Lloyd-Max Scalar Quantizer:

- for a signal with given PDF $f_x(x)$, find a quantizer with M representative levels such that

$$d = MSE = E[(x-\hat{x})^2] \rightarrow \min$$

Solution:

(1) M-1 decision thresholds exactly halfway between representation levels.

(2) M representative levels in the centroid of the PDF between two successive decision threshold.

(3) Necessary condition:

① $t_q = \frac{1}{2}(\hat{x}_{q-1} + \hat{x}_q)$, $q = 1, 2, \cdots M-1$

② $$\hat{x}_q = \frac{\int_{t_q}^{t_{q+1}} x f(x)\, dx}{\int_{t_q}^{t_{q+1}} f(x)\, dx}, \quad q = 0, 1, \cdots M-1$$

Iterative design:

(1) Guess initial set of representative levels $\hat{x}_q$, $q = 0, 1, \cdots M-1$

(2) Calculate decision thresholds: $t_q = \frac{1}{2}(\hat{x}_{q-1} + \hat{x}_q)$, $q = 1, 2, \cdots M-1$

(3) Calculate new representative levels:

$$\hat{x}_q = \frac{\int_{t_q}^{t_{q+1}} x f_x(x)\, dx}{\int_{t_q}^{t_{q+1}} f_x(x)\, dx}, \quad q = 0, 1, \cdots M-1$$

(4) Repeat (2), (3) until no further distortion reduction.

Properties:

- zero-mean quantization error $E[x-\hat{x}] = 0$
- Quantization error and reconstruction decorrelated $E[(x-\hat{x})\hat{x}] = 0$
- Variance subtraction $\sigma_{\hat{x}}^2 = \sigma_x^2 - E[(x-\hat{x})^2]$
- Equal MSE contributions:

$$Pr\{x \in [t_i, t_{i+1})\} E\{(x-\hat{x})^2 \mid x \in [t_i, t_{i+1})\}$$
$$= Pr\{x \in [t_j, t_{j+1})\} E\{(x-\hat{x})^2 \mid x \in [t_j, t_{j+1})\}$$

for all $i, j$.

Lloyd-Max Quantization :

$$D = E[(X-Q(X))^2] = \int_{-\infty}^{\infty} (x-Q(x))^2 f(x)\,dx = \sum_{k=1}^{M} \int_{t_k}^{t_{k+1}} (x-y_k)^2 f(x)\,dx$$

① $\dfrac{\partial D}{\partial t_k} = 0 \Rightarrow t_k = \dfrac{y_k + y_{k-1}}{2}$

$\dfrac{\partial D}{\partial y_k} = 0 \Rightarrow y_k = \dfrac{\int_{t_k}^{t_{k+1}} x f(x)\,dx}{\int_{t_k}^{t_{k+1}} f(x)\,dx}$

① $\dfrac{d}{dx}\int_a^x f(t)\,dt = f(x), \quad \dfrac{d}{dx}\int_x^a f(t)\,dt = -f(x).$

$$\frac{\partial D}{\partial t_k} = \frac{\partial}{\partial t_k}\left(\int_{t_{k-1}}^{t_k} (x-y_{k-1})^2 f(x)\,dx + \int_{t_k}^{t_{k+1}} (x-y_k)^2 f(x)\,dx\right)$$

$$= (t_k - y_{k-1})^2 f(t_k) - (t_k - y_k)^2 f(t_k) = 0$$

$$\Leftrightarrow (t_k - y_{k-1})^2 - (t_k - y_k)^2 = 0$$

$$\Leftrightarrow (y_k - y_{k-1})(2t_k - y_{k-1} - y_k) = 0$$

$$\Leftrightarrow t_k = \frac{y_{k-1} + y_k}{2}$$

② $$\frac{\partial D}{\partial y_k} = \frac{\partial}{\partial y_k}\left(\int_{t_k}^{t_{k+1}} (x-y_k)^2 f(x)\,dx\right) = \int_{t_k}^{t_{k+1}} \frac{\partial}{\partial y_k}(x-y_k)^2 f(x)\,dx$$

$$= \int_{t_k}^{t_{k+1}} -2(x-y_k) f(x)\,dx = 0 \Leftrightarrow \int_{t_k}^{t_{k+1}} (x-y_k) f(x)\,dx = 0$$

$$\Leftrightarrow \int_{t_k}^{t_{k+1}} x \cdot f(x)\,dx - \int_{t_k}^{t_{k+1}} f(x) y_k\,dx = 0$$

$$\Leftrightarrow y_k = \frac{\int_{t_k}^{t_{k+1}} x \cdot f(x)\,dx}{\int_{t_k}^{t_{k+1}} f(x)\,dx}$$

## Lagrangian Multiplier Method:

Consider an optimization problem with equality constraint:

minimize $f(x,y)$

subject to $g(x,y)=0$

Assumption: both $f$ and $g$ have continuous first partial derivatives.

