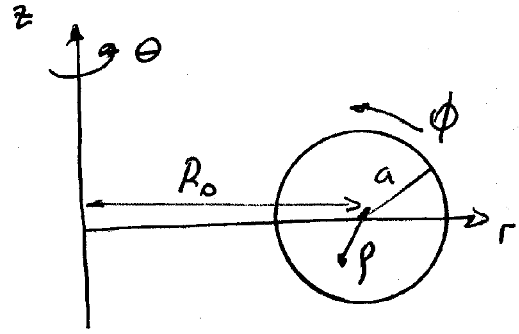
Analytic toroidal equilibrium, large aspect ratio, circular cross section

cylindrical coordinates: r, θ, z

toroidal coordinates ρ, ϕ, θ

$$\begin{cases} r = R_0 + \rho\cos\phi \\ z = \rho\sin\phi \end{cases}$$



Grad Shafranov equation, cylindrical coordinates, gaussian units

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) + \frac{\partial^2\psi}{\partial z^2} = -16\pi^3 r^2 \frac{dp}{d\psi} - \frac{8\pi^2}{c^2}\frac{dI^2}{d\psi}$$

ψ: poloidal flux I: poloidal current, such that:

$$\underline{B} = \frac{1}{2\pi r}\frac{\partial\psi}{\partial r}\hat{e}_z - \frac{1}{2\pi r}\frac{\partial\psi}{\partial z}\hat{e}_r + \frac{2I}{cr}\hat{e}_\theta$$

Transformation to toroidal coordinates.

$$dr = \frac{\partial r}{\partial\rho}d\rho + \frac{\partial r}{\partial\phi}d\phi = \cos\phi\, d\rho - \rho\sin\phi\, d\phi$$

$$dz = \frac{\partial z}{\partial\rho}d\rho + \frac{\partial z}{\partial\phi}d\phi = \sin\phi\, d\rho + \rho\cos\phi\, d\phi$$

From this we get

$$d\rho = \cos\phi\, dr + \sin\phi\, dz \Rightarrow \frac{\partial\rho}{\partial r}dr + \frac{\partial\rho}{\partial z}dz = \cos\phi\, dr + \sin\phi\, dz$$

$$d\phi = \frac{1}{\rho}(\cos\phi\, dz - \sin\phi\, dr) \Rightarrow \frac{\partial\phi}{\partial r}dr + \frac{\partial\phi}{\partial z}dz = \frac{1}{\rho}(\cos\phi\, dz - \sin\phi\, dr)$$

$$\frac{\partial\rho}{\partial r} = \cos\phi;\ \frac{\partial\rho}{\partial z} = \sin\phi;\ \frac{\partial\phi}{\partial r} = -\frac{1}{\rho}\sin\phi;\ \frac{\partial\phi}{\partial z} = \frac{\cos\phi}{\rho}$$

In the G-S equation we substitute

$$\frac{\partial}{\partial r} \equiv \frac{\partial}{\partial \rho}\frac{\partial \rho}{\partial r} + \frac{\partial}{\partial \phi}\frac{\partial \phi}{\partial r} \; ; \; \frac{\partial}{\partial z} \equiv \frac{\partial}{\partial \rho}\frac{\partial \rho}{\partial} + \frac{\partial}{\partial \phi}\frac{\partial \phi}{\partial z}$$

After some algebra

$$\frac{\partial^2 \Psi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \Psi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \Psi}{\partial \phi^2} - \frac{1}{(R_0+\rho\cos\phi)}\left[\cos\phi\frac{\partial \Psi}{\partial \rho} - \frac{\operatorname{sen}\phi}{\rho}\frac{\partial \Psi}{\partial \phi}\right] = -16\pi^3(R_0+\rho\cos\phi)^2 \times \frac{dp}{d\Psi} - \frac{8\pi^2}{c^2}\frac{dI^2}{d\Psi}$$

