Applied Stochastics With Applications in Security and Privacy - Excercises

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Exercise 1. (Author: Gabriel Tański)
Derive the formulas for the expectation and variance of a r.v. with $Geo(p)$ distribution.

Solution:

1) Expected value

The definition of expected value is given by $E[X] = \sum_{i=1}^{\infty} x_i * Pr[X = x_i]$. In case of Geometric distribution that will be:

$$\begin{aligned}
E[X] &= \sum_{i=1}^{\infty} i * p(1-p)^{i-1} \\
&= p(-\frac{d}{dp}\sum_{i=1}^{\infty}(1-p)^i) \\
&= p(-\frac{d}{dp}(\frac{1}{p})) \qquad \text{from the Sum of Infinite Geometric Sequence} \\
&= p(-(-\frac{1}{p^2})) \\
&= \frac{1}{p} \quad \square
\end{aligned} \tag{1}$$

2) Variance

$Var[X] = E[X^2] - E^2[X]$. The second term is easy to derive from the first part of the solution $E^2[X] = (\frac{1}{p})^2 = \frac{1}{p^2}$. $E[X^2]$ is trickier. In order to solve it, first let's demonstrate the following for $q = 1 - p$:

$$\begin{aligned}
\sum_{i=1}^{\infty} i^2(i+1)pq^{i-1} &= p\frac{d}{dq}\sum_{i=1}^{\infty} q^i(i+1) \\
&= p\frac{d}{dq}(\frac{1}{(1-q)^2} - 1) \quad \text{from the Corollary of Derivative of Geometric Sequence} \\
&= p(\frac{2}{(1-q)^3}) \\
&= p\frac{2}{p^3} \\
&= \frac{2}{p^2}
\end{aligned} \tag{2}$$

From definition $E[X^2] = \sum_{i=1}^{\infty} i^2 p(1-p)^{i-1}$. A simple expansion yields: $\sum_{i=1}^{\infty} i^2 p(1-p)^{i-1} = \sum_{i=1}^{\infty} i(i+1)p(1-p)^{i-1} - \sum_{i=1}^{\infty} ip(1-p)^{i-1}$. The first term is $\frac{2}{p^2}$, as demonstrated above, and the second term is identical to what was calculated in the first part of the solution, $\frac{1}{p}$. Therefore, $E[X^2] = \frac{2}{p^2} - \frac{1}{p} = \frac{2-p}{p^2}$.

$$Var[X] = E[X^2] - E^2[X] = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2} \quad \square$$

Exercise 2. (Author: Michał Budnik)
Let $X \sim Geo(p)$. Prove that for any $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}_+$ the following holds:

$$Pr[X = n + k | X > k] = Pr[X = n]$$

Solution:
First let's calculate the value of $Pr[X > k]$. From the definition that would be:

$$\begin{aligned}
Pr[X > k] &= \sum_{i=k+1}^{\infty} p(1-p)^{i-1}) \\
&= \sum_{i=k}^{\infty} p(1-p)^{i}) \\
&= p(1-p)^k * [1 + (1-p) + (1-p)^2 + \dots] \\
&= p(1-p)^k \sum_{i=0}^{\infty} (1-p)^i \\
&= (1-p)^k \sum_{i=0}^{\infty} p(1-p)^i \qquad \text{Note that the sum is now equivalent to 1} \\
&= (1-p)^k
\end{aligned} \tag{1}$$

Having calculated this, let's proceed with the definition of conditional probability:

$$Pr[X = n + k | X > k] = \frac{Pr[X = n + k]}{Pr[X > k]} = \frac{p(1-p)^{n+k-1}}{(1-p)^k} = p(1-p)^{n-1} = Pr[X = n] \quad \square$$

Exercise 3. (Author: Mateusz Jachniak) Find the probability mass function of a r.v. $X \sim NB(k, p)$

