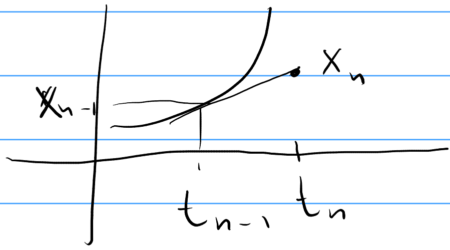
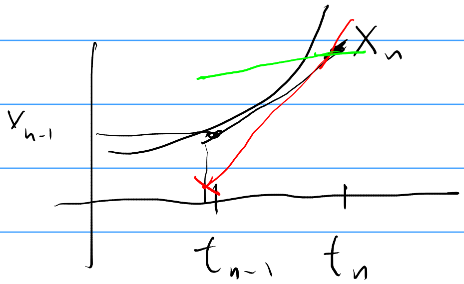
Método de Euler

explícito: $x_n = x_{n-1} + h f(t_{n-1}, x_{n-1})$



Implícito: $x_n = x_{n-1} + h f(\underbrace{t_{n-1}+h}_{t_n}, x_n)$

x_n es la solución de esta ecuación



Es más complicado computacionalmente
¿Por qué usarlo?

¿Cómo se resuelve la ecuación?

¿ Métodos de pto fijo

$$x_n = x_{n-1} + h\,\phi(h, t_{n-1}, x_{n-1})$$

$$e_n \left[\begin{array}{l} \\ \epsilon_n \left[\begin{array}{l} \tilde{x}_n = x(t_{n-1}) + h\,\phi(h, t_{n-1}, x(t_{n-1})) \\ x(t) \end{array} \right. \end{array} \right.$$

$$E_n = |x(t) - x_n| \leq |x(t) - \tilde{x}_n| + |\tilde{x}_n - x_n| = |\epsilon_n| + |\tilde{x}_n - x_n|$$

$$\tilde{x}_n - x_n = x(t_{n-1}) - x_{n-1} + h\left[\phi(h, t_{n-1}, x(t_{n-1})) - \phi(h, t_{n-1}, x_{n-1})\right]$$

$$|\tilde{X}_n - x_n| \leq E_{n-1} + h L E_{n-1} \leq (1+Lh) E_{n-1}$$

$$E_n \leq |\epsilon_n| + (1+Lh) E_{n-1}$$

$$E_1 \leq |\epsilon_1|$$

$$E_2 \leq |\epsilon_2| + (1+Lh)|\epsilon_1|$$

$$E_3 \leq |\epsilon_3| + (1+Lh)\left[|\epsilon_2| + (1+Lh)|\epsilon_1|\right]$$

$$E_3 \leq |\epsilon_3| + (1+Lh)|\epsilon_2| + (1+Lh)^2|\epsilon_1|$$

$$E_n \leq |\epsilon_n| + (1+Lh)|\epsilon_{n-1}| + \dots + (1+Lh)^{n-1}|\epsilon_1|$$

$$\epsilon = \max\{|\epsilon_1|, |\epsilon_2|, \dots, |\epsilon_n|\} = O(h^{p+1})$$

$$E_n \leq \epsilon \underbrace{\left(1 + \dots + (1+Lh)^{n-1}\right)}_{\frac{(1+Lh)^n - 1}{\cancel{1}+Lh\cancel{-1}} \leq \frac{e^{L\overbrace{nh}^{t_n - t_0}} - 1}{Lh}}$$

$$E_n \leq \frac{e^{L(t_n - t_0)} - 1}{L} \left(\frac{\epsilon}{h}\right) \leftarrow O(h^p)$$

$$t_0 \leq t \leq t_0 + T \Rightarrow \max_n E_n \leq \frac{e^{LT} - 1}{L} \cdot O(h^p)$$

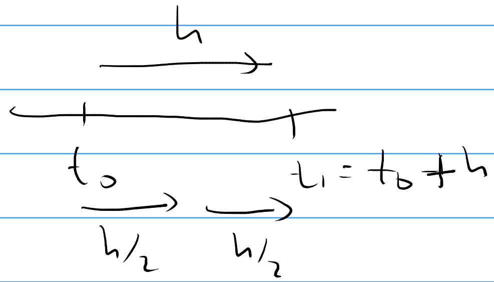
¿Cómo se relaciona L (cte de Lips de ϕ) con ~~f~~ la constante de Lips. de f?

