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Step-by-step analitical solutions of the Lane-Emden equation with polytropic index 0, 1 and 5 using SymPy

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The Lane-Emden Equation

The Lane-Emden equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \tag{1}$$

is of interes in physics due to its aplications in various fields as astrophysics, quantum mechanics and kinetic theory.

In astrophysics, the Lane-Emden equation provides us with a detailed explanation of the astrophysical properties of these stars based on newtonian self-gravitating, spherically symmetric and polytropic fluid [1]. To derive it, we begin with the equations of mass continuity and of hydrostatic equilibrium. Since there are three unknowns (pressure, density, and mass as a function of radius) and only two equations, we need to introduce an additional equation. For polytropes such equation is provided by the pressure-density relation

$$P = K\rho^{1+\frac{1}{n}}. \tag{2}$$

This adds a third equation, and the set of three equations can then be reduced to a single differential equation whose terms depend on n and on K, which after an scale transform gives us ec. 1

Sympy

Sympy is a Python library for symbolic mathematics. It aims to become a full-featured computer algebra system (CAS) while keeping the code as simple as possible in order to be comprehensible and easily extensible [2].Although Sympy is not a necessary tool to solve the Lane-Emdem equation, in this work we use it to show the capacity of the tool as a support in solving other problems that may require the use of a CAS due to the density of the algebraic manipulations required to address them.

With Sympy we can define the Lane-Endem equation as follows.

```
lhs = simplify(
    (1 / xi ** 2) * Derivative(
        (xi ** 2) * Derivative(
            theta(xi), xi
        ), xi
    ).doit()
)
rhs = -theta(xi) ** n
lane_endem_eq = Equation(lhs, rhs)
```

$$\frac{d^2}{d\xi^2}\theta(\xi) + \frac{2\frac{d}{d\xi}\theta(\xi)}{\xi} = -\theta^5(\xi)$$

Solutions for $n = 0$

```
lane_endem_eq_0 = lane_endem_eq.subs(n, 0)
solution = dsolve(lane_endem_eq_0, theta(xi))
solution = solution.subs(solve(
    [
        simplify(xi * solution.rhs).subs(xi, 0),
        Derivative(
            simplify(xi * solution.rhs), xi
        ).doit().subs(xi, 0) - 1,
    ],
    symbols('C1 C2'),
)).simplify()
```

$$\theta(\xi) = 1 - \frac{\xi^2}{6}$$

Solutions for $n = 1$

```
lane_endem_eq_0 = lane_endem_eq.subs(n, 0)
solution = dsolve(lane_endem_eq_0, theta(xi))
solution = solution.subs(solve(
    [
        simplify(xi * solution.rhs).subs(xi, 0),
        Derivative(
            simplify(xi * solution.rhs), xi
        ).doit().subs(xi, 0) - 1,
    ],
    symbols('C1 C2'),
)).simplify()
```

$$\theta(\xi) = \frac{\sin(\xi)}{\xi}$$

Solutions for $n = 5$

Although for $n = 0$ and $n = 1$, sympy solutions are straightforward, for $n = 5$ the process is longer. For such reason here we only present the real solutions we obtained. To proceed we need to transform ec. 1 to its autonomous form (ec. 3) which dependes on a parameter C [3].

$$\left(\frac{dz}{dt}\right)^2 = \frac{1}{12}\left(-z^6 + 3z^2 + C\right) \tag{3}$$

- $C = -2$

$$\theta(\xi) = \pm\left(\sqrt{2\xi}\right)^{-1}$$

- $-2 < C < 0$

$$\theta(\xi) = \pm\sqrt{\frac{aby^2}{2\xi(b(y^2-1)+a)}}, y = \mathrm{dc}\left(\frac{1}{2}\sqrt{\frac{(a+c)b}{3}}\ln(B\xi), \sqrt{\frac{(b-a)c}{(a+c)b}}\right)$$

- $C = 0$

$$\theta(\xi) = \pm\left(\sqrt{1+\xi^2/3}\right)^{-1}$$

- $0 < C < 2$

$$\theta(\xi) = \pm\sqrt{\frac{acy^2}{2\xi(a(y^2+1)+c)}}, y = \mathrm{dc}\left(\frac{1}{2}\sqrt{\frac{(a+c)b}{3}}\ln(B\xi), \sqrt{\frac{(b-a)c}{(a+c)b}}\right)$$

- $C = 2$

$$\theta(\xi) = \pm\sin\left(\ln\sqrt{\xi}\right)\left(\sqrt{3\xi + 2\xi\sin^2\left(\ln\sqrt{\xi}\right)}\right)^{-1}$$

- $2 < C$

$$\theta(\xi) = \pm\sqrt{C}\left(\sqrt{2\xi}\sqrt{\rho\left(\ln(B\xi)/\left(2\sqrt{3}\right); 12, 4(C^2-2)\right) - 1}\right)^{-1}$$

Where both dc and sc are subsidiary Jacobian elliptic functions, ρ is the Weierstrass elliptic function, B is an integration constant and

$$a = 2\sin\left(\frac{1}{3}\arcsin\left(\frac{|C|}{2}\right)\right), b = 2\cos\left(\frac{1}{3}\arccos\left(-\frac{|C|}{2}\right)\right),$$
$$c = 2\cos\left(\frac{1}{3}\arccos\left(\frac{|C|}{2}\right)\right),$$

For $C < -2$ there are no real solutions.

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