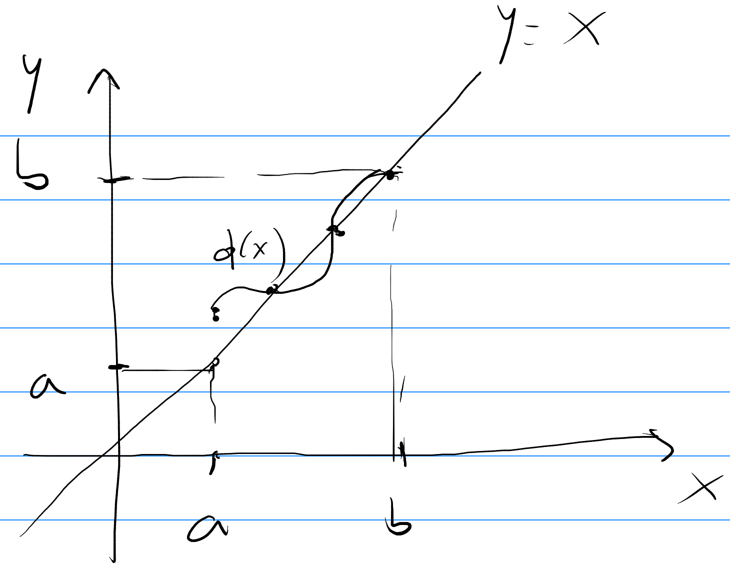
$\phi : [a,b] \longrightarrow [a,b]$

$\phi$ continua

$\Downarrow$

Existe pto fijo




$$|\phi(x) - \phi(\tilde{x})| \leq \gamma |x - \tilde{x}| \quad 0 < \gamma < 1$$

Si $x_0 \in [a,b]$ $\quad x_1 = \phi(x_0) \quad x_2 = \phi(x_1), \; \_\_ \; \_\_$

$$|x_2 - x_1| = |\phi(x_1) - \phi(x_0)| \leq \gamma |x_1 - x_0|$$

$$|x_3 - x_2| \leq \gamma |x_2 - x_1| \leq \gamma^2 |x_1 - x_0|$$

$$|x_{n+1} - x_n| \leq \gamma^n |x_1 - x_0|$$

Af: $\{x_n\}$ suc. de Cauchy : $\quad |x_n - x_m| < \varepsilon \quad$ si $n, m \geq n_0(\varepsilon)$

$$\underset{1000}{\downarrow} \qquad \underset{200 \times 10^{12}}{\downarrow}$$

$n \leq m$

$$|x_n - x_m| = \sum_{j=0}^{m-n-1} |x_{n+j} - x_{n+j+1}| \leq \sum_{j=0}^{m-n+1} \gamma^{j+n} |x_1 - x_0| =$$

$$= \gamma^n |x_1 - x_0| \sum_{j=0}^{m-n-1} \gamma^j \leq \frac{\gamma^n}{1-\gamma} |x_1 - x_0| \qquad \gamma^n \to 0 \quad n \to \infty$$

$$\sum_{j=0}^{m-n-1} \gamma^j \leq \frac{1}{1-\gamma} \qquad \frac{1}{1-\gamma} \text{ fijo}$$

Suc de Cauchy ($\mathbb{R}$ completo) $\Rightarrow x_n \to x_*$ pto fijo

$x_* = \phi(x_*)$

$x_{**} = \phi(x_{**})$

$$|x_* - x_{**}| \leq \gamma |x_* - x_{**}| \Rightarrow |x_* - x_{**}| = 0 \qquad \gamma < 1$$

Usamos $\phi$ Lipschitz continua

$\mathbb{R}$ completo

Se extiende a otras situaciones

por ej $\phi: \mathbb{R}^d \to \mathbb{R}^d$ $\|\phi(x) - \phi(y)\| \leq \gamma \|x - y\|$, $0 < \gamma < 1$

$$C([0,1]) = \{ f : [0,1] \to \mathbb{R} \text{ continua} \}$$

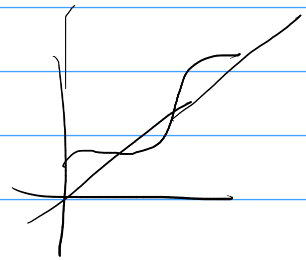
$$\|f-g\| \leq \max_{0 \leq x \leq 1} |f(x) - g(x)|$$

$$y = \phi(x) = x(0) + \int_0^t F(t', x(t'))\, dt' \qquad \phi(x) \in C([0,1])$$

$x(t)$ función continua

$$\begin{cases} y'(t) = F(t, x(t)) \\ y(0) = x(0) \end{cases}$$

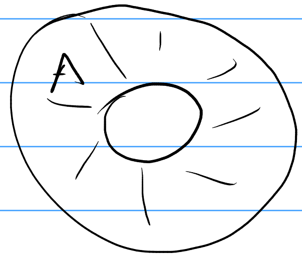
Si $y = x \Rightarrow \begin{cases} x'(t) = F(t, x(t)) \\ x(0) = x_0 \ \checkmark \end{cases}$



<u>Obs:</u> $\phi : [a,b] \to [a,b]$ continua $\Rightarrow$ tiene por lo menos un pto fijo

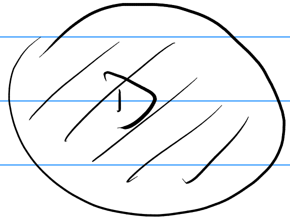
¿Vale para $d > 1$?

