Linear Programming:

Duality.

1. Introduction

For every Linear Programming Problem there exists a Dual that expresses the problem with a resource orientation. The original problem is called the **Primal Problem**. The solution of a Linear Programming Problem can be read from the final simplex tableau of either **Primal** or its **Dual** for all the variables.

The importance of duality in linear programming is twofold. Firstly, it is often possible to solve the dual linear problem in place of the original linear problem (Primal), to reap advantage of computational efficiency. Secondly, it helps in understanding the shadow price interpretation of the optimal simplex solution.

2. Formulation of a Dual Problem

- The **objective** in the Dual is the **opposite of the objective** in the Primal. If the Primal is a Maximization Problem then the Dual will be a Minimization Problem and vice versa.
- The **coefficients of the decision variables** in the Objective function of the Primal become the quantities on the **RHS of the resource constraints** of the Dual and the vice versa
- The **Technological Coefficients** in the constraints of the dual problem are **the transpose of the technological coefficients** of the constraints in the Primal.
- **'≥'constraints** in the Primal correspond to **'≤' constraints** in the Dual and vice versa.
- **Number of decision variables** in the Primal is equal to the **number of constraints** in the Dual and vice versa.

2.1. Dual of a Maximization Problem:

Let the primal problem be:

Primal Problem:

$$\text{Maximize:} \boldsymbol{Z} = \boldsymbol{c_1 x_1 + c_2 x_2 + \cdots + c_n x_n}$$

Subject to the constraints:

$$\boldsymbol{a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1}$$

$$\boldsymbol{a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}a_n \le b_2}$$

$$\vdots$$

$$\boldsymbol{a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m}$$

$$\boldsymbol{x_1, x_2, \dots, x_n \ge 0}$$

Where, $x_j(x_1, x_2, x_3, \dots, x_n)$ are decision variables,

$c_j(c_1, c_2, c_3, \dots, c_n)$ are the cost or profit coefficients,
$a_{ij}(i = 0 \ to \ n, j = 1 \ to \ m)$ are the exchange coefficients, and

$b_i(b_1, b_2, b_3, \dots, b_m)$ are resource values.

The associated **dual problem** can be written as:

Dual Problem

Minimize: $Z = b_1 w_1 + b_2 w_2 + \cdots + b_m w_m$

Subject to the constraints:

$$a_{11}w_1 + a_{12}w_2 + \cdots + a_{m1}w_m \le c_1$$

$$a_{21}w_1 + a_{22}w_2 + \cdots + a_{m2}w_m \le c_2$$

$$\vdots$$

$$a_{1n}w_1 + a_{2n}w_2 + \cdots + a_{mn}w_m \le c_n$$

$$w_1, w_2, \dots, w_n \ge 0$$

Where, $w_i(w_1, w_2, w_3, \dots, w_m)$ are the shadow prices of resources,

$c_j(c_1, c_2, c_3, \dots, c_n)$ are the cost or profit coefficients,
$a_{ij}(i = 0 \ to \ n, j = 1 \ to \ m)$ are the exchange coefficients, and

$b_i(b_1, b_2, b_3, \dots, b_m)$ are resource values.

2.2. Dual of Minimization Problem:

Let the primal problem be:

Primal Problem:

Minimize: $Z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$

Subject to the constraints:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}a_n \le b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m$$

$$x_1, x_2, \dots, x_n \ge 0$$

Where, $x_j(x_1, x_2, x_3, \dots, x_n)$ are decision variables,

$\boldsymbol{c_j(c_1, c_2, c_3, \dots, c_n)}$ are the cost or profit coefficients,

$\boldsymbol{a_{ij}(i = 0\ to\ n, j = 1\ to\ m)}$ are the exchange coefficients, and

$\boldsymbol{b_i(b_1, b_2, b_3, \dots, b_m)}$ are resource values.

