ECON 4360: Empirical Finance

Risk-Neutral Probabilities, Risk-Sharing, State Diagrams

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Theory Lecture #08

What are we doing today?

- Risk-Neutral Probabilities
- Consumers and Risk-Sharing
- State Diagrams

Risk-Neutral Probabilities

- Now, we're going to define a new type of probability - a *risk-neutral* probability, $\pi^*(s)$

$$\pi^*(s) = R^f m(s)\,\pi(s) = R^f \phi(s)$$

  - (Recall, $m(s) = \phi(s)/\pi(s)$)

- Are they a legitimate set of probabilities?

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    - (Recall, $R^f = 1/E(m) = 1/\sum \phi(s)$)

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- These probabilities $\pi^*(s)$ are called risk-neutral because we can use them to take the expected value of a payoff and discount at the risk-free rate to get prices.

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- Why would we want to do this?
  - It allows us to think of asset pricing *as if* people are risk-neutral, but with probabilities $\pi^* (s)$ in place of $\pi (s)$...

Risk-Neutral Probabilities

- Look again at

$$\pi^*(s) = R^f m(s) \pi(s)$$

- Think about what the probabilities $\pi^*(s)$ are...
  - They are a way to re-weight the true probabilities to give higher weight to states that have higher marginal utilities
  - (What does this mean in terms of consumption?)

- What does this mean if we put more weight on states with higher marginal utilities?

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- So with $\pi^*(s)$, we pay attention to states that are highly likely to occur (large $\pi(s)$), or may not be likely to occur, but may have disastrous consequences if they do occur (large $m(s)$).

Exercise 3: Risk-Neutral Probabilities

- Given the state-prices and corresponding probabilities, what are the risk-neutral probabilities $\pi^*(s)$?

State (s)	$\phi(s)$	$\pi(s)$
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- So

$$\begin{aligned} \pi^*(1) &= (1.11)(0.2) = 0.222 \\ \pi^*(2) &= (1.11)(0.4) = 0.444 \\ \pi^*(3) &= (1.11)(0.3) = 0.333 \end{aligned}$$

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  - Use $p(x) = \Sigma_s \phi(s) x(s)$

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Bringing the Consumer Back In

- Even though we don't need utility functions in the contingent claims context, we would like to see how this matches up with our consumer-investor's FOCs.
- Let's look at the definition of the SDF based on marginal utility

$$m(s) = \beta \frac{u'(c(s))}{u'(c)}$$

- Here, s represents some future state, and $m(s)$ is the SDF - a random variable whose value depends on the realization of state s
  - Note that this equation has to hold for any future state.
- Note that we can write

$$m(s) = \beta \frac{u'(c(s))}{u'(c)} = \frac{\phi(s)}{\pi(s)}$$

Bringing the Consumer Back In

- Consider two different possible future states

$$m(s_1) = \beta \frac{u'(c(s_1))}{u'(c)}, \text{ and } m(s_2) = \beta \frac{u'(c(s_2))}{u'(c)}$$

- Take the ratio of the SDFs to get

$$\frac{m(s_1)}{m(s_2)} = \frac{u'(c(s_1))}{u'(c(s_2))}$$

- Recall that $m(s) = \phi(s) / \pi(s)$, so we can write the previous equation as

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- Thinking in terms of contingent claims, what this says is that the price of giving up a unit of consumption in state 1 for an additional unit in state 2, $\left(\frac{\phi(s_1)}{\phi(s_2)}\right)$, has to equal the ratio of expected happiness lost in state 1 to expected happiness gained in state 2.

Exercise 4: Consumers Again

- Suppose $\pi(s_1) = 0.2$, $\pi(s_2) = 0.4$, $u'(c(s_1)) = 0.1$, $u'(c(s_2)) = 0.2$, $\phi_1 = 0.1$, and $\phi_2 = 0.8$. Does the first-order condition hold?

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- With the usual condition that the MRS should equal the price ratio, what should the consumer do?
  - Shift consumption from $s = 2$ to $s = 1$.
- So we see that m can give us the MRS between both *date* and *state* contingent claims

Risk-Sharing

- If markets are complete, it is possible to insure against any state by using contingent claims. Why? And what does that get us?
  - In complete contingent claims markets, all investors share all risks...
  - We just found that the MRS (for anyone) equals the contingent claims price ratio.
  - Since prices are the same for everyone, we find that marginal utility growth should be the same for everyone.

$$m\left(s_{t+1}\right) = \frac{\phi\left(s_{t+1}\right)}{\pi\left(s_{t+1}\right)} = \beta \frac{u'\left(c^i_{t+1}\right)}{u'\left(c^i_t\right)} = \beta \frac{u'\left(c^j_{t+1}\right)}{u'\left(c^j_t\right)}$$

  - What does this say? What doesn't this say?
  - In reality, markets are not complete. Keep in mind, though, that a big part of financial innovation is to bring products to market that better enable people to share risks.