Introduce $\quad x = \frac{\rho}{a} \quad \varepsilon = \frac{a}{R_0}$ (inverse aspect ratio, $\varepsilon = A^{-1}$)

$$\frac{\partial}{\partial \rho} = \frac{1}{a}\frac{\partial}{\partial x} \qquad \frac{1}{(R_0+\rho\cos\phi)} = \frac{1}{R_0(1+\varepsilon x\cos\phi)}$$

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{1}{x}\frac{\partial \Psi}{\partial x} + \frac{1}{x^2}\frac{\partial^2 \Psi}{\partial \phi^2} - \frac{\varepsilon}{(1+\varepsilon x\cos\phi)}\left[\cos\phi\frac{\partial \Psi}{\partial x} - \frac{\sin\phi}{x}\frac{\partial \Psi}{\partial \phi}\right] =$$

$$= -16\pi^3 a^2 R_0^2 (1+\varepsilon x\cos\phi)^2 \frac{dp}{d\Psi} - \frac{8\pi^2 a^2}{c^2}\frac{dI^2}{d\Psi}$$

Introduce the following dimensionless quantities

$$\hat{\Psi} = \frac{\Psi}{B_0 \pi a^2} \qquad \hat{p} = \frac{p}{(B_0^2/8\pi)} \qquad \hat{I} = \frac{2I}{B_0 c R_0}$$

where B_0 is the vacuum toroidal field at R_0. The G-S equation results

$$\frac{\partial^2 \hat{\Psi}}{\partial x^2} + \frac{1}{x}\frac{\partial \hat{\Psi}}{\partial x} + \frac{1}{x^2}\frac{\partial^2 \hat{\Psi}}{\partial \phi^2} - \frac{\varepsilon}{(1+\varepsilon x\cos\phi)}\left[\cos\phi\frac{\partial \hat{\Psi}}{\partial x} - \frac{\sin\phi}{x}\frac{\partial \hat{\Psi}}{\partial \phi}\right] = -\frac{2}{\varepsilon^2}(1+\varepsilon x\cos\phi)^2\frac{d\hat{p}}{d\hat{\Psi}} - \frac{2}{\varepsilon^2}\frac{d\hat{I}^2}{d\hat{\Psi}}$$

Specify $\hat{p}$ and $\hat{I}$ as functions of $\hat{\Psi}$

$$\hat{p} = p_1 \hat{\Psi}^2 \Rightarrow \frac{d\hat{p}}{d\hat{\Psi}} = 2p_1\hat{\Psi} \qquad \hat{I}^2 = I_0^2 + I_1^2\hat{\Psi}^2 \Rightarrow \frac{d\hat{I}^2}{d\hat{\Psi}} = 2I_1^2\hat{\Psi}$$

The RHS of the G-S eq. becomes

$$-\frac{4}{\varepsilon^2}(1+\varepsilon x\cos\phi)^2 p_1\hat{\Psi} - \frac{4}{\varepsilon^2} I_1^2\hat{\Psi}$$

Then,

$$\frac{\partial^2\hat{\Psi}}{\partial x^2} + \frac{1}{x}\frac{\partial\hat{\Psi}}{\partial x} + \frac{1}{x^2}\frac{\partial^2\hat{\Psi}}{\partial\phi^2} - \frac{\varepsilon}{(1+\varepsilon x\cos\phi)}\left[\cos\phi\frac{\partial\hat{\Psi}}{\partial x} + \frac{\sin\phi}{x}\frac{\partial\hat{\Psi}}{\partial\phi}\right] = -\frac{4}{\varepsilon^2}(1+\varepsilon x\cos\phi)^2 \times p_1\hat{\Psi} - \frac{4}{\varepsilon^2}I_1^2\hat{\Psi}$$

