Continuidad:

$f: \mathbb{R} \to \mathbb{R}$ es continuo en x_0 si $\forall \varepsilon > 0 \; \exists \delta > 0$

/ si $0 < |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

Lipschitz:

$f: \mathbb{R} \to \mathbb{R}$ es lipschitz si $\exists M > 0$ /

$|f(x) - f(y)| \leq M|x - y| \quad \forall x, y \in \mathbb{R}$

Derivable:

f es derivable en x_0 si $\exists \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$.

Lip. $\Rightarrow$ continuo: $\qquad \delta \leq \varepsilon/M$

$\varepsilon > 0$ x_0, $\quad |f(x) - f(x_0)| \leq M \overline{|x - x_0|}$

$< \cancel{M} \cdot \varepsilon/\cancel{M} = \varepsilon.$

Si f es derivable y f' acotado $\Rightarrow f$ Lip.

$|f(x) - f(y)| = \underbrace{|f'(\xi)|}_{\leq M} |x - y| \leq M|x - y|.$

$f(x) = \sqrt{x}$

$f'(x) = \frac{1}{2\sqrt{x}} \qquad x \neq 0$

$f'(x) \xrightarrow[x \to 0^+]{} +\infty$

si fuera lip $\Rightarrow \exists M > 0$/

$|f(x) - f(y)| \leq M|x - y|$

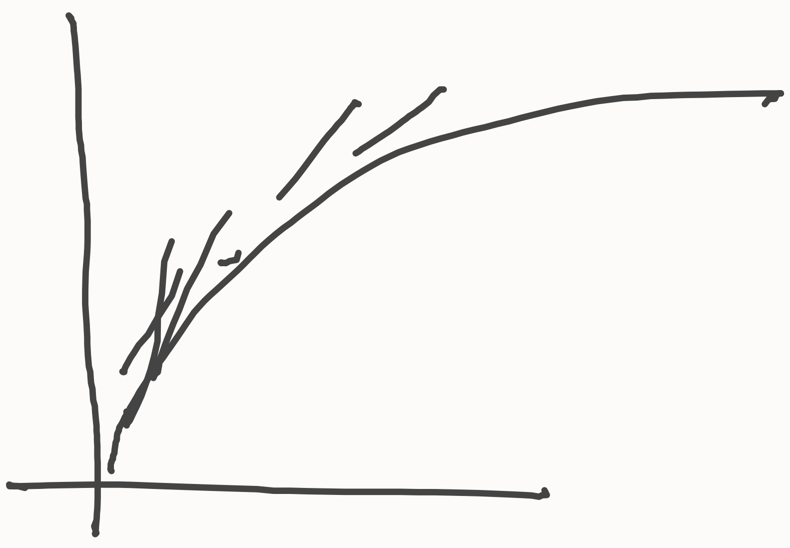
En particular

$|f(x) - \underbrace{f(0)}_{=0}| \leq M|x - 0|$

$|f(x)| = f'(\xi)\cdot\cancel{|x|} \le M\cancel{|x|} \Rightarrow f'(\xi) \le M.$

ABS!

$\xi \in (0,x)$



• $F(t,x): \underbrace{I}_{\subseteq \mathbb{R}} \times \underbrace{\Omega}_{\subseteq \mathbb{R}^n \text{ abto}} \longrightarrow \mathbb{R}^n$

F lip. en la variable x si F continua y $\exists L>0$

$/ \quad \|F(t,x) - F(t,y)\| \le L\|x-y\|$

$\forall t \in I, \ \forall x,y \in \Omega.$

• $F(t,x) = \underbrace{A(t)}_{\text{matriz}} X + b(t)$ $\quad b(t) \in \mathbb{R}^n$ con entradas cont.

$A(t) \in \mathbb{R}^{n\times n}$ con entradas cont.

