In [22]:
```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
import seaborn as sns
from math import sqrt
from tqdm import tqdm
from timeit import default_timer as timer
```

Intro to ML

Ex0

1. Algebra, Random variables

$\Omega =$ The sample space (the set of outcomes)
$\omega =$ The outcome
$P(\omega) > 0 =$ The probability of an outcome
$\sum_{\omega \in \Omega} P(\omega) = 1$ So sum of probabilities is 1

$X =$ a real-valued random variable (satunnaismuuttuja)

$X(\omega)$ creates a mapping between real numbers and the possible outcomes

a)

An operator L is linear if it obeys:

$$L(x + y) = Lx + Ly$$

where x, y are eg. functions

and

$$L(\lambda x) = \lambda Lx$$

where $\lambda \in \Re$

Show that the operator E is linear:

$E[X(\omega)] = \sum_\omega P(\omega)X(\omega)$

$$E[F(\omega)+G(\omega)] = \sum_{\omega} P(\omega)\,(F(\omega)+G(\omega))$$
$$E[F(\omega)+G(\omega)] = \sum_{\omega} P(\omega)F(\omega) + \sum_{\omega} P(\omega)G(\omega)$$
$$E[F(\omega)+G(\omega)] = E[F(\omega)] + E[G(\omega)]$$

$$E[F+G] = E[F] + E[G]$$

and:

$$E[cF(\omega)] = \sum_{\omega} P(\omega)cF(\omega)$$
$$E[cF(\omega)] = c\sum_{\omega} P(\omega)F(\omega)$$
$$E[cF(\omega)] = cE[F(\omega)]$$

$\mathrm{Var}[X(\omega)] = E[(X(\omega) - E[X(\omega)])^2]$

b)

$$\mathrm{Var}[X] = E[(X - E[X])^2]$$
$$\mathrm{Var}[X] = E[X^2 - 2XE[X] + E[X]^2]$$
$$\mathrm{Var}[X] = E[X^2] - 2E[X]E[X] + E[X]^2$$
$$\mathrm{Var}[X] = E[X^2] - E[X]^2$$

b)

$$E[X] = \sum_{\omega} PX$$

$$\mathrm{Var}[X] = E[(X - E[X])^2]$$
$$\mathrm{Var}[X] = E[X^2 - 2XE[X] + E[X]^2]$$
$$\mathrm{Var}[X] = E[X^2] - E[2XE[X]] + E[E[X]^2]$$
$$\mathrm{Var}[X] = E[X^2] - 2E[X]E[X] + E[E[X]E[X]]$$
$$\mathrm{Var}[X] = \sum_{\omega} PX^2 - 2\sum_{\omega} PX \sum_{\omega} PX + \sum_{\omega} P(\sum_{\omega} PX)^2$$
$$\mathrm{Var}[X] = \sum_{\omega} PX^2 - 2\sum_{\omega} PX \sum_{\omega} PX + \sum_{\omega} P\sum_{\omega} PX\sum_{\omega} PX$$
$$\mathrm{Var}[X] = \sum_{\omega} PX^2 - (\sum_{\omega} PX)^2$$
$$\mathrm{Var}[X] = E[X^2] - E[X]^2$$

2. Bayes Rule

a) Derivation

The probability of event A given B is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

The probability of event B given A is:

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

Noting that:

$$P(B \cap A) = P(A \cap B)$$

We get:

$$P(A|B)P(B) = P(B|A)P(A)$$

Rearranging gives Bayes:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

b) Medical test

$A =$ allergy
$\neg A =$ no allergy
$T =$ positive test
$\neg T =$ negative test

false positive in 1 % of the cases:

$$P(T|\neg A) = 0.01$$

false negative in 10 % of the cases:

$$P(\neg T|A) = 0.1$$

15 % of the population in Finland have allergy:

$$P(A) = 0.15$$
$$P(\neg A) = 0.85$$

the probability that a person is allergic, if the test is positive:

$$P(A|T) = \frac{P(A \cap T)}{P(T)}$$
$$P(A|T) = \frac{P(T|A)P(A)}{P(T)}$$

Write:

$$P(T) = P(A \cap T) + P(\neg A \cap T)$$
$$P(T) = P(T|A)P(A) + P(T|\neg A)P(\neg A)$$

Bayes with the marginal:

$$P(A|T) = \frac{P(T|A)P(A)}{P(T|A)P(A) + P(T|\neg A)P(\neg A)}$$

	A	$\neg A$	**SUM**
T		0.01	
$\neg T$	0.1		
SUM	0.15	0.85	1

	A	$\neg A$	**SUM**
T	0.05	0.01	0.06
$\neg T$	0.1	0.84	0.94
SUM	0.15	0.85	1

Bayes with the marginal:

$$P(A|T) = \frac{0.05 \times 0.15}{0.05 \times 0.15 + 0.01 \times 0.85}$$
$$P(A|T) = 0.46875$$

3. Matrix calculus

Matrix $\mathrm{A} \in \mathbb{R}^{n \times n}$
Eigenvalue $\lambda \in \mathbb{R}$ of A
Eigenvector $x \in \mathbb{R}^n$ if
Eigenfunction $\mathrm{A}x = \lambda x$