Expand $\hat{\Psi}$ in powers of ε, assuming that the zeroth order solution corresponds to the cylindrical case (no ϕ dependence)

$$\hat{\Psi}(x,\phi) = \Psi_0(x) + \varepsilon\Psi_1(x,\phi) + \varepsilon^2\Psi_2(x,\phi)\dots$$

Order zero

$$\frac{\partial^2\Psi_0}{\partial x^2} + \frac{1}{x}\frac{\partial\Psi_0}{\partial x} + k^2\Psi_0 = 0 \qquad \text{where } k^2 = \frac{4}{\varepsilon^2}(p_1 + I_1^2) \text{ is considered of order 0}$$

Note that, in k^2, there is a factor ε^{-2} but p_1 and I_1^2 are generally small. (for ex. $\hat{p}$ has to be of order β)

The solution is $\boxed{\Psi_0 = C\,J_0(kx)}$ where C is a constant, J_0: Bessel function

order 1

$$\frac{\partial^2\Psi_1}{\partial x^2} + \frac{1}{x}\frac{\partial\Psi_1}{\partial x} + \frac{1}{x^2}\frac{\partial^2\Psi_1}{\partial\phi^2} - \cos\phi\frac{\partial\Psi_0}{\partial x} + k^2\Psi_1 + 8\left(\frac{p_1}{\varepsilon^2}\right)x\cos\phi\,\Psi_0 = 0$$

$$\frac{\partial^2 \Psi_1}{\partial x^2}+\frac{1}{x}\frac{\partial \Psi_1}{\partial x}+\frac{1}{x^2}\frac{\partial^2 \Psi_1}{\partial \phi^2}+k^2\Psi_1 = \cos\phi\left\{\frac{\partial \Psi_0}{\partial x}-2\alpha x \Psi_0\right\} \qquad \alpha \equiv \frac{4p_1}{\varepsilon^2}$$

The solution to this eq. is:

$$\Psi_1(x,\phi)=\frac{\cos\phi}{2}\left\{C\left[xJ_0(kx)-\frac{\alpha x^2}{k}J_1(kx)\right]+DJ_1(kx)\right\} \qquad D: \text{constant}$$

The constants, C and D, must be determined using the boundary conditions. Assume the plasma is inside a perfect conductor, then $\Psi(x=1)=cte$. In particular, we can take $\hat{\Psi}(x=1)=0$

$$\hat{\Psi}_0(1)+\varepsilon\hat{\Psi}_1(1)=0 \Rightarrow \begin{cases}\hat{\Psi}_0(1)=0 \Rightarrow J_0(k)=0 \Rightarrow \boxed{k: \text{zero of } J_0}\\ \hat{\Psi}_1(1)=0\end{cases}$$

$\hat{\Psi}(1)=0 \Rightarrow C\left[J_0(k)-\frac{\alpha}{k}J_1(k)\right]+DJ_1(k)=0$, but $J_0(k)=0$ and

$$\boxed{D=\frac{C\alpha}{k}}$$

C: is determined by fixing $B_\phi(x=1)$. This is the same as fixing the toroidal current; α is proportional to p_1 and appears to be a free parameter related to β.

Substituting D in Ψ_1

$$\Psi_1(x,\phi)=C\frac{\cos\phi}{2}\left\{xJ_0(kx)+\frac{\alpha}{k}J_1(kx)(1-x^2)\right\}$$

Fields

$$B_z = \frac{1}{2\pi r}\frac{\partial \Psi}{\partial r} \; ; \; B_r = -\frac{1}{2\pi r}\frac{\partial \Psi}{\partial z} \; ; \; B_\theta = \frac{2I}{cr}$$

$$\frac{\partial \Psi}{\partial r} = \frac{\partial \Psi}{\partial \rho}\frac{\partial \rho}{\partial r} + \frac{\partial \Psi}{\partial \phi}\frac{\partial \phi}{\partial r} = \frac{\partial \Psi}{\partial \rho}\cos\phi + \frac{\partial \Psi}{\partial \phi}\left(-\frac{\sin\phi}{\rho}\right)$$

$$\frac{\partial \Psi}{\partial z} = \frac{\partial \Psi}{\partial \rho}\frac{\partial \rho}{\partial z} + \frac{\partial \Psi}{\partial \phi}\frac{\partial \phi}{\partial z} = \frac{\partial \Psi}{\partial \rho}\sin\phi + \frac{\partial \Psi}{\partial \phi}\frac{\cos\phi}{\rho}$$

