1 Risk-Neutral Probabilities

1.1 Defining Risk-Neutral Probabilities

- Now, we're going to define a new type of probability - a *risk-neutral* probability, $\pi^*(s)$

$$\pi^*(s) = R^f m(s)\pi(s) = R^f \phi(s)$$

  - (Recall, $m(s) = \phi(s)/\pi(s)$)

- Are they a legitimate set of probabilities?

  - Yes, the $\pi^*(s)$ are all positive, between 0 and 1, and add to 1.

    * (Recall, $R^f = 1/E(m) = 1/\sum \phi(s)$)

- These probabilities $\pi^*(s)$ are called risk-neutral because we can use them to take the expected value of a payoff and discount at the risk-free rate to get prices.

- From

$$\pi^*(s) = R^f \phi(s)$$

- We can write

$$p(x) = \Sigma_s \phi(s) x(s) = \frac{1}{R^f}\Sigma_s \pi^*(s) x(s) = \frac{1}{R^f}E^*[x]$$

- The notation E^* refers to the fact that we are taking an expectation w.r.t. risk-neutral probabilities (not real probabilities).

- Why would we want to do this?

  - It allows us to think of asset pricing *as if* people are risk-neutral, but with probabilities $\pi^*(s)$ in place of $\pi(s)$...

- Look again at

$$\pi^*(s) = R^f m(s)\pi(s)$$

- Think about what the probabilities $\pi^*(s)$ are...

  - They are a way to re-weight the true probabilities to give higher weight to states that have higher marginal utilities

  - (What does this mean in terms of consumption?)

- What does this mean if we put more weight on states with higher marginal utilities?

- What is this equivalent to?

- So with $\pi^* (s)$, we pay attention to states that are highly likely to occur (large $\pi (s)$), or may not be likely to occur, but may have disastrous consequences if they do occur (large $m (s)$).

1.2 Examples

1.2.1 Exercise 3: Risk-Neutral Probabilities

- Given the state-prices and corresponding probabilities:

State (s)	$\phi (s)$	$\pi (s)$
1	0.2	0.25
2	0.4	0.35
3	0.3	0.40

- What are the risk-neutral probabilities $\pi^* (s)$?

1.2.2 Example 5: Risk-Neutral Probabilities

- Using the risk-neutral probabilities we just found, value a security that pays off \$7 in state 1, \$10 in state 2, and \$3 in state 3.
  - Since $p(x) = E^*(x)/R^f$

$$\begin{aligned} p(x) &= E^*(x)/R^f \\ &= \frac{(0.222)(7) + (0.444)(10) + (0.333) * (3)}{1.11} \\ &= 6.3 \end{aligned}$$

- Do we get the same thing if we use the state prices to value the security?
  - Use $p(x) = \Sigma_s \phi(s) x(s)$

$$\begin{aligned} p(x) &= \Sigma_s \phi(s) x(s) \\ &= (0.2)(7) + (0.4)(10) + (0.3)(3) \\ &= 6.3 \end{aligned}$$

2 Consumers Again

2.1 Bringing the Consumer Back In

- Even though we don't need utility functions in the contingent claims context, we would like to see how this matches up with our consumer-investor's FOCs.
- Let's look again at the definition of the SDF based on marginal utility

$$m(s) = \beta \frac{u'(c(s))}{u'(c)}$$

- Here, s represents some future state, and $m(s)$ is the SDF - a random variable whose value depends on the realization of state s
  - Note that this equation has to hold for any future state.
- Note that we can write

$$m(s) = \beta \frac{u'(c(s))}{u'(c)} = \frac{\phi(s)}{\pi(s)}$$

- Now, consider two different possible future states

$$m(s_1) = \beta \frac{u'(c(s_1))}{u'(c)}, \text{ and } m(s_2) = \beta \frac{u'(c(s_2))}{u'(c)}$$

- Take the ratio of the SDFs to get

$$\frac{m(s_1)}{m(s_2)} = \frac{u'(c(s_1))}{u'(c(s_2))}$$

- Recall that $m(s) = \phi(s)/\pi(s)$, so we can write the previous equation as

$$\frac{\phi(s_1)}{\phi(s_2)} = \frac{\pi(s_1)\, u'(c(s_1))}{\pi(s_2)\, u'(c(s_2))}$$

- Thinking in terms of contingent claims, what this says is that the price of giving up a unit of consumption in state 1 for an additional unit in state 2, $\left(\frac{\phi(s_1)}{\phi(s_2)}\right)$, has to equal the ratio of expected happiness lost in state 1 to expected happiness gained in state 2.

2.2 Examples

2.2.1 Exercise 4: Consumers Again

- Suppose $\pi(s_1) = 0.2$, $\pi(s_2) = 0.4$, $u'(c(s_1)) = 0.1$, $u'(c(s_2)) = 0.2$, $\phi_1 = 0.1$, and $\phi_2 = 0.8$. Does the first-order condition hold?

