1 Equivalent Representations

1.1 Intro

- So far in this course, we have seen three basic representations of asset pricing models:
  - The β Representation: The expected return of any asset can be expressed as $E\left(R^i\right) = \gamma + \beta_{i,m}\lambda_m$
  - Mean-Variance Frontier: If R^{mv} is on the mean-variance frontier, then $m = a + R^{mv}$ can price all assets.
  - Discount Factor Representation: Given m such that $p = E(mx)$, then m, x^*, R^*, or $R^* + wR^{e*}$ can be used to price all assets
- What we're going to show now is that these representations are "equivalent"
- Credit for the connections:
  - Roll (1976): Between the mean-variance frontier and beta pricing (the CAPM)
  - Ross (1978) and Dybvig and Ingersoll (1982): Between linear discount factors and beta pricing
  - Hansen and Richard (1987): Between a discount factor and the mean-variance frontier

1.2 Discount Factors to Betas

- From Discount Factors to Betas: m, x^*, and R^* can all be the single factor in a single-beta representation

1.2.1 Using m

- Given m, $p = E(mx)$ implies $E\left(R^i\right) = \gamma + \beta_{i,m}\lambda_m$
  - To see this...
  - Start with
  $$1 = E\left(mR^i\right) = E(m)\,E\left(R^i\right) + cov\left(m, R^i\right)$$
  - So
  $$E\left(R^i\right) = \frac{1}{E(m)} - \frac{cov\left(m, R^i\right)}{E(m)}$$
  - Now, define $\gamma = 1/E(m)$, so
  $$\begin{aligned} E\left(R^i\right) &= \gamma + \left(\frac{cov\left(m, R^i\right)}{var(m)}\right)\left(-\frac{var(m)}{E(m)}\right) \\ &= \gamma + \beta_{i,m}\lambda_m \end{aligned}$$
  - What this says is that expected returns should be linear in the regression betas of asset returns on m.
    * For example, m could be $\left(c_{t+1}/c_t\right)^{-\gamma}$, where γ is risk aversion.

1.2.2 Using x^*

- Given x^*, $p = E(x^*x)$ implies $E(R^i) = \gamma + \beta_{i,x^*}\lambda_{x^*}$
  - To see this...
  - Recall that we can write the price of an asset in terms of the mimicking portfolio, x^* as $p = E(x^*x)$
  - Therefore,
  $$1 = E(mR^i) = E(x^*R^i) = E(x^*)E(R^i) + cov(x^*, R^i)$$
  - So as before, we can use x^* instead of m and write
  $$\begin{aligned} E(R^i) &= \gamma + \left(\frac{cov(x^*, R^i)}{var(x^*)}\right)\left(-\frac{var(x^*)}{E(x^*)}\right) \\ &= \gamma + \beta_{i,x^*}\lambda_{x^*} \end{aligned}$$
  - Note: the term $1/E(x^*)$ is the zero-beta rate and applies when there is no riskfree asset.

1.2.3 Using R^*

- Given R^*, $p = E(mx)$ implies $E(R^i) = \gamma + \beta_{i,R^*}\lambda_{R^*}$
  - To see this...
  - Recall the definition of R^* is
  $$R^* = \frac{x^*}{E(x^{*2})}$$
  - Multiply both sides by R^* and take expectations
  $$E(R^{*2}) = \frac{E(R^*x^*)}{E(x^{*2})}$$
  - And since the price of R^* is one, $1 = E(R^*x^*)$ and
  $$E(R^{*2}) = \frac{1}{E(x^{*2})}$$
  - Now we can re-arrange and combine this last expression with the first to get
  $$x^* = R^*E(x^{*2}) = \frac{R^*}{E(R^{*2})}$$
  - Now, we can substitute this expression for x^* into
  $$E(R^i) = \gamma + \left(\frac{cov(x^*, R^i)}{var(x^*)}\right)\left(-\frac{var(x^*)}{E(x^*)}\right)$$

  to get
  $$\begin{aligned} E(R^i) &= \gamma + \left(\frac{cov(R^*, R^i)}{var(R^*)}\right)\left(-\frac{var(R^*)}{E(R^*)}\right) \\ &= \gamma + \beta_{i,R^*}\lambda_{R^*} \end{aligned}$$

- Now this expression

$$E\left(R^i\right) = \gamma + \beta_{i,R^*}\lambda_{R^*}$$

has to hold for any R^i, including R^*, so we can write

$$E\left(R^*\right) = \gamma + \beta_{R^*,R^*}\lambda_{R^*} = \gamma + \lambda_{R^*}$$

- So the price of risk can be written as

$$\lambda_{R^*} = E\left(R^*\right) - \gamma$$

- And the return on any asset can be written as

$$E\left(R^i\right) = \gamma + \beta_{i,R^*}\left[E\left(R^*\right) - \gamma\right]$$

- Recall that the traditional CAPM states that

$$E\left(R^i\right) = R^f + \beta_{i,R^m}\left[E\left(R^m\right) - R^f\right]$$

and recall that $\gamma = 1/E\left[m\right] = R^f$, so we are close to the CAPM here.

