Investments TA Session

Duration and Convexity

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Introduction

Questions

Any questions before we start?

Plan for today

- Yield-to-Maturity
- Duration
- Convexity
- Return Attribution

Yield-to-Maturity

Yield-to-Maturity: Definition

A bond's **yield-to-maturity**, or **yield**, is the single rate that when used to discount a bond's cash flows produces the bond's market price.

Solving for Yield-to-Maturity

Consider a 10-year U.S. Treasury Note with price P and coupon C.

Yield-to-maturity y solves the equation

$$P = \sum_{n=1}^{19} \frac{C/2}{(1+y/2)^{2n}} + \frac{100 + C/2}{(1+y/2)^{20}}.$$

In R, we can search for the yield that equates the price function to the market price.

```
price <- function(yield, coupon, maturity) {
  t <- 1:(2 * maturity)
  df <- 1/(1+yield/2)^t
  cf <- c(rep(coupon / 2, 2*maturity - 1), 100 + coupon / 2)
  sum(df * cf)
}
```

Example: A 10-year U.S. Treasury Note

TREASURY NEWS

Department of the Treasury • Bureau of the Fiscal Service

For Immediate Release
February 08, 2017

CONTACT: Treasury Securities Services
202-504-3550

TREASURY AUCTION RESULTS

Term and Type of Security	10-Year Note
CUSIP Number	912828V98
Series	B-2027
Interest Rate	2-1/4%
High Yield [1]	2.333%
Allotted at High	5.23%
Price	99.263516
Accrued Interest per $1,000	None
Median Yield [2]	2.260%
Low Yield [3]	2.201%
Issue Date	February 15, 2017
Maturity Date	February 15, 2027
Original Issue Date	February 15, 2017
Dated Date	February 15, 2017

Example: Caculating T-Note's Price and Yield

For the auction results, we are told the price is 99.263516 and the yield is 2.333%.

```r
options(digits = 8)
price <- function(yield, coupon, maturity) {
  t <- 1:(2 * maturity)
  df <- 1/(1+yield/2)^t
  cf <- c(rep(coupon / 2, 2 * maturity - 1), 100 + coupon / 2)
  sum(df * cf)
}
price(0.02333, 2.25, 10)
```

```
## [1] 99.263516
```

```r
f <- function(yield, coupon, maturity){
 price(yield, coupon, maturity) - 99.263516
}
uniroot(f,c(0,10),2.25, 10, tol=10^-9)$root
```

```
## [1] 0.02333
```

What are Yields used for?

A bond's yield is a convenient way to talk about its price.

- Sometimes bonds are traded on yield, or yield spreads, rather than prices.

Yields can be used for **return attribution**.

- If a bond's yield-to-maturity remains unchanged over a *short period* of time, that bond's realized total return equals it's yield.

Yields can be Misleading

A yield is a useful summary of bond pricing, but you need to know exactly how to calculate the bond's price.

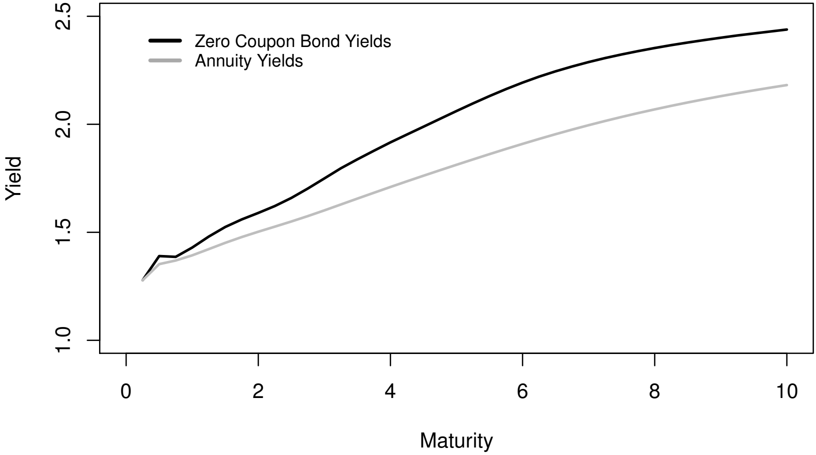
- There are many different yield and day count conventions.

