ECON 4360: Empirical Finance

Captial Asset Pricing Model (CAPM)

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Theory Lecture #11

What are we doing today?

- CAPM - Capital Asset Pricing Model

Introduction to CAPM

- Portfolio Theory provides a natural introduction to CAPM
  - Continuation of mean-variance analysis
- Recall, when we use Portfolio Theory, we take prices and returns as givens
- Now, we're going to introduce the CAPM, which is a model of asset pricing
  - It is an equilibrium model - we can derive equilibrium prices, returns, and risk-premiums
  - It relates the expected return of a stock to its risk

CAPM: A Simple Asset Pricing Model

- The CAPM is a very simple model of asset pricing.
  - But that simplicity comes at a cost...
  - And there are a number of assumptions that we need to keep in mind.

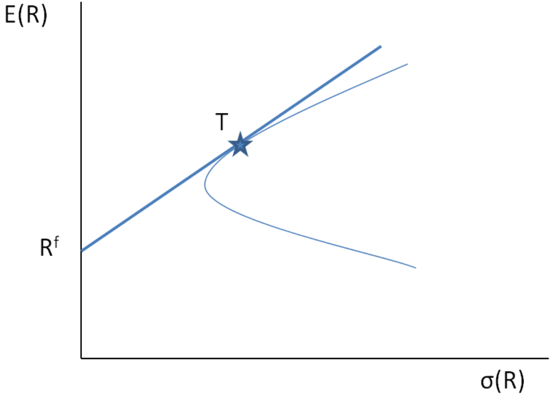
Assumptions for CAPM

- Investors are risk-averse
- All investors have homogeneous expectations about means and variances
- Either
    - Investors only care about mean and variance
    - Stock returns are normally distributed (I.e., Returns only *have* a mean and variance)
- All investors can invest in the same risk-free asset
- Assets are infinitely divisible
- Investors have a one-period time-horizon
- Perfect Markets: E.g., no taxes, no transactions costs, have access to short sale proceeds, etc.

Recall...

- The Efficient Frontier for 1 Risk-Free asset and N Risky Assets
  - Portfolio T is the portfolio with the highest Sharpe ratio

$$\left(E\left[\tilde{R}^p\right]-R^f\right)/\sigma^p$$



As a Consequence of Our Assumptions...

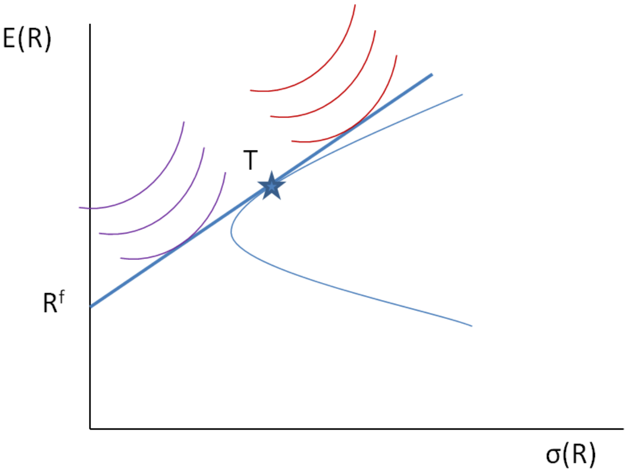
- Since all investors have the same expectations, they all want to hold the same tangency portfolio
- Since the CAPM is an equilibrium model
    - Prices, and therefore expected returns, adjust until supply = demand
    - This is the equilibrium condition for the model: supply = demand for the market portfolio

Result: Two Fund Separation

- Though investors may have different utility functions and levels of risk-aversion, they all choose the same 2 assets
    - The risk-free asset and the tangency portfolio T.
- Powerful result: We don't need to determine 100 million utility functions

Two Fund Separation

- Two investors with difference levels of risk aversion have different optimal portfolios, but they are composed of the same two assets



What does this tell us?

- We know that all investors want to hold 2 assets in their portfolio
  - The market portfolio, call its return R^m, and the risk-free asset R^f
- Because of this, all risk-return combinations can be plotted on a straight line that goes through these two assets
  - This line is called the Capital Market Line (CML)

What do we already know?

- We already know
  - How to find the efficient portfolio frontier of N risky assets.
  - Given the Risk-Free rate, how to solve for the tangency portfolio or market portfolio
- Do we know the slope of the CML? Sure we do.
  - It's easy to read this off the graph

$$\text{slope} = \frac{E\left[R^m\right] - R^f}{\sigma_m}$$

  - As well, the y-intercept of the CML is just R^f

Yay.

