Linear programming.

Linear programming is general problem of optimizing a linear function of several variables subject to a number of constraints that are linear in these variables and a subset of which restrict the variables to be non-negative.

A mathematical program (1.1) is linear if $f(x_1, x_2, \dots, x_n)$ and each $g_i(x_1, x_2, \dots, x_n)$ $(i = 1,2, \dots, m)$ are linear in each of their arguments – that is, if

$$\boldsymbol{f(x_1, x_2, \dots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n} \qquad (1.2)$$

and

$$\boldsymbol{g_i(x_1, x_2, \dots, x_n) = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n} \qquad (1.3)$$

Where c_j and $a_{ij}(i = 1,2, \dots, m; j = 1,2, \dots, n)$ are known constraints.

Any other mathematical program is nonlinear.

INTEGER PROGRAMS

An integer program is a linear program with the additional restriction that the input variables be integers. It is not necessary that the coefficients in (1:2) and (1.3), and the constants in (1.1), also be integers, but this will very often be the case.

QUADRATIC PROGRAMS

A quadratic program is a mathematical program in which each constraint is linear—that is, each constraint function has the form (1.3)—but the objective is of the form.

$$\boldsymbol{f(x_i, x_2, \dots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}x_ix_j + \sum_{i=1}^{n} d_ix_i} \qquad \textbf{(1.4)}$$

Where c_{ij} and d_i are known constants.

The program given in Example 1.1 is quadratic. Both constraints are linear, and the objective has the form (1.4), with n = 2 (two variables),$\boldsymbol{c_{11} = 1}$,

$\boldsymbol{c_{12} = c_{21} = 0, c_{22} = 1, d_1 = d_2 = 0.}$

SLACK VARIABLES AND SURPLUS VARIABLES

A linear constraint of the form $\sum a_{ij}x_j \le b_i$ can be converted into an equality by adding a new, nonnegative variable to the left-hand side of the inequality. Such a variable is numerically equal to the difference between the right- and left-hand sides of the inequality and is known as a slack variable. It represents the waste involved in that phase of the system modeled by the constraint.

GENERATING AN INITIAL FEASIBLE SOLUTION

After all linear constraints (with nonnegative right-hand sides) have been transformed into equalities by introducing slack and surplus variables where necessary, add a new variable, called an artificial variable, to the left-hand side of each constraint equation that does not contain a slack variable. Each constraint equation will then contain either one slack variable or one artificial variable. A nonnegative initial solution to this new set of constraints is obtained by setting each slack variable and each artificial variable equal to the right-hand side of the equation in which it appears and setting all other variables, including the surplus variables, equal to zero.

PENALTY COSTS

The introduction of slack and surplus variables alters neither the nature of the constraints nor the objective. Accordingly, such variables are incorporated into the objective function with zero coefficients. Artificial variables, however, do change the nature of the constraints. Since they are added to only one side of an equality, the new system is equivalent to the old system of constraints if and only if the artificial variables are zero. To guarantee such assignments in the optimal solution (in contrast to the initial solution), artificial variables are incorporated into the objective function with very large positive coefficients in a minimization program or very large negative coefficients in a maximization program. These coefficients, denoted by either M or —M, where M is understood to be a large positive number, represent the (severe) penalty incurred in making a unit assignment to the artificial variables.

STANDARD FORM

A linear program is in standard form if the constraints are all modeled as equalities and if one feasible solution is known. In matrix notation, standard form is

$$\text{optimize}: z = C^T X$$

$$\text{subject to: } AX = B$$

$$\text{with: } x \geq 0$$

Where X is the column vector of unknowns, including all slack, surplus, and artificial variables; C^T is the row vector of the corresponding costs; A is the coefficient matrix of the constraint equations; and B is the column vector of the right-hand sides of the constraint equations.

For example:

Put the following program in standard form:

$$\text{maximize: } Z = 5x_1 + 2x_2$$

$$\text{subject to: } 6x_1 + x_2 \geq 0$$

$$4x_1 + 3x_2 \geq 12$$

$$x_1 + 2x_2 \geq 4$$

with: x_1 and x_2 nonnegative

Subtracting surplus variables x_3, x_4, and x_5 respectively, from the left-hand sides of the constraints, and including each new variable with a zero cost coefficient in the objective, we obtain

$$\text{maximize: } Z = 5x_1 + 2x_2 + 0x_3 + 0x_4 + 0x_5$$

$$\begin{aligned} \text{subject to: } 6x_1 + x_2 - x_3 \qquad\qquad &= 6 \\ 4x_1 + 3x_2 \qquad - x_4 \qquad &= 12 \\ x_1 + 2x_2 \qquad\qquad - x_5 &= 4 \end{aligned}$$

with: all variables nonnegative

Since no constraint equation contains a slack variable, we next add artificial variables x_6, x_7, and x_8, respectively, to the left-hand sides of the equations. We also include these variables with very large negative cost coefficients in the objective. The program becomes

$$\text{maximize: } Z = 5x_1 + 2x_2 + 0x_3 + 0x_4 + 0x_5 - Mx_6 - Mx_7 - Mx_8$$

$$\begin{aligned} \text{subject to: } 6x_1 + x_2 - x_3 \qquad\qquad + x_6 \qquad\qquad &= 6 \\ 4x_1 + 3x_2 \qquad - x_4 \qquad\qquad + x_7 \qquad &= 12 \\ x_1 + 2x_2 \qquad\qquad - x_5 \qquad\qquad + x_8 &= 4 \end{aligned}$$

with: all variables nonnegative

This program is in standard form, with an initial feasible solution

$$x_6 = 6,\ x_7 = 12,\ x_8 = 4,\ x_1 = x_2 = x_3 = x_4 = x_5 = 0$$