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A review and comparison on some rubber elasticity models

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This study reviews several classical continuum mechanics models for incompressible and isotropic materials based on strain energy potential and then compares ability of neo-Hookean, Yeoh, Mooney-Rivlin and Ogden models in predicting uniaxial deformation states based on experimental data from dumb-bell test specimen under uniaxial loading.

Keywords: Curve fitting, Hyperelasticity, Rubber, Strain energy density

Introduction

Rubber material usually has long chain molecules as polymers. Elastomer is combination of elastic and polymer and is often used interchangeably with rubber[1]. Rubber can withstand very large strains with no permanent deformation or fracture[2]. Elastomers have special physical properties (flexibility, extensibility, resiliency and durability), which are unmatched by other types of materials[3], however, it still presents behavior in common with other material[4]. This notable characteristics change with fatigue, light, heat, oxygen and ozone, during passing of time[5]. Elastomers present avery complicated mechanical behavior that exceed linear elastic theory and contain large deformations, plastic and viscoelastic properties and stress softening[6,7]. This characteristic presents complications to modeling of elastomers compared with other traditional engineering materials[8]. Under three physical states of a polymer[9,10], a glassy polymer is brittle. A crystalline polymer pass sequence of changes consist of, elastic deformation, yield, plastic flow, necking, strain hardening and strain fracture. Rubbers are unique in being soft, very extensible and very elastic.

This study reviews and compares recent models, which are offered in hyperelastic materials and discuss their abilities in predicting uniaxial deformation states based on experimental data from dumb-bell test specimen under uniaxial loading.

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Elasticity

By a simple assumption of linear stress-strain relationship, rubber can be considered as a linearly elastic material at small strains. However, for analyzing rubber behavior in large deformation, large elastic deformation theory should be considered[11].

Elastic Properties at Large Deformations

According to Rivlin's phenomenological theory, rubber is assumed isotropic in elastic behavior and very nearly incompressible. Elastic properties of rubber can be explained as a strain energy function based on strain invariants (I_1, I_2 and I_3). Stress and strain analysis problems may be solved independent of microscopic system or molecular concepts and elasticity theory can be starting point of any kind of modeling effort as[12-14]

$$W = f(I_1, I_2, I_3) \qquad ...(1)$$

where W (or U) is strain energy density, and I_1, I_2 and I_3 are invariants of Green deformation tensor given in terms of principle extension ratios as $I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$, $I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2$ and $I_3 = \lambda_1^2\lambda_2^2\lambda_3^2$. Eq. (1) can be given as

$$W = \sum_{i+j+k=1}^{\infty} C_{ijk} (I_1 - 3)^i \cdot (I_2 - 3)^j \cdot (I_3 - 1)^k \qquad ...(2)$$

A simple extension is defined by $\lambda_1 = \lambda$, $\lambda_2 = \lambda_3 = \lambda^{-1/2}$. Incompressibility condition is assumed for this material that volume remains unchanged on deformation and $\lambda_1\lambda_2\lambda_3 = 1$, thus Eq. (2) decreases to

$$W = \sum_{i+j=1}^{\infty} C_{ij}(I_1 - 3)^i \cdot (I_2 - 3)^j \qquad ...(3)$$

Hyperelastic Models

Choice of model depends to its application, corresponding variables and its available data to determine material parameters[15,16]. There are various models on elastic response of rubbers. However, only few describe complete behavior of these materials, especially, for different loading conditions with experimental data[17,18]. Boyce & Arruda[12] compared five and Seibert & Schoche[19] compared six different models for describing of deformation behaviour with experimental data. Markmann & Verron[17] compared 20 hyperelastic models and classified with respect to their ability to fit experimental data. There are various forms of strain energy potentials for modeling of incompressible and isotropic elastomers (Ogden model, Arruda and Boyce model, Yeoh model, Van der Waals model, Mooney-Rivlin model, neo-Hookean model, polynomial model, and reduced polynomial model). Reduced polynomial and Mooney-Rivlin models can be considered as particular cases of polynomial model, whereas Yeoh and neo- Hookean potentials can be considered as special cases of reduced polynomial model[20].

