Ejercicio 12. Sean $v_1, \dots, v_k \in \mathbb{R}^n$. Probar que $\{v_1, \dots, v_k\}$ es linealmente independiente sobre $\mathbb{R}$ si y solo si $\{v_1, \dots, v_k\}$ es linealmente independiente sobre $\mathbb{C}$.

$\Rightarrow$ $\{v_i\}$ es li sobre $\mathbb{R}$

$$\Rightarrow \quad \alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \dots + \alpha_k \cdot v_k = 0 \qquad \alpha_i \in \mathbb{R}$$

$$\Leftrightarrow \alpha_i = 0 \quad \forall i \in \{1, \dots, k\}$$

Supongo que existen $\beta_i \neq 0$:

$$\beta_1 \cdot v_1 + \beta_2 \cdot v_2 + \dots + \beta_k \cdot v_k = 0 \qquad \text{con } \beta_i \in \mathbb{C}$$

Sin p. d. g.:
Si $\beta_1 \neq 0$

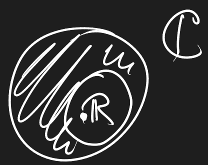
$$v_1 + \frac{\beta_2}{\beta_1} \cdot v_2 + \dots + \frac{\beta_k}{\beta_1} \cdot v_k = 0$$

$$v_1 = -\frac{\beta_2}{\beta_1} \cdot v_2 - \dots - \frac{\beta_k}{\beta_1} \cdot v_k$$

$$\underbrace{\overbrace{\begin{matrix} a_2 v_2 \\ + a_3 v_3 \\ + a_4 v_4 \end{matrix}}^{\in \mathbb{R}} \quad \begin{matrix} + \\ + \\ + \end{matrix} \quad \overbrace{\begin{matrix} i \cdot b_2 v_2 \\ i \cdot b_3 v_3 \\ i \cdot b_4 v_4 \end{matrix}}^{\in \mathbb{C}}}_{=0}$$

Si me quedo con la parte Real de cada $\frac{\beta_2}{\beta_1}$, tengo una comb. lineal en $\mathbb{R}$.

(deben existir pues $v_1 \in \mathbb{R}$)



$\Leftarrow$) $\{v_i\}$ es li sobre $\mathbb{C}$

$$\Rightarrow \beta_1 \cdot v_1 + \beta_2 \cdot v_2 + \dots + \beta_k \cdot v_k = 0 \qquad \text{con } \beta_i \in \mathbb{C}$$

con $\beta_i = 0 \quad \forall i \in \{1, \dots, k\}$

$\Rightarrow$ Como $\mathbb{R} \subset \mathbb{C}$, cualquier $\alpha_i \in \mathbb{R} \subset \mathbb{C}$

$$\Rightarrow \alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \dots + \alpha_k \cdot v_k = 0 \qquad \forall \alpha_i \in \mathbb{R}$$

Ejercicio 13. Sean m, n y $r \in \mathbb{N}$.

(a) Probar que si $\boldsymbol{A} \in K^{m\times n}$ satisface que $\boldsymbol{Ax} = 0\ \forall\, \boldsymbol{x} \in K^n$, entonces $\boldsymbol{A} = 0$. Deducir que si $\boldsymbol{A}, \boldsymbol{B} \in K^{m\times n}$ satisfacen que $\boldsymbol{Ax} = \boldsymbol{Bx}\ \forall\, \boldsymbol{x} \in K^n$, entonces $\boldsymbol{A} = \boldsymbol{B}$.

(b) Probar que si $\boldsymbol{A} \in K^{m\times n}$, $\boldsymbol{B} \in K^{n\times r}$ con $\boldsymbol{B} = (b_{ij})$ y, para $1 \le j \le r$, $\boldsymbol{B}_j = \begin{pmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{pmatrix}$ es la columna j-ésima de $\boldsymbol{B}$, entonces $\boldsymbol{AB} = (\,\boldsymbol{AB}_1 \mid \cdots \mid \boldsymbol{AB}_r)$ (es decir, $\boldsymbol{AB}_j$ es la columna j-ésima de $\boldsymbol{AB}$).

a) Supongo que no vale:

$Ax = 0 \quad \forall x$ pero existe un a_{ij} en A que no es cero

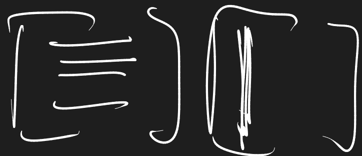
$$\rightarrow \begin{bmatrix} a_{11} \cdots a_{1n} \\ \vdots \quad \vdots \\ a_{m1}\ a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow \underbrace{a_{ij}}_{\neq 0}\ \underbrace{x_j}_{} = 0$$

x_j debe ser cero

(no vale $\forall x$)