Ejemplos:

i) $F(t,x) = t\cdot x \qquad I = [a,b] \subseteq \mathbb{R}.$

$\Omega = \mathbb{R}$

F es lip. en x:

• continua.

• $\exists L>0 \ / \ |F(t,x) - F(t,y)| \le L|x-y|.$

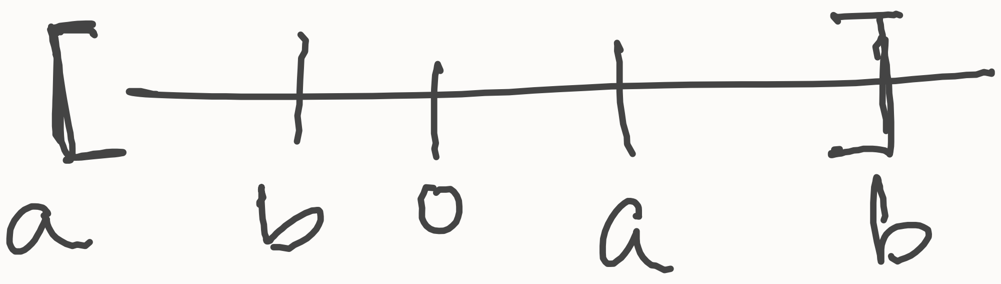
Cont:

$|F(t,x) - F(t_0,x_0)| = |tx - t_0x_0|$

$= |\underbrace{tx - tx_0} + \underbrace{tx_0 - t_0x_0}|$

$\leq |t(x-x_0)| + |x_0(t-t_0)|$

$\leq |t|\underbrace{|x-x_0|}_{\leq \|(t,x)-(t_0,x_0)\| \geq} + |x_0||t-t_0|$



$\leq (\underbrace{|t|}+|x_0|)\,\underbrace{\|(t,x)-(t_0,x_0)\|}$

$\leq (C+|x_0|)\,\delta < \varepsilon \quad$ si $\quad \delta < \dfrac{\varepsilon}{C+|x_0|}.$

↳ sólo depende de I

· Lip:

$|F(t,x)-F(t,y)| = |tx-ty| = |t|\,|x-y|$

$\leq C\,|x-y|$

↓ sólo depende de I.

· $F(t,x) = tx^2$ es localmente lip. en x pero no lipschitz.

$I = \mathbb{R} = \Omega$.

$J \subseteq \mathbb{R}$ intervalo cerrado $\wedge$ $\Omega' \subseteq \mathbb{R}$ cerrado y acotado. Tomo $t \in J$, $x, y \in \Omega'$.

$|F(t,x)-F(t,y)| = |tx^2-ty^2|$

$= |t|\,|x^2-y^2|$

$= |t| . 2|\xi|\,|x-y|$

ξ entre x e y

$x<y \Rightarrow \xi \in (x,y)$

$\leq |t| . 2M . |x-y|$

$\leq C . 2 . M . |x-y|$

$x, y \in \Omega'$ acotado $\Rightarrow |x|, |y| \leq M$

$\Rightarrow |\xi| \leq M$.

C es una const. q' depende sólo de J.

No es lip.

$$|F(t,x) - F(t,y)| = \underbrace{|t|\, 2|\xi|\, |x-y|}_{1} \leq \underbrace{L|x-y|}$$

Si existiera $L>0$ const lip.

$$\Downarrow$$

$$|t||\xi| \leq \frac{L}{2}$$

$t, \xi \in \mathbb{R}$ ABS!

<u>Ejemplo:</u> $n=2$

$$F(t,X) = A(t)X + b(t)$$

$$A(t) = (a_{ij}(t))_{i,j=1}^{2} = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \underline{\underline{b(t)}} = \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix}$$

$a_{ij}: \mathbb{R} \to \mathbb{R}$ cont.

