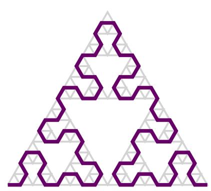
Color By Group Theory

Math is about more than just numbers. In this "book" the story of math is visual, told in shapes and patterns.



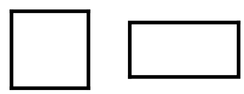
Group theory is a mathematical study with which we can explore symmetry.

This is a coloring book.

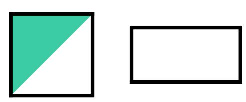
Table of Contents // Coloring Guide

Shapes & Symmetries

We have intuitive ideas of symmetry. Some shapes look "more symmetric" than others. For example, a square is "more symmetric" than a rectangle,

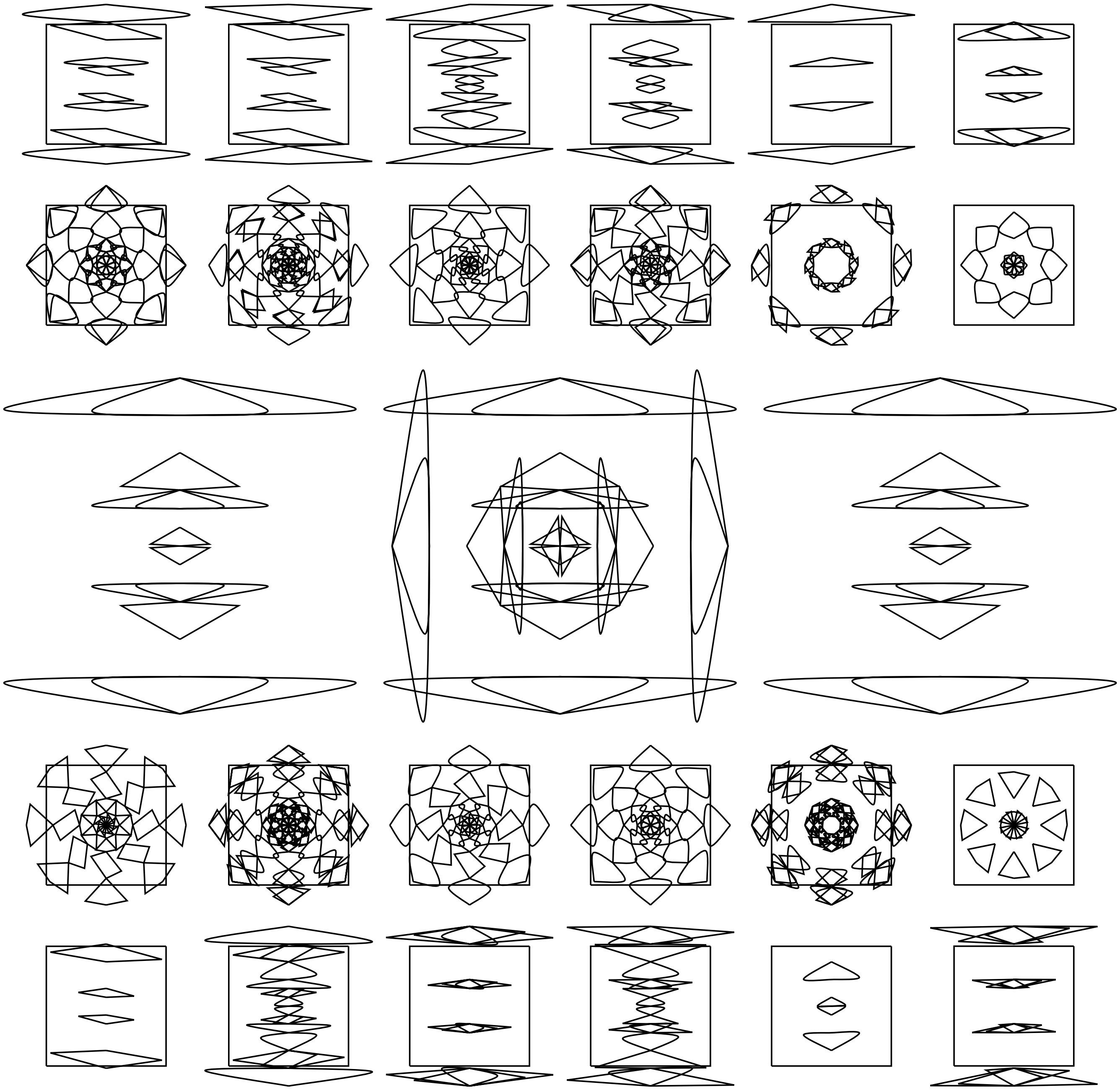


but this can change once color is added.



