Acceleration, momentum and forces

The acceleration is related to the time derivative of the particle momentum and to the forces acting on the particle. For a particle of inertial mass m the momentum is

$$\mathbf{p} = m\,\mathbf{v},$$

and Newton's second law states that the rate of change of momentum equals the total force $\mathbf{F}$ acting on the particle:

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}.$$

If the mass m is constant this gives the familiar relation between acceleration and force:

$$\mathbf{a} \equiv \frac{d\mathbf{v}}{dt} = \frac{1}{m}\mathbf{F}.$$

Below are common examples of forces and the corresponding acceleration:

- **Gravitational (conservative) force.** If $\Phi(\mathbf{x}, t)$ denotes the gravitational potential per unit mass (the usual convention in stellar dynamics), the force on a particle of mass m is

  $$\mathbf{F}_{\text{grav}} = -m\,\nabla\Phi(\mathbf{x}, t),$$

  and therefore the acceleration is mass-independent,

  $$\mathbf{a}_{\text{grav}} = \frac{\mathbf{F}_{\text{grav}}}{m} = -\nabla\Phi(\mathbf{x}, t).$$

- **Lorentz force (electromagnetic).** The full electromagnetic force on a charged particle moving with velocity $\mathbf{v}$ in fields $\mathbf{E}$ and $\mathbf{B}$ is

  $$\mathbf{F}_{\text{Lorentz}} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right),$$

  so

  $$\mathbf{a}_{\text{Lorentz}} = \frac{q}{m}\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right).$$

  Note that: 1) unlike gravity this acceleration depends on the charge-to-mass ratio q/m, 2) this force is non-conservative in general and depends explicitly on the particle velocity.

Collisionless Boltzmann equation and its moments

We consider the one-particle distribution (phase-space density) $f(\mathbf{x}, \mathbf{v}, t)$, such that $f(\mathbf{x}, \mathbf{v}, t)\, d^3x\, d^3v$ is the number of stars in the phase-space volume $d^3x\, d^3v$. In the absence of collisional encounters (i.e. when the mean field dominates) the distribution function satisfies the *collisionless Boltzmann equation* (CBE), also known as the Vlasov equation. For particles moving under a potential $\Phi(\mathbf{x}, t)$ the equations of motion are $\dot{\mathbf{x}} = \mathbf{v}$ and $\dot{\mathbf{v}} = \mathbf{a} = -\nabla\Phi$. The CBE expresses conservation of phase-space density along characteristics, meaning that the expansion of the total (Lagrangian) derivative is zero:

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_x f + \mathbf{a} \cdot \nabla_v f = 0, \qquad \mathbf{a}(\mathbf{x}, t) = -\nabla\Phi(\mathbf{x}, t). \tag{1}$$

Definitions

Define the mass (or number) density and velocity moments

$$\rho(\mathbf{x}, t) = \int f(\mathbf{x}, \mathbf{v}, t)\, d^3v, \tag{2}$$

$$\rho(\mathbf{x}, t)\, \langle v_i \rangle(\mathbf{x}, t) = \int v_i\, f(\mathbf{x}, \mathbf{v}, t)\, d^3v, \tag{3}$$

$$\rho(\mathbf{x}, t)\, \langle v_i v_j \rangle(\mathbf{x}, t) = \int v_i v_j\, f(\mathbf{x}, \mathbf{v}, t)\, d^3v, \tag{4}$$

and the peculiar velocity (mean velocity) $u_i(\mathbf{x}, t) \equiv \langle v_i \rangle$. Introduce the velocity dispersion tensor

$$\sigma^2_{ij}(\mathbf{x}, t) \equiv \langle (v_i - u_i)(v_j - u_j) \rangle = \langle v_i v_j \rangle - u_i u_j. \tag{5}$$

Zeroth moment: continuity equation

Integrate the CBE over velocity space d^3v. The ∇_v term vanishes at infinity (assume $f \to 0$ fast enough), so

$$\begin{aligned} \int \frac{\partial f}{\partial t}\, d^3v + \int v_j \frac{\partial f}{\partial x_j}\, d^3v &= 0 \\ \frac{\partial}{\partial t} \int f\, d^3v + \frac{\partial}{\partial x_j} \int v_j f\, d^3v &= 0 \\ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) &= 0. \end{aligned} \tag{6}$$

Equation (6) is the continuity equation expressing local conservation of mass (or number).