- So we actually already know the equation for the CML

$$E\left[R^p\right] = R^f + \left(\frac{E\left[R^m\right] - R^f}{\sigma_m}\right)\sigma_p$$

  - Where $E\left[R^p\right]$ = The expected return on an efficient portfolio
  - And where σ_p = The standard deviation of an efficient portfolio
- The quantity $E\left[R^m\right] - R^f$ is called the *market risk premium*

So where does that get us?

- The CML relates the expected return of any efficient portfolio to the portfolio's "risk" - i.e., standard deviation.
  - But the CML says nothing about risk and return for individual securities
- To relate risk and return for individual securities, we have to use the Security Market Line, or SML...
  - Reminder of where we're trying to go: *Asset Pricing*

SML

- Claim: If portfolio M is on the mean-variance frontier, then

$$E\left[R^i\right] = R^f + \left(\frac{E\left[R^m\right] - R^f}{\sigma_m^2}\right)\sigma_{im}$$

  - (The converse is also true.)

- Proof: Say you hold M. Consider making a small change to your portfolio composition.
  - Let's add a fraction ε of your wealth into asset i by selling ε of the risk-less asset.
  - Call this new portfolio p...

Continuation of Proof

- The (random) return of your new portfolio is

$$\tilde{R}^p = \tilde{R}^m + \varepsilon \left(\tilde{R}^i - R^f \right)$$

- The expected return of your new portfolio is

$$E\left[\tilde{R}^p\right] = E\left[\tilde{R}^m\right] + \varepsilon \left(E\left[\tilde{R}\right]^i - R^f \right)$$

A Digression...

- We're going to need to figure out the variance of our new portfolio as well.
- So as a reminder, here are some useful math facts:
  - If a and b are constants and x and y are random variables...

$$
\begin{aligned}
var(a) &= 0 \\
var(x+b) &= var(x) \\
var(ax) &= a^2 var(x) \\
var(ax+by) &= a^2 var(x) + b^2 var(y) + 2abcov(x,y)
\end{aligned}
$$

Continuation of Proof

- The variance of your new portfolio is

$$
\begin{aligned}
\sigma_p^2 &= var\left(\tilde{R}^p\right) \\
&= var\left(\tilde{R}^m + \varepsilon\left(\tilde{R}^i - R^f\right)\right) \\
&= \sigma_m^2 + \varepsilon^2\sigma_i^2 + 2\varepsilon cov\left(\tilde{R}^m, \tilde{R}^i\right) \\
&\approx \sigma_m^2 + 2\varepsilon cov\left(\tilde{R}^m, \tilde{R}^i\right)
\end{aligned}
$$

Continuation of Proof

- How much did the expected return and variance change from your old portfolio M?

$$E\left[\tilde{R}^p\right] - E\left[\tilde{R}^m\right] = \varepsilon\left(E\left[\tilde{R}^i\right] - R^f\right)$$

and

$$\sigma_p^2 - \sigma_m^2 \approx 2\varepsilon cov\left(\tilde{R}^m, \tilde{R}^i\right)$$

- Intuition: When you add a little more of one stock to a well-diversified portfolio, it's the stock's covariance that is important in determining the risk of the new portfolio.

Reward-to-Risk Ratio?

- The extra return we get for taking on extra portfolio risk is

$$
\begin{aligned}
\frac{E\left[\tilde{R}^p\right] - E\left[\tilde{R}^m\right]}{\sigma_p^2 - \sigma_m^2} &= \frac{\varepsilon\left(E\left[\tilde{R}^i\right] - R^f\right)}{2\varepsilon cov\left(\tilde{R}^m, \tilde{R}^i\right)} \\
&= \frac{\left(E\left[\tilde{R}^i\right] - R^f\right)}{2cov\left(\tilde{R}^m, \tilde{R}^i\right)} \\
&= \frac{\left(E\left[\tilde{R}^i\right] - R^f\right)}{2\sigma_{im}}
\end{aligned}
$$

- Now, we can also think of a portfolio of stocks as simply another asset. What happens if we sell a small fraction ε of the risk-less asset and add a small fraction ε of the market portfolio?

$$
\frac{E\left[\tilde{R}^p\right] - E\left[\tilde{R}^m\right]}{\sigma_p^2 - \sigma_m^2} \approx \frac{\left(E\left[\tilde{R}^m\right] - R^f\right)}{2\sigma_m^2}
$$

Reward-to-Risk Ratio?