Polynomial Model

Polynomial model, in compressible form, is based on 1st and 2nd invariant $\bar{I}_1$ and $\bar{I}_2$ of deviatoric Cauchy-Green tensor, as

$$U = \sum_{i+j=1}^{N} C_{ij}(\bar{I}_1 - 3)^i (\bar{I}_2 - 3)^j + \sum_{i=1}^{N} \frac{1}{D_i}(J_{el} - 1)^{2i} \qquad ...(4)$$

where U is strain energy density (strain /unit of reference volume), J_{el} is elastic volume ratio, $\bar{I}_1$ and $\bar{I}_2$ are 1st and 2nd invariants of deviatoric strain, and C_{ij} and D_i are material constant, while N is a positive determining number of terms in strain energy function $(N = 1,2,3)$. C_{ij} describes shear behavior of material, D_i introduces compressibility and is set equal to zero for fully incompressible materials. This model is usually used in modeling stress-strain behavior of filled elastomers, with 4-5 terms[21].

Reduced Polynomial Model

This model follows simple form of polynomial model by just omitting 2nd invariant of left Cauchy Green tensor[22]; by doing this, U becomes

$$U = \sum_{i=1}^{N} C_{i0}(\bar{I}_1 - 3)^i + \sum_{i=1}^{N} \frac{1}{D_i}(J_{el} - 1)^{2i} \qquad ...(5)$$

where j is always zero.

Ogden Model

This model proposes strain energy function of stretches λ_1, λ_2 and λ_3 in the form and for incompressible materials $\lambda_1\lambda_2\lambda_3 = 1$. Ogden strain energy potential is given as

$$U = \sum_{i=1}^{N} \frac{2\mu_i}{\alpha_i^2}\left(\bar{\lambda}_1^{\alpha_i} + \bar{\lambda}_2^{\alpha_i} + \bar{\lambda}_3^{\alpha_i} - 3\right) + \sum_{i=1}^{N} \frac{1}{D_i}(J_{el} - 1)^{2i} \qquad ...(6)$$

where $\bar{\lambda}_i = J^{\frac{-1}{3}}\lambda_i$, $J = \lambda_1\lambda_2\lambda_3$, λ_i are principal stretches, J is Jacobean determinant and J_{el} is elastic volume ratio, μ_i and α_i describe shear and D_i compressibility.

Calculation of invariant derivatives of Ogden's energy function is more used and computationally intensive than that of polynomial form[21]. Ogden model can be more accurate in fitting, when data from multiple experimental tests are available[23].

Mooney-Rivlin Model

Strain energy potential is proposed as

$$U = \sum_{i,j=0}^{N} C_{ij}(\bar{I}_1 - 3)^i (\bar{I}_2 - 3)^j + \sum_{i=1}^{N} \frac{1}{D_i}(J_{el} - 1)^{2i} \qquad ...(7)$$

where C_{ij} are material parameter and $C_{00} = 0$[17]. By setting $N = 1$ in polynomial model or $N = 2$, $\alpha_1 = 2$

and $\alpha_2 = -2$ in Ogden model, Mooney-Rivlin model is obtained as[19]

$$U = C_{10}\left(\bar{I}_1 - 3\right) + C_{01}\left(\bar{I}_2 - 3\right) + \frac{1}{D_i}(J-1)^2 \quad ...(8)$$

Most favorite ones of constitutive models are Mooney-Rivlin and Ogden models; disadvantage is that material parameters (not physically-based) must be obtained by experiments. Fitting method can be complicated if number of parameters is large[24].

neo-Hookean Model

This model is pre-programmed into ABAQUS package. If $N = 1$, reduced Polynomial Model change to neo-Hookean Model as

$$W = C_{10}\left(\bar{I}_1 - 3\right) + \frac{1}{D_1}(J_{el} - 1)^2 \quad ...(9)$$

This model is offered only in terms of first deviatoric invariant[25]. Moreover, this model is simplest hyperelastic model and used for elastomeric materials when material data is insufficient. This model is significant because of statistical theory of rubber elasticity appears at strain energy function as

$$W = \frac{1}{2} NKT\,(I_1 - 3) \quad ...(10)$$

where N is number of network chains per unit volume, K is Boltzmann's constant, and T is absolute temperature. Although, statistical and phenomenological begin from quite various premises, Eq. (10) is of same form as Eq. (9)[13].

