# ECON 4360: Empirical Finance

## Prices, Payoffs, and Notation

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Theory Lecture #04

# What are we doing today?

- Using the SDF Methodology
  - Prices, Payoffs, and Notation
- A Basic Pricing Example (MATLAB) with the SDF Methodology.

# Basic Pricing Equation

- Recall that last time, we set up a basic problem that derived the central asset pricing equation from the consumption-based model.
  - (We wanted to find the **price** of the asset that set the first order condition to zero.)
- Our basic pricing equation was

$$p_t = E_t \left[ \beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1} \right]$$

that implied, in equilibrium, the decrease in utility from buying a share of the asset today has to just equal the increase in expected discounted utility that results from having one more share tomorrow.

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- by defining

$$m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)}$$

- The variable $m_{t+1}$ is a random variable, called the SDF, that maps future payoffs into today's price.

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$$p = E\left(mx\right)$$

shows that all correction for risk can be captured by **a single random variable** put inside the expectation.

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shows that all correction for risk can be captured by **a single random variable** put inside the expectation.

  - Unlike $\beta$ in a model like the CAPM, the stochastic discount factor $m$ is the **same** for **all** assets...

# Example One

- Suppose investor preferences are $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ with $\gamma = 2$ and $\beta = 0.99$.
- There are three possible states of nature next period: good, average, and bad.
- The investor consumes 7 units of the consumption good today.
- Given the possible payoffs of the security $\left(x^A\right)$ in the table that follows, we're going to figure out the price the investor would pay for the security today.

# Example One: Asset A

- First, what is the **expected payoff** of the security?

| State | Prob | Future C | Payoff |
|---|---|---|---|
| $s$ | $\pi_{t+1}^{s}$ | $c_{t+1}$ | $x_{t+1}^{A}$ |
| Good | 0.3 | 12 | 20 |
| Average | 0.4 | 9 | 15 |
| Bad | 0.3 | 6 | 10 |

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- In our case,

$$
\begin{aligned}
E_t\left(x_{t+1}\right) &= \sum_s \pi_{t+1}^s x_{t+1}^A \\
&= (0.3)(20) + (0.4)(15) + (0.3)(10) \\
&= 15
\end{aligned}
$$

# Example One: Asset A

- Next, how do we start figuring out the **price** the investor would pay for the security today?

| State $s$ | Prob $\pi^s_{t+1}$ | Future C $c_{t+1}$ | Payoff $x^A_{t+1}$ |
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- Do you think an investor would pay more or less than $15 for it? Why?

# Example One: Asset A

- Recall our basic pricing equation,

$$p_t = E_t \left[ m_{t+1} x_{t+1} \right] = E_t \left[ \beta \frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)} x_{t+1} \right]$$

- In order to figure out the price, we really just have another expectation to calculate. Given the information in the table, do we have enough information to figure it out?

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# Example One: Asset A

- From $p_t = E_t \left[ \beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1} \right]$, we just need to calculate

$$p_t = \sum_s \pi_{t+1}^s m_{t+1}^s x_{t+1}^s$$

with

$$m_{t+1}^s = \beta \frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)}$$

- Let's see if we can do this in MATLAB...

# Example One: Asset A

- Answer: The price of asset A is

$$\sum_{s} \pi^{s} m_{t+1}^{s} x_{t+1}^{s} = 9.66$$

| State | Prob | Fut C | Payoff A | Marg Util | SDF |
|---|---|---|---|---|---|
| $s$ | $p_{t+1}^{s}$ | $c_{t+1}$ | $x_{t+1}^{A}$ | $u'\left(c_{t+1}\right)$ | $m_{t+1}$ |
| Good | 0.3 | 12 | 20 | 0.007 | 0.337 |
| Ave | 0.4 | 9 | 15 | 0.012 | 0.599 |
| Bad | 0.3 | 6 | 10 | 0.028 | 1.348 |

# Example Two: Asset B

- Assume again that we have the same investor as in Example One
  - He has the same preferences, consumption today, and faces the same three states of nature as before.
- Now, however, he can only invest in another asset, Asset B...

