Ejercicio 9 (a) Sea $\sigma(t) = (t; t^2)\ \forall\, t \in [0;1]$, calcular la longitud de σ entre 0 y 1. Para eso, notemos que σ es inyectiva y suave en $[0;1]$, además de regular pues $\sigma'(t) = (1; 2t)$, así, $\|\sigma'(t)\| = \sqrt{1+4t^2}\ \forall\, t \in [0;1]$

Así, $\mathrm{Long}_{[0;1]}(\sigma) = \displaystyle\int_0^1 \|\sigma'(t)\|\, dt = \int_0^1 \sqrt{1+4t^2}\, dt$

La resolveremos de varias maneras. Primero:

•) Funciones hiperbólicas: Defino $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$

$\mathrm{senh}(x) = \dfrac{e^x - e^{-x}}{2}\ \forall\, x \in \mathbb{R}$. Vale (ejercicio) $\cosh' = \mathrm{senh}$ y $\mathrm{senh}' = \cosh$ y $\cosh^2 - \mathrm{senh}^2 = 1$. Ahora, sea $t = \dfrac{\mathrm{senh}\, x}{2}$

$\Rightarrow dt = \dfrac{\cosh x}{2}\, dx$ y, ejercicio, senh es biyectiva.

resolvamos los límites de integración

si $t=0$, $0 = \dfrac{\mathrm{senh}\, x}{2} \Rightarrow \mathrm{senh}\, x = 0 \Rightarrow x = 0$.

si $t = 1$, $\mathrm{senh}(x) = 2 \Rightarrow x = \mathrm{arsenh}(2) = \ln(2+\sqrt{5}) \Rightarrow$

$$\int_0^1 \sqrt{1+4t^2}\, dt = \int_0^{\ln(2+\sqrt{5})} \sqrt{1+\mathrm{senh}^2(x)}\, \frac{\cosh(x)}{2}\, dx = \int_0^{\ln(2+\sqrt{5})} \sqrt{\cosh^2(x)}\, \frac{\cosh(x)}{2}\, dx$$

$$= \int_0^{\ln(2+\sqrt{5})} |\cosh(x)|\, \frac{\cosh(x)}{2}\, dx = \int_0^{\ln(2+\sqrt{5})} \frac{\cosh^2(x)}{2} = \int_0^{\ln(2+\sqrt{5})} \frac{\left(\frac{e^x + e^{-x}}{2}\right)^2}{2}\, dx =$$

$$\int_0^{\ln(2+\sqrt{5})} \frac{e^{2x}+2+e^{-2x}}{8}\,dx = \left(\frac{e^{2x}}{16}+\frac{1}{4}x-\frac{e^{-2x}}{16}\right)\Bigg|_0^{\ln(2+\sqrt{5})} =$$

$$\frac{(2+\sqrt{5})^2}{16}+\frac{\ln(2+\sqrt{5})}{4}-\frac{\cancel{\ln}\,1}{16(2+\sqrt{5})^2}-\cancelto{0}{\left(\frac{1}{16}+0-\frac{1}{16}\right)} =$$

$$\underset{\text{cuentitas}}{\ldots} = \boxed{\boxed{\frac{1}{4}\ln(2+\sqrt{5})+\frac{1}{2}\sqrt{5}}}$$

•) Funciones trigonométricas: Recordemos la identidad $1+tg^2(x)=\frac{1}{\cos^2(x)}=\sec^2(x)$ y $tg'=1+tg^2$

