1 Beta Representations

1.1 Motivation: Empirical Work

- Beta Representations
  - Expected Return-Beta Representations will be seen as equivalent to a linear model for the discount factor
$$m = b'f$$
  - We can derive models like the CAPM, ICAPM, and APT as factor models
    - Coming up: We will discuss what assumptions we need to express the discount factor as a linear function of factors f
- Mean-Variance Frontier
  - State-space representation provides useful framework; valid in infinite-dimensional payoff spaces
  - Many asset-pricing ideas and test statistics have interpretations in terms of the MV Frontier

1.2 Expected Return-Beta Representations

- The model:
  - Expected Return-Beta Representation of linear factor pricing models
$$E\left(R^i\right) = \gamma + \beta_{i,a}\lambda_a + \beta_{i,b}\lambda_b + \cdots,\ i = 1 \ldots N$$
- The general idea:
  - Explain the variation in average returns across asset's due to each assets exposure - as measured by the betas - to various risks, as priced by the lambdas.
- But let's start simpler...

1.3 Where to Start

- Recall how our central asset pricing equation gets us to a basic pricing equation for returns
$$1 = E\left[mR^i\right]$$
- And recall how we can manipulate this expression to re-write it as
$$\begin{aligned}
1 &= E\left[mR^i\right] \\
1 &= E(m)\,E\left(R^i\right) + cov\left(m, R^i\right) \\
E\left(R^i\right) &= R^f - R^f cov\left(m, R^i\right) \\
E\left(R^i\right) &= R^f + \left(\frac{cov\left(R^i, m\right)}{var(m)}\right)\left(-\frac{var(m)}{E(m)}\right)
\end{aligned}$$

- Or, equivalently

$$E\left(R^i\right) = R^f + \beta_{i,m}\lambda_m$$

- Now we actually have a beta pricing model...

$$E\left(R^i\right) = R^f + \beta_{i,m}\lambda_m$$

  - What does this equation tell us?

  - What is λ_m?

  - What is $\beta_{i,m}$?

- Confused? Don't be. Let's think about this within the context of the consumption-based model...

  - What can we tell from this equation?

$$E\left(R^i\right) = R^f + \beta_{i,m}\lambda_m$$

  - Go back to

$$E\left(R^i\right) = R^f + \left(\frac{cov\left(R^i, m\right)}{var\left(m\right)}\right)\left(-\frac{var\left(m\right)}{E\left(m\right)}\right)$$

  - First, recall that in the consumption-based model $m_{t+1} = \beta\frac{u'(c_{t+1})}{u'(c_t)}$, so

$$-\frac{var\left(m\right)}{E\left(m\right)} \propto -\frac{u'\left(c_t\right)}{u'\left(c_{t+1}\right)}$$

    * Suppose consumption growth is high. What do we then know about the term $-var\left(m\right)/E\left(m\right)$?

- Second, what do we know about the $cov\left(R^i, m\right)$ term?

- So what have we learned?
- Using:

$$\begin{aligned} E\left(R^i\right) &= R^f + \left(\frac{cov\left(R^i, m\right)}{var\left(m\right)}\right)\left(-\frac{var\left(m\right)}{E\left(m\right)}\right) \\ E\left(R^i\right) &= R^f + \beta_{i,m}\lambda_m \end{aligned}$$

- From the first equation?

- From the second equation?

1.4 Multiple Time-Series Regression

- Now, back to the Expected-Return Beta Model...

- The first step is to use time-series data on assets to find the betas in a regression of returns on factors

$$R_t^i = a_i + \beta_{i,a} f_t^a + \beta_{i,b} f_t^b + \cdots \varepsilon_t^i,\ t = 1 \ldots T$$

- The "factors" f are proxies for the growth rate in marginal utility
  - E.g., the CAPM is a one-factor model that uses $f = R_t^m$, the return on the market portfolio
- Again, assets that have high returns when consumption/the market is already high (and therefore, low returns when consumption is low) are the risky ones
  - Investors demand higher returns for holding them
  - This is reflected in higher betas, with $\beta_{i,a}$ interpreted as the amount of exposure of the i^{th} asset for factor a risks
- From this regression, we get *estimates of the betas* for each asset
- Now, from a multiple regression of returns on factors, what do we have?
  - We have estimates of the coefficients, the betas, for each asset i
    - So now we have a set of $\left\{R_t^i\right\}_{t=1}^T$ and $\left(\beta_{i,a}, \beta_{i,b}, \ldots\right)$ for each $i = 1...N$ asset
- But what we really want to explain is how *average* returns vary across assets...
  - So let's use $\left\{R_t^i\right\}_{t=1}^T$ to construct $E\left(R^i\right)$ and see what we can do with that...

