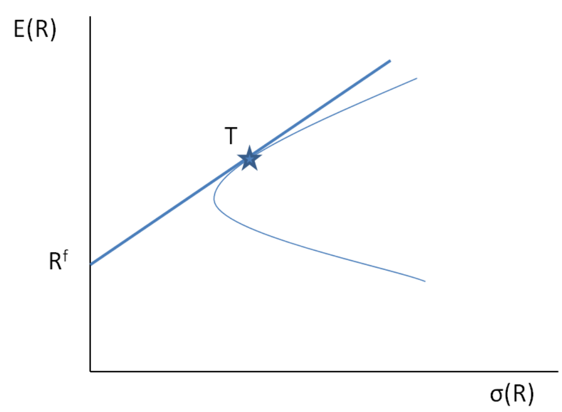
1 CAPM

1.1 Introduction

- Portfolio Theory provides a natural introduction to CAPM
  - Continuation of mean-variance analysis
- Recall, when we use Portfolio Theory, we take prices and returns as givens
- Now, we're going to introduce the CAPM, which is a model of asset pricing
  - It is an equilibrium model - we can derive equilibrium prices, returns, and risk-premiums
  - It relates the expected return of a stock to its risk
- The CAPM is a very simple model of asset pricing.
  - But that simplicity comes at a cost...
  - And there are a number of assumptions that we need to keep in mind.
- Assumptions for CAPM
  - Investors are risk-averse
  - All investors have homogeneous expectations about means and variances
  - Either
    * Investors only care about mean and variance
    * Stock returns are normally distributed (I.e., Returns only *have* a mean and variance)
  - All investors can invest in the same risk-free asset
  - Assets are infinitely divisible
  - Investors have a one-period time-horizon
  - Perfect Markets: E.g., no taxes, no transactions costs, have access to short sale proceeds, etc.
- Recall... The Efficient Frontier for 1 Risk-Free asset and N Risky Assets
  - Portfolio T is the portfolio with the highest Sharpe ratio

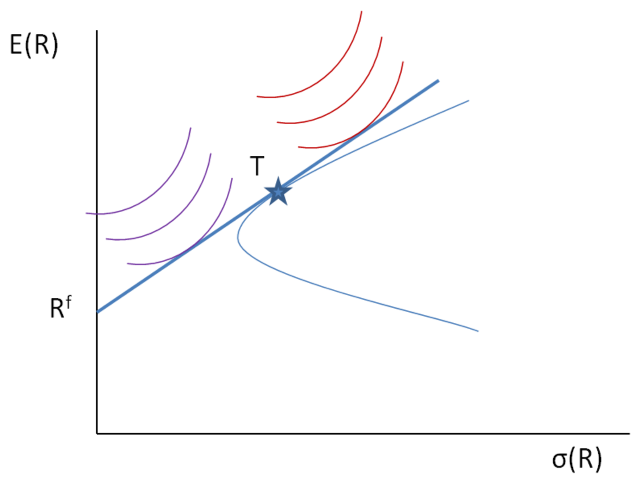
$$\left(E\left[\tilde{R}^p\right] - R^f\right)/\sigma^p$$



- As a Consequence of Our Assumptions...
  - Since all investors have the same expectations, they all want to hold the same tangency portfolio
  - Since the CAPM is an equilibrium model
    - Prices, and therefore expected returns, adjust until supply = demand
    - This is the equilibrium condition for the model: supply = demand for the market portfolio

1.2 Two Fund Separation

- Though investors may have different utility functions and levels of risk-aversion, they all choose the same 2 assets
  - The risk-free asset and the tangency portfolio T.
- Powerful result: We don't need to determine 100 million utility functions
- Two investors with difference levels of risk aversion have different optimal portfolios, but they are composed of the same two assets



1.3 CML

- What does this tell us?
  - We know that all investors want to hold 2 assets in their portfolio
    - The market portfolio, call its return R^m, and the risk-free asset R^f
  - Because of this, all risk-return combinations can be plotted on a straight line that goes through these two assets
    - This line is called the Capital Market Line (CML)
- What do we already know?
  - We already know

    * How to find the efficient portfolio frontier of N risky assets.
    * Given the Risk-Free rate, how to solve for the tangency portfolio or market portfolio
  - Do we know the slope of the CML? Sure we do.
    * It's easy to read this off the graph

$$\text{slope } = \frac{E\left[R^m\right] - R^f}{\sigma_m}$$

    * As well, the y-intercept of the CML is just R^f

- Yay!
  - So we actually already know the equation for the CML

$$E\left[R^p\right] = R^f + \left(\frac{E\left[R^m\right] - R^f}{\sigma_m}\right)\sigma_p$$

    * Where $E\left[R^p\right] =$ The expected return on an efficient portfolio
    * And where $\sigma_p =$ The standard deviation of an efficient portfolio
  - The quantity $E\left[R^m\right] - R^f$ is called the *market risk premium*

- So where does that get us?
  - The CML relates the expected return of any efficient portfolio to the portfolio's "risk" - i.e., standard deviation.
    * But the CML says nothing about risk and return for individual securities
  - To relate risk and return for individual securities, we have to use the Security Market Line, or SML...
    * Reminder of where we're trying to go: *Asset Pricing*

1.4 SML

- Claim: If portfolio M is on the mean-variance frontier, then

$$E\left[R^i\right] = R^f + \left(\frac{E\left[R^m\right] - R^f}{\sigma_m^2}\right)\sigma_{im}$$

  - (The converse is also true.)

