1 Prices and Payoffs

1.1 Model Setup

- Recall that last time, we set up a basic problem that derived the central asset pricing equation from the consumption-based model.
  - (We wanted to find the **price** of the asset that set the first order condition to zero.)
- Our basic pricing equation was
$$p_t = E_t\left[\beta\frac{u'(c_{t+1})}{u'(c_t)}x_{t+1}\right]$$
that implied, in equilibrium, the decrease in utility from buying a share of the asset today has to just equal the increase in expected discounted utility that results from having one more share tomorrow.
- We then wrote the pricing equation
$$p_t = E_t\left[\beta\frac{u'(c_{t+1})}{u'(c_t)}x_{t+1}\right]$$
as
$$p_t = E_t\left[m_{t+1}x_{t+1}\right]$$
by defining
$$m_{t+1} = \beta\frac{u'(c_{t+1})}{u'(c_t)}$$
- The variable m_{t+1} is a random variable, called the SDF, that maps future payoffs into today's price.

1.2 Some Stochastic Examples

- Recall that last time we did a nonstochastic example to illustrate the basic intuition
  - If the first-order condition is not met, an investor can increase his utility by purchasing of an asset that allow him to shift / smooth consumption across time
- Today, we're going to do a stochastic example that illustrates - in a simple way - how the equation
$$p = E(mx)$$
shows that all correction for risk can be captured by **a single random variable** put inside the expectation.
  - Unlike β in a model like the CAPM, the stochastic discount factor m is the **same** for **all** assets...

1.2.1 Example A

- Suppose investor preferences are $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ with $\gamma = 2$ and $\beta = 0.99$.
- There are three possible states of nature next period: good, average, and bad.
- The investor consumes 7 units of the consumption good today.

- Given the possible payoffs of the security $\left(x^A\right)$ in the table that follows, we're going to figure out the price the investor would pay for the security today.

State	Prob	Future C	Payoff
s	π_{t+1}^s	c_{t+1}	x_{t+1}^A
Good	0.3	12	20
Average	0.4	9	15
Bad	0.3	6	10

- First, what is the **expected payoff** of the security?

- Next, how do we start figuring out the **price** the investor would pay for the security today?
  - Recall our basic pricing equation,

$$p_t = E_t\left[m_{t+1}x_{t+1}\right] = E_t\left[\beta\frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)}x_{t+1}\right]$$

  - Answer: The price of asset A is

Price $=$

State	Prob	Fut C	Payoff A	_______	_______
s	p_{t+1}^s	c_{t+1}	x_{t+1}^A		
Good	0.3	12	20		
Ave	0.4	9	15		
Bad	0.3	6	10		

1.2.2 Example B

- Assume again that we have the same investor as in Example One
  - He has the same preferences, consumption today, and faces the same three states of nature as before.

- Now, however, he can only invest in another asset, Asset B...

State	Prob	Future C	Payoff
s	π^s_{t+1}	c_{t+1}	x^B_{t+1}
Good	0.3	12	10
Average	0.4	9	15
Bad	0.3	6	20

- First, what is the **expected payoff** of the security?

- And what is its **price**?

  - Answer: The price of asset B is

$$\text{Price} \ =$$

State	Prob	Fut C	Payoff A	_______	_______
s	p^s_{t+1}	c_{t+1}	x^B_{t+1}		
Good	0.3	12	10		
Ave	0.4	9	15		
Bad	0.3	6	20		

- What's the Difference between A and B?

- Why, though the expected payoffs are identical, does asset B sell for a higher price than asset A?

- Can you see from this example how m_{t+1} is a random variable that discounts payoffs to prices?

1.3 An Aside: Modelling Expectations

- What about modeling future expectations?
- This model uses expectations of the future to set prices today.
  - Is this useful if we can't observe future consumption?
- We **can** observe consumption historically and test the model if it correctly explains the relationship between consumption and prices.
  - Often the "market" - for example, the return on the S&P 500 - is used as a proxy for consumption.
  - (This gives us a model like the CAPM.)

2 Notation

- You may be thinking that an asset with price p_t and payoff x_{t+1} is a very restrictive security, but actually this notation is very general...
  - We can easily accommodate many different asset pricing equations with this notation.

2.1 Stocks

- We can easily use this notation to describe stocks.
  - Think about what the "payoff" is for owning a stock...
  - How would you describe this in a today-tomorrow sense?

2.2 Returns

- If an asset's price is 100 and it pays off 106 in a year's time, then what is its gross return R_{t+1}?

- How do we get gross returns?