Solution: Introduce a new variable $\lambda$ (Lagrangian multiplier)

Study the Lagrangian:

$\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda g(x,y)$

Solve $\nabla_{x,y,\lambda} \mathcal{L}(x,y,\lambda)=0$, where $\nabla_{x,y,\lambda} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial \lambda}\right)$

Note that $\nabla_\lambda \mathcal{L}(x,y,\lambda)=0$ gives $g(x,y)=0$.

$\nabla_{x,y} \mathcal{L}(x,y,\lambda)=0$ gives $\nabla_{x,y} f(x,y) = \lambda \nabla_{x,y} g(x,y)$.

Intepretation of $\lambda$: the rate of change of quantity being optimized as a function of the constraint parameter.

## Karush-Kuhn-Tucker Conditions:

Consider a non-linear optimization problem with equality & inequality constraint:

minimize $f_0(x)$

subject to $f_i(x) \le 0$, $i=1,\dots,m$

$h_i(x)=0$, $i=1\cdots p$.

Define Lagrangian: $\mathcal{L}(x,\lambda,\upsilon) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \upsilon\, h_i(x)$

KKT conditions: $f_i(x_{opt}) \le 0$, $i=1\cdots m$

$h_i(x_{opt})=0$, $i=1\cdots p$

$\lambda_i^* \ge 0$, $i=1,\dots,m$.

$\lambda_i^* f_i(x_{opt})=0$. $i=1,\dots,m \Rightarrow$ complementary slackness condition

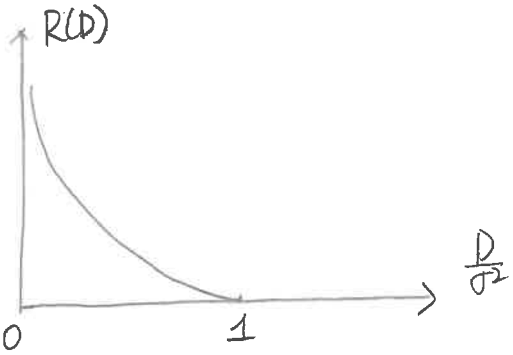
$\nabla_x \mathcal{L}\big|_{x=x_{opt}}=0$

## Exercise # 8 : Compression

### Problem 8.11 :

$$R(D) = \begin{cases} \frac{1}{2}\log\frac{\sigma^2}{D}, & 0 \le D \le \sigma^2 \\ 0, & D \geqslant \sigma^2 \end{cases}$$

(a) Plot $R(D)$ =




(b) At $D = D_{max}$, rate $R = 0$.

Therefore, $R(D_{max}) = \frac{1}{2}\log\frac{\sigma^2}{D_{max}}$.

$$0 = \frac{1}{2}\log\frac{\sigma^2}{D_{max}}$$

$$D_{max} = \sigma^2$$

(c) $R(D) = \frac{1}{2}\log\frac{\sigma^2}{D} \iff D = \sigma^2 2^{-2R}$.

$$10\log_{10} D = 10\log_{10}\sigma^2 2^{-2R} = 10\log_{10}\sigma^2 - 2R \cdot 10\log_{10} 2$$

$$= 10\log_{10}\sigma^2 - (2R)\cdot(3.0103)$$

$$\approx 10\log_{10}\sigma^2 - 6\cdot R \quad (dB)$$

Hence, increase 1 bit, distortion decreases 6 dB.

(d) $D \le 0.75\sigma^2 \iff \sigma^2 \cdot 2^{-2R} \le 0.75\sigma^2 \iff 2^{-2R} \le 0.75$

$\iff R \geqslant -\frac{1}{2}\log_2 0.75 \approx 0.21$ bit

Therefore, at least 0.21 bits per source symbol has to be used to achieve the fidelity objective, which is the maximum possible information compression under this criterion.

