LINEALIZACIÓN

CONSIDEREMOS EL PROBLEMA $X' = F(X)$, $F: \mathbb{R}^2 \to \mathbb{R}^2$ DE CLASE C^1

QUEREMOS ENCONTRAR LOS PUNTOS DE EQUILIBRIO Y DIBUJAR EL DIAGRAMA DE FASE, PERO NO SIEMPRE ES FACIL

USAREMOS EL SIGUIENTE TEOREMA DE ESTABILIDAD PARA DIBUJAR EL DIAGRAMA DE FASES CERCA DE LOS PTOS DE EQUILIBRIO

<u>TEOREMA</u>: SEA F UN CAMPO C^1 EN $\mathbb{R}^2$ Y SEA X_0 UN CERO DE F.

SI $DF(X_0)$ NO TIENE AUTOVALORES CON PARTE REAL NULA, ENTONCES EXISTE UN ENTORNO DE X_0 TAL QUE EL DIAGRAMA DE FASES DEL SISTEMA

$$X' = F(x)$$

ES "PARECIDO" AL DIAGRAMA DE FASES DEL SISTEMA

$$Y' = DF(x_0)\, Y \longrightarrow \text{LINEAL}$$

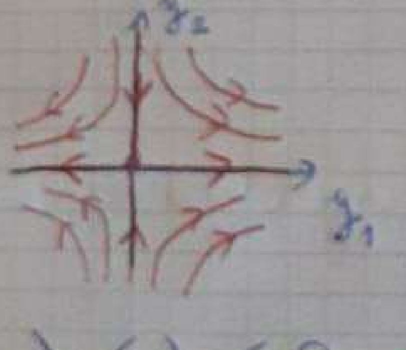
CERCA DE $Y_0 = 0$.

ESTE TEOREMA NOS PERMITE DIBUJAR APROXIMADAMENTE LOS DIAGRAMAS DE FASE ALREDEDOR DE UN PUNTO DE EQUILIBRIO QUE CUMPLA LAS HIPÓTESIS A PARTIR DE UN SISTEMA LINEAL.

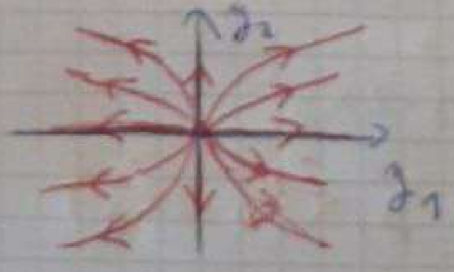
RECORDEMOS LOS SISTEMAS LINEALES:

$X' = AX$, λ_1, λ_2 AUTOVALORES DE A (CON PARTE REAL $\neq 0$)
v_1, v_2 AUTOVECTORES ASOCIADOS

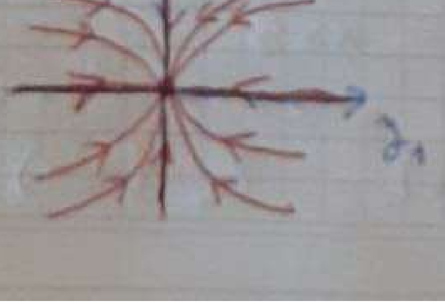
- $\lambda_2 < 0 < \lambda_1$



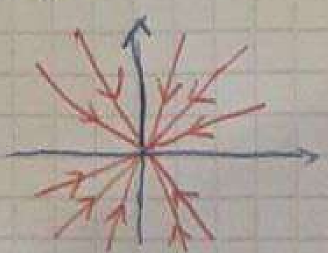
- $0 < \lambda_2 < \lambda_1$



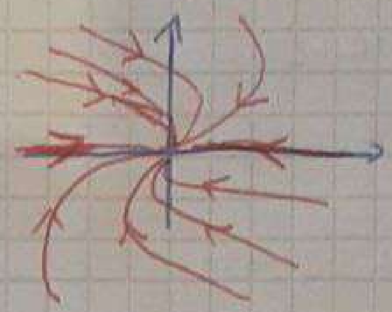
- $\lambda_1 < \lambda_2 < 0$



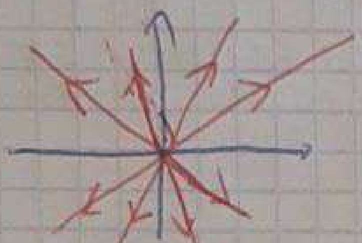
- $\lambda_1 = \lambda_2 = \lambda < 0$
  $\dim(S_\lambda) = 2$