- From

$$\frac{\phi(s_1)}{\phi(s_2)} = \frac{\pi(s_1)\, u'(c(s_1))}{\pi(s_2)\, u'(c(s_2))}$$

- We get that

$$\begin{array}{ccc} \frac{0.1}{0.8} & = & \frac{0.2 * 0.1}{0.4 * 0.2} \\ 0.125 & \neq & 0.250 \end{array}$$

- With the usual condition that the MRS should equal the price ratio, what should the consumer do?

- So we see that m can give us the MRS between both *date* and *state* contingent claims

2.2.2 Example 6: Risk-Sharing

- If markets are complete, it is possible to insure against any state by using contingent claims. Why? And what does that get us?
  - In complete contingent claims markets, all investors share all risks...
  - We just found that the MRS (for anyone) equals the contingent claims price ratio.
  - Since prices are the same for everyone, we find that marginal utility growth should be the same for everyone.

$$m(s_{t+1}) = \frac{\phi(s_{t+1})}{\pi(s_{t+1})} = \beta \frac{u'\left(c_{t+1}^i\right)}{u'\left(c_t^i\right)} = \beta \frac{u'\left(c_{t+1}^j\right)}{u'\left(c_t^j\right)}$$

– What does this say? What doesn't this say?

– In reality, markets are not complete. Keep in mind, though, that a big part of financial innovation is to bring products to market that better enable people to share risks.

3 State Diagrams and Price Functions

3.1 Intro to State-Space Geometry

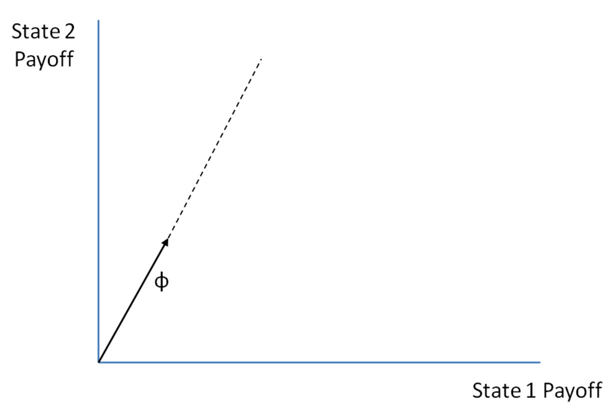
- Now, we're going to talk a bit about state-space geometry.
- Random variables can be represented by vectors, with each element representing a different possible outcome.
  - We can think of contingent claims prices and asset payoffs as vectors in R^S, where S is the total number of states.
  - Let's work in R^2 for now...
  - Ex: A payoff of \$7 in state 1 and \$3 in state 2 can be represented by the following vector in R^2

$$x = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

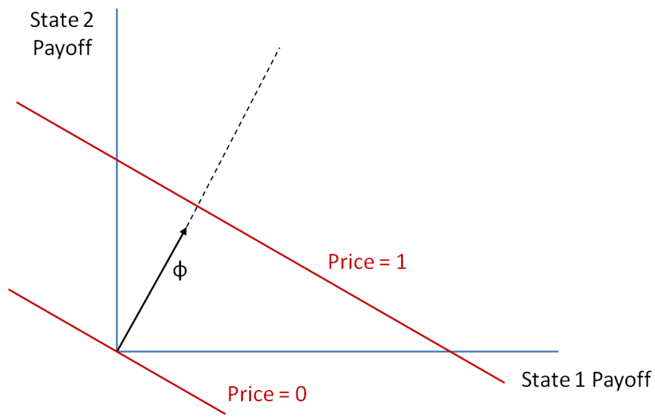
  - Ex: Contingent claims prices can be represented by the following

$$\phi = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix}$$

- Note that the contingent claims vector always points in the positive orthant, since marginal utility is always non-negative: $\phi(s) = m(s)\,\pi(s)$



- Think about the equation $p(x) = \Sigma\phi(s)x(s)$. If we interpret ϕ and x as vectors, we can interpret price as the inner product of the contingent claims prices and the payoffs.
- Recall, that two orthogonal vectors (vectors points out from the origin at right angles) have an inner product of zero.
- Where does the set of all zero-price payoffs lie?
- The plane of price = 0 payoffs is the plane of excess returns.
- Similarly, the plane of price = 1 payoffs is the plane of returns.



- Since we can write the price of any risky payoff as an inner product,

$$\begin{aligned} p(x) &= \Sigma_s\phi(s)x(s) \\ &= \phi \cdot x \\ &= |\phi| \times |proj(x|\phi)| \\ & \quad |\phi| \times |x| \times \cos(\theta) \end{aligned}$$

- This gives us the result that the set of all payoffs with the same price lie in a plane that is perpendicular to the contingent claims vector
- We also get the result that $p(x)$ is a linear pricing function:

$$p(ax+by) = ap(x) + bp(y)$$

since, e.g.,

$$p(2x) = \Sigma_s\phi(s)2x(s) = 2p(x)$$

- Where does the risk-free return lie?

- What about a state-contingent claim to a 1 unit payoff in state 1?