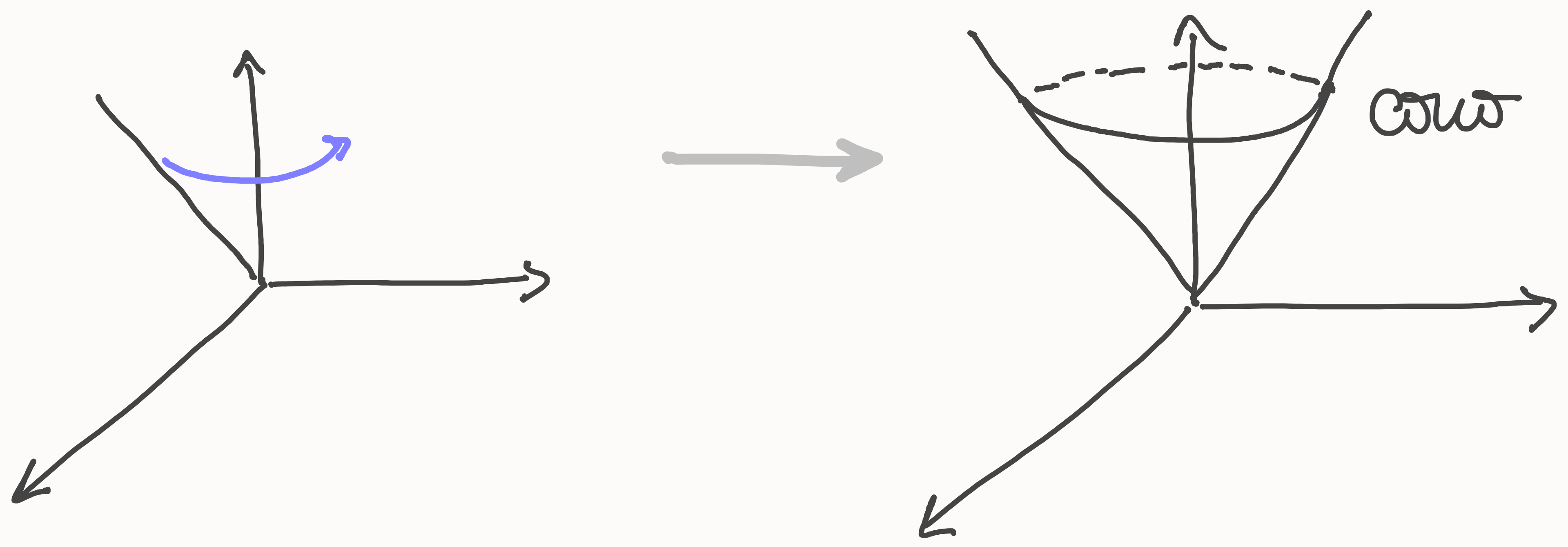
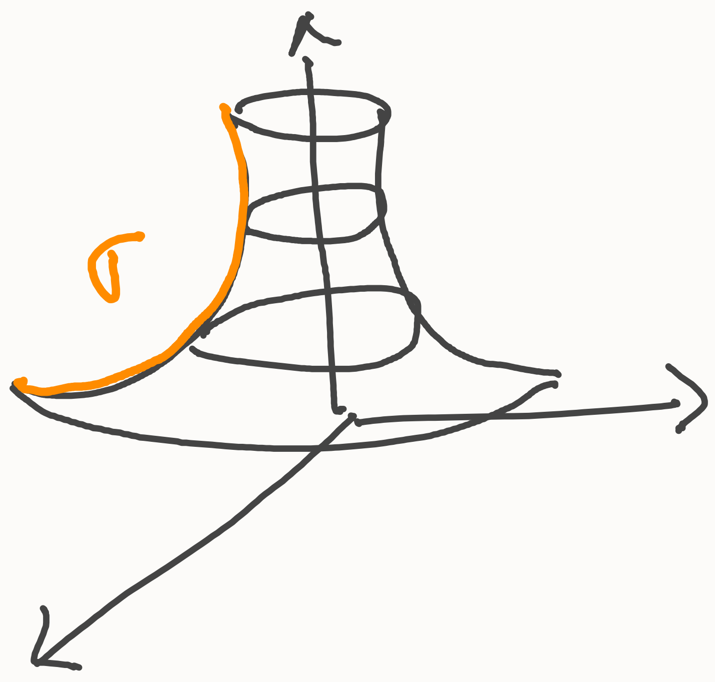
Superficies de revolución:

