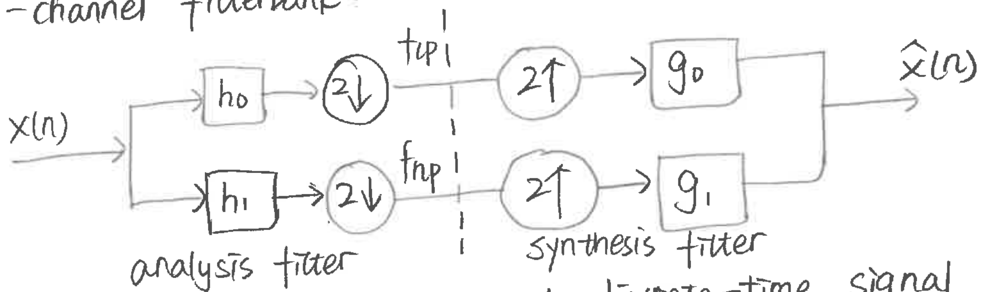
Subband Coding : (multiresolution analysis)

decomposition + reconstruction

Two-channel filterbank :



Input $x(n)$ = 1D , band-limited discrete-time signal

Output $\hat{x}(n)$: reconstruction

Analysis filter bank = h_0 , h_1

h_0 = Low-pass filter

h_1 = high-pass filter

f_{LP} = approximation of $x(n)$

f_{HP} = high-frequency / detail part

Synthesis filter bank = g_0, g_1

combine f_{LP} , f_{HP} to produce $\hat{x}(n)$

Goal of subband-coding :

select h_0, h_1, g_0, g_1 so that $\hat{x}(n) = x(n)$ (perfect reconstruction)

Z-transform (generalization of discrete Fourier Transform) :

$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$, z complex variable such as if $z = e^{jw}$

$2\downarrow$ in time domain :

$$x_{2\downarrow}(n) = x(2n) \Leftrightarrow X_{down}(z) = \frac{1}{2}\left[X(\sqrt{z}) + X(-\sqrt{z})\right]$$

$2\uparrow$ in time domain :

$$x_{2\uparrow}(n) = x\left(\frac{n}{2}\right) \text{ for } n = 0, 2, 4, \cdots \quad \Leftrightarrow \quad X^{up}(z) = X(z^2)$$

$$x_{2\uparrow}(n) = 0 \quad \text{for } n = 1, 3, 5, \cdots$$

System's output:

$$\hat{X}(z) = \frac{1}{2} G_0(z) \left[H_0(z) X(z) + H_0(-z) X(-z) \right]$$
$$+ \frac{1}{2} G_1(z) \left[H_1(z) X(z) + H_1(-z) X(-z) \right]$$

$$= \frac{1}{2} \left[H_0(z) G_0(z) + H_1(z) G_1(z) \right] X(z)$$
$$+ \frac{1}{2} \left[H_0(-z) G_0(z) + H_1(-z) G_1(z) \right] X(-z)$$

For error-free reconstruction, $\hat{x}(n) = x(n) \Leftrightarrow \hat{X}(z) = X(z)$

Therefore,
$$H_0(z) G_0(z) + H_1(z) G_1(z) = 2$$
$$H_0(-z) G_0(z) + H_1(-z) G_1(z) = 0$$

Exercise #6 Wavelet.

Problem 7.4 :

(a) For Quadrature Mirror Filter, from the table, it satisfies

$H_1(z) = H_0(-z)$ ①

$G_0(z) = H_0(z)$ ②

$G_1(z) = -H_0(-z)$ ③

Consider $H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0$, ④

Plug ①, ②, ③ into the left hand side of ④ such that it is expressed with $H_0(z)$, $H_0(-z)$, we obtain that

$H_0(-z)H_0(z) + H_0(z)\cdot(-H_0(-z)) = 0.$

$0 = 0.$

consider $H_0(z)\cdot G_0(z) + H_1(-z)G_1(z) = 2$, Plug ①, ②, ③, we obtain that

$H_0(z)H_0(z) + H_0(-z)(-H_0(-z)) = 2$

$\Leftrightarrow H_0^2(z) - H_0^2(-z) = 2,$

which is the design equation for the $H_0(z)$ prototype filter in the row 1 of the table.

(b) A orthonormal filter satisfies :

$H_0(z) = G_0(z^{-1})$ ⑤

$G_1(z^{-1}) = H_1(z)$ ⑥

$G_1(z) = -z^{-2k+1}G_0(-z^{-1})$ ⑦

Consider $H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0$ ⑧ Plug ⑤, ⑥, ⑦ into the left hand side of ⑧ such that it is expressed with $G_0(z)$ we obtain that

$H_0(-z)\,G_0(z) + H_1(-z)\,G_1(z)$

$= G_0(-z^{-1})\,G_0(z) + G_1(-z^{-1})\,G_1(z)$

$= G_0(-z^{-1})\,G_0(z) - (-z^{-1})^{2k+1}\,G_0(-(-z^{-1})^{-1})(-z^{-2k+1})\,G_0(-z^{-1})$

$= G_0(-z^{-1})\,G_0(z) - (-1)^{-2k+1}\,z^{2k-1}\,(-1)\,z^{-2k+1}\,G_0(z)\,G_0(-z^{-1})$

