ECON 4360: Empirical Finance

Discounting for Time and Risk: Contingent Claims

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Theory Lecture #07

What are we doing today?

- Run-Up to Contingent Claims Markets
  - Options Fundamentals
  - Discounting for Time and Risk
- Contingent Claims in Complete Markets

Introduction to Options

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  - Calls - The option to buy.
  - Puts - The option to sell.

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  - Buy a call option to purchase Apple at \$385 per share in October (or a later month).
- Why might you choose one of these strategies over the other?

Call Options: Some Definitions

- A call option to purchase Apple at \$385 per share in October has the following features:
  - The \$385 is called the strike price (or exercise price).
  - The call gives you the right - **but not the obligation** - to buy the stock.
  - The option's payoff is the cash flow received $(+)$ or paid $(-)$ when the option is optimally exercised.
  - The option's profit is the option's payoff less the initial cost of buying the option

Call Options - Example

- You own a call option (the right to buy) AAPL for \$385 in October 2011. Given the possibilities, fill in the table:

Price(AAPL) in Oct11	Exercise? Y/N	Payoff
\$185		
\$285		
\$385		
\$485		
\$585		

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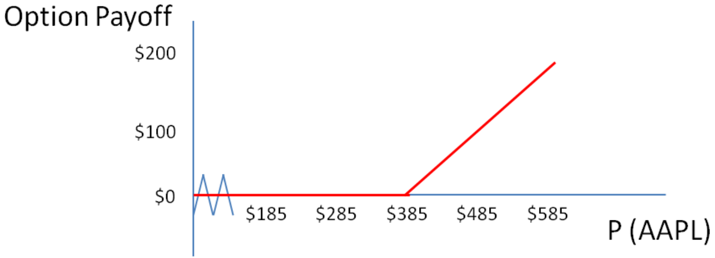
$285	N	$0
$385	Y/N	$0
$485	Y	$100
$585	Y	$200

Graphing a Call Option

- What have we learned?
  - The payoff of a call option with strike price K when the stock is priced at S_T at the expiry is $\max\left[0, S_T - K\right]$
- We can plot the payoffs of owning the option as a function of the price of AAPL:



- A call limits downside risk, while retaining potential upside gains

Option Premiums

- Of course, you can't get something for nothing.
  - We can find the option premiums (or option prices) from a listing like the one below
- Options Quotes for AAPL:

AAPL	Strike	EXP	Call (Last)	Vol	Put (Last)	Vol
\$384.62	\$375	OCT	\$22.90	1177	\$12.87	1881
\$384.62	\$385	OCT	\$16.70	3920	\$17.00	13388
\$384.62	\$395	OCT	\$11.65	2172	\$21.80	547

- Note: Options contracts are written multiples of 100, so if you buy "1" call option at \$16.70, you are buying a contract to buy 100 shares of AAPL at \$385 per share - you pay $\$16.70 * 100 = \$1670.00$ for this call.

Call Options - Example, Continued...

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    - Buy a put option to sell Apple at \$385 per share in October.
- The put gives you the right - **but not the obligation** - to sell the stock.
- If the price of AAPL rose to \$400 per share, would you exercise your option to sell a share at \$385?

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- The more the price drops, the more money you make.

AAPL Today?

- AAPL traded on Tuesday, 5th February for \$457.84...
    - Guess we should have bought those call options...

Discounting for Time and Risk

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- It's called the "state preference" model of uncertainty.
  - This model describes future risks in terms of occurrences of a finite number of possible "states".
  - We're going to see how - through the framework of this model - discounting for time and discounting for risk are analogous.

Discounting for Time and Risk

- Example: We could index states of nature by some economic variable, like GDP.
- A three state model could have the following outcomes:
  - good (GDP grows by 4%),
  - average (GDP grows by 2%), or
  - bad (GDP shrinks by 3%).

Arrow-Debreu Securities

- An Arrow-Debreu security is a very special type of (theoretical) security.
  - It pays \$1 if a certain future state occurs and \$0 otherwise.
  - Why it is also known as a state contingent claim.
- The following are synonymous:
  - The price of an Arrow-Debreu security
  - The price of a state contingent claim
  - State price

Arrow-Debreu Securities

- Why are AD securities useful?
  - If we know the prices of the state contingent claims, we can value any other security that has payoffs as a function of states
- Key: Any risky security can be viewed as a portfolio of state-contingent claims
  - We can see this in the example that follows...

Example: Using AD Securities

- Suppose there exists 3 Arrow-Debreu securities with prices $\phi_1 = \$0.2$, $\phi_2 = \$0.4$, and $\phi_3 = \$0.3$ for claims that payoff in states 1, 2, and 3, respectively.
- And suppose that AAPL may take on 3 possible future values, \$80, \$90, or \$100 in each of these states.

	State 1	State 2	State 3
	AAPL $=$ \$80	AAPL $=$ \$90	AAPL $=$ \$100
A-D 1	1	0	0
A-D 2	0	1	0
A-D 3	0	0	1

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  - The second pays off \$5 in State 3

Example, Continued...

