Elementos de Cálculo Numérico/Cálculo Numérico

Clase 8

Primer Cuatrimestre 2021

Método de Jacobi: ejemplo

El sistema

$$\left\{\begin{array}{rl} 3x_1 + x_2 + x_3 & = 8, \\ -x_1 + 4x_2 - x_3 & = 9, \\ x_1 - 3x_2 + 5x_3 & = 2, \end{array}\right.$$

tiene solución $\boldsymbol{x}^* = (1, 3, 2)$
Despejamos una variable de cada ecuación

$$\left\{\begin{array}{ll} x_1 & = (8 - x_2 - x_3)/3, \\ x_2 & = (9 + x_1 + x_3)/4, \\ x_3 & = (2 - x_1 + 3x_2)/5. \end{array}\right.$$

El problema original se convierte en una ecuación de punto fijo

$$\boldsymbol{x} = \mathrm{B}.\boldsymbol{x} + \boldsymbol{c}$$

Método de Jacobi: ejemplo

Si definimos $\mathrm{B} \in \mathbb{R}^{3\times 3}$ y $\boldsymbol{c} \in \mathbb{R}^3$

$$\mathrm{B} = \begin{bmatrix} 0 & -1/3 & -1/3 \\ 1/4 & 0 & 1/4 \\ -1/5 & 3/5 & 0 \end{bmatrix}, \quad \boldsymbol{c} = \begin{bmatrix} 8/3 \\ 9/4 \\ 2/5 \end{bmatrix}$$

La ecuación de punto fijo es

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & -1/3 & -1/3 \\ 1/4 & 0 & 1/4 \\ -1/5 & 3/5 & 0 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 8/3 \\ 9/4 \\ 2/5 \end{bmatrix}$$

Iteración k-ésima: $\boldsymbol{x}^{(k)} = \mathrm{B}.\boldsymbol{x}^{(k-1)} + \boldsymbol{c}$

Error del paso k: $\boldsymbol{\epsilon}^{(k)} = \|\boldsymbol{x}^* - \boldsymbol{x}^{(k)}\|$

Método de Jacobi: ejemplo

Iteraciones y errores

k	$\boldsymbol{x}^{(k)}$	$\boldsymbol{\epsilon}^{(k)}$	$\boldsymbol{\epsilon}^{(k)}/\boldsymbol{\epsilon}^{(k-1)}$
0	(0.00000, 0.00000, 1.00000)	3.317	
1	(2.33333, 2.50000, 0.40000)	2.142	0.646
2	(1.70000, 2.93333, 1.43333)	0.903	0.421
3	(1.21111, 3.03333, 1.82000)	0.279	0.309
4	(1.04889, 3.00778, 1.97778)	0.543×10^{-1}	0.194
5	(1.00481, 3.00667, 1.99489)	0.968×10^{-2}	0.178
$\vdots$	$\vdots$	$\vdots$	$\vdots$
33	(1.00000 , 3.00000 , 2.00000)	0.111×10^{-11}	0.455
34	(1.00000 , 3.00000 , 2.00000)	0.506×10^{-12}	0.455

Tabla: Iteraciones de $\boldsymbol{x}^{(k)} = \mathrm{B}.\boldsymbol{x}^{(k-1)} + \boldsymbol{c}$ con $\boldsymbol{x}^{(0)} = (0., 0., 1.)$

Método de Jacobi

Si $a_{i,i} \neq 0$ para $i = 1, \ldots, n$

$$x_1^{(k)} = a_{1,1}^{-1} \left(b_1 - \sum_{j=2}^{n} a_{1,j}\, x_j^{(k-1)} \right)$$

$$x_2^{(k)} = a_{2,2}^{-1} \left(b_2 - a_{2,1}\, x_1^{(k-1)} - \sum_{j=3}^{n} a_{2,j}\, x_j^{(k-1)} \right)$$

$$\vdots$$

$$x_n^{(k)} = a_{n,n}^{-1} \left(b_n - \sum_{j=1}^{n-1} a_{n,j}\, x_j^{(k-1)} \right)$$

Método de Jacobi: forma matricial

En general $\mathrm{A} = \mathrm{L} + \mathrm{D} + \mathrm{U}$

$$\mathrm{A}.\boldsymbol{x} = \boldsymbol{b} \Longleftrightarrow \boldsymbol{x} = -\mathrm{D}^{-1}.(\mathrm{L} + \mathrm{U}).\boldsymbol{x} + \mathrm{D}^{-1}.\boldsymbol{b}$$

