Exercise #7 : Compression

Problem 8.1 :

(a) Can variable-length coding procedures be used to compress a histogram equalized image with 2^n gray levels? Explain.

Ans: r_k = discrete random variable represents the gray levels

n_k = number of pixels with gray level k

n = total number of pixels

Each r_k occurs with probability $P_k(r_k) = \frac{n_k}{n}$

$l(r_k)$ = Average number of bits used to represent r_k

Average number of bits required to represent each pixel

$$L_{avg} = \sum_{k=0}^{L-1} P_r(r_k)\, l(r_k)$$

A ideally histogram equalized image has uniform intensity distribution

$$P_r(r_k) = \frac{1}{2^n}.$$

Therefore $L_{avg} = \frac{1}{2^n} \sum_{k=0}^{L-1} l(r_k)$

Since equal probable, no advantage by assigning different bits.

Therefore, assign each the fewest possible bits required to cover 2^n levels.

$$L_{avg} = \frac{1}{2^n} \sum_{k=0}^{L-1} n = \frac{2^n}{2^n} \cdot n = n.$$

(b). Spatial redundancy is associated with the geometric arrangement of the intensities in the image, it is possible for a histogram equalized image to contain a high level of spatial redundancy, or none at all.

Problem 8.7 : prove maximum value of entropy is achieved iff all symbols equiprobable.

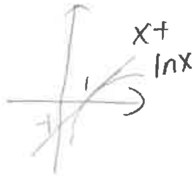
Ans: Source symbols: $\{a_1, a_2, \cdots, a_q\}$

Corresponding probabilities: $z = [P(a_1), P(a_2), \cdots, P(a_q)]^T$

Entropy = $H(z) = -\sum_{i=1}^{q} P(a_i) \log P(a_i)$

$$\log q - H(z) = \log q + \sum_{i=1}^{q} P(a_i) \log P(a_i)$$

$$= \sum_{i=1}^{q} P(a_i) \log q + \sum_{i=1}^{q} P(a_i) \log P(a_i)$$



$$= \sum_{i=1}^{q} P(a_i) \log (q P(a_i))$$

$$= \log e \sum_{i=1}^{q} P(a_i) \ln (q P(a_i)).$$

Then, multiplying the inequality $\ln x \leq x-1$ by -1 to get $\ln \frac{1}{x} \geq 1-x$

$$\log q - H(z) \geq \log e \sum_{i=1}^{q} P(a_i) \left(1 - \frac{1}{q P(a_i)}\right)$$

$$= \log e \left[\sum_{i=1}^{q} P(a_i) - \frac{1}{q} \sum_{i=1}^{q} \frac{P(a_i)}{P(a_i)} \right]$$

$$= \log e [1 - 1] = 0$$

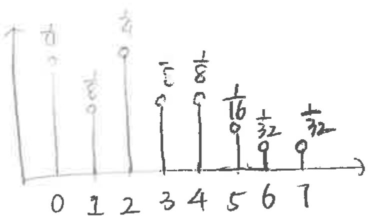
$$\log q - H(z) \geq 0, \quad \log q \geq H(z).$$

Therefore, $H(z)$ is always less than, or equal to, $\log q$.

The equality "=" is achieved when $\ln q P(a_i) = 1 - \frac{1}{q P(a_i)}$ for all i.

$\Rightarrow q P(a_i) = 1 \Rightarrow P(a_i) = \frac{1}{q}$.

3. (a)



(b) Entropy: $H = -\sum_{j=0}^{7} P(a_j) P(a_j) = 2.6875$ bits/pixel.

If histogram more flat, the entropy is higher.

picky, lower.

(c). pmf uniform $\longrightarrow$ highest entropy 3 bits/pixel

(d) Example: uniform pmf can be encoded at a lower rate than entropy.

pixel 1&2 are highly correlated, only one has to be encoded.

1.5 bits/pixel

(e). Describe a procedure for encoding at bit-rate lower than entropy.

- Calculate joint probabilities of the images.
- bit rate/pixel of joint coding < entropy of a pixel.

4. Exam March 2010; Wavelets.

(a) Given $h_\varphi = \frac{1}{4\sqrt{2}}[1+\sqrt{3} \quad 3+\sqrt{3} \quad 3-\sqrt{3} \quad 1-\sqrt{3}]$, and orthogonality condition

$$h_\psi(n) = (-1)^n h_\varphi(N-1-n),$$ we obtain that

$$h_\psi = \frac{1}{4\sqrt{2}}[1-\sqrt{3} \quad -3+\sqrt{3} \quad 3+\sqrt{3} \quad -1-\sqrt{3}].$$

