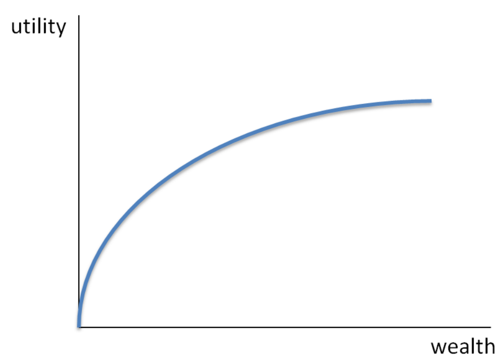
1 Preliminaries: Utility Functions and Risk Aversion

1.1 Utility Functions

- How can we characterize investment possibilities?
- We can think about all the possibilities of investing in different assets as essentially coming down to different gambles with money prizes.
  - For example, suppose there are two investment possibilities:
    - 1.) There is a 50 percent probability you get a payoff of x, and a 50 percent chance you get a payoff of y; where $x < y$.
    - 2.) You get a sure payoff of $1/2 * (x + y)$
- What do you prefer?
  - If we know the correct representation of our (or an investor's) utility function, we should be able to describe behavior over these (and all other) such money gambles
- Since a key objective in asset pricing is to describe investor behavior, we start with a description of the investor's utility function
- We tend to think that preferences are specified by a utility function $u(c)$ where
  - Marginal utility increases with c (i.e., people prefer more to less) $u'(c) > 0$,
  - but at a decreasing rate (diminishing rate of marginal utility) $u''(c) < 0$.
  - This can be shown graphically through concavity of the utility function.



- Again, consider the two investment possibilities:
  - 1.) There is a 50 percent probability you get a payoff of x, and a 50 percent chance you get a payoff of y; where $x < y$.
  - 2.) You get a sure payoff of $1/2 * (x + y)$

- *Which of the two gambles do you prefer if the graph above represents your utility function?*

1.2 Risk Aversion

- In our example, the investor prefers the second choice
  - This first choice is the "gamble"; the second choice is the expected value of that gamble.
  - I.e., our investor prefers to get the expected value of the gamble - or the "sure thing", rather than take the gamble itself.
- When the utility of a gamble is less than the utility of its expected value, we have investor behavior that is characterized as risk aversion.
  - Graphically, if a chord drawn between any two points of the graph lies below the function, we have a concave function.
  - So concavity is equivalent to risk aversion.

1.2.1 Measuring Risk Aversion

- We often want to have a measure of risk aversion.
  - Intuitively, the more concave the utility function, the higher risk aversion.
  - So we measure the degree of risk aversion by the curvature of the utility function.
- Arrow-Pratt measure of ARA

$$r(w) = -\frac{u''(w)}{u'(w)}$$

  - (Why can't we just use the second derivative?)
  - Note that our measure of risk aversion depends on current consumption, or current wealth.

- *How do you think absolute risk aversion might vary with wealth?*

- What about relative gambles?
  - The idea is that with some probability p, you receive x percent of your current wealth; and with probability $(1-p)$ you receive y percent.
  - E.g., returns on investments are usually stated relative to the level of investment.
- The appropriate measure is the Arrow-Pratt measure of RRA

$$\rho = -w\frac{u''(w)}{u'(w)}$$

- *What do you think happens to relative risk aversion as your wealth increases? Would you be more or less willing to risk a specific fraction of it?*

1.3 Common Utility Functions

- We will commonly use power utility, and you'll see why below...
- Power utility

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

  - With first derivative

$$u'(c) = c^{-\gamma}$$

  - And second derivative

$$u'(c) = -\gamma c^{-\gamma-1}$$

- Power Utility provides reasonable properties about risk aversion and this form is very easy to work with, so it is the utility function most often used in empirical work.