Yield is **not** a measure of relative value or realized return to maturity.

- If one bond has a higher yield than another, it is not a better value.
- A bond purchased at a particular yield and held to maturity will not necessarily earn that yield.

Yields depend on the Timing of Cash Flows

With an upward-sloping yield curve, annuities have lower yields than zero-coupon bonds, but this does not mean that zero coupon bonds are cheap to annuities.



Yields and Returns

Between coupon payments, if a bond's yield remains unchanged, then the total return of the bond over that period equals its yield-to-maturity.

$$
\begin{aligned}
P_0 &= \frac{C/2}{1+y/2} + \frac{C/2}{(1+y/2)^2} + \cdots + \frac{100 + C/2}{(1+y/2)^{2T}} \\
P_{1/2} &= \frac{C}{2} + \frac{C/2}{1+y/2} + \cdots + \frac{100 + C/2}{(1+y/2)^{2T-1}} \\
&\Rightarrow P_{1/2} = (1+y/2)\, P_0 \\
&\Rightarrow y = 2\left(\frac{P_{1/2}}{P_0} - 1\right)
\end{aligned}
$$

Calculating Yields between Coupon Payments

Calculating Yields between Coupon Payments

The **Street Convention** and the **Treasury Convention** are two methods to calculate the yield of a bond between coupon payment dates.

- Yields using the Treasury Convention are shown on the Treasury auction results.
- The Street Convention is generally used for pricing and it is shown on Bloomberg.

Calculating Yields between Coupon Payments (Con't)

First, we need to determine the fraction w of time between the settlement date and the next coupon date.

Using the appropriate day count convention, this is determined as follows:

$$w = \frac{\text{Days between settlement date and next coupon payment date}}{\text{Days in the coupon period}}$$

Consider a bond with N semi-annual coupon payments remaining.

Using the **Street Convention**, the yield solves

$$P_{\text{Full}} = \sum_{i=1}^{N-1} \frac{C/2}{(1+y/2)^{i+w-1}} + \frac{100}{(1+y/2)^{N+w-1}}$$

Calculating Yields between Coupon Payments (Con't)

The **Street Convention** assumes that the first discount factor is

$$D(w) = \frac{1}{\left(1 + \frac{y}{2}\right)^{w}}.$$

The **Treasury Convention** assumes simple interest in the first period, so that the first discount factor is

$$D(w) = \frac{1}{\left(1 + \frac{y}{2}w\right)}.$$

Therefore, in the **Treasury Convention** the yield solves

$$P_{\text{Full}} = \sum_{i=0}^{N-1} \frac{C/2}{\left(1 + \frac{y}{2}w\right)\left(1 + \frac{y}{2}\right)^{i-1}} + \frac{100}{\left(1 + \frac{y}{2}w\right)\left(1 + \frac{y}{2}\right)^{N-1}}$$

Treasury Convention Example

TREASURY NEWS

Department of the Treasury • Bureau of the Fiscal Service

For Immediate Release
October 11, 2017

CONTACT: Treasury Securities Services
202-504-3550

TREASURY AUCTION RESULTS

Term and Type of Security	9-Year 10-Month Note
CUSIP Number	9128282R0
Series	E-2027
Interest Rate	2-1/4%
High Yield [1]	2.346%
Allotted at High	17.96%
Price	99.158502
Accrued Interest per $1,000	$3.79076
Median Yield [2]	2.300%
Low Yield [3]	2.231%
Issue Date	October 16, 2017
Maturity Date	August 15, 2027
Original Issue Date	August 15, 2017
Dated Date	August 15, 2017

