1 The Discount Factor (LOP and Existence)

1.1 Motivation

- We're going to show the bare minimum of what's necessary for a discount factor to exist and be unique.
  - We're going to look at $p = E[mx]$ to figure out if we can always find such a discount factor, m.
- We can describe aspects of the payoff space through restrictions on the discount factor.
  - All we need is the law of one price and the absence of arbitrage opportunities
  - We don't need all the structure of utility functions and complete markets to be able to use the representation $p = E[mx]$.
- Why might this be useful?

1.2 Complete Markets

- If markets are complete, then a discount factor exists and is unique. Why?

1.3 Incomplete Markets

- In real markets, we have more states (S) than securities (N)
  - When this is the case, we say that markets are *incomplete*.
- For example, we might have three states, $S = 3$, but only two securities, $N = 2$ with payoff matrix

	1 ($\pi = 1/3$)	2 ($\pi = 1/3$)	3 ($\pi = 1/3$)
A ($P_A = 2.3$)	0	2	5
B ($P_B = 2.0$)	1	3	2

- What are the $S \times 1$ payoff vectors for each asset?

- How is the concept of "states of nature" useful?
  - We can think of the states as things like "good", "average", and "bad"
  - But we can also think about the states as returns observed in January, February, and March
- This is how models are implemented...
  - We assume that a different state is revealed by nature once per month (etc.), and we will use time series of gross (monthly) returns to test our asset pricing models.

1.4 Portfolio Formation

- The payoff space, $\underline{X}$, is the space of all possible security and portfolio payoffs that an investor can form
  - If markets are complete, $\underline{X} = R^S$
  - If markets are incomplete, $\underline{X}$ is a proper subset of R^S
- Assumption A1: Portfolio Formation
  - Investors can form portfolios of any assets that are traded
  - Mathematically: $x_A, x_B \in \underline{X} \Rightarrow ax_A + bx_B \in \underline{X}$ for any real a, b.
- For example, if you buy 5 units of asset A and 2 units of asset B, the payoff vector representing the portfolio's payoff would be

$$5\begin{bmatrix} 0 & 2 & 5 \end{bmatrix} + 2\begin{bmatrix} 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 16 & 29 \end{bmatrix}$$

- We can, of course, use matrix notation as follows:

$$x = \begin{bmatrix} x_A \\ x_B \end{bmatrix} = \begin{bmatrix} 0 & 2 & 5 \\ 1 & 3 & 2 \end{bmatrix}$$

- So that x is an $N \times S$ matrix of asset payoffs, where each row represents a security and each column represents a state.
- Represent the price vector and the vector of portfolio weights, respectively, as

$$p = \begin{bmatrix} 2.3 \\ 2.0 \end{bmatrix}, c = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

- Then the payoff of any portfolio is $x'c$, which gives you a $Sx1$ payoff vector

1.5 Law of One Price

- The Law of One Price (LOP) states that if two securities have exactly the same payoff, then they must have the same price.
  - What does this imply graphically, in terms of the state-space geometry we did last time?

- Assumption A2: LOP
  - Mathematically: $p\left(ax_A + bx_B\right) = ap\left(x_A\right) + bp\left(x_B\right)$
  - What this implies is that investors can't make instant profits simply by re-packaging portfolios.
  - Note: This is an equilibrium condition. What does that mean? It means that if there are any violations, traders will act quickly to eliminate them.

1.6 The Theorem

- Theorem (Easy direction): The existence of a discount factor implies the law of one price.
  - The proof is fairly obvious. Say there is some payoff, $x = y + z$, then

$$p\left(x\right) = E\left[mx\right] = E\left[m\left(y+z\right)\right] = E\left[my\right] + E\left[mz\right] = p\left(y\right) + p\left(z\right)$$

- Theorem (Hard(er) direction): The law of one price implies the existence of a discount factor.
  - The proof is in the book, but we will demonstrate what it means with an example.

- Let's use the same securities and states as our previous example, so that the payoff space is a plane in R^3.

- Start with $p = E\left[mx\right]$

- If markets are incomplete, there are an infinite number of m's that will work. Do you see why?
  - We basically have a system of two equations in three unknowns:

$$\begin{aligned} 2.3 &= \left(1/3\right)\left(0\right)m_1 + \left(1/3\right)\left(2\right)m_2 + \left(1/3\right)\left(5\right)m_3 \\ 2.0 &= \left(1/3\right)\left(1\right)m_1 + \left(1/3\right)\left(1\right)m_2 + \left(1/3\right)\left(2\right)m_3 \end{aligned}$$

1.6.1 The Mimicking Portfolio for m

- However, there is a special m that is a linear combination of the payoffs of the securities that we have (i.e., a portfolio in $\underline{X}$) that is *unique*.
  - Let's call this special portfolio x^*.
- We already know (as seen previously) that any portfolio payoff can be represented by a linear combination of the x's according to $x'c$.
  - Let's represent this special portfolio as $x^* = x'c$.
  - Here, we need the LOP that forces the price of a portfolio payoff to be equal to the price of its constituents. Let's see if we can figure out what the c is to make our special portfolio...
- Our special portfolio m is $x^* = x'c$. Now, use $p = E[mx]$ to write

$$\begin{aligned}
p &= E[xx^*] \\
p &= E[xx'c] \\
p &= E[xx']c \\
E[xx']^{-1} p &= E[xx']^{-1} E[xx'] c \\
E[xx']^{-1} p &= c
\end{aligned}$$

- Now, we can substitute to find

$$x^* = x'c = x' E[xx']^{-1} p = p' E[xx']^{-1} x$$

- What is x^*? A vector perpendicular to all the iso-price planes. This special discount factor is known as the *mimicking portfolio* for m.