$= G_0(-z^{-1})\,G_0(z) - G_0(z)\,G_0(-z^{-1})$

$= 0$

Consider $H_0(z)\,G_0(z) + H_1(z)\,G_1(z)$, plug ⑤⑥⑦ in, we obtain

$G_0(z^{-1})\,G_0(z) + G_1(z^{-1})\,G_1(z)$

$= G_0(z^{-1})\,G_0(z) + (-1)(z^{-1})^{-2k+1}\,G_0(-(z^{-1})^{-1})\,(-1)\cdot z^{-2k+1}\,G_0(-z^{-1})$

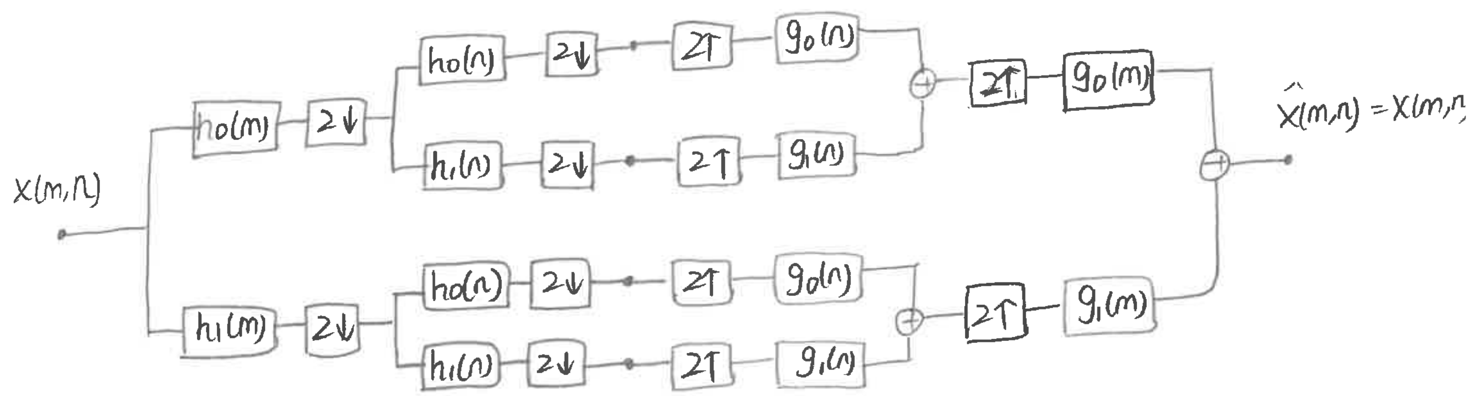
$= G_0(z^{-1})\,G_0(z) + G_0(-z)\,G_0(-z^{-1})$

$= 2$,

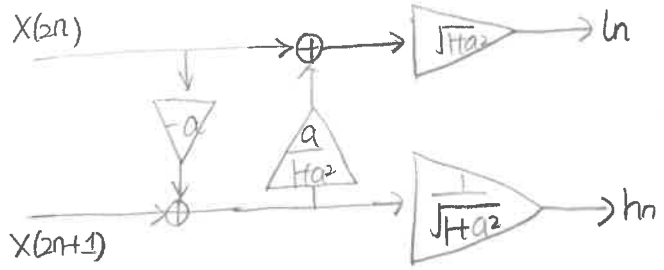
which is the design equation for the $G_0(z)$ prototype filter in the row 3 of the table.

Problem 7.7:

Reconstruction is performed by reversing the decomposition process, i.e., replacing the downsamplers with upsamplers and the analysis filter by their synthesis filter counterparts.



Exam March 2011 (Multiresolution Processing)



(a) Yes. The resulting multiresolution representation is critically sampled. The number of low pass and high pass samples are the same as the number of input pixels.

(b) $l_n = \left[x(2n) + \frac{a}{1+a^2}\left(x(2n+1) - a x(2n)\right)\right] \sqrt{1+a^2} = \frac{a}{\sqrt{1+a^2}} x(2n+1) + \frac{1}{\sqrt{1+a^2}} x(2n)$

$h_n = \left[-a x(2n) + x(2n+1)\right] \frac{1}{\sqrt{1+a^2}} = -\frac{a}{\sqrt{1+a^2}} x(2n) + \frac{1}{\sqrt{1+a^2}} x(2n+1)$

$$\begin{bmatrix} l_n \\ h_n \end{bmatrix} = \frac{1}{\sqrt{1+a^2}} \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \begin{bmatrix} x(2n) \\ x(2n+1) \end{bmatrix}$$

$$T = \frac{1}{\sqrt{1+a^2}} \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}$$

$$TT^T = \frac{1}{1+a^2} \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \begin{bmatrix} 1 & -a \\ a & 1 \end{bmatrix} = \frac{1}{1+a^2} \begin{bmatrix} 1+a^2 & 0 \\ 0 & 1+a^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$T^T T = I$

(c) Let $X = [x_{2n} \ x_{2n+1}]^T$, $Y = [l_n \ h_n]^T$. Then let $YY^T = \Lambda$, where $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$YY^T = TXX^T T^T = TCT^T = \Lambda \Leftrightarrow TC = \Lambda T$

Assume $\lambda_1 = 1$, then $\frac{1}{\sqrt{1+a^2}} \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 14 & -18 \\ -18 & 41 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \frac{1}{\sqrt{1+a^2}} \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}$

$$\frac{1}{5}(14 - 18a) = 1 \quad \Rightarrow \quad a = \frac{1}{2}.$$

(d): Energy compaction means pushing energy into the smallest number of efficients.

Here, Haar $(a=1)$ does not decorrelate the source, hence, Haar is inferior.

(e).