(e) Total distortion: $D = D_1 + D_2$

Total rate : $R = R_1 + R_2$

The optimization problem is:

$$\min \; D_1 + D_2$$
$$\text{subject to} \quad R_1 + R_2 \le \bar{R}$$
$$\left.\begin{array}{l} D_1 \le \sigma_1^2 \\ D_2 \le \sigma_2^2 \end{array}\right\} \text{ obtained from (a)}$$

Consider the Lagrangian :

$$J = D_1 + D_2 + \lambda_0 (R_1 + R_2 - \bar{R}) + \lambda_1 (D_1 - \sigma_1^2) + \lambda_2 (D_1 - \sigma_2^2)$$

Differentiate with respect to $D_1$, $D_2$, we obtain

$$\frac{\partial J}{\partial D_1} = 1 - \frac{\lambda_0}{2D_1} + \lambda_1$$

$$\frac{\partial J}{\partial D_2} = 1 - \frac{\lambda_0}{2D_2} + \lambda_2$$

Due to KKT condition, we obtain that for $i = 1, 2$

$\lambda_i = 0$ 计 $D_i < \sigma_i^2$, and $\lambda_i \ge 0$ 计 $D_i = \sigma_i^2$

$\Leftrightarrow$ $1 - \frac{\lambda_0}{2D_i} = 0$ 计 $D_i < \sigma_i^2$ ①

$1 - \frac{\lambda_0}{2D_i} \le 0$ 计 $D_i = \sigma_i^2$ ②

① $\Rightarrow$ when $D_1 < \sigma_1^2$, $D_2 < \sigma_2^2$, we have $D_1 = D_2 = \frac{\lambda_0}{2}$

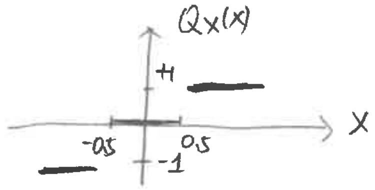
## 2. Uniform Quantizer:

(a) A valid pdf integrates to 1: $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$\int_{-1.5}^{1.5} f_X(x)\,dx = a + 2a + 4a = 7a$

Therefore $7a = 1$ and $a = \frac{1}{7}$.

(b) Plot $Q_X(x) = \begin{cases} -1, & -\infty < x < -0.5 \\ 0, & -0.5 \le x < 0.5 \\ 1, & 0.5 \le x < +\infty \end{cases}$




(c) Mean-square error of the quantization on source:

$$E[(X-\hat{X})^2] = \int_{-\infty}^{\infty} (x-\hat{x})^2 f_X(x)\,dx$$

$$= \int_{-1.5}^{-0.5} (x-\hat{x})^2 f_X(x)\,dx + \int_{-0.5}^{0.5} (x-\hat{x})^2 f_X(x)\,dx + \int_{0.5}^{1.5} (x-\hat{x})^2 f_X(x)\,dx$$

$$= \int_{-1.5}^{-0.5} (x-(-1))^2 \cdot \frac{1}{7}\,dx + \int_{-0.5}^{0.5} (x-0)^2 \cdot \frac{2}{7}\,dx + \int_{0.5}^{1.5} (x-1)^2 \cdot \frac{4}{7}\,dx$$

$$= \frac{1}{7}\cdot\frac{1}{12} + \frac{2}{7}\cdot\frac{1}{12} + \frac{4}{7}\cdot\frac{1}{12}$$

$$= \frac{1}{12}$$

(d) Given $f_X(x) = \begin{cases} 2, & x \in [-0.25, 0.25] \\ 0, & \text{ow} \end{cases}$, we obtain that

$$E[(X-\hat{X})^2] = \int_{-\infty}^{\infty} (x-\hat{x})^2 f_X(x)\,dx = \int_{-0.25}^{0.25} (x-0)^2 \cdot 2\,dx = \frac{1}{48}.$$

## Problem 8.21:

Define initial representation levels $\hat{x}_0 = -A$, $\hat{x}_1 = A$

Threshold update: $t_1 = \frac{1}{2}(\hat{x}_0 + \hat{x}_1) = \frac{1}{2}(-A + A) = 0$

Now we update $\hat{x}_0, \hat{x}_1$ as follows:

$$\hat{x}_0 = \frac{\int_{t_0}^{t_1} x\,p(x)\,dx}{\int_{t_0}^{t_1} p(x)\,dx} = \frac{\int_{-\infty}^{0} x\,p(x)\,dx}{\int_{-\infty}^{0} p(x)\,dx} = \frac{\int_{-A}^{0} x \cdot \frac{1}{2A}\,dx}{\int_{-A}^{0} \frac{1}{2A}\,dx} = \frac{-\frac{x^2}{4A}\Big|_{-A}^{0}}{\frac{1}{2}} = -\frac{A}{2}.$$

Due to symmetry $\hat{x}_1 = \frac{A}{2}$.

(Result converged. Stop here.)

## 4. Lloyd-Max Scalar Quantizer:

Experimental data $\{0, 0, 1, 2, 2, 6, 6, 6, 8, 8, 8\}$

Initial condition of codebook $\{0, 1, 2, 3\}$.