# Example Two: Asset B

- First, what is the **expected payoff** of the security?
- And what is its **price**?

| State $s$ | Prob $\pi^s_{t+1}$ | Future C $c_{t+1}$ | Payoff $x^B_{t+1}$ |
|---|---|---|---|
| Good | 0.3 | 12 | 10 |
| Average | 0.4 | 9 | 15 |
| Bad | 0.3 | 6 | 20 |

# Example Two: Asset B

- Answer: The price of asset B is

$$\sum_s \pi^s m^s_{t+1} x^s_{t+1} = 12.69$$

| State | Prob | Fut C | Payoff B | Marg Util | SDF |
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# What's the Difference between A and B?

- Why, though the expected payoffs are identical, does asset B sell for a higher price than asset A?

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- Why, though the expected payoffs are identical, does asset B sell for a higher price than asset A?
  - Because it covaries negatively with consumption - it pays off well when you are feeling poorly.

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- Can you see from this example how $m_{t+1}$ is a random variable that discounts payoffs to prices?

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  - Often the "market" - for example, the return on the S&P 500 - is used as a proxy for consumption.
  - (This gives us a model like the CAPM.)

# Prices, Payoffs and Notation

- You may be thinking that an asset with price $p_t$ and payoff $x_{t+1}$ is a very restrictive security, but actually this notation is very general...
  - We can easily accommodate many different asset pricing equations with this notation.

# Stocks

- We can easily use this notation to describe stocks.
    - Think about what the "payoff" is for owning a stock...
    - How would you describe this in a today-tomorrow sense?

# Stocks

- The payoff tomorrow is the price you get plus its dividends,

$$x_{t+1} = p_{t+1} + d_{t+1}$$

so for a stock we can write

$$\begin{aligned} p_t &= E_t\left[m_{t+1}x_{t+1}\right] \\ &= E_t\left[m_{t+1}\left(p_{t+1} + d_{t+1}\right)\right] \end{aligned}$$

# Returns

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$$R_{t+1} = \frac{x_{t+1}}{p_t}$$

- We often talk and work in terms of returns, since the gross return is what tells us what our dollar payoff is for each dollar invested.

# Returns

- Since a gross return is the dollar payoff per dollar invested,
  - You can think of a return as a payoff with a price of one.
  - Then we can write
    $$1 = E_t\left(m_{t+1}R_{t+1}\right)$$
    simply by re-writing the fundamental asset pricing equation $p_t = E_t\left(m_{t+1}x_{t+1}\right)$
- Note how the SDF discounts gross returns to their price; which, by definition, is 1.

# Notes about Returns

- Note that
  - Capital letters denote gross returns
  - Lowercase letters denote net returns $r = R - 1$
  - The return can also be defined in continuous compounding terms as $r = \ln(R)$
  - In our example, the net return $r$ is $r = R - 1 = 0.06$, or 6%
- Returns are "nice" and useful in empirical work because they are "stationary" over time in the sense that they don't have trends and you can meaningfully take averages

# Price-Dividend Ratio

- We use returns a lot in empirical work, but often we would prefer a stationary variable that lets us think in terms of prices...
- Return to our definition of the payoff for a stock, $x_{t+1} = p_{t+1} + d_{t+1}$
- First, write the asset pricing equation as

$$p_t = E_t\left[m_{t+1}(p_{t+1} + d_{t+1})\right]$$

- If we divide by today's dividend, we get the present value of the price/dividend ratio...

