1 Quantitative Model

We introduce here the full quantitative model. There are a finite number of perfectly competitive sectors indexed by $j = 1, ..., N$. A representative household consumes goods and supplies labor to firms in each sector. Time is discrete and infinite.

1.1 Households

The representative household has the following preferences over consumption of each good j, which we denote C_{jt}, and labor on industry j, which we denote L_{jt}:

$$U = \sum_{t=0}^{\infty} \beta^t \left[\frac{1}{1-\epsilon_c^{-1}} \left(C_t - \theta \frac{L_t^{1+\epsilon_l^{-1}}}{1+\epsilon_l^{-1}} \right)^{1-\epsilon_c^{-1}} \right] \quad \text{where}$$

$$C_t = \left(\sum_{j=1}^{N} \xi_j^{\frac{1}{\sigma_c}} (C_{jt})^{1-\sigma_c^{-1}} \right)^{\frac{1}{1-\sigma_c^{-1}}}, \quad \sum_{j=1}^{N} \xi_j = 1 \text{ and } L_t = \left(\sum_{j=1}^{N} (L_{jt})^{1+\sigma_l^{-1}} \right)^{\frac{1}{1+\sigma_l^{-1}}}$$

where β is the discount factor, ϵ_c is the intertemporal elasticity of substitution (or the inverse of the relative risk aversion), ϵ_l is the Frisch elasticity of labor, ξ_j captures the time-invariant preference for good j, σ_c is the elasticity of substitution across goods, σ_l controls the degree of labor reallocation between sectors, and θ is a normalization constant.

1.2 Firms

The representative firm in sector j produces gross output Q_{jt} using capital K_{jt}, labor L_{jt}, and intermediate inputs M_{jt}.[1] The production function is:

$$Q_{jt} = \left[(\mu_j)^{\sigma_q^{-1}} (Y_{jt})^{1-\sigma_q^{-1}} + (1-\mu_j)^{\sigma_q^{-1}} (M_{jt})^{1-\sigma_q^{-1}} \right]^{\frac{1}{1-\sigma_q^{-1}}}, \text{ where}$$

$$Y_{jt} = A_{jt} \left[(\alpha_j)^{\sigma_y^{-1}} (K_{jt})^{1-\sigma_y^{-1}} + (1-\alpha_j)^{\sigma_y^{-1}} (L_{jt})^{1-\sigma_y^{-1}} \right]^{\frac{1}{1-\sigma_y^{-1}}}.$$

Variable Y_{jt} denotes value-added production, α_j captures the share of capital in value-added, μ_j parametrizes the share of materials in gross output, σ_q is the elasticity of substitution between primary outputs (e.g. capital and labor) and materials, and A_{jt} is an industry-specific shock to value added productivity that follows the process

$$\log A_{jt+1} = \rho_j \log A_{jt} + \varepsilon_{jt+1}^A$$

where ρ_j represents industry-specific persistence and the shocks ε_{jt+1}^A are distributed multivariate normal with mean 0 and variance-covariance matrix Σ^A. The industry-specific productivity shocks may be correlated, that is, the variance-covariance matrix may not be diagonal.

[1] This variable is also labeled as material in some papers.

Firms can accumulate capital by producing an industry-specific investment good I_{jt} facing capital adjustment costs denoted by Φ_{jt}:

$$K_{jt+1} = (1-\delta_j)K_{jt} + I_{jt} - \Phi_{jt},$$
$$\Phi_{jt} = \frac{\phi}{2}\left(\frac{I_{jt}}{K_{jt}} - \delta_j\right)^2 K_{jt}$$

where δ_j is the industry-specific depreciation rate, and ϕ parametrize the adjustment cost function.

The investment good is produced by bundling goods produced by other industries:

$$I_{jt} = \left(\sum_{i=1}^{N} \left(\gamma_{ij}^{I}\right)^{\sigma_I^{-1}} \left(I_{ijt}\right)^{1-\sigma_I^{-1}}\right)^{\frac{1}{1-\sigma_I^{-1}}}, \quad \text{where} \quad \sum_{i=1}^{N} \gamma_{ij}^{I} = 1$$

where γ_{ij}^{I} represents the importance of good i in the production of the investment good for sector j, and σ_I is the elasticity of substitution between inputs of the investment bundle. In the same vein, the intermediate input is produced using the following bundle:

$$M_{jt} = \left(\sum_{i=1}^{N} \left(\gamma_{ij}^{m}\right)^{\sigma_m^{-1}} \left(M_{ijt}\right)^{1-\sigma_m^{-1}}\right)^{\frac{1}{1-\sigma_m^{-1}}}, \quad \text{where} \quad \sum_{i=1}^{N} \gamma_{ij}^{m} = 1.$$

Parameters γ_{ij}^{m} and σ_m are analogous to the parameters γ_{ij}^{I} and σ_I discussed for the investment bundle.

1.3 Market Clearing and the Planner's First Order Conditions

The market clearing conditions for each good is:

$$Q_{jt} = C_{jt} + \sum_{i=1}^{N} \left(M_{jit} + I_{jit}\right),$$

which implies that gross output equals final consumption, intermediate inputs, and investment goods.

In order to obtain the first order conditions, we will use the fact that the model satisfies the first welfare theorem, so we can formulate the problem as a planning problem. The planner's Lagragian is given by:

$$\begin{aligned}
\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ & \frac{1}{1-\epsilon_c^{-1}} \left(C_t - \theta \frac{L_t^{1+\epsilon_l^{-1}}}{1+\epsilon_l^{-1}}\right)^{1-\epsilon_c^{-1}} \\
& + \sum_{j=1}^{N} P_{jt}^{\mathrm{k}} \left[I_{jt} + (1-\delta_j)K_{jt} - \Phi_{jt} - K_{jt+1}\right] \\
& + \sum_{j=1}^{N} P_{jt} \left[Q_{jt} - C_{jt} - \sum_{i=1}^{N} \left[M_{jit} + I_{jit}\right]\right] \Bigg\}
\end{aligned}$$

where P_{jt}^{k} is the Lagrange multiplier associated to the capital accumulation constraint, P_{jt} is the Lagrange multiplier associated to the market clearing condition.

