Primer Parcial

1 a) $Cond_\infty(A_n) \geq \dfrac{\|A_n\|_\infty}{\|A_n - B\|_\infty}$ $\forall$ B singular

$\|A_n\|_\infty = \max\left\{ \frac{1}{n}+n^2, \frac{1}{n}, 1+\frac{1}{n} \right\}$. Como $n \geq 1 \Rightarrow \|A_n\|_\infty = n^2+\frac{1}{n}$ $\forall n \in \mathbb{N}$.

Considero $B_n = \begin{pmatrix} 1/n & 0 & \cdots & n^2 \\ 0 & 1/n & \cdots & 0 \\ \vdots & & & \\ 0 & & 1/n & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ (Como B_n tiene una fila de ceros B_n es singular.)

$\rightarrow \begin{cases} B_i = A_i \quad \forall\ 1 \leq i \leq n-1 \\ \cancel{(B_n)_j = 0 \ \forall\ j} \\ B_n = (0 \cdots 0) \end{cases}$

$A_n - B_n = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \\ 1 & 0 \cdots 0 & 1/n \end{pmatrix} \Rightarrow \|A_n - B_n\|_\infty = 1+\frac{1}{n}$

$\Rightarrow Cond_\infty(A_n) \geq \dfrac{n^2+1/n}{1+1/n} = \dfrac{n^3+1}{n+1} \xrightarrow[n\to\infty]{} \infty$

b) $Cond_2(A_n) \geq \dfrac{\|A_n\|_2}{\|A_n - B_n\|_2}$

Por el ejcio 10 de la P2 $\|A_n\|_2 \geq \frac{1}{\sqrt{n}}\|A_n\|_\infty$ y $\|A_n - B_n\|_2 \leq \sqrt{n}\,\|A_n - B_n\|_\infty$

$\Rightarrow Cond_2(A_n) \geq \dfrac{\frac{1}{\sqrt{n}}\|A_n\|_\infty}{\sqrt{n}\,\|A_n - B_n\|_\infty} = \dfrac{1}{n}\left(\dfrac{n^3+1}{n+1}\right) = \dfrac{n^3+1}{n^2+n} \xrightarrow[n\to\infty]{}$ ∞

2) a)

$$P = \begin{array}{c} \begin{matrix} B & L & N & T \end{matrix} \\ \begin{pmatrix} a & 0,5 & 0,5 & 0 \\ b & 0,5 & 0 & 0 \\ c & 0 & 0,5 & 0 \\ d & 0 & 0 & 1 \end{pmatrix} \end{array}$$

P de Markov $\Leftrightarrow$ $[a+b+c+d=1]$ $(a,b,c,d \geq 0)$ ⊛

Como $v = \frac{1}{8}\begin{pmatrix} 3 \\ 2 \\ 2 \\ 1 \end{pmatrix}$ es estado de equilibrio $\Rightarrow P.v = v$.

$$\Rightarrow P\begin{pmatrix} 3 \\ 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 2 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} 3a+1+1 = 3 \\ 3b+1 = 2 \\ 3c+1 = 2 \\ 3d+1 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} 3a = 1 \rightarrow a = 1/3 \\ 3b = 1 \Rightarrow b = 1/3 \\ 3c = 1 \Rightarrow c = 1/3 \\ 3d = 0 \rightarrow d = 0 \end{cases}$$ (verifica ⊛ ✓)

$$\left[P = \begin{pmatrix} 1/3 & 1/2 & 1/2 & 0 \\ 1/3 & 1/2 & 0 & 0 \\ 1/3 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right]$$

Busco avals. $\chi_P(\lambda) = \begin{vmatrix} 1/3-\lambda & 1/2 & 1/2 & 0 \\ 1/3 & 1/2-\lambda & 0 & 0 \\ 1/3 & 0 & 1/2-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1/3-\lambda & 1/2 & 1/2 \\ 1/3 & 1/2-\lambda & 0 \\ 1/3 & 0 & 1/2-\lambda \end{vmatrix} =$

$$\cancel{= (1-\lambda)\left[\frac{1}{2}\begin{vmatrix} 1/3-\lambda & 1/2 \\ 1/3 & 1/2-\lambda \end{vmatrix} + (1/2-\lambda)\begin{vmatrix} 1/3-\lambda & 1/2 \\ 1/3 & 1/2-\lambda \end{vmatrix}\right] = (1-\lambda)\left[\frac{1}{2}(\lambda^2-\frac{5}{6}\lambda) + (1/2-\lambda)(\lambda^2-5\right.}$$

