1 Baseline framework

- N^h households indexed by i choose how much to consume and how much to invest in order to maximize $\sum_{t=0}^{\infty} \beta^t U(C_{i,t})$.

- The final good is produced using N^c capital goods indexed by j according to technology

$$Y_{i,t} = Z_{i,t}^h \left(\Pi_{j=1}^{N^c} K_{i,j,t}^{1/N^c} \right)^{\alpha} = Z_{i,t}^h \Pi_{j=1}^{N^c} K_{i,j,t}^{\alpha/N^c}$$

  where total factor productivity $Z_{i,t}^h$ follows the stochastic process $Z_{i,t+1}^h \sim \mathcal{P}(Z_{i,t}^h)$.

- The agent allocates $Y_{i,t}$ between consumption and investment in new capital goods. The evolution of the stock of capital is

$$K_{i,j,t+1} = (1-\delta)K_{i,j,t} + I_{i,j,t}^h$$

  where $I_{i,j,t}^h$ represents investment in capital good j by household i in period t[1].

- **Adjustment costs:** The cost in term of the final good of producing $I_{j,t}^h$ new units of capital good j is $\frac{\phi}{2}\left(I_{j,t}^h\right)^2$.

- In order to frame the problem, we define the saving rate $s_{j,t}$ as the expenditure on new capital goods j in terms of the final good. Thus, consumption is

$$C_{i,t} = (1 - \sum_{j=1}^{N^c} s_{i,j,t})Y_{i,t}$$

- We will consider both a competitive market's formulation and a planner's formulation of the problem:
  - First, we assume that there is a market for investment goods and we introduce capital good firms that pay the adjustment cost and sell the investment good at price $p_{j,t}^k$. Thus, from the point of view of the household, investment is $I_{i,j,t}^h = s_{i,j,t}/p_{j,t}^k$.
  - Second, in order to formulate the planner's problem, we assume that all the resources that are committed by households for investment on capital good j are actually employed in production, and then total production of capital good j is distributed among housholds according to their contribution. In this case, total production is defined implicitly by:

$$\sum_{i=1}^{N^h} s_{i,j,t}Y_{i,t} = \frac{\phi}{2} I_{j,t}^2 \quad \text{for } j \in [1, ..., N^h]$$

[1] The superscript h is added because we will consider both planner and market solutions to the problem, and in the latter it is useful to keep track of the agent that is choosing the variable I.

Solving for $I_{j,t}$ we get:

$$I_{j,t} = \sqrt{\frac{2}{\phi}\sum_{i=1}^{N^h} s_{i,j,t}Y_{i,t}}$$

We can then distribute it among households according to

$$I_{i,j,t} = \frac{s_{i,j,t}Y_{i,t}}{\sum_{i=1}^{N^h} s_{i,j,t}Y_{i,t}} I_{j,t} = \frac{s_{i,j,t}Y_{i,t}}{\sqrt{\frac{\phi}{2}\sum_{i=1}^{N^h} s_{i,j,t}Y_{i,t}}}$$

1.1 Competitive Market's Formulation

- Now we have two types of agents. The household purchases capital goods and consume the final good. The capital good firm, which produces the capital good subject to a quadratic cost.

1.1.1 Household

- The recursive formulation of the problem for household i is:

$$\begin{aligned} V_i\left(\{K_{i,j,t}\}_{i,j}\right) = \max_{\{s_{i,j,t},K_{i,j,t+1}\}_j} & U(C_{i,t}) + \beta \mathbb{E}_t V_i(\{K_{i,j,t+1}\}_{i,j}) \qquad \text{s.t.} \\ & C_{i,t} = (1 - \sum_{j=1}^{N^c} s_{i,j,t})Y_{i,t} \\ & Y_{i,t} = Z_{i,t}^h \Pi_{j=1}^{N^c} K_{i,j,t}^{\alpha/N^c} \\ & K_{i,j,t+1} \leq (1-\delta)K_{i,j,t} + \frac{s_{i,j,t}Y_{i,t}}{p_{j,t}^k} \quad \text{for } j \in [1, ..., N^c] \end{aligned}$$