- We often talk and work in terms of returns, since the gross return is what tells us what our dollar payoff is for each dollar invested.
- Since a gross return is the dollar payoff per dollar invested,
  - You can think of a return as a payoff with a price of one.
  - Then we can write
    $$1 = E_t\left(m_{t+1}R_{t+1}\right)$$
    simply by re-writing the fundamental asset pricing equation $p_t = E_t\left(m_{t+1}x_{t+1}\right)$
- Note how the SDF discounts gross returns to their price; which, by definition, is 1.
- Notes about Returns:
  - Capital letters denote gross returns
  - Lowercase letters denote net returns $r = R - 1$
  - The return can also be defined in continuous compounding terms as $r = \ln(R)$
  - In our example, the net return r is $r = R - 1 = 0.06$, or 6%
- Returns are "nice" and useful in empirical work because they are "stationary" over time in the sense that they don't have trends and you can meaningfully take averages

2.3 P/D Ratio

- We use returns a lot in empirical work, but often we would prefer a stationary variable that lets us think in terms of prices...
- Return to our definition of the payoff for a stock, $x_{t+1} = p_{t+1} + d_{t+1}$
- First, write the asset pricing equation as
  $$p_t = E_t\left[m_{t+1}(p_{t+1} + d_{t+1})\right]$$
- If we divide by today's dividend, we get the present value of the price/dividend ratio...
- Next, dividing by today's dividend
  $$\begin{aligned}
  \frac{p_t}{d_t} &= \frac{E_t\left[m_{t+1}(p_{t+1} + d_{t+1})\right]}{d_t} \\
  \frac{p_t}{d_t} &= E_t\left[m_{t+1}(\frac{p_{t+1}}{d_{t+1}}\frac{d_{t+1}}{d_t} + \frac{d_{t+1}}{d_t})\right] \\
  \frac{p_t}{d_t} &= E_t\left[m_{t+1}(\frac{p_{t+1}}{d_{t+1}} + 1)\left(\frac{d_{t+1}}{d_t}\right)\right]
  \end{aligned}$$

- This gets us back to thinking about asset **prices**, but we are still looking at stationary variables
  - The price is p_t/d_t
  - The payoff is $x_{t+1} = \left(1 + \frac{p_{t+1}}{d_{t+1}}\right) \frac{d_{t+1}}{d_t}$

2.4 Excess Returns

- What is an "excess return"?
  - Generally speaking, it is the difference between two returns
  - Also called a "zero-cost" portfolio...
- It is often common to study equity strategies where you sell short one stock/portfolio and invest the proceeds in another to generate an "excess" return
- Think about this... You can borrow a dollar today at R^f and invest it in an asset with return R.
  - You pay no money out of pocket today, but you get the payoff $R - R^f$
  - This is a payoff with zero price.
- Example: Consider a strategy of borrowing \$100 to buy a share of Apple stock for \$100. This costs you \$0 today (its price is zero) since you put none of your own money into the investment.
  - But the payoff is not zero! You could make or lose money on the investment.
  - The payoff of such a long-short strategy with price today of zero is an "excess return" where $R^e_{t+1} = R^a_{t+1} - R^f$
- You can see that you don't have to just borrow at the risk-free rate - this can be done with any two assets, say $R^e_{t+1} = R^a_{t+1} - R^b_{t+1}$
  - This strategy is equivalent to "going long" stock A and "going short" stock B
- Notes on "shorting" a stock:
  - Going short is the practice of selling assets that have been borrowed from a third party, with the intention to buying identical assets back at a later date to return to that third party.
  - A short seller hopes to profit from a decline in the asset's price, since the seller will pay less to buy the assets back than it received for earlier selling them.
- You can see that - mathematically - short selling is equivalent to buying a negative amount of an asset

$$\begin{aligned} 1 &= E_t\left(m_{t+1} R^a_{t+1}\right) \\ -1 &= -E_t\left(m_{t+1} R^b_{t+1}\right) \end{aligned}$$

- So our asset pricing equation for excess returns becomes

$$0 = E_t\left(m_{t+1}\left(R^a_{t+1} - R^b_{t+1}\right)\right) = E_t\left(m_{t+1} R^e_{t+1}\right)$$

- Why Excess Returns?
- Why might excess returns be a useful thing to look at?
  - You can always borrow at R^f, so the interesting thing is the return you get **over and above** the risk-free rate.
  - Interest rate variation has little to do with our understanding of risk-premia, so we want to look at interest rates and risk premia separately

2.5 Managed Portfolios

- A managed portfolio is simply one where the weight on each asset varies through time.
- Let z_t be the amount (in dollars) of an asset owned at time t. I.e., the "price" of such an asset is the amount invested (in dollars).
- Then the payoff is $z_t R_{t+1}$ or
$$z_t = E_t\left[m_{t+1}\left(z_t R_{t+1}\right)\right]$$
- Example: a value-oriented timing strategy might make investments proportional to p_t/d_t ratios, investing less when prices are high relative to dividends such that
$$z_t = a - b\left(\frac{p_t}{d_t}\right)$$

2.6 Bonds

- What is a risk-free bond?
  - Essentially, it is a claim to a \$1 unit payoff in every possible state.
- Using our key equation, the price of a bond is given by
$$p_t = E_t\left[m_{t+1}1\right]$$
- If we go back to the same setup we used in our earlier Excel example,
  - What is the expected payoff of a risk-free bond?
  - What is its price?

- What is its return?