EQ2330 Image and Video Processing
Tutorial #5: Unitary Transforms

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Project

- Zero Padding (Pad 0s or replicate the boundary pixel values)
- Wiener Filter Wiener filter: $R_W = \frac{H^*(u,v)}{|H(u,v)|^2 + \frac{|N(u,v)|^2}{|F(u,v)|^2}}$, where where $H(u,v)$ is the degradation filter.

Unitary Transform

Matrix Formulation

- Image $f(x, y)$ with size $M \times N$.
- Sort $f(x, y)$ to a column vector $\mathbf{f}$ with length MN.
- A linear transformation can be expressed as $\mathbf{c} = \mathbf{A}\mathbf{f}$, where $\mathbf{A}$ is a matrix of size $MN \times MN$.

$\mathbf{A}$ is unitary iff $\mathbf{A}^{-1} = \mathbf{A}^{*T} = \mathbf{A}^{H}$

- H is hermitian conjugate.
- If $\mathbf{A}$ is real-valued, $\mathbf{A}^{-1} = \mathbf{A}^{T}$. Transform is orthonormal.
- E.g. DCT, KLT, Haar
- Energy conservation: $||\mathbf{c}||_2^2 = \mathbf{c}^H\mathbf{c} = \mathbf{f}^H\mathbf{A}^H\mathbf{A}\mathbf{f} = \mathbf{f}^H\mathbf{f} = ||\mathbf{f}||_2^2$.
- Unitary transform can be interpreted as a rotation of the coordinate system.

Karhunen-Loeve Transform

- Covariance matrix $\mathbf{R} = E[XX^H]$ is Hermitian. Therefore $\mathbf{R}$ can be diagonalized, i.e., $\mathbf{\Phi}^H\mathbf{R}\mathbf{\Phi} = \Lambda$, where Λ is a diagonal matrix with eigenvalues λ_i.
- Define KL transform as $\mathbf{Y} = \mathbf{\Phi}^H\mathbf{X}$, where columns of $\mathbf{\Phi}$ are eigenvectors ordered according to decreasing eigenvalues.
- Inverse transform $\mathbf{X} = \mathbf{\Phi}\mathbf{Y}$.
- Correlation matrix of $\mathbf{Y}$:
  $E[\mathbf{Y}\mathbf{Y}^H] = E[\mathbf{\Phi}^H\mathbf{X}\mathbf{X}^H\mathbf{\Phi}] = \mathbf{\Phi}^H E[\mathbf{X}\mathbf{X}^H]\mathbf{\Phi} = \mathbf{\Phi}^H\mathbf{R}\mathbf{\Phi} = \Lambda.$
  - KL transform totally decorrelates the signal.
  - KL transform is optimal in energy concentration.

Haar Transform

- Image $\mathbf{f}$ with size $N \times N$
- Haar transformation matrix $\mathbf{H}$ with size $N \times N$
- Haar transform $\mathbf{T} = \mathbf{H}\mathbf{F}\mathbf{H}^T$
- Inverse transform $\mathbf{F} = \mathbf{H}^T\mathbf{T}\mathbf{H}$
- E.g. when $N = 2$, $\mathbf{H}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Assignment 5

1. A transform is separable if the transform can process the image in each dimension, independent of the other dimension, i.e., $g(x, y, u, v) = g_1(x, u)g_2(y, v)$.
Consider the matrices formulation. Let $\mathbf{f}$ and $\mathbf{g}$ denote the input image and the output image. The the transformation is separable if the transform process can be written as

$$\mathbf{g} = \mathbf{H}_y^T \mathbf{f} \mathbf{H}_x, \tag{1}$$

where $\mathbf{H}_y$ and $\mathbf{H}_x$ are 1D transform in x and y directions, respectively.

