ECON 4360: Empirical Finance

Consumption-Based Model and Overview

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Theory Lecture #02

What are we doing today?

- Introduction of the Consumption-Based Model and SDF Methodology
- Review of Utility Functions and Risk Aversion

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- What do you prefer?
  - If we know the correct representation of our (or an investor's) utility function, we should be able to describe behavior over these (and all other) such money gambles.

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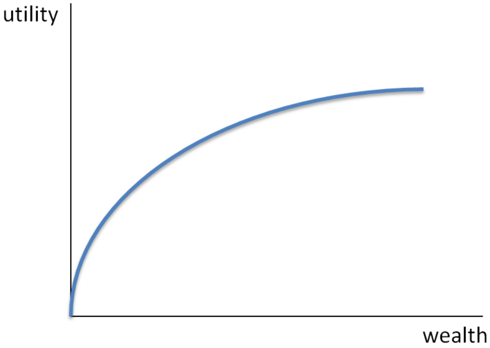
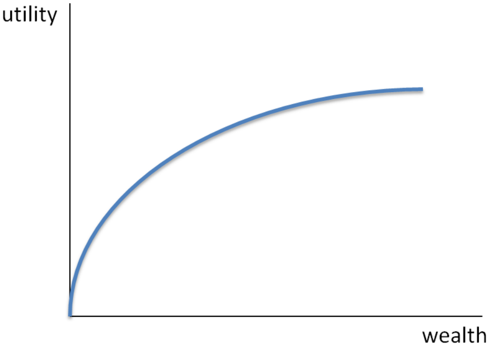
  - but at a decreasing rate (diminishing rate of marginal utility) $u''(c) < 0$.
  - This can be shown graphically through concavity of the utility function.

A Concave Utility Function



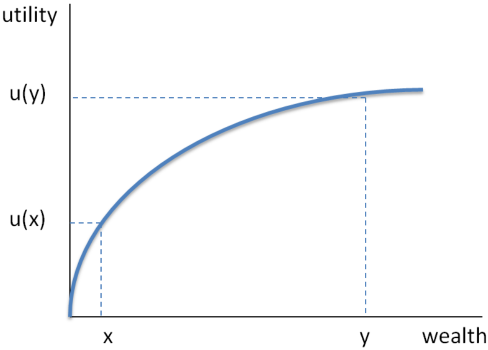
Example

- Does our investor prefer the gamble or the sure thing if the graph below represents his utility function?



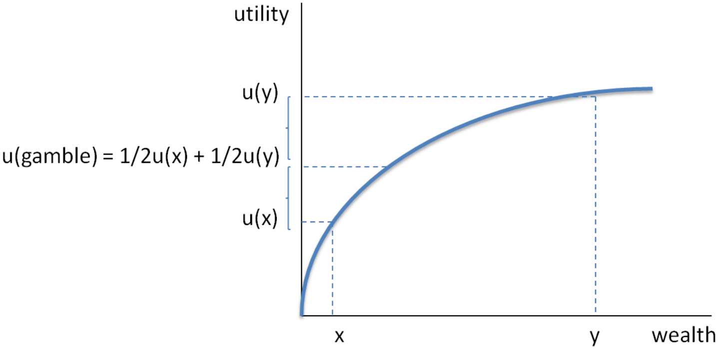
Solution

- First, let's graph the utilities of each possible outcome...



Solution

- Now, for the gamble...



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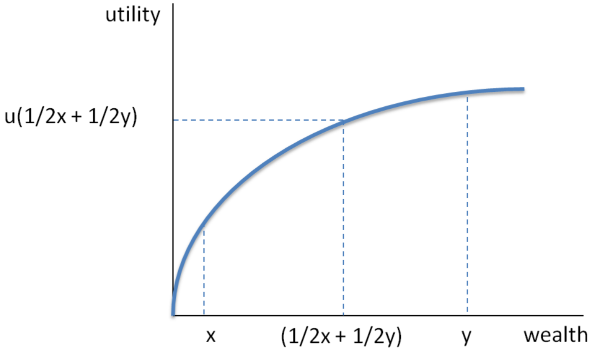
- And for the sure thing...



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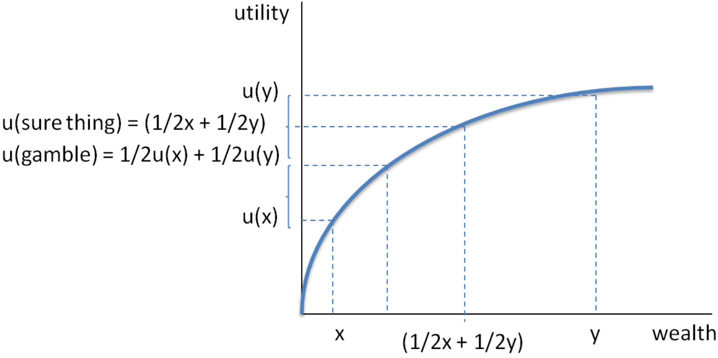
- So we see that...