The associated dual problem can be written as:

Dual Problem:

Maximize: $\boldsymbol{Z = b_1 w_1 + b_2 w_2 + \cdots + b_m w_m}$

Subject to the constraints:

$$\boldsymbol{a_{11}w_1 + a_{12}w_2 + \cdots + a_{m1}w_m \le c_1}$$
$$\boldsymbol{a_{21}w_1 + a_{22}w_2 + \cdots + a_{m2}w_m \le c_2}$$
$$\vdots$$
$$\boldsymbol{a_{1n}w_1 + a_{2n}w_2 + \cdots + a_{mn}w_m \le c_n}$$
$$\boldsymbol{w_1, w_2, \dots, w_n \ge 0}$$

Where, $\boldsymbol{x_j(x_1, x_2, x_3, \dots, x_n)}$ are decision variables,

$\boldsymbol{c_j(c_1, c_2, c_3, \dots, c_n)}$ are the cost or profit coefficients,

$\boldsymbol{a_{ij}(i = 0\ to\ n, j = 1\ to\ m)}$ are the exchange coefficients,

$\boldsymbol{b_i(b_1, b_2, b_3, \dots, b_m)}$ are resource values.

- Symmetric Form of Duality

Primal Problem

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{Ax} \ge \boldsymbol{b} \\ & \boldsymbol{x} \ge \boldsymbol{0} \end{aligned}$$

Dual Problem

$$\begin{aligned} \max \quad & \boldsymbol{b}^T \boldsymbol{\lambda} \\ \text{s.t.} \quad & \boldsymbol{A}^T \boldsymbol{\lambda} \le \boldsymbol{c} \\ & \boldsymbol{\lambda} \ge \boldsymbol{0} \end{aligned}$$

- Asymmetric form of Duality

Primal Problem

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{Ax} = \boldsymbol{b} \\ & \boldsymbol{x} \ge \boldsymbol{0} \end{aligned}$$

Dual Problem

$$\begin{aligned} \max \quad & \boldsymbol{b}^T \boldsymbol{\mu} \\ \text{s.t.} \quad & \boldsymbol{A}^T \boldsymbol{\mu} \le \boldsymbol{c} \end{aligned}$$

Primal Problem

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

Dual Problem

$$\begin{aligned} \max \quad & b^T \lambda \\ \text{s.t.} \quad & A^T \lambda \leq c \\ & \lambda \geq 0 \end{aligned}$$

For linear programs, the dual of the dual is the primal problem.

Primal Problem

$$\begin{aligned} -\min \quad & -b^T \lambda \\ \text{s.t.} \quad & -A^T \lambda \geq -c \\ & \lambda \geq 0 \end{aligned}$$

Dual Problem

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

Consider the following primal and dual problems:

Primal Problem (P)

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Dual Problem (D)

$$\begin{aligned} \max \quad & b^T \mu \\ \text{s.t.} \quad & A^T \mu \leq c \end{aligned}$$

Weak Duality Theorem

If x and μ are primal and dual feasible respectively, then $c^T x \geq b^T \mu$.

Strong Duality Theorem

If either of the problems **P** or **D** has a finite optimal solution, so does the other, and the corresponding optimal objective function values are equal. If any of these two problems is unbounded, the other problem has no feasible solution.

Mathematical Explanation:

If x_1 has unrestricted sign, then we express it as:

$$x_1 = x_2 - x_3 \text{ where, } x_2, x_3 \geq 0$$

It implies:

- if $x_2 > x_3$ then x_1 will be positive
- if $x_2 < x_3$ then x_1 will be negative and
- if $x_2 = x_3$ then x_1 will be zero

Example 1

Write the Dual of the following Linear Programming Problem:

Maximize $Z = 15x_1 + 29x_2 + 11x_3 + 37x_4$

Subject to constraints:

$$10x_1 + 15x_2 + \qquad 14x_4 \leq 250$$

$$22x_1 + 20x_2 + 20x_3 \qquad \geq 540$$

$$15x_1 + 13x_2 + 12x_3 + 13x_4 = 600$$

$x_2, x_3, x_4 \geq 0$ and $x_1: unrestricted$

Solution:

Step 1: Express the Linear Programming Problem in Standard Form.

