Universidad de Buenos Aires - Facultad de Ciencias Exactas y Naturales - Depto. de Matemática

Álgebra Lineal Computacional

2do Cuatrimestre 2023

Práctica N° 5: Matrices hermitianas. Valores singulares.

Matrices hermitianas

Ejercicio 1. Dada la matriz

$$A = \begin{pmatrix} 13 & 8 & 8 \\ -1 & 7 & -2 \\ -1 & -2 & 7 \end{pmatrix}.$$

(a) Hallar una descomposición de Schur $\boldsymbol{A} = \boldsymbol{UTU}^*$, con $\boldsymbol{U}$ unitaria y $\boldsymbol{T}$ triangular superior con los autovalores de la matriz $\boldsymbol{A}$ en la diagonal.

(b) Descomponer a la matriz $\boldsymbol{T}$ hallada en el ítem anterior como suma de una matriz diagonal $\boldsymbol{D}$ y una matriz triangular superior $\boldsymbol{S}$ con ceros en la diagonal. Probar que $\boldsymbol{S}^j = 0$ para todo $j \geq 2$.

(c) Usar los ítems anteriores para calcular $\boldsymbol{A}^{10}$.

a) $A = UTU^*$

$$AU = UT$$

$$= U \begin{bmatrix} \lambda_1 & * & * \\ 0 & \lambda_2 & * \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$A \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & * & * \\ 0 & \lambda_2 & * \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$A q_1 = \lambda_1 \cdot q_1 \leftarrow$ solo para $i = 1$

1°) Autovalores de A

$$\chi_A(\lambda) = \det \begin{bmatrix} \lambda - 13 & -8 & -8 \\ 1 & \lambda - 7 & 2 \\ 1 & 2 & \lambda - 7 \end{bmatrix}$$

$$= (\lambda-13)\underbrace{\begin{vmatrix} \lambda-7 & 2 \\ 2 & \lambda-7 \end{vmatrix}}_{\substack{(\lambda-7)^2-4 \\ \lambda^2-14\lambda+49-4 \\ (\lambda^2-14\lambda+45)}} - 1.\underbrace{\begin{vmatrix} -8 & -8 \\ 2 & \lambda-7 \end{vmatrix}}_{\substack{-(-8.(\lambda-7)+16) \\ -(-8\lambda+56+16) \\ 8\lambda-72}} + 1.\underbrace{\begin{vmatrix} -8 & -8 \\ \lambda-7 & 2 \end{vmatrix}}_{\substack{-16+8(\lambda-7) \\ -16+8\lambda-56 \\ 8\lambda-72}}$$

$$\lambda^3-14\lambda^2+45\lambda-13\lambda^2+13.14\lambda-13.45$$

$$\lambda^3-27\lambda^2+227\lambda-585$$

$$= \lambda^3-27\lambda^2+227\lambda-585+8\lambda-72+8\lambda-72$$

$$= \lambda^3-27\lambda^2+243\lambda-729 = 0$$

Raíces (calc)

λ: 9 (multiplicidad 3)

Avecs:

$$Nu\begin{bmatrix} 9-13 & -8 & -8 \\ 1 & 9-7 & 2 \\ 1 & 2 & 9-7 \end{bmatrix} = Nu\begin{bmatrix} -4 & -8 & -8 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{matrix} \times -4 \\ \text{iguales} \end{matrix}$$

$$\begin{pmatrix} -4 & -8 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad -4x-8y-8z=0$$

$$\Rightarrow x+2y+2z=0$$

$$x = -2y-2z$$

$$(-2y-2z,\, y,\, z) = y(-2,1,0) + z(-2,0,1)$$

$$E_9 = \langle (-2,1,0), (-2,0,1) \rangle$$

$$\Rightarrow q_1 = \frac{1}{\sqrt{5}}(-2,1,0)$$

$\Rightarrow$ Hasta ahora tengo

$$A = UTU^*$$

$$A = \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} 9 & * & * \\ 0 & 9 & * \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} - & q_1 & - \\ - & q_2 & - \\ - & q_3 & - \end{bmatrix}$$

A continuación, una resolución incorrecta. Más abajo corregido.

$$A = \underbrace{\begin{bmatrix} -2/\sqrt{5} & | & | \\ 1/\sqrt{5} & q_2 & q_3 \\ 0 & | & | \end{bmatrix}} \begin{bmatrix} 9 & * & * \\ 0 & 9 & * \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ - & q_2 & - \\ - & q_3 & - \end{bmatrix}$$

Mal! Completo con una BON y llamo U^1 Mal

$$U_1 = \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\leftarrow$ elijo q_2 y q_3 a ojo. sino uso G.S.