$\boxed{\text{Idea}}$: una curva que gira alrededor de un eje



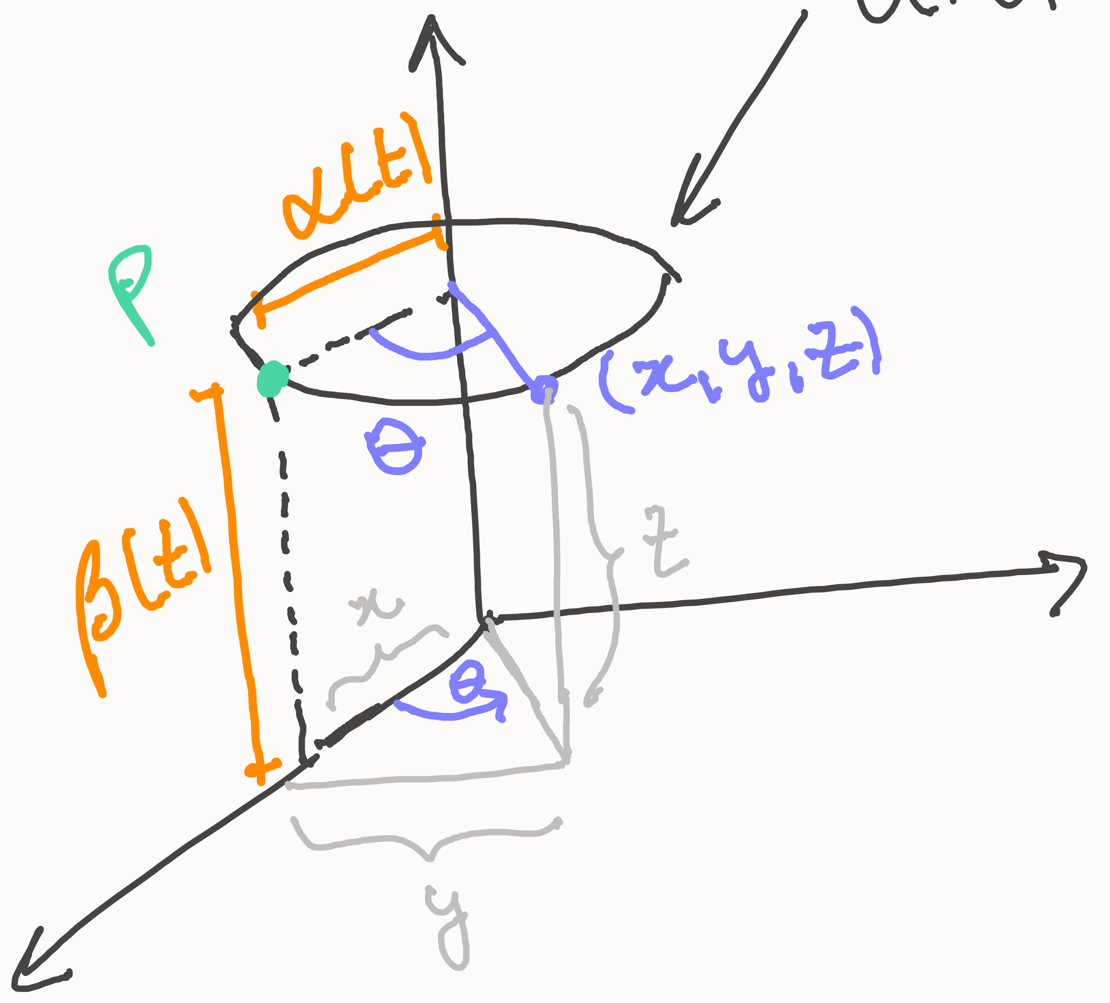
. Queremos parametrizar superficies de revolución.

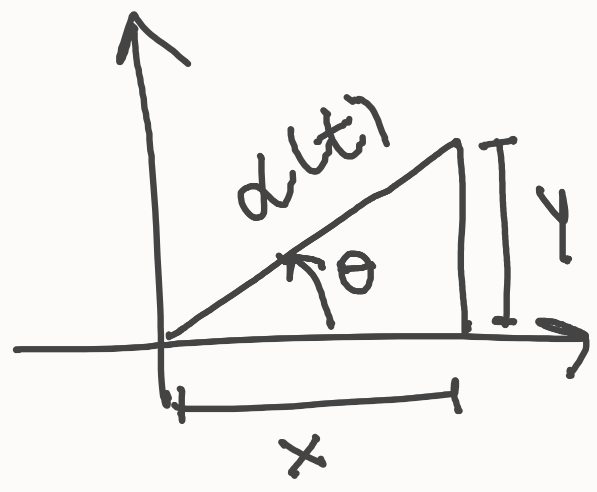
→ Tenemos una curva $\sigma(t) = (\alpha(t), \beta(t))$ en el plano xz y la hacemos girar alrededor del eje z. (como en el ejemplo del cono).