- In equilibrium, ratios of extra reward to extra risk should be the same:

$$\frac{\left(E\left[\tilde{R}^{i}\right]-R^{f}\right)}{2\sigma_{im}}=\frac{\left(E\left[\tilde{R}^{m}\right]-R^{f}\right)}{2\sigma_{m}^{2}}$$

- Or, re-arranging

$$E\left[\tilde{R}^{i}\right]=R^{f}+\frac{\left(E\left[\tilde{R}^{m}\right]-R^{f}\right)}{\sigma_{m}^{2}}\sigma_{im}$$

- And this completes our sketch of the proof...

Think about this...

- What if I get more return for the extra portfolio risk with the market portfolio than for asset i?
  - I.e., what if

$$\frac{\left(E\left[\tilde{R}^i\right]-R^f\right)}{2\sigma_{im}}<\frac{\left(E\left[\tilde{R}^m\right]-R^f\right)}{2\sigma_m^2}?$$

SML Logic

- Investors all hold M (The Market Portfolio), so they are concerned with the variance of M.
- The contribution of each security to the variance of M is directly proportional to the covariance of the security with the market.
  - Therefore, the relevant measure of risk for stock i is the covariance of its returns with those of the market, σ_{im}.
- Securities with larger covariances with M contribute more to the risk of the market portfolio, and must provide investors with larger expected returns

$$E\left[\tilde{R}^i\right] = R^f + \frac{\left(E\left[\tilde{R}^m\right] - R^f\right)}{\sigma_m^2}\sigma_{im}$$

  - Therefore, the equation for the SML provides an answer to the question: How much extra return over and above the risk-free rate we should expect, given a security's covariance with the market?

Questions...

- Using the equation for the SML

$$E\left[\tilde{R}^i\right] = R^f + \frac{\left(E\left[\tilde{R}^m\right] - R^f\right)}{\sigma_m^2}\sigma_{im}$$

  - What happens in the special case where $\sigma_{im} = \sigma_m^2$?
  - What about where $\sigma_{im} = 0$?

CAPM Derivation from the Mean-Variance Frontier

- The CAPM follows directly from the SML
- Let's rewrite the equation for the SML

$$E\left[\tilde{R}^i\right] = R^f + \frac{\left(E\left[\tilde{R}^m\right] - R^f\right)}{\sigma_m^2}\sigma_{im}$$

- As follows

$$E\left[\tilde{R}^i\right] = R^f + \frac{\sigma_{im}}{\sigma_m^2}\left(E\left[\tilde{R}^m\right] - R^f\right)$$

  - Where we define "Beta" as $\beta_{im} = \frac{\sigma_{im}}{\sigma_m^2}$
  - We can see that betas represents the sensitivity of expected individual asset returns to the expected market premium

SML in CAPM Form

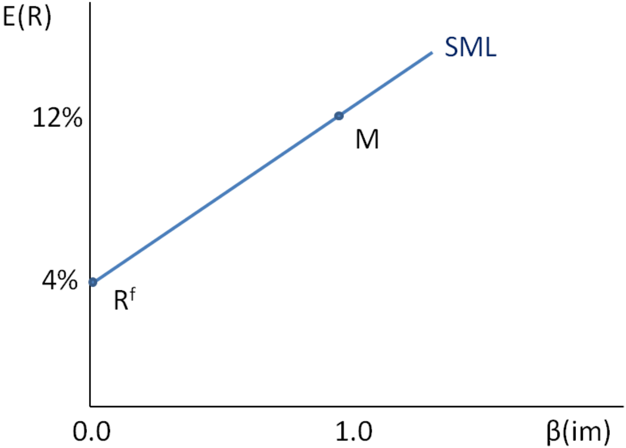
- Now, we can write the SML in CAPM form

$$E\left[\tilde{R}^i\right] = R^f + \beta_{im}\left(E\left[\tilde{R}^m\right] - R^f\right)$$

  - We can easily see here how higher expected returns go with higher risk, defined as its risk in a portfolio context
    - An asset's contribution to portfolio risk matters, idiosyncratic risk does not
- Now, we can plot the SML relating expected individual asset returns to their risk, i.e., their covariance with the market.

SML in CAPM Form

- Say we have a risk-free rate of 4% per year and the expected return on the market is 12% per year.