Solution:
Consider the series $X_1, X_2, \dots$ of independent samples from Bernoulli trials with the probability p. Let's determine the number r (the number of failures). Let's observe a series until the r'th failure is found. Let's call this moment T. We say that the random variable $T - r$ has a negative binomial distribution $NB(r, p)$ with the parameters r and p. Let $X \sim NB(r, p)$. Then $X = k$ (where $k = 0, 1, 2 \dots$) if in $r + k$'th trial a failure occurred and within $X_1, \dots,$ X_{r+k-1} there were $r - 1$ failures.
In terms of math formulas we have:

$$P(X = k) = \binom{r+k-1}{r-1} (1-p)^{r-1} p^{(r+k-1)} (1-p)$$

It simplifies to:

$$P(X = k) = \binom{r+k-1}{r-1} (1-p)^{r} p^{k}$$

Exercise 4. (Author: Mateusz Jachniak) Calculate the expected value and variance of a r.v. $X \sim NB(k, p)$

1) Expected value:

Solution:
From the previous task we know that negative binomial distribution can be described in that way: we take a series of independent random variables $Y_1, \dots Y_r$ following a geometric distribution with the success parameter $1 - p$. This series is bounded by $r - 1$'th failure (exclusive) and r'th failure (inclusive). Let $Y = Y_1 \dots Y_r$. Then the random variable $X = Y - r$, which counts only successes, has a negative binomial distribution with the parameters r and p. From this, we get a formula for the expected value of a random variable with this distribution.

We have:

$$E(X) = \frac{r}{1-p} - r$$

Which transforms to:

$$E(X) = \frac{rp}{1-p}$$

2) Variance:

Solution:

After some attempt of calculating and proving the formula for the variance, I think that the best way to learn how to calculate a such variance is described at proofwiki. Also StackExchange has a nice solution, but in a way to another representation of NB.

At the end we get:

$$Var(X) = \frac{rp}{(1-p)^2}$$

Exercise 5. (Author: Jakub Czyszczonik)

Derive the formulas for the expectation and variance of a r.v. with Po(λ) distribution.

$$X \sim Po(\lambda), \lambda > 0 \equiv Pr[X = k] = \frac{e^{-\lambda}\lambda^k}{k!}, k \in \mathbb{N}$$

Solution:

Expected Value

Let's take the expectation formula:

$$E[X] = \sum_{x=1}^{\infty} xPr(x)$$

Using Poisson distribution we had:

$$\begin{aligned} E[X] = \sum_{x=1}^{\infty} x\frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_{x=1}^{\infty} x\frac{\lambda\lambda^{x-1}}{x(x-1)!} &= \\ = \lambda e^{-\lambda}\sum_{x=1}^{\infty}\frac{\lambda^{x-1}}{(x-1)!} &\overset{*}{=} \qquad \text{Change indexing(*): } i = x-1 \\ \overset{*}{=} \lambda e^{-\lambda}\sum_{i=0}^{\infty}\frac{\lambda^i}{i!} &\overset{**}{=} \qquad \text{Using formula(**): } \sum_{n=0}^{\infty}\frac{x^n}{n!} = e^x \\ \overset{**}{=} \lambda e^{-\lambda}e^{\lambda} = \lambda e^{-\lambda+\lambda} = \lambda e^0 &= \lambda \end{aligned}$$

Variance

Let's take variance formula:

$$\sigma^2 = E[X^2] - E[X]^2$$

Previously we compute $E[X]$, so now we can easy calculate $E[X]^2 = \lambda^2$. Now we will just compute $E[X^2]$

$$
\begin{aligned}
E[X^2] &= \sum_{x=1}^{\infty} x^2 \frac{e^{-\lambda}\lambda^x}{x!} = \\
&= e^{-\lambda} \sum_{x=1}^{\infty} x^2 \frac{\lambda\lambda^{x-1}}{x(x-1)!} = \lambda e^{-\lambda} \sum_{x=1}^{\infty} x \frac{\lambda^{x-1}}{(x-1)!} = \\
&= \lambda e^{-\lambda} \left(\lambda \sum_{x=2}^{\infty} (x-1) \frac{\lambda^{x-2}}{(x-1)(x-2)!} + \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} \right) = \\
&= \lambda e^{-\lambda} \left(\lambda \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} \right) \overset{*}{=} \quad \text{Change indexing(*): } i = x-1 \text{ and } j = x-2 \\
&\overset{*}{=} \lambda e^{-\lambda} \left(\lambda \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} + \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} \right) \overset{**}{=} \quad \text{Using formula(**): } \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \\
&\overset{**}{=} \lambda e^{-\lambda}(\lambda e^{\lambda} + e^{\lambda}) = \lambda(\lambda+1) = \lambda^2 + \lambda
\end{aligned}
$$