State Diagrams and Price Functions

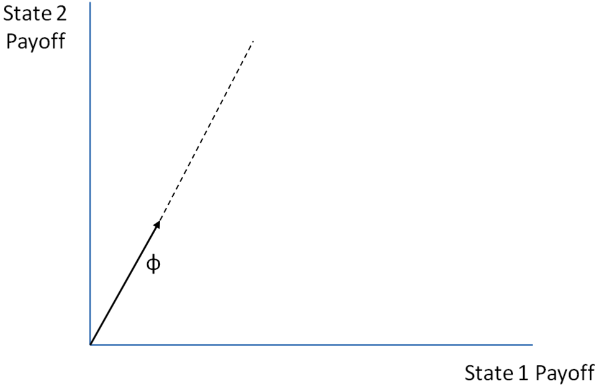
- Now, we're going to talk a bit about state-space geometry.
- Random variables can be represented by vectors, with each element representing a different possible outcome.
  - We can think of contingent claims prices and asset payoffs as vectors in R^S, where S is the total number of states.
  - Let's work in R^2 for now...
  - Ex: A payoff of \$7 in state 1 and \$3 in state 2 can be represented by the following vector in R^2

$$x = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

  - Ex: Contingent claims prices can be represented by the following

$$\phi = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix}$$

- Note that the contingent claims vector always points in the positive orthant, since marginal utility is always non-negative: $\phi(s) = m(s)\pi(s)$

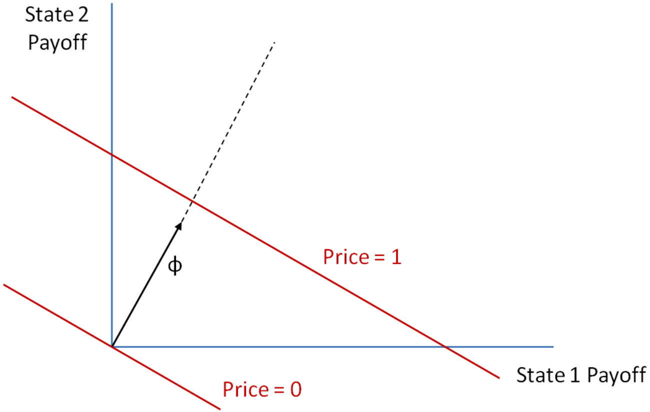


State Diagrams and Price Functions

- Think about the equation $p(x) = \Sigma\phi(s)\,x(s)$. If we interpret ϕ and x as vectors, we can interpret price as the inner product of the contingent claims prices and the payoffs.
- Recall, that two orthogonal vectors (vectors points out from the origin at right angles) have an inner product of zero.
- Where does the set of all zero-price payoffs lie?

State Diagrams and Price Functions

- The plane of price $= 0$ payoffs is the plane of excess returns.
- Similarly, the plane of price $= 1$ payoffs is the plane of returns.



State Diagrams and Price Functions

- Since we can write the price of any risky payoff as an inner product,

$$
\begin{aligned}
p(x) &= \Sigma_s \phi(s) x(s) \\
&= \phi \cdot x \\
&= |\phi| \times |proj(x|\phi)| \\
& \quad |\phi| \times |x| \times \cos(\theta)
\end{aligned}
$$

- This gives us the result that the set of all payoffs with the same price lie in a plane that is perpendicular to the contingent claims vector
- We also get the result that $p(x)$ is a linear pricing function:

$$p(ax + by) = ap(x) + bp(y)$$

since, e.g.,

$$p(2x) = \Sigma_s \phi(s) 2x(s) = 2p(x)$$

- Where does the risk-free return lie?

State Diagrams and Price Functions

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- What about a state-contingent claim to a 1 unit payoff in state 1?

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- Where does the risk-free return lie?
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- What about a state-contingent claim to a 1 unit payoff in state 1?
  - At the intersection of the horizontal axis and the price $= \phi_1$ plane.

End of Today's Lecture.

- That's all for today. Today's material corresponds roughly to Chapter 3 in Cochrane (2005).