$b_j: \mathbb{R} \to \mathbb{R}$

$$A(t) = \begin{pmatrix} t^2 & t+1 \\ 2t & t^3 \end{pmatrix} \qquad b(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Prq F es localmente lip. Sea $J \subseteq \mathbb{R}$ $\wedge$ $\Omega' \subseteq \mathbb{R}^2$ cerrado y acotado.

Tomamos $t \in J$, $x, y \in \Omega'$.

Como a_{ij} con $i,j = 1,2$ son cont. en $\mathbb{R}$ en particular $a_{ij}: J \to \mathbb{R}$ siguen siendo cont.

Como J es un intervalo cerrado y acotado cada a_{ij} $j,i=1,2$ es acotada. $\exists\, M_{11}, M_{12}, M_{22}, M_{21} > 0$ /

$|a_{11}(t)| \leq M_{11} \quad \forall t \in J$

$|a_{12}(t)| \leq M_{12} \quad \forall t \in J$

$|a_{21}(t)| \leq M_{21} \quad \forall t \in J$

$|a_{22}(t)| \leq M_{22} \quad \forall t \in J.$

$\Rightarrow M = \max\{M_{11}, M_{12}, M_{21}, M_{22}\}$

$\Rightarrow |a_{ij}(t)| \leq M \quad \forall t \in J, \; ij = 1,2.$

$$\|F(t,x) - F(t,y)\| = \|A(t)(x-y)\|$$

$$= \left\| \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix} \begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \end{pmatrix} \right\|$$

$$= \| (a_{11}(t)(x_1-y_1) + a_{12}(t)(x_2-y_2),$$

$$a_{21}(t)(x_1-y_1) + a_{22}(t)(x_2-y_2)) \|$$

$(\langle (a_{11}, a_{12}), (x_1-y_1, x_2-y_2) \rangle)^2 \leq \|(a_{11}, a_{12})\|_2^2 \, \|(x_1-y_1, x_2-y_2)\|^2$

$$= \left(\left[(a_{11}(t)(x_1-y_1) + a_{12}(t)(x_2-y_2))^2 + (a_{21}(t)(x_1-y_1) + a_{22}(t)(x_2-y_2))^2 \right] \right)^{1/2}$$

$$\underset{C\text{-}S \; (*)}{\leq} \left[(|a_{11}(t)|^2 + |a_{12}(t)|^2)(|x_1-y_1|^2 + |x_2-y_2|^2) + (|a_{21}(t)|^2 + |a_{22}(t)|^2)(|x_1-y_1|^2 + |y_2-x_2|^2) \right]^{1/2}$$

$$= \Big[(|x_1-y_1|^2 + |x_2-y_2|^2)\,(\underbrace{|a_{11}(t)|^2}_{\leq \Pi^2} + \underbrace{|a_{12}(t)|^2}_{\leq \Pi^2} + \underbrace{|a_{21}(t)|^2}_{\leq \Pi^2} + \underbrace{|a_{22}(t)|^2}_{\leq \Pi^2}) \Big]^{1/2}$$

$$\leq \left[\|x-y\|^2 \cdot 4\Pi^2\right]^{1/2} = \|x-y\| \cdot \underbrace{2\,\Pi}_{L}.$$

