ECON 4360: Empirical Finance

Volatility Bounds

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Empirics Lecture #10

What are we doing today?

- Variance Bounds
  - Bounds on the variance of stock prices
  - Bounds on the variance of SDF's
    - The Hansen-Jagannathan Bound: Empirical Implementation

Variance Bounds

- Diagnostic of Model Fit
  - Helps pinpoint where the model is failing
- 2 Types of Variance Bounds
  - First one comes from a very old literature that tries to put bounds on the variance of stock prices
  - Second one is the H-J Bound that takes prices as givens to figure out what SDF would work for the data
- So we're going to have two different kinds of variance bounds for examining the volatility of stock prices...

Variance Bounds for Stock Prices

- Q: Do our models explain the volatility of stock prices? And the answer is going to be no.
  - Paper by Shiller and LeRoy
  - Do lots of versions; but in the end, the story tends to be the same, so we're going to go with the simplest derivation
  - We're not going to go through all the details (e.g., see a recent paper by Charles Engels), but will try to get a flavor for what this literature is trying to do.
- Uses a simple model of stock prices: present discounted value of expected future dividends
  - So we're going to look at a very simple perfect foresight model
  - (Still do this - typical way of asset pricing (valuing a stock) in DSGE production economy)
  - This is an easy way of deriving a bounds: a bound where we're going to try and learn something about volatility...

A Simple Model (Shiller, 1981)

- Start with a perfect foresight stock price

$$p_t^* = \sum_{j=0}^{\infty} \beta^j d_{t+j}$$

  - p^* is the perfect foresight stock price
  - It's "perfect foresight" because we know what all future dividends are going to be
  - β is some constant, fixed discount factor

- The actual stock price, which is going to be the market stock price, is the market's "best guess" of the perfect foresight price, so...

Market (Actual) Price

- The market is always trying to figure out what the perfect foresight stock price actually is
  - So it's going to take an expectation of that, given some "information set", I_t
- That information set that the market has is going to be a very big information set - whatever is observed out there in the real world
- So the market price is

$$p_t = E\left[p_t^* | I_t\right]$$

Results: Shiller (1981); LeRoy and Porter (1981)

- We're going to show that the variance of p_t^* has to be bigger than the variance of p_t
- We'll show this directly based on a derivation for the inequality

$$var\left(p_t^*\right) \geq var\left(p_t\right)$$

- This inequality says that if you take dividends from the data, and calculate a present-discounted sum of those, the variance of that should be greater than the variance of actual stock prices.
  - The perfect foresight price should move more than observed prices
- But...if you actually do this, you find that the stock prices we observe in the data are way more volatile than stock prices predicted by the model!
  - This is the price volatility puzzle!
- This is where we're heading, but we haven't shown this just yet...

Intuitive Explanation

- Why? A forecast is always smoother than the actual realization of the variable itself
- Let's looks at an example from time-series forecasting:
- Say we want to forecast an AR(1)

$$y_t = \phi y_{t-1} + \varepsilon_t$$

- What's the forecast in period t for tomorrow? The only info you have is y_t, so the forecast is

$$E_t \left[y_{t+1} | I_t \right] = \phi y_t$$

Intuitive Explanation

- So the variance of the forecasted time-series is

$$var\left(E_t\left[y_{t+1}|I_t\right]\right) = \phi^2 var\left(y_t\right)$$

- But the variance of the realized time-series is of course $var\left(y_{t+1}\right)$
- So assuming stationarity, and with $\phi < 1$,

$$\phi^2 var\left(y\right) < var\left(y\right)$$

- So the variance of the forecast (LHS) is less volatile than the variance of the realization (RHS).
- Why? The actual time-series has the forecast in it which has movement, plus an error term (that is unforecastable and also has volatility in it)

Stock Price Forecasts

- Now, going back to our asset pricing model:
- The actual time-series realization is going to be our forecast plus an error term, so - here, switching back to stock prices -

$$p_t^* = p_t + \text{ forecast error}$$

  - So the perfect foresight price is equal to the market forecast + a forecast error (because of some randomness in future dividends).
- The market's forecast is not going to be 'perfect'; but we are going to assume that it is going to be 'optimal'
  - What that means is that there is no systematic forecasting error.
  - If you could predict the forecast error, you could use that information to come up with a better forecast, so the forecasting error is going to be i.i.d.