Ahora, sea $t=\frac{tg(x)}{2}$, así $dt=\frac{1}{2}(1+tg^2(x))dx$ y si $t=0$, $tg(x)=0$, tomemos $x=0$, si $t=1$, $tg(x)=2$, como tg es inyectiva en $[0;\frac{\pi}{2})$ y toma todos los valores en $\mathbb{R}_{\geq 0}$, elijamos $x=arctg(2) \Rightarrow$

$$\int_0^1\sqrt{1+4t^2}\,dt = \int_0^{arctg(2)}\sqrt{1+tg^2(x)}\,\frac{1}{2}(1+tg^2(x))\,dx = \int_0^{arctg(2)}\sqrt{\frac{1}{\cos^2(x)}}\,\frac{1}{2}\,\frac{1}{\cos^2(x)}\,dx$$

$$= \int_0^{arctg(2)}\frac{1}{2|\cos(x)|\cos^2(x)}\,dx = \int_0^{arctg(2)}\frac{dx}{2\cos^3(x)} = \int_0^{arctg(2)}\frac{\cos(x)}{2\cos^4(x)}\,dx =$$

↳ es >0 porque $\cos(x)\geq 0$ si $x\in[0;\frac{\pi}{2})\ni arctg(2)$

$$= \int_0^{\operatorname{arctg}(2)} \frac{\cos(x)}{2(1-\operatorname{sen}^2(x))^2}\,dx \underset{\downarrow}{=} \int_0^{\operatorname{sen}(\operatorname{arctg}(2))} \frac{du}{2(1-u^2)^2}$$

$u = \operatorname{sen}(x) \to$ sen es iny en $[0;\frac{\pi}{2})$

$du = \cos(x)dx$

Resta hacer fracciones <u>no</u> <u>tan</u> simples...

Quiero escribir $\frac{1}{(1-u^2)^2} = \frac{A}{1-u} + \frac{B}{(1-u)^2} + \frac{C}{1+u} + \frac{D}{(1+u)^2}$

$$\Rightarrow 1 = A(1-u)(1+u)^2 + B(1+u)^2 + C(1+u)(1-u)^2 + D(1-u)^2$$

$$= A(1-u)(1+2u+u^2) + B(1+2u+u^2) + C(1+u)(1-2u+u^2) + D(1-2u+u^2)$$

$$= A(1+u-u^2-u^3) + B(1+2u+u^2) + C(1-u-u^2+u^3) + D(1-2u+u^2)$$

$$= (A+B+C+D) + u(A+2B-C-2D) + u^2(-A+B-C+D) + u^3(-A+C)$$

$$\Rightarrow \begin{cases} A+B+C+D = 1 \Rightarrow 4A = 1 \Rightarrow A = \frac{1}{4} \\ A+2B-C-2D = 0 \longrightarrow 2B = 2D \Rightarrow B = D \\ -A+B-C+D = 0 \longrightarrow -2A+2B = 0 \Rightarrow A = B \\ -A+C = 0 \longrightarrow A = C \end{cases}$$

$C = D$

$$\Rightarrow A = B = C = D = \frac{1}{4}$$

¡Ahora sí! ¡¡Por finnnn!!

$$\int_0^{sen(arctg(2))} \frac{du}{2(1-u^2)^2} = \int_0^{sen(arctg(2))} \frac{1}{2}\left(\frac{1/4}{1-u} + \frac{1/4}{(1-u)^2} + \frac{1/4}{1+u} + \frac{1/4}{(1+u)^2}\right) du$$

$$= \frac{1}{8}\left(-\ln|1-u| + \frac{1}{1-u} + \ln|1+u| - \frac{1}{1+u}\right)\Bigg|_0^{sen(arctg(2))}$$ antes de evaluar

miremos $\frac{1}{1-u} - \frac{1}{1+u} = \frac{1+u-(1-u)}{1-u^2} = \frac{2u}{1-u^2}$

$$\Rightarrow \frac{2sen(arctg(2))}{1-sen^2(arctg(2))} = \frac{2sen(arctg(2))}{cos^2(arctg(2))} =$$

$$2sen(arctg(2)) \cdot (1+tg^2(arctg(2))) =$$

$$2sen(arctg(2)) \cdot (1+2^2) = 10\,sen(arctg(2))$$ y

$$\ln|1+u| - \ln|1-u| = \ln\left|\frac{1+u}{1-u}\right| \Rightarrow$$

$$\Rightarrow \ln\left|\frac{1+u}{1-u}\right| = \ln\left|\frac{(1+u)^2}{1-u^2}\right| = \ln\left|\frac{1+2u+u^2}{1-u^2}\right|$$

