1 Run-Up to Contingent Claims Markets...

1.1 Options Fundamentals

- What is an option?
  - A type of derivative security.
  - The holder has the *option* to buy or sell some *underlying* security at some time in the future.
- There are two main types of options
  - Calls - The option to buy.
  - Puts - The option to sell.

1.1.1 Call Options

- Apple (NasdaqGS: AAPL) traded on the 13th of September 2011 for $384.62.
- Suppose you thought Apple was going to go up in price. How do you make money?
  - Buy Apple stock.
  - Buy a call option to purchase Apple at $385 per share in October (or a later month).
- Why might you choose one of these strategies over the other?

- A call option to purchase Apple at $385 per share in October has the following features:
  - The $385 is called the strike price (or exercise price).
  - The call gives you the right - **but not the obligation** - to buy the stock.
  - The option's payoff is the cash flow received (+) or paid (−) when the option is optimally exercised.
  - The option's profit is the option's payoff less the initial cost of buying the option
- Example: You own a call option (the right to buy) AAPL for $385 in October 2011. Given the possibilities, fill in the table:

Price(AAPL) in Oct11	Exercise? Y/N	Payoff
$185		
$285		
$385		
$485		
$585		

- What have we learned?
  - The payoff of a call option with strike price K when the stock is priced at S_T at the expiry is $\max[0, S_T - K]$
- We can plot the payoffs of owning the option as a function of the price of AAPL:

- A call limits downside risk, while retaining potential upside gains
- Of course, you can't get something for nothing.
  - We can find the option premiums (or option prices) from a listing like the one below.
- Options Quotes for AAPL:

AAPL	Strike	EXP	Call (Last)	Vol	Put (Last)	Vol
$384.62	$375	OCT	$22.90	1177	$12.87	1881
$384.62	$385	OCT	$16.70	3920	$17.00	13388
$384.62	$395	OCT	$11.65	2172	$21.80	547

- Note: Options contracts are written multiples of 100, so if you buy "1" call option at $16.70, you are buying a contract to buy 100 shares of AAPL at $385 per share - you pay $\$16.70 * 100 = \$1670.00$ for this call.
- The call option for (the right to buy) AAPL for $385 in October 2011 cost you $16.70. What are the potential **profits** per option?

Price(AAPL) in Oct11	Exercise? Y/N	Payoff	Profit
$185			
$285			
$385			
$485			
$585			

1.1.2 Put Options

- Apple (NasdaqGS: AAPL) last traded on the 13th of September 2011 for $384.62.
- Suppose you think Apple is going to go down in price. How do you make money?
  - Short the stock.
  - Buy a put option to sell Apple at $385 per share in October.
    - The put gives you the right - **but not the obligation** - to sell the stock.
    - If the price of AAPL rose to $400 per share, would you exercise your option to sell a share at $385?
- Example: The put option for (the right to sell) AAPL for $385 in October 2011 cost you $17.00. What are the potential **payoffs** and **profits** per option?

Price(AAPL) in Oct11	Exercise? Y/N	Payoff	Profit
$185			
$285			
$385			
$485			
$585			

- The more the price drops, the more money you make.

2 Discounting for Time and Risk

2.1 State Preference Model

- Now, we're going to switch gears a bit and talk about a model that serves as the basis for option pricing.
- It's called the "state preference" model of uncertainty.
  - This model describes future risks in terms of occurrences of a finite number of possible "states".
  - We're going to see how - through the framework of this model - discounting for time and discounting for risk are analogous.
- Example: We could index states of nature by some economic variable, like GDP.
  - A three state model could have the following outcomes: good (GDP grows by 4%), average (GDP grows by 2%), or bad (GDP shrinks by 3%).
- An Arrow-Debreu security is a very special type of (theoretical) security.
  - It pays $1 if a certain future state occurs and $0 otherwise.

  - Why it is also known as a state contingent claim.
  - The following are synonymous:
    * The price of an Arrow-Debreu security
    * The price of a state contingent claim
    * State price

- Why are AD securities useful?
  - If we know the prices of the state contingent claims, we can value any other security that has payoffs as a function of states

- Key: Any risky security can be viewed as a portfolio of state-contingent claims
  - We can see this in the example that follows...

