Ejercicio 12. Sea $B = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ una base de K^n ($K = \mathbb{R}$ ó $\mathbb{C}$).

(a) Probar que si B es ortogonal, entonces

$$\mathbf{C}_{EB} = \begin{pmatrix} \cdots & \frac{\boldsymbol{v}_1^*}{\|\boldsymbol{v}_1\|_2^2} & \cdots \\ \cdots & \frac{\boldsymbol{v}_2^*}{\|\boldsymbol{v}_2\|_2^2} & \cdots \\ & \vdots & \\ \cdots & \frac{\boldsymbol{v}_n^*}{\|\boldsymbol{v}_n\|_2^2} & \cdots \end{pmatrix}$$

(b) Probar que si B es ortonormal, entonces $\mathbf{C}_{EB} = \mathbf{C}_{BE}^*$.

(c) Concluir que si B es ortonormal, entonces las coordenadas de un vector $\boldsymbol{v}$ en base B son:

$$(\boldsymbol{v})_B = (\boldsymbol{v}_1^*\boldsymbol{v}, \boldsymbol{v}_2^*\boldsymbol{v}, \dots, \boldsymbol{v}_n^*\boldsymbol{v}).$$

(d) Calcular $(\boldsymbol{v})_B$ siendo $\boldsymbol{v} = (1, -i, 3)$, $B = \{(\frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), (-\frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), (0, 0, i)\}$.

a) Para armar C_{EB} busco C tal que

$$v_i = \alpha_1 \cdot e_1 + \alpha_2 \cdot e_2 + \cdots + \alpha_n \cdot e_n \qquad \forall i \in \{1, \dots, n\}$$

$$\begin{bmatrix} C_{EB} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} v_i \end{bmatrix}$$

$$\begin{bmatrix} C_{EB} \end{bmatrix} \underbrace{\begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}}_{C_{BE}}$$

$\Rightarrow$ Como C_{BE} tiene columnas ortogonales

$$C_{BE}^t \, C_{BE} = D^2 \qquad \left(\begin{array}{l} \text{si fueran de norma 1} \\ \text{sería} = I \end{array} \right)$$

con D diagonal cuyos elementos $d_{ii} = \|v_i\|$

$\Rightarrow$ Como $C_{EB} = C_{BE}^{-1}$

$$C_{BE} = \left(C_{BE}^{t}\right)^{-1} D^2$$

$$C_{BE}^{-1} = \left(\left(C_{BE}^{t}\right)^{-1} D^2\right)^{-1}$$

$$C_{BE}^{-1} = \left(D^2\right)^{-1} \cdot C_{BE}^{t}$$

$$\parallel$$

$$C_{EB} = D^{-2} \cdot C_{BE}^{t}$$

$$C_{EB} = \begin{bmatrix} \frac{1}{\|v_1\|_2^2} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\|v_n\|_2^2} \end{bmatrix} \cdot \begin{bmatrix} - & v_1 & - \\ & \vdots & \\ - & v_n & - \end{bmatrix}$$

$$C_{EB} = \begin{bmatrix} - & \frac{v_1}{\|v_1\|_2^2} & - \\ - & \frac{v_n}{\|v_n\|_2^2} & - \end{bmatrix} \quad \checkmark$$

(b) Probar que si B es ortonormal, entonces $\mathbf{C}_{EB} = \mathbf{C}_{BE}^{*}$.

Si b es ortonormal

$\Rightarrow$ C_{BE} es ortogonal

$\Rightarrow$ $C_{BE}^{t} C_{BE} = I$

$$C_{BE} = \left(C_{BE}^{t}\right)^{-1}$$

Como $C_{BE}^{-1} = C_{EB}$

$$\Rightarrow \quad C_{BE}^{-1} = \left(\left(C_{BE}^{t} \right)^{-1} \right)^{-1}$$

$\therefore$

$$C_{EB} = C_{BE}^{t}$$

(c) Concluir que si B es ortonormal, entonces las coordenadas de un vector $\boldsymbol{v}$ en base B son:

$$(\boldsymbol{v})_B = (\boldsymbol{v}_1^* \boldsymbol{v}, \boldsymbol{v}_2^* \boldsymbol{v}, \ldots, \boldsymbol{v}_n^* \boldsymbol{v}).$$

$$(v)_B = C_{EB} \cdot v$$

$$= C_{BE}^{t} \cdot v$$

$$= \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}^{t} \cdot v$$

$$= \begin{bmatrix} - & v_1^t & - \\ & \vdots & \\ - & v_n^t & - \end{bmatrix} v$$

$$(v)_B = \left(v_1^t \cdot v, \; v_2^t \cdot v, \; \ldots, \; v_n^t \cdot v \right)$$