Si $p = (\alpha(t), 0, \beta(t))$ y roto un ángulo θ obtengo un punto (x, y, z):

circunf. de radio $\alpha(t)$





$$\cos\theta = \frac{x}{\alpha(t)}$$

$$\operatorname{sen}\theta = \frac{y}{\alpha(t)}$$

$$T \begin{cases} x = \alpha(t)\cos(\theta) \\ y = \alpha(t)\operatorname{sen}(\theta) \\ z = \beta(t) \end{cases}$$

$t \in [a,b] = \operatorname{dom}(\sigma)$

$\theta \in [0, 2\pi]$.

. Supongamos que σ es regular $\Rightarrow$

1) $T(t,\theta)$ es inyectiva en $[a,b]\times[0,2\pi)$

2) T es $\mathcal{C}^1$.

Calculemos $T_t \times T_\theta$:

$$T_t = (\alpha'(t)\cos\theta,\ \alpha'(t)\sin\theta,\ \beta'(t))$$

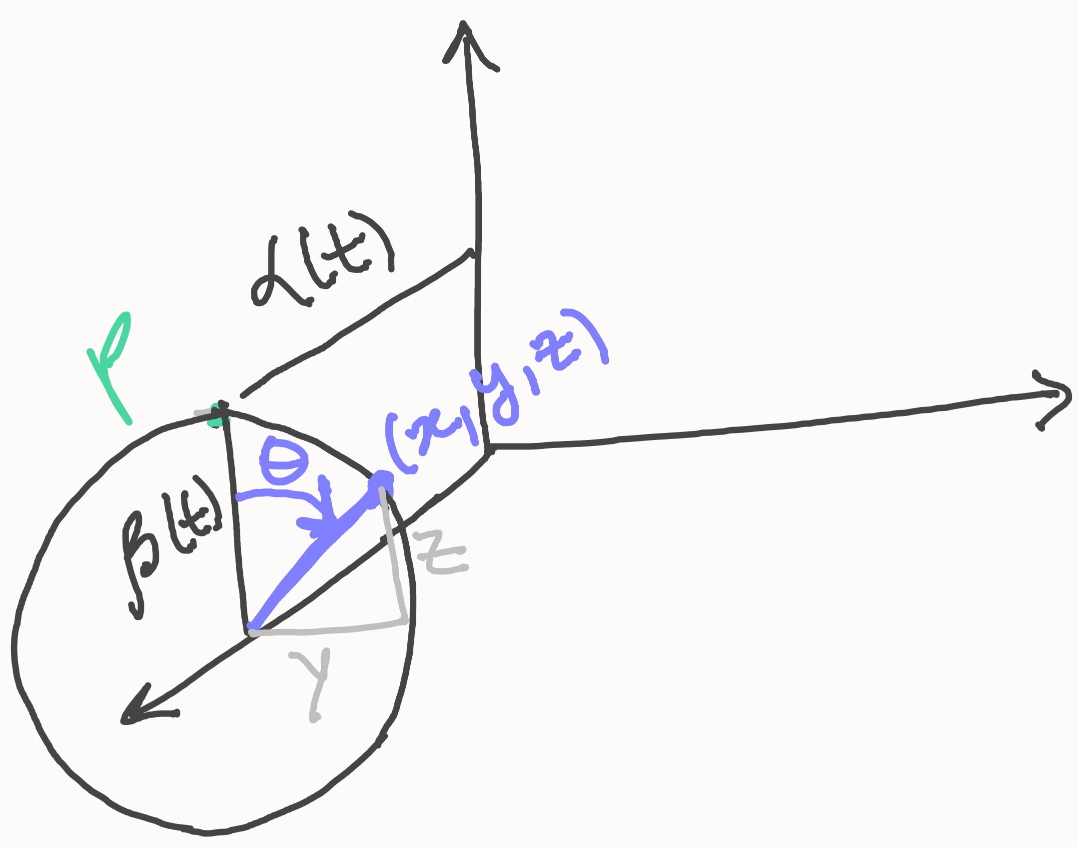
$$T_\theta = (-\alpha(t)\sin\theta,\ \alpha(t)\cos\theta,\ 0)$$

$$\Rightarrow T_t \times T_\theta = (-\alpha(t)\beta'(t)\cos\theta,\ -\alpha(t)\beta'(t)\sin\theta,\ \alpha(t)\alpha'(t))$$

y $\|T_t \times T_\theta\| = \sqrt{(\alpha(t)\beta'(t))^2 + (\alpha(t)\alpha'(t))^2} = |\alpha(t)|\,\|\sigma'(t)\|.$

$\Rightarrow \|T_t \times T_\theta\| \neq 0 \Leftrightarrow \alpha(t) \neq 0\ \forall t \in [a,b].$

→ ¿Si ahora giramos alrededor del eje x?



