1 Portfolio Theory

1.1 Overview

- Harry Markowitz
  - 1952 Article: "Portfolio Selection" in the Journal of Finance
  - 1990 Nobel Prize
- Investors are risk-averse, so they want to minimize risk while maximizing return. How do they do this?
- What two portfolio statistics are most important to investors?
- Portfolio Theory relates concepts of diversification to investing, using only a handful of simple statistics.
  - What about, e.g., firm earnings, dividend policies, financial statements, etc?

1.2 Expected Portfolio Return

- How do we find the expected return on a portfolio?
- Let's start simple: Suppose I invest half my money in the stock market $E\left(R^m\right) = 8\%$ and half my money in bonds $E\left(R^f\right) = 2\%$. What's the expected return on my portfolio?
- Generally, we find expected portfolio return by

$$E\left(\tilde{R}^p\right) = \Sigma_{i=1}^{n} w^i E\left(\tilde{R}^i\right)$$

where $E\left(\tilde{R}^p\right)$ is the expected return on the portfolio, $E\left(\tilde{R}^i\right)$ is the expected return on asset i, w^i is the weight (percent) invested in asset i, and n is the number of assets in the portfolio.

  - Note that $\tilde{R}^p$ is the actual return on the portfolio - it is a random variable.
  - $E\left(\tilde{R}^p\right)$ is the expected return...it is not random.

1.2.1 Example

- The expected portfolio return is simply a weighted average of the individual expected returns of the stocks.
- But what about portfolio risk?
- To see this, suppose we have two companies AAPL and HP, both selling at \$300 per share
  - Next year AAPL will be at \$375 or \$250 will equal probability
  - HP is exactly like AAPL, except when AAPL is \$375/share, HP is \$250 per share (and vice-versa).
- What is $E\left(\tilde{R}^i\right)$ for $i = AAPL, HP$?
- Each company is risky, but think about what we can do here...

1.3 Portfolio Risk

- Knowing how stocks move together is important in determining the risk of a portfolio
  - This is where we need correlation coefficients...
- To measure the "risk" - i.e., standard deviation - of the portfolio, we have to account for how stocks move together.
  - For two stocks, X and Y,
$$\sigma_p = \sqrt{w_x^2\sigma_x^2 + w_y^2\sigma_y^2 + 2w_x w_y \sigma_{xy}}$$
  - Note that σ_{xy} is the covariance between the returns of X and Y, and
$$\sigma_{xy} = \rho_{xy}\sigma_x\sigma_y$$
- You can easily see that as the covariance gets more negative, the portfolio is made less "risky"
  - In general, the lower the correlation between the stocks, the lower the risk of the portfolio

1.3.1 Example

- As an example, suppose that we invest half our money in BBB stock and half in Wal-Market.
  - Suppose that $\sigma_{BBB} = 15\%$ and $\sigma_{WAL} = 11\%$, and that the correlation between the two stocks is -0.85
- How "risky" is the portfolio?

1.4 Efficient Portfolio Frontier

- If an investor's goal is to maximize return and minimize risk
  - Investors can choose the weights of different stocks in their portfolio to attain different risk-return combinations.
- What are the possible risk-return combinations for two stocks, X and Y, if

$E\left(R^X\right) = 7\%$	$E\left(R^Y\right) = 9\%$
$\sigma_X = 15\%$	$\sigma_Y = 11\%$

  and the correlation is $\rho = -0.85$?
  - What would this look like graphically?

- Graphically, what part is the Feasible Set?
  - I.e., what region on the graph tells us the possible risk-return combinations that are available as a portfolio of the two assets?

- Graphically, what part is the Efficient Portfolio Frontier?
  - I.e., what part of the graph tells you for any given level of risk, what is the maximum expected return you can get?

- How do investors choose a portfolio?
  - I.e., how do investors make mean-variance trade-offs?

- What do you think the graph would look like if everything stays the same, except
  - The correlation coefficient is now $\rho_{X,Y} = 1$? What about $\rho_{X,Y} = -1$?

1.5 Linear Algebra Representation for N Securities

1.6 A Portfolio of N Securities

- It's easy to see how diversification can reduce risk for a two security portfolio
  - This general result still holds for N securities: Diversification reduces risk

- Again, the expected return of a portfolio of N securities is

$$E\left(\tilde{R}^p\right) = \Sigma_{i=1}^{N} w^i E\left(\tilde{R}^i\right)$$

- And the standard deviation of the portfolio's return is

$$\begin{aligned} \sigma_p &= \sqrt{\Sigma_i \Sigma_j w_i w_j \rho_{ij} \sigma_i \sigma_j} \\ &= \sqrt{\Sigma_i \Sigma_j w_i w_j \sigma_{ij}} \end{aligned}$$

- Where σ_{ij} is the covariance between stocks i and j

1.6.1 Example: A Portfolio of 3 Securities

- For example, let's look at the expression for the portfolio standard deviation when there are 3 securities
  - We can write $\sigma_p = \sqrt{\Sigma_i \Sigma_j w_i w_j \sigma_{ij}}$ as

$$\begin{aligned} \sigma_p^2 &= w_1 w_1 \sigma_{11} + w_1 w_2 \sigma_{12} + w_1 w_3 \sigma_{13} \\ &+ w_2 w_1 \sigma_{21} + w_2 w_2 \sigma_{22} + w_2 w_3 \sigma_{23} \\ &+ w_3 w_1 \sigma_{31} + w_3 w_2 \sigma_{32} + w_3 w_3 \sigma_{33} \end{aligned}$$

- This gets cumbersome....
- So let's re-write this using linear algebra... Can you see how?