- Suppose there exists 3 Arrow-Debreu securities with prices $\phi_1 = \$0.2$, $\phi_2 = 0.4$, and $\phi_3 = \$0.3$ for claims that payoff in states 1, 2, and 3, respectively.

- And suppose that AAPL may take on 3 possible future values, \$80, \$90, or \$100 in each of these states.

	State 1	State 2	State 3
	AAPL = \$80	AAPL = \$90	AAPL = \$100
A-D 1	1	0	0
A-D 2	0	1	0
A-D 3	0	0	1

- Suppose we want to find the price of two AAPL call options with strike prices of \$85 and \$95.

- First, let's figure out the payoffs of each of these calls. What are they?

- So we know what each call option would payoff in any given state. How much should we pay for them?

- Let's consider an equivalent strategy:

- What portfolio of state-contingent claims would replicate the payoffs of the call options?

- So finally, we now have a way to price the calls...
- What are the prices of the portfolios of state-contingent claims?

- So what have we learned from this example?
- The price of any security x can be found by

$$p(x) = \Sigma_s \phi_s x_s$$

2.2 Using Linear Algebra

- We will soon see that linear algebra is going to be very useful...
- Given the payoffs of the four securities listed below, how can we express the prices of the three securities using linear algebra?

Security	State 1	State 2	State 3
AAPL	\$80	\$90	\$100
AAPL Call, $K = \$85$	\$0	\$5	\$15
AAPL Call, $K = \$95$	\$0	\$0	\$5

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  - We know that $p(x) = \Sigma_s \phi_s x_s$, so let
  - $C = \begin{bmatrix} 80 & 90 & 100 \\ 0 & 5 & 15 \\ 0 & 0 & 5 \end{bmatrix}$, $\phi = \begin{bmatrix} 0.2 \\ 0.4 \\ 0.3 \end{bmatrix}$, $P = \begin{bmatrix} P_A \\ P_B \\ P_C \end{bmatrix}$
  - And we get from $P = C * \phi$ that
  - $P_A = \$82$, $P_B = \$6.5$, and $P_C = \$1.5$.
- We can also use state prices to value a risk-free security by forming a replicating portfolio that pays \$1 in every state

Security	State 1	State 2	State 3
AAPL	\$80	\$90	\$100
AAPL Call, $K = \$85$	\$0	\$5	\$15
AAPL Call, $K = \$95$	\$0	\$0	\$5
Risk-free \$100 Bond	\$100	\$100	\$100

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  - Adding a risk-free security gives us

- $C = \begin{bmatrix} 80 & 90 & 100 \\ 0 & 5 & 15 \\ 0 & 0 & 5 \\ 100 & 100 & 100 \end{bmatrix}$, $\phi = \begin{bmatrix} 0.2 \\ 0.4 \\ 0.3 \end{bmatrix}$, $P = \begin{bmatrix} P_A \\ P_B \\ P_C \\ P_D \end{bmatrix}$
- And we get from $P = C * \phi$ that $P_D = \$90$
- What else do you notice about the price of a risk-free security?

- State contingent claims do not really trade, but we can still use them for the purposes of valuation
- This is the process
  - Use the prices of the securities that do trade to imply what the state prices (ϕ) should be
  - Using ϕ, value new securities and/or check for arbitrage opportunities
  - Since we have from before that $P = C * \phi$, just use

$$\phi = C^{-1}P$$

- Number of Securities = Number of States
  - If we have the same number of securities as states, our system $\phi = C^{-1}P$ is exactly identified.
    * For example, since we have three states, if we only have AAPL stock and the two AAPL calls, we have a system of three equations and three unknowns.
  - We can then solve for the state price vector ϕ, and then determine the price of any other security.
  - Note: Rows and columns of the payoff matrix C have to be linearly independent, or else the matrix C cannot be inverted.
    * If state prices exist and are unique, markets are said to be *complete*
    * This is the case when the number of securities with linearly independent payoffs equals the number of states

3 Contingent Claims in Complete Markets

3.1 Contingent Claims and the SDF

- What do we mean by complete markets vis-a-vis contingent claims?
  - In a complete market, investors can buy any contingent claim - i.e., there have to be enough securities to "span" (synthesize) any contingent claim or any portfolio of contingent claims.
- Where are we going with this?