$$\underbrace{\cos\left(\tfrac{\pi}{2}-\theta\right)}^{\sin(\theta)} = \frac{y}{\beta(t)}$$

$$\underbrace{\sin\left(\tfrac{\pi}{2}-\theta\right)}_{\cos\theta} = \frac{z}{\beta(t)}$$

$$T: \begin{cases} x = \alpha(t) \\ y = \beta(t)\sin\theta \\ z = \beta(t)\cos\theta \end{cases}$$

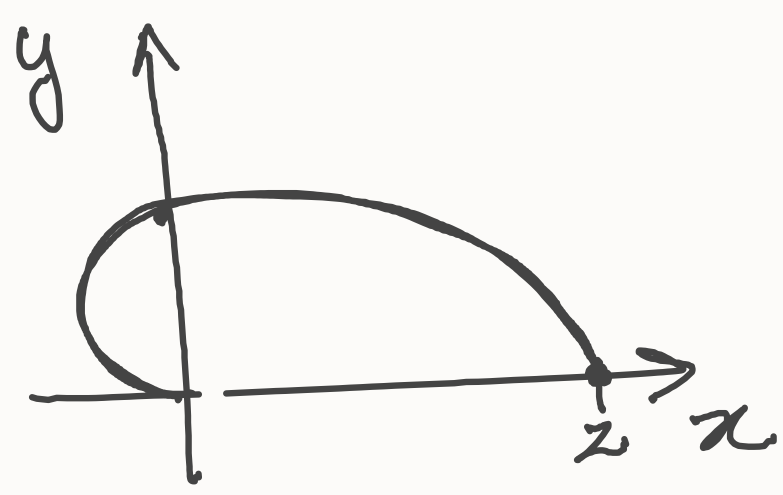
→ Se gira la curva $f(x) = z$, $x \in [a,b]$

al rededor del eje z.

$\Rightarrow \sigma(t) = (t, f(t))\quad t \in [a,b]$

y así $\quad T(t,\theta) = \big(t\cos(\theta), t\,\mathrm{sen}(\theta), f(t)\big)$.

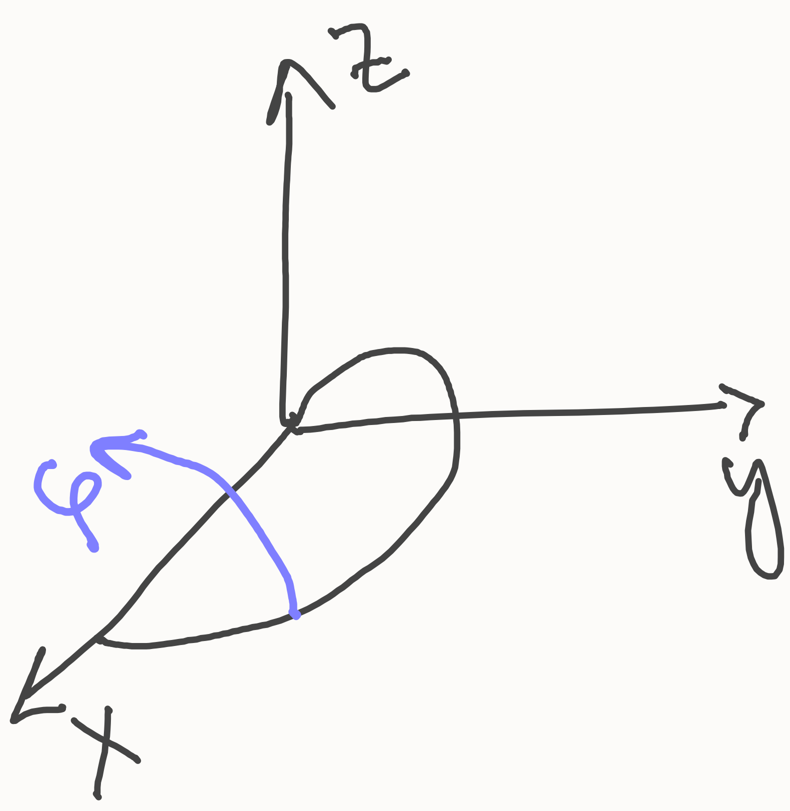
Ejemplo: Sea $\mathcal{C}$ el cardioide : $r = 1 + \cos\theta$
$\theta \in [0,\pi]$.



1) Parametrizamos en plano xy

$$\sigma(\theta): \begin{cases} x = r\cos\theta = (1+\cos\theta)\cos\theta = \alpha(\theta) \\ y = r\,\mathrm{sen}\,\theta = (1+\cos\theta)\,\mathrm{sen}\,\theta = \beta(\theta) \\ \theta \in [0,\pi]. \end{cases}$$

2) Parametrizamos el giro:



$$\begin{cases} x = \alpha(\theta) \\ y = \beta(\theta)\cos\varphi \\ z = \beta(\theta)\,\mathrm{sen}\,\varphi \end{cases} \qquad \varphi \in [0,2\pi].$$

——— x ——— x ——— x ———

Flujo:

- $S \subseteq \mathbb{R}^3$ superficie
- $T: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ parametrización regular de S que orienta S.
- $F(x,y,z)$ un campo vectorial continuo def. en S.