$$|f(t,x) - f(t,y)| \leq C|x-y|$$

$$C \leftarrow \max_{(t,x)\in Q} |f_x|$$

$X_1 = X_0 + h f(t_0, X_0)$ M Euler con paso h

$$\left.\begin{aligned}\tilde{X}_{1/2} &= X_0 + \frac{h}{2} f(t_0, X_0)\\ \tilde{X}_1 &= \tilde{X}_{1/2} + \frac{h}{2} f\left(t_0 + \frac{h}{2}, \tilde{X}_{1/2}\right)\end{aligned}\right| \text{ 2 pasos de M Euler con } \frac{h}{2}$$



$$2(\tilde{X}_1 - X_1) = \epsilon_1 + O(h^3)$$

$$X_1 + 2(\tilde{X}_1 - X_1) = 2\tilde{X}_1 - X_1$$

Método de orden 2 $\epsilon_1 = O(h^3)$

M Euler modificado

$$X_n = X_{n-1} + h\Big[\underbrace{f\Big(t_{n-1} + \frac{h}{2}, \underbrace{X_{n-1} + \frac{h}{2}\underbrace{f(t_{n-1}, X_{n-1})}_{k_1}}_{\tilde{X}}\Big)}_{k_2}\Big]$$

$t = t_0$

$x = x_0$

for $j = 1$ to n

$k_1 = f(t, x)$

$\tilde{x} = x + \frac{h}{2} k_1$ | Evaluamos f 2 veces

$k_2 = f\left(t + \frac{h}{2}, \tilde{x}\right)$

$t = t + h$

$x = x + h k_2$

$$\phi(h, t, x) = f\left(t + \frac{h}{2}, x + \frac{h}{2} f(t, x)\right)$$

$$|\phi(h,t,x) - \phi(h,t,y)| = \left|f\left(t + \frac{h}{2}, x + \frac{h}{2} f(t,x)\right) - f\left(t + \frac{h}{2}, y + \frac{h}{2} f(t,y)\right)\right|$$

$$\leq L_f \left|\left(x + \frac{h}{2} f(t,x)\right) - \left(y + \frac{h}{2} f(t,y)\right)\right| \leq$$

$$\leq L_f\left(|x - y| + \frac{h}{2}|f(t,x) - f(t,y)|\right) \leq L_f\left(|x-y| + \frac{h}{2} L_f |x - y|\right)$$

Para R-k 4

$$L_\phi \cong \frac{e^{L_f h}-1}{h} \xrightarrow[h \to 0]{(L'H)} L_f$$

Pero si $L_f h >> 1$ L_ϕ puede ser muy grande

$$\Downarrow$$

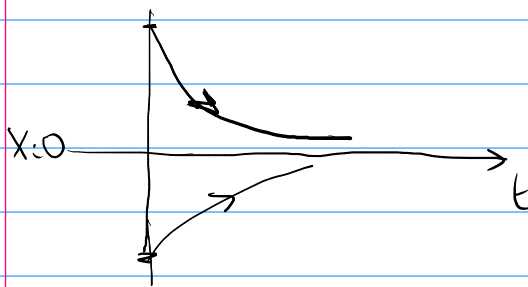
$$\bar{E}_n \cong \frac{e^{L_\phi (t_n - t_0)} - 1}{L_\phi}$$ crece rápido con t_n

Si L_f es grande los explícitos pueden ser ineficientes

Probemas rígidos (stiff)

$\dot{x} = -\lambda x$ $\quad \lambda >> 1$ $\quad$ Ej: $\lambda = 100$

$x(t) = e^{-\lambda t} x_0$



$f(t,x) = -\lambda x$

$L_f = \lambda >> 1$

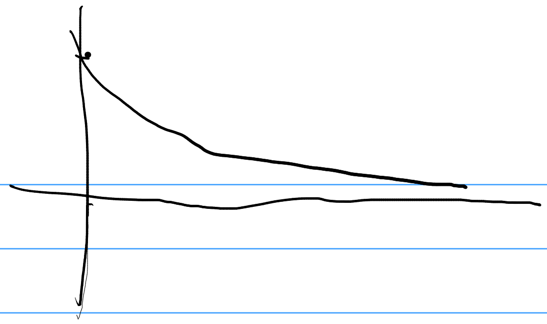
M. Euler explícito: $x_n = x_{n-1} + h(-\lambda x_{n-1})$