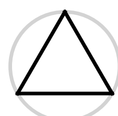
We can be more precise as to what this means. We can even count how much symmetry a shape has.

Challenge: Can you color the shapes to give them the same "amount of symmetry"?

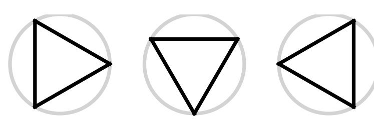


Rotations

An equilateral triangle can be rotated $\frac{1}{3}$ of the way around a circle and appear unchanged.

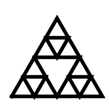


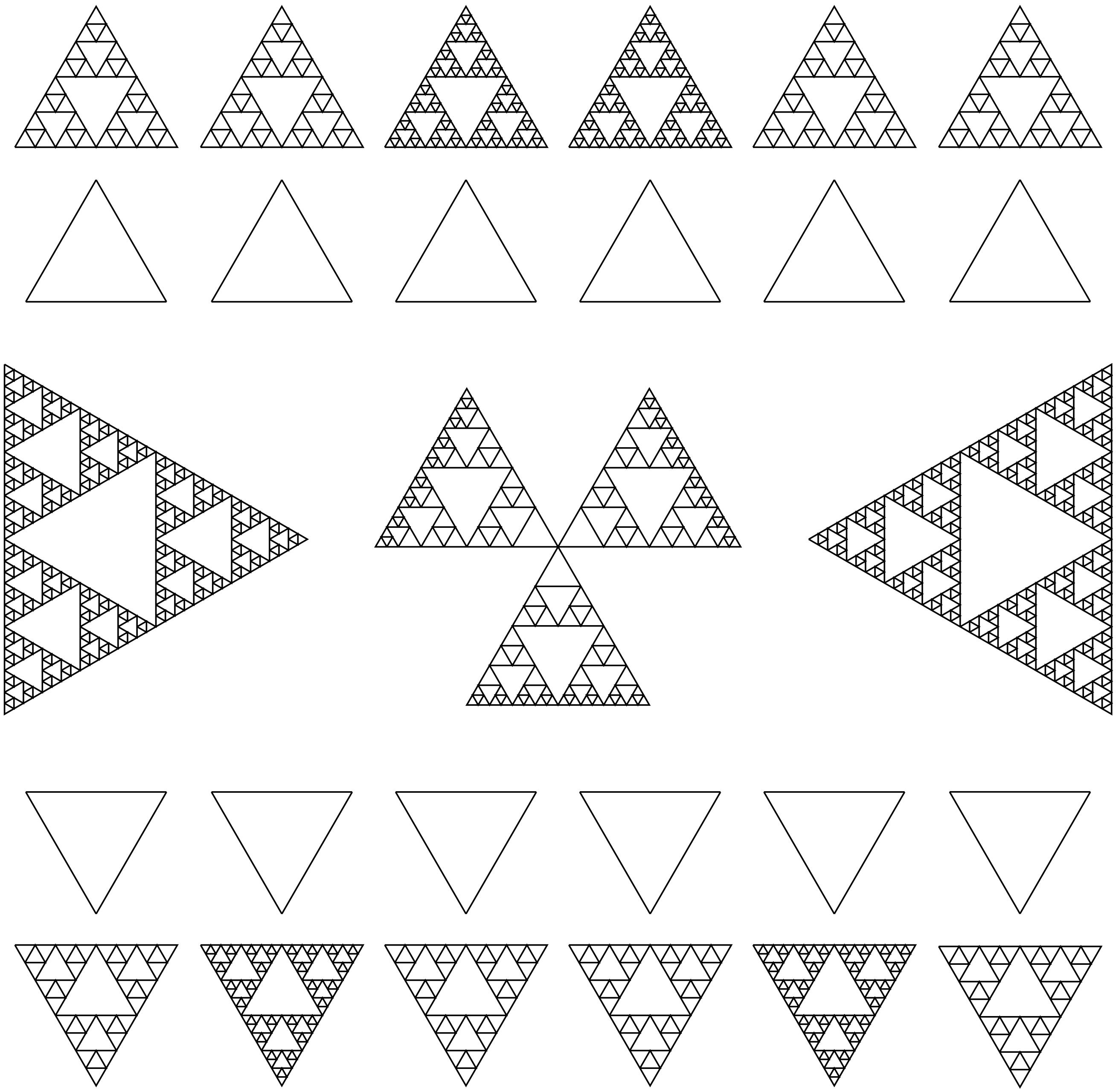
If the triangle were instead rotated by an arbitrary amount, like $\frac{1}{4}$ of the way around a circle, it would then appear changed, since it is oriented differently.