si ahora $u = senx$, $x \in [0; \frac{\pi}{2})$, $1-u^2 = 1-sen^2x = cos^2x$

y $1+2u+u^2 = 1+2senx+sen^2x = 1+2senx+1-cos^2x =$

$2+2senx-cos^2x = 2+2senx - \frac{1}{1+tg^2(x)}$ y como $x \in [0; \frac{\pi}{2})$, $\operatorname{sen} x \geq 0$ y $\cos x > 0$

$$\Rightarrow \operatorname{sen} x = \cos x \, tg(x) = \sqrt{\cos^2(x)} \, tg(x)$$

$$= \sqrt{\frac{1}{1+tg^2(x)}} \, tg(x) = \frac{tg(x)}{\sqrt{1+tg^2(x)}}$$

$$\Rightarrow \operatorname{sen}(arctg(2)) = \frac{tg(arctg(2))}{\sqrt{1+tg^2(arctg(2))}} = \frac{2}{\sqrt{5}}$$

$\Rightarrow$ Juntando todo tenemos

$$\frac{1}{1-u} - \frac{1}{1+u} + \ln|1+u| - \ln|1-u| \text{ evaluado en } u = \operatorname{sen}(arctg(2))$$

$$\frac{2u}{1-u^2} = 10 \operatorname{sen}(arctg(2)) = 10 \cdot \frac{2}{\sqrt{5}} = \frac{20}{\sqrt{5}} = 4\sqrt{5}$$

$$\ln\left|\frac{1+2u+u^2}{1-u^2}\right| = \ln\left|\frac{2+2\operatorname{sen}(arctg(2)) - \frac{1}{1+tg^2(arctg(2))}}{\cos^2(arctg(2))}\right|$$

$$\hookrightarrow \frac{1}{1+tg^2(arctg(2))} = \frac{1}{5}$$

$$= \ln\left|5 \cdot \left(2 + 2\frac{2}{\sqrt{5}} - \frac{1}{5}\right)\right| = \ln\left(9 + \frac{20}{\sqrt{5}}\right) = \ln(9+4\sqrt{5})$$

$$= \ln((2+\sqrt{5})^2) = 2\ln(2+\sqrt{5})$$

$$\Rightarrow \frac{1}{8}\left(\frac{1}{1-u}-\frac{1}{1+u}+\ln|1+u|-\ln|1-u|\right)\Big|_{0}^{\operatorname{sen}(\operatorname{arctg}(2))}$$

$$= \frac{1}{8}\left(4\sqrt{5}+2\ln(2+\sqrt{5})\right)-\frac{1}{8}\cancel{\left(1-1+0-0\right)}^{0} =$$

$$\boxed{\boxed{\frac{1}{2}\sqrt{5}+\frac{1}{4}\ln(2+\sqrt{5})}}$$ (Dio lo mismo, omg!!)

Listo el ejercicio ⓐ, pasamos al ⓑ

ⓑ Sea $\sigma:[10;20]\to\mathbb{R}^3 / \sigma(t)=(\sqrt{t};t+1;t)$, veamos que si dados $a,b\in[10;20] / \sigma(a)=\sigma(b) \Rightarrow$

$(\sqrt{a};a+1,a)=(\sqrt{b};b+1;b) \Rightarrow \begin{cases}\sqrt{a}=\sqrt{b}\\ a+1=b+1\\ a=b\end{cases} \Rightarrow a=b$ y σ es inyectiva. También, $\sigma'(t)=\left(\frac{1}{2\sqrt{t}};1;1\right)\neq\vec{0}\ \forall t\in[10,20]$ y así, σ es suave, simple y regular $\Rightarrow$ Ahora, a resolver la integral de varias maneras. Primero:

•) <u>Sustitución completa</u>: Acá queremos sustituir con todo lo que está en la integral, para eso el integrando necesita ser inyectivo, para eso notemos que dados $a,b\in[10,20] / \sqrt{2+\frac{1}{4a}}=\sqrt{2+\frac{1}{4b}} \Rightarrow$

$\cancel{2}+\frac{1}{4a}=\cancel{2}+\frac{1}{4b} \Rightarrow 4a=4b \Rightarrow a=b$, y listo, empezamos:

$$\int_{10}^{20} \|\sigma'(t)\|\,dt = \int_{10}^{20} \sqrt{2+\frac{1}{4t}}\,dt = \int_{9/\sqrt{40}}^{\sqrt{161/80}} u \frac{-u}{2(u^2-2)^2}\,du =$$

$$u = \sqrt{2+\frac{1}{4t}}$$

$$du = -\frac{1}{4t^2}\frac{1}{2\sqrt{2+\frac{1}{4t}}}\,dt \Rightarrow \frac{-u\,du}{2(u^2-2)^2} = dt$$

y si $t=10$, $u=\frac{9}{\sqrt{40}}$, si $t=20$, $u=\sqrt{\frac{161}{80}}$

$\int_{9/\sqrt{40}}^{\sqrt{161/80}} \frac{-u^2}{2(u^2-2)^2}\,du$, con lo que ahora hay que hacer fracciones simples. Como antes, hacen la cuenta y queda que

$$\frac{-u^2/2}{(u^2-2)^2} = \frac{-\sqrt{2}/16}{u-\sqrt{2}} + \frac{-1/8}{(u-\sqrt{2})^2} + \frac{\sqrt{2}/16}{u+\sqrt{2}} + \frac{-1/8}{(u+\sqrt{2})^2} \Rightarrow$$

$$\int_{9/\sqrt{40}}^{\sqrt{161/80}} \frac{-u^2}{2(u^2-2)^2}\,du = \left(-\frac{\sqrt{2}}{16}\ln|u-\sqrt{2}| + \frac{1/8}{u-\sqrt{2}} + \frac{\sqrt{2}}{16}\ln|u+\sqrt{2}| + \frac{1/8}{u+\sqrt{2}}\right)\Bigg|_{9/\sqrt{40}}^{\sqrt{161/80}}$$

Si juntamos logaritmos y fracciones

$$\frac{\sqrt{2}}{16}\ln|u+\sqrt{2}| - \frac{\sqrt{2}}{16}\ln|u-\sqrt{2}| = \frac{\sqrt{2}}{16}\ln\left|\frac{u+\sqrt{2}}{u-\sqrt{2}}\right| = \frac{\sqrt{2}}{16}\ln\left|\frac{(u+\sqrt{2})^2}{u^2-2}\right|$$

$$\text{y } \frac{1}{8}\frac{1}{u+\sqrt{2}} + \frac{1}{8}\frac{1}{u-\sqrt{2}} = \frac{1}{8}\left(\frac{u-\sqrt{2}+u+\sqrt{2}}{u^2-2}\right) = \frac{1}{8}\frac{2u}{u^2-2} = \frac{u}{4(u^2-2)} \Rightarrow$$

$$\int_{\frac{9}{\sqrt{40}}}^{\sqrt{\frac{161}{80}}} \frac{-u^2}{(u^2-2)^2\,2}\,du = \frac{\sqrt{2}}{16}\ln\left(\frac{\left(\sqrt{\frac{161}{80}}+\sqrt{2}\right)^2}{1/80}\right) + \frac{\sqrt{\frac{161}{80}}}{4\cdot 1/80} - \frac{\sqrt{2}}{16}\ln\left(\frac{\left(\frac{9}{\sqrt{40}}+\sqrt{2}\right)^2}{1/40}\right)$$

$$- \frac{\frac{9}{\sqrt{40}}}{4\cdot 1/40}$$

$$= \boxed{\frac{\sqrt{2}}{16}\ln\left(2\left(\frac{\sqrt{\frac{161}{80}}+\sqrt{2}}{\frac{9}{\sqrt{40}}+\sqrt{2}}\right)^2\right) + 20\sqrt{\frac{161}{80}} - 10\frac{9}{\sqrt{40}}}$$