  - The idea is that if there are complete markets, a discount factor exists and is equal to the contingent claims price divided by the probabilities.
  - Neat thing: We won't need any utility functions for the derivation

3.2 Deriving the SDF from Contingent Claims

- Think again about constructing an asset's payoff x as a bundle of contingent claims $x(s)$

$$p(x) = \Sigma_s \phi(s) x(s)$$

- Re-write this expression by multiplying by $1 = (\pi(s)/\pi(s))$, where $\pi(s)$ is just the probability of state s occuring

$$p(x) = \Sigma_s \pi(s) \frac{\phi(s)}{\pi(s)} x(s)$$

- Now, we have something that looks like an expectation....
- We can rewrite expression $p(x) = \Sigma_s \pi(s) \frac{\phi(s)}{\pi(s)} x(s)$ as

$$p(x) = \Sigma_s \pi(s) m(s) x(s)$$

- By defining

$$m(s) = \frac{\phi(s)}{\pi(s)}$$

- We now have

$$p(x) = \Sigma_s \pi(s) m(s) x(s) = E(mx)$$

- Conclusion: In complete markets, the SDF exists and can be interpreted as the state price per unit probability
- What are the contingent claims prices?
  - How do we determine state prices if these Arrow-Debreu securities don't actually trade?
  - We can figure them out by forming portfolios of securities that actually do trade. We can use real securities as building blocks to construct a portfolio that can "replicate" any state-contingent payoff.

3.3 Examples

3.3.1 Example 1: Complete Markets with Three States

- Assume there are three possible states tomorrow $(1, 2, 3)$ and three securities (A, B, C) that have prices $(p_A = 0.6, p_B = 2.3, p_C = 2.0)$.
- The payoffs of each security in each future state are given in the table below

State / Security	A	B	C
1	0	0	1
2	0	2	3
3	2	5	2

- What is the state-price for state 1, $\phi(1)$? (How can we express this problem using linear algebra?)

3.3.2 Example 2: Complete Markets with Three States

- So that's nice, but A-D securities don't really trade...
- Let's come up with a replicating portfolio of these securities that will replicate the payoffs for a state contingent claim that only pays off in state 1:
  - To do this, we need to figure out how many of each securities to buy
$$n = \begin{bmatrix} n_A & n_B & n_C \end{bmatrix}$$
  - To replicate payoffs
$$Z = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$
  - Now, we need to find n such that
$$nC = Z$$
- So, using linear algegra, we can find the replicating portfolio:

$$\begin{aligned} nC &= Z \\ nCC^{-1} &= ZC^{-1} \\ n &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2.75 & -1.5 & 1 \\ -1.25 & 0.5 & 0 \\ 0.5 & 0 & 0 \end{bmatrix} \\ n &= \begin{bmatrix} 2.75 & -1.50 & 1.0 \end{bmatrix} \end{aligned}$$

- Therefore, to replicate a state contingent claim that pays off \$1 in state 1, we buy 2.75 units of asset A, short 1.5 units of asset B, and buy 1 unit of asset C.

- Let's check to make sure the replicating portfolio works:

$$\begin{aligned} nC &= \begin{bmatrix} 2.75 & -1.50 & 1.0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 5 \\ 1 & 3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \end{aligned}$$

- How much does the replicating portfolio cost us?

$$\begin{aligned} nP &= \begin{bmatrix} 2.75 & -1.50 & 1.0 \end{bmatrix} \begin{bmatrix} 0.6 \\ 2.3 \\ 2.0 \end{bmatrix} \\ &= 0.2 \end{aligned}$$

- So the price of the replicating portfolio for the state contingent claim is 0.2, which matches what we got for the state price $\phi(1) = 0.2$.