- Proof: Say you hold M. Consider making a small change to your portfolio composition.
  - Let's add a fraction ε of your wealth into asset i by selling ε of the risk-less asset.
  - Call this new portfolio p...
  - The (random) return of your new portfolio is

$$\tilde{R}^p = \tilde{R}^m + \varepsilon\left(\tilde{R}^i - R^f\right)$$

- The expected return of your new portfolio is

$$E\left[\tilde{R}^p\right] = E\left[\tilde{R}^m\right] + \varepsilon\left(E\left[\tilde{R}\right]^i - R^f\right)$$

- A Digression...
  - We're going to need to figure out the variance of our new portfolio as well.
  - So as a reminder, here are some useful math facts:
    * If a and b are constants and x and y are random variables...

$$\begin{aligned} var(a) &= 0 \\ var(x+b) &= var(x) \\ var(ax) &= a^2 var(x) \\ var(ax+by) &= a^2 var(x) + b^2 var(y) + 2ab cov(x,y) \end{aligned}$$

- Now, back to the proof...
  - The variance of your new portfolio is

$$\begin{aligned} \sigma_p^2 &= var\left(\tilde{R}^p\right) \\ &= var\left(\tilde{R}^m + \varepsilon\left(\tilde{R}^i - R^f\right)\right) \\ &= \sigma_m^2 + \varepsilon^2\sigma_i^2 + 2\varepsilon cov\left(\tilde{R}^m, \tilde{R}^i\right) \\ &\approx \sigma_m^2 + 2\varepsilon cov\left(\tilde{R}^m, \tilde{R}^i\right) \end{aligned}$$

  - How much did the expected return and variance change from your old portfolio M?

$$E\left[\tilde{R}^p\right] - E\left[\tilde{R}^m\right] = \varepsilon\left(E\left[\tilde{R}^i\right] - R^f\right)$$

and

$$\sigma_p^2 - \sigma_m^2 \approx 2\varepsilon cov\left(\tilde{R}^m, \tilde{R}^i\right)$$

  - What's the intuition here?

  - What about the reward-to-risk ratio?
  - The extra return we get for taking on extra portfolio risk is

$$\begin{aligned} \frac{E\left[\tilde{R}^p\right] - E\left[\tilde{R}^m\right]}{\sigma_p^2 - \sigma_m^2} &= \frac{\varepsilon\left(E\left[\tilde{R}^i\right] - R^f\right)}{2\varepsilon cov\left(\tilde{R}^m, \tilde{R}^i\right)} \\ &= \frac{\left(E\left[\tilde{R}^i\right] - R^f\right)}{2cov\left(\tilde{R}^m, \tilde{R}^i\right)} \\ &= \frac{\left(E\left[\tilde{R}^i\right] - R^f\right)}{2\sigma_{im}} \end{aligned}$$

- Now, we can also think of a portfolio of stocks as simply another asset. What happens if we sell a small fraction ε of the risk-less asset and add a small fraction ε of the market portfolio?

$$\frac{E\left[\tilde{R}^p\right]-E\left[\tilde{R}^m\right]}{\sigma_p^2-\sigma_m^2}\approx\frac{\left(E\left[\tilde{R}^m\right]-R^f\right)}{2\sigma_m^2}$$

- In equilibrium, ratios of extra reward to extra risk should be the same:

$$\frac{\left(E\left[\tilde{R}^i\right]-R^f\right)}{2\sigma_{im}}=\frac{\left(E\left[\tilde{R}^m\right]-R^f\right)}{2\sigma_m^2}$$

- Or, re-arranging

$$E\left[\tilde{R}^i\right]=R^f+\frac{\left(E\left[\tilde{R}^m\right]-R^f\right)}{\sigma_m^2}\sigma_{im}$$

- And this completes our sketch of the proof...

- Think about this...
  - What if I get more return for the extra portfolio risk with the market portfolio than for asset i?
    - I.e., what if

$$\frac{\left(E\left[\tilde{R}^i\right]-R^f\right)}{2\sigma_{im}}<\frac{\left(E\left[\tilde{R}^m\right]-R^f\right)}{2\sigma_m^2}?$$

- What's the logic behind the SML?
  - Investors all hold M (The Market Portfolio), so they are concerned with the variance of M.
  - The contribution of each security to the variance of M is directly proportional to the covariance of the security with the market.
    - Therefore, the relevant measure of risk for stock i is the covariance of its returns with those of the market, σ_{im}.
  - Securities with larger covariances with M contribute more to the risk of the market portfolio, and must provide investors with larger expected returns

$$E\left[\tilde{R}^i\right]=R^f+\frac{\left(E\left[\tilde{R}^m\right]-R^f\right)}{\sigma_m^2}\sigma_{im}$$

    - Therefore, the equation for the SML provides an answer to the question: How much extra return over and above the risk-free rate we should expect, given a security's covariance with the market?