$$= (1-\lambda)\left[\frac{1}{2}\begin{vmatrix} 1/3 & 1/2-\lambda \\ 1/3 & 0 \end{vmatrix} + (1/2-\lambda)\begin{vmatrix} 1/3-\lambda & 1/2 \\ 1/3 & 1/2-\lambda \end{vmatrix}\right] = (1-\lambda)\left[\frac{1}{2}\left(-\frac{1}{3}\left(\frac{1}{2}-\lambda\right)\right) + \left(\frac{1}{2}-\lambda\right)\left(\lambda^2-\frac{5}{6}\lambda\right)\right]$$

$$= (1-\lambda)\left[\left(\frac{1}{2}-\lambda\right)\left[-\frac{1}{6}+\lambda^2-\frac{5}{6}\lambda\right]\right] = (1-\lambda)\left(\frac{1}{2}-\lambda\right)\left(\lambda^2-\frac{5}{6}\lambda-\frac{1}{6}\right)$$

$\lambda = 1$ $\quad \lambda = \frac{1}{2}$ $\quad \lambda = 1$ $\quad \lambda = -\frac{1}{6}$

$$\boxed{\text{Avals} = \{ \lambda = 1 \text{ (doble)}, \lambda = -\tfrac{1}{6}, \lambda = \tfrac{1}{2} \}}$$

Existe P^∞ ~~porque~~ P es diagonalizable y $\lambda = 1$ es el único aval de módulo 1

Busco $E_1 = \{ v / Pv = v \}$

$Nu(P - I)$:

$$\left(\begin{array}{cccc|c} -2/3 & 1/2 & 1/2 & 0 & 0 \\ 1/3 & -1/2 & 0 & 0 & 0 \\ 1/3 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \\ 2F_2 + F_1 \\ 2F_3 + F_1 \\ \\ \end{array} \left(\begin{array}{cccc|c} -2/3 & 1/2 & 1/2 & 0 & 0 \\ 0 & -1/2 & 1/2 & 0 & 0 \\ 0 & -1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$-\frac{2}{3}x + \frac{1}{2}y + \frac{1}{2}z = 0 \Rightarrow -\frac{2}{3}x = -z$

$y = z$

$x = \frac{3}{2}z$

$X = (\frac{3}{2}z, z, z, \omega) = z(\frac{3}{2}, 1, 1, 0) + \omega(0001)$

$E_1 = \langle (3, 2, 2, 0)(0001) \rangle$ Como $\dim(E_1) = \text{mult}(1, \chi_P) \Rightarrow P$ es diag

$\dim(E_1) = 2$

$$P = C \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & -1/6 \end{pmatrix} C^{-1} \rightarrow P^\infty = C \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} C^{-1}$$

b) $V^{(0)} = \begin{pmatrix} 300 \\ 100 \\ 300 \\ 0 \end{pmatrix}$ 20 minutos son 10 transiciones

$\Rightarrow$ busco $P^{10} \cdot V^{(0)} = V^{(10)}$

Busco base de avecs

$\lambda = 1/2$ $E_{1/2} = \{ v / Pv = \frac{1}{2}v \} = Nu(P - \frac{1}{2}I)$

$$\left(\begin{array}{cccc|c} -1/6 & 1/2 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 \end{array}\right)$$

$\frac{1}{2}y + \frac{1}{2}z = 0 \Rightarrow y = -z$ $X = (0, -z, z, 0)$

$x = 0$

$\omega = 0$

$E_{1/2} = \langle (0, -1, 1, 0) \rangle$

$\lambda = -1/6$

$$\left(\begin{array}{cccc|c} 1/2 & 1/2 & 1/2 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 7/6 & 0 \end{array}\right) \begin{array}{l} \\ \frac{1}{2}F_2 - \frac{1}{3}F_1 \\ \frac{1}{2}F_3 - \frac{1}{3}F_1 \\ \\ \end{array} \left(\begin{array}{cccc|c} 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1/6 & -1/6 & 0 & 0 \\ 0 & -1/6 & 1/6 & 0 & 0 \\ 0 & 0 & 0 & 7/6 & 0 \end{array}\right)$$

$\frac{1}{2}x + z = 0 \Rightarrow x = -2z$

$y = z$

$X = (-2z, z, z, 0)$

$\omega = 0$

$E_{1/6} = \langle(-2,1,1,0)\rangle$

$B = \{(3,2,2,0)(0001)(0,-1,1,0)(-2,1,1,0)\}$ es base de $\mathbb{R}^4$ formada por avecs de P.