(d) Calcular $(\boldsymbol{v})_B$ siendo $\boldsymbol{v} = (1, -i, 3)$, $B = \{(\frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), (-\frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), (0, 0, i)\}$.

$$(1, -i, 3)_B = C_{EB} \cdot (1, -i, 3)^t$$

$$= C_{BE}^{*} \cdot (1, -i, 3)^t$$

$$= \begin{bmatrix} \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & i \end{bmatrix}^* \begin{bmatrix} 1 \\ -i \\ 3 \end{bmatrix}$$

$$= \underset{3 \times 3}{\begin{bmatrix} -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -i \end{bmatrix}} \underset{3 \times 1}{\begin{bmatrix} 1 \\ -i \\ 3 \end{bmatrix}}$$

$\underline{CA}$:

$$\frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\cancel{2}\sqrt{2}}{\cancel{2}}$$

$$= \begin{bmatrix} -\frac{i}{\sqrt{2}} - \frac{i}{\sqrt{2}} + 0 \\ \frac{i}{\sqrt{2}} - \frac{i}{\sqrt{2}} + 0 \\ -3i \end{bmatrix} = \begin{bmatrix} -\sqrt{2}\, i \\ 0 \\ -3i \end{bmatrix}$$

$$(1, -i, 3)_B = \begin{bmatrix} -\sqrt{2}\, i \\ 0 \\ -3i \end{bmatrix}$$

Ejercicio 13. Aplicar el algoritmo de Gram-Schmidt para calcular bases ortonormales de los subespacios generados por las siguientes bases:

(a) $B = \{(1,0,1),(0,1,1),(0,0,1)\}$

(b) $B = \{(i,1-i,0),(i,1,0)\}$

(c) $B = \{(1,-1,0,1),(0,1,1,0),(-1,0,1,1)\}$.

a) 1°) $\tilde{a} = a = (1,0,1) \Rightarrow q_1 = \frac{\tilde{a}}{\|\tilde{a}\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

2°) $\tilde{b} = b - \frac{\tilde{a}^t \cdot b}{\|\tilde{a}\|^2} \cdot \tilde{a} \Rightarrow q_2 = \frac{\tilde{b}}{\|\tilde{b}\|}$

$$\tilde{b} = (0,1,1) - \frac{\cancel{1.0} + \cancel{0.1} + 1.1}{2} \cdot (1,0,1)$$

$$\tilde{b} = \left(-\frac{1}{2}, 1, 1-\frac{1}{2}\right)$$

$$q_2 = \frac{\left(-\frac{1}{2}, 1, \frac{1}{2}\right)}{\sqrt{\frac{1}{4}+1+\frac{1}{4}}} \qquad \sqrt{\frac{1}{2}+\frac{2}{2}} = \sqrt{\frac{3}{2}}$$

$$q_2 \overset{\text{wolfran}}{=} \left(-\frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}, \frac{1}{\sqrt{6}}\right)$$

3°) $\tilde{c} = c - \frac{\tilde{b}^t \cdot c}{\|\tilde{b}\|^2} \cdot \tilde{b} - \frac{\tilde{a}^t \cdot b}{\|\tilde{a}\|^2} \cdot \tilde{a}$

CA:

$$\tilde{c} = \left(-\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right) \qquad \frac{\frac{1}{3}}{\frac{\sqrt{3}}{3}} \Rightarrow \frac{1}{\cancel{3}} \cdot \frac{\cancel{3}}{\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\|\tilde{c}\| = \sqrt{\frac{3}{9}} = \sqrt{\frac{1}{3}} = \frac{\sqrt{3}}{3}$$

$$q_3 = \left(-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$$

emathhelp.net/en/calculators/linear-algebra/gram-schmidt-calculator/

YOUR INPUT

Orthonormalize the set of the vectors $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ using th

Gram-Schmidt process.