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- When the utility of the gamble is less than the utility of its expected value, we have investor behavior that is characterized as risk aversion.
  - Graphically, if a chord drawn between any two points of the graph lies below the function, we have a concave function.
  - So concavity is equivalent to risk aversion.

Measuring Risk Aversion

- We often want to have a measure of risk aversion.
  - Intuitively, the more concave the utility function, the higher risk aversion.
  - So we measure the degree of risk aversion by the curvature of the utility function.
- Arrow-Pratt measure of ARA

$$r(w) = -\frac{u''(w)}{u'(w)}$$

  - (Why can't we just use the second derivative?)

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    - (Utility functions are invariant to continuous monotonic transformations; Expected utility functions are unique up to affine transformations)
  - So dividing by the first derivative gives us a normalization.

Absolute Risk Aversion

- Note that our measure of risk aversion, $r(w) = -\frac{u''(w)}{u'(w)}$, depends on current consumption, or current wealth.
- How do you think absolute risk aversion might vary with wealth?

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- How do you think absolute risk aversion might vary with wealth?
  - It's reasonable to assume that ARA decreases with wealth: as you become more wealthy, you would be willing to accept more / higher valued dollar gambles.

What about Relative Gambles?

- With a relative gamble, the idea is that with some probability p, you receive x percent of your current wealth; and with probability $(1-p)$ you receive y percent.
  - E.g., returns on investments are usually stated relative to the level of investment.
- The appropriate measure is the Arrow-Pratt measure of RRA

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- What do you think happens to relative risk aversion as your wealth increases? Would you be more or less willing to risk a specific fraction of it?

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  - Intuiting the behavior is RRA is more problematic
  - We tend to assume that CRRA is reasonable - at least for small changes in wealth.

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- (Why can we do this?)

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- (Use the chain rule and the rule that $\frac{d}{dx} b^x = b^x \ln(b)$).

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- and increasing relative risk aversion (IRRA)

$$\rho(c) = -c\frac{u''(c)}{u'(c)} = -c\frac{-\alpha^2 e^{-\alpha c}}{\alpha e^{-\alpha c}} = c\alpha$$

Investor Preferences

- Summary: We model investors as having preferences over consumption of the form $u(c)$, which is increasing in consumption $u'(c) > 0$, but at a decreasing rate $u''(c) < 0$.
  - Examples: Power utility $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ or log utility $u(c) = \ln(x)$.
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  - We like to think that the fundamental goal is consumption.
  - This also implies that investors really care about consumption, not intermediate objectives like mean and variance.

A Consumption-Based Model

- We're going to set up a basic problem that derives the central asset pricing equation that we'll use throughout the course from a consumption-based model.
- The way we think about this is to focus on the basic trade-off for any investor: consumption now versus consumption later.
    - Because investors face a budget constraint, they have to give up a little consumption today to get a little more consumption tomorrow.
- Now, we're going to set up a simple problem for our investor: to maximize his utility...

Basic Model Setup (Two Period)

- Suppose our investor lives for two periods - today and tomorrow - and receives an endowment that he can consume each period.
  - An investor can smooth consumption over time by purchasing n units of a risky security that has price p_t and gives an (uncertain) payoff next period of x_{t+1}.
  - Given p_t, the investor must choose n.
- The investor's problem is to maximize utility by choosing the number of risky securities to purchase

$$\max_{n, c_t, c_{t+1}} \quad u(c_t) + E_t\left[\beta u(c_{t+1})\right]$$

s.t.

$$\begin{aligned} c_t &= e_t - np_t \\ c_{t+1} &= e_{t+1} + nx_{t+1}, \end{aligned}$$

where e_t is the investor's endowment at time t.

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  - Notice how the problem has simplified...we only have to choose one variable n now...

Solving the Problem

- So take first-order conditions and set them equal to zero

$$
\begin{aligned}
\frac{d}{dn}\left[u\left(e_t - np_t\right) + E_t\left[\beta u\left(e_{t+1} + nx_{t+1}\right)\right]\right] &= 0 \\
u'\left(c_t\right)\left(-p_t\right) + E_t\left[\beta u'\left(c_{t+1}\right)x_{t+1}\right] &= 0 \\
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- Think about what this says -
    - The decrease in utility from buying a share of the asset today has to just equal the increase in expected discounted utility that results from having one more share tomorrow.

The Key Equation

- We will usually re-arrange this key FOC

$$u'(c_t)\,p_t = E_t\left[\beta u'(c_{t+1})\,x_{t+1}\right]$$

  to write the equation as

$$p_t = E_t\left[\beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1}\right]$$

- This is THE CENTRAL ASSET PRICING EQUATION.