$\langle q_1, q_2 \rangle = 0$ $\quad \langle q_1, q_3 \rangle = 0$

con $\langle q_2, q_3 \rangle = 0$

$\Rightarrow$ Como $A = U_1 \, T \, U_1^*$

Mal !

$$U_1^* A \, U_1 = T$$

$$U_1^* A \, U_1 = \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 13 & 8 & 8 \\ -1 & 7 & -2 \\ -1 & -2 & 7 \end{bmatrix} \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Input

$$T = \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{pmatrix}^T . \begin{pmatrix} 13 & 8 & 8 \\ -1 & 7 & -2 \\ -1 & -2 & 7 \end{pmatrix} . \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Exact result

$$T = \frac{1}{5} \begin{pmatrix} 45 & -45 & -18\sqrt{5} \\ 0 & 55 & 4\sqrt{5} \\ 0 & -5\sqrt{5} & 35 \end{pmatrix}$$

$$T = \begin{bmatrix} 9 & -9 & -18\frac{\sqrt{5}}{5} \\ 0 & 11 & 4\frac{\sqrt{5}}{5} \\ 0 & -\sqrt{5} & 7 \end{bmatrix}$$

Como se ve en la matriz resultante T, en la diagonal NO quedaron los autovalores de la matriz original (lo cual SIEMPRE debe suceder).

Eso es porque no usé ambos autovectores para armar la BON, sino que usé solo 1.

Por lo tanto, debo asegurarme que debo usar todos los autovectores que definen el autoespacio de cada autovalor.

La forma correcta de resolver el ejercicio es usar TODOS los autovectores que definen el autoespacio del autovalor 9:

$$E_q = \langle (-2,1,0), (-2,0,1) \rangle$$

$$\Rightarrow q_1 = \frac{1}{\sqrt{5}}(-2,1,0)$$

Uso Gram-Smi.

$$u_2 = (-2,0,1) - \frac{\langle (-2,1,0), (-2,0,1) \rangle}{\underbrace{\|(-2,1,0)\|_2^2}_{=5}} \cdot (-2,1,0)$$

$$= (-2,0,1) - \frac{4}{5} \cdot (-2,1,0)$$

$$= \left(\underset{-\frac{10}{5}+\frac{8}{5}}{-2+\frac{8}{5}}, -\frac{4}{5}, 1\right)$$

$$= \left(-\frac{2}{5}, -\frac{4}{5}, 1\right)$$

$$u_2 = (-2,-4,5) \Rightarrow \|u_2\| = \sqrt{4+16+25} = 3\sqrt{5}$$

$$q_2 = \frac{1}{3\sqrt{5}} \cdot (-2,-4,5)$$

$$q_2 = \left(-\frac{2}{3\sqrt{5}}, -\frac{4}{3\sqrt{5}}, \frac{5}{3\sqrt{5}}\right)$$

Para q_3 no tengo v_3, entonces

$$q_1 = \frac{1}{\sqrt{5}}\underbrace{(-2,1,0)}_{u_1}$$

$$q_3 = \frac{u_1 \times u_2}{\|u_1 \times u_2\|} = \begin{vmatrix} i & j & k \\ -2 & 1 & 0 \\ -2 & -4 & 5 \end{vmatrix}$$

$$u_3 = \left(5, -(-10), 8+2\right)$$

$$u_3 = (5, 10, 10) \Rightarrow \|u_3\| = \sqrt{225}$$
$$= 15$$

$$q_3 = \frac{1}{15} \cdot (5, 10, 10)$$

$$q_3 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$q_1 = \frac{1}{\sqrt{5}} \underbrace{(-2, 1, 0)}_{u_1}$$

