The power $P = 2\omega H_1^2\chi''$ dissipated in the coil and provided by the nuclear spin system is calculated from (79) and (80) and found to be

$$P = |M_d|\frac{H_0}{T_1}\left(1-\frac{Q_0}{Q}\right). \tag{81}$$

By using the formula (68) rather than the more general formula (67) for the impedance of the tuned circuit containing the nuclear spins, we implicitly assumed that the frequency $\omega_c = (LC)^{-\frac{1}{2}}$ of the tuned circuit was equal to the frequency ω of the oscillation, that is, to $|\omega_0|$. If $\omega_c \neq |\omega_0|$, the frequency of oscillation ω will not be equal to ω_0 either, a phenomenon known as 'pulling'. Let us write in (67):

$$C\omega = \frac{1}{L\omega(1-\delta)}, \quad \text{whence} \quad \delta = \frac{\omega^2-\omega_c^2}{\omega^2} \cong 2\frac{\omega-\omega_c}{\omega} \cong 2\frac{(\omega-\omega_c)}{\omega_c}.$$

Eqn. (67) can be rewritten

$$Z = L\omega Q(1-\delta)\left[\frac{1+j4\pi Q\eta\chi+jQ\delta}{1+4\pi\eta\chi-j/Q}\right]^{-1}. \tag{82}$$

The condition $Z^{-1} = 0$ becomes

$$4\pi Q\eta\chi''(\omega) = -1, \qquad 4\pi\eta\chi'(\omega) = -\delta. \tag{83}$$

From (22) we obtain

$$\chi'(\omega) = -T_2(\omega-|\omega_0|)\chi''(\omega) \cong -T_2(\omega-|\omega_0|)\chi''(\omega_0)$$

and, from (83),

$$T_2(\omega-|\omega_0|) = -2Q\frac{\omega-\omega_c}{\omega_c} \cong 2Q\frac{\omega_c-|\omega_0|}{\omega_c}. \tag{84}$$

From (84) we see that $|\omega-|\omega_0||$ is smaller than $|\omega-\omega_c|$ by the ratio $2Q/\omega_c T_2$ which except for very low frequencies is very small. The 'pulling' is thus almost always negligible.

We shall conclude by remarking that the nuclear spin oscillator is simpler than most oscillators from a theoretical point of view, since its behaviour, independent of the characteristics of electronic tubes, is entirely calculable from first principles even in the non-linear region.

Further study (**14**) of properties of nuclear spin maser oscillators such as band width, noise figure, dynamic behaviour in reaching the steady-state oscillation when the conditions for it have been created, that could be pursued by an extension of the methods outlined here, are outside the scope of this book.

As already stated at the beginning of this section, this review of experimental methods in nuclear magnetism is far from complete. Thus no mention has been made of super-regenerative methods, nor of some interesting but rather special methods such as observation of

the resonance in low fields through a change in the longitudinal component M_z of the nuclear magnetization (1).

We conclude with a few words about the production of the magnetic fields of several thousands gauss, currently used in nuclear magnetism. The two main requirements to be met by the magnets producing these fields are homogeneity in space and stability in time. We shall not discuss the engineering problems connected with these requirements and will simply mention two rather elegant devices that have been used in that connexion. The first, aimed at solving the inhomogeneity problem, is that of the 'spinning' sample. Suppose that the inhomogeneity ΔH of the field over the sample is smaller than, say, one milligauss, so that the Larmor frequencies of any two nuclei differ by less than $\Delta\nu = \gamma/2\pi \times 10^{-3}$ c/s (4 c/s for protons). A macroscopic motion ('spinning') of the whole sample at a frequency much higher than $\Delta\nu$, will cause each nuclear spin to 'see' all the values of the applied field within the interval ΔH and will produce an appreciable narrowing of the resonance line, the effective field 'seen' by each proton being the average of all those 'seen' during the motion. Fig. III, 19 shows two signals obtained with and without spinning.

In connexion with the stability problem it can be remarked that in order to keep the resonance condition $\omega = |\gamma H_0|$ with a relative accuracy of, say, 10^{-8} over long periods of time it may not be necessary that H_0 and ω be separately stable to the same extent, if a device locking them together can be imagined. Several schemes have been tried with varying success. One of the simplest in principle, if possibly not in practice, would be the use of a maser oscillator operating in the same magnet since its frequency is proportional to the applied field.

APPENDIX

Proof of the Kramers–Krönig relations

The relations (8′) between the real and the imaginary parts of the complex susceptibility $\chi = \chi'-i\chi''$ are very general and apply to many systems besides assemblies of nuclear spins. Consider a physical system S, with an input at which a time-dependent excitation $\mathscr{E}(t)$ can be applied, and an output where a response $R(t)$ can be observed. $\mathscr{E}$ and R may be physical quantities of the same nature or of different nature. In nuclear magnetism $\mathscr{E}$ is a magnetic field and R a magnetization but many other examples can be imagined: $\mathscr{E}$ electric field and R electric polarization, $\mathscr{E}$ input and R output voltages, $\mathscr{E}$ incoming flux of particles on a scatterer and R outgoing flux, etc.