(b) To show the orthonormality, we find the inner products of h_φ, h_ψ,

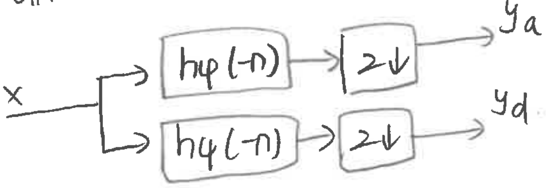
$$h_\varphi h_\varphi^T = \frac{1}{32}[1+\sqrt{3} \quad 3+\sqrt{3} \quad 3-\sqrt{3} \quad 1-\sqrt{3}][1+\sqrt{3} \quad 3+\sqrt{3} \quad 3-\sqrt{3} \quad 1-\sqrt{3}]^T = 1,$$

$$h_\psi h_\psi^T = \frac{1}{32}[1-\sqrt{3} \quad -3+\sqrt{3} \quad 3+\sqrt{3} \quad -1-\sqrt{3}][1-\sqrt{3} \quad -3+\sqrt{3} \quad 3+\sqrt{3} \quad -1-\sqrt{3}]^T = 1.$$

$$h_\varphi h_\psi^T = \frac{1}{32}[1+\sqrt{3} \quad 3+\sqrt{3} \quad 3-\sqrt{3} \quad 1-\sqrt{3}][1-\sqrt{3} \quad -3+\sqrt{3} \quad 3+\sqrt{3} \quad -1-\sqrt{3}] = 0.$$

Therefore h_φ and h_ψ are orthonormal vectors.

(c) One-dimensional two-band analysis filter bank:



(d) Apply $h_\varphi(-n)$ to $[x\, x\, x]$, i.e., find convolution between $h_\varphi(-n) \cdot [x, x, x] =$

$4\sqrt{2}\ 4\sqrt{2} \cdots 4\sqrt{2}$ ($\# 4\sqrt{2} = 18$)

$\frac{1}{4\sqrt{2}}[1+\sqrt{3} \quad 3+\sqrt{3} \quad 3-\sqrt{3} \quad 1-\sqrt{3}]$ → Mirror $h_\varphi(-n)$

$\frac{1}{4\sqrt{2}}[1+\sqrt{3} \quad 3+\sqrt{3} \quad 3-\sqrt{3} \quad 1-\sqrt{3}]$

$\frac{1}{4\sqrt{2}}[1+\sqrt{3} \quad 3+\sqrt{3} \quad 3-\sqrt{3} \quad 1-\sqrt{3}]$

$\frac{1}{4\sqrt{2}}[1+\sqrt{3} \quad 3+\sqrt{3} \quad 3-\sqrt{3} \quad 1-\sqrt{3}]$

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$$\boxed{f * g(n) = \sum_m f(m)\, g(n-m)}$$

The convolution is $[1-\sqrt{3} \quad 4-2\sqrt{3} \quad 7-\sqrt{3} \quad \underbrace{8 \cdots 8}_{\#8 = 15} \quad 7+\sqrt{3} \quad 4+2\sqrt{3} \quad 1+\sqrt{3}]$

Truncating and downsampling gives $y_a = [8 \ 8 \ 8]$

(e) k is chosen such that the first sample is kept after downsampling is filtered with all filter taps. For example,

— — — $4\sqrt{2}\ 4\sqrt{2}$

$\frac{1}{4\sqrt{2}}[1+\sqrt{3} \quad 3+\sqrt{3} \quad 3-\sqrt{3} \quad 1-\sqrt{3}]$

. . .

(without zero padding $4\sqrt{2}\ 4\sqrt{2}\ 4\sqrt{2}\ 4\sqrt{2}$

$\frac{1}{4\sqrt{2}}[1+\sqrt{3} \quad 3+\sqrt{3} \quad 3-\sqrt{3} \quad 1-\sqrt{3}]$)

Hence, $k = N-1$ in general. In this example, $k=3$ samples have to be added on each side.

(f) The computational complexity lies in the convolution of a length $L+2(N-1)$ signal with a length N filter.

Each output required N multiplications, $N-1$ additions.

There are $L+2(N-1)$ output. And there is downsampling operation.

Therefore total $\frac{1}{2}N(L+2(N-1))$ multiplications

$\frac{1}{2}(N-1)(L+2(N-1))$ additions.

(g) Distortion between x and y_a, y_d.

$\sum_n x(n)^2 - \sum_n y_a^2(n) = 192 - 192 = 0.$

Follow the analysis in (d), we obtain that $y_d = [0\ 0\ 0]$.

$\sum_n x(n)^2 - \sum_n y_d^2(n) = 192 - 0 = 192.$

(f) can also be answered in terms of complexity order $O(\cdot)$.

Each output requires N "x".

There are $O(L)$ output, and there is downsampling operation.

Therefore total $O(\frac{LN}{2})$ "x".

For $O(\cdot)$, constant can be dropped, thus $O(LN)$

Similaly, for addition, it is also $O(LN)$