Si $\mathrm{B} = -\mathrm{D}^{-1}.(\mathrm{L} + \mathrm{U})$, $\boldsymbol{c} = \mathrm{D}^{-1}.\boldsymbol{b} \ \Rightarrow \boldsymbol{x} = \mathrm{B}.\boldsymbol{x} + \boldsymbol{c}$

Iteración del método de Jacobi: $\boldsymbol{x}^{(k)} = \mathrm{B}.\boldsymbol{x}^{(k-1)} + \boldsymbol{c}$

Si $r = \#\{a_{i,j} \neq 0 : 1 \leq i, j \leq n\}$ cada iteración: $2\,r + n$ flops

Método de Gauss-Seidel

Usa los valores $x_j^{(k)}$ ya calculados:

$$x_1^{(k)} = a_{1,1}^{-1}\left(b_1 - \sum_{j=2}^{n} a_{1,j}\, x_j^{(k-1)}\right)$$

$$x_2^{(k)} = a_{2,2}^{-1}\left(b_2 - a_{2,1}\, x_1^{(k)} - \sum_{j=3}^{n} a_{2,j}\, x_j^{(k-1)}\right)$$

$$\vdots$$

$$x_n^{(k)} = a_{n,n}^{-1}\left(b_n - \sum_{j=1}^{n-1} a_{n,j}\, x_j^{(k)}\right)$$

Se espera mejorar la aproximación

Método de Gauss-Seidel: forma matricial

En general $\mathrm{A} = \mathrm{L} + \mathrm{D} + \mathrm{U}$

$$\mathrm{A}.\boldsymbol{x} = \boldsymbol{b} \Longleftrightarrow \boldsymbol{x} = -(\mathrm{D}+\mathrm{L})^{-1}.\mathrm{U}.\boldsymbol{x} + \mathrm{D}^{-1}.\boldsymbol{b}$$

Si $\mathrm{B} = -(\mathrm{D}+\mathrm{L})^{-1}.\mathrm{U}$, $\boldsymbol{c} = \mathrm{D}^{-1}.\boldsymbol{b} \;\Rightarrow \boldsymbol{x} = \mathrm{B}.\boldsymbol{x} + \boldsymbol{c}$

Iteración del método de Gauss-Seidel: $\boldsymbol{x}^{(k)} = \mathrm{B}.\boldsymbol{x}^{(k-1)} + \boldsymbol{c}$

Si $r = \#\{a_{i,j} \neq 0 : 1 \leq i, j \leq n\}$ cada iteración: $2\,r + n$ flops

Proposición

Si $\mathrm{B} \in \mathbb{R}^{n \times n}$ *verifica* $\|\mathrm{B}\| < 1$ *para alguna norma matricial inducida, entonces la ecuación* $\mathrm{B}.\boldsymbol{x} + \boldsymbol{c} = \boldsymbol{x}$ *tiene única solución* $\boldsymbol{x}^* \in \mathbb{R}^n$
Dado $\boldsymbol{x}^{(0)} \in \mathbb{R}^n$ *arbitrario, la sucesión definida por*

$$\boldsymbol{x}^{(k)} = \mathrm{B}.\boldsymbol{x}^{(k-1)} + \boldsymbol{c}$$

converge a $\boldsymbol{x}^*$

Demostración.

- Unicidad:

  Si $\boldsymbol{x}^*, \boldsymbol{x}^{**} \in \mathbb{R}^n$ son dos soluciones

  $$\boldsymbol{x}^* - \boldsymbol{x}^{**} = \mathrm{B}.(\boldsymbol{x}^* - \boldsymbol{x}^{**})$$

  $$\|\boldsymbol{x}^* - \boldsymbol{x}^{**}\| \leq \|\mathrm{B}\| \, \|\boldsymbol{x}^* - \boldsymbol{x}^{**}\| \ \Rightarrow (1 - \|\mathrm{B}\|) \, \|\boldsymbol{x}^* - \boldsymbol{x}^{**}\| \leq 0$$

  $$\|\boldsymbol{x}^* - \boldsymbol{x}^{**}\| = 0 \ \Rightarrow \boldsymbol{x}^* = \boldsymbol{x}^{**}$$

Demostración.