Overview: The Consumption-Based Model

- An investor must decide how much to save and how much to consume, and what portfolio of assets to hold.
  - The most basic pricing equation (the one that characterizes the SDF methodology) comes from the FOC for that decision.
- The basic idea is that the marginal utility loss of consuming a little less today (and buying a little more of the asset) should equal the marginal utility gain of consuming a little more of the asset's payoff in the future.
  - If prices and payoffs do not satisfy this relationship, the investor should buy more or less of the asset.

Intuition in a Non-Stochastic Example

- Let's see if we can get the basic intuition from a non-stochastic case.
- Now we have
$$u'(c_t)\, p_t = \beta u'(c_{t+1})\, x_{t+1}$$
- And suppose $\beta = 1$, $p_t = 2$, $x_{t+1} = 6$, $u'(c_t) = 0.2$, and $u'(c_{t+1}) = 0.1$.
- Is the first-order condition met? If not, explain what the investor should do and how that would affect his utility.

Answer

- The left-hand side is less than the right-hand side.

$$
\begin{aligned}
u'(c_t)\,p_t &= \beta u'(c_{t+1})\,x_{t+1} \\
0.2 * 2 &\neq 1 * 0.1 * 6 \\
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  - This says that if we decrease c_t today by increasing the number of shares purchased, marginal utility today $u'(c_t)$ would increase.
  - (In the same way, consumption tomorrow would increase, so marginal utility tomorrow would decrease.)
- Our investor would increase his utility by shifting consumption more towards tomorrow.
- Since shifting consumption to tomorrow increases marginal utility today and decreases it tomorrow, the investor should buy more shares today.

Answer

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- Think about what would happen if the price were $p_t = 4$ instead...

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  - Now, the LHS $>$ RHS
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- Does this accord with your intuition?

- So what have we learned?
  - We know that an asset's price should equal the expected discounted value of the asset's payoff, but:
- What should we use to discount the payoff?

The Consumption-Based Model and Marginal Utility

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  - Marginal utility (not consumption) is the fundamental measure of how you feel.
- Most of the theory of asset pricing is about how to go from marginal utility to observable indicators.
  - There is a relationship between consumption and marginal utility (inverse); so consumption - of course - may be a useful indicator.

Central Asset Pricing Equation

- This brings us back to the central asset pricing equation we introduced last time.
- We can use the pricing equation we just derived

$$p_t = E_t\left[\beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1}\right]$$

by defining

$$m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)}$$

and writing

$$p_t = E_t\left[m_{t+1} x_{t+1}\right]$$

Stochastic Discount Factor

- In our key equation
$$p_t = E_t\left[m_{t+1}x_{t+1}\right]$$
- The variable m_{t+1} is a random variable, called the SDF.
  - It maps future payoffs into today's price
  - It is a generalization of standard discount factor ideas, as you will see next...

Let's Think About Discount Factors...

- Consider an environment where there is no uncertainty.
- What is the price (i.e., present value) of a payoff tomorrow of x_{t+1} if the interest rate is $R^f = \left(1 + r^f\right)$?

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  - Here, the discount factor is $\frac{1}{R^f}$
  - See how the payoff tomorrow sells "at a discount"

Discounting a Risky Asset

- Now, let's think about generalizing this idea to risky assets:

$$p_t^i = \frac{1}{R^i} E_t \left(x_{t+1}^i \right)$$

- The idea is that risky assets have lower prices than risk-free assets, so they use asset-specific risk-adjusted discount factors.

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    - E.g., these can come from a model like the CAPM

But We can be Even More General...

- The generalization is $p_t = E_t \left[m_{t+1} x_{t+1} \right]$
  - It says that by putting the discount factor inside the expectation, we can use a single discount factor (the same one for each asset) to incorporate all risk corrections
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- Back-pocket: The SDF is also the IMRS (the intertemporal marginal rate of substitution) - the rate at which an investor is willing to substitute consumption at time $t+1$ for consumption at time t

Why is this Useful?

- All asset pricing models are just different ways of connecting the SDF m_{t+1} to the data.
- We will soon see that we can use equation $p_t = E_t\left[m_{t+1}x_{t+1}\right]$ in alternative ways to come up with different empirical approaches.
- By separating the model into these two pieces, we can skip a lot of steps and elaboration for each asset pricing model.
    - That is, all asset pricing models simply use a different m_{t+1}.
    - E.g., $p_t = E_t\left[m_{t+1}x_{t+1}\right]$ is valid for different utility functions, etc.

End of Today's Lecture.

- That's all for today. Today's material corresponds to parts of Chapter 1 in Cochrane (2005).