- This is not quite the CAPM, since though R^* is on the mean-variance frontier, it is not the market portfolio.
  - We will see later that any return on the MV Frontier can be used in place of R^*. So if the market portfolio is on the frontier, then the CAPM will hold.
- Note that because of this, the CAPM will **only** work if the market portfolio is mean-variance efficient (this is Roll's critique).

1.3 From MV Frontier to m and beta

- Now, we'll go in the other direction...
- If R^{mv} is any return on the mean-variance frontier, then

$$m = a + bR^{mv}$$

will price assets via

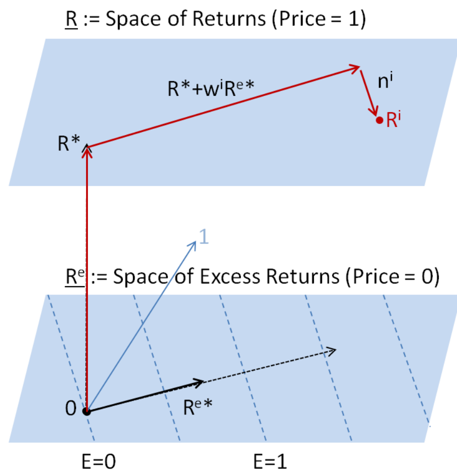
$$p = E\left(mx\right)$$

and so we can express the expected return of any asset as

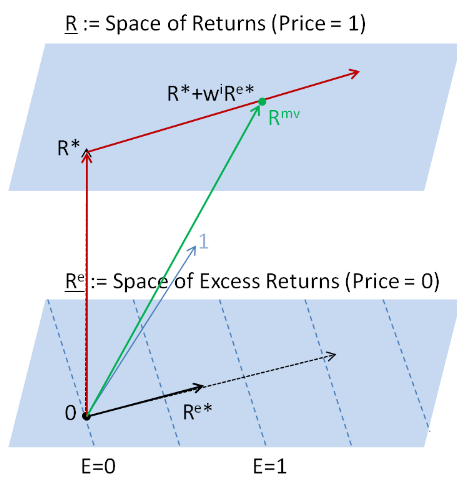
$$E\left(R^i\right) = \gamma + \beta_{i,R^{mv}}\left[E\left(R^{mv} - \gamma\right)\right]$$

- *Theorem: There is a discount factor of the form $m = a + bR^{mv}$ iff R^{mv} is on the mean-variance frontier.*
- Proof by pictures...

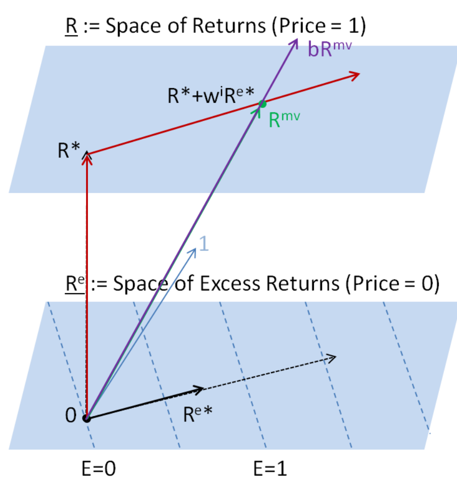
– Recall our characterization of the MV Frontier



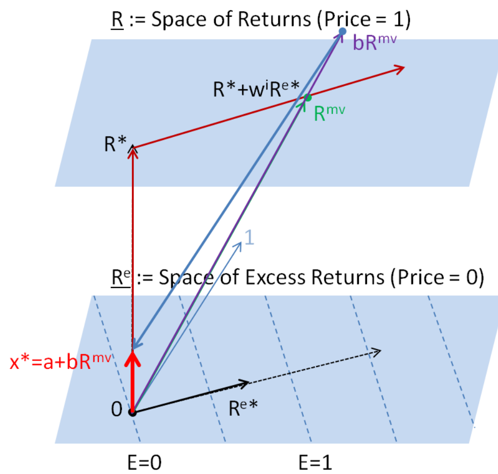
– The MV Frontier is swept out by $R^{mv} = R^* + wR^{e*}$ as real number w varies.