- Existencia:

  $\|\boldsymbol{x}^{(l)} - \boldsymbol{x}^{(l-1)}\| \leq \|\mathrm{B}\| \, \|\boldsymbol{x}^{(l-1)} - \boldsymbol{x}^{(l-2)}\|$

  Inductivamente: $\|\boldsymbol{x}^{(l)} - \boldsymbol{x}^{(l-1)}\| \leq \|\mathrm{B}\|^{l-1} \, \|\boldsymbol{x}^{(1)} - \boldsymbol{x}^{(0)}\|$

  Para $1 \leq m \leq l$

$$\begin{aligned} \|\boldsymbol{x}^{(l)} - \boldsymbol{x}^{(m)}\| &\leq \sum_{k=m}^{l-1} \|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}\| \\ &\leq \sum_{k=m}^{l-1} \|\mathrm{B}\|^{k} \, \|\boldsymbol{x}^{(1)} - \boldsymbol{x}^{(0)}\| \leq \frac{\|\mathrm{B}\|^{m}}{1 - \|\mathrm{B}\|} \, \|\boldsymbol{x}^{(1)} - \boldsymbol{x}^{(0)}\| \end{aligned}$$

Demostración.

Como $\|\mathrm{B}\| < 1$, $\|\mathrm{B}\|^m \to 0$

$\{\boldsymbol{x}^{(m)}\}_{m\geq 1}$ es una sucesión de Cauchy

$\boldsymbol{x}^{(m)} \to \boldsymbol{x}^*$ cuando $m \to \infty$

$\boldsymbol{x}^{(m)} = \mathrm{B}.\boldsymbol{x}^{(m-1)} + \boldsymbol{c} \to \mathrm{B}.\boldsymbol{x}^* + \boldsymbol{c}$

$\mathrm{B}.\boldsymbol{x}^* + \boldsymbol{c} = \boldsymbol{x}^*$

Corolario

Si $\mathrm{B} \in \mathbb{R}^{n\times n}$ *verifica* $\rho(\mathrm{B}) < 1$*, entonces para* $\boldsymbol{x}^{(0)} \in \mathbb{R}^n$ *arbitrario la sucesión definida por*

$$\boldsymbol{x}^{(k)} = \mathrm{B}.\boldsymbol{x}^{(k-1)} + \boldsymbol{c}$$

converge a $\boldsymbol{x}^*$

Demostración.

Si $\rho(\mathrm{B}) < 1$, existe una norma inducida tal que $\|\mathrm{B}\| < 1$

En el ejemplo 3×3, B tiene autovalores:

$$\lambda = -0.454597, 0.227298 \pm i0.147175 \Rightarrow \rho(\mathrm{B}) = 0.454597$$

$\epsilon^{(k)}/\epsilon^{(k-1)} \to \rho(\mathrm{B})$

Teorema de Gershgorin

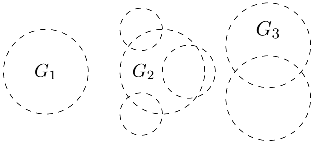
Teorema

Si $\mathrm{A} \in \mathbb{C}^{n\times n}$, *entonces todos sus autovalores* $\lambda_1, \ldots, \lambda_n$ *satisfacen*

$$\lambda_k \in G = \bigcup_{i=1}^{n} \bar{D}_i$$

donde $D_i \subset \mathbb{C}$ *es el disco de centro* $a_{i,i}$ *y radio* $r_i = \sum_{j\neq i} |a_{i,j}|$

Si $G' = \bar{D}_{i_1} \cup \cdots \cup \bar{D}_{i_k}$ *es una componente conexa de* G, *entonces* G' *contiene exactamente* k *autovalores de* A



Teorema de Gershgorin

Demostración.

- Primera parte:

Si $\mathrm{A}.\boldsymbol{v} = \lambda \boldsymbol{v}$ y $|v_i| = \|\boldsymbol{v}\| > 0$, entonces $\sum_{j \neq i} a_{i,j} v_j = (\lambda - a_{i,i})\, v_i$

$$|\lambda - a_{i,i}|\, |v_i| \leq \sum_{j \neq i} |a_{i,j}||v_j| \leq \sum_{j \neq i} |a_{i,j}|\|v\|_\infty$$

$|\lambda - a_{i,i}| \leq \sum_{j \neq i} |a_{i,j}|$

Teorema de Gershgorin

Demostración.