Treasury Convention Example in R

```r
priceTC <- function(yld, cpn, matdt, setdt) {
  pmtdts <- seq(from = matdt, to = setdt, by = "-6 months") %>% sort
  n <- pmtdts %>% length
  ncd <- pmtdts[1]
  lcd <- seq(from = matdt, len = n + 1, by = "-6 months")[n + 1]
  cf <- c(rep(cpn / 2, n-1), cpn / 2 + 100)
  basis <- (ncd - lcd) %>% as.numeric
  accrual <- (setdt - lcd) %>% as.numeric
  ncdays <- basis - accrual
  df <- c(1/(1 +yld/2*ncdays / basis), rep(1/(1 + yld/2), n-1))
  df <- cumprod(df)
  px <- sum(df * cf)
  ai <- accrual/basis * cpn / 2
  px - ai
}
priceTC(0.02346, 2.25, as.Date("2027-08-15"), as.Date("2017-10-16"))
```

```
## [1] 99.158502
```

Street Convention Example in R

```r
priceSC <- function(yld, cpn, matdt, setdt) {
  pmtdts <- seq(from = matdt, to = setdt, by = "-6 months") %>% sort
  n <- pmtdts %>% length
  ncd <- pmtdts[1]
  lcd <- seq(from = matdt, len = n + 1, by = "-6 months")[n + 1]
  cf <- c(rep(cpn / 2, n-1), cpn / 2 + 100)
  basis <- (ncd - lcd) %>% as.numeric
  accrual <- (setdt - lcd) %>% as.numeric
  ncdays <- basis - accrual
  df <- c(1/(1 +yld/2)^(ncdays / basis), rep(1/(1 + yld/2), n-1))
  df <- cumprod(df)
  px <- sum(df * cf)
  ai <- accrual/basis * cpn / 2
  px - ai
}
priceSC(0.02346, 2.25, as.Date("2027-08-15"), as.Date("2017-10-16"))
```

```
## [1] 99.160012
```

Street Convention on Bloomberg

```r
# 98 + 20.5 / 32 = 98.640625
priceSC(0.02406444, 2.25, as.Date("2027-08-15"), as.Date("2017-10-30"))
```

```
## [1] 98.640625
```

Duration

Duration

Duration is a *linear* measure of risk of a fixed-income investment to changes in yields.

There are many duration measures. The most important are:

- **Macaulay Duration** (D)
  - the time-weighted present value of cash flows
  - useful for intuition
- **Modified Duration** (D^*)
  - the percentage change in price for a unit change in yields
  - This is what we actually use in calculations

⚠ Both Macaulay and Modified duration are called "duration."

- This can create a lot of confusion!

Macaulay Duration

Introduced by Frederic Macaulay in 1938, "duration" was intended to be a better measure of the "length" of liabilities for insurance companies that had previously relied on maturity as their primary measure.

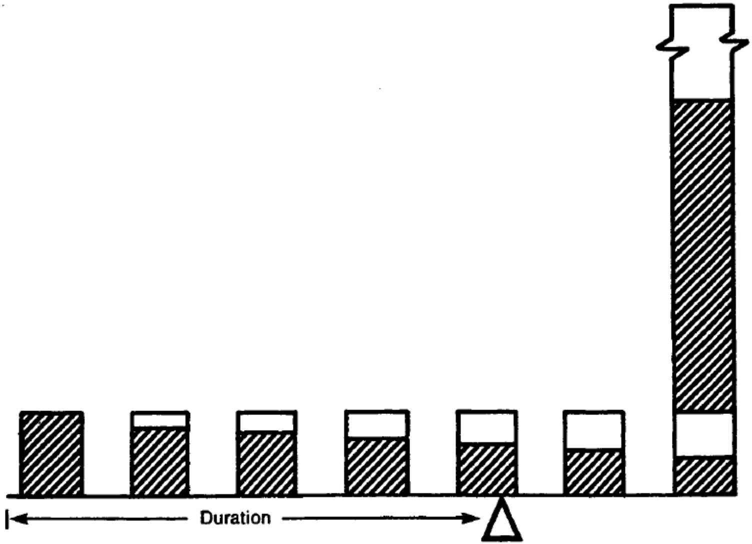
Macaulay duration is calculated as the **present-value-weighted time to receipt of cash flows**, which is why we tend to measure it in years.