El flujo de F a través de S es

$$\iint_S F \cdot dS = \iint_S \langle F, \eta \rangle \, dS.$$

Para calcularlo:

$$\boxed{\iint_S F.dS = \iint_D \langle F(T(u,v)), T_u \times T_v(u,v) \rangle \, du\,dv}$$

Ejemplo 1: Sea $F(x,y,z) = (0,0,4-x^2-y^2)$ y S la sup dada por:

$$S := \begin{cases} x^2+y^2 \leq 4 \\ y+z=1 \end{cases}$$

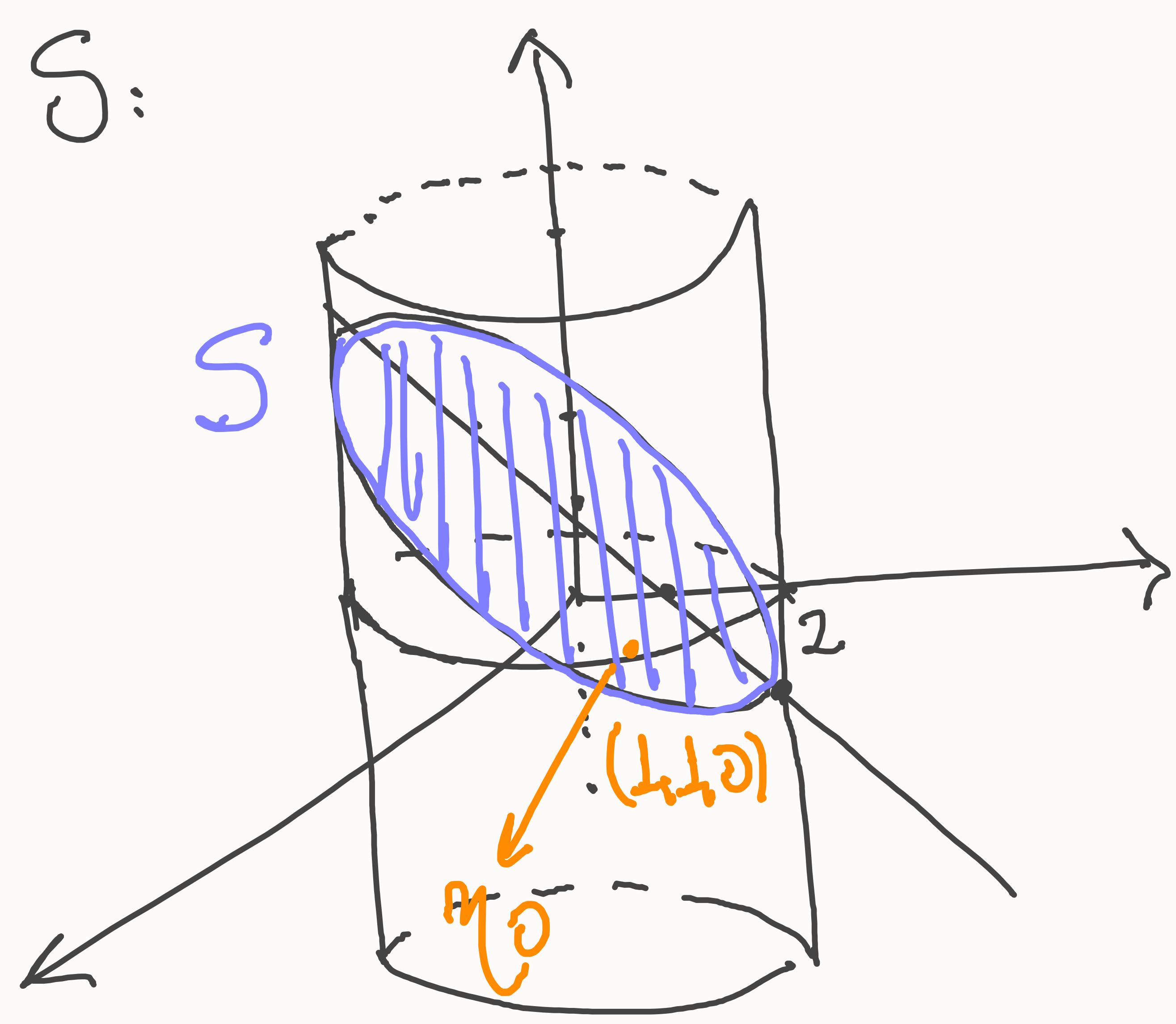
orientada de manera tal que en $p_0 = (1,1,0) \in S$ la normal sea $n_0 = \left(0, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

Hallar $\iint_S F.dS$.