$$q_2 = \frac{1}{3\sqrt{5}} \cdot (-2, -4, 5)$$

$$q_3 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$U_1 = \begin{bmatrix} -2/\sqrt{5} & -2/3\sqrt{5} & 1/3 \\ 1/\sqrt{5} & -4/3\sqrt{5} & 2/3 \\ 0 & 5/3\sqrt{5} & 2/3 \end{bmatrix}$$

$$A = U T U^t$$

$$\Rightarrow U^t A U = \underbrace{U^t U}_{I} \, T \, \underbrace{U^t U}_{I}$$

Input interpretation

simplify	$\begin{pmatrix} -\frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} & \frac{1}{3} \\ \frac{1}{\sqrt{5}} & -\frac{4}{3\sqrt{5}} & \frac{2}{3} \\ 0 & \frac{\sqrt{5}}{3} & \frac{2}{3} \end{pmatrix}^T \cdot \begin{pmatrix} 13 & 8 & 8 \\ -1 & 7 & -2 \\ -1 & -2 & 7 \end{pmatrix}$

Expanded form

$$\begin{pmatrix} -\frac{27}{\sqrt{5}} & -\frac{9}{\sqrt{5}} & -\frac{18}{\sqrt{5}} \\ -\frac{9}{\sqrt{5}} & -\frac{18}{\sqrt{5}} & \frac{9}{\sqrt{5}} \\ 3 & 6 & 6 \end{pmatrix}$$

Input interpretation

simplify	$\begin{pmatrix} -\frac{27}{\sqrt{5}} & -\frac{9}{\sqrt{5}} & -\frac{18}{\sqrt{5}} \\ -\frac{9}{\sqrt{5}} & -\frac{18}{\sqrt{5}} & \frac{9}{\sqrt{5}} \\ 3 & 6 & 6 \end{pmatrix} \cdot \begin{pmatrix} -\frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} & \frac{1}{3} \\ \frac{1}{\sqrt{5}} & -\frac{4}{3\sqrt{5}} & \frac{2}{3} \\ 0 & \frac{\sqrt{5}}{3} & \frac{2}{3} \end{pmatrix}$

Result

$$\begin{pmatrix} 9 & 0 & -\frac{27}{\sqrt{5}} \\ 0 & 9 & -\frac{9}{\sqrt{5}} \\ 0 & 0 & 9 \end{pmatrix}$$

$$T = \underbrace{\begin{bmatrix} 9 & 0 & -27/\sqrt{5} \\ 0 & 9 & -9/\sqrt{5} \\ 0 & 0 & 9 \end{bmatrix}}_{\text{9's en la diagonal}} \checkmark$$

En este caso ya terminé, pero si T no hubiera sido una matriz triangular debería seguir con:

Paso a 2×2. Me quedo con:

$$T = \begin{bmatrix} 9 & 0 & -27/\sqrt{5} \\ 0 & 9 & -9/\sqrt{5} \\ 0 & 0 & 9 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 9 & -9/\sqrt{5} \\ 0 & 9 \end{bmatrix} = A_2$$

Y repetir todo el procedimiento con A_2 de la misma forma que hice con la matriz A original.

$$A = U \cdot \begin{bmatrix} 9 & 0 & -27/5 \\ 0 & \multicolumn{2}{c}{\boxed{U_2 \cdot A_2 \cdot U_2^t}} \\ 0 & & \end{bmatrix} \cdot U^t$$

Finalmente obtuve

$$A = U T U^*$$

$$A = \begin{bmatrix} -2/\sqrt{5} & -2/3\sqrt{5} & 1/3 \\ 1/\sqrt{5} & -4/3\sqrt{5} & 2/3 \\ 0 & 5/3\sqrt{5} & 2/3 \end{bmatrix} \begin{bmatrix} 9 & 0 & -27/\sqrt{5} \\ 0 & 9 & -9/\sqrt{5} \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} -2/\sqrt{5} & -2/3\sqrt{5} & 1/3 \\ 1/\sqrt{5} & -4/3\sqrt{5} & 2/3 \\ 0 & 5/3\sqrt{5} & 2/3 \end{bmatrix}^t$$