Yeoh Model

When $N = 3$, reduced Polynomial model[22] changes to Yeoh model as

$$U = \sum_{i=1}^{3} C_{i0}\left(\bar{I}_1 - 3\right)^i + \sum_{i=1}^{3} \frac{1}{D_i}(J_{el} - 1)^{2i} \quad ...(11)$$

Initial shear modulus $\mu_0 = 2C_{10}$ and bulk modulus $K_0 = \frac{2}{D_1}$.

This model has been chosen to describe hyperelastic properties of rubber compounds[26] because: i) It is applicable for a much wider range of deformation; and ii) It is able to predict stress-strain behaviour in different deformation modes from data gained in one simple deformation mode like uniaxial extension.

Arruda and Boyce Model

Physical models (Arruda & Boyce) are based on an explanation of a molecular chains network. Strain energy is assumed to be equal to the sum of strain energies of individual chains oriented in space randomly[27]. This model is given as

$$U = \mu \sum_{i=1}^{5} \frac{C_i}{\lambda_m^{2i-2}}\left(\bar{I}_1^{\,i} - 3^i\right) + \frac{1}{D}\left[\frac{J_{el}^{\,2} - 1}{2} - Ln\left(J_{el}\right)\right] \quad ...(12)$$

where $C_1 = \frac{1}{2}$, $C_2 = \frac{1}{20}$, $C_3 = \frac{11}{1050}$, $C_4 = \frac{19}{7000}$, $C_5 = \frac{519}{673750}$; μ is initial shear modulus and λ_m locking stretch, at which upturn of stress-strain curve would rise significantly. D is double the inverse bulk modulus at small strain as $D = \frac{2}{K}$; and D is set to zero for incompressible material[19].

This model, in the range of smaller strains, helps to make accurate solutions with neglecting second invariant of left Cauchy-Green tensor. With increasing locking stretch parameter, a sufficient accuracy in both small and large strain is obtained [22].

Strain energy function is independent of second stretch invariant for several hyperelastic constitutive models (neo-Hookean model, Yeoh model and Arruda-Boyce model[28]). Each of these models has a set of mathematical form with different parameters that are established by using algorithm based curve-fitting of experimental data[29].

Van der Waals Model

This model is known as Kilian model and introduced as[19]

$$U = \mu\left\{-\left(\lambda_m^2 - 3\right)\left[Ln(1-\eta) + \eta\right] - \frac{2}{3} a\left(\frac{\tilde{I} - 3}{2}\right)^{\frac{3}{2}}\right\} + \frac{1}{D}\left(\frac{J^2 - 1}{2} - Ln(J)\right) \quad ...(13)$$

