1. *True or False Statements (10p)*

   Are the statements below true or false? You get 1 point if your answer is correct, 1 point is deducted if your answer is wrong, and 0 points if no answer is given. The total score for this problem will not be smaller than zero. Answer only true, false, or no answer. Do not provide any motivation (only this problem).

   (a) ($\pm$1p) Two different images cannot have the same histogram. **F**

   (b) ($\pm$1p) The Karhunen-Loéve transform only requires first order statistics. **F**

   (c) ($\pm$1p) P frames are encoded independently of the other frames. **F**

   (d) ($\pm$1p) The Hough transform of a point, using the line detection model of the template (m, c) such that $y = mx + c$, is a point. **F**

   (e) ($\pm$1p) Histogram equalization always yields uniform *pmf*. **F**

   (f) ($\pm$1p) The iterative Lloyd-Max quantizer design method always converges to the global optimum. **F**

   (g) ($\pm$1p) A McMillan sum of one is a sufficient condition for decodability. **F**

   (h) ($\pm$1p) The Wiener filter minimizes the MSE of the reconstruction error. **T**

   (i) ($\pm$1p) The intensity component contains no information about the color. **T**

   (j) ($\pm$1p) Consider lossy coding of a memoryless Gaussian source using the MSE distortion. An increase in the bit-rate by 1 bit can increase the SNR by at most 6.02 dB. **T**

2. *Histogram (15p)*

   In this problem, we consider transforms that equalize images.

   We represent the pixel values of an image by a continuous Beta random variable $x_{\alpha,\beta} \sim \mathrm{B}(\alpha, \beta)$ with the pdf:

$$p_{x,\alpha,\beta}(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} (1-x)^{\beta-1} x^{\alpha-1};\ x \in [0,1],\ \alpha > 0,\ \beta > 0. \tag{1}$$

   where Γ is the Gamma function, $\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}\mathrm{d}t$. Note that for a natural number n, $\Gamma(n) = (n-1)!$. Note $\Gamma(1) = 1$

   In this problem, the continuous histogram transform is denoted T, and the discrete histogram transform H.

   (a) (2p) Let $\forall(\alpha,\beta) \in \mathbb{R}_+^2$, $x_{\alpha,\beta} \sim \mathrm{B}(\alpha,\beta)$. Let be $y = T(x_{\alpha,\beta})$, the pixel values after the continuous histogram transform T. Show that these values are distributed according to $y \sim \mathrm{B}(1,1)$.
   Solution: We know that the histogram transform of an absolutely continuous random variable of compact support has an uniform pdf. Hence,

$$y = T(x_{\alpha,\beta}) \sim \mathcal{U}[0,1].\ (1p)$$

   A fortiori

$$\begin{aligned} p_y(y) &= \mathbb{1}_{[0,1]}(y) \\ &= \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)} (1-y)^{1-1} y^{1-1} \mathbb{1}_{[0,1]}(y) \\ &= p_{y,11}(y) \end{aligned}$$

   Hence $y \sim \mathrm{B}(1,1)$ $(1p)$

   (b) (2p) Consider $a = 1$, $b = 2$.Sketch the pdf of the random variable $x_{a,b} \sim \mathrm{B}(a,b)$. Describe an image having such a pdf.
   (1p) What if $a = 2$, $b = 1$?
   Solution:

   - a=1, b=2. $p_{x,1,2}(x) = 2(1-x)\mathbb{1}_{[0,1]}(x)$, (sketch 1p), the image has mostly dark tones.(1p)
   - a=2,b=1. $p_{x,2,1}(x) = 2x\mathbb{1}_{[0,1]}(x)$ and the image has mostly clear/light tones.(1p)

   (c) (2p) Now we consider the following two-bit quantizer:

$$Q(x) = \begin{cases} \frac{1}{8} \text{ for } 0 \leq x < \frac{1}{4} \\ \frac{3}{8} \text{ for } \frac{1}{4} \leq x < \frac{1}{2} \\ \frac{5}{8} \text{ for } \frac{1}{2} \leq x < \frac{3}{4} \\ \frac{7}{8} \text{ for } \frac{3}{4} \leq x \leq 1 \end{cases}. \tag{2}$$