$\therefore$

Si $Ax = 0 \quad \forall x \Rightarrow A = 0$ ◼



ii) $Ax = Bx \quad \forall x \Rightarrow A = B$

$Ax - Bx = 0$

$\underbrace{(A-B)}\, x = 0$

$A - B = 0 \Rightarrow A = B$

b)

$$\underbrace{\underset{m\times n}{A} \cdot \underset{n\times r}{B}}_{m\times r} = \left[\underset{m\times 1}{\underset{m\times n\ \ n\times 1}{AB_1}} \;\vdots\; \underset{m\times 1}{\underset{m\times n\ \ n\times 1}{AB_2}} \;\vdots\; \cdots \;\Big|\; \underset{m\times n\ \ n\times 1}{AB_r} \right]$$

Ejercicio 14. Sean las siguientes matrices de 3×3:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \quad \boldsymbol{B} = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 0 & 1 \\ 2 & 0 & 2 \end{pmatrix} \quad \boldsymbol{C} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

Y consideremos el producto $\boldsymbol{AB} = \boldsymbol{C}$ en bloques:

$$\left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right) \left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right) = \left(\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right)$$

Para cada una de las particiones en bloques mencionadas a continuación, indicar si es realizable el producto $\boldsymbol{C} = \boldsymbol{AB}$ en bloques. En caso de ser realizable, calcular cada bloque C_{ij} indicando sus dimensiones.

(a) $A_{11} = [a_{11}],\ A_{12} = [a_{12},\ a_{13}],\ A_{21} = \begin{bmatrix} a_{21} \\ a_{31} \end{bmatrix},\ A_{22} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$

$B_{11} = [b_{11}],\ B_{12} = [b_{12},\ b_{13}],\ B_{21} = \begin{bmatrix} b_{21} \\ b_{31} \end{bmatrix},\ B_{22} = \begin{bmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{bmatrix}$

(b) $A_{11} = [a_{11}\ \ a_{12}],\ A_{12} = [a_{13}],\ A_{21} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix},\ A_{22} = \begin{bmatrix} a_{23} \\ a_{33} \end{bmatrix}$

$B_{11} = [b_{11}],\ B_{12} = [b_{12}\ \ b_{13}],\ B_{21} = \begin{bmatrix} b_{21} \\ b_{31} \end{bmatrix},\ B_{22} = \begin{bmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{bmatrix}$

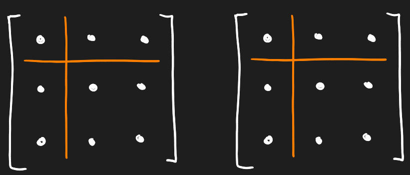
(c) $A_{11} = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix},\ A_{12} = \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix},\ A_{21} = [a_{31}],\ A_{11} = [a_{32}\ a_{33}]$

$B_{11} = [b_{11}],\ B_{12} = [b_{12}\ b_{13}],\ B_{21} = \begin{bmatrix} b_{21} \\ b_{31} \end{bmatrix},\ B_{22} = \begin{bmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{bmatrix}$

¿Qué otras particiones válidas son posibles?

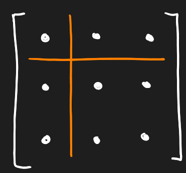
↳ ver pdf de particiones.

a)

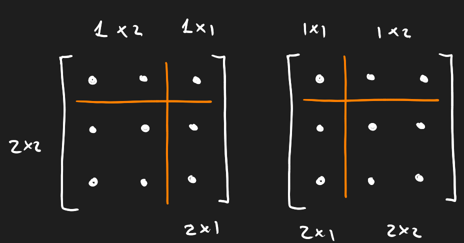


Se puede
(no voy a hacer la cuenta)

$$\left(\begin{array}{c|c} \overset{1\times1}{A_{11}}\cdot\overset{1\times1}{B_{11}} + \overset{1\times2}{A_{12}}\cdot\overset{2\times1}{B_{21}} & \overset{1\times1}{A_{11}}\cdot\overset{1\times2}{B_{12}} + \overset{1\times2}{A_{12}}\cdot\overset{2\times2}{B_{22}} \\ \hline \underbrace{\underset{2\times1}{A_{21}}\cdot\underset{1\times1}{B_{11}}}_{2\times1} + \underbrace{\underset{2\times2}{A_{22}}\cdot\underset{2\times1}{B_{21}}}_{2\times1} & \underbrace{\underset{2\times1}{A_{21}}\cdot\underset{2\times2}{B_{22}} + \underset{2\times2}{A_{22}}\cdot\underset{2\times2}{B_{22}}}_{2\times2} \end{array}\right) =$$