- The lagrangian of the problem is:

$$\begin{aligned} \mathcal{L}_t = \mathbb{E}_t \sum_{r=0}^{\infty} \beta^r & \left(U\left((1 - \sum_{j=1}^{N^c} s_{i,j,t+r}) Z_{i,t+r}^h \Pi_{j=1}^{N^c} K_{i,j,t+r}^{\alpha/N^c} \right) + \right. \\ & \left. \sum_{j=1}^{N^c} Q_{i,j,t+r} \left[(1-\delta)K_{i,j,t+r} + \frac{s_{i,j,t+r} Z_{i,t+r}^h \Pi_{j=1}^{N^c} K_{i,j,t+r}^{\alpha/N^c}}{p_{j,t+r}^k} - K_{i,j,t+r+1} \right] \right) \end{aligned}$$

To do:

- Rewrite FOCs for market

- The first order conditions are:

$$
\begin{aligned}
&[s_{i,j,t}] && Q_{i,j,t} = p^k_{j,t} U'(C_{i,t}) \\
&[K_{i,j,t+1}] && Q_{i,j,t} = \beta \mathbb{E}_t \Bigg(U'(C_{i,t+1})(1 - \sum_{j=1}^{N^c} s_{i,j,t+1}) \frac{\alpha}{N^c} \frac{Y_{i,t+1}}{K_{i,j,t+1}} \\
& && \quad + Q_{i,j,t+1} \left[(1-\delta) + \frac{s_{i,j,t+1}}{p^K_{j,t+1}} \frac{\alpha}{N^c} \frac{Y_{i,t+1}}{K_{i,j,t+1}} \right] \Bigg)
\end{aligned}
$$

- Combining both F.O.C.s we ge the inverse demand:

$$
p^k_{j,t} = \mathbb{E}_t \left(\beta \frac{U'(C_{i,t+1})}{U'(C_{i,t})} \left[(1 - \sum_{j=1}^{N^c} s_{i,j,t+1} + s_{i,j,t+1}) \frac{\alpha}{N^c} \frac{Y_{i,t+1}}{K_{i,j,t+1}} + p^k_{j,t+1}(1-\delta) \right] \right) \tag{1.1}
$$

1.1.2 Capital good firms (superscript c)

- Capital good firms maximize the discounted sum of profits. Profits in each period are

$$
\pi^c_{j,t} = p^k_{j,t} I^c_{j,t} - \frac{\phi}{2} \left(I^c_{j,t} \right)^2
$$

- The lagrangian of the problem for firm j is:

$$
\mathcal{L}_{j,t} = \mathbb{E}_t \sum_{r=0}^{\infty} \beta_f^r \left(p^k_{j,t+r} I^c_{j,t+r} - \frac{\phi}{2} \left(I^c_{j,t+r} \right)^2 \right)
$$

- The first order conditions is:

$$
\left[I^c_{j,t} \right] \qquad p^k_{j,t} = \phi I^c_{j,t} \tag{1.2}
$$

1.1.3 Market clearing and equilibrium definition

- Given that $I^h_{i,j,s} = s_{i,j,t} Y_{i,t} / p^k_{j,t}$, we can write the market clearing condition as:

$$
\sum_{i=1}^{N^h} \frac{s_{i,j,t} Y_{i,t}}{p^k_{j,t}} = I^c_{j,t} \quad \text{for } j \in [1, ..., N^c]
$$

- **Equilibrium:** A competitive equilibrium is a sequence of prices $\{p^k_{j,t}\}_{j,t}$ and allocations $\{s_{i,j,t}, I^c_{j,t}\}_{i,j,t}$ such that:
  - Given prices, household maximize the intertemporal utility of consumption and capital good firms maximize intertemporal profits.
  - Market for capital goods $j \in [1, ..., N^c]$ clear.

1.1.4 System of Equations

- The system of equations is:

$$
\begin{aligned}
p^k_{j,t} &= \mathbb{E}_t \left(\beta \frac{U'(C_{i,t+1})}{U'(C_{i,t})} \left[(1 - \sum_{j=1}^{N^c} s_{i,j,t+1} + s_{i,j,t+1}) \frac{\alpha}{N^c} \frac{Y_{i,t+1}}{K_{i,j,t+1}} + p^k_{j,t+1}(1-\delta) \right] \right) \quad \text{for } i, j \\
p^k_{j,t} &= \phi I^c_{j,t} \quad \text{for } j \\
I^c_{j,t} &= \sum_{i=1}^{N^h} \frac{s_{i,j,t} Y_{i,t}}{p^k_{j,t}} \quad \text{for } j \\
K_{i,j,t+1} &= (1-\delta) K_{i,j,t} + \frac{s_{i,j,t} Y_{i,t}}{p^k_{j,t}} \quad \text{for } i, j
\end{aligned}
$$