The very definition of χ implies that the response to a monochromatic excitation $\mathscr{E}$ is monochromatic. In order to prove the K.–K. relations (8′), we make a few general assumptions to be satisfied by the systems S which we consider.

(i) *The systems S are linear*

We mean thereby that if R_1 and R_2 are responses to excitations $\mathscr{E}_1$ and $\mathscr{E}_2$ the response to the excitation $c_1\mathscr{E}_1+c_2\mathscr{E}_2$ will be $c_1R_1+c_2R_2$. Saturation is thus explicitly excluded.

(ii) *The systems S are stationary*

If $R(t)$ is the response to $\mathscr{E}(t)$, the response to $\mathscr{E}(t-t_0)$ will be $R(t-t_0)$. The condition of a monochromatic response to a monochromatic excitation follows from (i) and (ii). Let $f(t)$ be the response to the unit impulse excitation $\delta(t)$ and, according to (ii), $f(t-t')$ be the response to $\delta(t-t')$. A monochromatic excitation $e^{i\omega t}$ can be rewritten:

$$\mathscr{E}(t) = e^{i\omega t} = \int_{-\infty}^{\infty} e^{i\omega t'}\delta(t-t')\,dt'$$

with, because of (i) and (ii), a response

$$R(t) = \int_{-\infty}^{\infty} e^{i\omega t'}f(t-t')\,dt' = e^{i\omega t}\int_{-\infty}^{\infty} e^{-i\omega t'}f(t')\,dt' \qquad (85)$$

which is indeed monochromatic with the same frequency.

(iii) *The systems S obey the principle of causality*

If $\mathscr{E}(t) = 0$ for $t < t_0$, $R(t) = 0$ for $t < t_0$. This assumption is the key to the proof of the K.–K. relations. It implies in particular that $f(t)$ (which incidentally is real, being the response of a physical system to a real excitation $\delta(t)$) vanishes for $t < 0$. Equation (85) can thus be written:

$$e^{i\omega t}\int_{0}^{\infty} f(t')e^{-i\omega t'}\,dt',$$

whence, from the definition of χ,

$$\mathscr{E} = e^{i\omega t}, \qquad R = \chi(\omega)e^{i\omega t},$$

$$\chi = \int_0^\infty f(t)e^{-i\omega t}\,dt, \qquad \chi' = \int_0^\infty f(t)\cos\omega t\,dt, \qquad \chi'' = \int_0^\infty f(t)\sin\omega t\,dt. \qquad (86)$$

(iv) *Finite total response to a finite total excitation*

We define the total excitation and the total response by the relations:

$$E_T(t) = \int_{-\infty}^{t} |\mathscr{E}(t')|\,dt',$$

$$R_T(t) = \int_{-\infty}^{t} |R(t')|\,dt'.$$

The condition (iv) requires $R_T(t)$ to be finite if $\mathscr{E}_T(t)$ is finite. It follows immediately from (iv) that $\int_0^\infty |f(t)|\,dt$ is finite. One should not, however, conclude from it that $f(t)$ has no singularities, for in the simplest case where $R = \mathscr{E}$, $f(t) = \delta(t)$. It is still possible, however, to state that $|tf(t)| \to 0$ when $t \to 0$, for otherwise the integral $\int_0^\infty |f(t)|\,dt$ would diverge.

The imaginary part $\chi''(\omega)$ is expected on physical grounds to vanish when ω tends to infinity, for otherwise an infinite absorption of energy by the system would occur. That this is actually so is easily verified from:

$$\lim_{\omega\to\infty}\chi''(\omega) = \lim_{\omega\to\infty}\int_0^\infty f(t)\sin\omega t\,dt = \lim_{\omega\to\infty}\int_0^\infty dt\,f(t)\int_0^\omega t\cos(\omega' t)\,d\omega'$$

$$= \lim_{\omega\to\infty}\tfrac{1}{2}\int_0^\infty tf(t)\int_{-\omega}^{\omega} e^{i\omega' t}\,d\omega' \to \pi\int_0^\infty tf(t)\delta(t)\,dt = 0.$$

A similar calculation shows that $\chi'_\infty = \lim_{\omega\to\infty}\chi'(\omega)$ does not necessarily vanish. This was to be expected, for in the simplest case when $\mathscr{E} = R$, $\chi'(\omega) = 1$ for all excitation frequencies. The function $\psi(\omega) = \chi(\omega)-\chi_\infty$, where $\chi(\omega)$ is defined by

$$\chi(\omega) = \int_0^\infty f(t)e^{-i\omega t}\,dt, \qquad (86)$$

is a complex function of the real variable ω, which vanishes at both ends of the real axis. According to a general theorem in the theory of analytical functions of a complex variable, the function $\psi(z)$, where the complex variable z replaces ω

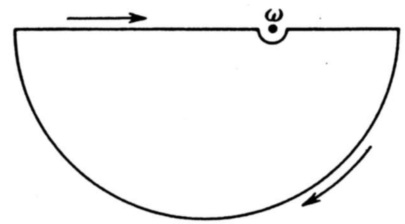

FIG. III, 20. Integration contour.