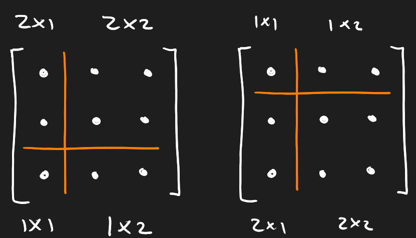


b)



$\underset{1\times2}{A_{11}}\cdot\underset{1\times1}{B_{11}}$ ✗ no funciona

c)



Funciona todo!

Ejercicio 15. Dadas las bases de $\mathbb{R}^3$, $B = \{(1,1,0),(0,1,1),(1,0,1)\}$ y $B' = \{(-1,1,1),(2,0,1),(1,-1,3)\}$

(a) Calcular $[(1,1,0)]_B$ y $[(1,1,0)]_{B'}$.

(b) Calcular la matriz de cambio de base $C(B,B')$.

(c) Comprobar que $C(B,B')[(1,1,0)]_B = [(1,1,0)]_{B'}$.

a) $(1,1,0)$ está en la base $\mathcal{E} = \{(1,0,0),(0,1,0),(0,0,1)\}$

Quiero escribir:

$$(1,1,0) = a.(1,1,0) + b.(0,1,1) + (1,0,1)$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{C(B,E)} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Busco a, b, c:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow{-F_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow{F_3-F_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right)$$

$$\begin{array}{c} F_1 - \frac{F_3}{2} \\ F_2 + \frac{F_3}{2} \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \begin{array}{l} \to a = 1 \\ \to b = 0 \\ \to c = 0 \end{array}$$

$$(1,1,0) = 1(1,1,0) + 0(0,1,1) + 0(1,0,1)$$

$$\therefore [(1,1,0)]_B = (1,0,0)$$

ii) $[(1,1,0)]_{B'}$

$$(1,1,0) = a(-1,1,1) + b(2,0,1) + c(1,-1,3)$$

$$\left(\begin{array}{ccc|c} \overbrace{-1 \quad 2 \quad 1}^{C(B',E)} & & & 1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 3 & 0 \end{array}\right) \begin{array}{c} \\ \cdot +F_1 \\ \cdot +F_1 \end{array} \left(\begin{array}{ccc|c} -1 & 2 & 1 & 1 \\ 0 & \cancel{2}' & 0 & \cancel{2}' \\ 0 & 3 & 4 & 1 \end{array}\right)$$

$$-3F_2\left(\begin{array}{ccc|c} -1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \cancel{4}\ 1 & -\cancel{2}\ \frac{-1}{2} \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right) \quad \begin{array}{l} a = \frac{1}{2} \\ b = 1 \\ c = \frac{-1}{2} \end{array}$$

$$\frac{1}{2}(-1,1,1) + 1\,(2,0,1) + \left(-\frac{1}{2}\right)(1,-1,3) =$$

$$= \Big(\underbrace{-\tfrac{1}{2} + 2 - \tfrac{1}{2}}_{=1},\ \underbrace{\tfrac{1}{2} + \tfrac{1}{2}}_{=1},\ \underbrace{\tfrac{1}{2} + 1 - \tfrac{3}{2}}_{=0}\Big) = (1,1,0) \ \checkmark$$

$$\therefore\ [(1,1,0)]_{B'} = \left(\frac{1}{2},\ 1,\ -\frac{1}{2}\right)$$

(b) Calcular la matriz de cambio de base $C(B, B')$.

