# Common mistakes in linear algebra handins

Matias Frank Jensen

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In this document you will find a small catalog of common mistakes and how to avoid them I have seen grading linear algebra handins. For every category of mistake listed, I have shown a few (more or less) specific instances of this mistake, why it is wrong and how to correct it. My hope is the reader will be able to generalize from these specific examples and in the future avoid similar errors. Everyone of the categories of errors I have listed could lose you points on the exam!

- **Always have clear conclusions.** Consider you want to show that $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is invertibe. Just writing

  $$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

  is not enough. Remember to write a conclusion that clearly states why you have solved the problem, e.g. *"By proposition 4.6 a matrix $A$ is invertible iff $A \sim I$, so the matrix is invertible."*

- **Write biimplication arrows between equivalent statements.** Say you want to show a function $L$ is linear. For the sake of presentation let $L(v) = 2v$ for all $v$ in the domain. Then the following is *not* a good way to write it up

  $$\begin{aligned} L(\alpha v + u) &= \alpha L(v) + L(u) \\ L(\alpha v + u) &= 2(\alpha v + u) \\ L(\alpha v + u) &= 2\alpha v + 2u \\ L(\alpha v + u) &= \alpha L(v) + L(u) \end{aligned}$$

  I have seen many variations on the following kind of deduction. I will describe what is wrong with this particular example and hopefully the reader will be able to generalize.
  First of all, the conclusion we want to prove is written as a statement in the beginning. This is fine if it is totally clear to the reader that is the intention. However, it is written without any justification and visual difference from the next 3 lines that are the actual deduction. Hence, it looks like one starts by assuming what one wants to prove. This is very bad!
  Second of all, these next 3 statements logically follows one another (which is the point, starting from a premise one knows to be true, a series of logically entailing deductions end up with the desired conclusion), but it isn't clearly shown. Instead one should use implication arrows to show the logical entailment.
  A final note, since all the expressions are equal, one could just write it as a series of equality statements.
  The revised formulation could look like this:
  *We want to show that $L(\alpha v + u) = \alpha L(v) + L(u)$.*

$$L(\alpha v + u) = 2(\alpha v + u) = \alpha 2v + 2u = \alpha L(v) + L(u)$$

(or)

$$L(v) = 2v \Rightarrow \alpha L(v) + L(u) = 2\alpha v + 2u = L(\alpha v + u)$$

*as desired. The first equality comes from the definition of L.*

- **Define the variables you use.** If you are working with some matricies $A, B$, state these are matrices, over what field and dimension, e.g. $A, B \in \mathrm{Mat}_m, n(F)$. Are you using a basis, write that $\mathcal{V} = \{v_1, ..., v_n\}$ is a basis for a vectorspace $V$. Do you need a matrix where the column vectors are a basis for some vectorspace? Define it precisely so. Doing this, there is no ambiguity for the reader and one can later refer to, for example, the elements of the basis as $v_1, v_2, ...$ etc.

- **Do not confuse elements and sets**. A set and single elements of sets are *not* the same thing! Make sure you do not make any of the following mistakes:

  If $v$ is an element of the set $V$, write $v \in V$, not $v = V$. Say you want to show a certain vector $v$ is in a span $span(v_1, ..., v_n)$, and manage to find scalars $a_1, ..., a_n$ so $v = a_1v_1 + ... + a_nv_n$, then do not conclude with

  $$v = a_1v_1 + ... + a_nv_n = span(v_1, ..., v_n)$$

  $span(v_1, ..., v_n)$ is a set of vectors, and what you have showed is that $v$ is a single element that is a member of this set. Hence write $v \in span(v_1, ..., v_n)$.

  In finding a solution set $S$ to a set of equations, you might conclude that every solution is a linear combination of a certain set of vectors, e.g. a solution could be on the form $v = a_1v1 + a_2v2$. Then the solution set $S$ does not equal $a_1v_1 + a_2v_2$. This expression is a single vector, a single element depending on what $a_1, a_2$ are. In stead, write

  $$S = \{a_1v_1 + a_2v_2 \mid a_1, a_2 \in F\}$$

  Then you have expressed every possible solution as the set of all linear combinations of the two vectors, not just a single linear combination dependent on a choice of scalars $a_1, a_2$.

  Expressing the intersection of two sets $U, V$ is *not* done by writing $U = V$. The mistake here is that the elements $x$ of interest are the ones where $x \in U$ and $x \in V$, but writing $U = V$ is the statement that the two sets $U$ and $V$ are equal. This is not the case! Instead one could write

  $$x \in U \cap V \Rightarrow x \in U \text{ and } x \in V$$

  or

  $$U \cap V = \{x|x \in U \text{ and } x \in V$$

  A set of a single element is not an element. If for a given linear transformation $L$ the only element mapping to zero is 0, $ker(L) \neq 0$, but the set consisting solely of 0, e.g. $ker(L) = \{0\}$.

- **In general, do not confuse elements, sets etc. to be what they are not.** If $\mathcal{V} = \{v_1, v_2, ..., v_n\}$ is a set of vectors, then it is *not* the span of these vectors. $\mathcal{V} \neq span(v_1, ..., v_n)$. $\mathcal{V}$ is neither a matrix, so even though constructing a matrix out of the vectors of $\mathcal{V}$ (assuming the vectors are elements of $F^d$) can be useful, do *not* write expressions like $\mathcal{V}x$, $x \in F^d$ as though you are doing matrix multiplication. Instead explicitly define a new matrix $V'$ where the column vectors are the vectors of $\mathcal{V}$.
  In the same way, if $V$ is a vectorspace, do not treat it as though it is basis of $V$.

- **Do not confuse row equivalence with equality.** If two matrices $A, B$ are row equivalent, we write $A \sim B$, however this does not mean that $A = B$. Row equivalence only means that the second matrix can be produced from the first with a sequence of row operations (and consequently that they have the same nullspace). For instance, consider the identity matrix $I$. If the matrix $A$ is invertible, it is row equivalent to the identity matrix, so if row equivalence entailed equality, $A = I$, hence every invertible matrix would be the identity matrix. This is clearly false!

- **Have a clear structure when writing induction proofs.** An induction proof contains 3 different elements that should all be stated clearly. In an induction proof, one wants to show a certain property $P$ holds for all $n \in \mathbb{N}$ (usaully this is the case, but one can do induction over a subsets of $\mathbb{N}$, $\mathbb{Z}$ or even $\mathbb{Q}$). This is done sequentially by

  - **Induction basis.** Showing that the desired property $P$ holds for the smallest $n$ which is of interest. Usually this means $n = 0$ or $n = 1$. For the most part this is not difficult.
  - **Induction hypothesis.** Here one assumes that for some arbitrary $n$ the property $P$ holds. This makes sense to do since one proved earlier that $P$ holds for the base case, so there exists at least one element $n$ so $P(n)$ is true.
  - **Induction step.** Here comes the magic. After assuming that $P(n)$ holds for some arbitrary $n$, one proves this entails that $P(n + 1)$ is also true.