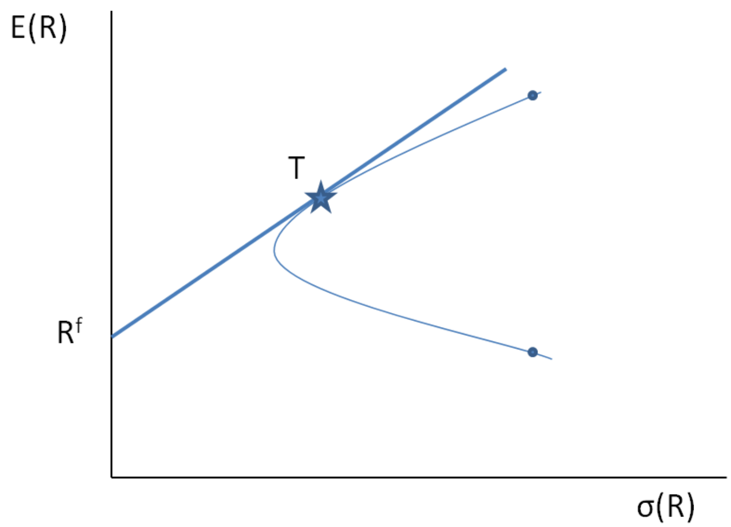
1 The Mean-Variance Frontier (Orthogonal Characterization)

1.1 A Reminder

- Now, we're going to switch gears and talk about the Mean-Variance Frontier again.
- Recall, that the MV Frontier - for a given set of assets - gives us a boundary of the set of means and variances for the returns on all portfolios of those assets.
  - We showed how a two-asset portfolio traces out a hyperbolic curve through those assets.
  - As the two assets become less correlated, the curve becomes sharper, since portfolio variance benefits from the increasing diversification.
- We also showed how adding a risk-free asset would give rise to a new MV Frontier as a straight line between the risk-free rate and a portfolio of the risky assets.



1.2 Orthogonal Characterization

- Now, we're going to think about a geometric interpretation of the MV Frontier.
  - We've already started to think about payoffs, discount factors, and other random variables as vectors in payoff space
- Why?
  - Analyzing the MV Frontier through portfolio problems that requires us to solve a minimization problem is cumbersome
- Where we're going:
  - We can describe (any) return as a three-way orthogonal decomposition
  - Then, we'll see how the MV Frontier "pops out" easily, expressed in terms of this decomposition

1.3 Some Definitions

1.3.1 Definition of R^*

- Recall that x^* was the "mimicking portfolio" that can act as a discount factor.
- Let's now consider a *payoff* x^* - we can think of it like any generic payoff x.
  - The price of x^* is then just
$$p(x^*) = E(x^* x^*) = E\left(x^{*2}\right)$$
- Now define R^* as the return corresponding to the payoff x^*, so that
$$R^* = \frac{x^*}{p(x^*)} = \frac{x^*}{E(x^{*2})}$$
- Note that R^* is a vector that points in the same direction as x^*, but has a price of 1.

1.3.2 Definition of R^{e*}

- Now, think of the space of all excess returns as $\underline{R^e}$, i.e., the set off all payoffs that have a price of zero.
- We define R^{e*} as
$$R^{e*} = proj\left(1|\underline{R^e}\right)$$
- What we're trying to find here are essentially means in the space of $\underline{R^e}$.
- Means (absenting a scale factor) in the space of $\underline{R^e}$ can be represented as an inner product, or projection, for a vector of ones on $\underline{R^e}$, so
$$E(R^e) = E(1 \times R^e) = E\left[proj\left(1|R^e\right) \times R^e\right] = E\left(R^{e*} R^e\right)$$
- Confused? Don't Be.
  - Remember how we represented prices as an inner product by constructing x^* from the payoff space to write
$$p(x) = E(mx) = E\left[proj(m|X)\,x\right] = E(x^* x)\,.$$
    - The mimicking portfolio x^* is just a payoff in $\underline{X}$
  - Now, we have *analogously* that
    - The excess return R^{e*} is just an excess return in the space of excess returns $\underline{R^e}$
    - Why? We're trying to get a handle on $E(R^e)$ so we can head towards the construction of a MV Frontier...
    - To do this, we need a variable construction that allows us to change means
$$E(R^e) = E\left(R^{e*} R^e\right)$$