In Appendix 2, we provide a detailed derivation of all first-order conditions and the resulting system of equations. We also calculate welfare, the deterministic steady state, and the closed form solution for the expenditure shares, to be used for the calibration.

2 First-order conditions and auxiliary results regarding the full model

In this appendix, we derive the first-order conditions of the full model. We also provide some additional results

The Lagrangian of the planner is:

$$
\begin{aligned}
\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ & \frac{1}{1-\epsilon_c^{-1}} \left(C_t - \theta \frac{L_t^{1+\epsilon_l^{-1}}}{1+\epsilon_l^{-1}} \right)^{1-\epsilon_c^{-1}} \\
& + \sum_{j=1}^{N} P_{jt}^{\mathrm{k}} \left[I_{jt} + (1-\delta_j) K_{jt} - \Phi_{jt} - K_{jt+1} \right] \\
& + \sum_{j=1}^{N} P_{jt} \left[Q_{jt} - C_{jt} - \sum_{i=1}^{N} \left[M_{jit} + I_{jit} \right] \right] \Bigg\},
\end{aligned}
$$

where

$$
\begin{aligned}
C_t &= \left(\sum_{j=1}^{N} \xi_j^{\sigma_c^{-1}} (C_{jt})^{1-\sigma_c^{-1}} \right)^{\frac{1}{1-\sigma_c^{-1}}}, \\
L_t &= \left(\sum_{j=1}^{N} (L_{jt})^{1+\sigma_l^{-1}} \right)^{\frac{1}{1+\sigma_l^{-1}}}, \\
Q_{jt} &= \left[(\mu_j)^{\sigma_q^{-1}} (Y_{jt})^{1-\sigma_q^{-1}} + (1-\mu_j)^{\sigma_q^{-1}} (M_{jt})^{1-\sigma_q^{-1}} \right]^{\frac{1}{1-\sigma_q^{-1}}}, \\
Y_{jt} &= A_{jt} \left[(\alpha_j)^{\sigma_q^{-1}} (K_{jt})^{1-\sigma_y^{-1}} + (1-\alpha_j)^{\sigma_y^{-1}} (L_{jt})^{1-\sigma_y^{-1}} \right]^{\frac{1}{1-\sigma_y^{-1}}}, \\
I_{jt} &= \left(\sum_{i=1}^{N} \left(\gamma_{ij}^{I} \right)^{\sigma_I^{-1}} (I_{ijt})^{1-\sigma_I^{-1}} \right)^{\frac{1}{1-\sigma_I^{-1}}}, \\
M_{jt} &= \left(\sum_{i=1}^{N} \left(\gamma_{ij}^{m} \right)^{\sigma_m^{-1}} (M_{ijt})^{1-\sigma_m^{-1}} \right)^{\frac{1}{1-\sigma_m^{-1}}}, \\
\Phi_{jt} &= \frac{\phi}{2} \left(\frac{I_{jt}}{K_{jt}} - \delta_j \right)^2 K_{jt}.
\end{aligned}
$$

We will start by writing down all the derivatives of the CES aggregators and the adjustment cost function. See sub-appendix 2.11 to see the details of the CES algebra involved:

$$\frac{\partial C_t}{\partial C_{jt}} = \left(\xi_j \frac{C_t}{C_{jt}}\right)^{\sigma_c^{-1}} \qquad \frac{\partial L_t}{\partial L_{jt}} = \left(\frac{L_{jt}}{L_t}\right)^{\sigma_l^{-1}}$$

$$\frac{\partial Q_{jt}}{\partial Y_{jt}} = \left(\mu_j \frac{Q_{jt}}{Y_{jt}}\right)^{\sigma_q^{-1}} \qquad \frac{\partial Q_{jt}}{\partial M_{jt}} = \left((1-\mu_j)\frac{Q_{jt}}{M_{jt}}\right)^{\sigma_q^{-1}}$$

$$\frac{\partial Y_{jt}}{\partial K_{jt}} = A_{jt}^{1-\sigma_y^{-1}}\left(\alpha_j \frac{Y_{jt}}{K_{jt}}\right)^{\sigma_y^{-1}} \qquad \frac{\partial Y_{jt}}{\partial L_{jt}} = A_{jt}^{1-\sigma_y^{-1}}\left((1-\alpha_j)\frac{Y_{jt}}{L_{jt}}\right)^{\sigma_y^{-1}}$$

$$\frac{\partial M_{jt}}{\partial M_{ijt}} = \left(\gamma_{ij}^M \frac{M_{jt}}{M_{ijt}}\right)^{\sigma_m^{-1}} \qquad \frac{\partial I_{jt}}{\partial I_{ijt}} = \left(\gamma_{ij}^I \frac{I_{jt}}{I_{ijt}}\right)^{\sigma_I^{-1}}$$

$$\frac{\partial \Phi_{jt}}{\partial I_{jt}} = \phi\left(\frac{I_{jt}}{K_{jt}} - \delta_j\right) \qquad \frac{\partial \Phi_{jt}}{\partial K_{jt}} = -\frac{\phi}{2}\left(\frac{I_{jt}^2}{K_{jt}^2} - \delta_j^2\right).$$

2.1 FOC for consumption

We start with

$$\frac{\partial \mathcal{L}}{\partial C_{jt}} = \beta^t \left(\left(C_t - \theta \frac{L_t^{1+\epsilon_l^{-1}}}{1+\epsilon_l^{-1}}\right)^{-\epsilon_c^{-1}} \frac{\partial C_t}{\partial C_{jt}} - P_{jt}\right) = 0.$$

