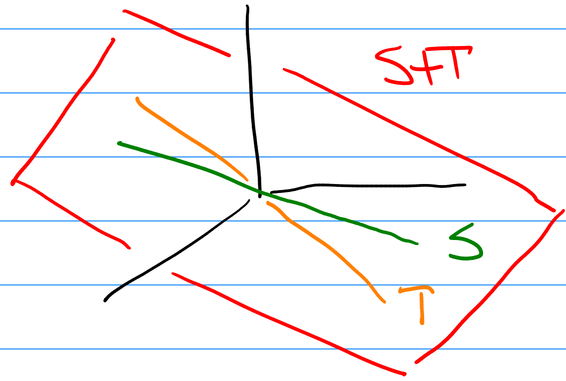
SUMA DE SUBESPACIOS

RECORDAR:

$$S+T = \{s+t : s\in S,\ t\in T\}$$
$$= \langle S \cup T\rangle$$
$$= \langle X \cup Y\rangle,$$

SI $S = \langle X\rangle,\ T = \langle Y\rangle$



EJERCICIOS:

1) SEA $S = \Big\langle \underbrace{\begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & 1\end{pmatrix}}_{M_1}, \underbrace{\begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 2\end{pmatrix}}_{M_2}, \underbrace{\begin{pmatrix} 1 & 3 & -1 \\ -1 & 0 & 3\end{pmatrix}}_{M_3} \Big\rangle \subseteq K^{2\times 3}$.

HALLAR $T \subseteq K^{2\times3}$ TAL QUE $\underbrace{S \oplus T = K^{2\times 3}}$.

T ES UN SUMANDO DIRECTO DE S

PROP: $S \subseteq V$. SI B ES BASE DE S Y B' ES TAL QUE $B \cup B'$ ES BASE DE V, ENTONCES $T = \langle B'\rangle$ ES SUMANDO DIRECTO DE S. //

¿ $B = \{M_1, M_2, M_3\}$ ES BASE DE S?

$0 = \alpha_1 M_1 + \alpha_2 M_2 + \alpha_3 M_3$ SII

$$\begin{cases} 0 = \alpha_1 + \alpha_3 & \to \text{COEF } (1,1) \\ 0 = 2\alpha_1 + \alpha_2 + 3\alpha_3 & \to \text{COEF } (1,2) \\ \vdots \end{cases}$$

⟶ CONSIDERAMOS EL SIST. HOM. EN $\alpha_1, \alpha_2, \alpha_3$:

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 0 & 1 & -1 \\ \cancel{-1} & \cancel{0} & \cancel{-1} \\ \cancel{0} & \cancel{0} & \cancel{0} \\ 1 & 2 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \cancel{0} & \cancel{-1} & \cancel{-1} \\ \cancel{0} & \cancel{2} & \cancel{2} \end{pmatrix}$$

TIENE SOL NO TRIVIALES; POR EJ

$\alpha_3 = 1$

$\alpha_2 = -\alpha_3 = -1$

$\alpha_1 = -\alpha_3 = -1$

ASÍ, $\quad -M_1 - M_2 + M_3 = 0$ "PUEDO SACAR CUALQUIERA"

$$\therefore \quad S = \langle \underbrace{M_1, M_2}_{} \rangle$$

SON INDEPTES $(M_1 \notin \langle M_2 \rangle, \; M_2 \notin \langle M_1 \rangle)$

⤳ SON BASE

BUSCO MATRICES $\underset{\text{(NUEVA)}}{M_3}, M_4, M_5, M_6$ TALES QUE

$$\sum_{i=1}^{6} \alpha_i M_i = 0 \Rightarrow \alpha_1 = \dots = \alpha_6 = 0$$

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \\ -1 & 0 \\ 0 & 0 \\ 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$M_1 \quad M_2 \qquad\qquad M_3 \quad M_4 \quad M_5 \quad M_6$

RTA: $\quad T = \left\langle \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$