1.4 Theorem: Return Decomposition

- Every return R^i can be expressed as

$$R^i = R^* + w^i R^{e*} + n^i$$

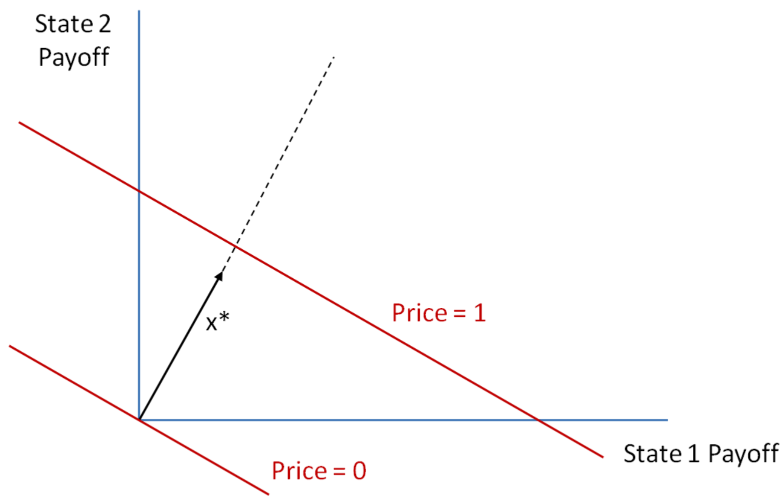
where w^i is a real number and n^i is an excess return with the property that $E\left(n^i\right) = 0$.

- The three components are orthogonal

$$E\left(R^* R^{e*}\right) = E\left(R^* n^i\right) = E\left(R^{e*} n^i\right) = 0.$$

1.4.1 Back to Basics: State Space Representation

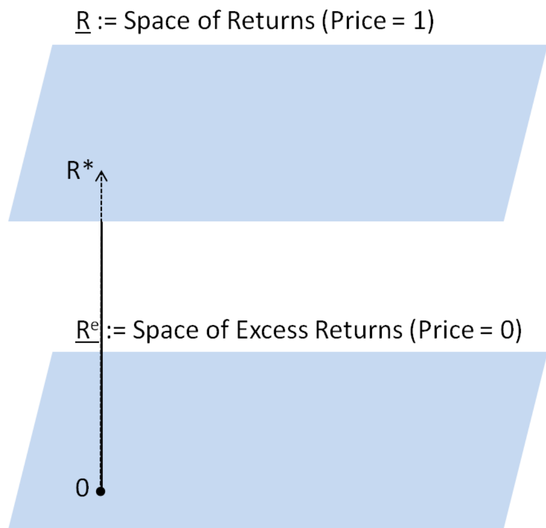
- To see what this means, recall our state-space representation



1.4.2 From the State-Space Representation to R^*

- Recall that
  - The discount factor, x^*, is a vector perpendicular to the constant price planes
  - The price = 1 plans corresponds to Returns
  - The price = 0 plane represents Excess Returns
- So if $R^* = \frac{x^*}{E(x^{*2})}$, then the R^* vector is also perpendicular to the constant price planes

- Now, let's re-draw and re-label our diagram...