# Price-Dividend Ratio

- Next, dividing by today's dividend

$$\begin{aligned}
\frac{p_t}{d_t} &= \frac{E_t\left[m_{t+1}(p_{t+1}+d_{t+1})\right]}{d_t} \\
\frac{p_t}{d_t} &= E_t\left[m_{t+1}\left(\frac{p_{t+1}}{d_{t+1}}\frac{d_{t+1}}{d_t}+\frac{d_{t+1}}{d_t}\right)\right] \\
\frac{p_t}{d_t} &= E_t\left[m_{t+1}\left(\frac{p_{t+1}}{d_{t+1}}+1\right)\left(\frac{d_{t+1}}{d_t}\right)\right]
\end{aligned}$$

- This gets us back to thinking about asset **prices**, but we are still looking at stationary variables
  - The price is $p_t / d_t$
  - The payoff is $x_{t+1} = \left(1+\frac{p_{t+1}}{d_{t+1}}\right)\frac{d_{t+1}}{d_t}$

# Excess Returns

- What is an "excess return"?
  - Generally speaking, it is the difference between two returns
  - Also called a "zero-cost" portfolio...
- It is often common to study equity strategies where you sell short one stock/portfolio and invest the proceeds in another to generate an "excess" return
- Think about this... You can borrow a dollar today at $R^f$ and invest it in an asset with return $R$.
  - You pay no money out of pocket today, but you get the payoff $R - R^f$
  - This is a payoff with zero price.

# Example: Excess Returns

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- Consider a strategy of borrowing \$100 to buy a share of Apple stock for \$100. This costs you \$0 today (its price is zero) since you put none of your own money into the investment.
    - But the payoff is not zero! You could make or lose money on the investment.
    - The payoff of such a long-short strategy with price today of zero is an "excess return" where $R^e_{t+1} = R^a_{t+1} - R^f$

# Excess Returns: Long-Short Strategy

- You can see that you don't have to just borrow at the risk-free rate - this can be done with any two assets, say $R^e_{t+1} = R^a_{t+1} - R^b_{t+1}$
  - This strategy is equivalent to "going long" stock A and "going short" stock B
- Notes on "shorting" a stock:

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  - This strategy is equivalent to "going long" stock A and "going short" stock B
- Notes on "shorting" a stock:
  - Going short is the practice of selling assets that have been borrowed from a third party, with the intention to buying identical assets back at a later date to return to that third party.
  - A short seller hopes to profit from a decline in the asset's price, since the seller will pay less to buy the assets back than it received for earlier selling them.

# Excess Returns: Long-Short Strategy

- You can see that - mathematically - short selling is equivalent to buying a negative amount of an asset

$$\begin{aligned} 1 &= E_t\left(m_{t+1}R^a_{t+1}\right) \\ -1 &= -E_t\left(m_{t+1}R^b_{t+1}\right) \end{aligned}$$

- So our asset pricing equation for excess returns becomes

$$0 = E_t\left(m_{t+1}\left(R^a_{t+1} - R^b_{t+1}\right)\right) = E_t\left(m_{t+1}R^e_{t+1}\right)$$

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- Why might excess returns be a useful thing to look at?
  - You can always borrow at $R^f$, so the interesting thing is the return you get **over and above** the risk-free rate.
  - Interest rate variation has little to do with our understanding of risk-premia, so we want to look at interest rates and risk premia separately

# Managed Portfolios

- A managed portfolio is simply one where the weight on each asset varies through time.
- Let $z_t$ be the amount (in dollars) of an asset owned at time $t$. I.e., the "price" of such an asset is the amount invested (in dollars).
- Then the payoff is $z_t R_{t+1}$ or

$$z_t = E_t \left[ m_{t+1} \left( z_t R_{t+1} \right) \right]$$

- Example: a value-oriented timing strategy might make investments proportional to $p_t / d_t$ ratios, investing less when prices are high relative to dividends such that

$$z_t = a - b \left( \frac{p_t}{d_t} \right)$$

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  - Essentially, it is a claim to a \$1 unit payoff in every possible state.
- Using our key equation, the price of a bond is given by

$$p_t = E_t \left[ m_{t+1} 1 \right]$$

# Bonds: Example

- If we go back to the same setup we used in our earlier example,

| $s$ | $\pi^s_{t+1}$ | ... | $m_{t+1}$ |
|---|---|---|---|
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- What is its return?
  - $R_{t+1} = 1/p_t = 1/0.75 = 1.34$, or 34 percent

# End of Today's Lecture.

- That's all for today. Today's material corresponds to parts of Chapter 1 in Cochrane (2005).