- So we know what each call option would pay off in any given state. How much should we pay for them?
- Let's consider an equivalent strategy:
- What portfolio of state-contingent claims would replicate the payoffs of the call options?

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    - Second: $5 * \$0.3 = \$1.5$
- So what have we learned from this example?
- The price of any security x can be found by

$$p(x) = \Sigma_s \phi_s x_s$$

Example: Using Linear Algebra

- We will soon see that linear algebra is going to be very useful...
- Given the payoffs of the four securities listed below, how can we express the prices of the three securities using linear algebra?

Security	State 1	State 2	State 3
AAPL	\$80	\$90	\$100
AAPL Call, $K = \$85$	\$0	\$5	\$15
AAPL Call, $K = \$95$	\$0	\$0	\$5

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- And we get from $P = C * \phi$ that
- $P_A = \$82$, $P_B = \$6.5$, and $P_C = \$1.5$.

Example: Pricing a Risk-Free Security

- We can also use state prices to value a risk-free security by forming a replicating portfolio that pays \$1 in every state

Security	State 1	State 2	State 3
AAPL	\$80	\$90	\$100
AAPL Call, $K = \$85$	\$0	\$5	\$15
AAPL Call, $K = \$95$	\$0	\$0	\$5
Risk-free \$100 Bond	\$100	\$100	\$100

Example: Pricing a Risk-Free Security

- Adding a risk-free security gives us
- $C = \begin{bmatrix} 80 & 90 & 100 \\ 0 & 5 & 15 \\ 0 & 0 & 5 \\ 100 & 100 & 100 \end{bmatrix}, \phi = \begin{bmatrix} 0.2 \\ 0.4 \\ 0.3 \end{bmatrix}, P = \begin{bmatrix} P_A \\ P_B \\ P_C \\ P_D \end{bmatrix}$
- And we get from $P = C * \phi$ that $P_D = \$90$
- What else do you notice about the price of a risk-free security?

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  - Use the prices of the securities that do trade to imply what the state prices (ϕ) should be
  - Using ϕ, value new securities and/or check for arbitrage opportunities
  - Since we have from before that $P = C * \phi$, just use

$$\phi = C^{-1} P$$

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  - If state prices exist and are unique, markets are said to be *complete*
  - This is the case when the number of securities with linearly independent payoffs equals the number of states

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  - In a complete market, investors can buy any contingent claim - i.e., there have to be enough securities to "span" (synthesize) any contingent claim or any portfolio of contingent claims.
- Where are we going with this?
  - The idea is that if there are complete markets, a discount factor exists and is equal to the contingent claims price divided by the probabilities.
  - Neat thing: We won't need any utility functions for the derivation

Deriving the SDF from Contingent Claims

- Think again about constructing an asset's payoff x as a bundle of contingent claims $x(s)$

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- Re-write this expression by multiplying by $1 = (\pi(s) / \pi(s))$, where $\pi(s)$ is just the probability of state s occuring

$$p(x) = \Sigma_s \pi(s) \frac{\phi(s)}{\pi(s)} x(s)$$

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- Now, we have something that looks like an expectation....

Deriving the SDF from Contingent Claims

- We can rewrite expression $p(x) = \Sigma_s \pi(s) \frac{\phi(s)}{\pi(s)} x(s)$ as

$$p(x) = \Sigma_s \pi(s)\, m(s)\, x(s)$$

- By defining

$$m(s) = \frac{\phi(s)}{\pi(s)}$$

- We now have

$$p(x) = \Sigma_s \pi(s)\, m(s)\, x(s) = E(mx)$$

- Conclusion: In complete markets, the SDF exists and can be interpreted as the state price per unit probability

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What are the contingent claims prices?

- How do we determine state prices if these Arrow-Debreu securities don't actually trade?
  - We can figure them out by forming portfolios of securities that actually do trade. We can use real securities as building blocks to construct a portfolio that can "replicate" any state-contingent payoff.

Example: Complete Markets with Three States

- Assume there are three possible states tomorrow $(1, 2, 3)$ and three securities (A, B, C) that have prices $(p_A = 0.6, p_B = 2.3, p_C = 2.0)$.
- The payoffs of each security in each future state are given in the table below

Security / State	1	2	3
A	0	0	2
B	0	2	5
C	1	3	2

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- What is the state-price for state 1, $\phi(1)$? How can we express this problem using linear algebra?
  - First write the payoff matrix as

$$C = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 5 \\ 1 & 3 & 2 \end{bmatrix}$$

  - And the vector of prices as

$$P = \begin{bmatrix} 0.6 \\ 2.3 \\ 2.0 \end{bmatrix}$$

Example 1: Complete Markets with Three States

- Now recall that from $P = C\phi$,

$$\begin{aligned}
\phi &= C^{-1}P \\
&= \begin{bmatrix} 2.75 & -1.5 & 1 \\ -1.25 & 0.5 & 0 \\ 0.5 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.6 \\ 2.3 \\ 2.0 \end{bmatrix} \\
&= \begin{bmatrix} 0.2 & 0.4 & 0.3 \end{bmatrix}'
\end{aligned}$$