Assignment 5

2. The 2D DCT is given by

$$F(u,v) = \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} f(x,y)\alpha_N(u)\alpha_M(v) \cos\left(\frac{(2x+1)u\pi}{2N}\right) \cos\left(\frac{(2y+1)v\pi}{2M}\right),$$

where

$$u \in \{0,1,\ldots,N-1\}, \alpha_N(0) = \sqrt{1/N}, \alpha(u>0) = \sqrt{2/N},$$
$$v \in \{0,1,\ldots,M-1\}, \alpha_M(0) = \sqrt{1/M}, \alpha(v>0) = \sqrt{2/M}$$

Assignment 5

2.(cont) The 2D DCT is separable as follows

$$F(u,v) = \sum_{x=0}^{N-1} \alpha_N(u)\Big(\sum_{y=0}^{M-1} f(x,y)\alpha_M(v)\cos\Big(\frac{(2y+1)v\pi}{2N}\Big)\Big)\cos\Big(\frac{(2x+1)u\pi}{2M}\Big)$$

$$= \sum_{x=0}^{N-1} \alpha_N(u)F(x,v)\cos\Big(\frac{(2x+1)u\pi}{2M}\Big)$$

where

$$F(x,v) = \sum_{y=0}^{M-1} f(x,y)\alpha_M(v)\cos\Big(\frac{(2x+1)v\pi}{2N}\Big).$$

Assignment 5

3. The first block $f_1(x, y)$ represent a noisy block (without obvious pattern). We associate $f_1(x, y)$ with F_3.

The other blocks show large correlation between pixels (with obvious patterns).

- f_2 and f_4 have constant pixels in the x direction.
  $\Longrightarrow F(u, v)$ in u direction only has DC component, i.e. $F(u > 0, v) = 0$.
- f_4 have larger pixel value
  $\Longrightarrow f_2$ matches F_4 and f_4 matches F_2
- f_3 has constant pixels in the y direction
  $\Longrightarrow F(u, v)$ in v direction only has DC component, i.e. $F(u, v > 0) = 0$.
  $\Longrightarrow f_3$ matches F_2

Assignment 5

4. They are equal since the transform is unitary.
   - Consider the matrix formulation.
     - $\mathbf{f}$ and $\mathbf{g}$: the vectorized images
     - $\mathbf{A}$: the DCT transformation matrix $\longrightarrow \mathbf{A}^T\mathbf{A} = \mathbf{A}^T\mathbf{A} = I$.
   - $\sum_{x,y}(f(x,y) - g(x,y))^2 = ||\mathbf{f} - \mathbf{g}||_2^2 = (\mathbf{f} - \mathbf{g})^T(\mathbf{f} - \mathbf{g}) = \mathbf{f}^T\mathbf{f} - \mathbf{f}^T\mathbf{g} - \mathbf{g}^T\mathbf{f} + \mathbf{g}^T\mathbf{f}$.
   - $\sum_{u,v}(F(u,v) - G(u,v))^2 = ||\mathbf{F} - \mathbf{G}||_2^2 = \mathbf{F}^T\mathbf{F} - \mathbf{F}^T\mathbf{G} - \mathbf{G}^T\mathbf{F} + \mathbf{G}^T\mathbf{F} = \mathbf{f}^T\mathbf{A}^T\mathbf{A}\mathbf{f} - \mathbf{f}^T\mathbf{A}^T\mathbf{A}\mathbf{g} - \mathbf{g}^T\mathbf{A}^T\mathbf{A}\mathbf{f} + \mathbf{g}^T\mathbf{A}^T\mathbf{A}\mathbf{f} = \mathbf{f}^T\mathbf{f} - \mathbf{f}^T\mathbf{g} - \mathbf{g}^T\mathbf{f} + \mathbf{g}^T\mathbf{f}$.

Assignment 5

5. Observation: $f_5(x,y) = f_1(x,y) - 10$.
Since the mean shift only affect the DC component, we only have to find the new DC component and keep the rest components the same. That is $F_5(u,v) = F_3(u,v)$ if $(u,v) \neq (0,0)$.
Use the result in problem 4, we have that

$$\sum_{u,v}(F_5(u,v) - F_3(u,v))^2 = (F_5(0,0) - F_3(0,0))^2 = \sum_{x,y}(f_5(x,y) - f_1(x,y))^2$$
$$= 16 \times 10^2 = 1600.$$

Therefore, we have that $F_5(0,0) = F_3(0,0) - \sqrt{1600} = 308.75$.

Assignment 5

6. The Karhunen-Loeve transform (KLT) is optimal in terms of energy concentration. It is not widely used for image coding as it is image dependent. Note, for the KLT, we require the 2-nd order image statistics (autocorrelation function) for computation.