Maximize $Z = 15x_1 + 29x_2 + 11x_3 + 37x_4$

Subject to constraints:

$$10x_1 + 15x_2 + \qquad 14x_4 \leq 250 \text{ ...(1)}$$

$$22x_1 + 20x_2 + 20x_3 \qquad \geq 540 \text{ ...(2)}$$

$$15x_1 + 13x_2 + 12x_3 + 13x_4 = 600 \text{ ...(3)}$$

$x_2, x_3, x_4 \geq 0$ and $x_1: unrestricted$

a. Convert (2) from ≥ to ≤:

$$-22x_1 + 20x_2 - 20x_3 \qquad \leq -540$$

b. Convert (3) from = into two ≤ constraints:

$$15x_1 + 13x_2 + 12x_3 + 13x_4 \leq 600$$

$$15x_1 + 13x_2 + 12x_3 + 13x_4 \geq 600$$

$$-15x_1 - 13x_2 - 12x_3 - 13x_4 \leq -600$$

(c) Convert $x_1: unrestricted$ into a non-negativity restriction:

Substitute $x_1 = x_5 - x_6$ where $x_5, x_6 \geq 0$

Maximize Z = $15(x_5 - x_6) + 29x_2 + 11x_3 + 37x_4$

Subject to the constraints:

$$10x_5 - 10x_6 + 15x_2 + \quad 14x_4 \leq 250$$
$$-22(x_5 - x_6) + 20x_2 - 20x_3 \quad \leq -540$$
$$15(x_5 - x_6) + 13x_2 + 12x_3 + 14x_4 \leq 600$$
$$-15(x_5 - x_6) - 13x_2 - 12x_3 - 13x_4 \leq -600$$
$$x_5, x_6, x_2, x_3, x_4 \geq 0$$

Primal in Standard Form:

Maximize Z = $15x_5 - 15x_6 + 29x_2 + 11x_3 + 37x_4$

Subject to the constraints:

$$10x_5 - 10x_6 + 15x_2 + 0x_3 + 14x_4 \leq 250 \ldots (1)$$
$$-22x_5 + 22x_6 + 20x_2 - 20x_3 + 0x_4 \leq -540 \ldots (2)$$
$$15x_5 - 15x_6 + 13x_2 + 12x_3 + 13x_4 \leq 600 \ldots (3)$$
$$-15x_5 + 15x_6 - 13x_2 - 12x_3 - 13x_4 \leq -600 \ldots (4)$$
$$x_5, x_6, x_2, x_3, x_4 \geq 0$$

Step 2: Write the Dual Problem.

(a) Let w_1, w_2, w_3, w_4 is the worth the resources represents by the constraints (1), (2), (3) and (4) respectively

Minimize Z=$250w_1 - 540w_2 + 600w_3 - 600w_4$

Subject to the constraints:

$$10w_1 - 22w_2 + 15w_3 - 15w_4 \geq 15$$
$$-10w_1 + 22w_2 - 15w_3 + 15w_4 \geq -15$$
$$15w_1 + 20w_2 + 13w_3 - 13w_4 \geq 29$$
$$0w_1 - 20w_2 + 12w_3 - 12w_4 \geq 11$$
$$14w_1 + 0w_2 + 13w_3 - 13w_4 \geq 37$$
$$w_1, w_2, w_3, w_4 \geq 0$$

(b) Substitute $w_5 = (w_3 - w_4)$ and combine $\geq$ and $\leq$ into an equation

The Dual can be expressed as:

Minimize Z=$250w_1 - 540w_2 + 600w_3$

Subject to the constraints:

$$10w_1 - 22w_2 + 15w_5 = 15$$

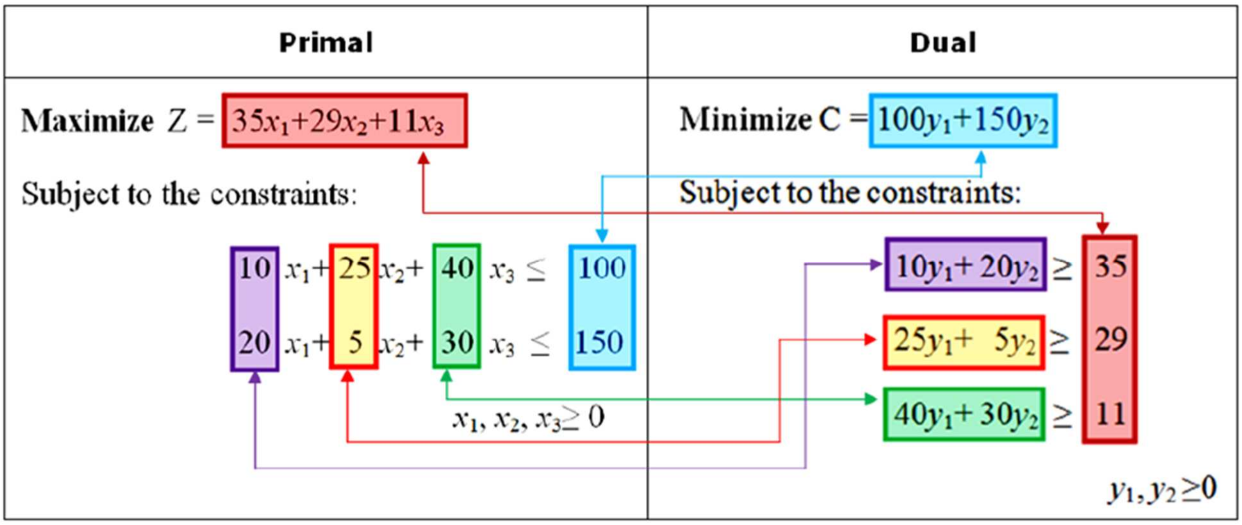
$$15w_1 + 20w_2 + 13w_5 \geq 29$$

$$-20w_2 + 12w_5 \geq 11$$

$$14w_1 \qquad + 13w_5 \geq 37$$

$$w_1, w_2 \geq 0 \; and \; y_5 : unrestricted \; sign$$

Example 2



Example 3

Determine the symmetric dual of the linear program

Primal Problem:	Solution: **Dual Problem** **Let y_1,y_2, y_3 be the dual variables**
Minimize: Z = $3x_1$ - $5x_2$ + x_3 **Subject to the constraints:** x_1 - $2x_3 \geq 4$ $2x_1 - x_2 + x_3 \geq 2$ $x_1, x_2, x_3 \geq 0$	**Maximize:** Z = $4y_1$ +$2y_2$ Subject to the constraints: $y_1 + 2y_2 \leq 3$ $- 5y_2 \leq -5$ $-2y_1 + y_2 \leq 1$ $y_1, y_2 \geq 0$

Example 3

Primal Problem:

Maximize: Z=$x_1 + 3x_2 - 2x_3$

Subject to the constraints:

$$4x_1 + 8x_2 + 6x_3 = 25$$

$$7x_1 + 5x_2 + 9x_3 = 30$$

$$x_1, x_2, x_3 \geq 0$$

Solution:

Dual Problem:

Let y_1, y_2, y_3 be the dual variables

Minimize: Z=$25y_1 + 30y_2$

Subject to the constraints:

$$4y_1 + 7y_2 \geq 1$$

$$8y_1 + 5y_2 \geq 2$$

$$6y_1 + 6y_2 \geq -2$$

$$y_1, y_2 \geq 0$$

Example 4

Determine the symmetric dual of the program.