1.5 Cross-Section Regression

- Given what we have now, we can get back to
  - The model for an Expected Return-Beta Representation of linear factor pricing models

$$E\left(R^i\right) = \gamma + \beta_{i,a}\lambda_a + \beta_{i,b}\lambda_b + \cdots,\ i = 1 \ldots N$$

- The general idea:
  - Explain the variation in average returns across assets due to each assets exposure - as measured by the betas - to various risks, as priced by the lambdas.
- To be clear:
  - The model says that assets with higher betas should get higher average returns
  - The betas are the explanatory variables - they vary across assets
  - The lambdas λ (coefficients) and γ (intercept) are what are are estimating - they are the same across assets

1.5.1 Interpretations

- From

$$E\left(R^i\right) = \gamma + \beta_{i,a}\lambda_a + \beta_{i,b}\lambda_b + \cdots,\ i = 1 \ldots N$$

- For any asset i,
  - $\beta_{i,a}$ represents the amount of risk due to factor a (which depends on the asset)
  - λ_a represents the extra return per unit of (factor-specific) risk that investors demand (which is the same for all assets)

- Think about a one factor model. What should you get if you plotted expected returns vs. betas?

1.5.2 CAPM Example

- For the CAPM

$$\begin{aligned} E\left(R^i\right) &= R^f + \beta_{i,m}\lambda_m,\ i = 1 \ldots N \\ R_t^i &= a_i + \beta_{i,m}R_t^m + \varepsilon_t^i,\ t = 1...T \end{aligned}$$

- The "factor" used in the CAPM is the return on the S&P 500, or some other market index.

- Interpretation: "For each unit of exposure $\beta_{i,m}$ to market risk, you must provide investors with an expected return premium of λ_m."

1.6 Testing

- One way to test an asset pricing model of this form:

- First, run time-series regressions to estimate the β's:

$$R_t^i = a_i + \beta_{i,a}f_t^a + \beta_{i,b}f_t^b + \cdots \varepsilon_t^i,\ t = 1...T$$

- Second, run a cross-section regression to see if expected returns are linearly related to the β's:

$$E\left(R^i\right) = \gamma + \beta_{i,a}\lambda_a + \beta_{i,b}\lambda_b + \cdots + \alpha_i,\ i = 1 \ldots N$$

- Model Predictions: the pricing errors, α_i, should be small and statistically insignificant.

1.6.1 Special Cases: Risk-Free Rate

- If there is a risk-free rate, for

$$E\left(R^f\right) = \gamma + \beta_{i,a}\lambda_a + \beta_{i,b}\lambda_b + \cdots + \alpha_i$$

- All of its betas are zero, which implies

$$\gamma = R^f$$

- Where that gets us:
  - We can estimate γ, or impose the condition that $\gamma = R^f$
  - Here, γ is called the expected *zero-beta rate*

1.6.2 Special Cases: Using Excess Returns

- If we use excess returns, where $E\left(R^{ei}\right) = E\left(R^i\right) - E\left(R^j\right)$

$$E\left(R^{ei}\right) = \beta_{i,a}\lambda_a + \beta_{i,b}\lambda_b + \cdots$$

- The intercept drops out.
- Note that if we use $E\left(R^{ei}\right) = E\left(R^i\right) - E\left(R^f\right)$, we are talking about a model of equity risk premium.

1.6.3 Special Cases: Factors are Returns/Excess Returns

- If the factors themselves are returns, like $f = R^m$ for the CAPM, the model should apply to the factor as well.
- For example, if the factor is the market return in excess of the risk-free rate, $R_t^{em} = R_t^m - R_t^f$
- Then the time-series regression is

$$\begin{aligned} R_t^i &= a_i + \beta_{i,a} f_t^a + \beta_{i,b} f_t^b + \cdots \varepsilon_t^i \\ R_t^{em} &= \beta_{i,m} R_t^{em} + \beta_{i,b} f_t^b + \cdots \varepsilon_t^i \end{aligned}$$

- So $\beta_{i,m} = 1$ (the factor has a beta of one on itself) and all other betas are zero.
- And for the excess market return,

$$E\left(R^{em}\right) = (1)\,\lambda_m + (0)\,\lambda_b + \cdots$$

- Now we can get that $\lambda_m = E\left(R^{em}\right) = E\left(R^m - R^f\right)$ and

$$\begin{aligned} E\left(R^{ei}\right) &= \beta_{i,m}\lambda_m + (0)\,\lambda_b + \cdots \\ &= \beta_{i,m} E\left(R^m - R^f\right) \end{aligned}$$

- Which is just the familiar CAPM

1.7 Note on Betas

- The betas β cannot be asset-specific firm characteristics, such as firm size or book-to-market
- The betas measure the sensitivity of a firm's return to a macroeconomic factor common to all firms
  - E.g., the return on small firms minus the return on big firms (SMB)
  - E.g., the return on high book-to-market firms minus the return on low book-to-market firms (HML)
- What matters is how a firm behaves (the sensitivity to the factor) rather than what the firm characteristic is.
- The idea is that a market equilibrium wouldn't otherwise survive simple repacking schemes, e.g.
  - We know that returns on small firms are larger than returns on big firms. What if you could form a company that buys small firms and holds them? Your firm is large, so you pay a low return; but you earn a large return from your small constituent firms. You get to pocket the difference!
  - What is firms whose names that start with a 'Z' command a higher return than firms that start with 'A'. Can a firm increase its market value by changing its name from Zoologic to Alphalogic?
- Why won't these schemes work?

- "Asset returns depend on how you behave, not on who you are"