(b) Descomponer a la matriz $\boldsymbol{T}$ hallada en el ítem anterior como suma de una matriz diagonal $\boldsymbol{D}$ y una matriz triangular superior $\boldsymbol{S}$ con ceros en la diagonal. Probar que $\boldsymbol{S}^j = 0$ para todo $j \geq 2$.

$$T = \begin{bmatrix} 9 & 0 & -27/\sqrt{5} \\ 0 & 9 & -9/\sqrt{5} \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}}_{=: D} + \underbrace{\begin{bmatrix} 0 & 0 & -27/\sqrt{5} \\ 0 & 0 & -9/\sqrt{5} \\ 0 & 0 & 0 \end{bmatrix}}_{=: S}$$

$$S^2 = \begin{bmatrix} 0 & 0 & -27/\sqrt{5} \\ 0 & 0 & -9/\sqrt{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -27/\sqrt{5} \\ 0 & 0 & -9/\sqrt{5} \\ 0 & 0 & 0 \end{bmatrix}$$

$$S^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \quad S^j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \forall j \geq 2$$

(c) Usar los ítems anteriores para calcular $\boldsymbol{A}^{10}$.

$$A = U T U^*$$

$$= U \cdot (D + S)\, U^*$$

$$A = U D U^* + U S U^*$$

$$\Rightarrow A^{10} = (U T U^*)^{10}$$

$$= U T \underbrace{U^* U}_{=I} T \underbrace{U^* U}_{=I} T \cdots T \underbrace{U^* U}_{=I} T U^*$$

Pues U es unitaria (ortogonal en $\mathbb{C}$)

$$A^{10} = U \cdot T^{10} \cdot U^*$$

$$A^{10} \overset{(b)}{=} U (D + S)^{10} \cdot U^*$$

Input interpretation

expand	$(D + S)^{10}$

Result

$$D^{10} + 10 D^9 S + 45 D^8 S^2 + 120 D^7 S^3 + 210 D^6 S^4 + 252 D^5 S^5 + 210 D^4 S^6 + 120 D^3 S^7 + 45 D^2 S^8 + 10 D S^9 + S^{10}$$

(11 terms)

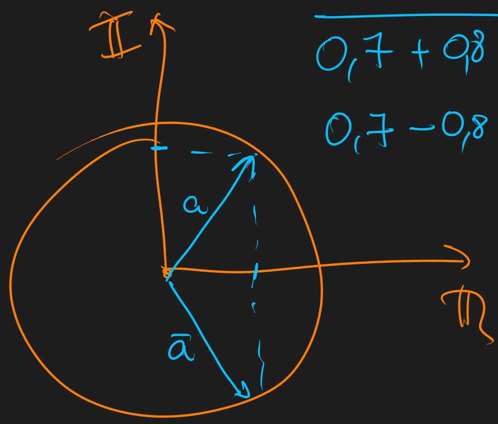
$$A^{10} = U \cdot \left(D^{10} + 10 D^{9} S \right) U^{*}$$

y listo ?

Ejercicio 2. Probar que si $\boldsymbol{A} \in K^{n\times n}$ es hermitiana, entonces los elementos de la diagonal $a_{ii} \in \mathbb{R}$.

Si A hermitiana

$$A = A^* = \overline{A}^{t}$$



$$\overline{0{,}7 + 0{,}8i} = 0{,}7 - 0{,}8i$$

$$\Rightarrow \quad A_{ii} = A^*_{ii} \qquad \forall i \in \{1, \dots, n\}$$

$$\Rightarrow \quad A_{ii} = \overline{A_{ii}}$$

$$\Rightarrow \quad A_{ii} \in \mathbb{R} \qquad \forall i \in \{1, \dots, n\}$$

pues $\quad c + di = \overline{c + di} = c - di \qquad c, d \in \mathbb{R}$

$$c + di = c - di$$

$$\Rightarrow di = 0$$

$$\Rightarrow A_{ii} = c \qquad \forall i \in \{1, \dots, n\}$$

$$\therefore \quad A_{ii} \in \mathbb{R} \qquad \forall i \in \{1, \dots, n\}$$

Ejercicio 3. Dada $\boldsymbol{A} \in K^{n\times n}$ hermitiana, probar que existen matrices $\boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}^{n\times n}$ con $\boldsymbol{B}$ simétrica y $\boldsymbol{C}$ antisimétrica ($\boldsymbol{C}^t = -\boldsymbol{C}$) tales que $\boldsymbol{A} = \boldsymbol{B} + i\boldsymbol{C}$.