- We want to reduce this system to $N^h * N^c$ equations that only depend on $\{s_{i,j,t}, s_{i,j,t+1}\}_{i,j}$. We start by replacing the market clearing condition in the supply curve:

$$
p^k_{j,t} = \sqrt{\phi \sum_{i=1}^{N^h} s_{i,j,t} Y_{i,t}}
$$

- We now replace this expression for $p^k_{j,t}$ in the transition law for capital $K_{i,j,t+1}$:

$$
K_{i,j,t+1} = (1-\delta) K_{i,j,t} + \frac{s_{i,j,t} Y_{i,t}}{\sqrt{\phi \sum_{i=1}^{N^h} s_{i,j,t} Y_{i,t}}} \tag{1.3}
$$

$$
\tag{1.4}
$$

- Finally, we can replace these expressions for $p^k_{j,t}$ and $K_{i,j,t+1}$ in the inverse demand function. Notice that $Y_{i,t+1}$ depends on the set of $\{K_{i,j,t+1}\}_j$ and $C_{i,t+1}$ is a known function of the set $\{s_{i,j,t+1}, K_{i,j,t+1}\}_j$, so we get $N^c * N^h$ differential equations whose only unknowns are $\{s_{i,j,t}, s_{i,j,t+1}\}_{i,j}$.

1.2 Deterministic Steady state

- In the deterministic steady state, we assume that $Z_i^h = 1, \forall i \in [1, ..., N^h]$. Then, all households have a symmetric problem and all capital goods are equivalent. Consequently, we can drop subscripts i and j on all variables. The steady state equations become:

$$\begin{aligned}
Y &= K^\alpha \\
p^k &= \sqrt{\phi N^h s Y} \\
\delta K &= \frac{sY}{\sqrt{\phi N^h s Y}} \\
p^k &= \frac{\beta}{1-(1-\delta)\beta}\left[(1-(N^c-1)s)\frac{\alpha}{N^c}K^{\alpha-1}\right]
\end{aligned}$$

- The solution is:

$$K = \left[\phi\delta N^h N^c\left(\frac{1-\beta*(1-\delta)}{\alpha\beta} + \frac{\delta(N^c-1)}{N^c}\right)\right]^{\frac{1}{\alpha-2}}$$

- For $N^f = 1$, $N^c = 1$, The steady state is:

$$K = \left(\frac{\beta_f}{1-(1-\delta)\beta_f}\frac{\alpha^f}{\phi\delta}\right)^{\frac{1}{2-\alpha^f}}$$

1.3 Planner's Formulation

- In order to formulate the planner's problem, we assume that all the resources that are committed by households for investment on the capital good are actually employed in production, and then total production is distributed among households according to their contribution. In this case, total production is defined implicitly by:

$$\sum_{i=1}^{N^h} s_{i,t}Y_{i,t} = \frac{\phi}{2}I_t^2 \quad \text{for } j \in [1, ..., N^h]$$

Solving for I_t we get:

$$I_t = \sqrt{\frac{2}{\phi}\sum_{i=1}^{N^h} s_{i,t}Y_{i,t}}$$

We can then distribute it among households according to

$$I_{i,t} = \frac{s_{i,t} Y_{i,t}}{\sum_{i=1}^{N^h} s_{i,t} Y_{i,t}} I_t = \frac{s_{i,t} Y_{i,t}}{\sqrt{\frac{\phi}{2} \sum_{i=1}^{N^h} s_{i,t} Y_{i,t}}}$$

- The recursive formulation of the problem is:

$$V\left(\{K_{i,t}\}_i\right) = \max_{\{s_{i,t}; K_{i,t+1}\}_i} \sum_{i=1}^{N^h} U(C_{i,t}) + \beta \mathbb{E}_t V(\{K_{i,t+1}\}_i) \qquad \text{s.t.}$$

$$\begin{aligned}
C_{i,t} &= (1 - s_{i,t}) Y_{i,t} \quad \text{for } i \in [1, ..., N^h] \\
Y_{i,t} &= Z_t^{agg} Z_{i,t}^{ind} K_{i,t}^{\alpha} \quad \text{for } i \in [1, ..., N^h] \\
K_{i,t+1} &\leq (1-\delta) K_{i,t} + \frac{s_{i,,t} Y_{i,t}}{\sqrt{\frac{\phi}{2} \sum_{i=1}^{N^h} s_{i,t} Y_{i,t}}} \quad \text{for } i \in [1, ..., N^h]
\end{aligned}$$

