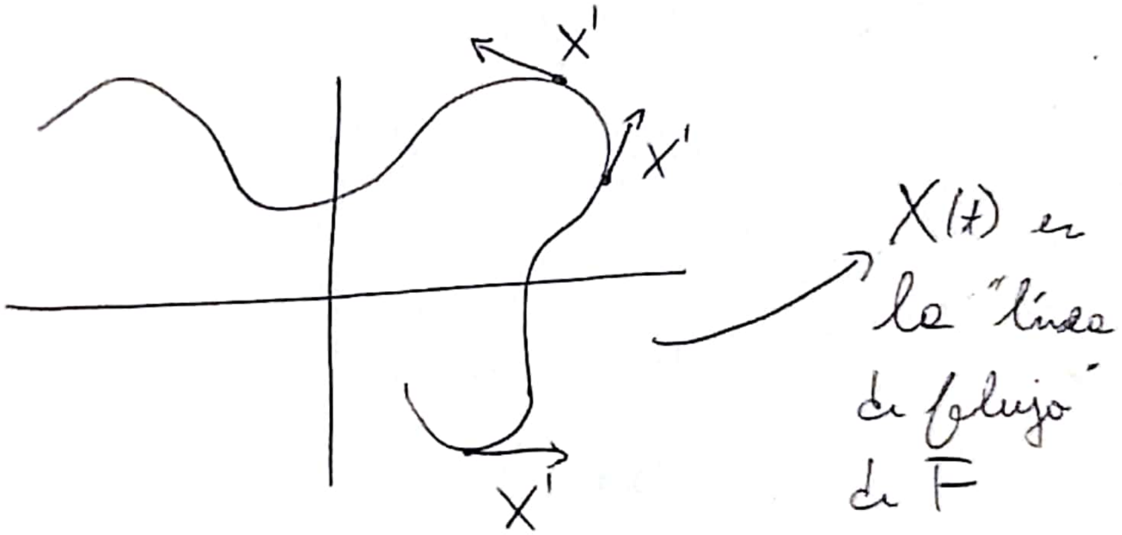
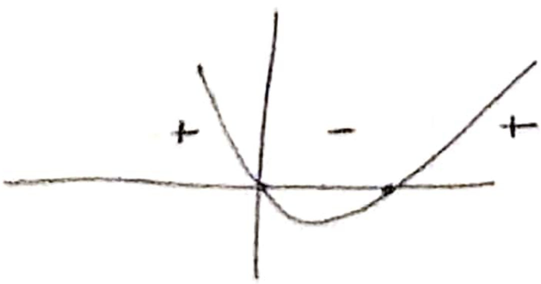
Diagrama de Fases

Sea $F:\mathbb{R}^n \to \mathbb{R}^n$. El problema a resolver es entender las soluciones de $X'(t) = F(X(t))$.

¡Tener en cuenta que $X:\mathbb{R} \to \mathbb{R}^n$ es una curva!



Ejemplo (n=1): $F(x) = x(x-1)$



- $X=0$ y $X=1$ son solución de $F(x)=0 \Rightarrow$ "PUNTOS DE EQUILIBRIO"

• El 0 es <u>ESTABLE</u>: Si $X(t) \in (-\varepsilon, \varepsilon)$, entonces

- X crece en $(-\varepsilon, 0)$ porque $X' = F(x) > 0$
- X decrece en $(0, \varepsilon)$ porque $X' = F(x) < 0$

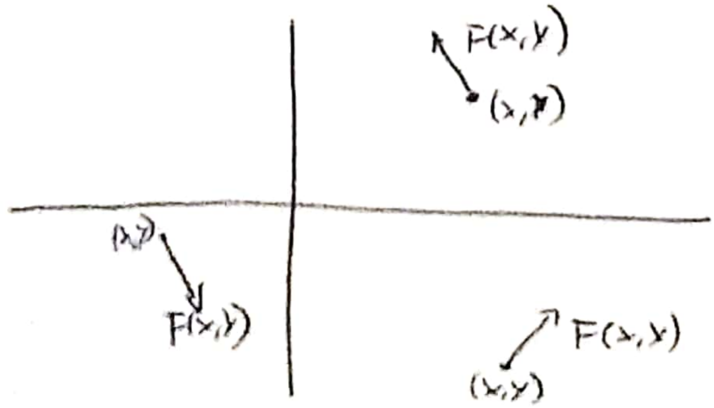
$\Rightarrow X(t) \to 0 \ ; \ t \to +\infty$

• El 1 es <u>INESTABLE</u>: si $X(t) > 0$, como $X'(t) = F(x(t)) > 0$, entonces X se aleja de 0 cuando t crece.