For a bond with N payments at times $t_1, t_2, \ldots, t_N$ (measured in years), the Macaulay duration D is given by

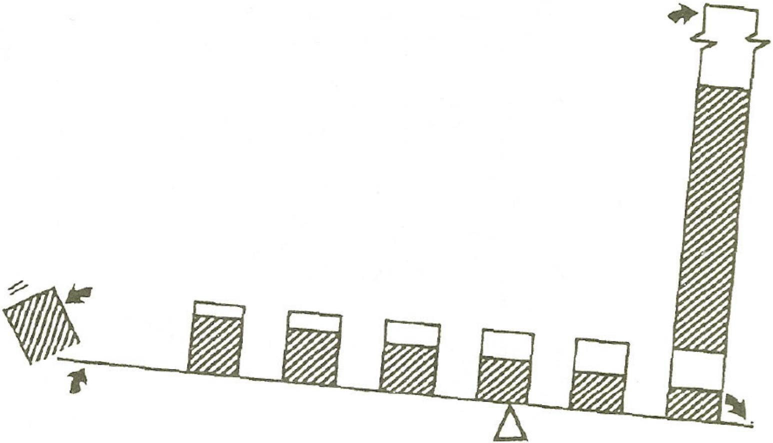
$$D = \frac{\sum_{i=1}^{N} t_i PV_i}{\sum_{i=1}^{N} PV_i}$$

where PV_i is the present value of the ith cash payment.

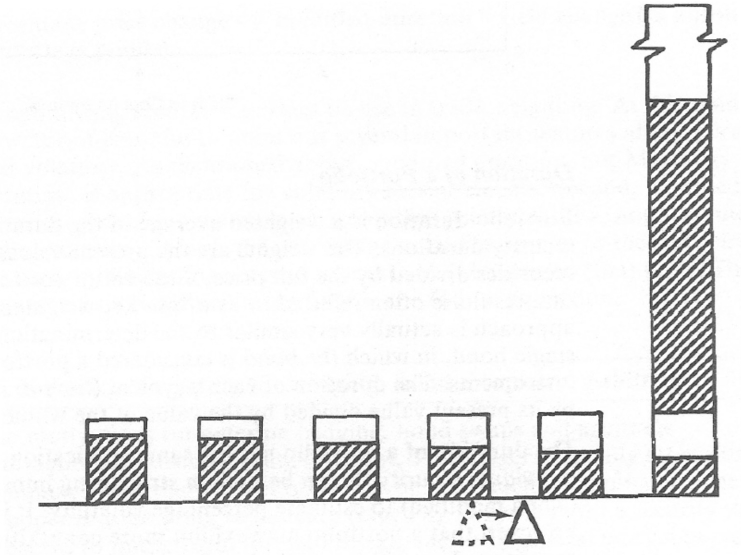
Macaulay Duration: Intuition



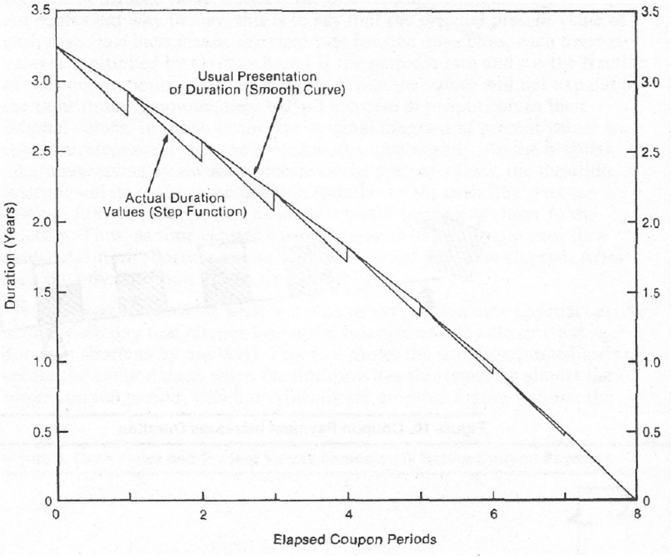
Macaulay Duration: Effect of a Coupon Payment



Receiving Coupon Payments Increases Duration



Duration over Time with Coupon Payments



Modified Duration

Modified Duration (D^*) is a measure of price sensitivity and is defined as

$$D^* = -\frac{1}{P}\frac{\partial P}{\partial y}$$

It turns out that Modified Duration and Macaulay Duration are related by

$$D^* = \frac{D}{1 + y_k/k}$$

where k is the compounding frequency ($k = 2$ for a semiannual yield).