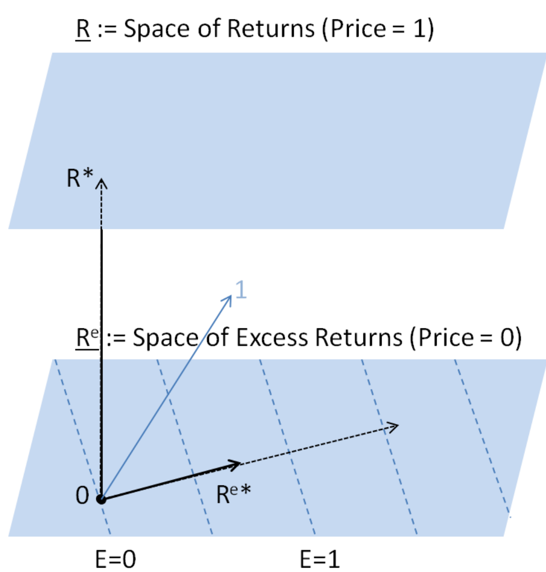


1.4.3 Now for R^{e*}

- Since we define R^{e*} as

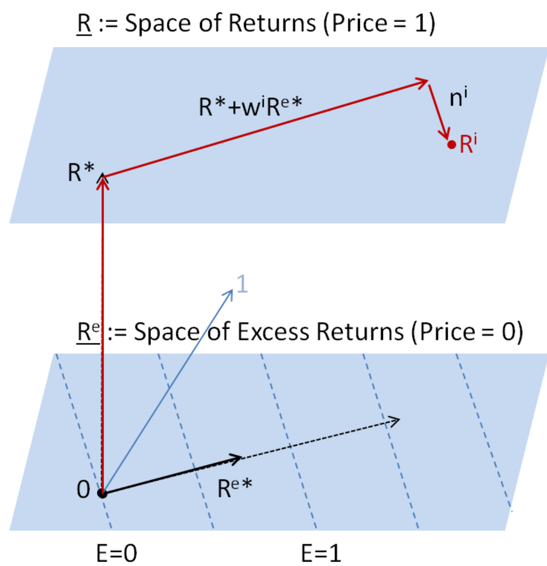
$$R^{e*} = proj\left(1|\underline{R^e}\right)$$

- R^{e*} is the excess return that is closest to the vector 1.
  - The vector R^{e*} lies at right angles to planes in $\underline{R^e}$ of constant *mean* return
  - (Like the return R^* lies at right angles to planes of constant price)
- And since we know that R^{e*} is an excess return, it is orthogonal to R^*



1.4.4 Finding a Return

- Now, to find a return in this space (i.e., get a return's orthogonal decomposition),
- We know that any return R^i begins at the origin and terminates in the hyperplane that represents the space of returns.
  - So, to get to the end of R^i
    - Start at the origin and travel to the end of R^*
    - Then travel along R^{e*} for a distance of w^i
    - Next, the excess return n^i is the little bit of extra (in an orthogonal direction) you need to reach R^i



1.5 The Mean-Variance Frontier

- Now, think about what the MV Frontier is:
  - Given an expected return, what is the portfolio that has the smallest variance?
- If we fix the expected return, then minimizing the variance is the same and minimizing the second moment, since

$$E\left(R^2\right) = \sigma^2\left(R\right) + E\left(R\right)^2$$

- Since the second moment is the same as the length of a vector, the vector that has the shortest length (for a given expected return) is on the MV Frontier.
  - The shortest vector for a given expected return, R^{mv}, is simply one with $n^i = 0$, or

$$R^{mv} = R^* + wR^{e*}$$

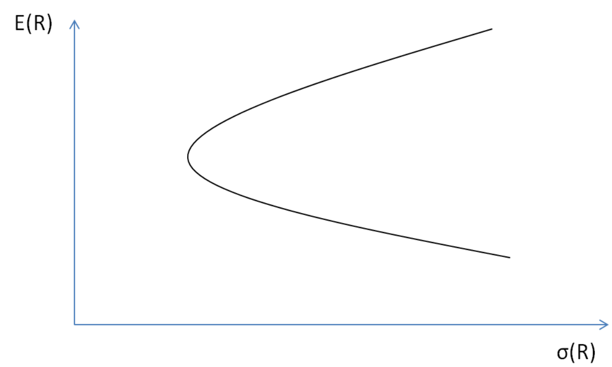
  All we have to do is vary the scalar w to sweep out the efficient frontier.

1.6 MV Frontier in $\mu - \sigma$ Space

- Now we have an orthogonal decomposition for returns.
  - We're going to show how we can also graph that decomposition in mean-standard deviation space.
- Recall that our decomposition is expressed

$$R^i = R^* + w^i R^{e*} + n^i$$

- And recall what a typical mean-variance frontier looks like in mean-standard deviation space...



- Let's first try to find the return with the smallest second moment.
  - Let's look at the second moment of a return expressed this way, found as

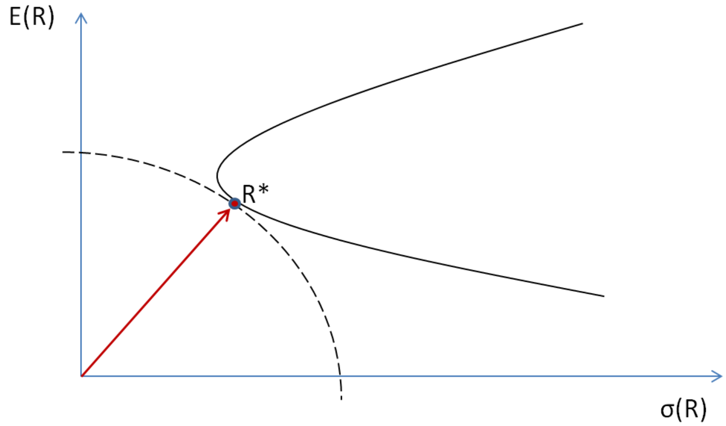
$$E\left(R^2\right) = E\left(R^{*2}\right) + w^2 E\left(R^{e*2}\right) + E\left(n^2\right)$$