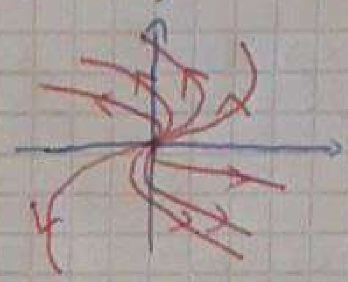
- $\lambda_1 = \lambda_2 = \lambda < 0$
  $\dim(S_\lambda) = 1$



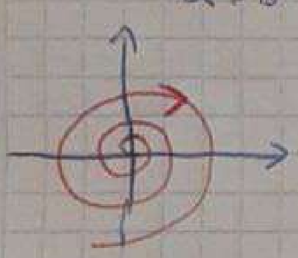
- $\lambda_1 = \lambda_2 = \lambda > 0$
  $\dim(S_\lambda) = 2$

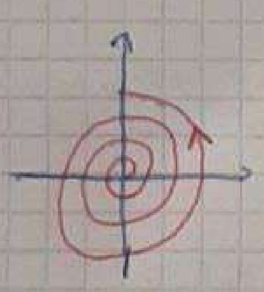


- $\lambda_1 = \lambda_2 = \lambda > 0$
  $\dim(S_\lambda) = 1$

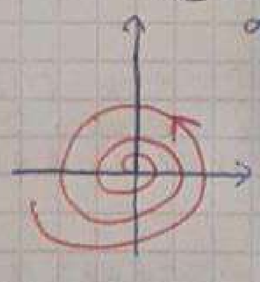


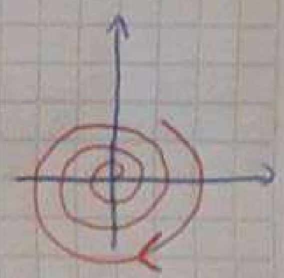
- $\lambda_1 = \alpha + \beta i$
  $\lambda_2 = \alpha - \beta i$
  $\alpha > 0$





- $\lambda_1 = \alpha + \beta i$
  $\lambda_2 = \alpha - \beta i$
  $\alpha < 0$





EJEMPLOS:

1) $\begin{cases} x' = x - y \\ y' = x + y - 2xy \end{cases}$

$F(x,y) = (x-y,\ x+y-2xy) \rightsquigarrow$ DE CLASE C^1

PUNTOS DE EQUILIBRIO: $F(x,y) = (0,0)$

$$\begin{cases} x - y = 0 \longrightarrow x = y \\ x + y - 2xy = 0 \end{cases} \hookrightarrow 2x - 2x^2 = 0 \begin{cases} x = 0 \\ x = 1 \end{cases}$$

$\rightsquigarrow P_1 = (0,0) \qquad P_2 = (1,1)$

$$DF(x,y) = \begin{pmatrix} 1 & -1 \\ 1-2y & 1-2x \end{pmatrix}$$

- $P_1 = (0,0)$

$$DF(0,0) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = A$$

$$\det\begin{pmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2 = 0 \begin{cases} \lambda = 1+i \\ \lambda = 1-i \end{cases}$$

$\rightsquigarrow$ CASO DE RAICES COMPLEJAS CON PARTE REAL POSITIVA:

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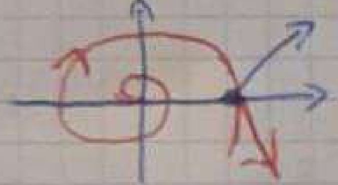
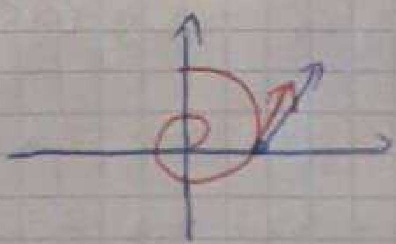
¿COMO SABEMOS CUAL?

$$A \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \longrightarrow$$



$\rightarrow$ NO ES



$\rightarrow$ ES ✓

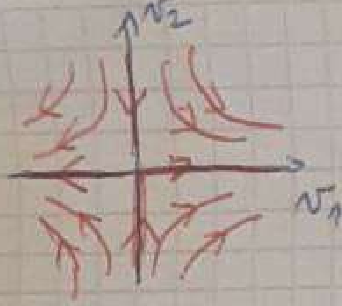
- $P_2 = (1,1)$

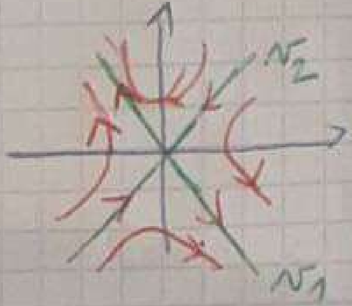
$$DF(1,1) = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} = A$$

$$\det\begin{pmatrix} 1-\lambda & -1 \\ -1 & -1-\lambda \end{pmatrix} = (1-\lambda)(-1-\lambda) - 1 = \lambda^2 - 2 = 0 \begin{cases} \lambda = \sqrt{2} \\ \lambda = -\sqrt{2} \end{cases}$$

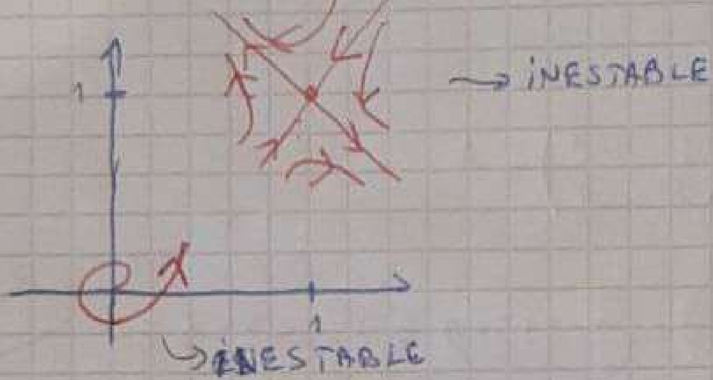
$\lambda_1 = \sqrt{2} \rightsquigarrow$ AUTOVECTOR $v_1 = (1, 1-\sqrt{2})$

$\lambda_2 = -\sqrt{2} \rightsquigarrow$ AUTOVECTOR $v_2 = (1, 1+\sqrt{2})$





FINALMENTE:



2) $\begin{cases} x' = 2x - y^2 \\ y' = -y + xy \end{cases}$

$F(x,y) = (2x - y^2, -y + xy) \longrightarrow$ DE CLASE C^1

$DF(x,y) = \begin{pmatrix} 2 & -2y \\ y & x-1 \end{pmatrix}$

PUNTOS DE EQUILIBRIO: $F(x,y) = (0,0)$

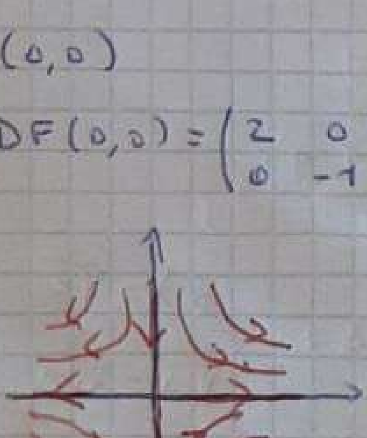
$\begin{cases} 2x - y^2 = 0 \\ -y + xy = 0 \end{cases} \longrightarrow y(x-1) = 0 \begin{cases} y = 0 \\ x = 1 \end{cases}$