Explícitamente las soluciones son:

$$X_c(t) = \begin{cases} (1 - ce^t)^{-1}, & t > \ln(1/c) \\ (1 - ce^t)^{-1}, & t < \ln(1/c) \\ (1 + ce^t)^{-1}, & t \in \mathbb{R} \end{cases} \qquad c_p > 0.$$

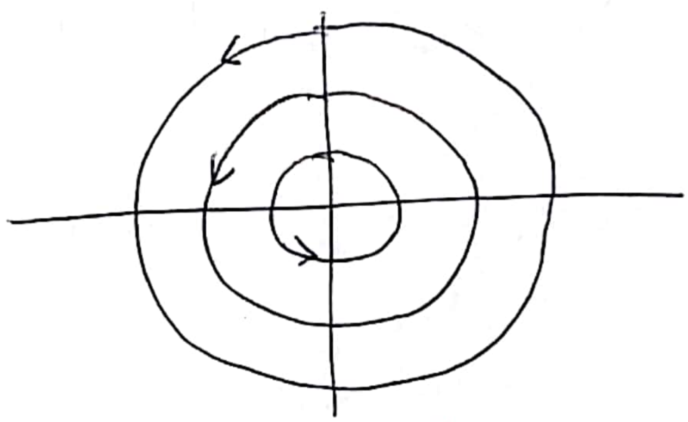
Ejemplo $(n=2)$. $F(x,y) = \dfrac{(-y, x)}{\|(x,y)\|}$. <u>Obs</u>: 1) No es lineal
2) $\|F\| \equiv 1$.



Soluciones: $X_r(t) = \frac{1}{r}\left(\cos(r^2 t), \operatorname{sen}(r^2 t)\right) = \frac{1}{r} e^{i r^2 t}$

es solución $\forall r > 0$

El diagrama queda de la siguiente forma:



MÁS EJEMPLOS: F lineal $\Rightarrow F(x,y) = A \cdot \begin{pmatrix} x \\ y \end{pmatrix}$, $A \in \mathbb{R}^{2\times 2}$

YA SABEMOS QUE LAS SOLUCIONES DE $X' = F(X)$ DEPENDEN de AUTOVALORES Y AUTOVECTORES DE A.

<u>Ejemplo 3</u> "$\lambda_1 > 0 > \lambda_2$"

$$A = \begin{pmatrix} -3 & 0 \\ 5 & 2 \end{pmatrix} \Rightarrow \begin{array}{l} \lambda_1 = 2, \; v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \lambda_2 = -3, \; v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{array}$$

$$\Rightarrow X(t) = c_1 e^{2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}}_{=:C} \underbrace{\begin{pmatrix} c_1 e^{2t} \\ c_2 e^{-3t} \end{pmatrix}}_{=:Y(t)}$$

¿Cuál es el plano? A. Gráfica Y (✱1)

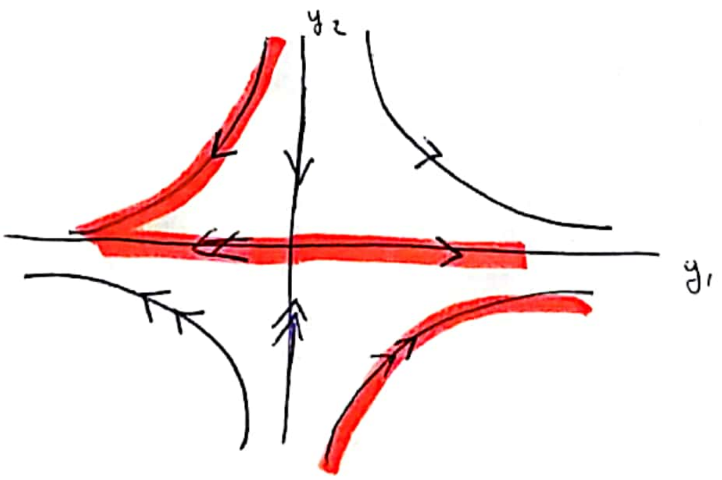
B. Transformo con C. (✱2)