A has n orthonormal eigenvectors $x_i \in \mathbb{R}^n$
and corresponding eigenvalue $\lambda_i \in \mathbb{R}$
where $i \in [1, n]$

Orthonormality means:
$x_i^\top x_i = 1$ and
$x_i^\top x_j = 0$ if $i \neq j$

A new matrix:
$\mathrm{B} = \sum_{j=1}^{n} \lambda_j x_j x_j^\top$ "spectral decomposition"

A matrix A can be expressed in the basis of it's eigenvectors $X = [x_1 + x_2 + \ldots + x_n]$ and $X^\top$

$$\mathrm{A} = X \Lambda X^\top$$

where Λ is a diagonal matrix with $\mathrm{Tr}(\Lambda) = \sum_i^n \lambda_i$

$$\mathrm{A} = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ x_1 & x_2 & \ldots & x_n \\ \downarrow & \downarrow & & \downarrow \end{pmatrix} \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \begin{pmatrix} \leftarrow & x_1 & \rightarrow \\ \leftarrow & x_2 & \rightarrow \\ \leftarrow & \vdots & \rightarrow \\ \leftarrow & x_n & \rightarrow \end{pmatrix}$$

Multiplying the matrices yields $\mathrm{A} = \sum_{i=1}^{n} \lambda_i x_i x_i^\top$ Hence $\mathrm{A} = \mathrm{B}$ and the eigenvectors and -values are the same for B as well.

4. Optimization

Constants:
$a \in \mathbb{R}$
$b \in \mathbb{R}$
$c \in \mathbb{R}$

A function:

$$f(x) = ax^4 + bx + c$$
$$f'(x) = 4ax^3 + b$$

a)

The extreme values are found when $f'(x) = 0$

$$\begin{aligned} 4ax^3 + b &= 0 \\ 4ax^3 &= -b \\ x^3 &= \frac{-b}{4a} \\ x &= \sqrt[3]{\frac{-b}{4a}} \end{aligned}$$

b)

Conditions: $a \neq 0$ and none of them can go to the infinity (and beyond)

5. Algorithms

The fibonacci numbers $F(i)$ are defined for $i \in \mathbb{N}$ as:
$F(i+2) = F(i+1) + F(i)$ when $F(1) = F(2) = 1$

Pseudocode for producing fibonacci numbers

```
func fibo(n) {
    i = 1
    j = 1
    for k in [1,n + 1] {
        i = j
        j = i + j
        print j
    }
}
```

In [1]:
```
def fibo1(n):
    """Function to calculate values from fibonacci sequence using r
ecursion. Scales O(2^n)"""
    if n<= 1: return n
    else: return fibo1(n-1)+fibo1(n-2)
```

In [2]:
```
def fibo2(n):
    """Function to calculate valuesfrom fibonacci sequence. Scales
"""
    return int(((1+sqrt(5))**n-(1-sqrt(5))**n)/(2**n*sqrt(5)))
```

In [14]:
```
length = 50
```

In [15]:
```
idx = np.arange(length)
```

In [32]:
```
fibo1_df = pd.DataFrame(index=idx,columns=["Time","F"])
fibo1_list = []
fibo1_times = []
```

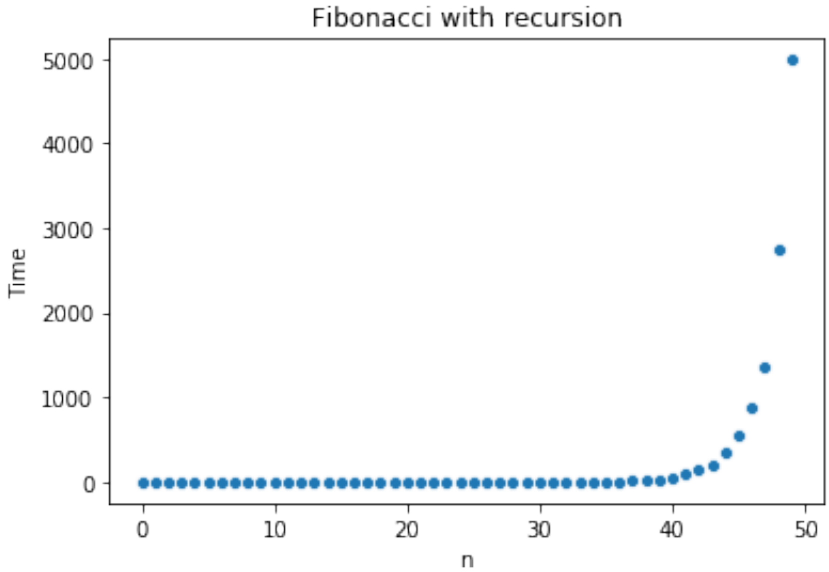
In [33]:
```
for n in tqdm(range(length)):
    start = timer()
    fibo1_list.append(fibo1(n))
    end = timer()
    time = end - start
    fibo1_times.append(time)
```

```
100%|██████████| 50/50 [3:11:21<00:00, 2436.57s/it]
```