- Therefore, Modified Duration is Macaulay Duration *modified* by the factor $1 + y_k/k$.
- If yields are continuously compounded, Modified Duration equals Macaulay Duration.

Modified Duration: k-Period Compounding

The value V of a bond is given by

$$V = \sum_{i=1}^{N} \frac{CF_i}{(1 + y_k/k)^{kt_i}}$$

Therefore,

$$\begin{aligned}
\frac{\partial V}{\partial y_k} &= -\frac{1}{1 + y_k/k} \sum_{i=1}^{N} t_i \cdot \frac{CF_i}{(1 + y_k/k)^{kt_i}} \\
&= -\frac{1}{1 + y_k/k} \sum_{i=1}^{N} t_i PV_i \\
&= -\frac{1}{1 + y_k/k} D \cdot V
\end{aligned}$$

Hence,

$$D^* = -\frac{1}{V} \frac{\partial V}{\partial y} = \frac{D}{1 + y_k/k}.$$

Modified Duration: Continuous Compounding

The value V of a bond is given by

$$V = \sum_{i=1}^{N} CF_i \cdot e^{-yt_i}$$

Therefore,

$$\begin{aligned}\frac{\partial V}{\partial y_k} &= -\sum_{i=1}^{N} t_i \cdot CF_i \cdot e^{-yt_i} \\ &= -\sum_{i=1}^{N} t_i \cdot PV_i \\ &= -D \cdot V\end{aligned}$$

Hence,

$$D^* = -\frac{1}{V}\frac{\partial V}{\partial y} = D.$$

Portfolio Duration

The duration of a portfolio D_P^* is the market-value weighted average of duration of the individual securities.

$$
\begin{aligned}
V_P &= \sum_{i=1}^{N} V_i \\
\Rightarrow \frac{\partial V_P}{\partial y} &= \sum_{i=1}^{N} \frac{\partial V_i}{\partial y} \\
\Rightarrow D_P^* &= -\frac{1}{V_P}\frac{\partial V_P}{\partial y} = -\frac{1}{V_P}\sum_{i=1}^{N}\frac{\partial V_i}{\partial y} \\
&= \sum_{i=1}^{N}\frac{V_i}{V_P}\left(-\frac{1}{V_i}\frac{\partial V_i}{\partial y}\right) \\
&= \sum_{i=1}^{N}\frac{V_i}{V_P} D_i^*
\end{aligned}
$$

Convexity

Convexity: Definition

Convexity (C) measures how interest rate sensitivity changes with interest rates.

Convexity is defined as

$$C = -\frac{1}{V}\frac{\partial^2 V}{\partial y^2}.$$

⚠ You need to be careful about the units convexity is quoted in.

- On Bloomberg and elsewhere, convexity is defined as

$$C = -\frac{1}{100}\frac{1}{V}\frac{\partial^2 V}{\partial y^2}.$$

Convexity

Bond prices are a non-linear function of yields.

- Duration is a *linear* measure of interest rate risk.
- Duration is a good approximation of the change in price of a bond for small changes in yields.
- However, the error grows in with the size of the yield change.

