

ECON 4360: Empirical Finance

GMM for Linear Factor Models in Discount Factor Form

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Empirics Lecture #08

What are we doing today?

- GMM for Linear Factor Models in Discount Factor Form

Linear Discount Factor Models

- A linear discount model can be expressed as

$$p = E(mx)$$

where

$$m =$$

- A GMM approach to estimating this model will use

Linear Discount Factor Models

- If we have the model

$$E(p) = E(mx)$$

and

$$m = b'f$$

- We can just write

$$E(p) =$$

for a vector of asset prices p and payoffs x that are $N \times 1$, a vector of factors f and parameters b that are $K \times 1$.

Linear Discount Factor Models

- If our model is

$$E(p) =$$

how would we implement GMM?

Linear Discount Factor Models

- To implement GMM, we need to pick a set of moments.
- What do we choose?
 - Easy choice: The pricing errors.

$$g_T(b) =$$

- The GMM estimate solves

$$\min_b$$

- The FOC of this problem is

$$= 0$$

where

$$d' = \frac{\partial g'_T(b)}{\partial b} =$$

- Note that d' is just the

- So the moment condition is
$$= 0$$
- For our first-stage estimates, we just use
- But for the second-stage estimates, we can use

- Recall, that when we were using GMM in a previous exercise to estimate the parameters β and γ in the consumption-based asset-pricing model, we solved for the estimated parameters
- In this model, we can solve for the estimates
 - For out first-stage estimates,

$$\begin{aligned}d' E_T (x f' b - p) &= 0 \\d' E_T (x f' b) &= d' E_T (p) \\E_T (f x') E_T (x f' b) &= d' E_T (p) \\\hat{b}_1 &= (d' d)^{-1} d' E_T (p)\end{aligned}$$

- Interpretation: The first-stage estimate is just

- For our second-stage estimates, by the same algebra, we get

$$\hat{b}_2 = (d' S^{-1} d)^{-1} d' S^{-1} E_T(p)$$

- Interpretation: The second-stage estimate is just a

GMM Methodology: Intuition

- What is this model doing?
- The model

$$E(p) =$$

says that average prices are

- So we
- The regressions operate across assets on sample averages.
 - Data are:
- GMM is finding b to make the model explain the

- The usual GMM standard error formulas apply:
 - For our first- and second-stage estimates

$$\text{cov}(\hat{b}_1) =$$

$$\text{cov}(\hat{b}_2) =$$

- What do you notice about these formulas?
 - They are
- Note that OLS standard errors have to be corrected for

What else?

- Standard GMM results for the covariance matrix of the pricing errors $\left(\text{cov} \left[g_T \left(\hat{b} \right) \right] \right)$ and for the model test (the chi-square test).
 - (See Cochrane, Ch. 13 for the formulas).

- Look again at

$$\hat{b}_1 = (d'd)^{-1}d'E_T(p)$$

- What if we are using excess returns?
 - We would have $\hat{b}_1 = (d'd)^{-1}d'E_T(p) =$
 - The idea here is that
 - What we've done up to now requires that at least one asset has a
- We want to look at the case of excess returns...

- If we write

$$m =$$

we cannot identify a and b separately, so we use a

- Then we have

$$\begin{aligned} g_T(b) &= -E_T(mR^e) \\ &= \end{aligned}$$

with

$$d = \frac{\partial g_T(b)}{\partial b'} =$$

as the second-moment matrix of

- The GMM estimate again solves

$$\min_b g_T(b)' W g_T(b)$$

- And with FOC $-d' W [E_T(R^e) - db] = 0$, we get for the GMM estimates

$$\begin{aligned}\hat{b}_1 &= (d'd)^{-1} d'E_T(R^e) \\ \hat{b}_2 &= (d'S^{-1}d) d'S^{-1}E_T(R^e)\end{aligned}$$

- These GMM estimates are just cross-section regressions of mean excess returns on the second moments of returns with factors; the distribution theory is

Other Normalizations

- We need a normalization to use GMM for Excess Returns, but
 - We can also use
- Why would we want to do that?
 - This normalization allows GMM to run cross-section regressions of mean excess returns on *covariances*... which gets us really close to betas.
 - This normalization yields

$$m = 1 - b' (f - E(f))$$

with

$$E(m) = 1.$$

Other Normalizations

- The derivations proceed the same way as before, but to be concrete

$$\begin{aligned} g_T(b) &= E_T(mR^e) \\ &= E_T(R^e) - E_T(R^e(f - E(f))')b \end{aligned}$$

with

$$d = \frac{\partial g_T(b)}{\partial b'} =$$

- And the moment condition is

$$= 0$$

Other Normalizations

- Note that d is now the covariance matrix,

$$d = \frac{\partial g_T(b)}{\partial b'}$$

- So if we were to write

$$E(R^e) = -cov(R^e, f') b$$

we just have the covariances entering in place of the betas.

- But what's the problem here?
 - We have a
 - We have to take this into account in our distribution theory, just as we did when

Other Normalizations

- For this case, it's important to note that the normalization does NOT matter for the pricing errors and the chi-square statistics.
 - What does change is the estimate of b .
 - Normalizations therefore only matter for
- The very ambitious may want to try the derivation!

- Can one set of factors drive out another set? How would we test this?
- For

$$m = b_1' f_1 + b_2' f_2$$

we just test if

$$b_2 = 0$$

- Note that b_2 can be a scalar (use a t-test) or a vector (use a chi-square test).
- We can also test a restricted v. unrestricted versions of the model (as before, with a chi-square difference test).
 - Remember how these work?

Testing for Priced Factors

- Should we use lambdas λ (the risk factor premia) or b 's b (the regression coefficient of m on f)?
 - Note that b 's and β 's are not the same! β 's are
- The lambdas λ capture whether a factor is
- The b 's b capture whether a factor is

lambdas and b's

- The two are related by $\lambda = E(ff') b$
- Here's the derivation

$$\begin{aligned} E(mR^e) &= 0 \\ E[R^e(1 - f'b)] &= 0 \end{aligned}$$

So

$$\begin{aligned} E(R^e) &= \text{cov}(R^e, f') b \\ &= \text{cov}(R^e, f') E(ff')^{-1} E(ff') b \\ &= \beta' \lambda \end{aligned}$$

- Note that it only matters whether you use b or λ if the factors are correlated.
 - If they are not, $E(ff')$ is diagonal, and each $\lambda_i = 0$ iff the corresponding $b_i = 0$.
- The idea is that λ_i is the single regression coefficient of m on f_i ; while b_i is the multiple regression coefficient of m on f_i , given the other factors.
 - Asking if $\lambda_i = 0$ asks
 - Asking if $b_i = 0$ asks

A Note on Methodologies

- Today, we've used GMM to estimate linear discount models of the form

$$\begin{aligned}E(p) &= E(mx) \\ m &= b'f\end{aligned}$$

- Do we *have* to use GMM to estimate this model?
- Can GMM *only* be used on models of this form?
- Estimation methodology is a choice; but it's important to keep in mind that the results you get depend on how you specify your model and which method you use!

End of Today's Lecture.

- That's all for today. Today's material corresponds roughly to Chapter 13 in Cochrane (2005).