ECON 4360: Empirical Finance

GMM for Linear Factor Models in Discount Factor Form

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Empirics Lecture #08

What are we doing today?

- GMM for Linear Factor Models in Discount Factor Form

Linear Discount Factor Models

- A linear discount model can be expressed as

$$p = E\left(mx\right)$$

where

$$m = b'f$$

- A GMM approach to estimating this model will use the pricing errors as the moments

Linear Discount Factor Models

- If we have the model
$$E\left(p\right) = E\left(mx\right)$$
and
$$m = b'f$$
- We can just write
$$E\left(p\right) = E\left(xf'\right)b$$
for a vector of asset prices p and payoffs x that are $N \times 1$, a vector of factors f and parameters b that are $K \times 1$.

Linear Discount Factor Models

- If our model is
$$E\left(p\right) = E\left(xf'\right) b,$$
how would we implement GMM?

Linear Discount Factor Models

- To implement GMM, we need to pick a set of moments.
- What do we choose?
  - Easy choice: The pricing errors.

$$g_T(b) = E_T\left(xf'b - p\right)$$

GMM Methodology

- The GMM estimate solves

$$\min_b g_T(b)' W g_T(b)$$

- The FOC of this problem is

$$d' W g_T(b) = 0$$

  where

$$d' = \frac{\partial g_T'(b)}{\partial b} = E_T\left(fx'\right)$$

- Note that d' is just the second-moment matrix of payoffs x and factors f. (Can you see that?)

GMM Methodology

- So the moment condition is

$$d'WE_T\left(xf'b - p\right) = 0$$

- For our first-stage estimates, we just use $W = I$
- But for the second-state estimates, we can use S^{-1}

GMM Methodology

- Recall, that when we were using GMM in a previous exercise to estimate the parameters β and γ in the consumption-based asset-pricing model, we solved for the estimated parameters numerically. (Why?)
- In this model, we can solve for the estimates analytically. (Why?)
  - We have a linear model!
  - For out first-stage estimates,

$$\begin{aligned} d' E_T \left(x f' b - p \right) &= 0 \\ d' E_T (x f' b) &= d' E_T (p) \\ E_T \left(f x' \right) E_T (x f' b) &= d' E_T (p) \\ \widehat{b_1} &= (d' d)^{-1} d' E_T (p) \end{aligned}$$

  - Interpretation: The first-stage estimate is just an OLS cross-section regression of average prices on the second moment of payoff with factors

GMM Methodology

- For our second-stage estimates, by the same algebra, we get

$$\widehat{b_2} = (d' S^{-1} d)^{-1} d' S^{-1} E_T(p)$$

- Interpretation: The second-stage estimate is just a GLS cross-section regression.

GMM Methodology: Intuition

- What is this model doing?
- The model

$$E(p) = E(xf')b$$

  says that average prices are a linear function of the second-moment of payoff with factors.
  - So we run simple linear regressions!
- The regressions operate across assets on sample averages.
  - Data are:
    - sample average prices (the y variable)
    - second-moments of payoffs with factors across assets (the x variable)
- GMM is finding b to make the model explain the *cross-section* of asset prices as well as possible.

Standard Errors

- The usual GMM standard error formulas apply:
  - For our first- and second-stage estimates

$$
\begin{aligned}
cov\left(\widehat{b}_1\right) &= \frac{1}{T}\left(d'd\right)^{-1} d'Sd\left(d'd\right)^{-1} \\
cov\left(\widehat{b}_2\right) &= \frac{1}{T}\left(d'S^{-1}d\right)^{-1}
\end{aligned}
$$

- What do you notice about these formulas?
  - They are identical to what you get with OLS and GLS with error covariance matrix S.
- Note that OLS standard errors have to be corrected for correlation, since the pricing errors are correlated across assets since the payoffs are correlated.

What else?

- Standard GMM results for the covariance matrix of the pricing errors $\left(cov\left[g_T\left(\widehat{b}\right)\right]\right)$ and for the model test (the chi-square test).
  - (See Cochrane, Ch. 13 for the formulas).

A Note

- Look again at
$$\widehat{b_1} = (d'd)^{-1} d' E_T(p)$$
- What if we are using excess returns?
  - We would have $\widehat{b_1} = (d'd)^{-1} d' E_T(p) = 0$!
  - The idea here is that the mean discount factor is not identified with $E(mR^e) = 0$.
  - What we've done up to now requires that at least one asset has a non-zero price.
- We want to look at the case of excess returns...

Excess Returns

- If we write

$$m = a - b' f,$$

we cannot identify a and b separately, so we use a normalization of $a = 1$.

- Then we have

$$\begin{aligned} g_T(b) &= -E_T(mR^e) \\ &= -E_T(R^e) + -E_T\left(R^e f'\right) b \end{aligned}$$

with

$$d = \frac{\partial g_T(b)}{\partial b'} = E\left(R^e f'\right)$$

as the second-moment matrix of excess returns and factors.