First moment: Jeans equations

Multiply the CBE by v_i and integrate over velocities. We obtain three contributions:

$$\frac{\partial}{\partial t}\int v_i f\, d^3v + \frac{\partial}{\partial x_j}\int v_i v_j f\, d^3v + \int v_i\, a_k \frac{\partial f}{\partial v_k}\, d^3v = 0.$$

Evaluate the last term by integrating by parts in v_k. Using $a_k = -\partial_k \Phi$ (where $\partial_k \equiv \partial/\partial x_k$) and assuming surface terms vanish,

$$\begin{aligned}\int v_i a_k \frac{\partial f}{\partial v_k}\, d^3v = a_k \int v_i \frac{\partial f}{\partial v_k}\, d^3v &= a_k \Big[\underbrace{(v_i f)_{v_k=\infty}}_{=0} - \int \delta_{ik} f\, d^3v \Big] \\ &= -a_k\, \delta_{ik}\, \rho = -\rho\, a_i = +\rho\, \partial_i \Phi, \end{aligned} \tag{7}$$

so the integrated v_i-moment equation becomes

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho \langle v_i v_j \rangle)}{\partial x_j} + \rho\, \partial_i \Phi = 0. \tag{8}$$

Rearranging to put the gravitational force on the right-hand side,

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho \langle v_i v_j \rangle)}{\partial x_j} = -\, \rho\, \partial_i \Phi. \tag{9}$$

We can separate the mean-velocity contribution from the random motions: $\langle v_i v_j \rangle = u_i u_j + \sigma_{ij}^2$. Using the continuity equation to convert to convective derivatives one obtains the familiar *Jeans equations* in convective form:

$$\rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = -\rho\, \partial_i \Phi - \frac{\partial(\rho\, \sigma_{ij}^2)}{\partial x_j}. \tag{10}$$

This is the momentum equation for a collisionless stellar system: the gravitational acceleration is balanced by the divergence of the pressure-like tensor $\rho\sigma_{ij}^2$ and the inertia of mean streaming.

Second moment: evolution of the velocity dispersion (and the third moment)

Multiply the CBE by $v_i v_j$ and integrate over velocities to get the second moment evolution:

$$\frac{\partial(\rho\langle v_i v_j\rangle)}{\partial t} + \frac{\partial(\rho\langle v_i v_j v_k\rangle)}{\partial x_k} + \rho\left(u_i \partial_j \Phi + u_j \partial_i \Phi\right) = 0, \tag{11}$$

where the gravitational term was integrated by parts similarly to the first moment (one obtains terms proportional to $\rho\partial_i\Phi$ and $\rho\partial_j\Phi$). It is convenient to express the third moment in terms of mean and peculiar velocities:

$$\langle v_i v_j v_k\rangle = u_i u_j u_k + u_i \sigma_{jk}^2 + u_j \sigma_{ik}^2 + u_k \sigma_{ij}^2 + Q_{ijk},$$

where the third central moment (flux of velocity dispersion) is

$$Q_{ijk} \equiv \langle (v_i - u_i)(v_j - u_j)(v_k - u_k)\rangle.$$

Substituting this decomposition into (11) and doing algebra using the continuity and Jeans equations yields an evolution equation for the dispersion tensor. One commonly used form is

$$\rho\left(\frac{\partial\sigma_{ij}^2}{\partial t} + u_k \frac{\partial\sigma_{ij}^2}{\partial x_k} + \sigma_{ik}^2 \frac{\partial u_j}{\partial x_k} + \sigma_{jk}^2 \frac{\partial u_i}{\partial x_k}\right) + \frac{\partial(\rho Q_{ijk})}{\partial x_k} = -\rho\left(u_i \partial_j \Phi + u_j \partial_i \Phi\right) + (\text{terms cancelled by Jeans/continuity}). \tag{12}$$

Equation (12) shows that the evolution of the second moments (the "pressure tensor") is coupled to the third central moment Q_{ijk}. To close the hierarchy one must adopt an approximation (e.g. assume $Q_{ijk} = 0$ or prescribe a closure relation), analogous to hydrodynamic closures for moments of the Boltzmann equation.