Exercise #5: Problem 1

Find the KL transform $Y = \Phi^H X$ of covariance matrix $\mathbf{R} = E[XX^T]$.
Hint:

1. Find eigenvalues λ_1, λ_2 and corresponding eigenvectors $\phi_1 = \begin{bmatrix} \phi_{11} \\ \phi_{21} \end{bmatrix}, \phi_2 = \begin{bmatrix} \phi_{12} \\ \phi_{22} \end{bmatrix}$
2. Let $\det(\mathbf{R} - \lambda \mathbf{I}) = 0$ and find the eigenvalues λ_1, λ_2
3. Let $(\mathbf{R} - \lambda \mathbf{I}) \begin{bmatrix} \phi_{11} \\ \phi_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to find the eigenvector $\phi_1 = \begin{bmatrix} \phi_{11} \\ \phi_{21} \end{bmatrix}$. And similarly find $\phi_2 = \begin{bmatrix} \phi_{12} \\ \phi_{22} \end{bmatrix}$
4. Columns of Φ are ordered according to decreasing eigenvalues.

Exercise #5: Problem 7.9

Hint: Haar transformation of $\mathbf{F}$ is $\mathbf{HFH}^T$, where $\mathbf{H} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Exercise #5: Exam Problem

Hint: (a) $\mathbf{A}$ is unitary iff $\mathbf{A}^{-1} = \mathbf{A}^H$

(d) Sum of eigenvalues is the trace, which is the sum of diagonals.

C is Hermitian matrix and can be rewritten as $\mathbf{C} = \mathbf{\Phi}\mathbf{\Lambda}\mathbf{\Phi}^H$.

Let $\mathbf{\Phi}^H = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Exercise #5: Problem 7.1,7.2

Hint: The decoding structure is the "opposite" of the encoding structure.

Exercise #5 Unitary Transform + Multiresolution Processing

1. $E[xx^T] = \begin{bmatrix} 4 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}$. Find KL transform.

Ans: Let R denote $R = E[XX^T] = \begin{bmatrix} 4 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}$.

Since R is Hermitian, then there exist a unitary matrix Φ such that $R = \Phi \Lambda \Phi^H$. $(\Phi^H R \Phi = \Phi^H \Phi \Lambda \Phi^H \Phi = \Lambda)$

Then Φ is the KLT matrix of X and KLT transform is

$Y = \Phi^H X$ and inverse transform is $X = \Phi Y$.

Y is the random vector in the transformed domain, with correlation matrix $E[YY^H] = E[\Phi^H X X^H \Phi] = \Phi^H E[XX^H] \Phi = \Phi^H R \Phi = \Lambda$

(This random sequence has no correlation.)

[About KL transform]

Now we look for eigenvalues and eigen vectors of R

Let $\det\left(\begin{bmatrix} 4-\lambda & \sqrt{2} \\ \sqrt{2} & 2-\lambda \end{bmatrix}\right) = 0 \Rightarrow \lambda^2 - 6\lambda + 6 = 0 \Rightarrow \lambda = \begin{cases} 3-\sqrt{3} & (1.2679) \\ 3+\sqrt{3} & (4.7321) \end{cases}$

when $\lambda = 1.2679$, $\begin{bmatrix} 4-\lambda & \sqrt{2} \\ \sqrt{2} & 2-\lambda \end{bmatrix} = \begin{bmatrix} 2.7321 & \sqrt{2} \\ \sqrt{2} & 0.7321 \end{bmatrix}$.

Let $\begin{bmatrix} \phi_{11} \\ \phi_{21} \end{bmatrix}$ denote the corresponding eigenvector of 1.2679, then

$$\begin{bmatrix} 2.7321 & \sqrt{2} \\ \sqrt{2} & 0.7321 \end{bmatrix} \begin{bmatrix} \phi_{11} \\ \phi_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 2.7321\phi_{11} + \sqrt{2}\phi_{21} = 0 \Rightarrow \phi_{11} = -0.5176\phi_{21}$ ①

combine ① with the fact that $\begin{bmatrix} \phi_{11} \\ \phi_{21} \end{bmatrix}$ is orthogonal eigenvector, we have $\phi_{11}^2 + \phi_{21}^2 = 1$ and therefore $\phi_{11} = -0.4597$, $\phi_{21} = 0.8881$.

when $\lambda = 4.7321$, $\begin{bmatrix} 4-\lambda & \sqrt{2} \\ \sqrt{2} & 2-\lambda \end{bmatrix} = \begin{bmatrix} -0.7321 & \sqrt{2} \\ \sqrt{2} & -2.7321 \end{bmatrix}$.