- The lagrangian of the problem is:

$$\begin{aligned}
\mathcal{L}_t = \mathbb{E}_t \sum_{r=0}^{\infty} \beta^r \Bigg(& \sum_{i=1}^{N^h} U\left((1 - s_{i,t+r}) Z_{i,t+r}^h \Pi_{j=1}^{N^c} K_{i,t+r}^{\alpha/N^c} \right) + \\
& \sum_{i=1}^{N^h} Q_{i,t+r} \left[(1-\delta) K_{i,t+r} + \frac{s_{i,,t+r} Y_{i,t+r}}{\sqrt{\frac{\phi}{2} \sum_{i=1}^{N^h} s_{i,t+r} Y_{i,t+r}}} - K_{i,t+r+1} \right] \Bigg)
\end{aligned}$$

- The first order conditions are:

$$[s_{i,t}] \qquad Q_{i,t} = \frac{U'(C_{i,t}) \sqrt{\frac{\phi}{2} s_{i,t} Y_{i,t}}}{\left[1 - \frac{1}{2} \frac{s_{i,t} Y_{i,t}}{\sum_{i=1}^{N^c} s_{i,t} Y_{i,t}}\right]} = U'(C_{i,t}) \frac{\phi I_t}{2 - \frac{I_{i,t}}{I_t}}$$

$$\begin{aligned}
[K_{i,t+1}] \qquad Q_{i,t} = \beta \mathbb{E}_t \Bigg[& U'(C_{i,t+1})(1 - s_{i,t+1}) \alpha \frac{Y_{i,t+1}}{K_{i,t+1}} \\
& + Q_{i,t+1} \left((1-\delta) + s_{i,t} \alpha \frac{Y_{i,t+1}}{K_{i,t+1}} \frac{\left[1 - \frac{1}{2} \frac{s_{i,t+1} Y_{i,t+1}}{\sum_{i=1}^{N^c} s_{i,t+1} Y_{i,t+1}}\right]}{\sqrt{\frac{\phi}{2} s_{i,t+1} Y_{i,t+1}}} \right) \Bigg]
\end{aligned}$$

- Combining both F.O.C.s, we can write the euler equation as:

$$\frac{\phi I_t}{2-\frac{I_{i,t}}{I_t}} = \beta \mathbb{E}_t \mathcal{M}_{t,t+1}\left[\alpha \frac{Y_{i,t+1}}{K_{i,t+1}} + (1-\delta)\frac{\phi I_{t+1}}{2-\frac{I_{i,t+1}}{I_{t+1}}}\right]$$

- With $N^c > 1$, the recursive formulation is:

$$\begin{aligned}
V\left(\{K_{i,j,t}\}_{i,j}\right) = \max_{\{s_{i,j,t},K_{i,j,t+1}\}_{i,j}} & \sum_{i=1}^{N^h} U(C_{i,t}) + \beta \mathbb{E}_t V(\{K_{i,j,t+1}\}_{i,j}) \qquad \text{s.t.} \\
& C_{i,t} = (1-\sum_{j=1}^{N^c} s_{i,j,t})Y_{i,t} \quad \text{for } i \in [1,...,N^h] \\
& Y_{i,t} = Z^h_{i,t}\Pi_{j=1}^{N^c} K_{i,j,t}^{\alpha/N^c} \quad \text{for } i \in [1,...,N^h] \\
& K_{i,j,t+1} \leq (1-\delta)K_{i,j,t} + \frac{s_{i,,j,t}Y_{i,j,t}}{\sqrt{\frac{\phi}{2}\sum_{i=1}^{N^h} s_{i,j,t}Y_{i,j,t}}} \quad \text{for } i \in [1,...,N^h] \text{ and } j \in [1,...,N^c]
\end{aligned}$$