SOLUTION

According to the Gram-Schmidt process, $\vec{u_k} = \vec{v_k} - \sum_{j=1}^{k-1} \mathrm{proj}_{\vec{u_j}}\left(\vec{v_k}\right)$, where $\mathrm{proj}_{\vec{u_j}}\left(\vec{v_k}\right) = \frac{\vec{u_j} \cdot \vec{v_k}}{|\vec{u_j}|^2} \vec{u_j}$ is a vector projection.

The normalized vector is $\vec{e_k} = \frac{\vec{u_k}}{|\vec{u_k}|}$.

Step 1

$$\vec{u_1} = \vec{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{e_1} = \frac{\vec{u_1}}{|\vec{u_1}|} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$ (for steps, see unit vector calculator).

Step 2

$$\vec{u_2} = \vec{v_2} - \mathrm{proj}_{\vec{u_1}}\left(\vec{v_2}\right) = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix}$$ (for steps, see vector projection calculator and vector subtraction calculator).

$$\vec{e_2} = \frac{\vec{u_2}}{|\vec{u_2}|} = \begin{bmatrix} -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \end{bmatrix}$$ (for steps, see unit vector calculator).

Step 3

$$\vec{u_3} = \vec{v_3} - \mathrm{proj}_{\vec{u_1}}\left(\vec{v_3}\right) - \mathrm{proj}_{\vec{u_2}}\left(\vec{v_3}\right) = \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$ (for steps, see vector projection calculator and vector subtraction calculator).

$$\vec{e_3} = \frac{\vec{u_3}}{|\vec{u_3}|} = \begin{bmatrix} -\frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{bmatrix}$$ (for steps, see unit vector calculator).

ANSWER

The set of the orthonormal vectors is $\left\{ \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{bmatrix} \right\} \approx$

$$\left\{ \begin{bmatrix} 0.707106781186548 \\ 0 \\ 0.707106781186548 \end{bmatrix}, \begin{bmatrix} -0.408248290463863 \\ 0.816496580927726 \\ 0.408248290463863 \end{bmatrix}, \begin{bmatrix} -0.577350269189626 \\ -0.577350269189626 \\ 0.577350269189626 \end{bmatrix} \right.$$

A

YOUR INPUT

Orthonormalize the set of the vectors $\mathbf{\vec{v_1}} = \left[\begin{array}{c}i\\1 - i\\0\end{array}\right]$, $\mathbf{\vec{v_2}} = \left[\begin{array}{c}i\\1\\0\end{array}\right]$ using the Gram-Schmidt process.

SOLUTION

According to the Gram-Schmidt process, $\mathbf{\vec{u_k}} = \mathbf{\vec{v_k}} - \sum_{j=1}^{k-1} \operatorname{proj}_{\mathbf{\vec{u_j}}}\left(\mathbf{\vec{v_k}}\right)$, where $\operatorname{proj}_{\mathbf{\vec{u_j}}}\left(\mathbf{\vec{v_k}}\right) = \frac{\mathbf{\vec{u_j}}\cdot \mathbf{\vec{v_k}}}{\left|\mathbf{\vec{u_j}}\right|^{2}} \mathbf{\vec{u_j}}$ is a vector projection.

The normalized vector is $\mathbf{\vec{e_k}} = \frac{\mathbf{\vec{u_k}}}{\left|\mathbf{\vec{u_k}}\right|}$.

Step 1

$$\mathbf{\vec{u_1}} = \mathbf{\vec{v_1}} = \left[\begin{array}{c}i\\1 - i\\0\end{array}\right]$$

$\mathbf{\vec{e_1}} = \frac{\mathbf{\vec{u_1}}}{\left|\mathbf{\vec{u_1}}\right|} = \left[\begin{array}{c}\frac{\sqrt{3} i}{3}\\\frac{\sqrt{3} \left(1 - i\right)}{3}\\0\end{array}\right]$ (for steps, see unit vector calculator).

Step 2

$\mathbf{\vec{u_2}} = \mathbf{\vec{v_2}} - \operatorname{proj}_{\mathbf{\vec{u_1}}}\left(\mathbf{\vec{v_2}}\right) = \left[\begin{array}{c}\frac{1}{3} + \frac{i}{3}\\\frac{i}{3}\\0\end{array}\right]$ (for steps, see vector projection calculator and vector subtraction calculator).

$\mathbf{\vec{e_2}} = \frac{\mathbf{\vec{u_2}}}{\left|\mathbf{\vec{u_2}}\right|} = \left[\begin{array}{c}\frac{\sqrt{3} \left(1 + i\right)}{3}\\\frac{\sqrt{3} i}{3}\\0\end{array}\right]$ (for steps, see unit vector calculator).