Finally:

$$\sigma^2 = E[X^2] - E[X]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Exercise 6. (Author: Gabriel Tański)
Let $X_i \sim Po(\lambda_i), i \in \{1, \ldots, n\}$ be independent Poisson random variables with means λ_i and let $X = \sum_{i=1}^{n} X_i$. Prove that $X \sim Po(\lambda)$ with $\lambda = \sum_{i=1}^{n} \lambda_i$.

Solution:
First, let's take the Probability Generation Function for Poisson Distribution:

$$\Pi_X(s) = e^{-\lambda(1-s)} \tag{1}$$

The proof can be found here. Therefore in this example $\Pi_{X_i}(s) = e^{-\lambda_i(1-s)}$. So for $X = \sum_{i=1}^{n} X_i$ we have:

$$
\begin{aligned}
\Pi_X(s) &= \Pi_{\sum_{i=1}^{n} X_i} \\
&= \prod_{i=1}^{n} \Pi_{X_i}(s) \qquad \text{Proven here} \\
&= \prod_{i=1}^{n} e^{-\lambda_i(1-s)} \\
&= e^{-\sum_{i=1}^{n} \lambda_i(1-s)} \\
&= e^{-\lambda(1-s)}
\end{aligned} \tag{2}
$$

This concludes that $\Pi_X(s) \sim Po(\lambda)$, and therefore any sum of independent Poisson random variables is a Poisson random variable, and its mean λ is the sum of the means of these random variables.

Exercise 7. (Author: Jakub Czyszczonik)
Find the probability generating functions (PGF's) and moment generating functions (MGF's) of random variables with distributions $Bin(n, p)$, $Geo(p)$ and $Po(\lambda)$.

Solution:
PGF formula:

$$G_X(s) = E[s^X] = \sum_{x=0}^{\infty} s^x Pr(X = x)$$

MGF formula:

$$M_X(s) = G_X(e^s) = E[e^{sX}]$$

Binomial Distribution
Formula:

$$X \sim Bin(n,p) \equiv Pr[X=k] = \binom{n}{k} p^k q^{n-k}, k \in \{0,1,...,n\} \qquad q = (1-p)$$

PGF

$$G_X(s) = \sum_{x=0}^{n} s^x \binom{n}{x} p^x q^{n-x} = \sum_{x=0}^{n} \binom{n}{x} (ps)^x q^{n-x} \overset{*}{=} \qquad \text{Substitution}(*) : ps = z$$

$$\overset{*}{=} \sum_{x=0}^{n} \binom{n}{x} z^x q^{n-x} \overset{**}{=} (z+y)^n \overset{*}{=} (ps+q)^n \quad \text{Using binomial identity}(**) : \sum_{k=0}^{n} \binom{n}{k} x^n y^{n-k} = (x+y)^n$$

MGF

$$M_X(s) = \sum_{x=0}^{n} e^{sx} \binom{n}{x} p^x q^{n-x} = \sum_{x=0}^{n} \binom{n}{x} (pe^s)^x q^{n-x} \overset{*}{=} \qquad \text{Substitution}(*) : pe^s = z$$

$$\overset{*}{=} \sum_{x=0}^{n} \binom{n}{x} z^x q^{n-x} \overset{**}{=} (z+q)^n \overset{*}{=} (pe^s+q)^n \quad \text{Using binomial identity}(**) : \sum_{k=0}^{n} \binom{n}{k} x^n y^{n-k} = (x+y)^n$$