A hermitiana : $A = A^* = \overline{A}^t$

$$\Rightarrow B + iC \overset{\substack{\text{quiero}\\ \downarrow}}{=} (B+iC)^*$$

$$B + iC = B^* + (iC)^*$$

- Si B es simétrica

$\Rightarrow B = B^t$

- Si $B \in \mathbb{R}$

$\Rightarrow B = \overline{B}$

$\Rightarrow B^* = B$

- Si C antisimétrica

$\Rightarrow C^t = -C$

- Si $C \in \mathbb{R}$

$\Rightarrow C = \overline{C}$

$$\Rightarrow (iC)^* = \overline{(iC)}^t$$

$$= \overline{iC^t} \underset{\substack{\uparrow\\ \text{antisim}}}{=} \overline{i.(-C)}$$

$$= \overline{i} . \overline{(-C)}$$

$$= (-i).(-C)$$

$$= i \cdot C$$

$$\Rightarrow B^* + (iC)^* = B + iC$$

∴ Si A es hermitiana, $\exists$ matrices $B, C \in \mathbb{R}$

con B simétrica y C antisimétrica

tal que

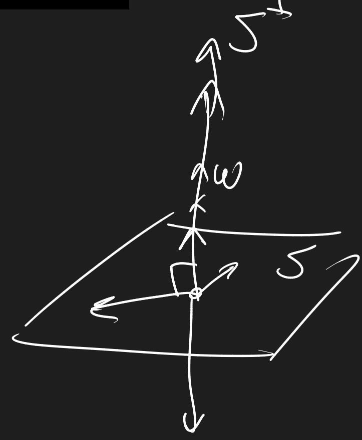
$$A = B + i C.$$

$\cancel{\boxtimes}$ Falta.

Completo en:

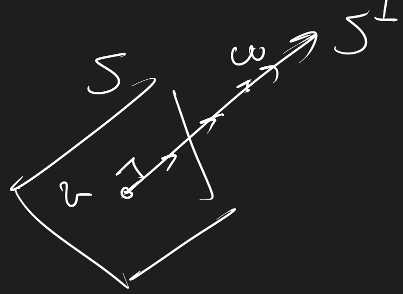
https://yutsumura.com/express-a-hermitian-matrix-as-a-sum-of-real-symmetric-matrix-and-a-real-skew-symmetric-matrix/

Ejercicio 4. Dada $\boldsymbol{A} \in K^{n\times n}$ hermitiana y $S \subset K^n$ un subespacio invariante por $\boldsymbol{A}$, es decir $\boldsymbol{A}\boldsymbol{v} \in S$ para todo $\boldsymbol{v} \in S$. Probar que $S^\perp$ es invariante por $\boldsymbol{A}$.



Si A hermitiana y $Av \in S \ \forall v \in S$

$\Rightarrow Aw \in S^\perp \ \forall w \in S^\perp$



Dem:

A hermitiana y $Av \in S \ \forall v \in S$

$\hookrightarrow A = A^*$

Si $Av \in S \ \forall v \in S$ y $w \in S^\perp$

$$\Rightarrow \langle \underbrace{Av}_{\in S}, \underbrace{w}_{\in S^\perp} \rangle = 0$$

$$\Rightarrow \langle v, A^* w \rangle = 0$$

$A = A^*$

$$\Rightarrow \langle \underbrace{v}_{\in S}, A w \rangle = 0$$

$\Rightarrow$ Como $v \in S$, $Aw \in S^\perp \ \forall w \in S^\perp$

Ejercicio 5. Probar que $\boldsymbol{A} \in K^{n\times n}$ es hermitiana y definida positiva si y solo si $\boldsymbol{A}$ es unitariamente semejante a una matriz diagonal real positiva con elementos de la diagonal positivos.

