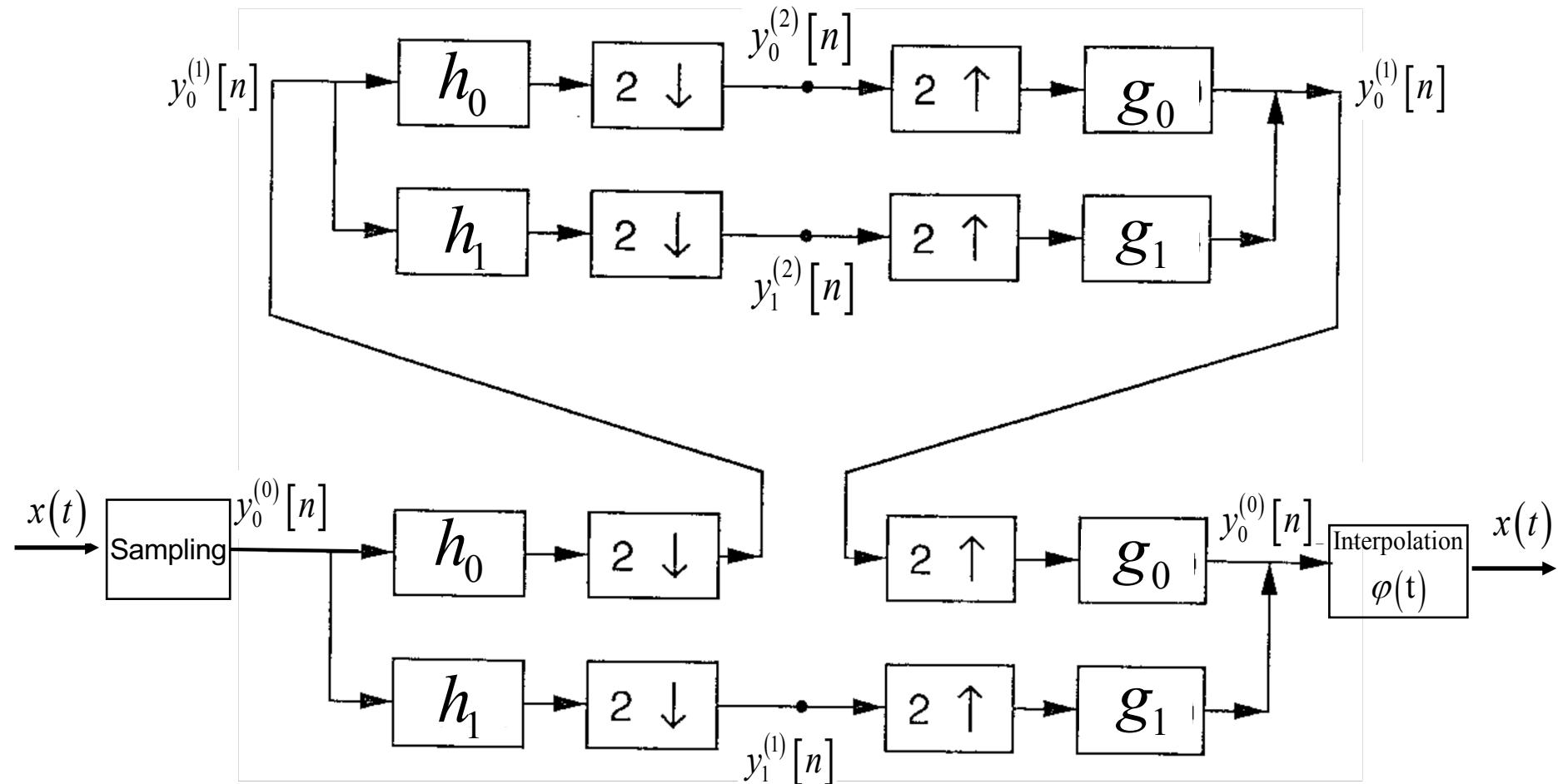
Write as superposition of $x^{(1)}(t) \in V^{(1)}$ and $w^{(1)}(t) \in W^{(1)}$

$$\begin{aligned} x^{(0)}(t) &= \underbrace{\sum_{i \in \mathbf{Z}} y_0^{(1)}[i] \varphi_n^{(1)}}_{x^{(1)}(t) \in V^{(1)}} + \underbrace{\sum_{j \in \mathbf{Z}} y_1^{(1)}[j] \psi_n^{(1)}}_{w^{(1)}(t) \in W^{(1)}} \\ &= \sum_{n \in \mathbf{Z}} \varphi_n^{(0)} \left(\underbrace{\sum_{i \in \mathbf{Z}} y_0^{(1)}[n] g_0[n - 2i] + \sum_{j \in \mathbf{Z}} y_1^{(1)}[j] g_1[n - 2j]}_{y_0^{(0)}[n]} \right) \end{aligned}$$