✱1: $y_1(t) = c_1 e^{2t}$, $y_2(t) = c_2 e^{-3t}$

$$\Rightarrow y_2(t) = \frac{c_2}{|c_1|^{-3/2}} \left(|c_1| e^{2t} \right)^{-3/2}$$

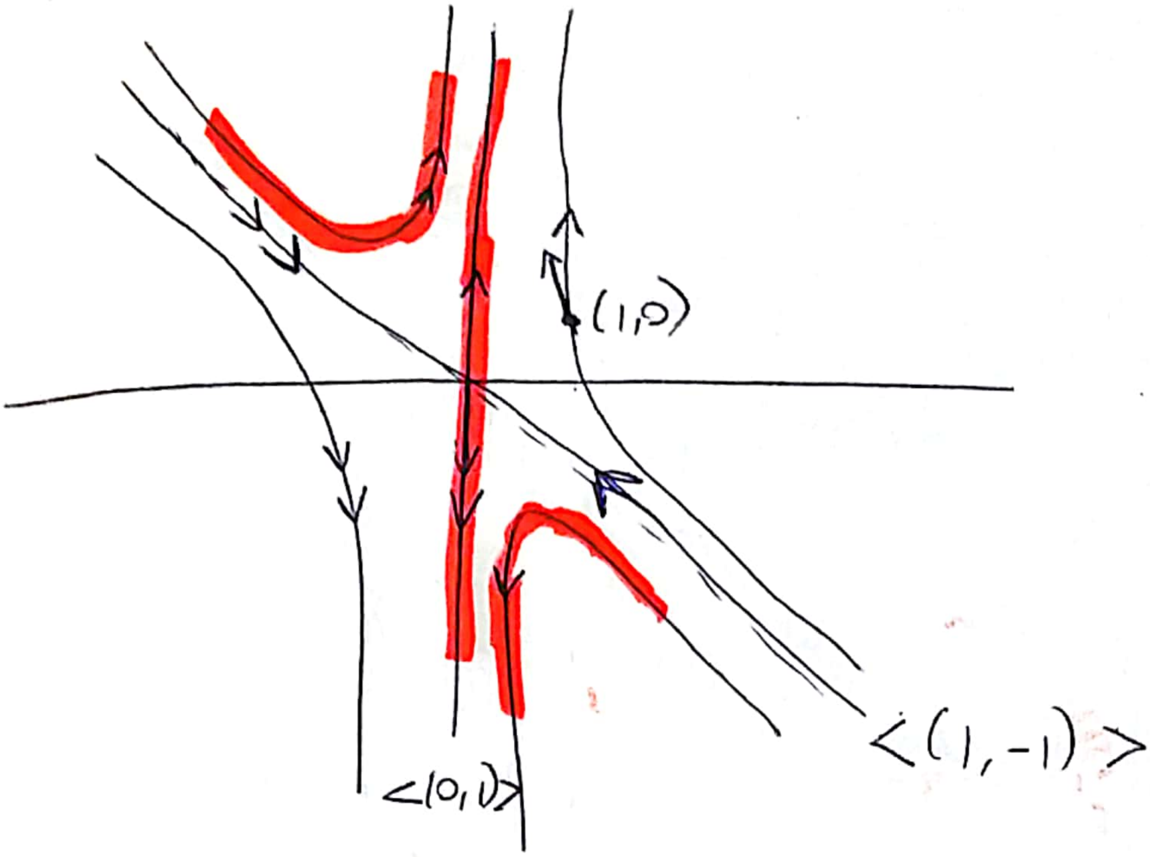
$$= \frac{c_2}{|c_1|^{-3/2}} |y_1(t)|^{-3/2}$$

Luego



✱2: Debemos "TRANSFORMAR" las soluciones con C.

$$C\left(\lambda \begin{pmatrix}1\\0\end{pmatrix}\right) = \lambda \begin{pmatrix}0\\1\end{pmatrix} \quad , \quad C\left(\lambda \begin{pmatrix}0\\1\end{pmatrix}\right) = \lambda \begin{pmatrix}1\\-1\end{pmatrix}$$