  - (Where did all the covariance terms go?)
  - The return with the smallest second moment is found where $w = 0$ and $n = 0$, which is just R^*
- Now, think about

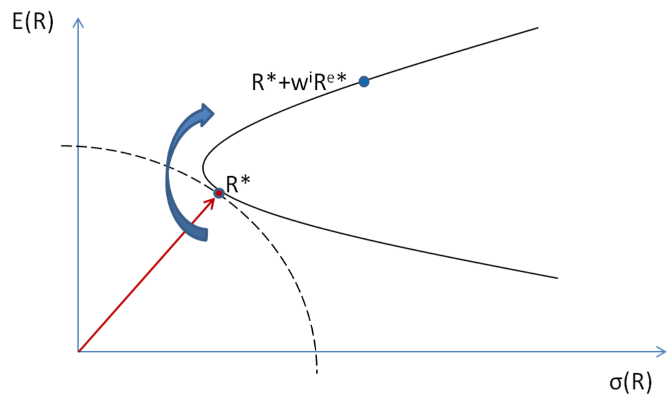
$$E\left(R^2\right) = \sigma^2\left(R\right) + E\left(R\right)^2$$

- Returns of constant second moment trace out a circle in mean-standard deviation space...
  - (E.g., $x^2 + y^2 = 1$ is the equation of a unit circle centered at the origin.)
- Now we know that R^* is on the MV Frontier and we know that it is the return with the smallest second moment...

- So we get that R^* is the return with the smallest "length" from the origin to the MV Frontier.



- Now what happens is that as you vary w, you move along the curve to trace out the MV Frontier
  - As you change w, you change $E(R)$ and $\sigma(R)$, but as long as $n = 0$, the returns are on the Frontier.



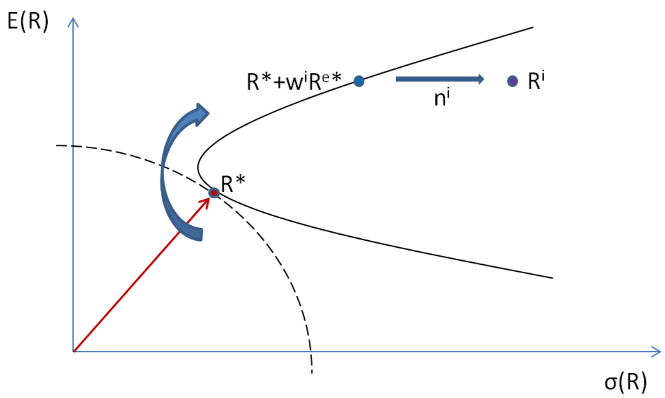
- What about n? We have that $E(n) = 0$, so adding more n doesn't change the expected return, but it will increase the second moment and return standard deviation.
  - This is the idiosyncratic, or diversifiable, risk.

2 Hansen-Jagannathan Bounds

2.1 Deriving Hansen-Jagannathan Bounds

- To find the mean-variance frontier, recall that we can write the expected return of an asset as follows

$$\begin{aligned} E(R) &= R^f - \frac{\rho\sigma(m)\sigma(R)}{E(m)} \\ \frac{E(R) - R^f}{\sigma(R)} &= -\frac{\rho\sigma(m)}{E(m)} \end{aligned}$$



- And since $-1 \leq \rho \leq 1$ and $R^e = R - R^f$ is an excess return

$$\frac{|E(R^e)|}{\sigma(R^e)} \leq \frac{\sigma(m)}{E(m)}$$

- Earlier, we interpreted this expression as providing a bounds on the set of possible returns that we should see (i.e., the mean-variance frontier), given a particular discount factor
  - How did we do this?

- Alternatively, use

$$\begin{aligned} 0 &= E(mR^e) \\ 0 &= E(m)E(R^e) + \rho_{m,R^e}\sigma(m)\sigma(R^e) \end{aligned}$$

  and re-arrange...