   Determine the *pmf* of the discrete random variable $y_{1,2} = Q(x_{1,2})$.
   Solution:

$$\begin{cases} P(y = \frac{1}{8}) = \int_0^{\frac{1}{4}} 2(1-x)dx = [2x - x^2]_0^{\frac{1}{4}} = \frac{7}{16} \\ P(y = \frac{3}{8}) = \frac{5}{16} \\ P(y = \frac{5}{8}) = \frac{3}{16} \\ P(y = \frac{7}{8}) = \frac{1}{16} \end{cases} \quad (2p).$$

(d) (2p) Determine the discrete histogram equalization $\tilde{y} = H(y_{1,2})$.
Solution:

$$\tilde{y} = H(y_{1,2}) = \begin{cases} \frac{7}{16} \text{ for } y = \frac{1}{8} \\ \frac{12}{16} \text{ for } y = \frac{3}{8} \\ \frac{15}{16} \text{ for } y = \frac{5}{8} \\ 1 \text{ for } y = \frac{7}{8} \end{cases} \quad (2p, 1p \text{ if the formula is correct}). \tag{3}$$

(e) (2p) We now reverse the order of the operations and define $y' = Q(T(x_{1,2}))$. What is the *pmf* of y'?
Solution: $P(y') = \frac{1}{4}$ (2p)

(f) (2p) Compute the entropy of $\tilde{y}$ and y'.
Solution: Remembering that $H(y) = -\sum p_i \log_2(p_i)$ (1p), then $H(\tilde{y}) = 1.75$, $H(y') = 2$ (1p)

(g) (2p) Comment on the differences between $\tilde{y}$ and y'.
Solution:
Discrete equalization cannot yield a uniform histogram since it is a one to one mapping (1p). Hence the operations 'Quantization' and 'Equalization' cannot be reversed. (1p)

3. *Frequency Domain Processing (15p)*

The discrete Fourier transform of an image $f(x,y)$ of size $M \times N$ is given by

$$F(u,v) = \mathcal{F}\{f\}(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M+vy/N)}. \tag{4}$$

The discrete convolution of two images $f(x,y)$ and $h(x,y)$ of size $M \times N$ is defined by the expression

$$f * h\,(x,y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) h(x-m, y-n). \tag{5}$$

Now, we model our images, e.g. f(x,y) and g(x,y), as two-dimensional wide-sense stationary random processes. Then the cross-correlation between f and g is defined by the expected value

$$\phi_{f,g}\,(x,y) = \mathbb{E}[f(x+m, y+n) g^*(m,n)], \tag{6}$$

where g^* denotes the complex conjugate of g. Eventually we write the power spectral density

$$\Phi_{f,h}(u,v) = \mathcal{F}\{\phi_{f,g}\}\,(u,v). \tag{7}$$

We consider the distortion model of Fig. 1, where f is the original uncorrupted image, η is a noise signal and q a convolutive kernel representing a distortion. For simplicity, let the noise signal η be statistically independent from the image f. Eventually, g is the observed signal from which we wish to recover the original image f.

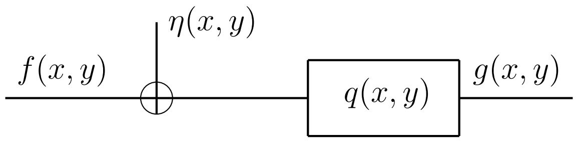

Figure 1: Distortion model

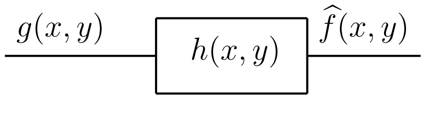

Figure 2: Recovery filter

(a) (3p) Recall and demonstrate the convolution theorem using the definitions (4) and (5).
Solution:
The convolution theroem states that

$$\mathcal{F}\{f * g\} = F \cdot G \text{ (1p)}.$$

$$\begin{aligned}
\mathcal{F}\{f * g\} &= \sum_x \sum_y \sum_m \sum_n f(m,n)h(x-m,y-n)\exp\left(-2\pi j\left(\frac{ux}{M}+\frac{vy}{N}\right)\right)\\
&= \sum_m \sum_n f(m,n)\exp\left(-2\pi j\left(\frac{um}{M}+\frac{vn}{N}\right)\right)\\
&\times \sum_x \sum_y h(x-m,y-n)\exp\left(-2\pi j\left(\frac{u(x-m)}{M}+\frac{v(y-n)}{N}\right)\right)\\
\mathcal{F}\{f * g\} &= F(u,v)G(u,v) \text{ (2p).}
\end{aligned}$$