Quiero C tal que

Si:

$$[(1,1,0)]_B = (1,0,0)$$

$$[(1,1,0)]_{B'} = \left(\frac{1}{2},\ 1,\ -\frac{1}{2}\right)$$

$$\Rightarrow \underset{3\times 3}{C_{BB'}} \underset{3\times 1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{bmatrix}$$

Tengo :

$$\underbrace{C(B, E)}_{\substack{\text{manda algo} \\ \text{de } B \text{ a } E}} \quad \text{y} \quad \underbrace{C(B', E)}_{\substack{\text{manda algo} \\ \text{de } B' \text{ a } E}}$$

Quiero

$C(\mathcal{B},\mathcal{B}')$: manda algo de $\mathcal{B}$ a $\mathcal{B}'$

Esta matriz tiene como columnas los vectores v_i de $\mathcal{B}$ expresados en la base $\mathcal{B}'$

$\Rightarrow$ Para cada v_i de $\mathcal{B}$ quiero $(v_i)_{\mathcal{B}'} = w_i$

$$C(\mathcal{B}',E) \underbrace{\begin{bmatrix} | \\ w_i \\ | \end{bmatrix}}_{\text{incógnita}} = \begin{bmatrix} | \\ v_i \\ | \end{bmatrix}$$

$$B = \{\underbrace{(1,1,0)}_{v_1}, \underbrace{(0,1,1)}_{v_2}, \underbrace{(1,0,1)}_{v_3}\}$$

$$B' = \{(-1,1,1),(2,0,1),(1,-1,3)\}$$

$$\underbrace{\begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 3 \end{bmatrix}}_{\text{Columnas de } \mathcal{B}'} \begin{bmatrix} | \\ w_i \\ | \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\text{Columna } i \text{ de } \mathcal{B}}$$

$$\begin{array}{cccc|ccc} & & & & v_1 & v_2 & v_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} -1 & 2 & 1 & 1 & 0 & 1 \\ 1 & 0 & -1 & 1 & 1 & 0 \\ 1 & 1 & 3 & 0 & 1 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & | & | & | \\ 0 & 1 & 0 & w_1 & w_2 & w_3 \\ 0 & 0 & 1 & | & | & | \end{array}\right)$$

$$\underbrace{\qquad\qquad}_{C(\mathcal{B},\mathcal{B}')}$$

Como $C(\mathcal{B}',E)$ es invertible

$$C(\mathcal{B}',E) \begin{bmatrix} | \\ w_i \\ | \end{bmatrix} = \begin{bmatrix} | \\ v_i \\ | \end{bmatrix}$$

$$\begin{bmatrix} | \\ w_i \\ | \end{bmatrix} = C(\mathcal{B}',E)^{-1} \begin{bmatrix} | \\ v_i \\ | \end{bmatrix}$$

Pero $C(\mathcal{B},\mathcal{B}')$ tiene los w_i como columna

$$C(\mathcal{B},\mathcal{B}') = \begin{bmatrix} | & | & | \\ w_1 & w_2 & w_3 \\ | & | & | \end{bmatrix} = C(\mathcal{B}',E)^{-1} \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}$$

Calculo $C(B', E)^{-1}$

$$C(B', E)^{-1} = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 3 \end{bmatrix}^{-1}$$

$$\left(\begin{array}{ccc|ccc} -1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} F_2 + F_1 \\ F_3 + F_1 \end{array} \left(\begin{array}{ccc|ccc} -1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 & 0 \\ 0 & 3 & 4 & 1 & 0 & 1 \end{array}\right)$$

$$\begin{array}{c} \\ 1/2 \\ \\ \end{array} \left(\begin{array}{ccc|ccc} -1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 4 & -\frac{1}{2} & -\frac{3}{2} & 1 \end{array}\right) \begin{array}{l} \times -1 \\ \\ \times 1/4 \end{array} \left(\begin{array}{ccc|ccc} 1 & -2 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & -\frac{1}{8} & -\frac{3}{8} & \frac{1}{4} \end{array}\right)$$

$$\begin{array}{c} +F_2 \\ \\ \\ \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & -\frac{1}{8} & -\frac{3}{8} & \frac{1}{4} \end{array}\right) \begin{array}{c} +F_3 \\ \\ \\ \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{8} & \frac{5}{8} & \frac{1}{4} \\ 0 & 1 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & -\frac{1}{8} & -\frac{3}{8} & \frac{1}{4} \end{array}\right)$$

$$C(B', E)^{-1} = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{8} & \frac{5}{8} & \frac{1}{4} \\ 1/2 & 1/2 & 0 \\ -\frac{1}{8} & -\frac{3}{8} & \frac{1}{4} \end{bmatrix} = C(E, B')$$