- Segunda parte (variable compleja):

  Si $\mathrm{A}(t) = \mathrm{D} + t\,(\mathrm{L} + \mathrm{U})$, entonces $\mathrm{A}(1) = \mathrm{A}$

  Autovalores de $A(0)$: $a_{1,1}, \ldots, a_{n,n}$

  $\bar{D}_i(t) \subset \bar{D}_i \quad \Rightarrow G'(t) \subset G'$

  $p(t, \lambda) = \det(\mathrm{A}(t) - \lambda\,\mathrm{I})$

  γ curva de Jordan $G' \subset \mathrm{int}(\gamma)$

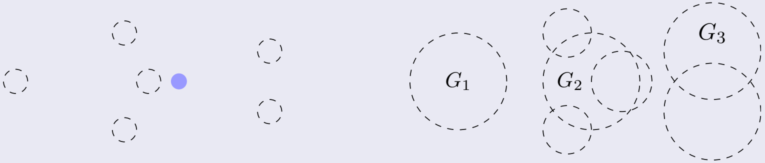
$$N(t) = \frac{1}{2\pi i} \int_\gamma \frac{p_\lambda(t, z)}{p(t, z)}\, dz$$

Teorema de Gershgorin

Demostración.

- Segunda parte (variable compleja):

  $N(t)$ cuenta las raíces de $p(t, \lambda)$ en $\mathrm{int}(\gamma)$



(a) Conjunto $G'(t)$ (b) Conjunto G'

Fig.: Componentes conexas de $G(t)$

Teorema de Gershgorin

Demostración.

- Segunda parte (variable compleja):

  $N(t) \in \mathbb{N}$ y es continua $\Rightarrow N(t)$ es constante

  $N(0) = k$ por lo tanto $N(1) = k$

Ejemplo: Jacobi no converge y Gauss-Seidel converge

Si $\mathrm{A} \in \mathbb{R}^{3\times 3}$

$$\mathrm{A} = \begin{bmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{bmatrix}$$

$$\mathrm{B_J} = \begin{bmatrix} 0 & -1/2 & -1/2 \\ -1/2 & 0 & -1/2 \\ -1/2 & -1/2 & 0 \end{bmatrix} \quad \mathrm{B_{GS}} = \begin{bmatrix} 0 & -1/2 & -1/2 \\ 0 & 1/4 & -1/4 \\ 0 & 1/8 & 3/8 \end{bmatrix}$$

$\rho(\mathrm{B_J}) = 1$

$\rho(\mathrm{B_{GS}}) = \sqrt{2}/4 < 1$

Ejemplo: Jacobi converge y Gauss-Seidel no converge

Si $\mathrm{A} \in \mathbb{R}^{3\times 3}$

$$\mathrm{A} = \begin{bmatrix} -5 & 2 & 1 \\ 5 & 1 & 2 \\ -3 & 2 & 2 \end{bmatrix}$$

$$\mathrm{B_J} = \begin{bmatrix} 0 & 2/5 & 1/5 \\ -5 & 0 & -2 \\ 3/2 & -1 & 0 \end{bmatrix} \quad \mathrm{B_{GS}} = \begin{bmatrix} 0 & 2/5 & 1/5 \\ 0 & -2 & -3 \\ 0 & 13/5 & 33/10 \end{bmatrix}$$

$$\rho(\mathrm{B_J}) = \left((100 - 30\sqrt{10})^{1/3} + (100 + 30\sqrt{10})^{1/3}\right)/10 \cong 0.752244$$

$$\rho(\mathrm{B_{GS}}) = \sqrt{6/5} > 1$$

Matrices diagonal dominante

$\mathrm{A} \in \mathbb{R}^{n\times n}$ es estrictamente diagonal dominante sii

$$|a_{i,i}| > \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{i,j}|, \qquad i = 1, \ldots, n$$

$$\mathrm{B}_t = (\mathrm{D} + t\,\mathrm{L})^{-1}.((1-t)\,\mathrm{L} + \mathrm{U}) \quad \Rightarrow \mathrm{B}_0 = \mathrm{B}_{\mathrm{J}}, \quad \mathrm{B}_1 = \mathrm{B}_{\mathrm{GS}}$$

Proposición

Si $\mathrm{A} \in \mathbb{C}^{n\times n}$ es estrictamente diagonal dominante, entonces $\|\mathrm{B}_t\|_\infty < 1$ para $t \in [0,1]$.