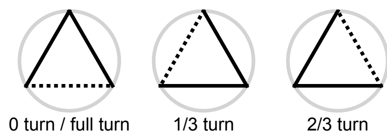
1/4 turn 2/4 turn 3/4 turn

Challenge: Can you color the shapes so that a $\frac{1}{3}$ *turn continues to leave their appearance unchanged?*

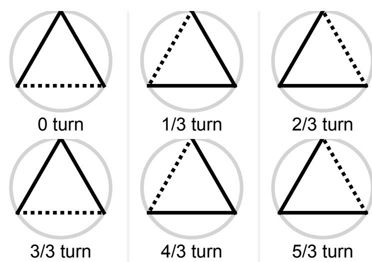




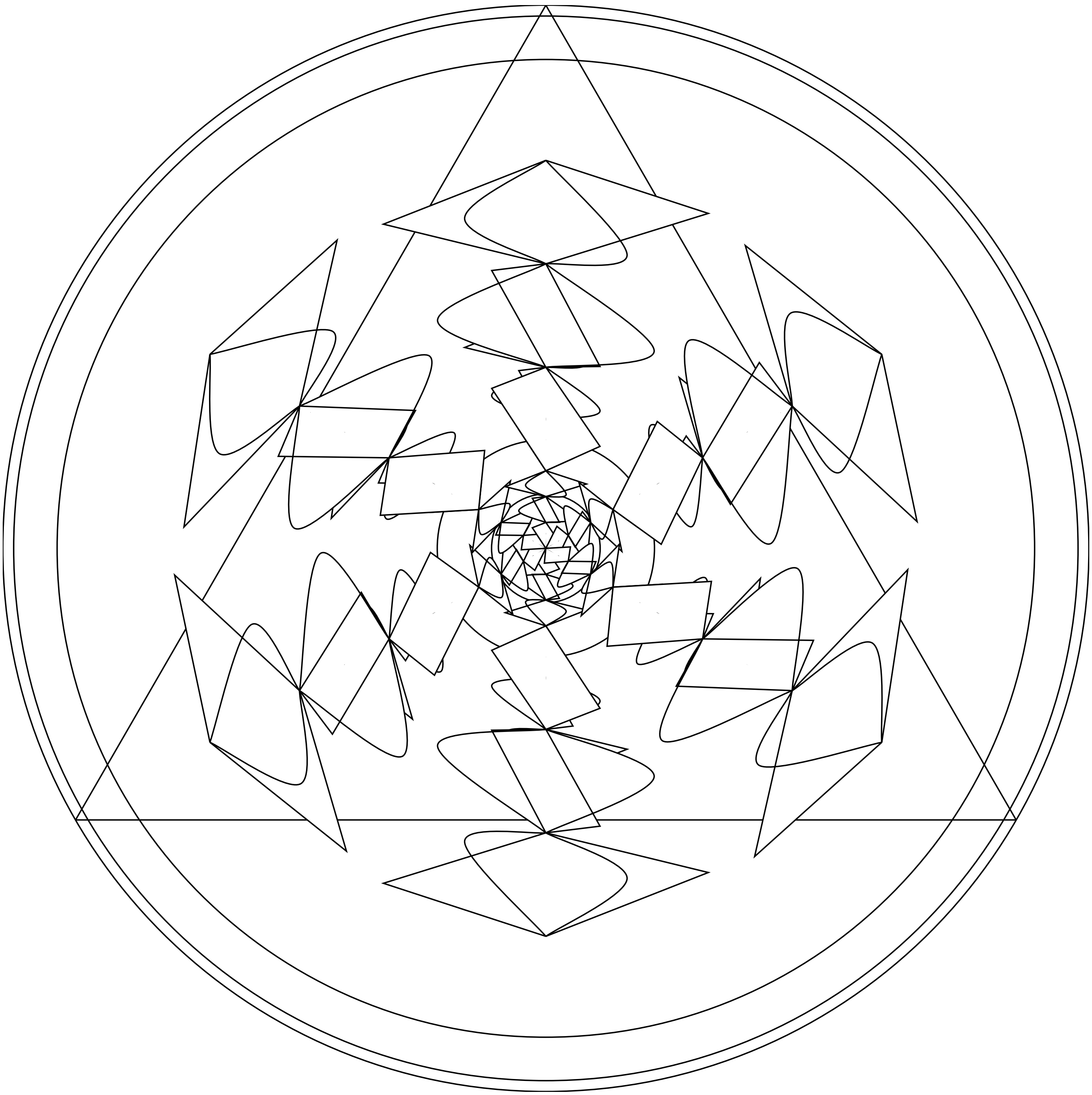
We can still rotate that triangle more than $\frac{1}{3}$ of the way around a circle without changing it. It can be rotated by twice that much - $\frac{2}{3}$ of the way around the circle - or by 3 times that much, which is all the way around the circle.



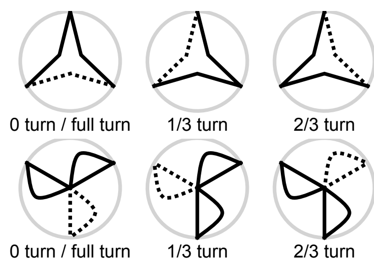
We can keep rotating - by 4 times that much, 5 times that much, 6 times... and keep going. The triangle seems to have an infinite number of rotations, but after rotating 3 times they become repetitive.