Let $\begin{bmatrix} \phi_{12} \\ \phi_{22} \end{bmatrix}$ denote the corresponding eigenvector of 4.7321, then

$$\begin{bmatrix} -0.7321 & \sqrt{2} \\ \sqrt{2} & -2.7321 \end{bmatrix} \begin{bmatrix} \phi_{12} \\ \phi_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -0.7321\phi_{12} + \sqrt{2}\phi_{22} = 0 \Rightarrow \left.\begin{aligned} \phi_{12} &= 1.9317\phi_{22} \\ \phi_{12}^2 + \phi_{22}^2 &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} \phi_{12} &= 0.8881 \\ \phi_{22} &= 0.4597 \end{aligned}$$

Therefore, $\Phi = \begin{bmatrix} 0.8881 & -0.4597 \\ 0.4597 & 0.8881 \end{bmatrix}$ and KL-transform is

$$Y = \Phi^H X = \begin{bmatrix} 0.8881 & 0.4597 \\ -0.4597 & 0.8881 \end{bmatrix} X.$$

Problem 7.9:

(a) Compute the Haar transform of 2×2 image: $F = \begin{bmatrix} 3 & -1 \\ 6 & 2 \end{bmatrix}$

Ans: The 2×2 Haar transformation matrix is given by as

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Therefore, the Haar transform of the image F is

$$\hat{F} = HFH^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 6 & 2 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ -3 & 0 \end{bmatrix}$$

(b) The inverse transform is $F = H^T \hat{F} H$, where $\hat{F}$ is the Haar transform of F and H^T is the matrix inverse of H. Show that $H_2^{-1} = H_2^T$ and use it to compute the inverse Haar transform of the result in (a).

Ans: Let the inverse Haar transform be $\hat{H} = \frac{1}{\sqrt{2}} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Therefore $\hat{H} H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$\Rightarrow \frac{1}{2} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \frac{1}{2} \begin{bmatrix} a+b & a-b \\ c+d & c-d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \hat{H} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Therefore $\hat{H} \hat{F} H^T$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2 & 4 \\ 8 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 6 & -2 \\ 12 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 6 & 2 \end{bmatrix} = F$$

Exam March 2019:

(a) Show $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is unitary.

Proof: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\cdot\sin\theta - \sin\theta\cdot\cos\theta \\ \sin\theta\cdot\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

It is unitary!

(b) Determine a rotation angle θ corresponding to the given Haar transform.

Ans: $\cos\theta = \frac{\sqrt{2}}{2} = -\sin\theta, \quad \theta = -\frac{\pi}{4}$

(c) $$a_{kl} = \begin{cases} \sqrt{\frac{1}{N}} & k=1, \quad 1 \le l \le N \\ -\sqrt{\frac{2}{N}} \cos\left(\frac{(2(l-1)+1)(k-1)\pi}{2N}\right) & 2 \le k \le N, \quad 1 \le l \le N \end{cases}$$

Ans: $N=2$

$a_{11} = \frac{1}{\sqrt{2}}, \quad a_{12} = \frac{1}{\sqrt{2}}$

$a_{21} = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}, \quad a_{22} = -\cos\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$

$\theta = -\frac{\pi}{4}.$

(d) Find KLT that diagonalizes $C = \frac{1}{4}\begin{bmatrix} 5 & \sqrt{3} \\ \sqrt{3} & 7 \end{bmatrix}$. Determine rotation angle.