Verifico que funcione

$$\begin{bmatrix} -\frac{1}{8} & \frac{5}{8} & \frac{1}{4} \\ 1/2 & 1/2 & 0 \\ -\frac{1}{8} & -\frac{3}{8} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4/8 \\ 1 \\ -4/8 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{bmatrix} \checkmark$$

$$C(B,B') = \begin{bmatrix} | & | & | \\ \omega_1 & \omega_2 & \omega_3 \\ | & | & | \end{bmatrix} = C(B',E)^{-1} \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}$$

$$\begin{pmatrix} -1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Solution

$$C(B,B') = \begin{pmatrix} \frac{1}{2} & \frac{7}{8} & \frac{1}{8} \\ 1 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{8} & \frac{1}{8} \end{pmatrix}$$

$$C(B,B') = \begin{bmatrix} \frac{1}{2} & \frac{7}{8} & \frac{1}{8} \\ 1 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{8} & \frac{1}{8} \end{bmatrix}$$

$$\underbrace{}_{(v_1)_{B'}} \qquad \underbrace{}_{(v_3)_{B'}}$$

(c) Comprobar que $C(B,B')[(1,1,0)]_B = [(1,1,0)]_{B'}$.

$$[(1,1,0)]_B = (1,0,0)$$

$$[(1,1,0)]_{B'} = \left(\frac{1}{2}, 1, -\frac{1}{2}\right)$$

$$\begin{bmatrix} \frac{1}{2} & \frac{7}{8} & \frac{1}{8} \\ 1 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{8} & \frac{1}{8} \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{[(1,1,0)]_B} = \underbrace{\begin{bmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{bmatrix}}_{[(1,1,0)]_{B'}} \quad \checkmark$$

Ejercicio 16. Sean $\boldsymbol{A}, \boldsymbol{A}' \in K^{m\times n}$; $\boldsymbol{B} \in K^{n\times r}$; $\boldsymbol{D}, \boldsymbol{D}' \in K^{n\times n}$; $\alpha \in K$. Probar:

(a) $(\boldsymbol{A}+\boldsymbol{A}')^t = \boldsymbol{A}^t + (\boldsymbol{A}')^t$

(b) $(\alpha \boldsymbol{A})^t = \alpha \boldsymbol{A}^t$

(c) $(\boldsymbol{AB})^t = \boldsymbol{B}^t\boldsymbol{A}^t$

(d) $\boldsymbol{AA}^t$ y $\boldsymbol{A}^t\boldsymbol{A}$ son matrices simétricas.

(e) $tr(\boldsymbol{D}+\boldsymbol{D}') = tr(\boldsymbol{D}) + tr(\boldsymbol{D}')$

(f) $tr(\alpha \boldsymbol{D}) = \alpha tr(\boldsymbol{D})$

(g) $tr(\boldsymbol{DD}') = tr(\boldsymbol{D}'\boldsymbol{D})$

a) La suma es coordenada a coordenada,

trasponer "mueve" las coordenadas de la misma manera para cualquier matriz.

b) Lo mismo que a)

c) Sabemos

$$(AB)^t_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk} B_{ki}$$

Además

$$(B^t A^t)_{ij} = \sum_{k=1}^{n} B^t_{ik} A^t_{kj} = \sum_{k=1}^{n} B_{ki} A_{jk}$$

$$= \sum_{k=1}^{n} A_{jk} B_{ki}$$

d) Por c) $(AA^t)^t = A^{tt}A^t = AA^t$ ✓

e) Lo mismo que a)

f) Lo mismo que a)

g)

by definition

$$\begin{aligned} trace(AB) &= (AB)_{11} + (AB)_{22} + \cdots + (AB)_{nn} \\ &= a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1k}b_{k1} \\ &+ a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2k}b_{k2} \\ &+ \vdots \\ &+ a_{n1}b_{1n} + a_{n2}b_{2n} + \cdots + a_{nk}b_{kn} \end{aligned}$$

$(BA)_{11}$

if you view the sum according to the columns, then you see that it is the $trace(BA)$. therefore,

$$trace(AB) = trace(BA).$$

Ejercicio 17. Decidir cuáles de los siguientes subconjuntos son subespacios de $\mathbb{R}^{n\times n}$ como $\mathbb{R}$-espacio vectorial:

(a) $S_1 = \{A \in \mathbb{R}^{n\times n} : A \text{ es triangular inferior}\}$ (b) $S_2 = \{A \in \mathbb{R}^{n\times n} : A \text{ es simétrica}\}$

a) 1) $A + B \in S_1$? ✓

2) $\alpha . A \in S_1$? ✓

3) $\vec{0} \in S_1$? ✓

b) También.