2.2 HV Bounds Interpretation

- Now we're going to use this expression and interpret it the other way around

$$\frac{|E(R^e)|}{\sigma(R^e)} \leq \frac{\sigma(m)}{E(m)}$$

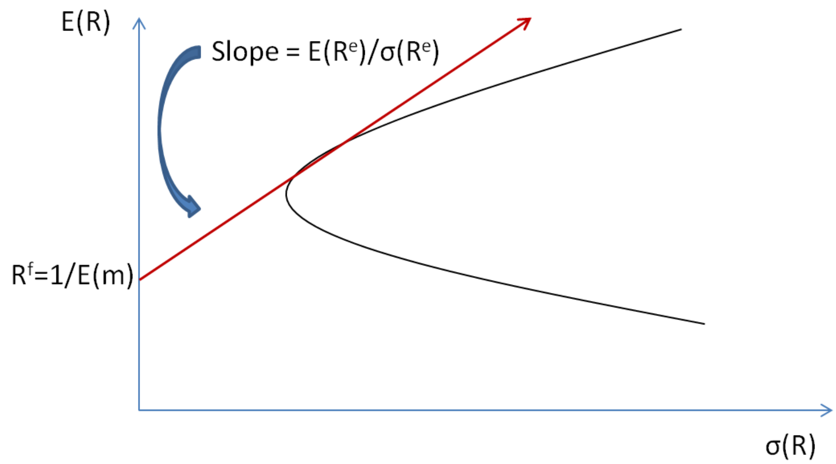
- We can also ask the following question:
  - Given a set of returns (and their means and variances), what are the bounds on all the possible discount factors?
- This is the interpretation for the H-J Bounds.
- Recall that if markets are incomplete, there is an infinite set of discount factors that can be used to price payoffs.
  - Why?
- We want to be able to narrow that down a bit... So

- What can we tell about the set of $[E(m), \sigma(m)]$ that is consistent with a given set of returns?

- Using
$$\frac{|E(R^e)|}{\sigma(R^e)} \leq \frac{\sigma(m)}{E(m)}$$

- We can get the following from this equation, using the H-J interpretation:
  - For a given risk-free rate, the tightest (most restrictive) bound on discount factors is obtained when the Sharpe ratio is the highest.

- The H-J Bounds are found by finding the smallest $\sigma(m)$ for any given $E(m)$ that prices assets
  - For any hypothetical risk-free rate...find the highest Sharpe ratio
  - The highest Sharpe ratio shows the slope of the line through that hypothetical risk-free rate and the tangency portfolio that gives the lowest bound on the volatility of m

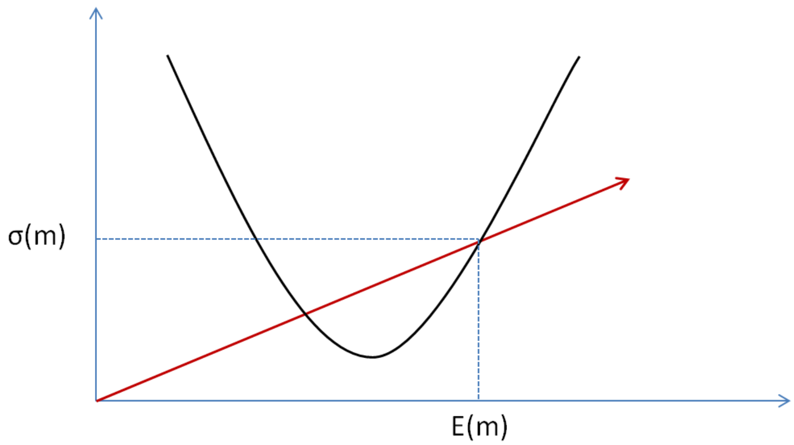


- If we know the risk-free rate, then the MV Frontier has a V shape, and H-J Bound is just a bound on variance.