The Variance Bound

- If the forecast is optimal, there is zero covariance between the market forecast and the forecast error, so

$$E_t\left[p_t,\ \text{forecast error}\right] = 0$$

- And we get that (just applying the variance operator)

$$var\left(p_t^*\right) = var\left(p_t\right) + var\left(\text{forecast error}\right)$$

- Therefore, we get the inequality

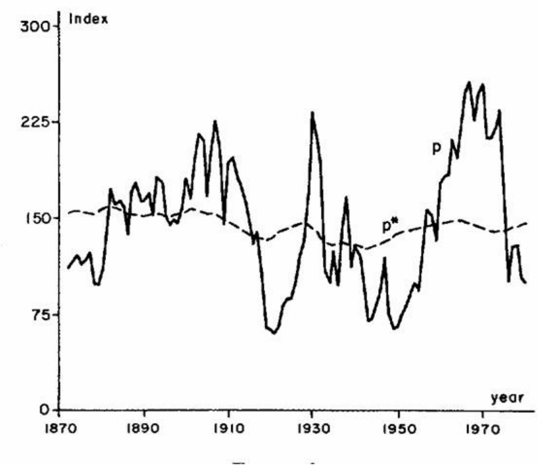
$$var\left(p_t^*\right) \geq var\left(p_t\right)$$

- And of course we don't know what the forecast error is going to be exactly, which is why we calculate a volatility *bound*...

Stock Price Volatility: Results

- So what can we do with this?
  - Note, we aren't doing formal estimation here; but our theory makes a clear prediction as to which has to be more volatile.
- We can go to the data, take a discounted sum of dividends, and calculate $var\left(p_t^*\right)$ and $var\left(p_t\right)$
  - Note that this is generally applied to a broad index, like the S&P 500.
- Dividends in data are very predictable; they are not very volatile, so a discounted sum of them is even less volatile
  - So $var\left(p_t^*\right)$ turns out to be pretty small
  - But $var\left(p_t\right)$ is way more volatile! And we violate our (theoretical) bound..
- Prices are way more volatile than what is predicted by our perfect foresight bounds.
  - Puzzle: Why are prices so volatile?

Plot from Shiller (1981)



What to do Next?

- The price volatility puzzle leads us to the conclusion that either something is wrong with our model, or that something is wrong with our test.
- Some ideas are to look at stationarity assumptions and/or information sets.
    - Another way to derive an alternative bounds is to assume an information set other than the market information set - e.g., the econometrician's information set, which is smaller. (But important to keep this relatively simple!)

What to do Next?

- A promising approach (that you are already familiar with!) is to drop the assumption of a constant discount factor.
  - The natural way is to use an agent's IMRS (using consumption data) as a discount factor...
  - Why? The IMRS varies over time, so using it as a discount factor puts more volatility into the discounted sum of dividends than just using a constant discount factor.
  - Of course, it turns out that this alone (e.g., using the CRRA discount factor) is not enough to solve the puzzle - it makes p^* more volatile, but still not volatile enough.

Hansen and Jagannathan

- HJ thought that an SDF constructed with the habit model or Epstein-Zin preferences might work
  - But it's difficult to keep testing the model in this way, because you don't know exactly how much volatility you need in a discount factor to rationalize what we see in prices, and this is what motivated HJ.
- HJ suggested that instead of using returns (prices) to check the reasonableness of a model,
  - We can instead take returns as givens and construct bounds on what the stochastic discount factor should look like.
  - Gives some direction on the construction of an SDF, once you know some of the properties that we think the SDF needs to have to rationalize the data
- The HJ Bound is of course also motivated by the equity premium puzzle, which is related to these sorts of questions.