Demostración.

Para $0 \leq t \leq 1$

$$|a_{i,i}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{i,j}| = t \sum_{j=1}^{i-1} |a_{i,j}| + (1-t) \sum_{j=1}^{i-1} |a_{i,j}| + \sum_{j=i+1}^{n} |a_{i,j}|$$

Se verifica $\beta < 1$, donde

$$\beta = \max_{1\leq i\leq N} \left(|a_{i,i}| - t \sum_{j=1}^{i-1} |a_{i,j}| \right)^{-1} \left((1-t) \sum_{j=1}^{i-1} |a_{i,j}| + \sum_{j=i+1}^{N} |a_{i,j}| \right)$$

Demostración.

Si $\boldsymbol{y} = \mathrm{B}_t.\boldsymbol{x} \Rightarrow (\mathrm{D} + t\,\mathrm{L}).\boldsymbol{y} = ((1-t)\,\mathrm{L} + \mathrm{U}).\boldsymbol{x}$

$$a_{i,i}\, y_i + t \sum_{j=1}^{i-1} a_{i,j}\, y_j = (1-t) \sum_{j=1}^{i-1} a_{i,j}\, x_j + \sum_{j=i+1}^{n} a_{i,j}\, x_j$$

$$|a_{i,i}|\,|y_i| - t \sum_{j=1}^{i-1} |a_{i,j}|\,|y_j| \leq (1-t) \sum_{j=1}^{i-1} |a_{i,j}|\,|x_j| + \sum_{j=i+1}^{N} |a_{i,j}|\,|x_j|.$$

Matrices tridiagonales

Proposición

Sea $\mathrm{A} \in \mathbb{R}^{n \times n}$ *una matriz tridiagonal, entonces* $\rho(\mathrm{B_{GS}}) = \rho^2(\mathrm{B_J})$, *donde* $\mathrm{B_J} = -\mathrm{D}^{-1}.(\mathrm{L} + \mathrm{L}^{\mathrm{T}})$ *y* $\mathrm{B_{GS}} = -(\mathrm{D} + \mathrm{L})^{-1}.\mathrm{L}^{\mathrm{T}}$

Corolario

Sea $\mathrm{A} \in \mathbb{R}^{n \times n}$ *una matriz tridiagonal, el método de Jacobi converge si y sólo si el método de Gauss-Seidel converge*

Demostración.

$\mathrm{A} = \mathrm{D} + \mathrm{L} + \mathrm{U}$ es equivalente a $\mathrm{A}_\alpha = \mathrm{D} + \alpha^{-1}\mathrm{L} + \alpha\mathrm{U}$ con $\alpha \neq 0$

$\mathrm{A}_\alpha = \mathrm{P}_\alpha^{-1}.\mathrm{A}.\mathrm{P}_\alpha$

$$\mathrm{P}_\alpha = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & \alpha & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha^{n-1} \end{bmatrix}$$

Demostración.

μ autovalor de $\mathrm{B_J}$ si y solo si

$$0 = \det\left(-\mathrm{D}^{-1}.(\mathrm{L}+\mathrm{U}) - \mu\,\mathrm{I}\right) = \det\left(-\mathrm{D}^{-1}\right)\det(\mathrm{L}+\mathrm{U}+\mu\,\mathrm{D})$$

μ autovalor de $\mathrm{B_J}$ si y solo si $\det(\mathrm{L}+\mathrm{U}+\mu\,\mathrm{D}) = 0$

λ autovalor de $\mathrm{B_{GS}}$ si y solo si

$$\begin{aligned} 0 &= \det\left(-(\mathrm{D}+\mathrm{L})^{-1}.\mathrm{U} - \lambda\,\mathrm{I}\right) \\ &= \det\left(-(\mathrm{D}+\mathrm{L})^{-1}\right)\det(\mathrm{U}+\lambda\,\mathrm{D}+\lambda\,\mathrm{L}) \end{aligned}$$

λ autovalor de $\mathrm{B_{GS}}$ si y solo si $\det(\mathrm{U}+\lambda\,(\mathrm{D}+\mathrm{L})) = 0$

Demostración.