In [34]:
```
fibo1_df["F"]=fibo1_list
fibo1_df["Time"]=fibo1_times
```

In [44]:
```
sns.scatterplot(x=idx,y=fibo1_df.Time)
plt.title("Fibonacci with recursion")
plt.xlabel("n")
plt.ylabel("Time")
```

Out[44]:
```
Text(0, 0.5, 'Time')
```



So it almost doubles the time every time
Well the big-O is the worst case scenario here (not like one usually thinks)

In [23]:
```
fibo2_df = pd.DataFrame(index=idx,columns=["Time","F"])
fibo2_list = []
fibo2_times = []
```

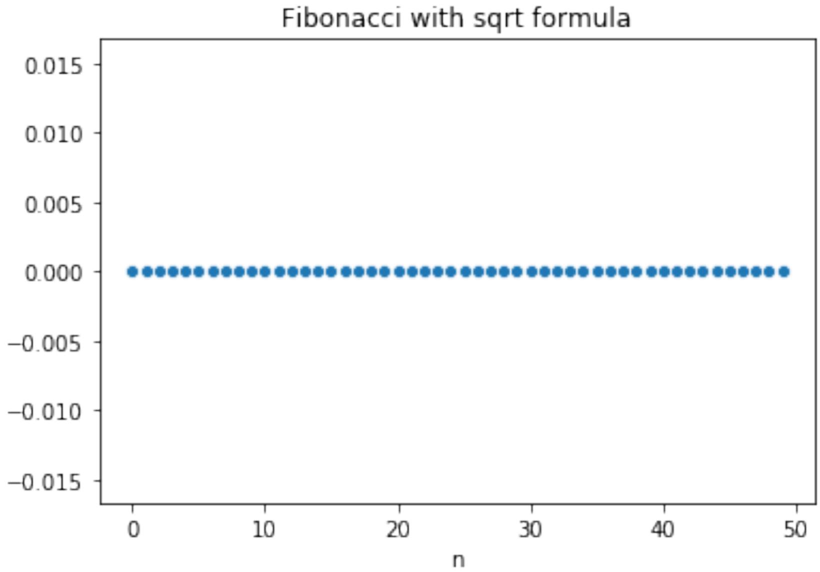
In [24]:
```
for n in tqdm(range(length)):
    start = timer()
    fibo2_list.append(fibo2(n))
    end = timer()
    time = end - start
    fibo2_times.append(time)
```

```
100%|██████████| 50/50 [00:00<00:00, 9364.79it/s]
```

In [25]:
```
fibo2_df["F"]=fibo2_list
fibo2_df["Time"]=fibo2_times
```

In [27]:
```
sns.scatterplot(x=idx, y=fibo2_df.Time)
plt.title("Fibonacci with sqrt formula")
plt.xlabel("n")
plt.ylabel("Time")
```

Out[27]: `Text(0, 0.5, '')`



This follows basically $O(1)$ scaling

6. Data analysis

In [37]:
```
x_data = pd.read_csv("x.csv")
x_data.sample(4)
```

Out[37]:

	V1	V2	V3	V4	V5	V6	V7	V8	
708	-4.597123	-4.220368	7.404075	-4.136433	5.526760	-1.680865	7.044002	-8.015283	
704	-5.544900	-4.872460	5.765845	-4.165147	7.296932	-1.894360	7.509021	-8.786945	
931	-1.741200	-1.065827	7.687956	-2.579892	7.887600	-0.035358	6.411963	-5.078061	1
877	-1.234139	-0.703955	8.137113	-5.114361	7.608299	-2.379770	7.233044	-4.648962	

4 rows × 32 columns

In [38]:
```
x_data.shape
```

Out[38]: `(1000, 32)`

In [39]:
```python
x_data.describe()
```
Out[39]:

	V1	V2	V3	V4	V5	V6	
count	1000.000000	1000.000000	1000.000000	1000.000000	1000.000000	1000.000000	100
mean	-5.493603	-4.848361	6.748369	-3.592207	5.432545	-1.009983	
std	1.860624	1.868029	1.002521	1.798006	1.561889	1.801655	
min	-12.051620	-11.449646	3.402454	-9.883042	0.650691	-7.255290	
25%	-6.720468	-6.091409	6.098733	-4.771207	4.390488	-2.164160	
50%	-5.483653	-4.871369	6.761059	-3.571100	5.406136	-0.987092	
75%	-4.240198	-3.555943	7.438199	-2.368798	6.432138	0.173440	
max	0.765007	1.443165	10.269498	1.833573	10.671700	4.451196	1

8 rows × 32 columns

In [40]:
```python
indices = np.where(x_data.describe().loc["std",:] > 2)[0]
```

In [41]:
```python
use = x_data.iloc[:,indices]
```

In [42]:
```python
use.sample(3)
```
Out[42]:

	V13	V21
606	1.084029	5.581020
937	5.314569	-1.139744
349	2.604948	1.017301

In [43]:
```python
plt.scatter(use.V13,use.V21)
```
Out[43]: <matplotlib.collections.PathCollection at 0x1a1a203278>