No $\quad A = \{(x,y) : 1 \le x^2 + y^2 \le 2\}$




$\phi : A \to A \quad \phi$ es giro de $90°$

No existe $x^* \in A \;/\; \phi(x^*) = x^*$




$x^* = 0$ pto fijo

Teorema Brouwer

$D \subseteq \mathbb{R}^d$ cerrado, acotado y convexo

$\phi : D \to D \Rightarrow$ existe por lo menos un pto fijo

continua $\qquad \phi(x^*) = x^*$

$\phi_\lambda(x) = \lambda x(1-x)$ $\quad$ $\phi_\lambda : [0,1] \to [0,1]$ $\quad$ $\lambda \in (0,4)$

$\phi_\lambda(0) = 0$

$$e_n = -\frac{f''(\xi_n)}{2f[x_{n-1},x_{n-2}]}\, e_{n-1} e_{n-2}$$

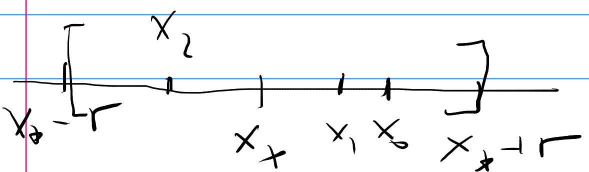
$$|e_n| = \frac{|f''(\xi_n)|}{2|f[x_{n-1},x_{n-2}]|}\, |e_{n-1}||e_{n-2}| \le \frac{C_2}{2C_1} |e_{n-1}||e_{n-2}|$$

$|f''(\xi_n)| \le C_2$

$|f[\ ,\ ]| = |f'(\eta)| \ge C_1$

$< 1$

$$|e_0|, |e_1| < r \Rightarrow |e_2| \le \frac{C_2}{2C_1} r^2 = \frac{C_2 r}{2C_1}\, r < r$$




Inductivamente

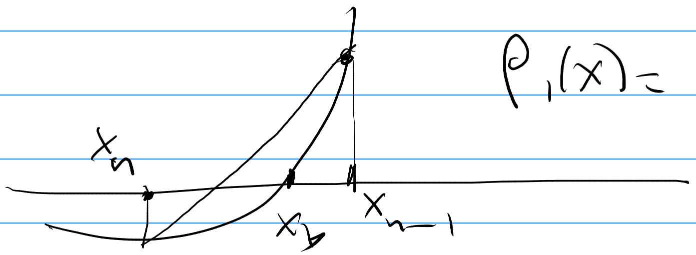
$$|e_3| < \frac{C_2 r}{2C_1}\, r < r$$

$$|e_n| \le \overbrace{\frac{C_2 r}{2C_1}}^{\gamma} |e_{n-1}| \le \gamma\, |e_{n-1}| \qquad \gamma < 1$$

$$|e_n| \le \gamma^{n-1} |e_1| \longrightarrow 0 \quad \Rightarrow \quad x_n \longrightarrow x_*$$

Obtención de la formula de error

$$0 = f(x_*) = \underbrace{p_1(x_*)}_{\text{polinomio lineal interpolador}} + R(x_*)$$




$$p_1(x) = \underbrace{f(x_{n-1})}_{a} + \underbrace{\frac{f(x_{n-1}) - f(x_{n-2})}{x_{n-1} - x_{n-2}}}_{m} (x - x_{n-1})$$

$R(x)$ error de interpolación $\quad f(x) - p_1(x)$

$$R(x_*) = \frac{f''(\xi)}{2} W(x_*) = \frac{f''(\xi)}{2} \underbrace{(x_* - x_n)}_{e_n} \underbrace{(x_* - x_{n-1})}_{e_{n-1}}$$

$x_n$ por def metodo secante

$$\cancel{\alpha} + m(x_* - x_{\cancel{n-1}})$$

$$\cancel{\alpha} + m(x_n - \cancel{x_{n-1}})$$

$$\underbrace{m(x_* - x_n)}_{e_n}$$

$$p_1(x\ ) = 0$$

$$0 = p_1(x_*) - p_1(x_n) + R(x^*)$$

$$0 = f[x_{n-1}, x_{n-2}]\underbrace{(x_* - x_n)}_{e_n} + \underbrace{R(x^*)}_{\frac{f''(\xi)}{2} e_n e_{n-1}}$$

$$e_n = -\frac{f''(\xi_n)}{2f[x_{n-1}, x_{n-2}]} e_n e_{n-1}$$

$f''(\xi_n) \to f''(x_*)$

$2f[x_{n-1}, x_{n-2}] \to f'(x_*)$

$$e_n \to 0$$

$$x_n \to x_*$$

$$\xi_n \to x_*$$

$$f[x_{n-1}, x_{n-2}] \to$$

$$e_n \cong -\frac{f''(x_*)}{2f'(x_*)} e_{n-2} e_{n-1} \Rightarrow \frac{|e_n|}{|e_{n-1}|^{\varphi}} \to \left(\frac{|f''(x_*)|}{2|f'(x_*)|}\right)^{\varphi - 1}$$

$$\varphi = \frac{1+\sqrt{5}}{2}$$

$$|e_n| = C\,|e_{n-1}|\,|e_{n-1}| \overset{\Delta f}{\Rightarrow} |e_n| = C^{\varphi-1}|e_{n-1}|^{\varphi}$$