Next, we replace the derivatives:

$$\left(C_t - \theta \frac{L_t^{1+\epsilon_l^{-1}}}{1+\epsilon_l^{-1}}\right)^{-\epsilon_c^{-1}} \frac{\partial C_t}{\partial C_{jt}} = P_{jt},$$

$$\left(C_t - \theta \frac{L_t^{1+\epsilon_l^{-1}}}{1+\epsilon_l^{-1}}\right)^{-\epsilon_c^{-1}} \left(\xi_j \frac{C_t}{C_{jt}}\right)^{\sigma_c^{-1}} = P_{jt}.$$

2.2 FOC for labor

We start with

$$\frac{\partial \mathcal{L}}{\partial L_{jt}} = \beta^t \left(-\left(C_t - \theta \frac{L_t^{1+\epsilon_l^{-1}}}{1+\epsilon_l^{-1}}\right)^{-\epsilon_c^{-1}} \theta \left(L_t\right)^{\epsilon_l^{-1}} \frac{\partial L_t}{\partial L_{jt}} + P_{jt}\frac{\partial Q_{jt}}{\partial L_{jt}}\right) = 0.$$

Next, we replace the derivatives, using the chain rule when needed:

$$\left(C_t - \theta \frac{L_t^{1+\epsilon_l^{-1}}}{1+\epsilon_l^{-1}}\right)^{-\epsilon_c^{-1}} \theta \left(L_t\right)^{\epsilon_l^{-1}} \frac{\partial L_t}{\partial L_{jt}} = P_{jt}\frac{\partial Q_{jt}}{\partial Y_{jt}}\frac{\partial Y_{jt}}{\partial L_{jt}},$$

$$\left(C_t - \theta \frac{L_t^{1+\epsilon_l^{-1}}}{1+\epsilon_l^{-1}}\right)^{-\epsilon_c^{-1}} \theta \left(L_t\right)^{\epsilon_l^{-1}} \left(\frac{L_{jt}}{L_t}\right)^{\sigma_l^{-1}} = P_{jt} A_{jt}^{1-\sigma_y^{-1}} \left(\mu_j \frac{Q_{jt}}{Y_{jt}}\right)^{\sigma_q^{-1}} \left((1-\alpha_j)\frac{Y_{jt}}{L_{jt}}\right)^{\sigma_y^{-1}}.$$

2.3 FOC with respect to capital in next period

We start with

$$\frac{\partial \mathcal{L}}{\partial K_{jt+1}} = \beta^t \left(-P^k_{jt}\right) + \beta^{t+1}\mathbb{E}_t \left(P^k_{jt+1} \left((1-\delta_j) - \frac{\partial \Phi_{jt+1}}{\partial K_{jt+1}} \right) + P_{jt+1} \frac{\partial Q_{jt+1}}{\partial K_{jt+1}} \right) = 0.$$

Next, we replace the derivatives, using the chain rule when needed:

$$\begin{aligned}
P^k_{jt} &= \beta \mathbb{E}_t \left(P^k_{jt+1} \left((1-\delta_j) - \frac{\partial \Phi_{jt+1}}{\partial K_{jt+1}} \right) + P_{jt+1} \frac{\partial Q_{jt+1}}{\partial Y_{jt+1}} \frac{\partial Y_{jt+1}}{\partial K_{jt+1}} \right), \\
P^k_{jt} &= \beta \mathbb{E}_t \left[P^k_{jt+1} \left((1-\delta_j) + \frac{\phi}{2} \left(\frac{I^2_{jt+1}}{K^2_{jt+1}} - \delta^2_j \right) \right) \right. \\
&\quad \left. + P_{jt+1} A^{1-\sigma_y^{-1}}_{jt+1} \left(\mu_j \frac{Q_{jt+1}}{Y_{jt+1}} \right)^{\sigma_q^{-1}} \left(\alpha_j \frac{Y_{jt+1}}{K_{jt+1}} \right)^{\sigma_y^{-1}} \right].
\end{aligned}$$

2.4 FOC for intermediates and system reduction

We start with

$$\frac{\partial \mathcal{L}}{\partial M_{ijt}} = \beta^t \left(P_{jt} \frac{\partial Q_{jt}}{\partial M_{ijt}} - P_{it} \right) = 0.$$

Next, we replace the derivatives, using the chain rule when needed:

$$\begin{aligned}
P_{jt} \frac{\partial Q_{jt}}{\partial M_{jt}} \frac{\partial M_{jt}}{\partial M_{ijt}} &= P_{it}, \\
P_{jt} \left((1-\mu_j) \frac{Q_{jt}}{M_{jt}} \right)^{\sigma_q^{-1}} \left(\gamma^M_{ij} \frac{M_{jt}}{M_{ijt}} \right)^{\sigma_m^{-1}} &= P_{it}
\end{aligned}$$

We want to use this FOC to get rid of M_{ijt}. First, we solve for M_{ijt}:

$$M_{ijt} = \left(\frac{P_{jt}}{P_{it}} \right)^{\sigma_m} \left((1-\mu_j) \frac{Q_{jt}}{M_{jt}} \right)^{\frac{\sigma_m}{\sigma_q}} \gamma^m_{ij} M_{jt}$$

We solve for M_{jt} by aggregating from the solution for M_{ijt}. First, we construct the term $\left(\gamma^m_{ij}\right)^{\frac{1}{\sigma_m}} M^{1-\sigma_m^{-1}}_{ijt}$ that is inside the aggregator:

$$\begin{aligned}
M_{ijt} &= \left(\frac{P_{jt}}{P_{it}} \right)^{\sigma_m} \left((1-\mu_j) \frac{Q_{jt}}{M_{jt}} \right)^{\frac{\sigma_m}{\sigma_q}} \gamma^m_{ij} M_{jt}, \\
\left(\gamma^m_{ij}\right)^{\frac{1}{\sigma_m}} M^{1-\sigma_m^{-1}}_{ijt} &= \left(\gamma^m_{ij}\right)^{\frac{1}{\sigma_m}} \left(\frac{P_{it}}{P_{jt}} \right)^{1-\sigma_m} \left((1-\mu_j) \frac{Q_{jt}}{M_{jt}} \right)^{\frac{\sigma_m - 1}{\sigma_q}} \left(\gamma^m_{ij} M_{jt} \right)^{1-\sigma_m^{-1}}, \\
\left(\gamma^m_{ij}\right)^{\frac{1}{\sigma_m}} M^{1-\sigma_m^{-1}}_{ijt} &= \gamma^m_{ij} \left(\frac{P_{it}}{P_{jt}} \right)^{1-\sigma_m} \left((1-\mu_j) \frac{Q_{jt}}{M_{jt}} \right)^{\frac{\sigma_m - 1}{\sigma_q}} \left(M_{jt} \right)^{1-\sigma_m^{-1}}.
\end{aligned}$$