Lloyd Algorithm with training data:

1. Guess initial set of representative levels $\hat{x}_q$; $q = 0, 1, \cdots M-1$
2. Assign Each Sample $X_i$ in training set T to closest representative $\hat{x}_q$.

$$B_q = \{x \in T : Q(x) = q\}, \quad q = 0, 1, 2, \cdots, M-1$$

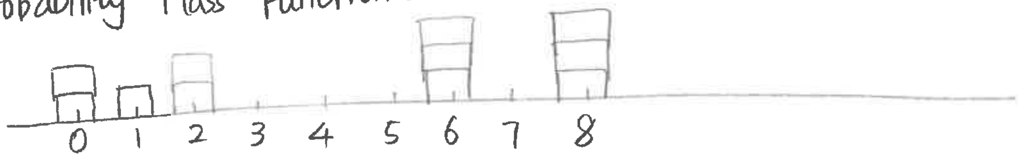
3. Calculate new representative levels:

$$\hat{x}_q = \frac{1}{\|B_q\|} \sum_{x \in B_q} x, \quad q = 0, 1, \cdots M-1 \qquad (\|\cdot\| \text{ cardinality of a set})$$

4. Repeat 2&3 until no further distortion reduction.

(a)

Probability Mass Function:



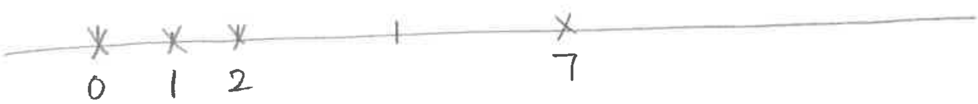

* representative levels

| thresholds

Initial Codebook:


Representative Update:


Threshold Update:




Converged. STOP

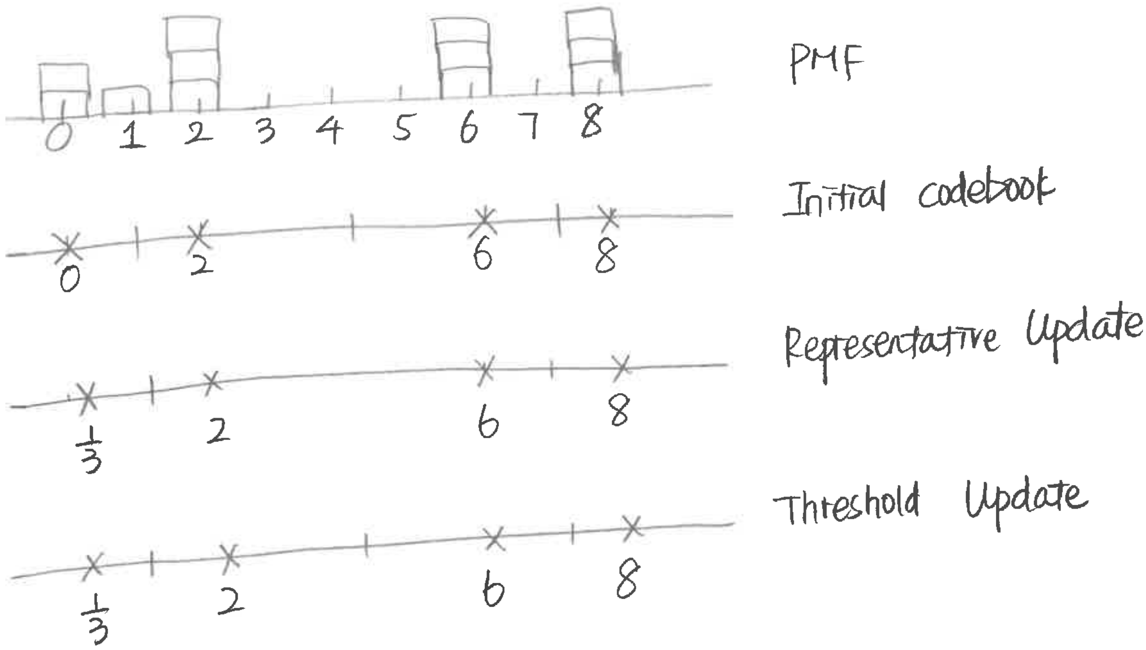
(b) It is locally optimal, i.e., any small change in the training data results in a degradation in performance.

(c) Total error =

# data points
↑

$$D = \sum_{i=1}^{4} d_i = \sum_{i=1}^{4} d(x_i, c_i) = 2\cdot 0 + 1\cdot 0 + 2\cdot 0 + 3\cdot 1 + 3\cdot 1 = 6$$

(d). Initial codebook $\{0, 2, 6, 8\}$




Converged. STOP

Total distortion:

$$D = 2\cdot\left(\tfrac{1}{3}\right)^2 + 1\cdot\left(\tfrac{2}{3}\right)^2 + 3\times 0 + 3\cdot 0 = \tfrac{2}{3} << 6.$$

Therefore this initial codebook is better than the previous one.