3.3.3 Exercise 1: Pricing Other Securities

- Now, from example one we got that $\phi = \begin{bmatrix} 0.2 & 0.4 & 0.3 \end{bmatrix}'$.
- So given these state (contingent claims) prices, we can easily price any other security.
- What is the price of a security that pays off \$7 in state 1, \$10 in state 2, and \$3 in state 3?

3.3.4 Exercise 2: The Risk-Free Rate

- Again, we want to use $\phi = \begin{bmatrix} 0.2 & 0.4 & 0.3 \end{bmatrix}'$.

- What is the risk-free rate?

3.3.5 Example 3: The SDF from Contingent Claims

- Given the state-prices and corresponding probabilities, what is $m(s)$?

State (s)	$\phi(s)$	$\pi(s)$
1	0.2	0.25
2	0.4	0.35
3	0.3	0.40

- Since $m(s) = \phi(s)/\pi(s)$,

$$\begin{aligned} m(1) &= \frac{0.2}{0.25} = 0.8 \\ m(2) &= \frac{0.4}{0.35} = 1.1429 \\ m(3) &= \frac{0.3}{0.40} = 0.75 \end{aligned}$$

3.3.6 Example 4: The SDF from Contingent Claims

- Using what you got in the previous example for $m(s)$, use our central asset pricing equation to determine the price of a security that pays off \$7 in state 1, \$10 in state 2, and \$3 in state 3.
- From $p = E[mx]$ and the definition of an expectation

$$\begin{aligned} p &= E[mx] \\ &= \Sigma_s \pi(s) m(s) x(s) \\ &= (0.25)(0.8)(7) + (0.35)(1.1429)(10) + (0.40)(0.75)(3) \\ &= 6.3 \end{aligned}$$

- What do notice about this example?
  - Can you see why the SDF can also be referred to as a state-price density?

4 Summary

- When Discounting for Time and Risk...
  - Contingent claims prices for risky securities are similar to the discount factors for riskless securities.
  - In a risk-free world, discount factors tell us the PV of \$1 received at **some time** t in the future.
    * Once we know the discount factors d_t, we can price a risk-free security by multiplying the cash flows received at time t by the discount factor and add

$$p(x) = \Sigma_t d_t x_t$$

  - In a world with risk, contingent claims prices tell us the PV of \$1 received at **some time and some state** in the future.
    * Once we know ϕ_s, we price a risky security by multiplying the cash flows received in each state s by the contingent claim price for that state and add

$$p(x) = \Sigma_s \phi_s x_s$$

  - From

$$p(x) = \Sigma_s \phi_s x_s$$

  - We can view the payoff of any security as a bundle (or portfolio) of contingent claims that pay \$1 in state s and \$0 otherwise.
  - Why do we care?
    * Using contingent claims, we can value not only riskless securities, but also securities that have payoffs *dependent on* or *derived from* a particular state that occurs (like a call option).

- Useful Properties and the Way Ahead...
  - When the number of securities with linearly independent payoffs equals the number of states
    * There is a unique ϕ and markets are complete
      · If the state prices are all positive, there are no arbitrage opportunities
  - Today, we've talked about "complete markets"...
  - Suppose instead that the number of securities (with linearly independent payoffs) is larger than the number of states.
    * In this case, we can always find a state-price vector
    * There is a unique ϕ and markets are complete
  - However, there is a chance that some securities may be mis-priced relative to others and there may be chances to make arbitrage profits
    * If the state prices are all positive and correctly price all securities, there are no arbitrage opportunities

- Suppose that the number of securities with linearly independent payoffs is less than the number of states
  - A unique set of state prices does not exist and markets are said to be *incomplete*.
  - There could still be an arbitrage opportunity if the LOP does not hold - i.e., two identical securities trade at different prices.
- This is why options are important. They provide us with a "kink" so that payoffs at different strike prices that are linearly independent can be used to "complete" the market to infer the state-price vector.
  - If the number of options with different strike prices equals the number of states, then we can price *any* derivative of the underlying stock.
  - In this case, option payoffs "span" the state space so that we can recover a set of unique state prices.