Next, we sum over all the goods in the aggregator:

$$
\begin{aligned}
\sum_{i=1}^{N} \left(\gamma_{ij}^{m}\right)^{\frac{1}{\sigma_m}} M_{ijt}^{1-\sigma_m^{-1}} &= \sum_{i=1}^{N} \gamma_{ij}^{m} \left(\frac{P_{it}}{P_{jt}}\right)^{1-\sigma_m} \left(\left(1-\mu_j\right)\frac{Q_{jt}}{M_{jt}}\right)^{\frac{\sigma_m-1}{\sigma_q}} M_{jt}^{1-\sigma_m^{-1}}, \\
M_{jt}^{1-\sigma_m^{-1}} &= \left(P_{jt}\right)^{\sigma_m-1} \left(\left(1-\mu_j\right)\frac{Q_{jt}}{M_{jt}}\right)^{\frac{\sigma_m-1}{\sigma_q}} M_{jt}^{1-\sigma_m^{-1}} \sum_{i=1}^{N} \gamma_{ij}^{m} \left(P_{it}\right)^{1-\sigma_m}, \\
M_{jt}^{\frac{\sigma_m-1}{\sigma_q}} &= \left(\left(1-\mu_j\right) Q_{jt}\right)^{\frac{\sigma_m-1}{\sigma_q}} \left(P_{jt}\right)^{\sigma_m-1} \sum_{i=1}^{N} \left(\gamma_{ij}^{m}\right) \left(P_{it}\right)^{1-\sigma_m}, \\
M_{jt} &= \left(\left(1-\mu_j\right) Q_{jt}\right) P_{jt}^{\sigma_q} \left(\sum_{i=1}^{N} \left(\gamma_{ij}^{m}\right) \left(P_{it}\right)^{1-\sigma_m}\right)^{\frac{\sigma_q}{\sigma_m-1}}.
\end{aligned}
$$

Following the CES algebra in sub-appendix 2.11, we define the price index for the M_{jt} bundle as:

$$
P_{jt}^{m} = \left(\sum_{i=1}^{N} \left(\gamma_{ij}^{m}\right) \left(P_{it}\right)^{1-\sigma_m}\right)^{\frac{1}{1-\sigma_m}}.
$$

so we can write the FOC for M_{jt} as:

$$
M_{jt} = \left(1-\mu_j\right) \left(\frac{P_{jt}^{m}}{P_{jt}}\right)^{-\sigma_q} Q_{jt}.
$$

We can use this expression for M_{jt} to simplify our solution for M_{ijt}:

$$
\begin{aligned}
M_{ijt} &= \left(\frac{P_{it}}{P_{jt}}\right)^{-\sigma_m} \left(\left(1-\mu_j\right)\frac{Q_{jt}}{M_{jt}}\right)^{\frac{\sigma_m}{\sigma_q}} \gamma_{ij}^{m} M_{jt}, \\
&= \left(\frac{P_{it}}{P_{jt}}\right)^{-\sigma_m} \left(\left(1-\mu_j\right)\frac{Q_{jt}}{\left(1-\mu_j\right)\left(\frac{P_{jt}^{m}}{P_{jt}}\right)^{-\sigma_q} Q_{jt}}\right)^{\frac{\sigma_m}{\sigma_q}} \gamma_{ij}^{m} M_{jt}, \\
&= \gamma_{ij}^{m} \left(\frac{P_{it}}{P_{jt}^{m}}\right)^{-\sigma_m} M_{jt}.
\end{aligned}
$$

Finally, using this solution for M_{ijt}, we can calculate the supply of intermediate goods of each sector, which we denote $M_{jt}^{out} = \sum_{i=1}^{N} M_{jit}$:

$$
M_{jt}^{out} = \sum_{i=1}^{N} M_{jit} = \sum_{i=1}^{N} \gamma_{ji}^{m} \left(\frac{P_{jt}}{P_{it}^{m}}\right)^{-\sigma_m} M_{it}.
$$

2.5 FOC for investment and system reduction

We start with

$$
\frac{\partial \mathcal{L}}{\partial I_{ijt}} = \beta^{t} \left(P_{jt}^{k}\left(\frac{\partial I_{jt}}{\partial I_{ijt}} - \frac{\partial \Phi_{jt}}{\partial I_{ijt}}\right) - P_{it}\right) = 0.
$$

Next, we replace the derivatives, using the chain rule when needed:

$$
\begin{aligned}
P^k_{jt}\left(\frac{\partial I_{jt}}{\partial I_{ijt}} - \frac{\partial \Phi_{jt}}{\partial I_{jt}}\frac{\partial I_{jt}}{\partial I_{ijt}}\right) &= P_{it}, \\
P^k_{jt}\frac{\partial I_{jt}}{\partial I_{ijt}}\left(1 - \frac{\partial \Phi_{jt}}{\partial I_{jt}}\right) &= P_{it}, \\
P^k_{jt}\left(\gamma^I_{ij}\frac{I_{jt}}{I_{ijt}}\right)^{\sigma_I^{-1}}\left(1 - \phi\left(\frac{I_{jt}}{K_{jt}} - \delta_j\right)\right) &= P_{it}.
\end{aligned}
$$