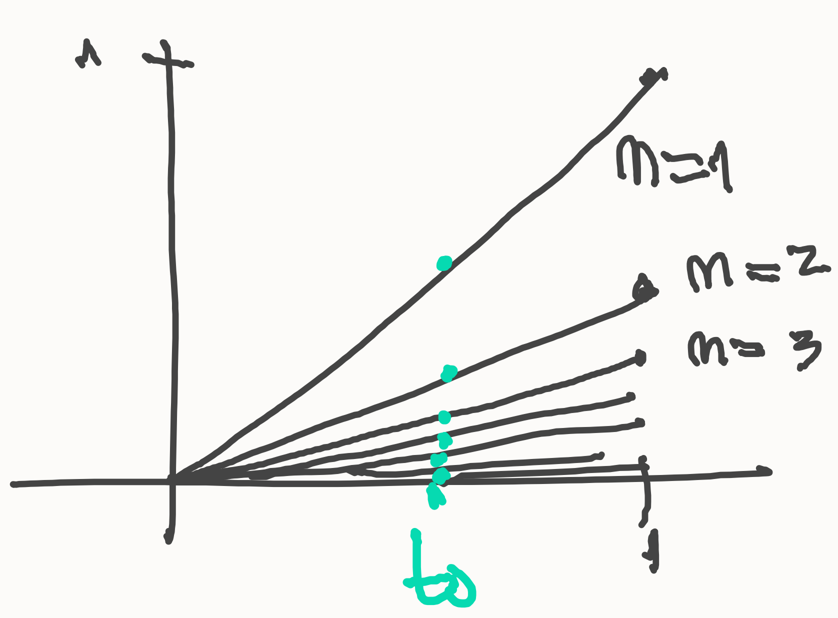
$n=2 \longrightarrow n=$ general.

$4 \rightsquigarrow n^2$

- $f_n(t)$ funciones de $\mathbb{R} \longrightarrow \mathbb{R}$.

$\{f_n(t)\}_{n\in\mathbb{N}}$ es una suc. de funciones.

Ejemplo $f_n(t) = \frac{t}{n}$. $t \in [0,1)$.



fijo t_0 $\{f_n(t_0)\}_{n\in\mathbb{N}}$ suc. en $\mathbb{R}$.

Si $(x_n)_n \subseteq \mathbb{R}$ decimos que es de Cauchy si $\forall \varepsilon > 0\ \exists\ n_0 \in \mathbb{N}$ / si $n, m \geq n_0 \Rightarrow |x_n - x_m| < \varepsilon$

$f_n(t): \mathbb{R} \longrightarrow \mathbb{R}$

$\{f_n\}_n$ es uniformemente de Cauchy si

$\forall \varepsilon > 0\ \exists\ n_0 = n_0(\varepsilon)$ / si $n, m \geq n_0$

$|f_n(t) - f_m(t)| < \varepsilon \quad \forall t \in \mathbb{R}.$

Ej $f_n(t) = \frac{t}{n}$ $t \in [0,1]$. ¿es unif. de Cauchy?

$\varepsilon > 0$

$|f_n(t) - f_m(t)| = |\frac{t}{n} - \frac{t}{m}| = |t|\,|\frac{1}{n} - \frac{1}{m}|$

$\leq |\frac{1}{n} - \frac{1}{m}| \leq \frac{1}{n} + \frac{1}{m} \leq \frac{2}{n} \leq \frac{2}{n_0} < \varepsilon$