(b) (1p) Show that $G(u,v) = Q(u,v)F(u,v) + Q(u,v)\eta(u,v)$.
Solution:

$$\begin{aligned}
g(x,y) &= (q * (f+\eta))(x,y)\\
\mathcal{F}(g)(u,v) &= Q(u,v)(F(u,v)+\eta(u,v))
\end{aligned}$$

(c) Now we want to recover the original image f from the observed image g, while minimizing the expected square error $\mathbb{E}\left[\left(f-\widehat{f}\right)^2\right]$.

i. (3p) Express $\Phi_{g,g}$ as a function of $Q(u,v)$, $\Phi_{f,f}$ and $\Phi_{\eta,\eta}$.
Solution: Under the assumption that the image and the noise are independent we can derive

$$\begin{aligned}
\Phi_{g,g}(u,v) &= \mathcal{F}(\phi_{g,g})(u,v)\\
&= \mathcal{F}\{\mathbb{E}\left[g^*(m,n)g(x+m,y+n)\right]\}\\
&= \mathcal{F}\{\mathbb{E}\left[q^*(m,n)(f^*(m,n)+\eta^*(m,n))\right.\\
&\quad \times q(m+x,n+y)(f(m+x,n+y)+\eta(x+m,n+y))]\}\\
\Phi_{g,g}(u,v) &= |Q(u,v)|^2\left(\Phi_{f,f}(u,v)+\Phi_{\eta,\eta}(u,v)\right) \qquad (8)
\end{aligned}$$

ii. (3p) We define the error image $e = f - \widehat{f}$ where $\widehat{f}$ is obtained as shown in Fig. 2 from a recovery filter h. Express the power spectral density $\Phi_{e,e}$ as a function of $H(u,v)$, $Q(u,v)$, $\Phi_{f,f}$ and $\Phi_{\eta,\eta}$.
Solution:

$$\begin{aligned}
\Phi_{e,e}(u,v) &= \Phi_{\hat{f},\hat{f}}(u,v) + \Phi_{f,f}(u,v) - \Phi_{f,\hat{f}}(u,v) - \Phi_{\hat{f},f}(u,v)\\
&= |H(u,v)|^2\Phi_{g,g}(u,v) + \Phi_{f,f}(u,v) - H(u,v)\Phi_{g,f} - H^*(u,v)\Phi_{f,g}\\
\Phi_{e,e}(u,v) &= |H(u,v)Q(u,v)|^2\left(\Phi_{f,f}(u,v)+\Phi_{\eta,\eta}(u,v)\right) + \Phi_{f,f}\\
&\quad - H(u,v)Q(u,v)\Phi_{f,f} - H^*(u,v)Q^*(u,v)\Phi_{f,f} \qquad (9)
\end{aligned}$$

iii. (3p) We recall that the Wiener filter minimizes the expected mean square error and is obtained as

$$H(u,v) = \frac{\Phi_{f,g}(u,v)}{\Phi_{g,g}(u,v)}. \qquad (10)$$

Express H in terms of $Q(u,v)$, $\Phi_{f,f}$ and $\Phi_{\eta,\eta}$.
Solution: It is easily found that

$$H(u,v) = \frac{\Phi_{f,f}}{Q(u,v)\left(\Phi_{f,f}+\Phi_{\eta,\eta}\right)}$$

iv. (1p) What happens if there exists a frequency $(\omega_{x0}, \omega_{y0})$ for which $Q(\omega_{x0}, \omega_{y0}) = 0$?
Solution: If there exist such pair, the Wiener filter has poles and is not stable. It also means that the distortion introduced by q cuts this frequency component and hence that it cannot be recovered.

v. (1p) Propose a better distortion model for an image.
Solution: A simple way to cope with the problem of the previous question is to introduce another noise component in our model after the filter q.