Suponemos

$$\boxed{|e_n| = K\,|e_{n-1}|^{r}}$$

Asumen esto

$$|e_n| = K\,|e_{n-1}|^{r} = K\left(K\,|e_{n-2}|^{r}\right)^{r} = K^{1+r}\,|e_{n-2}|^{r^2}$$

$$|e_{n-1}| = K\,|e_{n-2}|^{r} \Rightarrow |e_n| = K\left(K\,|e_{n-2}|^{r}\right)^{r}$$

$$= K^{1+r}\,|e_{n-2}|^{r^2}$$

$$|e_n| = CK\,|e_{n-2}|^{r}|e_{n-2}| = CK\,|e_{n-2}|^{r+1}$$

$$CK = K^{1+r}$$

$$\frac{1}{\varphi} = \varphi - 1$$

$$C = K^{r} \Rightarrow K = C^{1/r}$$

$$r^2 = r+1 \Rightarrow r^2 - r - 1 = 0 \quad \begin{cases} r = \dfrac{1+\sqrt{5}}{2} = \varphi \\[2ex] r = \dfrac{1-\sqrt{5}}{2} < 0 \end{cases}$$

$-\ln$ $|e_n| = \cancel{\alpha}_1 |e_{n-1}||e_{n-2}|$

$$\underbrace{-\ln|e_n|}_{\lambda_n} = \underbrace{-\ln|e_{n-1}|}_{\lambda_{n-1}} \underbrace{- \ln|e_{n-2}|}_{+\lambda_{n-2}}$$

$$\lambda_n \sim A\varphi^n \Longrightarrow |e_n| = e^{-\lambda_n}$$

$\lambda_n = \lambda_{n-1} + \lambda_{n-2}$

$\lambda_0, \lambda_1, \lambda_2 = \lambda_0 + \lambda_1$

$\lambda_3 = \lambda_2 + \lambda_1 - \ldots$

$\lambda_n = A\varphi^n + B\underbrace{(\cancel{1-}\varphi)}_{||<1}^n$

$\lambda_n \simeq A\varphi^n$

Aplicacion: método de disparo (shooting method)

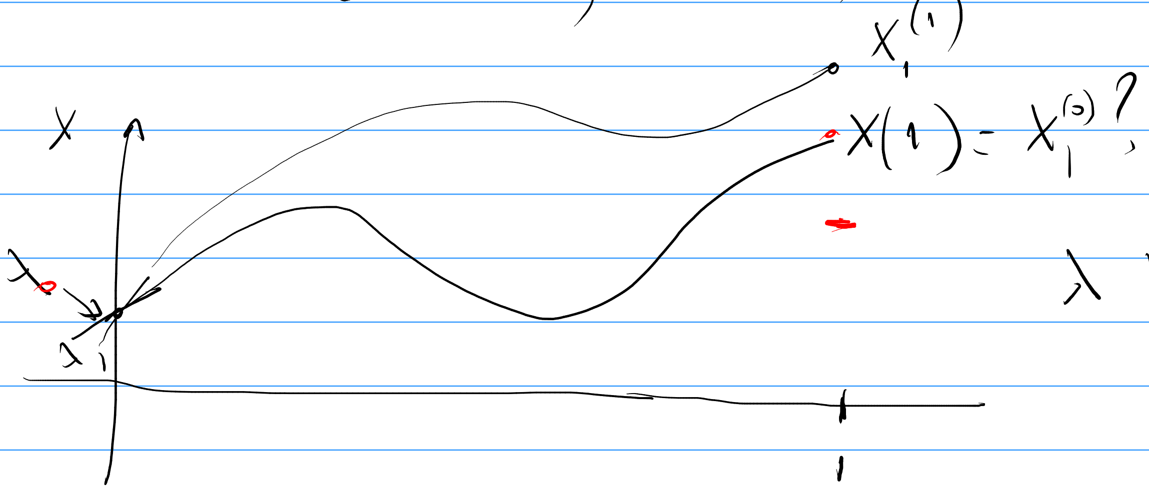
PVI $\begin{cases} \ddot{X} = f(t, X, \dot{X}) \\ X(0) = X_0 \\ X'(0) = X_0' \end{cases}$

$\begin{cases} \ddot{X} = f(t, X, \dot{X}) \\ X(0) = X_0 \quad X(1) = X_1 \end{cases}$

Problema de frontera

Tomamos $\dot{x}(0) = \lambda$ arbitrario

usamos Euler, RK, etc.




$\lambda \longmapsto x(1)$