– We can pick any vector R^{mv} on the frontier and then stretch it to length bR^{mv}



- Then we can add (subtract) an amount a of the 1 vector, which is simply the constant. This gets us back to the tip of x^*



- Therefore, we can find some a and b, such that $x^* = a + bR^{mv}$
- Now, we can get to m by adding an orthogonal piece ε to x^*:

$$p = E(mx) = E((x^* + \varepsilon)x) = E(x^* x)$$

- Recall that there are an infinite number of m's that can price assets: $m = x^* + \varepsilon$
- Perhaps some $a + bR$ with R not on the MV Frontier could be a discount factor?
- Turns out this can't be the case...
- For any R, we can write $R = R^* + wR^{e*} + n$
  * Recall that frontier returns have $n = 0$...
- If n is not zero, there is no way to express x^* as a linear form $a + bR$.
- To see this, pick some R not on the frontier where $n \neq 0$.
  * Now, if we stretch this R to any bR, there is no way to subtract the constant (the 1 vector) to get back to x^*.
- Hence, only returns that are on the frontier can be transformed by $a + bR^{mv}$ into a valid SDF.

1.4 Factor Models and Discount Factors

- We've shown that $p = E(mx)$ implies a single beta representation.
- Now let's go the other way. Given an expected-return beta model (e.g., CAPM), what discount factor does that imply?
- Theorem: *Given the model*

$$m = a + b'f,\ 1 = E\left(mR^i\right)$$

where b is a vector of constants and f is a vector of factors, we can find γ and λ such that

$$E\left(R^i\right) = \gamma + \lambda' \beta_i$$

where the β_i are multiple regression coefficients of R^i on f with a constant. Conversely, given γ and λ in a factor model, one can find a and b such that $m = a + b'f$.

- The most common example of this type of model is the CAPM:

$$E\left(R^i\right) = R^f + \beta_{i,R^m}\left[E\left(R^m\right) - R^f\right]$$

  - We would first regress a time-series of stock returns on our factor (the market portfolio) to get a β_i for each stock. Then we would see if β could explain stock returns in the cross-section: $E\left(R^i\right) = \gamma + \lambda\beta_i$

- To prove this theorem, we just have to find a relation between (λ, γ) and (a, b) and show that it works.
  - Start with $m = a + b'f$ and $1 = E\left(mR^i\right)$ and fold the means of f into a so that $E(f) = 0$.
  - We have

$$1 = E\left(mR^i\right) = E\left[\left(a + b'f\right)R^i\right] = aE\left(R^i\right) + b'E\left(fR^i\right)$$

  - Solve for $E\left(R^i\right)$:

$$E\left(R^i\right) = \frac{1}{a} - \frac{E\left(R^i f'\right)b}{a}$$

  - If we regress R^i on f, the fitted coefficient vector is $\beta_i = E\left(ff'\right)^{-1} E\left(fR^i\right)$, so we can incorporate β now and continue with...

$$\begin{aligned}
E\left(R^i\right) &= \frac{1}{a} - \frac{E\left(R^i f'\right)b}{a} \\
&= \frac{1}{a} - \frac{E\left(R^i f'\right)E\left(ff'\right)^{-1}E\left(ff'\right)b}{a} \\
&= \frac{1}{a} - \left[E\left(R^i f'\right)E\left(ff'\right)^{-1}\right]\frac{E\left(ff'\right)b}{a} \\
&= \frac{1}{a} - \beta_i \frac{E\left(ff'\right)b}{a}
\end{aligned}$$

  - And define γ and λ to make it work...
  - Define

$$\gamma := \frac{1}{E(m)} = \frac{1}{a}$$

$$\lambda := -\frac{1}{a}E\left(ff'\right)b = -\gamma E\left(ff'\right)b = -\gamma E(mf) = -\gamma p(f)$$

  - So

$$E\left(R^i\right) = \gamma + \lambda' \beta_i$$

- So we can:
  - Start with the discount factor (a, b) and get the factor model (λ, γ). For this, we have to rule out an infinite riskfree rate, $E(m) = 0$

- Or we can start with a factor model (λ, γ) and get the discount factor (a, b). Note that since $a = 1/\gamma$, we have to rule out the riskfree rate being zero $(\gamma = 0)$. And since $b = -\lambda a E(ff')^{-1}$, we have to rule out $E(ff')$ being singular.
- We can go from a multiple beta model to a single beta model, just use $m = a + b'f$ as the single factor.

2 Implications

- So what are the implications of these equivalences?
  - We're now going to discuss several of these in turn...