in (86), is in half the plane $\mathrm{im}(z) \leqslant 0$ an analytical regular function of the variable z. The important point for the validity of this statement is that the integral in (86) is over *positive* values of t only, which in turn is a consequence of the causality principle (iii). Then, applying the theorem of residues to the function

$$\xi(z) = \psi(z)/(z-\omega)$$

for the contour shown in Fig. III, 20, we find

$$\mathscr{P}\int_{-\infty}^{\infty}\frac{\chi(\omega')-\chi_\infty}{\omega'-\omega}\,d\omega' + \pi i\{\chi(\omega)-\chi_\infty\} = 0. \qquad (87)$$

The real and imaginary parts of (87) give respectively

$$\chi'(\omega)-\chi_\infty = \frac{1}{\pi}\mathscr{P}\int_{-\infty}^{\infty}\frac{\chi''(\omega')\,d\omega'}{\omega'-\omega},$$

$$\chi''(\omega) = -\frac{1}{\pi}\mathscr{P}\int_{-\infty}^{\infty}\frac{\chi'(\omega')-\chi_\infty}{\omega'-\omega}\,d\omega'.$$

These are the K.–K. formulae in the form (8′) most suitable for the study of nuclear magnetism. Using the fact that χ' is even and χ'' is odd they are sometimes rewritten as

$$\chi'(\omega)-\chi_\infty = \frac{2}{\pi}\mathscr{P}\int_0^\infty \frac{\chi''(\omega')\omega'\,d\omega'}{\omega'^2-\omega^2},$$

$$\chi''(\omega) = -\frac{2\omega}{\pi}\mathscr{P}\int_0^\infty \frac{\chi'(\omega')-\chi_\infty}{\omega'^2-\omega^2}\,d\omega'. \tag{88}$$

As a special case consider a system with a monochromatic absorption response:

$$\chi''(\omega) = \delta(\omega-\omega_0)-\delta(-\omega-\omega_0) = \delta(\omega-\omega_0)-\delta(\omega+\omega_0). \tag{89}$$

The relations (88) give $\quad \chi'(\omega)-\chi_\infty = \dfrac{2}{\pi}\dfrac{\omega_0}{\omega_0^2-\omega^2}, \quad (89')$

which is the response of an undamped harmonic oscillator. Conversely, it would have been possible to start from (89) and (89′) obtained as a limiting case of a slightly damped oscillator, and use these relations as a starting-point to demonstrate the K.–K. relations.

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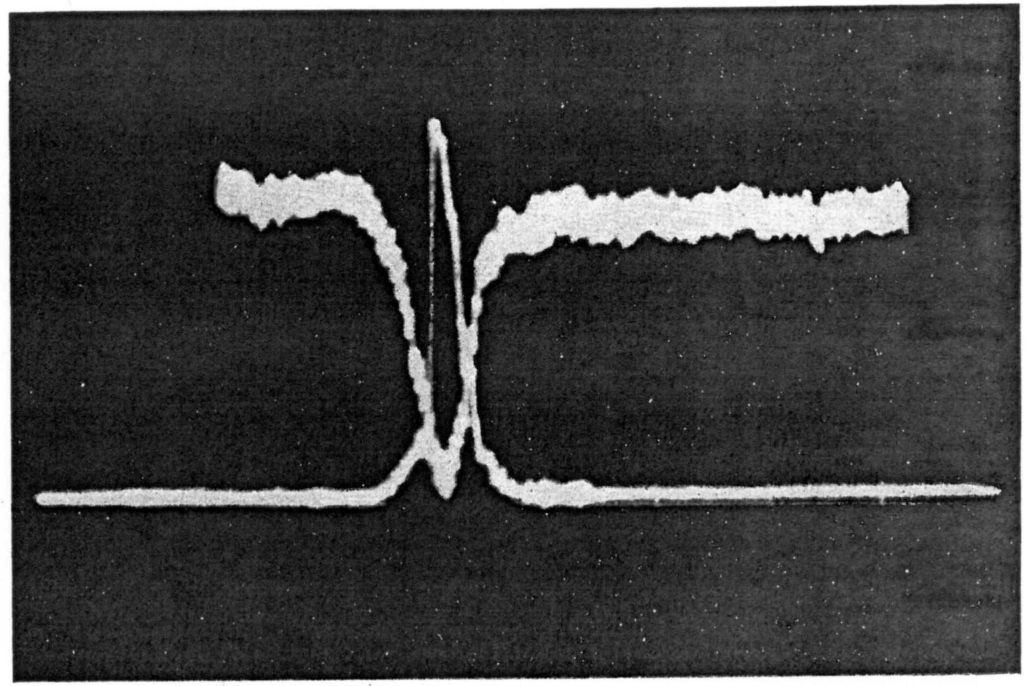

Fig. III, 18. Normal and enhanced signal of protons in water at 3000 gauss. The enhancement is of the order of -50. The scale is different for the two signals as can be seen from the reduced noise on the lower trace.