$$x_n = (1-\lambda h) x_{n-1}$$

$$x_n = (1-\lambda h)^n x_0$$

$$(1-\lambda h)^n \cong e^{-\lambda n h} = e^{-\lambda t_n}$$

$h = 0.05$ $\quad x_n = (1 - 100 \times 0.05)^n x_0 = (-4)^n x_0$



$h = 0.001$

$$(1-\lambda h)^n = (1-100\times 0.001)^n = 0.9^n$$

$$(1-0.1)^n \approx e^{-0.1n} = e^{-\lambda h n}$$

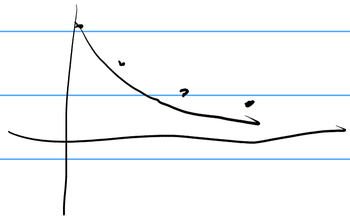
M. Euler implícito: $x_n = x_{n-1} + h(-\lambda x_n) = x_{n-1} - \lambda h x_n$

$$(1+\lambda h)x_n = x_n + \lambda h x_n = x_{n-1}$$

$$x_n = (1+\lambda h)^{-1} x_{n-1}$$

$$x_n = (1+\lambda h)^{-n} x_0$$

$\lambda = 100 \quad h = 0.05 \qquad x_n = 6^{-n} x_0$



Problemas de frontera

PVI $\begin{cases} \ddot{x} + a(t)\dot{x} + b(t)x = f(t) \\ x(0) = x_0 \\ \dot{x}(0) = x_1 \end{cases}$ $\quad \begin{cases} \dot{x} = y \\ \dot{y} = -ay - bx + f \end{cases}$

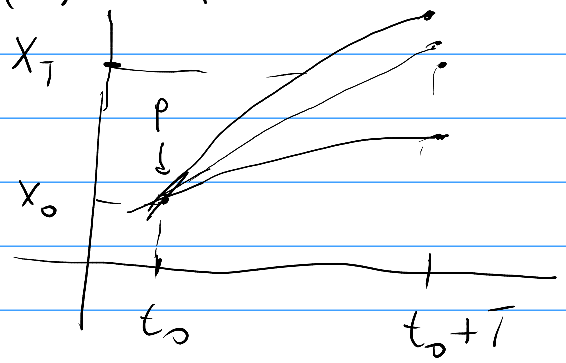
PVb $\quad x(0) = x_0 \quad x(L) = x_L \quad$ No podemos aplicar los métodos Euler, R-K

$x(0) = x_0 \quad x(L) = x_L$

$y(0) = ? \quad y(L) = ?$

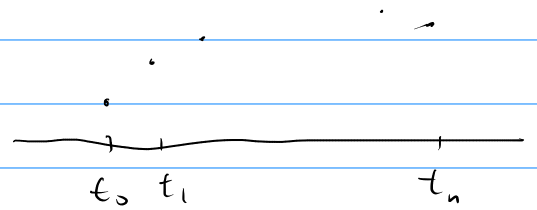
Métodos de disparo

$$\begin{cases} \ddot{x} + a\dot{x} + bx = 0 \\ x(0) = x_0 \\ \cancel{x(T) = X_T} \end{cases} \qquad \dot{x}(0) = p$$



$$x(T, p) = X_T$$

p: incógnita



$$\dot{x}(t_n) \approx \frac{x(t_{n+1}) - x(t_n)}{h}$$

$$\ddot{x}(t_n) \approx \text{------}$$

diferencias finitas

$$\dot{x}_n = \frac{x_{n+1} - x_{n-1}}{2h} \qquad \ddot{x}_n = \frac{x_{n+1} - 2x_n + x_{n-1}}{h^2}$$

$$a\ddot{x} + b\dot{x} + cx = 0$$

$n = 1, \ldots, N-1$

$$a\left(\frac{x_{n+1} - 2x_n + x_{n-1}}{h^2}\right) + b\,\frac{x_{n+1} - x_{n-1}}{h} + c\,x_n = 0$$

$x_0 = x_i$

$x_N = x_f$

sistema de $N-1$ ecuaciones lineales