Primal Problem	Dual Problem
Maximize: Z = $2x_1 + x_2$ Subject to the constraints: $x_1 + 5x_2 \leq 10$ $x_1 + 3x_2 \leq 6$ $2x_1 + 2x_2 \leq 8$ $x_1, x_2 \geq 0$	Minimize: Z = $10y_1 + 6y_2 + 8y_3$ Subject to the constraints: $y_1 + y_2 + 2y_3 \geq 2$ $5y_1 + 3y_2 + 2y_3 \geq 1$ $y_1, y_2, y_3 \geq 0$

Show that both the primal and dual programs in Exercise 4 have the same optimal value for z, and that the solution of each is imbedded in the final simplex tableau of the other. Introducing slack variables x_3, x_4, and xs, respectively, in the constraint inequalities of program (1) of Exercise 4, and then applying the simplex method to the resulting program, we generate sequentially Tableaux 1 and 2.

		x_1	x_2	x_3	x_4	x_5	
		2	1	0	0	0	
x_3	0	1	5	1	0	0	10
x_4	0	1	3	0	1	0	6
x_5	0	2*	2	0	0	1	8
$(z_j - c_j)$:		−2	−1	0	0	0	0

Tableau 1

			slack variables			
	x_1	x_2	x_3	x_4	x_5	
x_3	0	4	1	0	−1/2	6
x_4	0	2	0	1	−1/2	2
x_1	1	1	0	0	1/2	4
	0	1	0	0	1	8
			solution to the dual			

Tableau 2

The solution to the primal is obtained from Tableau 2 as x_1= 4, x_2 = 0, with z* = 8. The solution to the dual program is found in the last row of this tableau, in those columns associated with the slack variables for the primal. Here, y_1 = 0, y_2 = 0, and y_3 = 1. We can solve the dual directly by introducing surplus variables y_4 and y_5, and artificial variables w_6 and w_7, to program (2) of Exercise 4, and then applying the two-phase method, which generates Tableaux 1,…, 4.

		w_1	w_2	w_3	w_4	w_5	w_6	w_7	
		10	6	8	0	0	M	M	
w_6	M	1	1	2	−1	0	1	0	2
w_7	M	5*	3	2	0	−1	0	1	1
$(c_j - z_j)$:		10	6	8	0	0	0	0	0
		−6	−4	−4	1	1	0	0	−3

Tableau 1′

				surplus variables		
	w_1	w_2	w_3	w_4	w_5	
w_5	−4	−5	0	−1	1	1
w_3	1/2	1/2	1	−1/2	0	1
	6	2	0	4	0	−8
				solution to the primal		

Tableau 4′

The solution to the dual is read from Tableau 4' as y_1 = y_2 = 0, y_3 = 1, with

z* = -(-8) = 8. The solution to the primal is found in the last row of this tableau, in those columns associated with the surplus variables. It is the same solution as found previously.

Example 5:

A boat manufacturing company makes three different kinds of boats. All can be made profitably in this company, but the company’s production is constrained by the limited amount of labour, wood and screws available each month. The director will choose the combination of boats that maximizes his revenue in view of the information given in the following table:

Input	Row Boat	Canoe	Kayak	Monthly Availability
Labour	12	7	9	1260 hours
Wood(Board feet)	22	18	16	19008
Screws(kg)	2	4	3	396kg
Selling Price in Rs	4000	2000	5000	

I. Formulate the above as a linear programming problem and write its dual.
II. Solve the primal by Simplex Method.
III. What are the shadow prices?
IV. Which, if any, of the resources are not fully utilized? If so, how much is the spare capacity left?
V. How much wood will be used to make all the boats in the optimal solution?
VI. If 9 units of Canoe are produced, what will be the impact on the product mix and revenue?

Solution:

I. (a) Formulation of Primal Linear Programming Problem:

Let the number of Row Boat, Canoe and Kayak manufactured are manufactured are x_1, x_2 *and* x_3 respectively.