1.7 A Portfolio of N Securities with Linear Algebra

- For concreteness, if we have N securities, let

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix}; E = \begin{bmatrix} E\left(R^1\right) \\ E\left(R^2\right) \\ \vdots \\ E\left(R^N\right) \end{bmatrix}; \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N1} & \sigma_{N2} & \cdots & \sigma_{NN} \end{bmatrix}$$

- Then

$$E\left(\tilde{R}^p\right) = w'E$$

- And

$$\sigma_p^2 = w'\Sigma w$$

1.7.1 Example: A Portfolio of 2 Securities

- To remind you of how it works...

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}; E = \begin{bmatrix} E\left(R^1\right) \\ E\left(R^2\right) \end{bmatrix}; \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

- Then

$$E\left(\tilde{R}^p\right) = w'E = \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} E\left(R^1\right) \\ E\left(R^2\right) \end{bmatrix} = w_1 E\left(R^1\right) + w_2 E\left(R^2\right)$$

- And

$$\begin{aligned} \sigma_p^2 &= w'\Sigma w = \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\ &= \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} w_1\sigma_{11} + w_2\sigma_{12} \\ w_1\sigma_{21} + w_2\sigma_{22} \end{bmatrix} \\ &= w_1 w_1 \sigma_{11} + w_1 w_2 \sigma_{12} + w_2 w_1 \sigma_{21} + w_2 w_2 \sigma_{22} \end{aligned}$$

1.8 Application: Finding Optimal Weights

- Think about an investor who would like to find a portfolio of N assets - with the lowest possible standard deviation - that gives him an expected return of 8.5%.
  - How could you set up a problem to find the optimal weights?

- If we use our two asset example from earlier, we get that the optimal portfolio contains 25% of asset X and 75% of asset Y, with $\sigma_p = 5.43\%$.
  - Check this: Is this what our earlier graph says we should get?

1.9 Risk-Free Borrowing and Lending

- Let's continue to use our example with 2 risky securities, X and Y.
- Suppose, in addition, there is now a risk-free asset that has a return of 7.5%

  - How does the possibility of investing in the riskless asset change our risk-return opportunities?

- First, note that we can treat a portfolio of the risky stocks X and Y as a single risky asset.
- Let's take a particular portfolio of X and Y for an example...
  - E.g., if $w_X = 0.4$ and $w_Y = 0.6$, then the portfolio has an expected return of 8.2% and a standard deviation of 3.5%.
  - Call this portfolio c
- Second, note that we can treat the risk-free asset as a "risky" asset
  - Its expected return is 7.5% and it has a standard deviation of 0.0
  - Call this asset f
- Now, we can find the various risk-return tradeoffs by combining these two assets in a portfolio

$$\begin{aligned} E\left(\tilde{R}^p\right) &= w_c E\left(\tilde{R}^c\right) + w_f E\left(\tilde{R}^f\right) \\ \sigma_p &= \sqrt{w_c^2\sigma_c^2 + w_f^2\sigma_f^2 + 2w_c w_f \sigma_{cf}} \end{aligned}$$

- What can we infer about σ_f and σ_{cf}?

- Using this information, we can create a table of possible risk-return tradeoffs as we did before and plot them...
  - What is the shape of the portfolio risk-return tradeoff when one of the assets is riskless?
- Now, we have a new efficient frontier.

- What is the graphical distinction between borrowing and lending at the risk-free rate?

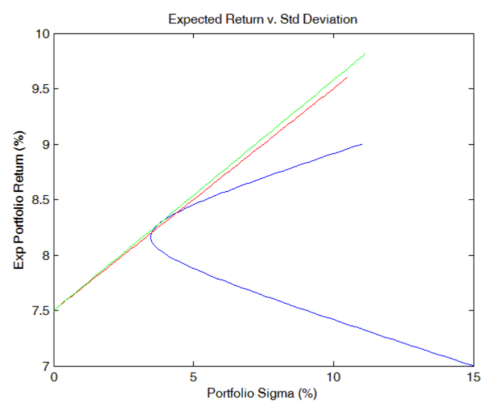
- Is there a "better" efficient frontier than this one?

1.9.1 Efficient Frontier with a Risk-Free Asset

- The "tangency portfolio" maximizes

$$\text{slope } = \frac{E(R^p) - R^f}{\sigma_p}$$

- In this example, it's found where $w_X = 0.3636$ and $w_Y = 0.6364$



- (Think about how you might be able to find this in MATLAB...)