Elementos de Cálculo Numérico/Cálculo Numérico

Clase 3

Primer Cuatrimestre 2021

Estimación del error

Por el desarrollo de Taylor

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Por el desarrollo de Taylor

si $a = f_t(t_0, x_0) + f_x(t_0, x_0)\, f(t_0, x_0)$

$$x(t_1) = x_0 + hf(t_0, x_0) + h^2/2\, a + O(h^3)$$

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Los errores verifican

$$\epsilon_1 = x(t_1) - x_1 = h^2/2\, a + O(h^3)$$

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Estimación del error

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Euler modificado

Por la estimación del error local de trucamiento

$$\epsilon_1 = x(t_1) - x_1 = h^2/2\,a + O(h^3)$$
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Obtenemos $\epsilon_1 = 2(\tilde{x}_1 - x_1) + O(h^3)$

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Método de Euler modificado:

$$\bar{x}_1 = x_1 + 2(\tilde{x}_1 - x_1) = 2\tilde{x}_1 - x_1$$

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Error de Euler modificado: $\bar{\epsilon}_1 = x(t_1) - \bar{x}_1 = O(h^3)$

Euler modificado: ejemplo

Problema: $\dot{x}(t) = 3\,x(t) - 3\,x^2(t)$, $x(0) = 0.01$

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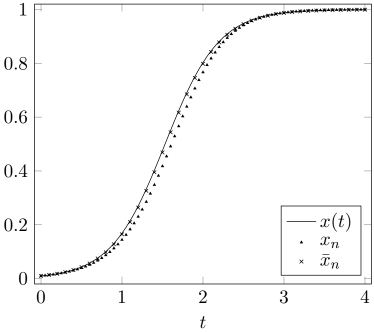
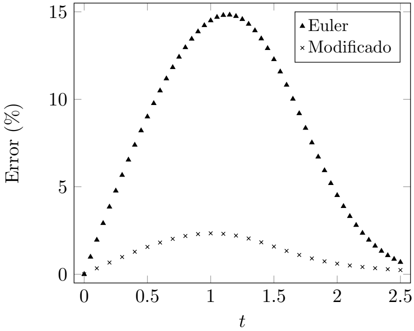
Soluciones numéricas y errores para $h = 0.05$





Método Runge-Kutta de segundo orden

$$k_1 = f(t_0, x_0)$$

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Desarrollo de Taylor

$$k_2 = f(t_0, x_0) + h\, f_t(t_0, x_0) + h\, f_x(t_0, x_0)\, k_1 + O(h^2)$$

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$$\frac{k_1 + k_2}{2} = f(t_0, x_0) + h/2\,(f_t(t_0, x_0) + f_x(t_0, x_0)\, f(t_0, x_0)) + O(h^2)$$

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Definimos $x_1 = x_0 + h(k_1 + k_2)/2$

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Definimos

$$x_1 = x_0 + \frac{h}{6}(k_1 + 2\,k_2 + 2\,k_3 + k_4)$$

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Problema: $\dot{x}(t) = x(t), x(0) = 1$

Solución: $x(t) = e^t$

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Problema: $\dot{x}(t) = x(t), x(0) = 1$

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$x(t_1) = x(h) = e^h = 1 + h + h^2/2 + h^3/6 + h^4/24 + O(h^5)$

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$x_1 = 1 + \frac{h}{6}(k_1 + 2\, k_2 + 2\, k_3 + k_4)$

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$x_1 = 1 + h + h^2/2 + h^3/6 + h^4/24$

Error local de truncamiento: $x(t_1) - x_1 = O(h^5)$

Errores de Runge-Kutta (orden 2 y 4)

Problema: $\dot{x}(t) = x(t), x(0) = 1,\ h = 0.1, 0.01$

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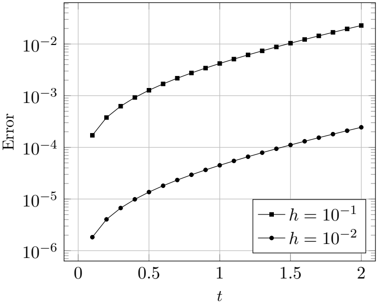
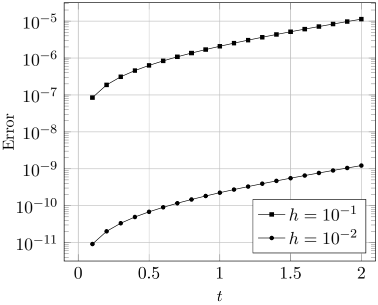
Gráfico de errores



(a) R-K de orden 2



(b) R-K de orden 4

Métodos de un paso

Método general: $x_n = x_{n-1} + h\,\Phi(h, t_{n-1}, x_{-1})$

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Condición de Lipschitz:

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Condición de Lipschitz:

$$|\Phi(h,t,x) - \Phi(h,t,\tilde{x})| \leq L|x - \tilde{x}|$$

Convergencia:

$$E_n = |x(t_n) - x_n| \leq \epsilon(h, t_{n-1}) + (1 + L\,h)E_{n-1}$$

$$\max_{0\leq n\leq N} E_n \leq \frac{e^{LT} - 1}{L} \max_{1\leq n\leq N} \frac{\epsilon(h, t_{n-1})}{h} = O(h^p)$$

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- Método de Euler:

$$\Phi(h,t,x) = f(t,x)$$
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$$\Phi(h,t,x) = f(t+h/2, x+h/2\, f(t,x))$$
$$\epsilon(h,t) = O(h^3),\ L_\Phi = L_f(1+h/2\, L_f)$$

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- Runge-Kutta de orden 2:

$$\Phi(h,t,x) = \frac{1}{2}\left(f(t,x) + f(t+h, x+h\, f(t,x))\right)$$
$$\epsilon(h,t) = O(h^3),\ L_\Phi = L_f(1+h/2\, L_f)$$

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- Runge-Kutta de orden 4:

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$$k_1 = \Phi_1(h,t,x) = f(t,x)$$
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- Constante de Lipschitz

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$$k_3 = \Phi_3(h,t,x) = f(t+h/2, x+h/2\,\Phi_2(h,t,x))$$
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$$L_{\Phi_1} = L_f$$

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Paso adaptativo

Métodos de un paso: $x_n = x_{n-1} + h\,\Phi(h, t_{n-1}, x_{n-1})$

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Si $x_n = x_{n-1} + h\,\tilde{\Phi}(h, t_{n-1}, x_{n-1})$ se define

$$x_{n-1/2} = x_{n-1} + \frac{h}{2}\,\Phi\left(\frac{h}{2}, t_{n-1}, x_{n-1}\right)$$
$$x_n = x_{n-1/2} + \frac{h}{2}\,\Phi\left(\frac{h}{2}, t_{n-1/2}, x_{n-1/2}\right)$$

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