ANSWER

The set of the orthonormal vectors is $\left\{\left[\begin{array}{c}\frac{\sqrt{3} i}{3}\\\frac{\sqrt{3} \left(1 - i\right)}{3}\\0\end{array}\right], \left[\begin{array}{c}\frac{\sqrt{3} \left(1 + i\right)}{3}\\\frac{\sqrt{3} i}{3}\\0\end{array}\right]\right\}$

YOUR INPUT

Orthonormalize the set of the vectors $\vec{\mathbf{v_1}} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{\mathbf{v_2}} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{\mathbf{v_3}} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ using the Gram-Schmidt process.

SOLUTION

According to the Gram-Schmidt process, $\vec{\mathbf{u_k}} = \vec{\mathbf{v_k}} - \sum_{j=1}^{k-1} \mathbf{proj}_{\vec{\mathbf{u_j}}}\left(\vec{\mathbf{v_k}}\right)$, where $\mathbf{proj}_{\vec{\mathbf{u_j}}}\left(\vec{\mathbf{v_k}}\right) = \frac{\vec{\mathbf{u_j}} \cdot \vec{\mathbf{v_k}}}{\left|\vec{\mathbf{u_j}}\right|^2} \vec{\mathbf{u_j}}$ is a vector projection.

The normalized vector is $\vec{\mathbf{e_k}} = \frac{\vec{\mathbf{u_k}}}{\left|\vec{\mathbf{u_k}}\right|}$.

Step 1

$$\vec{\mathbf{u_1}} = \vec{\mathbf{v_1}} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$\vec{\mathbf{e_1}} = \frac{\vec{\mathbf{u_1}}}{\left|\vec{\mathbf{u_1}}\right|} = \begin{bmatrix} \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \\ 0 \\ \frac{\sqrt{3}}{3} \end{bmatrix}$ (for steps, see unit vector calculator).

Step 2

$\vec{\mathbf{u_2}} = \vec{\mathbf{v_2}} - \mathbf{proj}_{\vec{\mathbf{u_1}}}\left(\vec{\mathbf{v_2}}\right) = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \\ \frac{1}{3} \end{bmatrix}$ (for steps, see vector projection calculator and vector subtraction calculator).

$\vec{\mathbf{e_2}} = \frac{\vec{\mathbf{u_2}}}{\left|\vec{\mathbf{u_2}}\right|} = \begin{bmatrix} \frac{\sqrt{15}}{15} \\ \frac{2\sqrt{15}}{15} \\ \frac{\sqrt{15}}{5} \\ \frac{\sqrt{15}}{15} \end{bmatrix}$ (for steps, see unit vector calculator).

Step 3

$\vec{\mathbf{u_3}} = \vec{\mathbf{v_3}} - \mathbf{proj}_{\vec{\mathbf{u_1}}}\left(\vec{\mathbf{v_3}}\right) - \mathbf{proj}_{\vec{\mathbf{u_2}}}\left(\vec{\mathbf{v_3}}\right) = \begin{bmatrix} -\frac{6}{5} \\ -\frac{2}{5} \\ \frac{2}{5} \\ \frac{4}{5} \end{bmatrix}$ (for steps, see vector projection calculator and vector subtraction calculator).

$\vec{\mathbf{e_3}} = \frac{\vec{\mathbf{u_3}}}{\left|\vec{\mathbf{u_3}}\right|} = \begin{bmatrix} -\frac{\sqrt{15}}{5} \\ -\frac{\sqrt{15}}{15} \\ \frac{\sqrt{15}}{15} \\ \frac{2\sqrt{15}}{15} \end{bmatrix}$ (for steps, see unit vector calculator).

ANSWER

The set of the orthonormal vectors is $\left\{ \begin{bmatrix} \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \\ 0 \\ \frac{\sqrt{3}}{3} \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{15}}{15} \\ \frac{2\sqrt{15}}{15} \\ \frac{\sqrt{15}}{5} \\ \frac{\sqrt{15}}{15} \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{15}}{5} \\ -\frac{\sqrt{15}}{15} \\ \frac{\sqrt{15}}{15} \\ \frac{2\sqrt{15}}{15} \end{bmatrix} \right\} \approx$