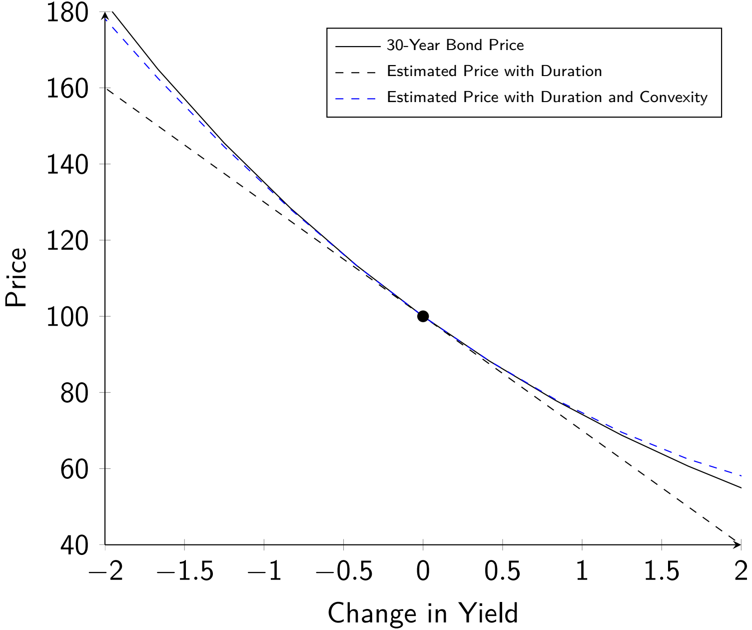
We can improve the estimate by adding a second-order term, known as **convexity**.

- Convexity measures how much the bond outperforms the linear estimate.

A lot of fixed income investing involves searching for cheap convexity.

- i.e. bonds that have a favorable price-yield relationship.

30-Year Zero Coupon Bond Prices



Example: Zero Coupon Bond (Semiannual Compounding)

$$
\begin{aligned}
P &= \frac{100}{(1 + y/2)^{2T}} \\
D &= T \\
D^* &= -\frac{1}{P}\frac{\partial P}{\partial y} = \frac{T}{1 + y/2} \\
C &= \frac{1}{P}\frac{\partial^2 P}{\partial y^2} = \frac{T(T+1)}{(1 + y/2)^2}
\end{aligned}
$$

Example: Zero Coupon Bond (Continuous Compounding)

$$
\begin{aligned}
P &= 100e^{-yT} \\
D &= T \\
D^* &= -\frac{1}{P}\frac{\partial P}{\partial y} = T \\
C &= \frac{1}{P}\frac{\partial^2 P}{\partial y^2} = T^2
\end{aligned}
$$

Understanding Price Changes

A second-order Taylor Approximation implies that

$$
\begin{aligned}
P(y+\Delta y) &\approx P(y) + \frac{dP}{dy}\Delta y + \frac{1}{2}\frac{d^2P}{dy}(\Delta y)^2 \\
&= P(y) - D^*P\Delta y + \frac{1}{2}CP(\Delta y)^2
\end{aligned}
$$

Over time (for zcb, continuous compounding):

$$
\begin{aligned}
\text{Because } \frac{dP}{dt} = \frac{d}{dt}e^{-y(T-t)} &= ye^{-y(T-t)} = yP, \\
\Delta P \approx P(y) + \frac{dP}{dy}\Delta y + \frac{1}{2}\frac{d^2P}{dy}(\Delta y)^2 + \frac{dP}{dt}\Delta t \\
= P(y) - D^*P\Delta y + \frac{1}{2}CP(\Delta y)^2 + yP\Delta t
\end{aligned}
$$

Example

Return attribution is method that attributes returns to different factors or risks. There are many ways to go about it and it works best over short time horizons. Duration and convexity are used to attribute instantaneous changes in yields. The time return (carry and roll down) act over time.

Example: The 10y yield today is 2.37 and last year it was 1.83. The 9y yield is 2.32 today and last year it was 1.76. We bought a 10y bond last year, and we want to decompose the return into various sources of risk. For simplicity, assume it is a ZCB. We really should be doing this analysis over shorter time horizons.

In any case, we now have a 9y bond. If yields did not change, the yield of the 9y bond would be 1.76 today. Therefore, the return from duration is $-D \times P \times (0.0232 - 0.0176)/P$ and the return from convexity is $C/2 \times P \times (0.0232 - 0.0176)^2/P$. The time return is

$$\frac{P_9(0.0176)}{P_{10}(0.0183)} - 1 = \frac{e^{-0.0176 \times 9}}{e^{-0.0183 \times 10}} - 1$$