$\hookrightarrow \approx 14{,}2032642$

•) Otra sustitución trigonométrica: Acá vamos a usar un truco parecido al que usamos en la resolución naranja que nos muestra la versatilidad de la tangente como función. Para esto voy a considerar el cambio $t = \frac{1}{8}\cot g^2(x)$ donde uso que, como tg era inyectiva, $\frac{1}{tg} = \cot g$ es inyectiva, y como $t \in [10;20], \Rightarrow \cot g^2(x) \in [80;160]$ $\Rightarrow$ si pedimos $x \in \mathbb{R} / \cot g(x) \in [\sqrt{80};\sqrt{160}]$ vamos a ser inyectivos. Así, veamos quienes son los extremos. $\cot g(x) = \sqrt{80} \Rightarrow tg(x) = \frac{1}{\sqrt{80}} = \frac{1}{4\sqrt{5}} \Rightarrow x = \operatorname{arctg}\frac{1}{4\sqrt{5}}$ y si $\cot g(x) = \sqrt{160} \Rightarrow tg(x) = \frac{1}{\sqrt{160}} \Rightarrow x = \operatorname{arctg}\left(\frac{1}{4\sqrt{10}}\right)$. Veamos el diferencial ahora. $dt = \frac{1}{8}2\cot g(x)\frac{-1}{sen^2 x}dx \Rightarrow$

$$\int_{10}^{20}\sqrt{2+\frac{1}{4t}}\,dt = \int_{\operatorname{arctg}\frac{1}{4\sqrt{5}}}^{\operatorname{arctg}\frac{1}{4\sqrt{10}}}\sqrt{2+2tg^2(x)}\left(-\frac{1}{4}\right)\frac{\cot g(x)}{sen^2 x}\,dx =$$

$$\int_{\operatorname{arctg}\frac{1}{4\sqrt{10}}}^{\operatorname{arctg}\frac{1}{4\sqrt{5}}} \sqrt{2}\,\frac{1}{4}\,\frac{1}{|\cos x|}\,\frac{\cos x}{\operatorname{sen}^3 x}\,dx = \frac{\sqrt{2}}{4}\int \frac{dx}{\operatorname{sen}^3 x}$$

$\operatorname{arctg}\left(\frac{1}{4\sqrt{5}}\right) \simeq 0{,}11134$

$\operatorname{arctg}\left(\frac{1}{4\sqrt{10}}\right) \simeq 0{,}07889$

$\left[\operatorname{arctg}\frac{1}{4\sqrt{10}};\ \operatorname{arctg}\frac{1}{4\sqrt{5}}\right] \subseteq \left[0, \frac{\pi}{2}\right] \Rightarrow \cos x \geq 0$

$$= \frac{\sqrt{2}}{4}\int_{\operatorname{arctg}\frac{1}{4\sqrt{10}}}^{\operatorname{arctg}\frac{1}{4\sqrt{5}}} \frac{\operatorname{sen} x}{(1-\cos^2 x)^2}\,dx$$

$u = -\cos x$

$du = \operatorname{sen} x\,dx$ y como vimos, cos es positivo e inyectivo

$$= \frac{\sqrt{2}}{4}\int_{-\cos\left(\operatorname{arctg}\frac{1}{4\sqrt{10}}\right)}^{-\cos\left(\operatorname{arctg}\frac{1}{4\sqrt{5}}\right)} \frac{du}{(1-u^2)^2}$$

y quedan las mismas fracciones simples de le resolución naranja

Y así, podemos reciclar hasta las integrales de esa resolución, queda

$$\int_{10}^{20}\sqrt{2+\frac{1}{4t}}\,dt = \frac{\sqrt{2}}{4}\left(+\frac{1}{4}\ln\left|\frac{1+u}{1-u}\right| + \frac{1}{4}\,\frac{2u}{1-u^2}\right)\Bigg|_{-\cos\left(\operatorname{arctg}\frac{1}{4\sqrt{10}}\right)}^{-\cos\left(\operatorname{arctg}\frac{1}{4\sqrt{5}}\right)}$$