We want to use this FOC to eliminate I_{ijt}. First, we solve for I_{ijt}:

$$
I_{ijt} = \gamma^I_{ij}\left(\frac{P_{it}}{P^k_{jt}}\right)^{-\sigma_I} I_{jt}\left(1 - \phi\left(\frac{I_{jt}}{K_{jt}} - \delta_j\right)\right)^{\sigma_I}.
$$

We solve for I_{jt} by aggregating up from the solution for I_{ijt}

$$
\begin{aligned}
I_{ijt} &= \gamma^I_{ij}\left(\frac{P_{it}}{P^k_{jt}}\right)^{-\sigma_I} I_{jt}\left(1 - \phi\left(\frac{I_{jt}}{K_{jt}} - \delta_j\right)\right)^{\sigma_I}, \\
\left(\gamma^I_{ij}\right)^{\sigma_I^{-1}} I_{ijt}^{1-\sigma_I^{-1}} &= \left(\gamma^I_{ij}\right)^{\sigma_I^{-1}}\left(\frac{P_{it}}{P^k_{jt}}\right)^{1-\sigma_I}\left(\gamma^I_{ij} I_{jt}\right)^{1-\sigma_I^{-1}}\left(1 - \phi\left(\frac{I_{jt}}{K_{jt}} - \delta_j\right)\right)^{\sigma_I - 1}, \\
I_{jt}^{1-\sigma_I^{-1}} &= \sum_{i=1}^{N}\left(\gamma^I_{ij}\right)\left(\frac{P_{it}}{P^k_{jt}}\right)^{1-\sigma_I}\left(I_{jt}\right)^{1-\sigma_I^{-1}}\left(1 - \phi\left(\frac{I_{jt}}{K_{jt}} - \delta_j\right)\right)^{\sigma_I - 1}. \\
1 &= \left(P^k_{jt}\right)^{\sigma_I - 1}\left(1 - \phi\left(\frac{I_{jt}}{K_{jt}} - \delta_j\right)\right)^{\sigma_I - 1}\sum_{i=1}^{N}\left(\gamma^I_{ij}\right)\left(P_{it}\right)^{1-\sigma_I}, \\
P^k_{jt} &= \left(1 - \phi\left(\frac{I_{jt}}{K_{jt}} - \delta_j\right)\right)^{-1}\left(\sum_{i=1}^{N}\left(\gamma^I_{ij}\right)\left(P_{it}\right)^{1-\sigma_I}\right)^{\frac{1}{1-\sigma_I}}.
\end{aligned}
$$

We define the frictionless price index of capital goods as

$$
\tilde{P}^k_{jt} = \left(\sum_{i=1}^{N}\left(\gamma^I_{ij}\right)\left(P_{it}\right)^{1-\sigma_I}\right)^{\frac{1}{1-\sigma_I}}.
$$

Then, we can write the FOC for I_{jt} as

$$
P^k_{jt} = \tilde{P}^k_{jt}\left(1 - \phi\left(\frac{I_{jt}}{K_{jt}} - \delta_j\right)\right)^{-1}.
$$

Next, we calculate the amount of goods of a sector that goes to other sectors as investment goods. We define

$$
I^{out}_{jt} = \sum_{i=1}^{N} I_{jit}.
$$

Using the FOC for I_{ijt}, we have

$$I_{jt}^{out} = \sum_{i=1}^{N} \gamma_{ji}^{I} \left(\frac{P_{jt}}{P_{it}^{k}} \right)^{-\sigma_I} I_{it} \left(1 - \phi \left(\frac{I_{it}}{K_{it}} - \delta_i \right) \right)^{\sigma_I} .$$

2.6 Full system of equations

The full system we get is:

$$
\begin{aligned}
\log A_{jt+1} &= \rho_j \log A_{jt} + \varepsilon^A_{jt+1}, \\
K_{jt+1} &= (1-\delta_j)K_{jt} + I_{jt} - \frac{\phi}{2}\left(\frac{I_{jt}}{K_{jt}} - \delta_j\right)^2 K_{jt}, \\
P_{jt} &= \left(C_t - \theta \frac{L_t^{1+\epsilon_l^{-1}}}{1+\epsilon_l^{-1}}\right)^{-\epsilon_c^{-1}} \left(\xi_j \frac{C_t}{C_{jt}}\right)^{\sigma_c^{-1}}, \\
\frac{\theta \left(L_t\right)^{\epsilon_l^{-1}} \left(\frac{L_{jt}}{L_t}\right)^{\sigma_l^{-1}}}{\left(C_t - \theta \frac{L_t^{1+\epsilon_l^{-1}}}{1+\epsilon_l^{-1}}\right)^{\epsilon_c^{-1}}} &= P_{jt} A_{jt}^{1-\sigma_y^{-1}} \left(\mu_j \frac{Q_{jt}}{Y_{jt}}\right)^{\sigma_q^{-1}} \left((1-\alpha_j)\frac{Y_{jt}}{L_{jt}}\right)^{\sigma_y^{-1}}, \\
P^k_{jt} &= \beta \mathbb{E}_t \Bigg[P_{jt+1} A_{jt+1}^{1-\sigma_y^{-1}} \left(\mu_j \frac{Q_{jt+1}}{Y_{jt+1}}\right)^{\sigma_q^{-1}} \left(\alpha_j \frac{Y_{jt+1}}{K_{jt+1}}\right)^{\sigma_y^{-1}} \Bigg], \\
&\quad + P^k_{jt+1}\left((1-\delta_j) + \frac{\phi}{2}\left(\frac{I^2_{jt+1}}{K^2_{jt+1}} - \delta_j^2\right)\right)\Bigg], \\
P^m_{jt} &= \left(\sum_{i=1}^N \left(\gamma^m_{ij}\right)\left(P_{it}\right)^{1-\sigma_m}\right)^{\frac{1}{1-\sigma_m}}, \\
M_{jt} &= (1-\mu_j)\left(\frac{P^m_{jt}}{P_{jt}}\right)^{-\sigma_q} Q_{jt}, \\
M^{out}_{jt} &= \sum_{i=1}^N \gamma^m_{ji}\left(\frac{P_{jt}}{P^m_{it}}\right)^{-\sigma_m} M_{it}, \\
P^k_{jt} &= \left(\sum_{i=1}^N \left(\gamma^I_{ij}\right)\left(P_{it}\right)^{1-\sigma_I}\right)^{\frac{1}{1-\sigma_I}} \left(1 - \phi\left(\frac{I_{jt}}{K_{jt}} - \delta_j\right)\right)^{-1}, \\
I^{out}_{jt} &= \sum_{i=1}^N \gamma^I_{ji}\left(\frac{P_{jt}}{P^k_{it}}\right)^{-\sigma_I} I_{it}\left(1 - \phi\left(\frac{I_{it}}{K_{it}} - \delta_i\right)\right)^{\sigma_I}, \\
Q_{jt} &= C_{jt} + M^{out}_{jt} + I^{out}_{jt}, \\
Q_{jt} &= \left[(\mu_j)^{\sigma_q^{-1}}\left(Y_{jt}\right)^{1-\sigma_q^{-1}} + (1-\mu_j)^{\sigma_q^{-1}}\left(M_{jt}\right)^{1-\sigma_q^{-1}}\right]^{\frac{1}{1-\sigma_q^{-1}}}, \\
Y_{jt} &= A_{jt}\left[(\alpha_j)^{\sigma_y^{-1}}\left(K_{jt}\right)^{1-\sigma_y^{-1}} + (1-\alpha_j)^{\sigma_y^{-1}}\left(L_{jt}\right)^{1-\sigma_y^{-1}}\right]^{\frac{1}{1-\sigma_y^{-1}}}, \\
C_t &= \left(\sum_{j=1}^N \xi_j^{\frac{1}{\sigma_c}}\left(C_{jt}\right)^{1-\sigma_c^{-1}}\right)^{\frac{1}{1-\sigma_c^{-1}}}, \\
L_t &= \left(\sum_{j=1}^N (L_{jt})^{1+\sigma_l^{-1}}\right)^{\frac{1}{1+\sigma_l^{-1}}}.
\end{aligned}
$$

2.7 Welfare

In order to calculate welfare, we can write the intertemporal utility of the representative household as:

$$V_t = \frac{1}{1-\epsilon_c^{-1}} \left(C_t - \theta \frac{L_t^{1+\epsilon_l^{-1}}}{1+\epsilon_l^{-1}} \right)^{1-\epsilon_c^{-1}} + \beta E_t V_{t+1}.$$

Steady state welfare is:

$$\bar{V} = \frac{1}{1-\beta} \frac{1}{1-\epsilon_c^{-1}} \left(\bar{C} - \theta \frac{\bar{L}^{1+\epsilon_l^{-1}}}{1+\epsilon_l^{-1}} \right)^{1-\epsilon_c^{-1}}.$$

Then, in a given period, we can get an interpretable measure of welfare by calculating the fraction of steady-state consumption that you would need to give up to achieve that level of welfare in the steady state. We denote such consumption-equivalent welfare as $\hat{V}_t^c$:

$$V_t = \frac{1}{1-\beta} \frac{1}{1-\epsilon_c^{-1}} \left(\bar{C}(1+\hat{V}_t^c) - \theta \frac{\bar{L}^{1+\epsilon_l^{-1}}}{1+\epsilon_l^{-1}} \right)^{1-\epsilon_c^{-1}}.$$

We can solve analytically for consumption-equivalent welfare $\hat{V}_t^c$:

$$\hat{V}_t^c = \frac{1}{\bar{C}} \left[\left(V_t(1-\beta)(1-\epsilon_c^{-1}) \right)^{\frac{1}{1-\epsilon_c^{-1}}} + \theta \frac{\bar{L}^{1+\epsilon_l^{-1}}}{1+\epsilon_l^{-1}} \right] - 1.$$

We will analyze how $\hat{V}_t^c$ is affected by productivity shocks.

2.8 Vectorizations

In terms of programming, it will be useful to vectorize the equations that involve sums over sectoral variables. We will use variables in bold to denote vectors where each element represents the corresponding sectoral value, and use $*$ to denote element by element multiplication. When we raise a vector to the power of a parameter, we mean element-to-element exponentiation. Then, we get the following equations:

$$\begin{aligned}
\boldsymbol{P_t^m} &= \left(\Gamma_M^{'} \boldsymbol{P_t}^{1-\sigma_m} \right)^{\frac{1}{1-\sigma_m}}, \\
\boldsymbol{M_t^{out}} &= \boldsymbol{P_t}^{-\sigma_m} * \Gamma_M \left(\boldsymbol{P_t^m} \right)^{\sigma_m} * \boldsymbol{M_t} \\
\tilde{\boldsymbol{P}}_{\boldsymbol{t}}^{\boldsymbol{k}} &= \left(\Gamma_I^{'} \boldsymbol{P_t}^{1-\sigma_I} \right)^{\frac{1}{1-\sigma_I}}, \\
\tilde{\boldsymbol{I}}_{\boldsymbol{t}}^{\boldsymbol{out}} &= \boldsymbol{P_t}^{-\sigma_I} * \Gamma_I \left(\boldsymbol{P_t^k} \right)^{\sigma_I} * \boldsymbol{I_t}.
\end{aligned}$$