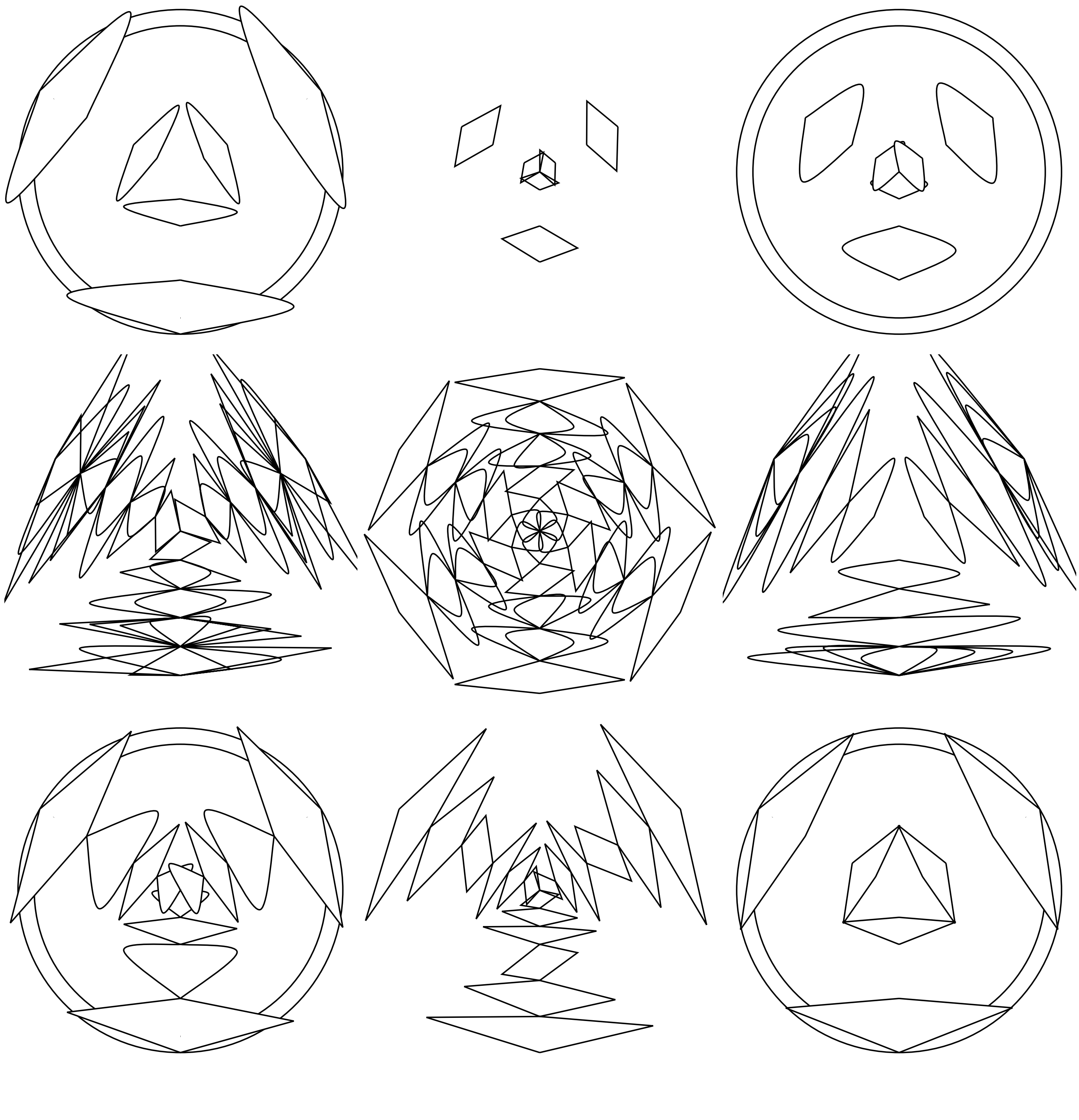


There are only 3 unique rotations for a triangle. We'll talk about them by referring to the rotations that are less than a full turn.

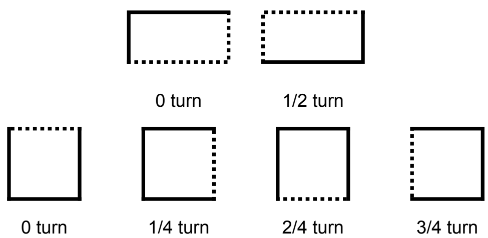


Other shapes, not just equilateral triangles, have the same 3 rotations. For this reason, we can say they all share the same group (C3).