$G_X(s) = (ps+q)^n$
$M_X(s) = (pe^s+q)^n$

Geometric distribution
Formula:

$$X \sim Geo(p), 0 \leqslant p \leqslant 1 \equiv Pr[X=k] = p(1-p)^k, k \in \mathbb{N}$$

PGF

$$G_X(s) = \sum_{x=0}^{\infty} s^x p(1-p) \overset{*}{=} \sum_{x=0}^{\infty} s^x p q^x = p \sum_{x=0}^{\infty} (sq)^x \overset{**}{=} \qquad \text{Substitution}(*) : q = (1-p)$$

$$\overset{**}{=} \frac{p}{1-sq} \quad \text{Using formula}(**) : \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

MGF

$$G_X(s) = \sum_{x=0}^{\infty} e^{sx} p(1-p) \overset{*}{=} \sum_{x=0}^{\infty} e^{sx} p q^x = p \sum_{x=0}^{\infty} (qe^s)^x \overset{**}{=} \qquad \text{Substitution}(*) : q = (1-p)$$

$$\overset{**}{=} \frac{p}{1-qe^s} \quad \text{Using formula}(**) : \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$G_X(s) = \frac{p}{1-sq}$
$M_X(s) = \frac{p}{1-eq^s}$

Poisson distribution
Formula:

$$X \sim Po(\lambda), \lambda > 0 \equiv Pr[X=k] = \frac{e^{-\lambda}\lambda^k}{k!}, k \in \mathbb{N}$$

PGF

$$G_X(s) = \sum_{x=0}^{\infty} s^x \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda s)^x}{x!} \stackrel{*}{=} \quad \text{Using formula(*): } \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$
$$\stackrel{*}{=} e^{-\lambda} e^{s\lambda} = e^{\lambda(s-1)}$$

MGF

$$M_X(s) = \sum_{x=0}^{\infty} e^{sx} \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^s)^x}{x!} \stackrel{*}{=} \quad \text{Using formula(*): } \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$
$$\stackrel{*}{=} e^{-\lambda} e^{e^s \lambda} = e^{\lambda(e^s-1)}$$

$G_X(s) = e^{s\lambda - \lambda}$
$M_X(s) = e^{\lambda(e^s-1)}$

Exercise 8. (Author: Jakub Czyszczonik) Prove two given lemmas. **Lemma 1.11.**
Let $X \sim Exp(\lambda)$. Then for any $s, t \in \mathbb{R}$

$$Pr[X > s + t | X > s] = Pr[X > t]$$

Lemma 1.12. Let $X_i \sim Exp(\lambda_i), i \in \{1, ..., n\}$, be independent exponential random variables with rates λ_i and let $X = min\{X_i : 1 \leqslant i \leqslant n\}$. Then, $X \sim Exp(\lambda)$, where $\lambda = \sum_{i=1}^{n} \lambda_i$.

Solution:
Lemma 1.11.

$$Pr(T > s + t | T > s) = \frac{Pr(T > s + t) \cap T > s}{Pr(T > s)} = \frac{Pr(T > s + t)}{Pr(T > s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = Pr(T > t) \quad \square$$

Lemma 1.12. TODO

Exercise 9. (Author: Michał Budnik)
For any discrete random variables $X : \Omega \to \mathcal{X}$ and $Y : \Omega \to \mathcal{Y}$ $E[X] = \sum_{y \in \mathcal{Y}} E[X|Y = y] Pr[Y = y]$, provided that the expectations $E[X|Y = y]$ are well defined.

Solution:

$$E[X] = \sum_{x \in \mathcal{X}} x Pr[X = x] = \sum_{x \in \mathcal{X}} x \sum_{y \in \mathcal{Y}} Pr[X = x, Y = y] = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} x * Pr[X = x, Y = y] =$$

Using fact that the series is well defined implies that the series is finite and we can switch the summations.

$$= \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} x * Pr[X = x, Y = y] = \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} x * Pr[X = x | Y = y] * Pr[Y = y] = \sum_{y \in \mathcal{Y}} E[X|Y = y] * Pr[Y = y] \quad \square$$

Exercise 10. (Author: Mateusz Jachniak)
Let X and Y be independent random variables that follow Poisson distribution with rates λ_1 and λ_2, respectively. Let $n \in N$. Calculate $Pr[X|X + Y = n]$, $E[X|X + Y = n]$ and $E[X|X + Y]$

TODO Don't know how to end it. Ask lecturer about it.