2.9 Steady state

There are three dynamic equations in the model:

$$
\begin{aligned}
K_{jt+1} &= (1-\delta_j)K_{jt} + I_{jt} - \frac{\phi}{2}\left(\frac{I_{jt}}{K_{jt}} - \delta_j\right)^2 K_{jt}, \\
a_{jt+1} &= \rho_j a_{jt} + \epsilon_{jt}, \\
P^k_{jt} &= \beta \mathbb{E}_t \Bigg[P_{jt+1} A_{jt+1}^{1-\sigma_y^{-1}} \left(\mu_j \frac{Q_{jt+1}}{Y_{jt+1}}\right)^{\sigma_q^{-1}} \left(\alpha_j \frac{Y_{jt+1}}{K_{jt+1}}\right)^{\sigma_y^{-1}} \\
&\quad + P^k_{jt+1}\left((1-\delta_j) + \frac{\phi}{2}\left(\frac{I^2_{jt+1}}{K^2_{jt+1}} - \delta_j^2\right)\right)\Bigg].
\end{aligned}
$$

First, notice that $I_{jt} = \delta_j K_{jt}$ implies $K_{jt+1} = K_{jt}$ and makes the adjustment costs equal to zero. Then, in the steady state we have:

$$
\begin{aligned}
\bar{K} &= \bar{I}_j/\delta_j, \\
\bar{P}^k_j &= \frac{\beta}{1-\beta(1-\delta)} \bar{P}_j \left(\mu_j \frac{\bar{Q}_j}{\bar{Y}_j}\right)^{\sigma_q^{-1}} \left(\alpha \frac{\bar{Y}_j}{\bar{K}_j}\right)^{\sigma_y^{-1}}.
\end{aligned}
$$

2.10 Intensity shares mapping to expenditure shares

In sub-appendix 2.11, we show that for standard CES aggregators, the intensity shares (e.g., ξ_j for consumption bundle or α_j for value-added function) do not correspond to expenditure shares. In the same sub-appendix, we show that how to map expenditures shares to intensity shares once you know the steady state equilibrium variables. We use tilde notation to refer to expenditures, so $\tilde{\xi}_j$ is the consumption share of good j. Then, for a given steady state equilibrium the relation between the expenditure share in steady state and the intensity share used to calculate the steady state is:

$$
\begin{aligned}
\tilde{\xi}_j &= \xi_j^{\sigma_c^{-1}} \left(\frac{\bar{C}_j}{\bar{C}}\right)^{1-\sigma_c^{-1}}, \\
\tilde{\mu}_j &= \mu_j^{\sigma_q^{-1}} \left(\frac{\bar{Y}_j}{\bar{Q}_j}\right)^{1-\sigma_q^{-1}}, \\
\tilde{\alpha}_j &= \alpha_j^{\sigma_y^{-1}} \left(\frac{\bar{K}_j}{\bar{Y}_j}\right)^{1-\sigma_y^{-1}}, \\
\tilde{\gamma}^m_{ij} &= \left(\gamma^m_{ij}\right)^{\sigma_m^{-1}} \left(\frac{\bar{M}_{ij}}{\bar{M}_j}\right)^{1-\sigma_m^{-1}}, \\
\tilde{\gamma}^I_{ij} &= \left(\gamma^I_{ij}\right)^{\sigma_I^{-1}} \left(\frac{\bar{I}_{ij}}{\bar{I}_j}\right)^{1-\sigma_I^{-1}}.
\end{aligned}
$$

Since we do not solve explicitly for $\bar{M}_{ij}$ and $\bar{I}_{ij}$ we are going to use the first order conditions to solve for $\bar{M}_{ij}/\bar{M}_j$ and $\bar{I_{ij}}/\bar{I}_j$. We get

$$\tilde{\gamma}^m_{ij} = \gamma^m_{ij}\left(\frac{P_{it}}{P^m_{jt}}\right)^{1-\sigma_m},$$
$$\tilde{\gamma}^I_{ij} = \gamma^I_{ij}\left(\frac{P_{it}}{P^k_{jt}}\right)^{1-\sigma_I}.$$

Given this mapping, a naive approach would be simply to replace the intensity shares with the equations that map the empirical shares with the model shares. Nevertheless, if we use this mapping equations as endogenous equations in the steady state system of equations, the output of each aggregator become indeterminate. To see why, we can replace the mapping in the consumption aggregator and rearrange to obtain:

$$C_t = \bar{C}\left(\sum_{j=1}^{N}\tilde{\xi}_j\left(\frac{C_{jt}}{\bar{C}_j}\right)^{1-\sigma_c^{-1}}\right)^{\frac{1}{1-\sigma_c^{-1}}}.$$

This equation for aggregate C cannot be used in steady state, since if we replace the time t values for steady-state values we get a tautology ($\bar{C} = \bar{C}$). Given this, the correct approach is to include in the steady state system the difference between the model-implied expenditure shares as additional expressions to minimize.

2.11 CES algebra

The objective of this appendix is to obtain a mapping between the intensity shares (the primitive parameters that appear in the CES aggregator) and the expenditure shares (input expenditure/aggregate expenditure). We start with the common CES aggregator:

$$X = A\left(\sum_{j=1}^{N}\xi_j^{\sigma_x^{-1}}{X_j}^{1-\sigma_x^{-1}}\right)^{\frac{1}{1-\sigma_x^{-1}}}.$$

where A is a constant (e.g., TFP in the value added aggregator). The derivative is:

$$\begin{aligned}
\frac{\partial X}{\partial X_j} &= \frac{A}{1-\sigma_x^{-1}}\left(\sum_{j=1}^{N}\xi_j^{\sigma_x^{-1}}(X_j)^{1-\sigma_x^{-1}}\right)^{\frac{\sigma_x^{-1}}{1-\sigma_x^{-1}}}\left(\xi_j^{\sigma_x^{-1}}\left(1-\sigma_x^{-1}\right)(X_j)^{-\sigma_x^{-1}}\right),\\
&= \frac{A}{A^{\sigma_x^{-1}}}A^{\sigma_x^{-1}}\left(\sum_{j=1}^{N}\xi_j^{\sigma_q^{-1}}(X_j)^{1-\sigma_x^{-1}}\right)^{\frac{\sigma_x^{-1}}{1-\sigma_x^{-1}}}\xi_j^{\sigma_x^{-1}}(X_j)^{-\sigma_x^{-1}},\\
&= A^{1-\sigma_x^{-1}}\left(A\left(\sum_{j=1}^{N}\xi_j^{\sigma_x^{-1}}(X_j)^{1-\sigma_x^{-1}}\right)^{\frac{1}{1-\sigma_x^{-1}}}\right)^{\sigma_x^{-1}}\xi_j^{\sigma_x^{-1}}(X_j)^{-\sigma_x^{-1}}.
\end{aligned}$$