- $y = 0 \longrightarrow 2x - 0 = 0 \longrightarrow x = 0 \qquad (0,0)$
- $x = 1 \longrightarrow 2 - y^2 = 0 \longrightarrow y = \pm\sqrt{2} \qquad (1,\sqrt{2})$, $(1,-\sqrt{2})$

- $P_1 = (0,0)$

$DF(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \longrightarrow \lambda_1 = 2, \quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = -1, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

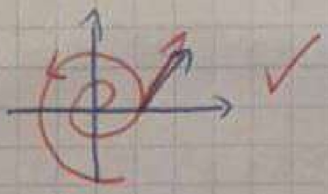


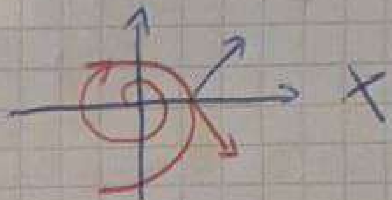
• $P_2 = (1, \sqrt{2})$

$$DF(1,\sqrt{2}) = \begin{pmatrix} 2 & -2\sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix} = A$$

$$\det\begin{pmatrix} 2-\lambda & -2\sqrt{2} \\ \sqrt{2} & -\lambda \end{pmatrix} = \lambda^2 - 2\lambda + 4 = 0 \left\langle \begin{array}{l} \lambda = 1+\sqrt{3}\,i \\ \lambda = 1-\sqrt{3}\,i \end{array} \right. \Bigg\}$$ AUTOVALORES COMP. CON PARTE REAL (+)

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ \sqrt{2} \end{pmatrix}$$





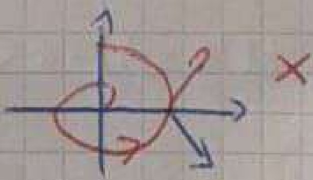
↳ ESTE

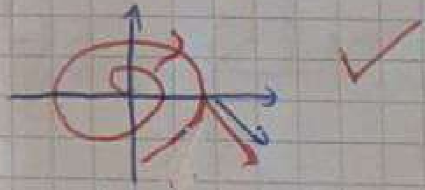
• $P_3 = (1, -\sqrt{2})$

$$DF(1,-\sqrt{2}) = \begin{pmatrix} 2 & 2\sqrt{2} \\ -\sqrt{2} & 0 \end{pmatrix} = A$$

$$\det\begin{pmatrix} 2-\lambda & 2\sqrt{2} \\ -\sqrt{2} & -\lambda \end{pmatrix} = \lambda^2 - 2\lambda + 4 \left\langle \begin{array}{l} \lambda = 1+\sqrt{3}\,i \\ \lambda = 1-\sqrt{3}\,i \end{array} \right. \Bigg\}$$ AUTOV. COMP. CON PARTE REAL (+)

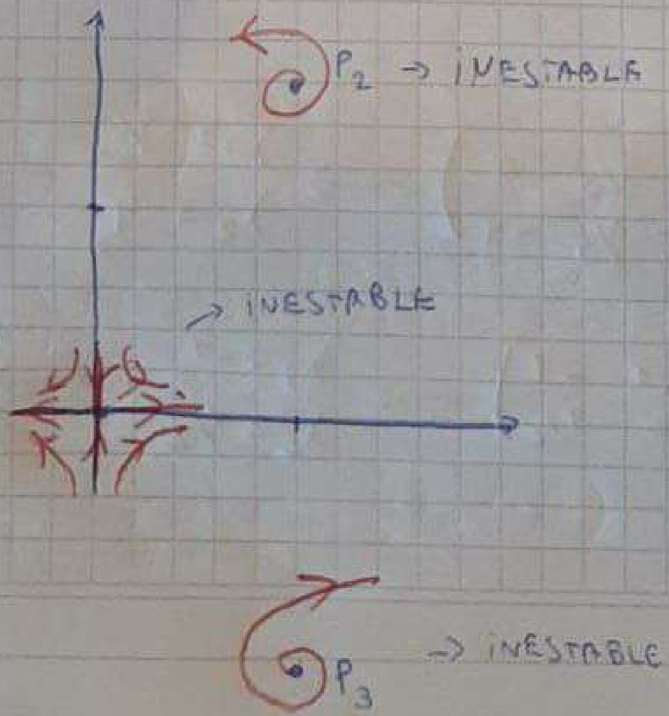
$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -\sqrt{2} \end{pmatrix}$$