  – DARA
$$r(c) = -\frac{u''(c)}{u'(c)} = -\frac{-\gamma c^{-\gamma-1}}{c^{-\gamma}} = \frac{\gamma}{c}$$

  – CRRA
$$\rho(c) = -c\frac{u''(c)}{u'(c)} = -c\frac{-\gamma c^{-\gamma-1}}{c^{-\gamma}} = \gamma$$

- Log utility $u(c) = \ln(c)$ is a special case of power utility.

  – It is the limit of power utility as $\gamma \to 1$.

- *You should be able to show this...*

- Also common: Exponential Utility
$$u(c) = -e^{-\alpha c}$$

  – With $u'(c) = \alpha e^{-\alpha c}$ and $u''(c) = -\alpha^2 e^{-\alpha c}$, so that we have

  – constant absolute risk aversion (CARA)
$$r(c) = -\frac{u''(c)}{u'(c)} = -\frac{-\alpha^2 e^{-\alpha c}}{\alpha e^{-\alpha c}} = \alpha$$

  – and increasing relative risk aversion (IRRA)
$$\rho(c) = -c\frac{u''(c)}{u'(c)} = -c\frac{-\alpha^2 e^{-\alpha c}}{\alpha e^{-\alpha c}} = c\alpha$$

- In summary, we model investors as having preferences over consumption of the form $u(c)$, which is increasing in consumption $u'(c) > 0$, but at a decreasing rate $u''(c) < 0$.

  – Examples: Power utility $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ or log utility $u(c) = \ln(x)$.

- You may have noticed that sometimes we write $u(w)$ and sometimes we write $u(c)$ to represent utility over wealth and consumption, resp.
  - We like to think that the fundamental goal is consumption.
  - This also implies that investors really care about consumption, not intermediate objectives like mean and variance.

2 The Consumption-Based Model

2.1 Overview

- We're going to set up a basic problem that derives the central asset pricing equation that we'll use throughout the course from a consumption-based model.
- The way we think about this is to focus on the basic trade-off for any investor: consumption now versus consumption later.
  - Because investors face a budget constraint, they have to give up a little consumption today to get a little more consumption tomorrow.
- Now, we're going to set up a simple problem for our investor: to maximize his utility...

2.2 Model Setup

- Suppose our investor lives for two periods - today and tomorrow - and receives an endowment that he can consume each period.
  - An investor can smooth consumption over time by purchasing n units of a risky security that has price p_t and gives an (uncertain) payoff next period of x_{t+1}.
  - Given p_t, the investor must choose n.
- The investor's problem is to maximize utility by choosing the number of risky securities to purchase

  $$\max_{n,c_t,c_{t+1}} \quad u(c_t) + E_t[\beta u(c_{t+1})]$$

  s.t.

  $$\begin{aligned} c_t &= e_t - np_t \\ c_{t+1} &= e_{t+1} + nx_{t+1}, \end{aligned}$$

  where e_t is the investor's endowment at time t.
- *What do we want to get out of this problem?*

- How do we solve this problem?
  - First, let's substitute out the budget constraints to simplify the problem...

$$\max_{n} \quad u\left(e_t - np_t\right) + E_t\left[\beta u\left(e_{t+1} + nx_{t+1}\right)\right]$$

  - Notice how the problem has simplified...we only have to choose one variable n now...

- So take first-order conditions and set them equal to zero

$$\begin{aligned}
\frac{d}{dn}\left[u\left(e_t - np_t\right) + E_t\left[\beta u\left(e_{t+1} + nx_{t+1}\right)\right]\right] &= 0 \\
u'\left(c_t\right)\left(-p_t\right) + E_t\left[\beta u'\left(c_{t+1}\right)x_{t+1}\right] &= 0 \\
u'\left(c_t\right)p_t &= E_t\left[\beta u'\left(c_{t+1}\right)x_{t+1}\right]
\end{aligned}$$

- *Think about what this says -*

2.3 Key Equation

- We will usually re-arrange this key FOC

$$u'\left(c_t\right)p_t = E_t\left[\beta u'\left(c_{t+1}\right)x_{t+1}\right]$$

  to write the equation as

$$p_t = E_t\left[\beta \frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)} x_{t+1}\right]$$

- This is THE CENTRAL ASSET PRICING EQUATION.
- In general, an investor must decide how much to save and how much to consume, and what portfolio of assets to hold.
  - The most basic pricing equation (the one that characterizes the SDF methodology) comes from the FOC for that decision.
- The basic idea is that the marginal utility loss of consuming a little less today (and buying a little more of the asset) should equal the marginal utility gain of consuming a little more of the asset's payoff in the future.