A hermitiana : $A = A^*$

A dp $\Rightarrow x^t A x > 0 \quad \forall x \in \mathbb{R}^n$

$\Rightarrow)$ Si A es hermitiana $\Rightarrow$ es diagonalizable

Pues como $A = A^*$

$$\Rightarrow \langle A v_i , v_i \rangle = \langle \lambda_i \cdot v_i , v_i \rangle$$

λ_i autovalor y v_i su autovector

$$= \lambda_i \langle v_i , v_i \rangle$$

$$= \lambda_i \underbrace{\| v_i \|_2^2}_{>0} \quad \text{con } v_i \neq \vec{0}$$

$\Rightarrow$ Como $\langle A v_i , v_i \rangle \overset{*}{\in} \mathbb{R}$: $\lambda_i \in \mathbb{R}$

$$* \ \langle Av , v \rangle \overset{A=A^*}{=} \langle v , Av \rangle = \overline{\langle Av , v \rangle}$$

$$\Rightarrow \langle Av , v \rangle = \overline{\langle Av , v \rangle}$$

$$\Rightarrow \langle Av , v \rangle \in \mathbb{R}$$

- Theorem (Properties of Hermitian Operators): Suppose V is a finite-dimensional inner product space and $T : V \to V$ is a Hermitian linear transformation. Then the following hold:

  1. For any $\mathbf{v} \in V$, $\langle T(\mathbf{v}), \mathbf{v}\rangle$ is a real number.
     - Proof: We have $\langle T(\mathbf{v}), \mathbf{v}\rangle = \langle \mathbf{v}, T^*(\mathbf{v})\rangle = \langle \mathbf{v}, T(\mathbf{v})\rangle = \overline{\langle T(\mathbf{v}), \mathbf{v}\rangle}$, so $\langle T(\mathbf{v}), \mathbf{v}\rangle$ is equal to its complex conjugate, hence is real.
  2. All eigenvalues of T are real numbers.
     - Proof: Suppose λ is an eigenvalue of T with eigenvector $\mathbf{v} \neq \mathbf{0}$.
     - Then $\langle T(\mathbf{v}), \mathbf{v}\rangle = \langle \lambda\mathbf{v}, \mathbf{v}\rangle = \lambda \langle \mathbf{v}, \mathbf{v}\rangle$ is real. Since $\mathbf{v}$ is not the zero vector we conclude that $\langle \mathbf{v}, \mathbf{v}\rangle$ is a nonzero real number, so λ is also real.
  3. Eigenvectors of T with different eigenvalues are orthogonal.
     - Proof: Suppose that $T\mathbf{v}_1 = \lambda_1\mathbf{v}_1$ and $T\mathbf{v}_2 = \lambda_2\mathbf{v}_2$.
     - Then $\lambda_1 \langle \mathbf{v}_1, \mathbf{v}_2\rangle = \langle T\mathbf{v}_1, \mathbf{v}_2\rangle = \langle \mathbf{v}_1, T^*\mathbf{v}_2\rangle = \langle \mathbf{v}_1, \lambda_2\mathbf{v}_2\rangle = \lambda_2 \langle \mathbf{v}_1, \mathbf{v}_2\rangle$ since λ_2 is real. But since $\lambda_1 \neq \lambda_2$, this means $\langle \mathbf{v}_1, \mathbf{v}_2\rangle = 0$.
  4. Every generalized eigenvector of T is an eigenvector of T.
     - Proof: We show by induction that if $(T - \lambda I)^k\mathbf{w} = \mathbf{0}$ then in fact $(T - \lambda I)\mathbf{w} = \mathbf{0}$.
     - For the base case we take $k = 2$, so that $(\lambda I - T)^2\mathbf{w} = \mathbf{0}$. Then since λ is an eigenvalue of T and therefore real, we have

$$\begin{aligned} \mathbf{0} = \left\langle (T - \lambda I)^2\mathbf{w}, \mathbf{w}\right\rangle &= \langle (T - \lambda I)\mathbf{w}, (T - \lambda I)^*\mathbf{w}\rangle \\ &= \left\langle (T - \lambda I)\mathbf{w}, (T^* - \overline{\lambda} I)\mathbf{w}\right\rangle \\ &= \langle (T - \lambda I)\mathbf{w}, (T - \lambda I)\mathbf{w}\rangle \end{aligned}$$