- ; $\lambda > 0$
- , $\lambda < 0$
- , $\lambda > 0$
- , $\lambda < 0$

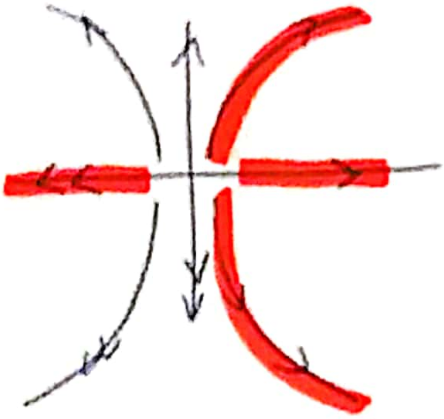
- $A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-3\\5\end{pmatrix}$
- $\det C < 0 \Rightarrow C$ invierte orientación

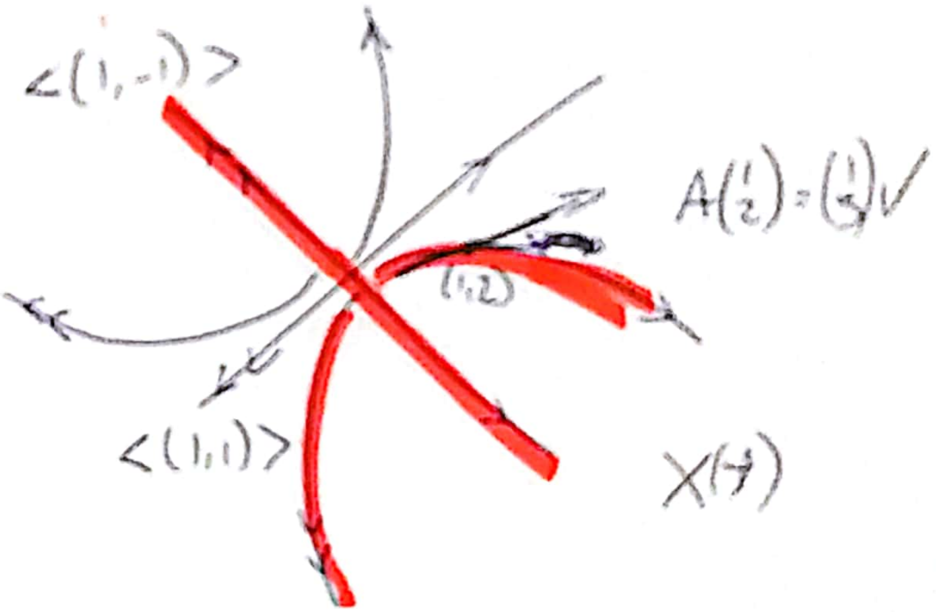
Ejemplo 4 $A=\begin{pmatrix}3 & -1\\ -1 & 3\end{pmatrix}$, "$\lambda_1 > \lambda_2 > 0$"

$\Rightarrow \lambda_1 = 4, \quad v_1 = \begin{pmatrix}1\\-1\end{pmatrix}$

$\lambda_2 = 2, \quad v_2 = \begin{pmatrix}1\\1\end{pmatrix}$

$\Rightarrow X(t) = \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}c_1 e^{4t}\\ c_2 e^{2t}\end{pmatrix}$. Obs: $\det(C) > 0$.