1.7 Example

- Using the current example securities and states, let's find x^*.

$$xx' = \begin{bmatrix} 0 & 2 & 5 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 29 & 16 \\ 16 & 14 \end{bmatrix}$$

- And since $E[xx'] = (1/3)[xx']$,

$$\begin{aligned}
x^* &= x' E[xx']^{-1} p \\
x^* &= 3 \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 0.0933 & -0.1066 \\ -0.1066 & 0.1933 \end{bmatrix} \begin{bmatrix} 2.3 \\ 2.0 \end{bmatrix} \\
x^* &= \begin{bmatrix} 0.4240 \\ 1.2800 \\ 0.8680 \end{bmatrix}
\end{aligned}$$

- Let's check and see if x^* prices our securities correctly. For x_A:

$$\begin{aligned} p &= E[mx] \\ &= E[xx^*] \\ &= (1/3)(0)(0.4240) + (1/3)(2)(1.2800) + (1/3)(5)(0.8680) \\ &= 2.3 \end{aligned}$$

- And for x_B:

$$\begin{aligned} p &= E[mx] \\ &= E[xx^*] \\ &= (1/3)(1)(0.4240) + (1/3)(3)(1.2800) + (1/3)(2)(0.8680) \\ &= 2.0 \end{aligned}$$

1.8 Use of the Mimicking Portfolio

- What's the use of x^*?
  - Now, we can price any payoff in X. Note that we have not made any assumptions about utility functions, payoff distributions, market completeness, etc.
- Can we price the payoff of $x_C = \begin{bmatrix} 3 & 13 & 16 \end{bmatrix}$?
  - Sure, it's just

$$\begin{aligned} p_D &= (1/3)(3)(0.4240) + (1/3)(13)(1.2800) + (1/3)(16)(0.8680) \\ &= 10.6 \end{aligned}$$

- Note that the theorem does not say that x^* is unique.
  - Unless markets are complete, there are an infinite number of random variables that satisfy $p = E[mx]$
- How can we see this?
  - Take some ε that is orthogonal to x (Meaning that $E[\varepsilon x] = 0$)...
  - Then $m + \varepsilon$ will also price the payoffs:

$$p = E[(m + \varepsilon)x] = E[mx] + E[\varepsilon x] = E[mx]$$

  - This representation, in fact, generates all possible discount factors.
  - Any discount factor m can be represented by $m = x^* + \varepsilon$, as long as $E[\varepsilon x] = 0$.
  - If markets are complete, there is "no where to go" orthogonal to the payoff space, so x^* would be the only discount factor.

- Again, for completeness, it is important to keep in mind that x^* lies in the payoff space.
- If we have incomplete markets, e.g., $S = 3$ but $N = 2$, then
  - The payoff space is a plane $\left(R^2\right)$ in R^3 and x^* lies in the R^2 plane.
  - If ε is perpendicular to that plane (e.g., coming out of the page), then a valid discount factor can be represented by $m = x^* + \varepsilon$.
- If markets are complete, e.g., $S = 3$ and $N = 3$, then
  - There is nowhere to go that is orthogonal to the payoff space.
- Note: x^* is the projection of any SDF m on the space of payoffs, $\underline{X}$.
- So what have we done?
  - We've basically used only one key assumption - that the law of one price holds - to show that a discount factor exists.
- What we're going to do now is to show that if we add to what we've already done an assumption of no arbitrage, we get not only the existence of a discount factor, but the existence of a positive discount factor.

2 The Discount Factor (No Arbitrage and Positive Discount Factors)

2.1 Arbitrage

- Absence of arbitrage means, generally speaking, you cannot get for free a portfolio that might payoff something but not cost you anything.
  - Note the difference between this definition and the colloquial version that basically only implies the violation of the LOP
  - Mathematically: If there is some payoff x that is greater than or equal to zero in all states, and strictly greater than zero in at least one state, then its price must be strictly positive.
  - E.g., if we had a payoff that looked like

$$x_1 = \begin{bmatrix} 0 & 0 & 0.01 \end{bmatrix}'$$

  - Could the price of x_1 be zero, given the absence of arbitrage?

2.2 The Theorem

- Theorem (Easy Direction)
  - Theorem: $p = E[mx]$ and $m > 0$ imply no arbitrage.
  - Proof: We know that $m > 0$ in all states since it is the IMRS and that $u'(c(s)) > 0$ since consumers always prefer more to less.
    * Then if $m > 0$ in all states and $x \geq 0$ in all states and $x > 0$ in at least one state, then $p = E[mx] > 0$.
- Theorem (Hard/Interesting Direction)
  - Theorem: No arbitrage and the LOP imply m exists and that $m > 0$.
  - The proof is in the book, but we'll illustrate the idea for the case of complete markets.
    * No arbitrage means that the price of any payoff in the positive orthant must be greater than zero.
    * The price = 0 line divides the region of negative prices from positive prices.
    * The price = 0 lines must lie only in the NW or SE regions - it cannot cross into the NE or SW resions.
    * Since m is perpendicular to the price = 0 line, the vector m must lie strictly in the positive orthant.