Solution:

$$
\begin{aligned}
Pr[X|X+Y=n] &= Pr[X=x|X+Y=n] \\
&= \frac{Pr[X=x \cap X+Y=n]]}{Pr[X+Y=n]} \\
&= \frac{Pr[X=x \cap Y=n-x]]}{Pr[X+Y=n]} \\
&= \frac{Pr[X=x]Pr[Y=n-x]]}{Pr[X+Y=n]} \\
&= \frac{Pr[X=x]Pr[Y=n-x]]}{\sum_{k=1} n-1 Pr[Y=n-k|X=k]Pr[X=k]} \\
&= \frac{Pr[X=x]Pr[Y=n-x]]}{\sum_{k=1}^{n-1} Pr[Y=n-k]Pr[X=k]} \\
&= \ldots
\end{aligned}
$$

TODO End it after do first exercise.

Solution:

From Theorem 2.6 of lecture notes, we know that $E[X] = E[E[X|Y]]$. Then $E[X|X+Y]$ has as support (X+Y)'s, so:

$$E[X|X+Y=n] = \sum_{x \in \mathcal{R}_{\S}} x * P(X=x|X+Y=n)$$

Then goes with result from the previous example, here comes:

TODO Don't know how to do it. Ask lecturer about it.

Solution:

$$E[X|X+Y] =?$$

Exercise 11. (Author: Mateusz Jachniak)

Let X and Y be independent random variables that follow geometric distribution with parameter p. Let $n \in N$, $n \geqslant 2$. Calculate $Pr[X|X+Y=n]$, $E[X|X+Y=n]$ and $E[X|X+Y]$

Solution:

$$
\begin{aligned}
Pr[X|X+Y=n] &= Pr[X=x|X+Y=n] \\
&= \frac{Pr[X=x \cap X+Y=n]]}{Pr[X+Y=n]} \\
&= \frac{Pr[X=x \cap Y=n-x]]}{Pr[X+Y=n]} \\
&= \frac{Pr[X=x]Pr[Y=n-x]]}{Pr[X+Y=n]} \\
&= \frac{Pr[X=x]Pr[Y=n-x]]}{\sum\limits_{k=1} n-1 Pr[Y=n-k|X=k]Pr[X=k]} \\
&= \frac{Pr[X=x]Pr[Y=n-x]]}{\sum\limits_{k=1}^{n-1} Pr[Y=n-k]Pr[X=k]} \\
&= \frac{p^{x-1}qp^{n-x-1}q}{\sum\limits_{k=1}^{n-1} p^{k-1}qp^{n-k-1}q} \\
&= \frac{p^{n-2}q^2}{\sum\limits_{k=1}^{n-1} p^{n-2}q^2} \\
&= \frac{1}{n-1}
\end{aligned}
$$

Solution:

From Theorem 2.6 of lecture notes, we know that $E[X] = E[E[X|Y]]$. Then $E[X|X+Y]$ has as support (X+Y)'s, so:

$$E[X|X+Y=n] = \sum_{x \in R_x} x * P(X=x|X+Y=n)$$

Then goes with result from the previous example, here comes:

$$\sum_{x \in R_x} x * P(X=x|X+Y=n) = \frac{1}{n-1} * \sum_{x \in R_x} x$$

TODO Don't know how to do it. Ask lecturer about it.

Solution:

$$E[X|X+Y] = ?$$

Exercise 12. (Authors: M. Budnik, J. Czyszczonik, M. Jachniak, G. Tański)
Compare the bounds on the tails of the binomial distribution provided by Markov's and Chebyshev's inequalities. Namely, for $X \sim Bin(n, \frac{1}{2})$ and for $n \in \{100, 1000, 10000\}$ calculate the exact value of $Pr[X > 1\frac{1}{5}E[X]$ and the estimates of $Pr[X > 1\frac{1}{5}E[X]]$ using Markov's and Chebyshev's inequalities. Perform similar calculations for $Pr[|X - E[X]| \geqslant \frac{1}{10}E[X]$.