$(300,100,300,0) = \overset{=100}{a}(3,2,2,0) + \overset{=0}{b}(0,0,0,1) + \overset{=100}{c}(0,-1,1,0) + \overset{=0}{d}(-2,1,1,0)$

$$\left(\begin{array}{cccc|c} 3 & 0 & 0 & -2 & 300 \\ 2 & 0 & -1 & 1 & 100 \\ 2 & 0 & 1 & 1 & 300 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} 3F_2-2F_1 \\ 3F_3-2F_1 \end{array} \left(\begin{array}{cccc|c} 3 & 0 & 0 & -2 & 300 \\ 0 & 0 & -3 & 7 & -300 \\ 0 & 0 & 3 & 7 & 300 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right) F_3+F_2 \left(\begin{array}{cccc|c} 3 & 0 & 0 & -2 & 300 \\ 0 & 0 & -3 & 7 & -300 \\ 0 & 0 & 0 & 14 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right)$$

$b=0,\ d=0$ $\quad \begin{cases} 3a - 2d = 300 & \Rightarrow a = 100 \\ -3c + 7d = -300 & \Rightarrow c = 100 \end{cases}$

$(300,100,300,0) = 100\,(3,2,2,0) + 100\,(0,-1,1,0)$

$P^{10} \cdot v^{(0)} = 100\,P^{10}(3220) + 100\,P^{10}(0-110) = 100 \cdot 1^{10}(3220) + 100\underbrace{\left(\tfrac{1}{2}\right)^{10}}_{0,1}(0-110)$

$\simeq (300,\ 200-0{,}1,\ 200+0{,}1,\ 0)$

$\downarrow$ B $\quad \downarrow$ L $\quad \downarrow$ N $\quad \downarrow$ T

Habrá aprox 200 usuarios escuchando Xlothy Peluso

3 a) $A = A^T \iff \alpha^2 = \alpha + 2 \iff \alpha^2 - \alpha - 2 = 0 \nearrow \alpha = -1$, $\searrow \alpha = 2$.

$\lambda = 0$ es aval $\iff (\exists v \neq 0 \,/\, Av = 0.v) \iff Nu(A) \neq \{(000)\}$.

$\underline{\alpha = -1}$ $\quad A = \begin{pmatrix} 4 & 1 & 2 \\ 1 & 4 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ $\quad (\det(A) \neq 0 \,.\, \circ\, P_A(0) \neq 0)$

$$\left(\begin{array}{ccc|c} 4 & 1 & 2 & 0 \\ 0 & 15 & 6 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right) \sim \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$ $\left[\alpha = -1 \ \underline{\text{No sirve}}\right]$

$\alpha = 2$

$$A = \begin{pmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{pmatrix} \begin{matrix} \\ F_2 - F_1 \\ 2F_3 - F_1 \end{matrix} \left(\begin{matrix} 4 & 4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \middle| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right) \quad E_0 = \langle (1,0,-2)(0,1,-2) \rangle$$

$2x + 2y + \cancel{2}z = 0$

$z = -2x - 2y$

$(X_1 = (x, y, -2x-2y)$

$\alpha = 2$ único q' sirve?

b) (Si se plantea $\chi_A(\lambda) = 0$ es un bajón...)

Como A simétrica $\exists$ BON de $\mathbb{R}^3$ formada con avec de A.

$\Rightarrow$ busco $E_0^{\perp} = \langle (1\,0\,-2)(0\,1\,-2) \rangle^{\perp} = \langle (2\,2\,1) \rangle$

$$A \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 18 \\ 18 \\ 9 \end{pmatrix} = 9 \cdot \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \Rightarrow \left[\lambda = 9 \text{ es aval y } E_9 = \langle (2,2,1) \rangle \right.$$

$B = \{(1,0,-2)(0,1,-2)(2,2,1)\}$ es Base de $\mathbb{R}^3$ formada con avecs de A

Aplico G-S a E_0:

$\tilde{w}_1 = (1,0,-2) \Rightarrow \left[w_1 = \frac{1}{\sqrt{5}}(1,0,-2) \right]$

$$\tilde{w}_2 = (0,1,-2) - \frac{(0,1,-2)(1,0,-2)}{(1,0,-2)(1,0,-2)}(1,0,-2) = (0,1,-2) - \frac{4}{5}(1,0,-2) = \left(-\frac{4}{5}, 1, -\frac{2}{5}\right) = \frac{1}{5}(-4,5,-2)$$