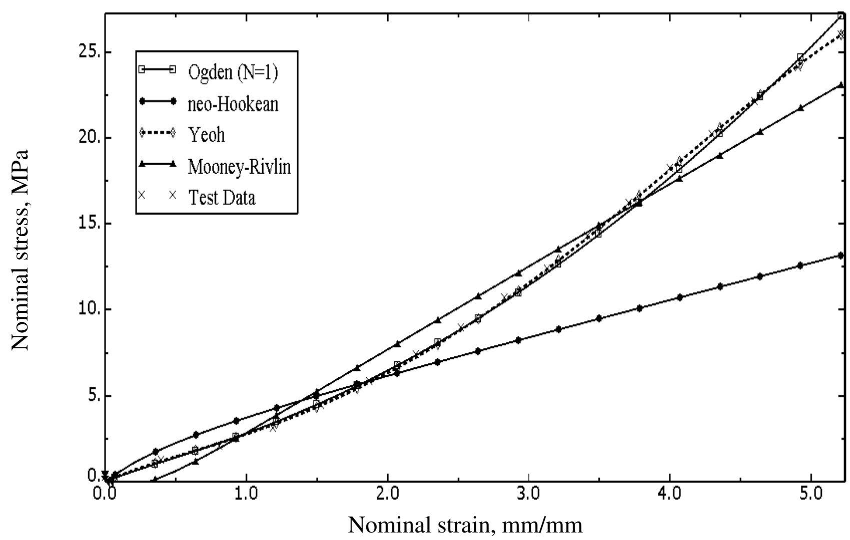

Fig. 1— Stress-strain curve with uniaxial test data

With β representing linear mixture parameter in $\tilde{I} = (1-\beta)I_1 + \beta\tilde{I}_2$ and $\eta = \sqrt{\frac{\tilde{I}-3}{\lambda_m^2 - 3}}$, so that it uses two invariants of left deviatoric Cauchy Green tensor in $\tilde{I}$. If $\beta = 0$, only effects of first invariant are considered and if there is only one type of test data, this parameter is recommended to be set to zero. Interaction parameter α is difficult to measure. It is usually between 0.1 and 0.3. Formula of this model cannot be used, when deformation of material makes stretches larger than locking stretch λ_m[22].

Experimental Tensile Condition

A static stress-strain test determines relationship between an applied force and resulting deformation. It is applied to different stress-strain or force-deformation measurements, i.e. ignored effects of repeated loading[30]. Generally, in rubber industry, tensile test is used for evaluating effect of various compounding ingredients, compound development and manufacturing control[1]. In this paper, data from a carbon filled natural rubber with IRHD 60, which is used in engine mount component application, is presented. Typical engine mount rubber formulation for natural rubber compound is as follows: SMR 10, 100; N550, FEF black, 45; processing oil, 10; zinc oxide, 5; stearic acid, 2; antioxidant, TMQ, 2; antidegradant, 6PPD, 2; sulfur, 2.25; MBTS, 1; and DPG, 0.2 Phr (parts per hundred rubber). For material characterization, uniaxial test was performed on Instron 5500 testing machine (capacity 1 KN). Type 2 dumb-bell test specimen according to ISO 37 with gauge (length, 20 mm; cross section, 2mm×4 mm) was used to determine uniaxial stress-strain relation[31]. Tests were carried out on 5 samples at feed rate of 500 mm/min at RT and median of values for each property was considered.

Table 1—Values of material parameters

Model	First parameter	Second parameter	Third parameter
Ogden	μ_1 =1.12	α_1 =2.9	
neo-Hookean	C_{10}=1.065		
Yeoh	C_{10} =0.645	C_{20} =3.66E-2	C_{30}= -3.03E-4
Mooney-Rivlin	C_{10} =2.368	C_{01}=-3.114	

Results and Discussion

Curve Fitting Procedure

Leading software suppliers use curve fitting procedures to determine material models by minimization of root mean square or least square fit[32]. Stress-strain data form experimental results were imported into ABAQUS and material curve fitting was performed for neo-Hookean (Reduced Polynominal form with N =1), Yeoh (Reduced Polynominal form with N =3), Mooney-Rivlin (Polynominal form with N =1) and Ogden with N =1 (Table 1, Fig. 1). Fitting of Ogden and Yeoh models for small and large strain responses with uniaxial data is acceptable while neo-Hookean and Mooney-Rivlin models perform very poor when calibrated with uniaxial data. Material behavior is defined with strain energy function based on experimental stress-strain data but material properties are sensitive to changes in manufacturing conditions including filler, plasticizer content, mixing and curing efficiency.

Conclusions

Rubber-like material can be explained by continuum based mechanical models. Conventional hyperelastic material models (Mooney-Rivlin or Ogden model) operate very well for many applications. However, Mooney-Rivlin model presents strain energy density based on principal strain invariants, whereas Ogden model offers strain energy density based on three principal stretches. Generally, classification of models is presented based on domain of validity for all modes of deformation, number of parameters and type of formulation used to derive models. So that, it depends on considered domain of deformation; neo-Hookean model, Mooney model and Ogden model can be used for small, moderate and large strain, respectively. Uniaxial test is generally applied to characterize material properties since it is easy to perform. However, reducing number of experiments lead to inaccurate material parameters in strain energy equation and it cannot replicate exactly the behavior in other modes but it will give a reasonable approximation.

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