H-J Bounds: Interpretation Review

- Recall, that we used the equation

$$\frac{|E(R^e)|}{\sigma(R^e)} \leq \frac{\sigma(m)}{E(m)}$$

- To Ask: Given a set of returns, what are the bounds on all possible discount factors?

Hansen and Jagannathan

- Recall, our key asset pricing equation is

$$E_t\left[m_{t+1}R_{t+1}\right]=1$$

  where R is a vector of risky returns and
  $m_{t+1}=\beta\left(u'\left(c_{t+1}\right)\right)/\left(u'\left(c_t\right)\right)$ is a scalar discount factor.
- HJ want to use the properties of R (i.e., mean, variance, and covariance) to bound the mean and variance of m.

Returns: Which Ones?

- Can typically take R to comprise a T-bill return (which has a low mean and low volatility) and the S&P 500 return (which has a higher mean and volatility)
    - How high the bound is depends on which returns you use to construct it - the previous two gives a large bound (i.e., the SDF has to be very volatile)
    - If instead you take a couple of returns that look very similar - e.g., the S&P 500 and Nasdaq indices - you might not come up with a very strong bound.
- In practice, want to find a SDF that prices all assets, but can't use all the returns in the world... So we pick returns that are very different from each other.

Where to Start

- HJ work with the unconditional version

$$E\left[mR\right] = 1$$

  and derive a bound by constructing a 'candidate' IMRS, which is called a bounding IMRS.
- The SDF m is the 'true' SDF in the world, which we don't observe; so we want to start learning about its properties.
- First, we construct a candidate SDF, m_v.
  - The idea is that m_v, the candidate IMRS, should be consistent with the unconditional asset pricing equation.
  - Consistent here means that the candidate m_v should provide a lower bound for the mean and variance of the true SDF, m, as a necessary condition.

What's the Point?

- Objective is to hypothesis about a model that can get into the bound.
- Note that just because you get into the HJ bound does not mean that you have the right SDF
  - It just means that you don't have the wrong SDF.
  - So it's just a necessary condition; but it allows us to get rid of a lot of models quickly.
- If you do satisfy the bound, then want to look at other things like formal tests and GMM.

The Derivation

- First, note that we don't have a risk-less asset in our vector R. (Why?)
  - If we had a truly riskless asset, that would be valuable, because it would tell us the true mean of the SDF
- So we're going to augment our return vector with a (fictitious) riskless asset with riskless return $1/v$, so

$$R_v = \begin{bmatrix} 1/v \\ R_1 \\ \vdots \\ R_n \end{bmatrix}, \; E\left[R_v m_v\right] = 1, \text{ and } Em_v = Em = v$$

- And then we look for many v's that seem reasonable (the possible riskless rates that are out there).
  - By varying the v's, we get (i.e., can trace out) the entire possible bound/frontier.

The Derivation

- We know we can construct the true discount factor as a linear combination of returns
  - So the goal is to try and construct a candidate m_v by figuring out what that linear function is.
  - And the candidate m_v is going to tell us something about the true one.
- So we start out with the fact that even though m_v is not the true SDF, we construct it to be related to the true SDF

$$m = m_v + \varepsilon$$

where ε is an error term with $E(\varepsilon) = 0$ and $E(R_v \varepsilon) = 0$.

- Why? ...

(Justification)

- Since we introduced a risk-less asset, we pin down the mean ensuring that $Em_v = Em = v$ holds, so

$$E\left(m - m_v\right) = 0$$

- This implies $E\left(\varepsilon\right) = 0$, and since $E\left[R_v m_v\right] = 1$,

$$E\left[R_v\left(m - m_v\right)\right] = 0$$

so

$$E\left[R_v \varepsilon\right] = 0$$

The True SDF...