Solución: Graficamos S:

$x^2+y^2 \leq 4 \to$ cilindro

$y+z=1 \to$ plano



Parametrizamos S (usamos polares)

$$T \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = 1 - r\sin\theta \end{cases} \qquad \theta \in [0, 2\pi] \qquad r \in [0,2]$$

. T inyectiva en $[0,2] \times [0, 2\pi)$

. T es C^1.

$T_r = (\cos\theta, \sin\theta, -\sin\theta)$

$T_\theta = (-r\sin\theta, r\cos\theta, -r\cos\theta)$

$T_r \times T_\theta = (0, +r, r) \to$ T invierte la orientación!

$p_0 = T\left(\frac{2}{\sqrt{2}}, \frac{\pi}{4}\right) = (1,1,0)$

$$\Rightarrow \iint_S F\cdot dS = -\int_0^{2\pi}\int_0^2 \langle F(r\cos\theta, r\sin\theta, 1-r\sin\theta), (0, r, r)\rangle dr\, d\theta$$

$$= -\int_0^{2\pi}\int_0^2 (4-r^2)\cdot r\, dr\, d\theta$$

$$= -2\pi \int_0^2 4r - r^3\, dr = -2\pi \left[2r^2 - \frac{r^4}{4} \Big|_0^2 \right]$$

$$= -2\pi [8-4] = \boxed{-8\pi}$$

$\boxtimes$

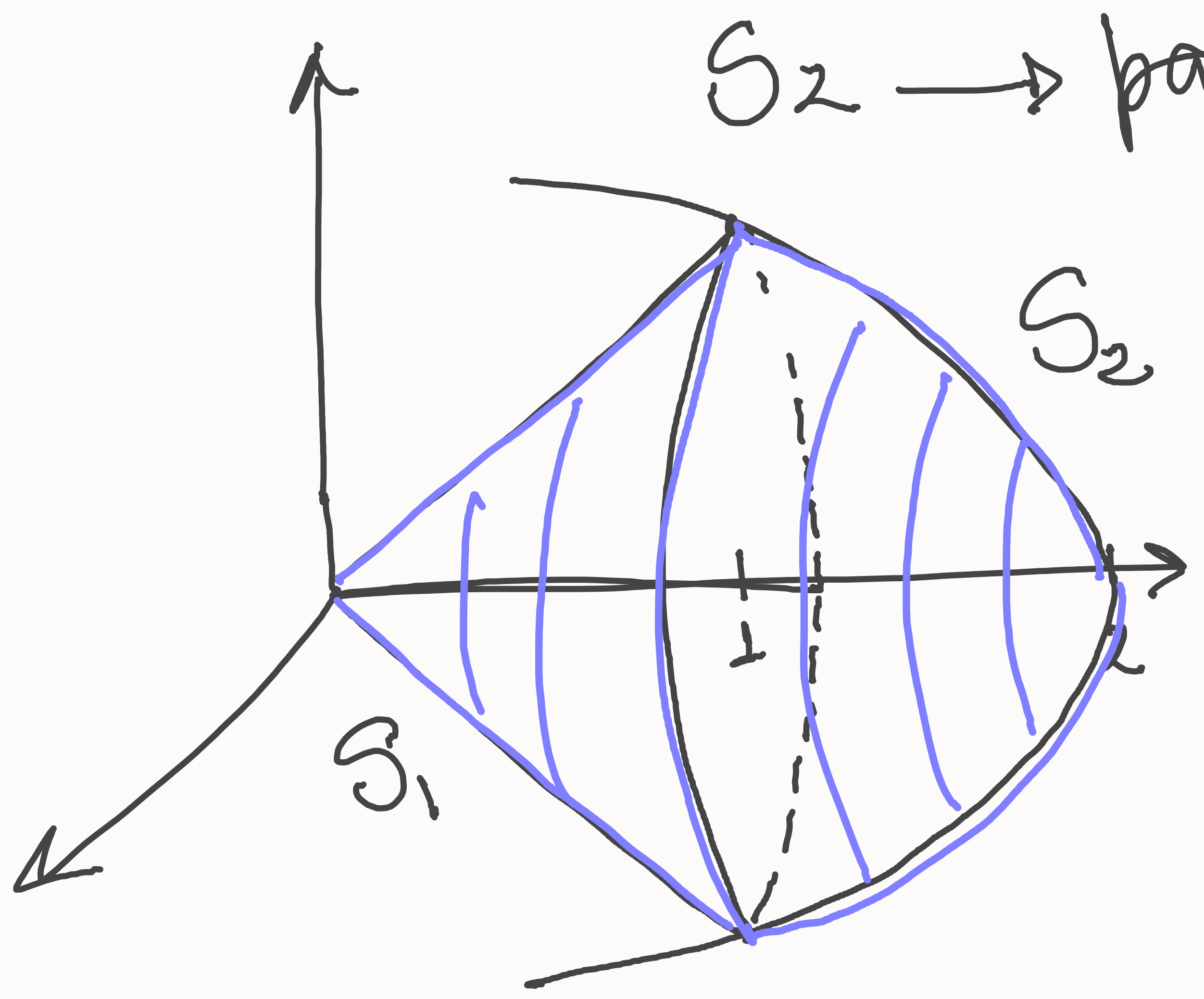
Ejemplo 2: Sea $S_1: \begin{cases} y^2 = z^2 + x^2 \\ 0 \le y \le 1 \end{cases}$ y $S_2: \begin{cases} y = 2 - x^2 - z^2 \\ y \ge 1 \end{cases}$