$\Psi = \pi a^2 B_0 \hat{\Psi}$; $\rho = a x$, $z = a\hat{z}$; $B_z = B_0 \hat{B}_z$; $B_r = B_0 \hat{B}_r$

$B_\theta = B_0 \hat{B}_\theta$ $\quad I = \hat{I}\,\frac{B_0 c R_0}{2}$; $r = R_0 + \rho\cos\phi$

$$\cancel{B_0}\hat{B}_z = \frac{1}{2\cancel{\pi} R_0(1+\varepsilon x\cos\phi)}\frac{\cancel{\pi} a^{\cancel{2}} \cancel{B_0}}{}\left\{\frac{\partial \hat{\Psi}}{\cancel{a}\partial x}\cos\phi - \frac{\sin\phi}{a x}\frac{\partial \hat{\Psi}}{\partial \phi}\right\}$$

$$\boxed{\hat{B}_z = \frac{\varepsilon}{2(1+\varepsilon x\cos\phi)}\left\{\frac{\partial \hat{\Psi}}{\partial x}\cos\phi - \frac{\partial \hat{\Psi}}{\partial \phi}\frac{\sin\phi}{x}\right\}}$$

$$\boxed{\hat{B}_r = -\frac{\varepsilon}{2(1+\varepsilon x\cos\phi)}\left\{\frac{\partial \hat{\Psi}}{\partial x}\sin\phi + \frac{\partial \hat{\Psi}}{\partial \phi}\frac{\cos\phi}{x}\right\}}$$

$$B_0\hat{B}_\theta = \frac{2\, B_0 c R_0 \hat{I}}{c R_0(1+\varepsilon x\cos\phi)\, 2} \Rightarrow \boxed{\frac{\hat{I}}{(1+\varepsilon x\cos\phi)} = \hat{B}_\theta}$$

$$\hat{\Psi}(x,\phi) = \Psi_0(x) + \varepsilon\Psi_1(x,\phi) ; \; \frac{\partial \hat{\Psi}}{\partial x} = \frac{\partial \Psi_0}{\partial x} + \varepsilon\frac{\partial \Psi_1}{\partial x} \; ; \; \frac{\partial \hat{\Psi}}{\partial \phi} = \frac{\partial \Psi_1}{\partial \phi}\varepsilon$$

$$\frac{\partial \Psi_0}{\partial x} = Ck J_0'(kx) = -Ck J_1(kx)$$

$$\frac{\partial \Psi_1}{\partial x} = C\frac{\cos\phi}{2}\left\{ J_0(kx) + kx\, J_0'(kx) + \frac{\alpha}{k}\left[J_1'(kx)\, k(1-x^2) - 2x\, J_1(kx)\right]\right\}$$

$$\frac{\partial \Psi_1}{\partial x} = C\frac{\cos\phi}{2}\left\{ J_0(kx) - J_1(kx)\, kx\left(1+\frac{\alpha 2}{k^3}\right) + \alpha(1-x^2)\, J_1'(kx)\right\}$$

$$\frac{\partial \Psi_1}{\partial \phi} = -C\frac{\sin\phi}{2}\left\{ x J_0(kx) + \frac{\alpha}{k} J_1(kx)\,(1-x^2)\right\}$$

$$\hat{B}_z = \frac{\varepsilon}{2(1+\varepsilon x\cos\phi)}\left\{\cos\phi\left[-CkJ_1 + \frac{\varepsilon C\cos\phi}{2}\left(J_0 - kxJ_1\left(1+\frac{\alpha 2}{k^3}\right) + \right.\right.\right.$$