- So the state price for state 1 is $\phi(1) = 0.2$

Example 2: Complete Markets with Three States

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  - Now, we need to find n such that

$$nC = Z$$

Example 2: Complete Markets with Three States

- So, using linear algegra, we can find the replicating portfolio:

$$
\begin{aligned}
nC &= Z \\
nCC^{-1} &= ZC^{-1} \\
n &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2.75 & -1.5 & 1 \\ -1.25 & 0.5 & 0 \\ 0.5 & 0 & 0 \end{bmatrix} \\
n &= \begin{bmatrix} 2.75 & -1.50 & 1.0 \end{bmatrix}
\end{aligned}
$$

Example 2: Complete Markets with Three States

- So, using linear algegra, we can find the replicating portfolio:

$$
\begin{aligned}
nC &= Z \\
nCC^{-1} &= ZC^{-1} \\
n &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2.75 & -1.5 & 1 \\ -1.25 & 0.5 & 0 \\ 0.5 & 0 & 0 \end{bmatrix} \\
n &= \begin{bmatrix} 2.75 & -1.50 & 1.0 \end{bmatrix}
\end{aligned}
$$

- Therefore, to replicate a state contingent claim that pays off \$1 in state 1, we buy 2.75 units of asset A, short 1.5 units of asset B, and buy 1 unit of asset C.

Example 2: Complete Markets with Three States

- Let's check to make sure the replicating portfolio works:

$$
\begin{aligned}
nC &= \begin{bmatrix} 2.75 & -1.50 & 1.0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 5 \\ 1 & 3 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

- How much does the replicating portfolio cost us?

$$
\begin{aligned}
nP &= \begin{bmatrix} 2.75 & -1.50 & 1.0 \end{bmatrix} \begin{bmatrix} 0.6 \\ 2.3 \\ 2.0 \end{bmatrix} \\
&= 0.2
\end{aligned}
$$

- So the price of the replicating portfolio for the state contingent claim is 0.2, which matches what we got for the state price $\phi(1) = 0.2$.

Exercise 1: Pricing Other Securities

- Now, from example one we got that $\phi = \left[\begin{array}{ccc} 0.2 & 0.4 & 0.3 \end{array} \right]'$.
- So given these state (contingent claims) prices, we can easily price any other security.
- What is the price of a security that pays off \$7 in state 1, \$10 in state 2, and \$3 in state 3?

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$$\begin{aligned} P &= C\phi \\ &= \begin{bmatrix} 7 & 10 & 3 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0.4 \\ 0.3 \end{bmatrix} \\ &= 6.3 \end{aligned}$$

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    - And recall, that this is the *price* of the risk-free security... If we want the (gross) risk-free rate,

$$R^f = 1/P = 1/0.90 = 1.11$$

Example 3: The SDF from Contingent Claims

- Given the state-prices and corresponding probabilities, what is $m(s)$?

State (s)	$\phi(s)$	$\pi(s)$
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- Since $m(s) = \phi(s) / \pi(s)$,

$$
\begin{aligned}
m(1) &= \frac{0.2}{0.25} = 0.8 \\
m(2) &= \frac{0.4}{0.35} = 1.1429 \\
m(3) &= \frac{0.3}{0.40} = 0.75
\end{aligned}
$$

Example 4: The SDF from Contingent Claims

- Using what you got in the previous example for $m(s)$, use our central asset pricing equation to determine the price of a security that pays off \$7 in state 1, \$10 in state 2, and \$3 in state 3.

Example 4: The SDF from Contingent Claims

- Using what you got in the previous example for $m(s)$, use our central asset pricing equation to determine the price of a security that pays off \$7 in state 1, \$10 in state 2, and \$3 in state 3.
- From $p = E[mx]$ and the definition of an expectation

$$\begin{aligned}
p &= E[mx] \\
&= \Sigma_s \pi(s)\, m(s)\, x(s) \\
&= (0.25)(0.8)(7) + (0.35)(1.1429)(10) + (0.40)(0.75)(3) \\
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- What do notice about this example?
    - Can you see why the SDF can also be referred to as a state-price density?

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- In a world with risk, contingent claims prices tell us the PV of $1 received at **some time and some state** in the future.
  - Once we know ϕ_s, we price a risky security by multiplying the cash flows received in each state s by the contingent claim price for that state and add
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Using State-Contingent Claims

- From

$$p(x) = \Sigma_s \phi_s x_s$$

- We can view the payoff of any security as a bundle (or portfolio) of contingent claims that pay \$1 in state s and \$0 otherwise.
- Why do we care?
  - Using contingent claims, we can value not only riskless securities, but also securities that have payoffs *dependent on* or *derived from* a particular state that occurs (like a call option).

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- Today, we've talked about "complete markets"...

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- If the number of options with different strike prices equals the number of states, then we can price *any* derivative of the underlying stock.
  - In this case, option payoffs "span" the state space so that we can recover a set of unique state prices.

End of Today's Lecture.

- That's all for today. Today's material provides some background and introduces Chapter 3 in Cochrane (2005), which we will be discussing more next time.