Solution:

First formula

The expected value for binomial distribution is denoted by $E[X] = n * p$. Therefore, for $n \in \{100, 1000, 10000\}$ and

$p = \frac{1}{2}$, $Pr[X > 1\frac{1}{5}E[X]]$ is:

$$\begin{aligned} &Pr[X > 60] && \text{for } n = 100 \\ &Pr[X > 600] && \text{for } n = 1000 \\ &Pr[X > 6000] && \text{for } n = 10000 \end{aligned} \tag{1}$$

Using the standard formula for binomial we get $Pr[X > a] = 1 - \sum_{k=0}^{n} Pr[X = k] = 1 - \sum_{k=0}^{a} \binom{n}{k} p^k (1-p)^{n-k}$. Since in this case $p = 1 - p = \frac{1}{2}$, this simplifies to $\frac{1}{2}^n \sum_{k=0}^{a} \binom{n}{k}$. The results are as follows:

$$\begin{aligned} &Pr[X > 60] \approx 1.76 \times 10^{-2} \\ &Pr[X > 600] \approx 9.00842 \times 10^{-11} \\ &Pr[X > 6000] \approx 1.0 \times 10^{-16} \end{aligned} \tag{2}$$

Instead of writing the loop yourself, you can use a Python library:

```
from scipy.stats import binom
print(binom.cdf(60, 100, .5))
```

or WolframAlpha formula:

```
1-(sum(100 choose k), k=0 to 60)*(1/2)^100
```

Calculating this sum can be cumbersome for any large n, and that's where Markov's and Chebyshev's inequalities prove invaluable. Markov's inequality can be written:

$$Pr[X \geqslant a] \leqslant \frac{E[X]}{a},$$

where $a \geqslant 0$. Chebyshev's inequality, on the other hand, is:

$$Pr[|X - E[X]| \geqslant b)] \leqslant \frac{Var(X)}{b^2}$$

For a distribution which is symmetric around the mean the two inequalities are two ways of describing almost the same phenomenon - the Markov inequality helps us calculate the probability of a random variable taking a value larger than some given a, whereas Chebyshev's inequality is useful when describing deviation from expected value. With minor changes, both can be used for the same purposes in case of binomial distribution.
Plugging the numbers into Markov's inequality we get:

$$\begin{aligned} &Pr[X > 60] \equiv Pr[X \geqslant 61] \leqslant \frac{50}{61} \approx 0.812 \\ &Pr[X > 600] \equiv Pr[X \geqslant 601] \leqslant \frac{500}{601} \approx 0.832 \\ &Pr[X > 6000] \equiv Pr[X \geqslant 6001] \leqslant \frac{5000}{6001} \approx 0.833 \end{aligned} \tag{3}$$

Although the inequalities clearly hold, these numbers seem pretty ~~shitty~~ unhelpful, so let's see what we'll get using Chebyshev's inequality. In order to turn $Pr[|X - E[X]| \geqslant b)]$ into $Pr[X \geqslant a]$, we just need to remember that $Pr[|X - E[X]| \geqslant b] \equiv Pr[X \leqslant E[X] - b] + Pr[X \geqslant E[X] + b]$. In case of Binomial distribution, both terms are equal, and that's great because we need only the latter. So in order to calculate $Pr[X \geqslant a]$, we just take $\frac{1}{2}Pr[|X - E[X] \geqslant a - E[X]]$. In order to calculate this, we will use $Var(X) = np(1-p) \equiv (\frac{1}{2})^2 n$

$$\begin{aligned} &Pr[X > 60] \equiv \frac{1}{2}Pr[X - 50 \geqslant 11] \leqslant \frac{1}{2} * \frac{25}{121} \approx 1.03 \times 10^{-1} \\ &Pr[X > 600] \equiv \frac{1}{2}Pr[X - 500 \geqslant 101] \leqslant \frac{1}{2} * \frac{250}{10201} \approx 1.22 \times 10^{-2} \\ &Pr[X > 6000] \equiv \frac{1}{2}Pr[X - 5000 \geqslant 1001] \leqslant \frac{1}{2} * \frac{2500}{1002001} \approx 1.25 \times 10^{-3} \end{aligned} \tag{4}$$

It is apparent that in this case Chebyshev's inequality yields much more useful results, and that the accuracy actually increases with n, as opposed to Markov's inequality.