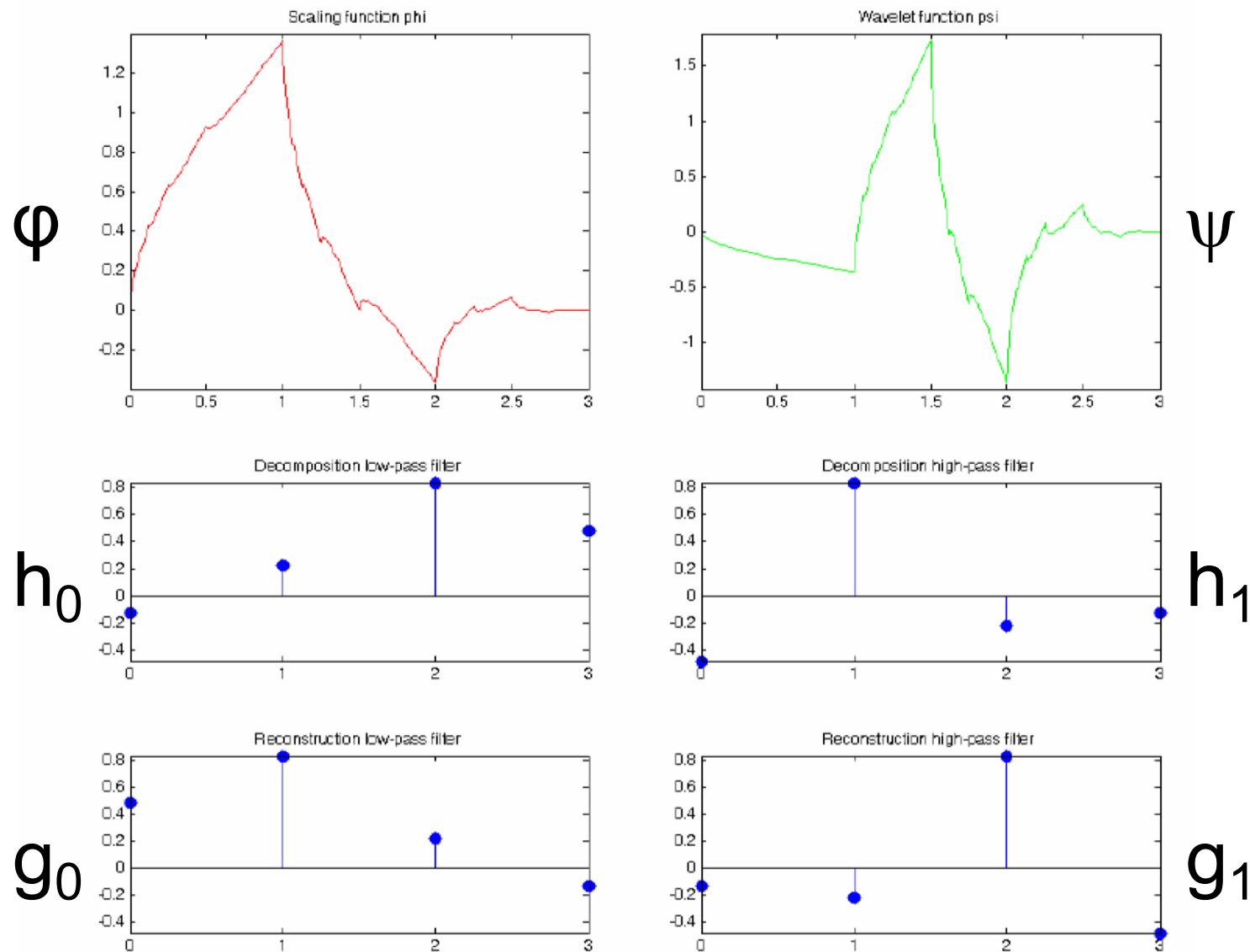
- Signal analysis by analysis filters $h_0[k], h_1[k]$
- Discrete wavelet transform



1-D Discrete Wavelet Transform



Example: Daubechies Wavelet, Order 2



Example: Daubechies Wavelet, Order 9