2) SEAN $S, T \subseteq \mathbb{Q}_3[X]$ DADOS POR

$$S = \langle \underbrace{X^3+(k-2)X^2+(k-3)}_{P_1},\ \underbrace{2X^3+(2k-4)X^2+(3-k)X+(2k-6)}_{P_2},\ \underbrace{(5-k)X^3+2X^2+(k-3)}_{P_3} \rangle,$$

$$T = \{ P \in \mathbb{Q}_3[X] : P(k) = P(-k) = 0 \}$$

HALLAR TODOS LOS $k \in \mathbb{Q}$ TALES QUE $S \oplus T = \mathbb{Q}_3[X]$

$$S \oplus T = V \underset{\text{TEO. DIM.}}{\Rightarrow} \dim S + \dim T = \underbrace{\dim V}_{4}$$

NOTAR:

$$T = \begin{cases} \{ (X^2-k^2)\cdot Q : Q \in \mathbb{Q}_1[X] \}, & \text{si } k \neq 0 \\ \{ X \cdot Q : Q \in \mathbb{Q}_2[X] \}, & \text{si } k = 0 \end{cases}$$

$$\leadsto \dim T = \begin{cases} 2, & k \neq 0 \\ 3, & k = 0 \end{cases}$$

ASÍ SI S ES UN SUM. DIRECTO DE T

$$\underbrace{\dim S}_{\text{A CALCULAR}} = \begin{cases} 2, & k \neq 0 \\ 1, & k = 0 \end{cases} ;$$

$$0 = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3$$

$\Rightarrow (\alpha_1, \alpha_2, \alpha_3)$ ES SOL DE

COEF. A COEF.

$$\begin{pmatrix} 1 & 2 & 5-k \\ k-2 & 2k-4 & 2 \\ 0 & 3-k & 0 \\ k-3 & 2k-6 & k-3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 5-k \\ 0 & 0 & 2-(5-k)(k-2) \\ 0 & 3-k & 0 \\ 0 & 0 & (k-3)-(k-3)(5-k) \end{pmatrix}$$

$2-(5-k)(k-2) = 0$ SII $k=3$ ó $k=4$

$(k-3)-(k-3)(5-k) = 0$ SII $k=3$ ó $k=4$;

SI $k \neq 3, 4$, $\dim S = 3$ $\rightsquigarrow$ A LO SUMO SIRVE $k=3$ ó $k=4$

- SI $k=3$, $S = \langle x^3+x^2, \cancel{2x^3+2x^2}, \cancel{2x^3+2x^2} \rangle = \langle x^3+x^2 \rangle$

$\rightsquigarrow$ $\dim S = 1$, NO SIRVE

- SI $k=4$, $S = \langle x^3+2x^2+1, 2x^3+4x^2-x+2, \cancel{x^3+2x^2+1} \rangle$

$= \langle x^3+2x^2+1, \cancel{-}x \rangle$

$\rightsquigarrow$ $\dim S = 2$. LUEGO $S \oplus T = \mathbb{R}_3[x]$ SII $S \cap T = \{0\}$.

SEA $P \in S$: $P = \alpha(x^3+2x^2+1) + \beta x$. ASÍ, $P \in T$ SII

- $0 = P(4) = 73\alpha + 4\beta$
- $0 = P(-4) = -71\alpha - 4\beta$

SII $\alpha = \beta = 0$. LUEGO, $S \cap T = \{0\}$.

RTA: $k=4$

3) SEA $V = k^{(\mathbb{N})}$

$= \{a = (a_m)_{m\geq 1} \subseteq k : (\exists N_a)\ a_m = 0 \ \forall m \geq N_a\}$,

Y SEA $S = \{a \in V : \sum_{m\geq 1} a_m = 0\}$.

PROBAR QUE SI $a \in V \setminus S$ ENTONCES $\underbrace{V = S \oplus \langle a \rangle}_{S \text{ ES UN HIPERPLANO}}$

SEA $a \in V \setminus S$, Y SEA $\sigma = \sum_{m\geq 1} a_m \neq 0$.

DADO $b \in V$, SEA $\tau = \sum_{m\geq 1} b_m$. ASÍ,

$$b = \underbrace{\left(b - \tau/\sigma\, a\right)}_{=c} + \underbrace{\tau/\sigma \cdot a}_{\in \langle a \rangle}$$

NOTAR QUE $\sum_{m\geq 1} c_m = \tau - \tau/\sigma \cdot \sigma = 0$,

POR LO QUE $c \in S$

$\therefore\ b \in S + \langle a \rangle$