Consideremos $S = S_1 \cup S_2$.

Dado $F(x,y,z) = (x,y,z)$, calcular el flujo saliente a través de S.

Solución: $S_1 \longrightarrow$ cono en eje y

$S_2 \longrightarrow$ paraboloide en eje y.



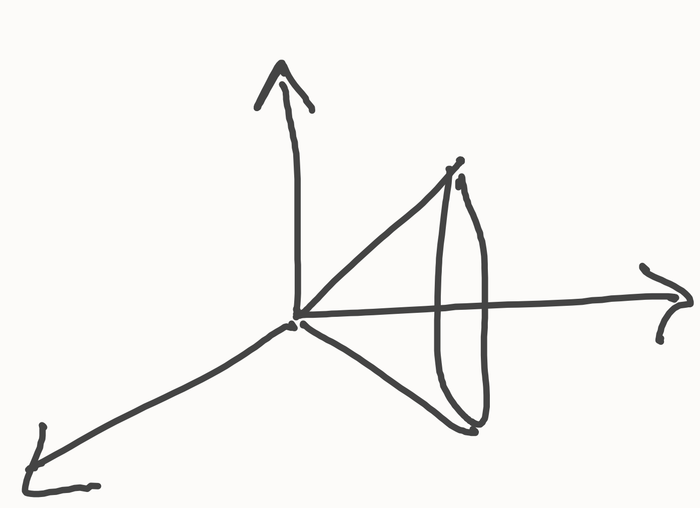
Flujo saliente

$\Downarrow$

S con normal exterior.

$$\iint_S F\cdot dS = \iint_{S_1} F\, dS + \iint_{S_2} F\, dS$$ con S_1 y S_2 orientados c/ normal ext.

Parametrizamos S_1:



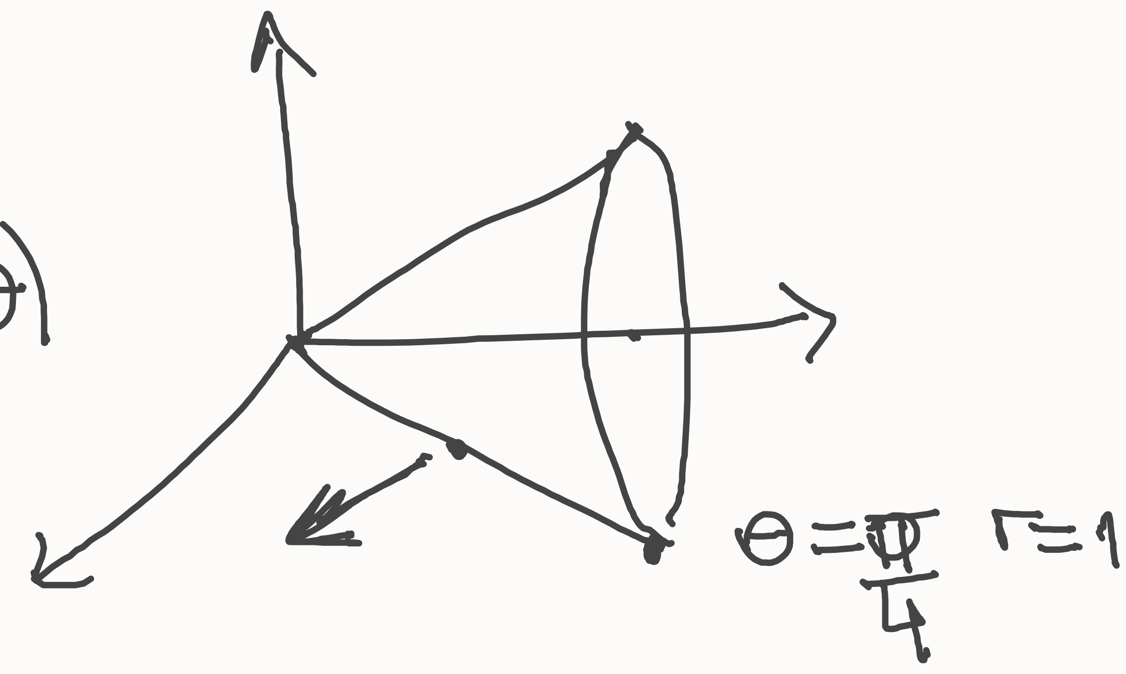
$$T(r,\theta) = (r\cos\theta,\ r,\ r\sin\theta)$$