  - If prices and payoffs do not satisfy this relationship, the investor should buy more or less of the asset.

- Let's see if we can get the basic intuition from a non-stochastic case.

  - Now we have that
$$u'(c_t)\, p_t = \beta u'(c_{t+1})\, x_{t+1}$$
  - And suppose $\beta = 1$, $p_t = 2$, $x_{t+1} = 6$, $u'(c_t) = 0.2$, and $u'(c_{t+1}) = 0.1$.
  - *Is the first-order condition met? If not, what should the investor do? And how would that affect his utility?*

2.4 Big Picture

- So what have we learned?

  - We know that an asset's price should equal the expected discounted value of the asset's payoff, but:

- What should we use to discount the payoff?

  - The investor's marginal utility. Why?
  - Marginal utility (not consumption) is the fundamental measure of how you feel.

- Most of the theory of asset pricing is about how to go from marginal utility to observable indicators.

  - There is a relationship between consumption and marginal utility (inverse); so consumption - of course - may be a useful indicator.

3 SDF Methodology

- This brings us back to the central asset pricing equation we introduced last time.
- We can use the pricing equation we just derived

$$p_t = E_t\left[\beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1}\right]$$

  by defining

$$m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)}$$

  and writing

$$p_t = E_t\left[m_{t+1} x_{t+1}\right]$$

- In our key equation, the variable m_{t+1} is a random variable, called the SDF.
  - It maps future payoffs into today's price
  - It is a generalization of standard discount factor ideas, as you will see next...

3.1 Discount Factor Ideas

- Let's Think About Discount Factors...
- Consider an environment where there is no uncertainty.
- What is the price (i.e., present value) of a payoff tomorrow of x_{t+1} if the interest rate is $R^f = \left(1 + r^f\right)$?
  - Using standard present value ideas, $p_t = \frac{1}{R^f} x_{t+1}$
  - Here, the discount factor is $\frac{1}{R^f}$
  - See how the payoff tomorrow sells "at a discount"?
- Now, let's think about generalizing this idea to risky assets:

$$p_t^i = \frac{1}{R^i} E_t\left(x_{t+1}^i\right)$$

  - The idea is that risky assets have lower prices than risk-free assets, so they use asset-specific risk-adjusted discount factors.
    - The risk that is inherent in asset i is already taken into account in its asset-specific return, R^i
  - This is a traditional view of asset pricing that uses R^i as a risk-adjusted rate of return particular to each asset i
    - E.g., these can come from a model like the CAPM
- But We can be Even More General...
  - The generalization is $p_t = E_t\left[m_{t+1} x_{t+1}\right]$

  - It says that by putting the discount factor inside the expectation, we can use a single discount factor (the same one for each asset) to incorporate all risk corrections

- *Can you see what will generate asset-specific risk corrections here?*

- So why is this useful?
  - All asset pricing models are just different ways of connecting the SDF m_{t+1} to the data.
  - We will soon see that we can use equation $p_t = E_t\left[m_{t+1}x_{t+1}\right]$ in alternative ways to come up with different empirical approaches.
  - By separating the model into these two pieces, we can skip a lot of steps and elaboration for each asset pricing model.
    * That is, all asset pricing models simply use a different m_{t+1}.
    * E.g., $p_t = E_t\left[m_{t+1}x_{t+1}\right]$ is valid for different utility functions, etc.