↳ ESTE

FINALMENTE:



3) CONSIDEREMOS EL SISTEMA

$$\begin{cases} x' = x - 2x^2 - 2xy \\ y' = 4y - 5y^2 - 7xy \end{cases}$$

ENCONTRAR LOS PTOS DE EQUILIBRIO Y ANALIZAR ESTABILIDAD

SOLUCIÓN: $F(x,y) = (x - 2x^2 - 2xy,\ 4y - 5y^2 - 7xy) \rightarrow$ DE CLASE C^1

$$DF(x,y) = \begin{pmatrix} 1-4x-2y & -2x \\ -7y & 4-10y-7x \end{pmatrix}$$

PUNTOS DE EQUILIBRIO: $F(x,y) = (0,0)$

$$\begin{cases} x - 2x^2 - 2xy = 0 \\ 4y - 5y^2 - 7xy = 0 \end{cases} \longrightarrow \begin{cases} x(1-2x-2y) = 0 \\ y(4-5y-7x) = 0 \end{cases}$$

$$x(1-2x-2y) = 0 \begin{cases} x = 0 \\ x = \frac{1}{2} - y \end{cases}$$

$\underline{x=0}$ $\quad y(4-5y-0) = 0 \begin{cases} y = 0 \\ y = 4/5 \end{cases} \rightarrow \begin{matrix} (0,0) \\ (0, 4/5) \end{matrix}$

$\underline{x = \frac{1}{2} - y}$ $\quad y(4-5y-7(1/2-y)) = 0 \begin{cases} y = 0 \rightarrow x = 1/2 \\ y = \frac{-1}{4} \rightarrow x = 3/4 \end{cases} \rightarrow \begin{matrix} (1/2, 0) \\ (3/4, -1/4) \end{matrix}$

- $P_1 = (0,0)$

$$DF(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \longrightarrow \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 4 \end{matrix} \longrightarrow \text{¡INESTABLE!}$$

- $P_2 = (0, 4/5)$

$$DF(0, 4/5) = \begin{pmatrix} -3/5 & 0 \\ -28/5 & -4 \end{pmatrix} \longrightarrow \begin{matrix} \lambda_1 = -3/5 \\ \lambda_2 = -4 \end{matrix} \longrightarrow \text{ESTABLE!}$$

- $P_3 = (1/2, 0)$

$$DF(1/2, 0) = \begin{pmatrix} -1 & -1 \\ 0 & 1/2 \end{pmatrix} \longrightarrow \begin{matrix} \lambda_1 = -1 \\ \lambda_2 = 1/2 \end{matrix} \longrightarrow \text{¡INESTABLE!}$$

HOJA Nº

FECHA

$\cdot P_4 = \left(3/4, -1/4\right)$

$$DF\left(3/4, -1/4\right) = \begin{pmatrix} -3/2 & -3/2 \\ 7/4 & 5/4 \end{pmatrix} \longrightarrow \begin{array}{l} \lambda_1 = \frac{-1}{8} + i\frac{\sqrt{47}}{8} \\ \lambda_2 = \frac{-1}{8} - i\frac{\sqrt{47}}{8} \end{array} \rightarrow \text{¡NESTABLE!}$$

RESUMIENDO:

$(0,0)$; $(1/2, 0)$; $(3/4, -1/4)$ LOCALMENTE INESTABLES

$(0, 4/5)$ LOCALMENTE ESTABLE