Si $\mu \neq 0$ tomamos $\alpha = \mu^{-1}$

$$
\begin{aligned}
0 &= \det(\mathrm{U} + \mu\,\mathrm{D} + \mathrm{L}) = \det\left(\mu^{-1}\mathrm{U} + \mu\mathrm{D} + \mu\,\mathrm{L}\right) \\
&= \mu^{-n}\det(\mathrm{U} + \mu^2\,(\mathrm{D} + \mathrm{L}))
\end{aligned}
$$

μ^2 autovalor de B_{GS}

Si $\lambda \neq 0$ tomamos $\alpha = \lambda^{1/2}$

$$
\begin{aligned}
0 &= \det(\mathrm{U} + \lambda\,\mathrm{D} + \lambda\,\mathrm{L}) = \det\left(\lambda^{1/2}\,\mathrm{U} + \lambda\,\mathrm{D} + \lambda^{1/2}\,\mathrm{L}\right) \\
&= \lambda^{n/2}\,\det(\mathrm{U} + \lambda^{1/2}\,\mathrm{D} + \mathrm{L})
\end{aligned}
$$

$\lambda^{1/2}$ autovalor de B_{J}

Matrices definidas positivas

Proposición

Sea $\mathrm{A} \in \mathbb{R}^{n \times n}$ *una matriz simétrica y definida positiva, entonces* $\rho(\mathrm{B_{GS}}) < 1$, *donde* $\mathrm{B_{GS}} = -(\mathrm{D} + \mathrm{L})^{-1}.\mathrm{L}^{\mathrm{T}}$

A simétrica y definida positiva $\not\Rightarrow \rho(\mathrm{B_J}) < 1$

Ejemplo:

$$\mathrm{A} = \begin{bmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{bmatrix}$$

Demostración.

Si $\lambda \in \mathbb{C}$ autovalor de B_{GS} con autovector $\boldsymbol{x} \in \mathbb{C}^n$, entonces

$$\lambda\,\mathrm{A}.\boldsymbol{x} + (1-\lambda)\,\mathrm{L}^{\mathrm{T}}.\boldsymbol{x} = \mathbf{0}$$

Multiplicando por $\boldsymbol{x}^*$: $\lambda\,\boldsymbol{x}^*.(\mathrm{A}.\boldsymbol{x}) + (1-\lambda)\,\boldsymbol{x}^*.(\mathrm{L}^{\mathrm{T}}.\boldsymbol{x}) = 0$

$$\frac{\lambda}{\lambda - 1} = \frac{\boldsymbol{x}^*.(\mathrm{L}^{\mathrm{T}}.\boldsymbol{x})}{\boldsymbol{x}^*.(\mathrm{A}.\boldsymbol{x})}$$

Demostración.

Como $\boldsymbol{x}^*.(\mathrm{L}^{\mathrm{T}}.\boldsymbol{x}) = \overline{\boldsymbol{x}^*.(\mathrm{L}.\boldsymbol{x})}$ y $\mathrm{L} + \mathrm{L}^{\mathrm{T}} = \mathrm{A} - \mathrm{D}$, tenemos

$$2\mathrm{Re}\left(\frac{\lambda}{\lambda - 1}\right) = \frac{\boldsymbol{x}^*.(\mathrm{L}.\boldsymbol{x}) + \boldsymbol{x}^*.(\mathrm{L}^{\mathrm{T}}.\boldsymbol{x})}{\boldsymbol{x}^*.(\mathrm{A}.\boldsymbol{x})} = \frac{\boldsymbol{x}^*.(\mathrm{A}.\boldsymbol{x}) - \boldsymbol{x}^*.(\mathrm{D}.\boldsymbol{x})}{\boldsymbol{x}^*.(\mathrm{A}.\boldsymbol{x})} < 1$$

Pero

$$\frac{1}{2} > \mathrm{Re}\left(\frac{\lambda}{\lambda - 1}\right) = \frac{|\lambda|^2 - \mathrm{Re}\,(\lambda)}{|\lambda|^2 - 2\mathrm{Re}\,(\lambda) + 1}$$

que es equivalente a $|\lambda| < 1$.