$\left[w_2 = \frac{1}{3\sqrt{5}}(-4,5,-2) \right]$

$B = \left\{ \frac{1}{\sqrt{5}}(1,0,-2), \frac{1}{3\sqrt{5}}(-4,5,-2), \frac{1}{3}(2,2,1) \right\}$ es BON de $\mathbb{R}^3$ formada con avecs de A

(4) a)

$$\left(\begin{array}{ccc|c} 1 & 0 & 10^{-3} & 0 \\ 1 & 10^{-3} & 1 & 1 \\ 1 & 1 & 2 & 1 \end{array}\right) \begin{array}{c} F_2 - F_1 \\ F_3 - F_1 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 10^{-3} & 0 \\ 0 & 10^{-3} & fl\boxed{1-10^{-3}} & 1 \\ 0 & 1 & fl\boxed{2-10^{-3}} & 1 \end{array}\right)$$

$$fl(1-10^{-3}) = fl(1-0{,}001) = fl(0{,}999) = 1$$
$$fl(2-10^{-3}) = \quad = 2$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 10^{-3} & 0 \\ 0 & 10^{-3} & 1 & 1 \\ 0 & 1 & 2 & 1 \end{array}\right) F_3 - 10^{3}F_2 \left(\begin{array}{ccc|c} 1 & 0 & 10^{-3} & 0 \\ 0 & 10^{-3} & 1 & 1 \\ 0 & 0 & fl\,\overline{2-10^{3}} & fl(1-10^{3}) \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 10^{-3} & 0 \\ 0 & 10^{-3} & 1 & 1 \\ 0 & 0 & -10^{3} & -10^{3} \end{array}\right)$$

$$fl(2-10^{3}) = fl(-998) = fl(-0{,}998 \times 10^{3}) = -1 \times 10^{3}$$

$$\begin{cases} x + 10^{-3}z = 0 & \Rightarrow x = -10^{-3} \\ 10^{-3}y + z = 1 & \Rightarrow y = 0 \\ -10^{3}z = -10^{3} & \Rightarrow z = 1 \end{cases} \qquad \boxed{\tilde{x} = \begin{pmatrix} -10^{-3} \\ 0 \\ 1 \end{pmatrix}}$$

$$\tilde{b} = A\tilde{x} = \begin{pmatrix} 1 & 0 & 10^{-3} \\ 1 & 10^{-3} & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} -10^{-3} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -10^{-3}+1 \\ -10^{-3}+2 \end{pmatrix}$$

$$\|b-\tilde{b}\|_\infty = \left\| \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ -10^{-3}+1 \\ -10^{-3}+2 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 \\ 10^{-3} \\ -1+10^{-3} \end{pmatrix} \right\|_\infty = |-1+10^{-3}| = |-1{,}001|$$

$$\|b-\tilde{b}\|_\infty > 1\,!$$

b) $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $PAx = Pb$ intercambia $F_2 \leftrightarrow F_3$ para pivotear con el mayor pivote posible.

Resuelvo:

$$\left(\begin{array}{ccc|c} 1 & 0 & 10^{-3} & 0 \\ 1 & 1 & 2 & 1 \\ 1 & 10^{-3} & 1 & 1 \end{array}\right) \begin{array}{c} F_2 - F_1 \\ F_3 - F_1 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 10^{-3} & 0 \\ 0 & 1 & fl[2-10^{-3}]=2 & 1 \\ 0 & 10^{-3} & fl(1-10^{-3}) & 1 \end{array}\right) F_3 - 10^{-3}F_2$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 10^{-3} & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & fl[1-2\times10^{-3}] & fl[1-10^{-3}] \end{array}\right) \qquad \begin{cases} x + 10^{-3} z = 0 \rightarrow x = -10^{-3} \\ y + 2z = 1 \rightarrow y = -1 \\ z = 1 \end{cases}$$

$$fl[1-2\times10^{-3}] = 1, \quad 1 - 0{,}002 = 0{,}998 \qquad fl[1-10^{-3}] = 1$$

$$\boxed{\tilde{x} = \begin{pmatrix} -10^{-3} \\ -1 \\ 1 \end{pmatrix}}$$

$$\tilde{b} = A\tilde{x} = \begin{pmatrix} 1 & 0 & 10^{-3} \\ 1 & 10^{-3} & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} -10^{-3} \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2\times10^{-3}+1 \\ -1\times10^{-3}+1 \end{pmatrix}$$

$$\left[\|\tilde{b} - b\|_\infty = \left\| \begin{pmatrix} 0 \\ -2\times10^{-3} \\ -1\times10^{-3} \end{pmatrix} \right\|_\infty = 2\times10^{-3} < 1\times10^{-2} \text{ como queríamos} \right]$$