$$T_r = (\cos\theta,\ 1,\ \sin\theta)$$

$$T_\theta = (-r\sin\theta,\ 0,\ r\cos\theta)$$

$$T_r \times T_\theta = (r\cos\theta,\ -r,\ r\sin\theta)$$

orientación correcta!



$\theta = \frac{\pi}{4}$ $r=1$

$$T_r \times T_\theta(1, \pi/4) = \left(\frac{\sqrt{2}}{2}, -1, \frac{\sqrt{2}}{2}\right)$$

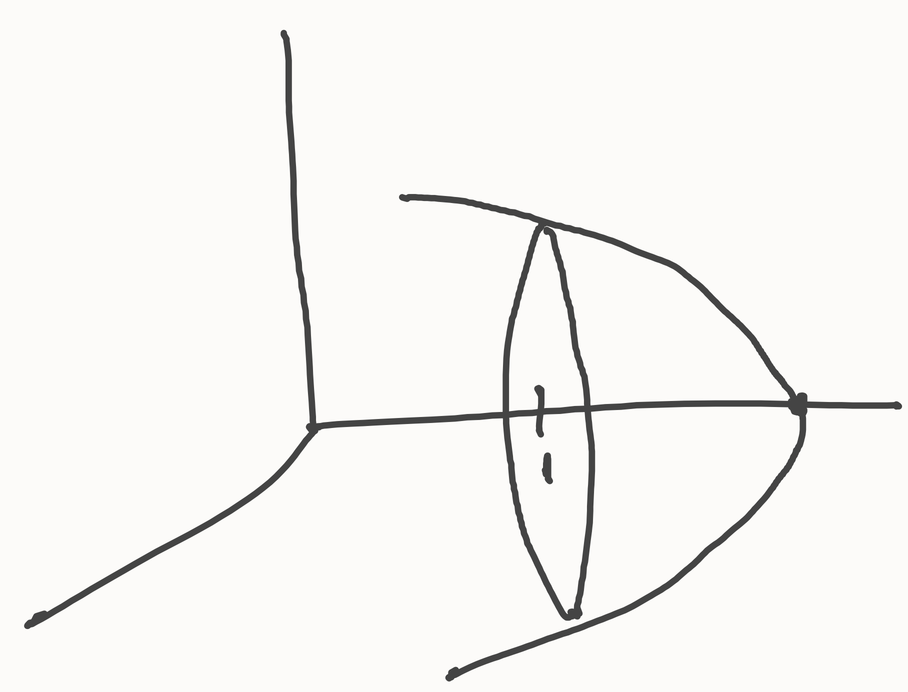
$$\iint_{S_1} F\,dS = \int_0^{2\pi}\int_0^1 \langle F(T(r,\theta)), T_r\times T_\theta\rangle\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \langle (r\cos\theta,\ r,\ r\sin\theta)(r\cos\theta, -r, r\sin\theta)\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^2 - r^2\, dr\, d\theta = 0.$$

Parametrizamos S_2:

(usamos gráfico)



$$T(x,z) = (x,\ 2-x^2-z^2,\ z)$$

D: $x^2+z^2 \leq 1$

$$T_x = (1, -2x, 0) \Rightarrow T_x \times T_z = (-2x, -1, -2z)$$

$$T_z = (0, -2z, 1)$$

$$T(0,0) = (0,2,0)$$

$$T_x \times T_z(0,0) = (0,-1,0)$$

apunta hacia adentro!

$$\iint_{S_2} F\cdot dS = -\iint_D \langle (x, 2-x^2-z^2, z), (-2x, -1, -2z)\rangle\, dx\,dz$$

$$= -\iint_D -2x^2 - 2 + x^2 + z^2 - 2z^2\, dx\,dz$$

$$= -\iint_D -2 - x^2 - z^2 \, dx\, dz$$

$$= \int_0^{2\pi} \int_0^1 (2+r^2)\, r \, dr\, d\theta$$

polares

$x = r\cos\theta$

$y = r\sin\theta$

$$= 2\pi \cdot \left(r^2 + \frac{r^4}{4} \right) \Big|_0^1$$

$$= \frac{5}{2}\pi.$$

$$\Longrightarrow \boxed{\iint_S F \cdot dS = \frac{5}{2}\pi}$$

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