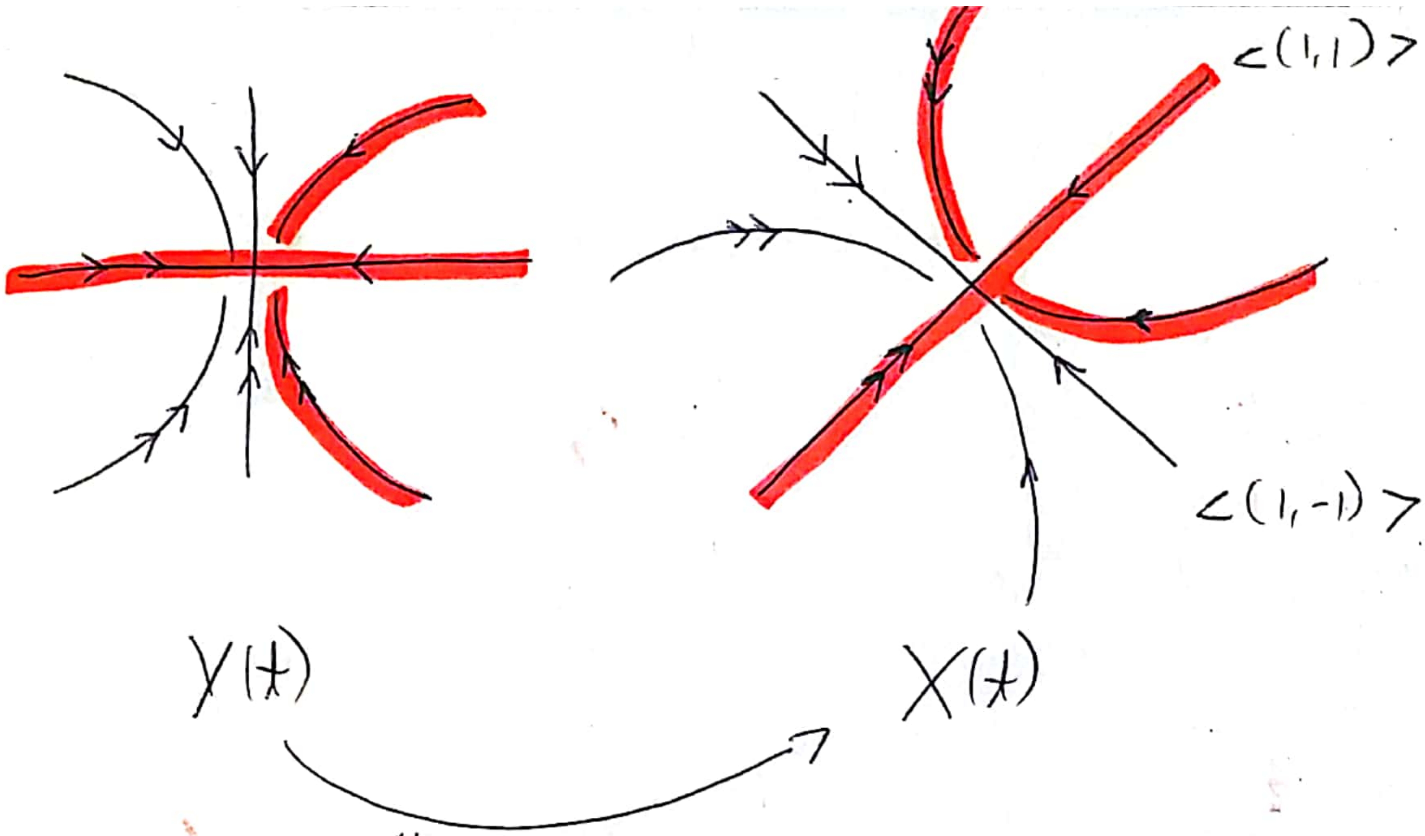


Ejemplo 5 $A=\begin{pmatrix}-3 & -1\\ -1 & -3\end{pmatrix}$, "$\lambda_1 < \lambda_2 < 0$"

$\Rightarrow \lambda_1 = -4, \quad v_1 = \begin{pmatrix}1\\1\end{pmatrix}$

$\lambda_2 = -2, \quad v_2 = \begin{pmatrix}1\\-1\end{pmatrix}$

$\Rightarrow X(t) = \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}c_1 e^{-4t}\\ c_2 e^{-2t}\end{pmatrix}$. Obs. $\det(C) < 0$