The Linear Programming Problem can be expressed as under:

Maximize Z = $4000x_1 + 2000x_2 + 5000x_3$**(Total Revenue)**

Subject to the constraints:

$$12x_1 + 7x_2 + 9x_3 \leq 1260 \quad \text{(Labour constraint)}$$

$$22x_1 + 18x_2 + 16x_3 \leq 19008 \text{ (Wood constraint)}$$

$$2x_1 + 4x_2 + 3x_3 \leq 396 \quad \text{(Screws constraint)}$$

$$x_1, x_2, x_3 \geq 0$$

(b) Dual Problem:

Let y_1, y_2 *and* y_3 be the shadow price of Labour (per Hour), Wood (per Board feet) and Screws (per kg.) used to manufacture the boats respectively. The Dual can be written as:

Minimize Z = $1260y_1 + 19008y_2 + 396y_3$ (Total shadow price of the resources)

Subject to the constraints:

$$12y_1 + 22y_2 + 2y_3 \geq 4000 \text{ (Row Boat constraint)}$$

$$7y_1 + 18y_2 + 4y_3 \geq 2000 \text{ (Canoe constraint)}$$

$$9y_1 + 16y_2 + 3y_3 \geq 5000 \text{ (Kayak constraint)}$$
$$y_1, y_2, y_3 \geq 0$$

II. Simplex solution of the primal:

Introducing slack variables S_1,S_2, S_3 the linear programming problem can be expressed as:

Maximize Z = $4000x_1 + 2000x_2 + 5000x_3 + 0S_1 + 0S_2 + 0S_3$

Subject to the constraints:

$$12x_1 + 7x_2 + 9x_3 + S_1 = 1260$$
$$22x_1 + 18x_2 + 16x_3 + S_2 = 19008$$
$$2x_1 + 4x_2 + 3x_3 + S_3 = 396$$
$$x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$$

Augments Linear Programming Problem:

Maximize Z = $4000x_1 + 2000x_2 + 5000x_3 + 0S_1 + 0S_2 + 0S_3$

Subject to the constraints:

$$12x_1 + 7x_2 + 9x_3 + S_1 + 0S_2 + 0S_3 = 1260$$
$$22x_1 + 18x_2 + 16x_3 + 0S_1 + S_2 + 0S_3 = 19008$$
$$2x_1 + 4x_2 + 3x_3 + 0S_1 + 0S_2 + S_3 = 396$$
$$x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$$

Initial Basic Feasible Solution:

An initial basic feasible solution is obtained by putting the decision variables equal to zero and calculating values of the slack variables.

Putting x_1, x_2 and x_3

→ S_1 =1260, $S_2 = 19008$, $S_3 = 396$ and Z=0

Accordingly, the initial tableau appears as under:

Simplex Tableau-I

C_j	Basic Variable	Solution b_i	x_1	x_2	x_3	S_1	S_2	S_3	Ratio=$\frac{b_i}{a_{ij}}$
		C_j	4000	2000	5000	0	0	0	
0	S_1	1260	12	7	9	1	0	0	$\frac{1260}{9} = 140$
0	S_2	19008	22	18	16	0	1	0	$\frac{19008}{16} = 1188$
0	S_3	396	2	4	**3**	0	0	1	$\frac{396}{3} = 132 \leftarrow$
	Z_j	**0**	0	0	0	0	0	0	
	$C_j - Z_j$		4000	2000	**5000**↑	0	0	0	

Incoming Variable: x_3 is the incoming variable because it has the highest $C_j - Z_j$ value in the Net Evaluation Row, i.e. 5000. The column containing it is the key column which is used to calculate the replacement ratios in the ratio column.

Outgoing Variable: S_3 is the outgoing basic variable corresponding to the minimum positive ratio in the ratio column, i.e. 132. The row containing it is the key row.

Pivot Value: The highlighted number at the intersection of the Key Column and the Key Row is the Pivot Value, i.e. 3 that will be used to perform the elementary row operation to obtain the improved solution. The simplex table-II containing the improved solution is obtained as under:

Simplex Tableau-II

C_j	Basic Variable	Solution b_i	x_1	x_2	x_3	S_1	S_2	S_3	Ratio=$\frac{b_i}{a_{ij}}$
		C_j	4000	2000	5000	0	0	0	
0	S_1	72	**6**	-5	0	1	0	-3	$\frac{72}{6} = 12 \leftarrow$
0	S_2	16896	$\frac{34}{3}$	$-\frac{10}{3}$	0	0	1	-16/3	$\frac{16896}{34/3} = \frac{25344}{17}$
5000	x_3	132	$\frac{2}{3}$	$\frac{4}{3}$	1	0	0	1/3	$\frac{132}{2/3} = 198$
	Z_j	**160,000**	$\frac{10000}{3}$	$\frac{20000}{3}$	5000	0	0	$\frac{5000}{3}$	
	$C_j - Z_j$		$\frac{2000}{3}$↑	$-\frac{14000}{3}$	0	0	0	$-\frac{5000}{3}$	

Incoming Variable: x_1 is the incoming variable because it has the highest $C_j - Z_j$ value in the Net Evaluation Row, i.e. $\frac{2000}{3}$. The column containing it is the key column which is used to calculate the replacement ratios in the ratio column.

Outgoing Variable: S_1 s is the outgoing basic variable corresponding to the minimum positive ratio in the ratio column, i.e. 12. The row containing it is the key row.

Pivot Value: The highlighted number at the intersection of the Key Column and the Key Row is the Pivot Value, i.e. 6 that will be used to perform the elementary row operation to obtain the improved solution. The simplex table-III containing the improved solution is obtained as under:

Simplex Tableau-III

		C_j	4000	2000	5000	0	0	0
C_j	Basic Variable	Solution b_i	x_1	x_2	x_3	S_1	S_2	S_3
4000	x_1	12	1	$-\frac{5}{6}$	0	$\frac{1}{6}$	0	$-\frac{1}{2}$
0	S_2	16760	0	$\frac{55}{9}$	0	$-\frac{17}{9}$	1	1/3
5000	x_3	124	0	$\frac{17}{9}$	1	$-\frac{1}{9}$	0	2/3
	Z_j	**668000**	4000	$\frac{55000}{9}$	5000	$\frac{1000}{9}$	0	$\frac{4000}{3}$
	$C_j - Z_j$		0	$-\frac{37000}{9}$	0	$-\frac{1000}{9}$	0	$-\frac{4000}{3}$

The Net Evaluation Row indicates that the C_j - Z_j values are non-positive for all the variables, thus the solution is optimal.

Optimal solution:

x_1 = 12 and x_2 = 40 and $x_3 = 124$

Maximum value of the objective function, Z = 668000.

Thus, 12 units of Row Boat and 124 units of Kayak will be manufactured yielding the maximum revenue amounting to Rs 668000.

III. Shadow prices of Labour, Wood and Screws are given by the absolute values of the slack variables in the $C_j - Z_j$ i.e. net evaluation row as Rs 1000/9 per hour, Rs 0 per board feet and Rs 4000/3 per kg respectively.

IV. Wood represented by the slack variable S_2 is unutilized as it remains as a basic variable. The spare capacity of wood is 16,760 Board Feet.

V. Wood used to make all the boats in the optimal solution is calculated as:
= 22 Board feet p.u. X 12units (Row Boat) +16 Board feet p.u. X 124 units (Kayak) = 2248 Board Feet.
Alternatively, it can be calculated as:
Utilization=Availability–Spare capacity =19008 –16760 = 2248 Board feet.

VI. For interpreting the impact of producing 9 units of Canoe represented by x_2, we take the $x_2$column of the final simplex tableau:

Basic Variable	Solution b_i	x_2
x_1	12	$-\frac{5}{6}$
S_2	16760	$\frac{55}{9}$
x_3	124	$\frac{17}{9}$
Z_j	668000	$\frac{55000}{9}$
$C_j - Z_j$		$-\frac{37000}{9}$

New Product Mix will be:

Units of Row Boat (x_1) = $12 - \left[-\frac{5}{6}(9)\right] = 19.5\ units$

Units of Canoe (x_2) = 9 units

Units of Kayak (x_3) = 124- $\left[-\frac{17}{9}(9)\right]$ = 107units

New Revenue = 6,68,000 – $\frac{37000}{9}$(9) = Rs 6,31,000