       and thus the inner product of $(T - \lambda I)\mathbf{w}$ with itself is zero, so $(T - \lambda I)\mathbf{w}$ must be zero.
     - For the inductive step, observe that $(T - \lambda I)^{k+1}\mathbf{w} = \mathbf{0}$ implies $(T - \lambda I)^k [(T - \lambda I)\mathbf{w}] = \mathbf{0}$, and therefore by the inductive hypothesis this means $(T - \lambda I) [(T - \lambda I)\mathbf{w}] = \mathbf{0}$, or equivalently, $(T - \lambda I)^2\mathbf{w} = \mathbf{0}$. Applying the result for $k = 2$ from above yields $(T - \lambda I)\mathbf{w} = \mathbf{0}$, as required.

- Using these results we can establish a fundamental result called the spectral theorem:

- Theorem (Spectral Theorem): Suppose V is a finite-dimensional inner product space over $\mathbb{R}$ or $\mathbb{C}$ and $T : V \to V$ is a Hermitian linear transformation. Then V has an orthonormal basis β of eigenvectors of T, so in particular, T is diagonalizable.
  - The equivalent formulation for Hermitian matrices is: every Hermitian matrix A can be written as $A = U^{-1}DU$ where D is a real diagonal matrix and U is a unitary matrix (i.e., satisfying $U^* = U^{-1}$).
  - Proof: By the theorem above, every eigenvalue of T is real hence lies in the scalar field.
  - Then every generalized eigenvector of T is an eigenvector of T, and so since V has a basis of generalized eigenvectors, it has a basis of eigenvectors and is therefore diagonalizable.
  - To finish the proof, start with a basis for each eigenspace, and then apply Gram-Schmidt, yielding an orthonormal basis for each eigenspace.
  - Since T is diagonalizable, the union of these bases is a basis for V: furthermore, each of the vectors has norm 1, and they are all orthogonal by the orthogonal result above.
  - By construction, each vector is orthogonal to the others in its eigenspace, and by the observation above it is also orthogonal to the vectors in the other eigenspaces, so we obtain an orthonormal basis β of eigenvectors of T.
  - Remark: In fact, the converse of this theorem is also true: if V has an orthonormal basis of eigenvectors of T, then T is Hermitian.

$\therefore$ $A = UDU^*$ con U unitaria matriz de autovectores ortonormales de A y D

matriz de autovalores $\in \mathbb{R}$

Falta probar $\lambda_i \geq 0$. Creo que o por acá:

Como $x^t A x > 0 \Rightarrow \lambda_i > 0$

$$x^t A x = x^t U D U^* x$$

Ejercicio 6. Sea $A = \begin{pmatrix} 4 & \alpha+2 & 2 \\ \alpha^2 & 4 & 2 \\ 2 & 2 & 1 \end{pmatrix}$.

(a) Hallar los valores de $\alpha \in \mathbb{R}$ para que A sea simétrica y $\lambda = 0$ sea autovalor de A.

(b) Para el valor de α hallado en (a), diagonalizar ortonormalmente la matriz A.

$$A = \begin{bmatrix} 4 & \alpha+2 & 2 \\ \alpha^2 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix} = A^t = \begin{bmatrix} 4 & \alpha^2 & 2 \\ \alpha+2 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$\alpha^2 - \alpha - 2 = 0$$

$$\Rightarrow \alpha = -1 \quad \text{ó} \quad \alpha = 2$$

Calculo Avd $\alpha = -1$

wolfram

$$\chi_{A_{-1}}(\lambda) = \det \begin{bmatrix} \lambda-4 & -1 & -2 \\ -1 & \lambda-4 & -2 \\ -2 & -2 & \lambda-1 \end{bmatrix} =$$

Input

eigenvalues $\begin{pmatrix} 4 & 1 & 2 \\ 1 & 4 & 2 \\ 2 & 2 & 1 \end{pmatrix}$

Results

$\lambda_1 = 3 + 2\sqrt{3}$

$\lambda_2 = 3$

$\lambda_3 = 3 - 2\sqrt{3}$

α no es -1

Calculo Avd $\alpha = 2$

$$\begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix} \rightsquigarrow$$