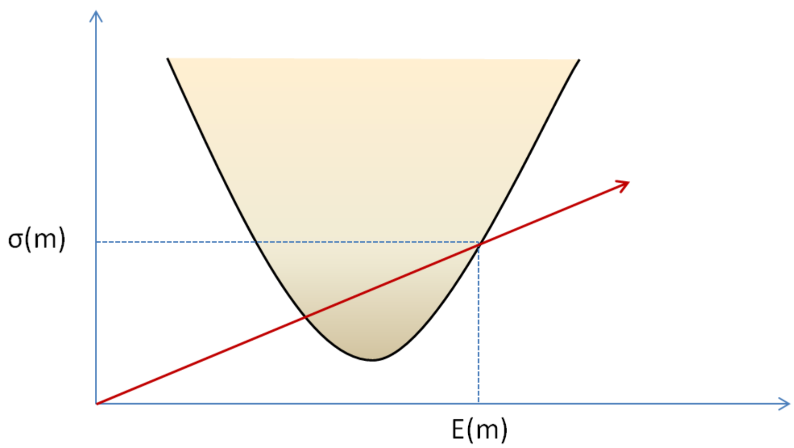
- But if we don't know R^f, then we can trace out all the possible $[E(m), \sigma(m)]$ possibilities and plot them...

- Now, since the question we want to ask is what is the smallest variance $\sigma(m)$ for any given $E(m)$ that prices assets, we flip the graph around

- (What is the slope of the red line pictured here?)



- What is feasible? What's not?



- KEY: Classical mean-variance analysis places bounds on the moments of portfolio payoffs, given m.
  - H-J analysis places bounds on the moments of m, given the portfolio payoffs.
- Given a set of data, we would like to be able to compute the H-J Bounds - that is, we want to compete the mean-variance frontier *for discount factors.*

2.3 Construction from Orthogonal Decomposition

- The first step is to decompose an SDF m into three orthogonal parts (just like we did with returns)

$$m = x^* + we^* + n$$

- What is this?
  - m is a discount factor, represented as a vector that starts at the origin and terminates in hyperplane $\underline{M}$, the space of all discount factors

  - x^* is what we've defined before, the projection of m onto the payoff space $\underline{X}$. (Do $\underline{M}$ and $\underline{X}$ intersect?)
  - e^* is the projection of the 1 vector onto the space spanned by $m - x^*$.
    - It generates means of m just as R^{e*} did for returns.

- To construct H-J Bounds from

$$m = x^* + we^* + n$$

- We need to find the discount factor that has the smallest second moment for any given $E(m)$ and w. So from

$$E\left(m^2\right) = E\left(x^{*2}\right) + w^2 E\left(e^{*2}\right) + E\left(n^2\right)$$

we see that points on the H-J Bounds have $n = 0$, so are given by

$$m = x^* + we^*$$

2.4 Equations (Linear Algebra) for H-J Bounds

- We would like to be able to construct the H-J Bounds using what we know...
  - Assume we have x as a $k \times T$ matrix of observed returns (k assets for T time periods)
- First, recall that we've already found x^* as

$$x^* = p' E\left(xx'\right)^{-1} x$$

- Now, we also have that

$$1 = e^* + proj\left(1|\underline{X}\right)$$

  where $proj\left(1|\underline{X}\right)$ is just a regression of the vector 1 onto the space spanned by x, $\underline{X}$.

- So, from

$$proj\left(1|\underline{X}\right) = E\left(x'\right) E\left(xx'\right)^{-1} x$$

- We get that

$$\begin{aligned} e^* &= 1 - proj\left(1|\underline{X}\right) \\ &= 1 - E\left(x'\right) E\left(xx'\right)^{-1} x \end{aligned}$$

- Now, we can write m^*, which has the smallest second moment for a given $E(m)$ in terms of observed returns

$$\begin{aligned} m &= x^* + we^* \\ &= p' E\left(xx'\right)^{-1} x + w[1 - E\left(x'\right) E\left(xx'\right)^{-1} x] \\ &= w + [p - wE\left(x\right)]' E\left(xx'\right)^{-1} x \end{aligned}$$

- So then we get our variance-minimizing discount factors with

$$
\begin{aligned}
E[m^*] &= w + [p - wE(x)]' E(xx')^{-1} x \\
\sigma^2(m^*) &= [p - wE(x)]' cov(xx')^{-1} [p - wE(x)]
\end{aligned}
$$

- So, these equations give bounds for the first and second moments of the m's that map payoffs into prices, $p = E[mx]$.
- The H-J Bounds are a useful diagnostic for asset pricing models
  - Given a set of returns, we know in what region any SDF that is able to price all the assets must be.

2.5 Questions for Understanding

- In the traditional mean-variance framework, what will adding assets do to the mean-variance bounds on returns?

- What does that mean for the H-J Bounds?