$m \geq n \geq n_0$

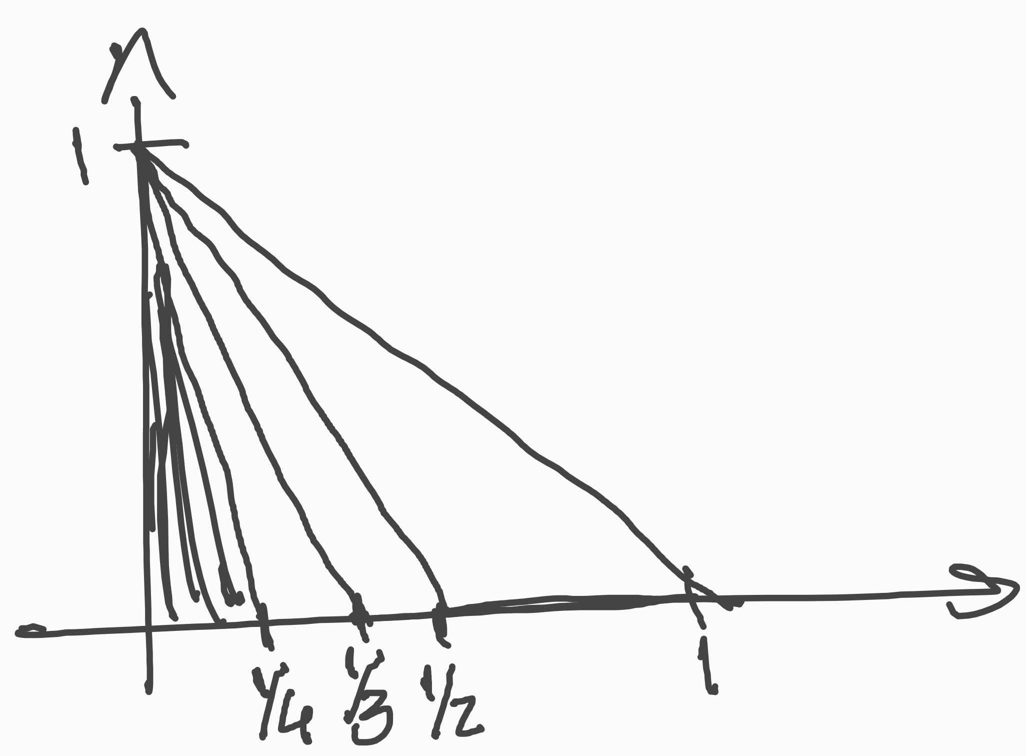
$\forall t \in [0,1]$

tomando $n_0 > 2/\varepsilon$ Ok ✓ es U de C.

Ej: $f_n : [0,1] \to \mathbb{R}$ $f_n(t) = \begin{cases} (1 - tn) & t \in [0, 1/n] \\ 0 & t \in [1/n, 1] \end{cases}$

Afirmo: no es U. de Cauchy.

Dem:



qvq $\exists \varepsilon > 0 \,/\, \forall n_0 \in \mathbb{N}$

$\exists t_{n_0} \in [0,1]$ y $n, m \geq n_0$

$/ \, |f_n(t_{n_0}) - f_m(t_{n_0})| \geq \varepsilon.$

$\varepsilon = \frac{1}{2}$ $t_{n_0} = \frac{1}{2n_0} \in [0, 1/n_0]$

$t_{n_0} = \frac{1}{2n_0} \in [0, \underbrace{1/2n_0}_{1/m}]$

$n = n_0 \quad m = 2n_0.$

$|f_n(t_{n_0}) - f_m(t_{n_0})| = |\cancel{1} - t_{n_0} n_0 - (\cancel{1} - t_{n_0} m)|$

$= \left|-\frac{1}{2\cancel{n_0}} \cdot \cancel{n_0} + \frac{1}{\cancel{2n_0}} \cancel{2n_0}\right| = \frac{1}{2}$

$P(x,y)\,dx + Q(x,y)\,dy = 0$

$\hookrightarrow P(x,y) + Q(x,y)\,y'(x) = 0 \quad (*)$

$\to Q_x - P_y = 0 \quad$ y $\quad F = (P,Q)$ es $\mathcal{C}^1$.

$\Rightarrow \exists\, \varphi \,/\, \nabla\varphi = F$.

$$\underbrace{\langle F(x,y(x)),\,(1,y'(x))\rangle}_{(*)} = 0$$

$$P(x,y(x))\cdot 1 + Q(x,y(x))\cdot y'(x) = 0$$

$$\to \underbrace{\langle \nabla\varphi(x,y(x)),\,(1,y'(x))\rangle}_{} = 0 \quad \forall x.$$

$$\frac{\partial\,\boxed{\varphi(x,y(x))}}{\partial x} = 0$$

$\varphi \circ \sigma(x)$

$\sigma(x) = (x, y(x))$

$\Rightarrow \varphi(x,y(x))$ es constante.

$\Rightarrow (x,y(x))$ vive en las curvas de nivel de φ $\{\varphi(x,y) = C\}$.