Ejercicio 6.

a) Mostrar que toda matriz $\boldsymbol{B} \in \mathbb{R}^{n\times n}$ con $|\det(\boldsymbol{B})| > 1$ tiene un autovalor λ, real o complejo, con $|\lambda| > 1$.

b) Decidir si el método de Jacobi converge o no para un sistema dado por la matriz

$$\boldsymbol{A} = \begin{pmatrix} -1 & 1 & 2 \\ 4 & -1 & 3 \\ 5 & 6 & -1 \end{pmatrix}.$$

Suppose that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A. Then the λs are also the roots of the characteristic polynomial, i.e.

$$\begin{aligned} \det(A - \lambda I) = p(\lambda) &= (-1)^n(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n) \\ &= (-1)(\lambda-\lambda_1)(-1)(\lambda-\lambda_2)\cdots(-1)(\lambda-\lambda_n) \\ &= (\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda) \end{aligned}$$

The first equality follows from the factorization of a polynomial given its roots; the leading (highest degree) coefficient $(-1)^n$ can be obtained by expanding the determinant along the diagonal.

Now, by setting λ to zero (simply because it is a variable) we get on the left side $\det(A)$, and on the right side $\lambda_1\lambda_2\cdots\lambda_n$, that is, we indeed obtain the desired result

$$\det(A) = \lambda_1\lambda_2\cdots\lambda_n$$

So the determinant of the matrix is equal to the product of its eigenvalues.

$$\|Ax\|_2 = \|U\Sigma V^* x\|_2$$

$$\|Ax\|_2 = \|\Sigma x\|_2$$

$$\frac{\|Ax\|_2}{\|\Sigma x\|_2} = 1 \qquad \|A\|_2 = \max_x \frac{\|Ax\|_2}{\|x\|_2}$$

$$=$$