Excess Returns

- The GMM estimate again solves

$$\min_b g_T(b)' W g_T(b)$$

- And with FOC $-d'W\left[E_T(R^e) - db\right] = 0$, we get for the GMM estimates

$$\begin{aligned}\widehat{b}_1 &= \left(d'd\right)^{-1} d' E_T(R^e) \\ \widehat{b}_2 &= \left(d'S^{-1}d\right) d'S^{-1} E_T(R^e)\end{aligned}$$

- These GMM estimates are just cross-section regressions of mean excess returns on the second moments of returns with factors; the distribution theory is the same.

Other Normalizations

- We need a normalization to use GMM for Excess Returns, but $a = 1$ isn't the only one we can use.
  - We can also use $a = 1 + b'E(f)$
- Why would we want to do that?
  - This normalization allows GMM to run cross-section regressions of mean excess returns on *covariances*... which gets us really close to betas.
  - This normalization yields

$$m = 1 - b'(f - E(f))$$

with

$$E(m) = 1.$$

Other Normalizations

- The derivations proceed the same way as before, but to be concrete

$$\begin{aligned} g_T(b) &= E_T(mR^e) \\ &= E_T(R^e) - E_T\left(R^e(f - E(f))'\right)b \end{aligned}$$

with

$$d = \frac{\partial g_T(b)}{\partial b'} = E_T\left(R^e(f - E(f))'\right)$$

- And the moment condition is

$$d'W\left[db + E_T(R^e)\right] = 0$$

Other Normalizations

- Note that d is now the covariance matrix,

$$d = \frac{\partial g_T(b)}{\partial b'} = E_T\left((f - E(f))\, R^{e\prime}\right)$$

- So if we were to write

$$E(R^e) = -cov\left(R^e, f'\right) b$$

we just have the covariances entering in place of the betas.

- But what's the problem here?
  - We have a "sample-dependent" normalization for a.
  - We have to take this into account in our distribution theory, just as we did when the betas were "generated regressors."

Other Normalizations

- For this case, it's important to note that the normalization does NOT matter for the pricing errors and the chi-square statistics.
  - What does change is the estimate of b.
  - Normalizations therefore only matter for the sampling variance of the estimated parameter b.
- The very ambitious may want to try the derivation!

Horse Races

- Can one set of factors drive out another set? How would we test this?
- For

$$m = b_1' f_1 + b_2' f_2$$

  we just test if

$$b_2 = 0$$

- Note that b_2 can be a scalar (use a t-test) or a vector (use a chi-square test).
- We can also test a restricted v. unrestricted versions of the model (as before, with a chi-square difference test).
  - Remember how these work?

Testing for Priced Factors

- Should we use lambdas λ (the risk factor premia) or b's b (the regression coefficient of m on f)?
  - Note that b's and β's are not the same! β's are regression coefficients of R^i on f!
- The lambdas λ capture whether a factor is *priced*
- The b's b capture whether a factor is useful in *pricing*

lambdas and b's

- The two are related by $\lambda = E\left(ff'\right) b$
- Here's the derivation

$$\begin{aligned} E\left(mR^e\right) &= 0 \\ E\left[R^e\left(1 - f'b\right)\right] &= 0 \end{aligned}$$

So

$$\begin{aligned} E\left(R^e\right) &= cov\left(R^e, f'\right) b \\ &= cov\left(R^e, f'\right) E\left(ff'\right)^{-1} E\left(ff'\right) b \\ &= \beta'\lambda \end{aligned}$$

lambdas and b's

- Note that it only matters whether you use b or λ if the factors are correlated.
  - If they are not, $E\left(ff'\right)$ is diagonal, and each $\lambda_i = 0$ iff the corresponding $b_i = 0$.
- The idea is that λ_i is the single regression coefficient of m of f; while b_i is the multiple regression coefficient of m on f, given the other factors.
  - Asking if $\lambda_i = 0$ asks "Is factor i correlated with the true discount factor?"
  - Asking if $b_i = 0$ asks "Should I include factor i, given the other factors?"

A Note on Methodologies

- Today, we've used GMM to estimate linear discount models of the form

$$\begin{aligned} E(p) &= E(mx) \\ m &= b'f \end{aligned}$$

- Do we *have* to use GMM to estimate this model?
  - No, we could have also used ML
- Can GMM *only* be used on models of this form?
  - No, we can also easily use it on expected return-beta models.
- Estimation methodology is a choice; but it's important to keep in mind that the results you get depend on how you specify your model and which method you use!

End of Today's Lecture.

- That's all for today. Today's material corresponds roughly to Chapter 13 in Cochrane (2005).