4. *Multiresolution Processing (15p)*

Consider the following lifting implementation for multiresolution image processing as illustrated in Figure 3.

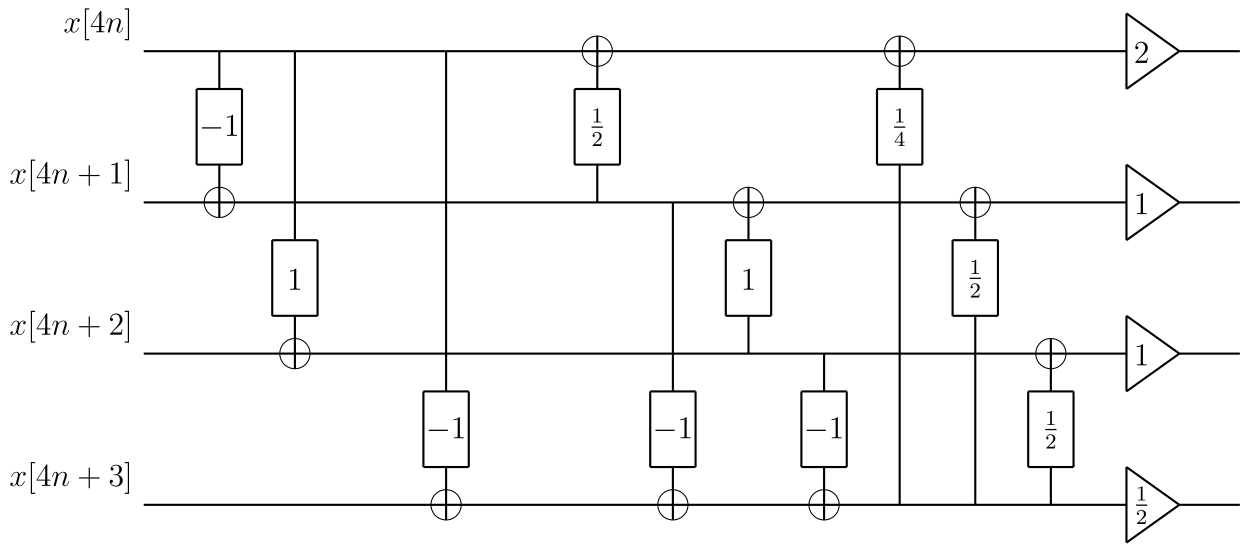

Figure 3: Lifting implementation of analysis filterbank

(a) (2p) Is the resulting representation critically sampled? Please explain.
Solution: The signal representation is critically sampled since it has the same number of inputs and outputs.

(b) (3p) Derive the transform matrix associated with this lifting scheme.

Hint: Your transform should be of the type $\frac{1}{\sqrt{a^2+b^2+c^2+d^2}}\begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}$.

Solution:

$$H = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}.$$

(c) (1p) Is the obtained transform energy preserving? (2p) Is it unitary?
Solution: The transform is unitary since $H^{\mathrm{T}}H = I_4$, it is then energy preserving.

(d) (5p) Constuct the corresponding synthesis filterbank.
Solution: This is obtain by reversing the lifting structure of the analysis filter bank.

(e) (2p) Would you achieve perfect reconstruction if you exchange the order of the analysis and synthesis filterbanks? Please explain.
Solution: The matrix associated to the synthesis operation is H^{T} it is also unitary and its synthesis operation is the analysis operation of the former scheme. Hence we still achieve perfect reconstruction.

5. *Lossless Image Coding (15p)*

We want to compress the following source X where each event $X = i$ has the probability P_i :

X	P_i
1	0.25
2	0.5
3	0.125
4	0.125

(a) (2p) What is the entropy of the source?
Solution: $H(X) = 1.75$ (2p, 1p if the formula is correct).

(b) (2p) Determine the Huffman code for the source.
Solution: This is a possible Huffman code for the source

X	Codeword
1	10
2	0
3	110
4	111

(c) (1p) Is the Kraft-McMillan inequality satisfied?
Solution: Yes.