NOTAR COMO SE INVIERTE LA ORIENTACIÓN

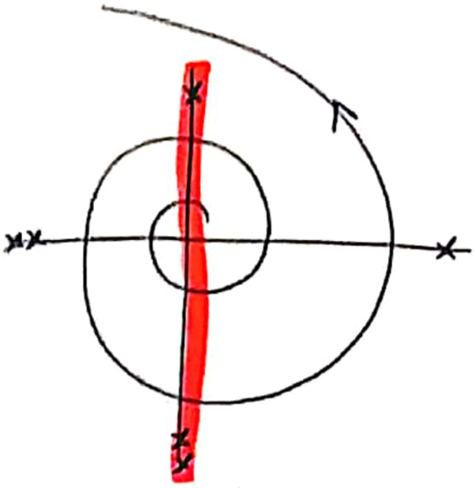
Ejemplo 6: $A = \begin{pmatrix} 0 & -2 \\ 1 & 2 \end{pmatrix}$, "AUTOVALORES COMPLEJOS"

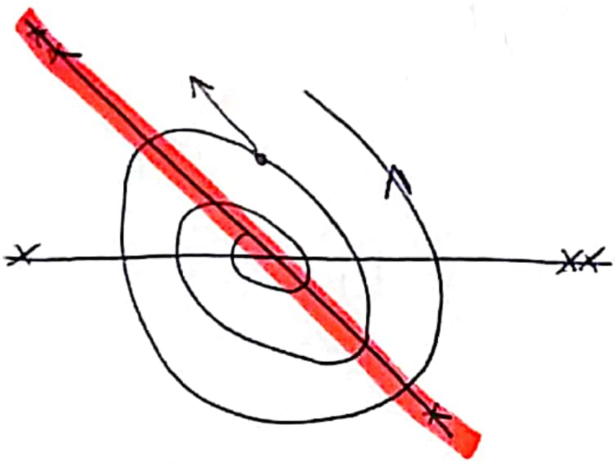
$$\Rightarrow \lambda = 1 - i, \quad v = \begin{pmatrix} -1+i \\ -i \end{pmatrix}$$

$$\Rightarrow X(t) = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c_1 e^t \cos(t) & c_2 e^t \operatorname{sen}(t) \\ -c_1 e^t \operatorname{sen}(t) & c_2 e^t \cos(t) \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} e^t \begin{pmatrix} \cos(t) & \operatorname{sen}(t) \\ -\operatorname{sen}(t) & \cos(t) \end{pmatrix} \underbrace{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}_{= r \begin{pmatrix} \cos\theta \\ \operatorname{sen}\theta \end{pmatrix}}$$

$$= \underbrace{\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}}_{=:C} \underbrace{r e^{t} \begin{pmatrix} \cos(\theta + t) \\ \sin(\theta + t) \end{pmatrix}}_{=: Y(t)} \qquad (\det C > 0)$$





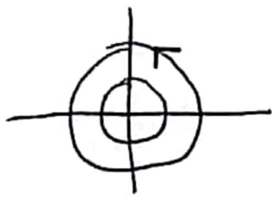
$$A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

En general, si $\lambda = \alpha + \beta i$, entonces

$$Y(t) = r\, e^{\alpha t} \begin{pmatrix} \cos(\theta - \beta t) \\ \sin(\theta - \beta t) \end{pmatrix}$$

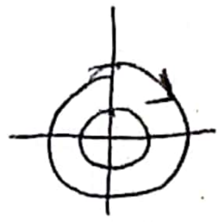
$$Y(t) \xrightarrow[t \to +\infty]{} \begin{cases} +\infty, & \alpha > 0 \\ 0, & \alpha < 0 \end{cases}, \quad Y(t) \text{ gira } \begin{cases} \text{ANTIHORARIO si } \beta < 0 \\ \text{HORARIO si } \beta > 0 \end{cases}$$

• $\alpha=0,\ \beta<0$:



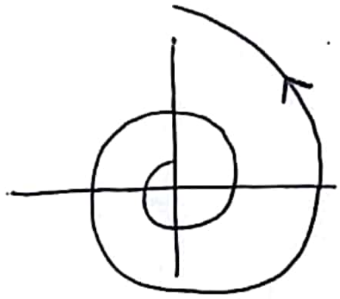
$Y(t)$

; $\alpha=0,\ \beta>0$

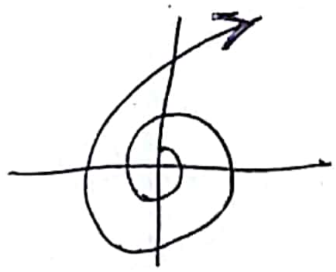


$Y(t)$

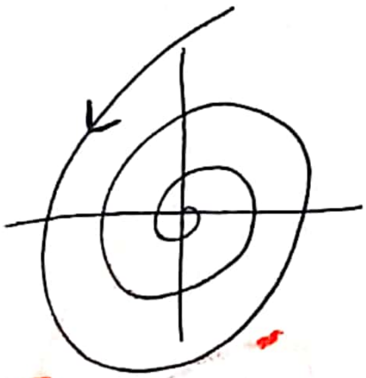
• $\alpha>0,\ \beta<0$:



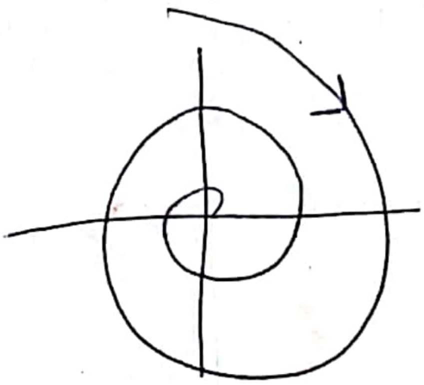
; $\alpha>0,\ \beta>0$



• $\alpha<0,\ \beta<0$:



; $\alpha<0,\ \beta>0$



Ejemplo 7: $A=\begin{pmatrix}0 & -4\\ 1 & 4\end{pmatrix}$, "NO DIAGONALIZABLE"

$$\Rightarrow \lambda=2,\quad v=\begin{pmatrix}-2\\ 1\end{pmatrix}$$

$$C^{-1}AC=\begin{pmatrix}2 & 0\\ 1 & 2\end{pmatrix},\quad C=\begin{pmatrix}1 & -2\\ 0 & 1\end{pmatrix}$$

LA SOLUCIÓN GENERAL ES

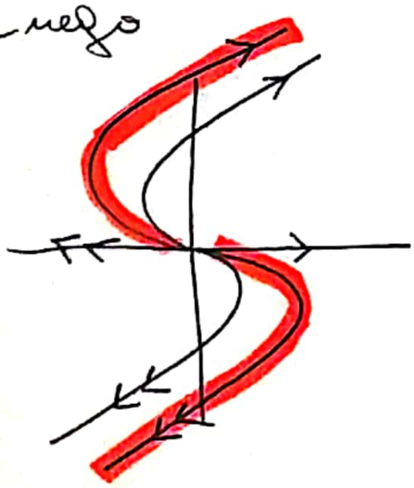
$$X(t) = c_1 e^{2t}\begin{pmatrix}-2\\1\end{pmatrix} + c_2 e^{2t}\left(t\begin{pmatrix}-2\\1\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix}\right)$$

$$= \underbrace{\begin{pmatrix}-2 & 1\\ 1 & 0\end{pmatrix}}_{\tilde{C}} \begin{pmatrix} e^{2t}(c_1 + c_2 t) \\ e^{2t} c_2 \end{pmatrix}$$

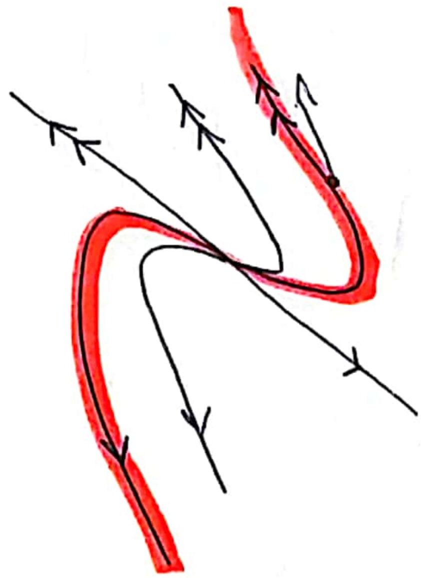
Vale que $y_1(t) = f(y_2(t))$, con

$$f(y) = y\left(\frac{c_1}{c_2} - \frac{1}{2}\ln|c_2| + \frac{1}{2}\ln|y|\right)$$

Luego



$$A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-4\\5\end{pmatrix}$$



$$X(t) \xrightarrow{\tilde{C}\,\cdot} X(t)$$

$\det \tilde{C} \neq 0 \Rightarrow$ INVERTIBLE