- Recall that we can construct a discount factor that prices assets through a linear combination of returns
  - (Remember x^*?)
  - (Originally, Hansen and Richards (1987) Econometrica paper)
  - But we can't get to the true m through a projection in practice. (Why?)
- So we revert to calculating a lower bound on m...

The Derivation: The Basic Equation

- Now go back to

$$m = m_v + \varepsilon$$

and apply the variance operator

$$var\left(m\right) = var\left(m_v\right) + var\left(\varepsilon\right)$$

- So

$$var\left(m\right) \geq var\left(m_v\right)$$

and m_v then provides a lower bound on m.

The Derivation: A Note

- Note that

$$var\left(m\right) = var\left(m_v\right) + var\left(\varepsilon\right) + 2cov\left(m_v, \varepsilon\right)$$

so if the covariance term was not zero, we wouldn't be able to construct a lower bound!

Constructing a Candidate

- How do we construct m_v?
  - We've already gone through one derivation (the hard work)...
  - (See Lecture 12 from the theory half of the course)
- But now, we've already pinned down the mean of m_v by assuming a risk-free rate, so all we have to worry about is the variance...
  - Once the bound is derived, using it is pretty easy...

Constructing a Candidate

- Using returns, we get for the HJ Bound

$$var(m_v) = (1 - vE(R))' \Omega^{-1} (1 - vE(R))$$

  where

  - $E(R)$ is the vector of mean returns
  - Ω is the variance-covariance matrix of returns
  - and $1/v$ is the candidate risk-free rate that we vary (to construct the whole frontier).

- So our lower bound requires knowledge of the mean and VCV matrix of asset returns.

Then what?

- Since we don't know v, we construct a bound for a *sequence* of v's.
  - Once we have a bound, we calculate the variance of the SDF
- Example: The power utility model with the T-bill and the S&P500 Returns
  - The SDF does not satisfy the HJ bounds (for reasonable parameters)
  - The distance to the bound is statistically significant
  - Note that since we have uncertainty, we will have confidence bounds for these point estimates we get.
    - GMM will let us estimate the distance between the model and the bound!

Why does Habit help?

- What's wrong with power utility for calculating the SDF?
  - The power utility SDF is not volatile enough; higher (too high) risk aversion increases the SDF volatility
- Habit gets us higher SDF volatility
- If the habit parameter is very big, it's like applying the first-difference operator to consumption
  - We know that filtering data in this way (getting to higher frequencies) makes a time series more volatile

Now what?

- For models that are rejected, we have some guidance on how to proceed in building new models
- Increase the volatility of the SDF
  - Habit Formation; Epstein-Zin (see Cochrane and Hansen, 1992)
  - Note that Habit model still not very good
    - If you look at volatility bounds in the frequency domain, it actually generates the wrong kind of volatility (high frequency instead of business cycle)
  - Campbell and Cochrane (1999) best version from this literature
    - But the utility function is very complicated (and bizarre) - i.e., too incredible to use.

What else?

- We can also make alterations to the model that lower the bound
- E.g., financial/trading frictions (see He and Modest, 1995; Luttmer, 1996)
  - For example, if there is a cost associated with buying and selling or something that restricts short sales, that gives us lower volatility.
    - But can get all the way down to zero, so in most cases these modifications aren't very interesting
  - Popular in the early 1990s, but people didn't really buy into it

Final Note of Caution...

- If you want to see how well the volatility bound does in the real world...
  - Generate data in a fake world and see how well it does.
- If you simulate data from an asset pricing model and then calculate the volatility bounds, does the model SDF actually get into those bounds?
  - Surprisingly enough, there are cases where you can simulate data from a true model, apply the HJ bounds, and reject the model most of the time
- So you want to be aware of those possibilities...
  - When that happens, you can make adjustments to statistics to make sure it doesn't happen too often.

End of Today's Lecture.

- That's all for today.