(d) Consider the following coding technique where we use the modified CDF:

$$\bar{F}(i) = \sum_{j \leq i} P_j - \frac{1}{2} P_i. \tag{11}$$

i. (1p) Show that the value of $\bar{F}$ can be used for uniquely encoding any source symbol x, i.e that is $\forall x, y$, we have $\bar{F}(x) \neq \bar{F}(y) \Leftrightarrow x \neq y$.
Solution: $\Leftarrow$ is trivial.
$\Rightarrow$: Without loss of generality we can assume $x > y$ we hence can write:

$$\bar{F}(x) = \sum_{i<y} P_i + \sum_{j=y}^{x} P_j - \frac{1}{2} P_x$$

Since we add stricly positive values of P_i:

$$\bar{F}(x) > \sum_{i<y} P_i + \frac{1}{2} P_y = \bar{F}(y). \text{ (1p)} \tag{12}$$

ii. (3p) Now we truncate the value of $\bar{F}(x)$ such that we can represent it by $\ell(x)$ bits. We denote the obtained value by $\lfloor \bar{F}(x) \rfloor_{\ell(x)}$. It has the property that $\bar{F}(x) - \lfloor \bar{F}(x) \rfloor_{\ell(x)} < \frac{1}{2^{\ell(x)}}$. With that show that using $\ell(x) = \lceil -\log_2(P_x) \rceil + 1$ bits is sufficient to uniquely encode x.

Solution:

$$\begin{aligned}
&\bar{F}(x) - \lfloor \bar{F}(x) \rfloor_{\ell(x)} < \frac{1}{2^{\ell(x)}} \\
\Leftrightarrow &\bar{F}(x) - \lfloor \bar{F}(x) \rfloor_{\ell(x)} < \frac{1}{2^{1-\lceil \log_2(P_x) \rceil}} \\
\Leftrightarrow &\bar{F}(x) - \lfloor \bar{F}(x) \rfloor_{\ell(x)} < \frac{P_x}{2} \\
\Leftrightarrow &\bar{F}(x) - \frac{P_x}{2} < \lfloor \bar{F}(x) \rfloor_{\ell(x)}
\end{aligned} \tag{13}$$

Assuming again without loss of generality $x > y$ we hence can write:

$$\begin{aligned}
&\sum_{i<x} P_i \geq \sum_{j \leq y} P_j \\
\Leftrightarrow &\lfloor \bar{F}(x) \rfloor_{\ell(x)} > \sum_{i<x} P_i \geq \lfloor \bar{F}(y) \rfloor_{\ell(y)}
\end{aligned}$$

We then have $\lfloor \bar{F}(y) \rfloor_{\ell(y)} \neq \lfloor \bar{F}(x) \rfloor_{\ell(x)} \Leftrightarrow x \neq y$.

iii. (1p) Show that, for any source $\mathcal{X}$, the average length of such code is $L < H(\mathcal{X}) + 2$.
Solution: The average length writes $L = \sum P_i \ell(i) = \sum P_i(\lceil -\log_2(\mathrm{P}_i) \rceil + 1)$, hence $L < \sum P_i((-\log_2(\mathrm{P}_i) + 1) + 1)$, leading to the desired formula. (1p)

iv. (2p) What is the average length of the code for the above source? Is the Kraft-McMillan inequality satisfied?
Solution: The length table is

X	Length
1	3
2	2
3	4
4	4

with an average length of 2.75 the McMillan sum is $1/2 < 1$.

v. (3p) Give the code table for the above source.
Hint: Recall that the binary representation of a number $z \in [0, 1[$ that uses $\ell(z)$ bits is simply

$$z = \sum_{i=1}^{\ell(z)} b_i 2^{-i}; \qquad b_i \in 0, 1. \tag{14}$$

Then we write $z := \{b_1 b_2 \cdots b_{\ell(z)}\}_2$. For example $0.75 = 2^{-1} + 2^{-2} = \{11\}_2$.
Solution:

X	$\bar{F}$	Codeword
1	0.125	001
2	0.5	10
3	0.8125	1101
4	0.9375	1111