Remarks

- The CBE (1) plus Poisson's equation for Φ (with the source given by ρ) form the collisionless Boltzmann–Poisson system used to describe self-gravitating stellar systems.
- The zeroth and first moments give mass conservation (6) and the Jeans equations (10). The second-moment equation (12) demonstrates the hierarchical structure: the n-th moment equation involves the $(n+1)$-th moment.
- In many practical applications one assumes steady-state ($\partial/\partial t = 0$), axisymmetry, or symmetry/anisotropy models for σ_{ij}^2 to simplify Jeans equations for modelling galaxies or clusters.

1 Jeans Equations: Spherical Systems

For a steady-state, spherically symmetric stellar system:

$$\frac{d(\rho\overline{v_r^2})}{dr} + \frac{2\rho}{r}(\overline{v_r^2} - \overline{v_\theta^2}) = -\rho\frac{d\Phi}{dr}$$

Define the anisotropy parameter:

$$\beta = 1 - \frac{\overline{v_\theta^2}}{\overline{v_r^2}}$$

Then the spherical Jeans equation becomes:

$$\frac{1}{\rho}\frac{d(\rho\overline{v_r^2})}{dr} + \frac{2\beta\overline{v_r^2}}{r} = -\frac{d\Phi}{dr}$$

2 Example: Isotropic Spherical System

For isotropy,

$$\beta = 0$$

:

$$\frac{d(\rho\overline{v_r^2})}{dr} = -\rho\frac{d\Phi}{dr}$$

If the system follows a singular isothermal sphere:

$$\rho(r) = \frac{\sigma^2}{2\pi G r^2}$$

Then the potential satisfies:

$$\frac{d\Phi}{dr} = \frac{2\sigma^2}{r}$$

and the rotation curve is flat:

$$v_c^2 = r\frac{d\Phi}{dr} = 2\sigma^2 = \text{const.}$$

3 Jeans Equations: Axisymmetric Disk

For an axisymmetric, steady-state disk (

$$\frac{\partial}{\partial t} = 0$$

,

$$\frac{\partial}{\partial \phi} = 0$$

): Radial Jeans equation:

$$\frac{\partial(\rho \overline{v_R^2})}{\partial R} + \frac{\partial(\rho \overline{v_R v_z})}{\partial z} + \rho \left(\frac{\overline{v_R^2} - \overline{v_\phi^2}}{R} \right) = -\rho \frac{\partial \Phi}{\partial R}$$

Vertical Jeans equation:

$$\frac{\partial(\rho \overline{v_z^2})}{\partial z} + \frac{\partial(\rho \overline{v_R v_z})}{\partial R} + \frac{\rho \overline{v_R v_z}}{R} = -\rho \frac{\partial \Phi}{\partial z}$$

4 Example: Thin Isothermal Disk

Assume negligible radial motions (

$$\overline{v_R v_z} \approx 0$$

) and constant vertical velocity dispersion (

$$\overline{v_z^2} = \sigma_z^2 = \text{const.}$$

): Then the vertical equilibrium becomes:

$$\frac{d(\rho \sigma_z^2)}{dz} = -\rho \frac{d\Phi}{dz}$$

Combine with Poisson's equation for a self-gravitating disk:

$$\frac{d^2\Phi}{dz^2} = 4\pi G \rho$$

Solution (Spitzer 1942):

$$\rho(z) = \rho_0 \,\mathrm{sech}^2\left(\frac{z}{z_0}\right)$$

with:

$$z_0 = \frac{\sigma_z}{\sqrt{2\pi G \rho_0}}$$