$$= \frac{\sqrt{2}}{16}\ln\left(81\left(\frac{161}{81}-\frac{2\sqrt{80}}{9}\right)\right) - \frac{\sqrt{2}}{16}\ln\left(161\left(\frac{321}{161}-2\sqrt{\frac{160}{161}}\right)\right)$$

$$-\frac{\sqrt{2}}{8}\,81\,\frac{\sqrt{80}}{9} + \frac{\sqrt{2}}{8}\,161\sqrt{\frac{160}{161}}$$

$$= \boxed{\frac{\sqrt{2}}{16}\ln\left(\frac{161-18\sqrt{80}}{321-8\sqrt{1610}}\right) + \sqrt{805} - 9\sqrt{\frac{5}{2}}} \approx 14{,}2032642\ldots$$

Integrales tricky.

Acá nos vamos a centrar en integrales que suelen aparecer que involucran sen y cos

1) $\int \frac{dx}{sen^n x}$, con n impar, i.e: $n=2K-1$, $K\in\mathbb{N}$

$$\Rightarrow \int \frac{dx}{sen^n x} = \int \frac{sen\,x}{sen^{2K} x}\,dx = \int \frac{sen\,x}{(1-\cos^2 x)^K} = \int \frac{du}{(1-u^2)^K}$$

$u = -\cos x$
$du = sen\,x\,dx$

Resta hacer fracciones simples y se resuelve, y es simétrico el problema, ya que si tuvieras $\cos^n$ dividiendo,

$$\int \frac{dx}{\cos^n x} = \int \frac{\cos x}{\cos^{2K} x}\,dx = \int \frac{\cos x\,dx}{(1-sen^2 x)^K} = \int \frac{du}{(1-u^2)^K}$$

$u = sen\,x$
$du = \cos x\,dx$

Ahora, ¿qué pasa si n es par, ie: $n=2K$, $K\in\mathbb{N}$

$$\Rightarrow \int \frac{dx}{\cos^n x} = \int \frac{dx}{\cos^2 x}\frac{1}{\cos^{2K-2} x} = \int \frac{dx}{\cos^2 x}(1+tg^2 x)^{K-1}$$

$$= \int (1+u^2)^{K-1}\,du = \int \sum_{i=0}^{K-1} \binom{K-1}{i} u^{2i}\,du = \sum_{i=0}^{K-1} \binom{K-1}{i} \int u^{2i}\,du$$

$$= \sum_{i=0}^{K-1} \binom{K-1}{i} \frac{(tg\,x)^{2i+1}}{2i+1}$$

$u = tg\,x$
$du = \frac{1}{\cos^2 x}\,dx$

Teorema del Binomio de Newton

¿Qué pasa si vemos sen y cos juntos?

¿$\int \frac{dx}{\cos^n x \operatorname{sen}^m x}$? Para eso vayamos con un ejemplo para inspirarlos:

$$\int \frac{dx}{\cos^2 x \operatorname{sen}^3 x} = \int \frac{\operatorname{sen} x \, dx}{\cos^2 x \operatorname{sen}^4 x} = \int \frac{\operatorname{sen} x \, dx}{\cos^2 x (1-\cos^2 x)^2}$$

$u = \cos x$

$du = -\operatorname{sen} x \, dx$

$$= -\int \frac{du}{u^2(1-u^2)^2} = -\int \frac{du}{u^2(1-u)^2(1+u)^2} \ldots$$

Fíjense que si n y m tienen paridades distintas este truco funciona siempre

¿Qué pasa si n y m son pares?

¿Y si fueran impares? Se los dejo de ejercicio, un tip: sale muy parecido a lo que vimos antes... :)