CAPM

- Now we have an *asset pricing model*
    - The CAPM relates expected return to risk - we just need to find the β's
- What do the β's tell you?
    - A β larger than 1.0 signifies more than "average" riskiness - its covariance with the market is higher, so its expected return must be raised
    - Note that the market itself has a $\beta = 1$
        - What is the "market" portfolio? Stock indices like the S&P are often used as proxies.

Negative Betas?

- What about a stock that has a negative beta?

Negative Betas?

- What about a stock that has a negative beta?
  - It moves opposite to the market's direction. A negative beta stock has an expected return of less than the risk-free rate, but it is still desirable for its potential to reduce risk.

Question for understanding: The Slope of the SML

- Think about the significance of the slope of the SML. What might change its slope over time?
  - Since the SML is a graph of expected return v. beta

    $$E\left[\tilde{R}^i\right] = R^f + \beta_{im}\left(E\left[\tilde{R}^m\right] - R^f\right),$$

    the slope of the SML is the market risk premium: $\left(E\left[\tilde{R}^m\right] - R^f\right)$, which represents the reward that investors get per unit of systematic risk.
  - So for example, if investors become more risk averse over time, they will require more compensation for bearing a given amount of risk, and the slope of the SML would increase.

Asset Pricing using the CAPM

- Let's think about how we might price an asset using the CAPM.
- Recall that our central asset pricing equation

$$p_t = E_t\left[m_{t+1} x_{t+1}\right]$$

  is really just a very general way of mapping future payoffs into today's price.
- This equation is just a generalization of standard discount factor ideas...

Recall: Standard Concept of Present Value

- Consider an environment where there is no uncertainty.
- What is the price (i.e., present value) of a payoff tomorrow of x_{t+1} if the interest rate is $R^f = \left(1 + r^f\right)$?
  - Using standard PV ideas,

$$p_t = \frac{1}{R^f} x_{t+1}$$

- Here, the discount factor is $\frac{1}{R^f}$
  - See how the payoff tomorrow sells "at a discount"

Recall: Present Value with Risky Assets

- Now generalize this idea to risky assets, i.e., where there is some uncertainty about the payoffs
- We could price an asset using

$$p_t^i = \frac{1}{R^i} E_t \left(x_{t+1}^i \right)$$

where R^i is an asset-specific risk-adjusted discount factor

- This is a traditional view of asset pricing that uses R^i as a risk-adjusted rate of return particular to each asset i from a model like the CAPM

A Pricing Example with CAPM

- You expect a stock i to sell for \$100 a year from now and pay a \$5 dividend during the year.
- Suppose the stock's correlation coefficient with the market portfolio is $\rho_{im} = 0.4$ and $\sigma_i = 0.5$.
- You also know that $E\left[R^m\right] = 15\%$ and $\sigma_m = 0.3$; and you know that $R^f = 6\%$.
- At what price should the stock sell for today?

A Pricing Example with CAPM

- First, find the beta of the stock

$$\beta_{im} = \frac{\sigma_{im}}{\sigma_m^2} = \frac{\rho_{im}\sigma_i\sigma_m}{\sigma_m^2} = \frac{(0.4)(0.5)(0.3)}{(0.3)^2} = 0.67$$

A Pricing Example with CAPM

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- Next find the expected rate of return

$$\begin{aligned} E\left[\tilde{R}^i\right] &= R^f + \beta_{im}\left(E\left[\tilde{R}^m\right] - R^f\right) \\ &= 1.06 + (0.67)(1.15 - 1.06) \\ &= 1.12 \end{aligned}$$

A Pricing Example with CAPM

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- Now, we have a number that we can use as a risk-adjusted rate of return for asset i, so

$$p_t^i = \frac{1}{R^i} E_t\left(x_{t+1}^i\right) = \frac{100+5}{1.12} = \$93.75$$

Where we're going...

- What we're going to see in the upcoming lectures is how CAPM fits into our discount factor framework...
  - Factor pricing models, like the CAPM, simply replace the consumption-based expression for marginal utility growth with something that looks like

$$m_{t+1} = \beta \frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)} \approx a + b' f_{t+1}$$

- The questions we're going to be asking are
  - How reasonable are the proxies for marginal utility growth?
  - Can we abide the assumptions - like the ones we listed earlier as necessary for the CAPM - to be comfortable using those models?

End of Today's Lecture.

- That's all for today.