Second formula

$Pr[|X - E[X]| \geqslant \frac{1}{10}E[X]]$ can be easily converted to $Pr[X \geqslant 1\frac{1}{10}E[X]] + Pr[X \leqslant \frac{9}{10}E[X]]$ or even $1 - Pr[\frac{9}{10}E[X] < X < 1\frac{1}{10}E[X]]$, to make the naïve calculations as straightforward as possible. The exact value would then be given by $1 - \frac{1}{2}^n \sum_{k=a+1}^{b-1} \binom{n}{k}$, where $a = \frac{9}{10}E[X], b = 1\frac{1}{10}E[X]$. The expected values are $50, 500, 5000$ for $n = 100, 1000, 10000$ respectively. Therefore, the exact results are:

$$
\begin{aligned}
&Pr[X \leqslant 45 \vee X \geqslant 55] \approx 3.20 \times 10^{-1} \\
&Pr[X \leqslant 450 \vee X \geqslant 550] \approx 1.56 \times 10^{-3} \\
&Pr[X \leqslant 4500 \vee X \geqslant 5500] \approx 1.11 \times 10^{-16}
\end{aligned} \tag{5}
$$

Since Markov's inequality handles only the form of $Pr[|X - E[X]| \geqslant b)] \leqslant \frac{Var(X)}{b^2}$, we need to convert the task so that it fits the formula. Due to the fact that we are dealing with binomial distribution, which is symmetric around the mean, we know that the following holds: $Pr[X \geqslant E[X] + a] = Pr[X \leqslant E[X] - a]$ for any reasonable (non-negative and smaller than $E[X]$) a. Therefore $Pr[\frac{9}{10}E[X] \leqslant X \geqslant 1\frac{1}{10}E[X]] = 2 * Pr[X \geqslant 1\frac{1}{10}E[X]]$. Plugging that into Markov's inequality yields:

$$
\begin{aligned}
&2Pr[X \geqslant 55] \leqslant 2 * \frac{50}{55} = 1.(81) \\
&2Pr[X \geqslant 550] \leqslant 2 * \frac{500}{550} = 1.(81) \\
&2Pr[X \geqslant 5500] \leqslant 2 * \frac{5000}{5500} = 1.(81)
\end{aligned} \tag{6}
$$

As with the previous example, using Markov's inequality yields exceptionally useless upper bounds. Let's try using Chebyshev's inequality. Analogically to previous examples, $Pr[|X - E[X]| \geqslant a] \equiv Pr[X \geqslant E[X] + a] + Pr[X \leqslant E[X] - a]$ for any a such that $0 \leqslant a \leqslant E[X]$. Therefore:

$$
\begin{aligned}
&Pr[|X - 50| \geqslant 5] \leqslant \frac{25}{25} = 1.0 \times 10^0 \\
&Pr[|X - 500| \geqslant 50] \leqslant \frac{250}{2500} = 1.0 \times 10^{-1} \\
&Pr[|X - 5000| \geqslant 500] \leqslant \frac{2500}{250000} = 1.0 \times 10^{-2}
\end{aligned} \tag{7}
$$

As we can see, Chebyshev's inequality can be inaccurate as well in some cases, but as n rises, it starts approaching a fairly decent ballpark guess. Markov's inequality, on the other hand, does not yield any helpful results in this task. What's important to remember, though, is that in this example for very large values of n, Chebyshev's inequality will be a satisfactory upper bound, whereas calculating the exact value will ~~kill your PC~~ take a long time.