✓

✓

✓

Ejercicio 18. Calcular el determinante de $\boldsymbol{A}$ en cada uno de los siguientes casos:

$$\text{(a) } \boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -2 & 3 & -1 \\ -1 & 0 & 1 & 4 \\ 0 & 1 & -2 & 0 \end{pmatrix} \qquad \text{(b) } \boldsymbol{A} = \begin{pmatrix} i & 0 & 2+i \\ -1 & 1-i & 0 \\ 2 & 0 & -1 \end{pmatrix}$$

$(-1)^{i+j}$

a) $$\det A = 1 \cdot \det \underbrace{\begin{bmatrix} -2 & 3 & -1 \\ 0 & 1 & 4 \\ 1 & -2 & 0 \end{bmatrix}}_{\underbrace{-2\cdot(0+8)}_{-16} + \underbrace{1\cdot(12+1)}_{13} = -3} - 2 \cdot \det \underbrace{\begin{bmatrix} 0 & -2 & 3 \\ -1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}}_{+1(4-3) = 1}$$

$$\det A = -3 - 2 = -5$$

Ejercicio 19. Determinar si las siguientes matrices son inversibles y en caso afirmativo exhibir sus inversas. Cuando sea posible, verificar utilizando `Python`, con el comando np.linalg.inv.

(a) $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ (b) $A = \begin{pmatrix} \cos\theta & -\operatorname{sen}\theta & 0 \\ \operatorname{sen}\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (c) $A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 1 & -2 & 3 \\ 3 & 1 & -1 & 3 \end{pmatrix}$

(d) $A = \begin{pmatrix} 2 & 1 & 3 & 1 & 2 \\ 0 & 5 & -1 & 8 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$ (e) $A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$

a) Sí

b) Sí

c) No : F4 es cl de F1, 2 y 3

d) No ; Dos Filas iguales , no son li.

e) Sí, si $a_{ii} \neq 0$ $\forall i \in \{1, \dots, n\}$

Ejercicio 20. Sean $A, B, C, D \in K^{n\times n}$ y $M \in K^{2n\times 2n}$ la matriz de bloques

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Probar que si A es inversible, entonces

(a) $M = \begin{pmatrix} A & 0 \\ C & I \end{pmatrix} \cdot \begin{pmatrix} I & A^{-1}B \\ 0 & D - CA^{-1}B \end{pmatrix}$.

(b) $\det(M) = \det(AD - ACA^{-1}B)$. Concluir que si $AC = CA$, $\det(M) = \det(AD - CB)$.

a) $$M = \left[\begin{array}{c|c} \overset{A}{AI + 0} & \overset{B}{AA^{-1}B + 0} \\ \hline \underset{C}{CI + 0} & \underset{D}{CA^{-1}B + ID - ICA^{-1}B} \end{array}\right] \checkmark$$

b) "Complemento de Schur"

$$\det(AD - ACA^{-1}B) = \det\left(A\left(D - CA^{-1}B\right)\right)$$

$$= \det(A) \cdot \underbrace{\det(D - CA^{-1}B)}_{?}$$

https://www.statlect.com/matrix-algebra/determinant-of-block-matrix

https://www.statlect.com/matrix-algebra/Schur-complement

https://math.stackexchange.com/questions/1811433/
let-m-big-beginsmallmatrix-a-b-c-d-endsmallmatrix-big-prove

Uso

$$\det \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = \det A \cdot \det D$$

$$\det \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} = \det A \cdot \det D$$

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \overset{M}{\begin{bmatrix} A & B \\ C & D \end{bmatrix}} = \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}$$

$\det\big($

$$\det I \cdot \det I \cdot \det M = \det A \cdot \det\left(D - CA^{-1}B\right)$$

$$\det M = \det A \cdot \det\left(D - CA^{-1}B\right)$$

Concluir que si $AC = CA$, $\det(M) = \det(AD - CB)$.

$$AD - \underbrace{AC}A^{-1}B$$

$$C\underbrace{AA^{-1}}B$$

$$\underbrace{C\,I\,B}$$

$$AD - CB$$

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$\Rightarrow \det M = \det\left(AD - CB\right)$$

Ejercicio 21. Escribir funciones de `Python` que realicen las siguientes operaciones:

(a) Calcular la traza de una matriz.

(b) Calcular la sumatoria de todos los elementos de una matriz.

a) `sum([A[i,i] for i in range(len(A))])`

Si $A \in$ numpy

`A[range(len(A)), range(len(A))].sum()`

b) `A.sum()`