- Using the equation for the SML

$$E\left[\tilde{R}^i\right] = R^f + \frac{\left(E\left[\tilde{R}^m\right] - R^f\right)}{\sigma_m^2}\sigma_{im}$$

  - What happens in the special case where $\sigma_{im} = \sigma_m^2$?

  - What about where $\sigma_{im} = 0$?

1.5 CAPM Derivation from the Mean-Variance Frontier

- The CAPM follows directly from the SML

- Let's rewrite the equation for the SML

$$E\left[\tilde{R}^i\right] = R^f + \frac{\left(E\left[\tilde{R}^m\right] - R^f\right)}{\sigma_m^2}\sigma_{im}$$

- As follows

$$E\left[\tilde{R}^i\right] = R^f + \frac{\sigma_{im}}{\sigma_m^2}\left(E\left[\tilde{R}^m\right] - R^f\right)$$

  - Where we define "Beta" as $\beta_{im} = \frac{\sigma_{im}}{\sigma_m^2}$
  - We can see that betas represents the sensitivity of expected individual asset returns to the expected market premium

- Now, we can write the SML in CAPM form

$$E\left[\tilde{R}^i\right] = R^f + \beta_{im}\left(E\left[\tilde{R}^m\right] - R^f\right)$$

  - We can easily see here how higher expected returns go with higher risk, defined as its risk in a portfolio context
    - An asset's contribution to portfolio risk matters, idiosyncratic risk does not

- Now, we can plot the SML relating expected individual asset returns to their risk, i.e., their covariance with the market.

- Say we have a risk-free rate of 4% per year and the expected return on the market is 12% per year. What does the SML look like?

1.6 The CAPM

- Now we have an *asset pricing model*
  - The CAPM relates expected return to risk - we just need to find the β's
- What do the β's tell you?

- What is the "market portfolio"?

- What about a stock that has a negative beta?

- To see if you understand, think about the slope of the SML.... Does it matter? Why might it change over time?

  - Since the SML is a graph of expected return v. beta

    $$E\left[\tilde{R}^i\right] = R^f + \beta_{im}\left(E\left[\tilde{R}^m\right] - R^f\right),$$

    the slope of the SML is the market risk premium: $\left(E\left[\tilde{R}^m\right] - R^f\right)$, which represents the reward that investors get per unit of systematic risk.

  - So for example, if investors become more risk averse over time, they will require more compensation for bearing a given amount of risk, and the slope of the SML would increase.

1.7 Asset Pricing using the CAPM

- Let's think about how we might price an asset using the CAPM.

- Recall that our central asset pricing equation

  $$p_t = E_t\left[m_{t+1}x_{t+1}\right]$$

  is really just a very general way of mapping future payoffs into today's price.

- This equation is just a generalization of standard discount factor ideas...

  - Recall: Standard Concept of Present Value...

    * Consider an environment where there is no uncertainty.
    * What is the price (i.e., present value) of a payoff tomorrow of x_{t+1} if the interest rate is $R^f = \left(1 + r^f\right)$?

      · Using standard PV ideas,

      $$p_t = \frac{1}{R^f}x_{t+1}$$

    * Here, the discount factor is $\frac{1}{R^f}$

      · See how the payoff tomorrow sells "at a discount"

  - Recall: Present Value with Risky Assets...

    * Now generalize this idea to risky assets, i.e., where there is some uncertainty about the payoffs
    * We could price an asset using

$$p_t^i = \frac{1}{R^i} E_t \left(x_{t+1}^i \right)$$

  - where R^i is an asset-specific risk-adjusted discount factor
    * This is a traditional view of asset pricing that uses R^i as a risk-adjusted rate of return particular to each asset i from a model like the CAPM

1.7.1 A Pricing Example with CAPM

- You expect a stock i to sell for \$100 a year from now and pay a \$5 dividend during the year.
- Suppose the stock's correlation coefficient with the market portfolio is $\rho_{im} = 0.4$ and $\sigma_i = 0.5$.
- You also know that $E[R^m] = 15\%$ and $\sigma_m = 0.3$; and you know that $R^f = 6\%$.
- At what price should the stock sell for today?

1.8 Where we're going...

- What we're going to see in the upcoming lectures is how CAPM fits into our discount factor framework...
  - Factor pricing models, like the CAPM, simply replace the consumption-based expression for marginal utility growth with something that looks like

$$m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)} \approx a + b' f_{t+1}$$

- The questions we're going to be asking are
  - How reasonable are the proxies for marginal utility growth?
  - Can we abide the assumptions - like the ones we listed earlier as necessary for the CAPM - to be comfortable using those models?