$$\left.\left.\left. + \alpha(1-x^2)J_1'\right)\right] + \frac{\varepsilon C\sin^2\phi}{2x}\left[xJ_0 + \frac{\alpha}{k}J_1(1-x^2)\right]\right\}$$

$$\boxed{\hat{B}_z = \frac{\varepsilon C}{2(1+\varepsilon x\cos\phi)}\left\{-kJ_1\cos\phi + \frac{\varepsilon}{2}\left[J_0 + J_1\left(\frac{\alpha}{kx}(1-x^2)\sin^2\phi - kx\left(1+\frac{\alpha 2}{k^2}\right)\cos^2\phi\right) + J_1'\alpha(1-x^2)\cos^2\phi\right]\right\}}$$

Introducing $B_w = B_z(x=1, \phi=0)$

$$B_w = \frac{\varepsilon C}{2(1+\varepsilon)}\left\{-kJ_1(k) - \frac{\varepsilon}{2}J_1(k)k\left(1+\frac{2\alpha}{k^2}\right)\right\}, \quad J_0(k)=0$$

$$\boxed{C = \frac{-2(1+\varepsilon)\,B_w}{\varepsilon J_1(k)k\left[1+\frac{\varepsilon}{2}\left(1+\frac{2\alpha}{k^2}\right)\right]}}$$

$$\hat{B}_r = -\frac{\varepsilon C}{2(1+\varepsilon x\cos\phi)}\left\{\sin\phi\left[-kJ_1 + \frac{\varepsilon}{2}\cos\phi\left(\cancel{J_0} - kxJ_1\left[1+\frac{2\alpha}{k^2}\right] + \alpha(1-x^2)J_1'\right)\right]\right.$$

$$\left. - \frac{\sin\phi\cos\phi}{2x}\varepsilon\left[x\cancel{J_0} + \frac{\alpha}{k}J_1(1-x^2)\right]\right\}$$

using $J_1'(kx) = J_0(kx) - \frac{1}{kx} J_1(kx)$

$$\boxed{\hat{B}_r = -\frac{\varepsilon C}{2(1+\varepsilon x\cos\phi)}\left\{-k J_1 \sin\phi + \frac{\varepsilon}{2}\sin\phi\cos\phi\left[J_1\left(-kx - \frac{2\alpha}{kx}\right) + J_0\alpha(1-x^2)\right]\right\}}$$

$$\boxed{\hat{B}_z = \frac{\varepsilon C}{2(1+\varepsilon x\cos\phi)}\left\{-kJ_1\cos\phi + \frac{\varepsilon}{2}\left[J_0\left(1+\alpha(1-x^2)\cos^2\phi\right) + J_1\left(\frac{\alpha}{kx}(1-x^2)\sin^2\phi - \cos^2\phi\left[kx + \frac{\alpha x}{k} + \frac{\alpha}{kx}\right]\right)\right]\right\}}$$

$$\hat{B}_\theta = \frac{\hat{I}}{(1+\varepsilon x\cos\phi)} \qquad \hat{I}^2 = I_0^2 + I_1^2\hat{\psi}^2$$

I_0^2 is related to the vacuum toroidal field, $I_0 = 1$

$$\boxed{\hat{B}_\theta = \frac{1}{(1+\varepsilon x\cos\phi)}\left[1 + I_1^2(\psi_0 + \varepsilon\psi_1)^2\right]^{1/2}}$$

$$k^2 = \frac{4}{\varepsilon^2}(p_1 + I_1^2) \Rightarrow \boxed{I_1^2 = \frac{k^2\varepsilon^2}{4} - p_1}$$

since k must be a zero of J_0, the value of I_1 is fixed, once p_1 is fixed ($k = 2.404$)