Input

eigenvalues $\begin{pmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{pmatrix}$

Results

$\lambda_1 = 9$

$\lambda_2 = 0$

$\lambda_3 = 0$

$\therefore \alpha = 2$ vale.

$$A = \begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

Corresponding eigenvectors
$v_1 = (2, 2, 1)$
$v_2 = (-1, 0, 2)$
$v_3 = (-1, 1, 0)$

Pues C unitaria $\Rightarrow C^* = C^{-1}$

$$A = C\, D\, C^*$$

con $D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{matrix} \leftarrow \lambda_1 \\ \leftarrow \lambda_2 \\ \leftarrow \lambda_3 \end{matrix}$

y $C = \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix}$ con q_i elementos de la BON a partir de los autovectores v_i

Por Prop sabemos que cada v_i correspondiente a algun autovalor λ_i es ortogonal a cualquier otro v_j correspondiente a algun $\lambda_j \neq \lambda_i$.

$\Rightarrow$ sé que $v_1 \perp v_2$

$v_1 \perp v_3$ pues $\lambda_1 = 9,\ \lambda_2 = \lambda_3 = 0$

$\Rightarrow$ quiero que v_3 sea ortonormal a v_2 (y lo siga siendo con v_1)

$$\Rightarrow \quad u_1 = v_1 \qquad \text{con} \qquad q_1 = \frac{u_1}{\|u_1\|} = \frac{1}{3}(2,2,1)$$

$$u_2 = v_2 \qquad \text{con} \qquad q_2 = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{5}}(-1,0,2)$$

Para q_3 uso GS:

$$u_3 = v_3 - \frac{\overbrace{\langle v_3, u_1 \rangle}^{\text{0 pues } v_3 \perp u_1}}{\|u_1\|^2} \cdot u_1 - \frac{\langle v_3, u_2 \rangle}{\|u_2\|^2} \cdot u_2$$

$$u_3 = (-1,1,0) - \frac{1}{5} \cdot 1 \cdot (-1,0,2)$$

$$= \left(-1 - \frac{1}{5},\ 1,\ -\frac{2}{5}\right)$$

$$u_3 = \left(-\frac{6}{5},\ 1,\ -\frac{2}{5}\right) \Rightarrow q_3 = \sqrt{\frac{5}{13}} \cdot \left(-\frac{6}{5},\ 1,\ -\frac{2}{5}\right)$$

$$q_3 = \left(-\frac{6}{\sqrt{65}},\ \sqrt{\frac{5}{13}},\ -\frac{2}{\sqrt{65}}\right)$$

$$\Rightarrow C = \begin{bmatrix} 2/3 & -1/\sqrt{5} & -6/\sqrt{65} \\ 2/3 & 0 & \sqrt{5}/\sqrt{13} \\ 1/3 & 2/\sqrt{5} & -2/\sqrt{65} \end{bmatrix}$$

Finalmente

$$A = \begin{bmatrix} 2/3 & -1/\sqrt{5} & -6/\sqrt{65} \\ 2/3 & 0 & \sqrt{5}/\sqrt{13} \\ 1/3 & 2/\sqrt{5} & -2/\sqrt{65} \end{bmatrix} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2/3 & -1/\sqrt{5} & -6/\sqrt{65} \\ 2/3 & 0 & \sqrt{5}/\sqrt{13} \\ 1/3 & 2/\sqrt{5} & -2/\sqrt{65} \end{bmatrix}^t$$

Reviso:

Input

$$\begin{pmatrix} \frac{2}{3} & -\frac{1}{\sqrt{5}} & -\frac{6}{\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{\sqrt{5}}{\sqrt{13}} \\ \frac{1}{3} & \frac{2}{\sqrt{5}} & -\frac{2}{\sqrt{65}} \end{pmatrix} . \begin{pmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \begin{pmatrix} \frac{2}{3} & -\frac{1}{\sqrt{5}} & -\frac{6}{\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{\sqrt{5}}{\sqrt{13}} \\ \frac{1}{3} & \frac{2}{\sqrt{5}} & -\frac{2}{\sqrt{65}} \end{pmatrix}^T$$

Result

$$\begin{pmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{pmatrix} \checkmark$$