- The lagrangian of the problem is:

$$\begin{aligned}
\mathcal{L}_t = \mathbb{E}_t \sum_{r=0}^{\infty} \beta^r & \left(\sum_{i=1}^{N^h} U\left((1-\sum_{j=1}^{N^c} s_{i,j,t+r}) Z^h_{i,t+r}\Pi_{j=1}^{N^c} K_{i,j,t+r}^{\alpha/N^c}\right) + \right. \\
& \left. \sum_{i=1}^{N^h}\sum_{j=1}^{N^c} Q_{i,j,t+r}\left[(1-\delta)K_{i,j,t+r} + \frac{s_{i,,j,t+r}Y_{i,j,t+r}}{\sqrt{\frac{\phi}{2}\sum_{i=1}^{N^h} s_{i,j,t+r}Y_{i,j,t+r}}} - K_{i,j,t+r+1}\right]\right)
\end{aligned}$$

- The first order conditions are:

$$[s_{i,j,t}] \qquad Q_{i,j,t} = \frac{U'(C_{i,t})\sqrt{\frac{\phi}{2}s_{i,j,t}Y_{i,t}}}{\left[1-\frac{1}{2}\frac{s_{i,j,t}Y_{i,t}}{\sum_{i=1}^{N^c} s_{i,j,t}Y_{i,t}}\right]} = U'(C_{i,t})\frac{\phi I_{j,t}}{2-\frac{I_{i,j,t}}{I_{j,t}}}$$

$$\begin{aligned}
[K_{i,j,t+1}] \qquad Q_{i,j,t} = \beta \mathbb{E}_t & \left[U'(C_{i,t+1})(1-\sum_{j=1}^{N^c} s_{i,j,t+1})\frac{\alpha}{N^c}\frac{Y_{i,t+1}}{K_{i,j,t+1}} \right. \\
& \left. + Q_{i,j,t+1}\left((1-\delta) + s_{i,j,t}\frac{\alpha}{N^c}\frac{Y_{i,t+1}}{K_{i,j,t+1}}\frac{\left[1-\frac{1}{2}\frac{s_{i,j,t+1}Y_{i,t+1}}{\sum_{i=1}^{N^c} s_{i,j,t+1}Y_{i,t+1}}\right]}{\sqrt{\frac{\phi}{2}s_{i,j,t+1}Y_{i,t+1}}}\right)\right]
\end{aligned}$$

- Combining both F.O.C.s, we can write the euler equation as:

$$\frac{\phi I_{j,t}}{2-\frac{I_{i,j,t}}{I_{j,t}}} = \beta \mathbb{E}_t \mathcal{M}_{t,t+1} \left[(1-\sum_{j=1}^{N^c} s_{i,j,t+1} + s_{i,j,t}) \frac{\alpha}{N^c} \frac{Y_{i,t+1}}{K_{i,j,t+1}} + (1-\delta) \frac{\phi I_{j,t+1}}{2-\frac{I_{i,j,t+1}}{I_{j,t+1}}} \right]$$

To do:

- Add cross term of the other Q's in F.O.C.
- Is there any way to show aggregation failure?
- Calculate steady state system of equations.
- Order the rest of the doc.

- The Euler conditions for s_t and K_{t+1} can be condensed into:

$$\phi I_t = \mathbb{E}_t \left[\beta \frac{\beta U'(C_{t+1})}{U'(C_t)} \left(\alpha \frac{Y_{t+1}}{K_{t+1}} + \phi I_{t+1}(1-\delta) \right) \right]$$

where:

$$I_t = \sqrt{\frac{2}{\phi} s_t Y_t}$$
$$Y_t = Z_t K_t^\alpha$$
$$K_{t+1} = (1-\delta)K_t + \sqrt{\frac{2}{\phi} s_t Y_t}$$

1.4 Steady State

- In the steady state, we have that $\sqrt{2sK^\alpha} = \delta K$ and so $sK^{\alpha-1} = \delta^2 K/2$.
- By replacing the first F.O.C. in the second F.O. an, getting rid of time subscript we get

$$\sqrt{2sK^\alpha} = \beta\alpha K^{\alpha-1} - \beta\alpha s K^{\alpha-1} + \beta\sqrt{2sK^\alpha}(1-\delta) + \beta\alpha s K^{\alpha-1} \tag{1.5}$$

- Collecting terms and replacing $\sqrt{2sK^\alpha} = \delta K$ we get:

$$\delta K = \beta\alpha K^{\alpha-1} + \beta\delta K(1-\delta) \tag{1.6}$$

- Finally, solving for K we get:

$$K = \left[\frac{\beta\alpha}{\delta[1-\beta(1-\delta)]}\right]^{\frac{1}{2-\alpha}}$$