Ans: KLT is a transformation that diagonalizes a covariance matrix.
We start by finding the eigenvalues λ_1 and λ_2.
Use the fact that $Tr\{C\} = \lambda_2 + \lambda_1 = 3$.
One of the eigenvalue is 1 according to the hint, the other is 2.
Now we express the KLT by a rotation matrix and solve trigonometric equations of a single variable. Use the fact that $C = \Phi \Lambda \Phi^H$, Let $\Phi^H = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$C = \frac{1}{4}\begin{bmatrix} 5 & \sqrt{3} \\ \sqrt{3} & 7 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

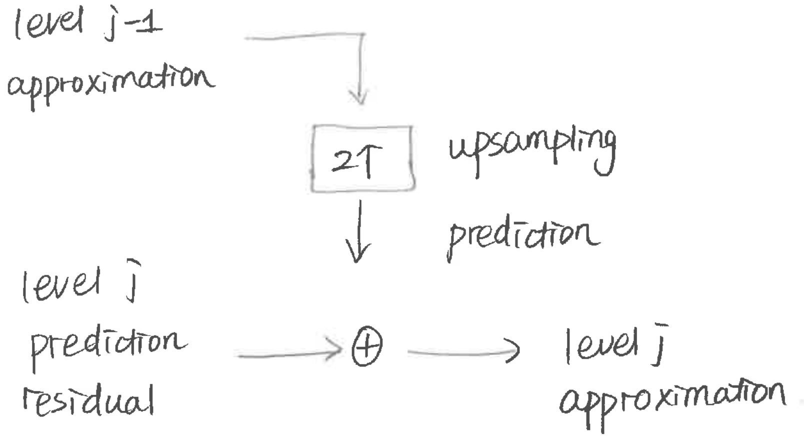
$$= \begin{bmatrix} 2\cos\theta & \sin\theta \\ -2\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 2\cos^2\theta + \sin^2\theta & \cos\theta\cdot\sin\theta \\ \cos\theta\cdot\sin\theta & 2\sin^2\theta + \cos^2\theta \end{bmatrix}$$

$\Rightarrow \cos\theta\cdot\sin\theta = \frac{\sqrt{3}}{4}, \quad \cos^2\theta = \frac{1}{4} \Rightarrow \cos\theta = \frac{1}{2}, \ \sin\theta = \frac{\sqrt{3}}{2}, \Rightarrow \theta = \frac{\pi}{3}$

(e) KLT rotate the space by an arbitrary angle depending on the covariance matrix. DCT rotate the space by a constant angle.

Problem 7.1:

Decoding system for a prediction residual pyramid:



Problem 7.2:

Consider the case with 3 levels, i.e., J=2.

level 2 input image (original image):

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

Downsampling using 2x2 block neighborhood averaging and obtain level 1 approximation image:

$$\begin{bmatrix} 3.5 & 5.5 \\ 11.5 & 13.5 \end{bmatrix}$$

Downsampling level 1 approximation and obtain level 0 approximation image:

$$[8.5].$$

Therefore the approximation pyramid is:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \begin{bmatrix} 3.5 & 5.5 \\ 11.5 & 13.5 \end{bmatrix} \begin{bmatrix} 8.5 \end{bmatrix}.$$

Since intepolation filter is omitted, we consider pixel replication in generation of prediction residual pyramid levels.

Upsampling level 1 approximation and subtract it from level 2 image to obtain level 2 predition residual:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} - \begin{bmatrix} 3.5 & 3.5 & 5.5 & 5.5 \\ 3.5 & 3.5 & 5.5 & 5.5 \\ 11.5 & 11.5 & 13.5 & 13.5 \\ 11.5 & 11.5 & 13.5 & 13.5 \end{bmatrix} = \begin{bmatrix} -2.5 & -1.5 & -2.5 & -1.5 \\ 1.5 & 2.5 & 1.5 & 2.5 \\ -2.5 & -1.5 & -2.5 & -1.5 \\ 1.5 & 2.5 & 1.5 & 2.5 \end{bmatrix}$$

Similarly, we can obtain level 1 prediction residual:

$$\begin{bmatrix} 3.5 & 5.5 \\ 11.5 & 13.5 \end{bmatrix} - \begin{bmatrix} 8.5 & 8.5 \\ 8.5 & 8.5 \end{bmatrix} = \begin{bmatrix} -5 & -3 \\ 3 & 5 \end{bmatrix}$$

Therefore the prediction residual pyramid is:

$$\begin{bmatrix} -2.5 & -1.5 & & \\ 1.5 & 2.5 & & \cdots \\ & & & \\ & \cdots & & \cdots \end{bmatrix} \begin{bmatrix} -5 & -3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 8.5 \end{bmatrix}$$