2.1 Ex-Ante and Ex-Post

- Roll showed that mean-variance efficiency implies a single-beta representation.
  - Problem: You can always find some mean-variance efficient return ex-post
    * So you can always find a single-beta representation that works.
- The proper way to test is not ex-post, but ex-ante.
- Theory should tell us what the reference portfolio should be
  - And then we should test to see if the reference portfolio is efficient
  - Example: The CAPM tells us that the reference portfolio should be the market portfolio. Therefore, the only test of the CAPM is if the market portfolio is mean-variance efficient.
- The expression $p = E(mx)$ is just an updated re-statement of Roll's theorem.
  - $p = E(mx)$ always works for some m, like x^*
- Key: Use theory to guide us in writing $m = f(\text{data})$
- Equivalent to this, if the s**ample** covariance matrix for a set of returns is non-singular, then there exists an **ex-post** mean-variance efficient portfolio where sample average returns line up exactly with sample betas.
  - Ex-post, you can always find a portfolio that makes the asset pricing model work.
  - You need theory to put restrictions on the reference portfolio (or the m). Then you can test the asset pricing model.
- Danger: A lot of asset pricing research proposes a set of ad hoc factors, tries them all, gets a few that work "pretty well" and then claims success, in that the model is not statistically rejected.

- Question: If x^* prices all returns correctly, then why not forget about the CAPM, marginal utility, etc., and simply price returns using x^*?

- What can be done? There are two possible solutions:
  - Use economic theory to guide us on what $m = f\,(\text{data})$ should look like
  - Use out-of-sample (new time periods) and cross-sample (e.g., different countries) tests to confirm the results

- Problems:
  - The factors that current theory states should work well don't. We need new theories
  - What about with the second?

2.2 Market Efficiency

- If investors are irrational and markets are inefficient (as some people claim), is finding an asset pricing model hopeless?

- Actually... All we need is the absence of arbitrage to give us the existence of m.
  - Since consistent arbitrage opportunities do not exist, we know that there is an SDF that can be used to price payoffs.

- Markets can appear to be irrational without generating arbitrage opportunities if and only if the discount factors that generate asset prices are disconnected from MRS's.

2.3 Number of Factors

- Some researchers have focused on the number of factors needed to explain the cross-section of returns.
  - For example, we could have $m = b'f$ where f is a 5×1 vector.

- We have already shown that this is really not that meaningful...
  - We can always reduce a model with 5 factors to a model with 1 factor, m, and find betas not with respect to the factors, but with respect to m.

- We may want to keep a multifactor model if the factors have economic meaning that adds richness to the model; but at the end of the day, using the 5 factors or using m as a single factor will price assets just the same.

2.4 Historical Context

- Historically, asset pricing began by putting means on the y-axis and standard deviations on the x-axis.
  - We then described consumer preferences over means and variances by way of a utility function $u(\mu, \sigma)$.
  - Given this framework, the expected return $E\left(R^i\right)$ measured the security's contribution to an overall portfolio expected return and its beta β measured the contribution to portfolio variance...
- From the CAPM, we know the relation between expected return and risk:

$$E\left(R^i\right) = R^f + \beta_{im}\left(E\left(R^m\right) - R^f\right)$$

- But while the form above deals with expectations, the following deals with actual outcomes:

$$R^i = R^f + \beta_{im}\left(R^m - R^f\right) + \varepsilon^i$$

- Note that R^i, R^m, and ε^i are the only random variables... everything else is a constant...
- Looking at the latter, we can figure out how the variance of a stock's return relates to the CAPM...
- Take the variance of both sides of the previous equation to get

$$var\left[R^i\right] = \beta_{im}^2 var\left[R^m\right] + var\left[\varepsilon^i\right]$$

  where
  - $var\left[R^i\right]$ is the variance of returns (total risk) for stock i
  - $\beta_{im}^2 var\left[R^m\right]$ is the market (systematic) risk for stock i
  - and $var\left[\varepsilon^i\right]$ is the non-market (unique, systematic) risk for stock i
- In words,

$$\begin{aligned}
\text{Total Risk} &= \text{(market risk)} + \text{(non-market risk)} \\
&= \text{(systematic risk)} + \text{(non-systematic risk)} \\
&= \text{(non-diversifiable risk)} + \text{(diversifiable risk)}
\end{aligned}$$

- In the $p = E(mx)$ framework, we have a much more direct mapping of finance into microeconomics.
  - Instead of dealing with heuristics like mean and variance, we can specify preference and budget constraints over state-contingent consumption.
- Why would we prefer this representation?
  - It is much simpler... We get all the classical results, plus the $p = E(mx)$ framework can be used to price *any* asset.