Notice that we can now plug the definition of the aggregator. We get:

$$\begin{aligned}\frac{\partial X_t}{\partial X_{jt}} &= A^{1-\sigma_x^{-1}} X^{\sigma_x^{-1}} \xi_j^{\sigma_x^{-1}} (X_j)^{-\sigma_x^{-1}}, \\ &= A^{1-\sigma_x^{-1}} \left(\xi_j \frac{X}{X_j} \right)^{\sigma_x^{-1}}\end{aligned}$$

From this first-order condition, we are going to get a price aggregator. We start from the budget constraint:

$$\sum_{j=1}^{N} P_j X_j = Y_j.$$

where Y_{jt} represents income. The optimization problem is

$$\max_{\{X_j\}_{j=1}^N} X \quad \text{s.t.} \quad \sum_{j=1}^{N} P_j X_j = Y_j,$$

The first-order conditions with respect to X_{jt} are:

$$\frac{\partial X}{\partial X_j} - \lambda P_j = 0.$$

So, for al goods $i \neq j$, we have:

$$\begin{aligned}\frac{\partial X}{\partial X_j}\frac{1}{P_j} &= \frac{\partial X}{\partial X_i}\frac{1}{P_i}, \\ A^{1-\sigma_x^{-1}} \left(\xi_j \frac{X}{X_j} \right)^{\sigma_x^{-1}} \frac{1}{P_j} &= A^{1-\sigma_x^{-1}} \left(\xi_i \frac{X}{X_i} \right)^{\sigma_x^{-1}} \frac{1}{P_i},\end{aligned}$$

and we get

$$X_j = \left(\frac{P_i}{P_j} \right)^{\sigma_x} \frac{\xi_j}{\xi_i} X_i.$$

Let's replace that into the aggregator:

$$\begin{aligned}X &= A \left(\sum_{j=1}^{N} \xi_j^{\sigma_x^{-1}} \left(\left(\frac{P_i}{P_j} \right)^{\sigma_x} \frac{\xi_j}{\xi_i} X_i \right)^{1-\sigma_x^{-1}} \right)^{\frac{1}{1-\sigma_x^{-1}}}, \\ &= \frac{1}{\xi_i} (P_i)^{\sigma_x} X_i A \left(\sum_{j=1}^{N} \xi_j (P_j)^{1-\sigma_x} \right)^{\frac{1}{1-\sigma_x^{-1}}}.\end{aligned}$$

We get an expression for X_i in terms of aggregates:

$$X_i = \xi_i \left(\frac{1}{P_i} \right)^{\sigma_x} X A^{-1} \left(\sum_{j=1}^{N} \xi_j^{\sigma_x} (P_j)^{1-\sigma_x} \right)^{\frac{-1}{1-\sigma_x^{-1}}}.$$

We calculate total expenditure:

$$P_i X_i = P_i^{1-\sigma_x} \xi_i X A^{-1} \left(\sum_{j=1}^{N} \xi_j \left(P_j\right)^{1-\sigma_x} \right)^{\frac{-1}{1-\sigma_x^{-1}}}.$$

Adding up all the goods:

$$\begin{aligned} Y &= X A^{-1} \left(\sum_{j=1}^{N} \xi_j \left(P_j\right)^{1-\sigma_x} \right) \left(\sum_{j=1}^{N} \xi_j \left(P_j\right)^{1-\sigma_x} \right)^{\frac{-1}{1-\sigma_x^{-1}}}, \\ &= X A^{-1} \left(\sum_{j=1}^{N} \xi_j \left(P_j\right)^{1-\sigma_x} \right)^{\frac{1}{1-\sigma_x}}. \end{aligned}$$

We can define the price aggregator

$$P = A^{-1} \left(\sum_{j=1}^{N} \xi_j \left(P_j\right)^{1-\sigma_x} \right)^{\frac{1}{1-\sigma_x}}.$$

such that $Y_t = X_t P_t$. Also, we can plug in the price aggregator in our expression for X_{it} to obtain:

$$\begin{aligned} X_i &= \xi_i \left(\frac{1}{P_i} \right)^{\sigma_x} X A^{-1} \left(\sum_{j=1}^{N} \xi_j \left(P_j\right)^{1-\sigma_x} \right)^{\frac{-1}{1-\sigma_x^{-1}}}, \\ &= \xi_i \left(\frac{P}{P_i} \right)^{\sigma_x} X. \end{aligned}$$

The problem with the standard formulation of the CES is that the taste parameters ξ_i do not correspond to expenditure shares unless $\sigma_x = 1$:

$$\xi_i = \left(\frac{X_i}{X} \right) \left(\frac{P_i}{P} \right)^{\sigma_x}$$

Given this, we propose a mapping between intensity shares ξ_i and expenditure shares $\tilde{\xi}_i$:

$$\begin{aligned} \xi_i &= \left(\frac{X_i}{X} \right) \left(\frac{P_i}{P} \right)^{\sigma_x}, \\ &= \left(\frac{X_i}{X} \right) \left(\frac{X}{X_i} \frac{X_i}{X} \frac{P_i}{P} \right)^{\sigma_x}, \\ &= \left(\frac{X_i}{X} \right)^{1-\sigma_x} \left(\tilde{\xi}_i \right)^{\sigma_x}. \end{aligned}$$

This implies the following mapping from intensity shares to expenditure shares:

$$\tilde{\xi}_i = \xi_i^{\sigma_x^{-1}} \left(\frac{X_i}{X} \right)^{1-\sigma_x^{-1}}.$$