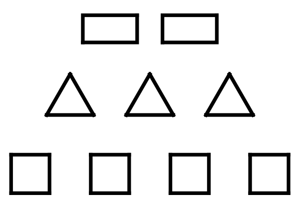


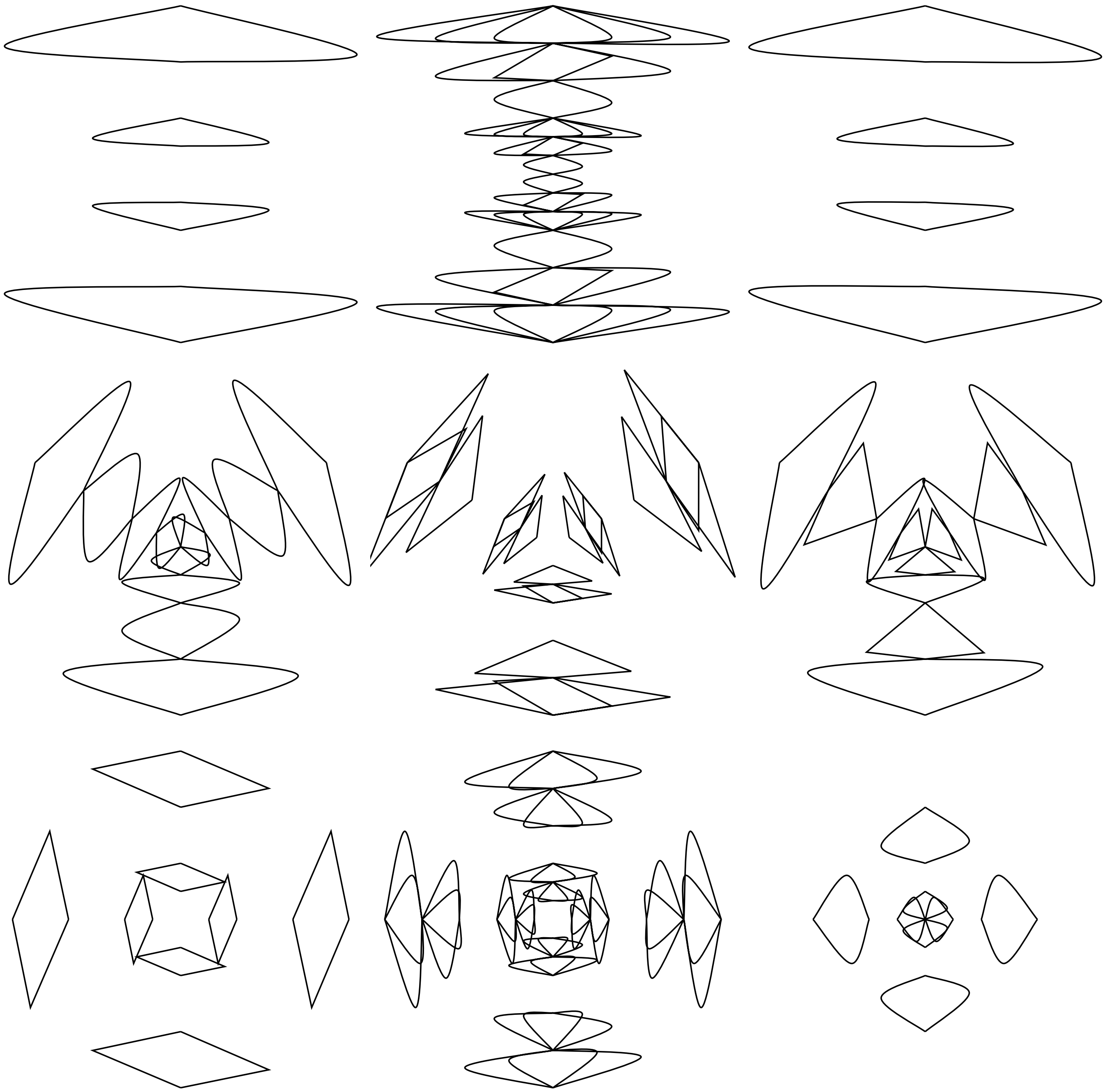


Now that we can count rotations, we can be more precise when we say a square has more symmetry than a rectangle.

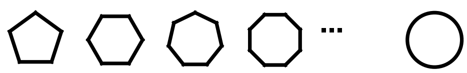


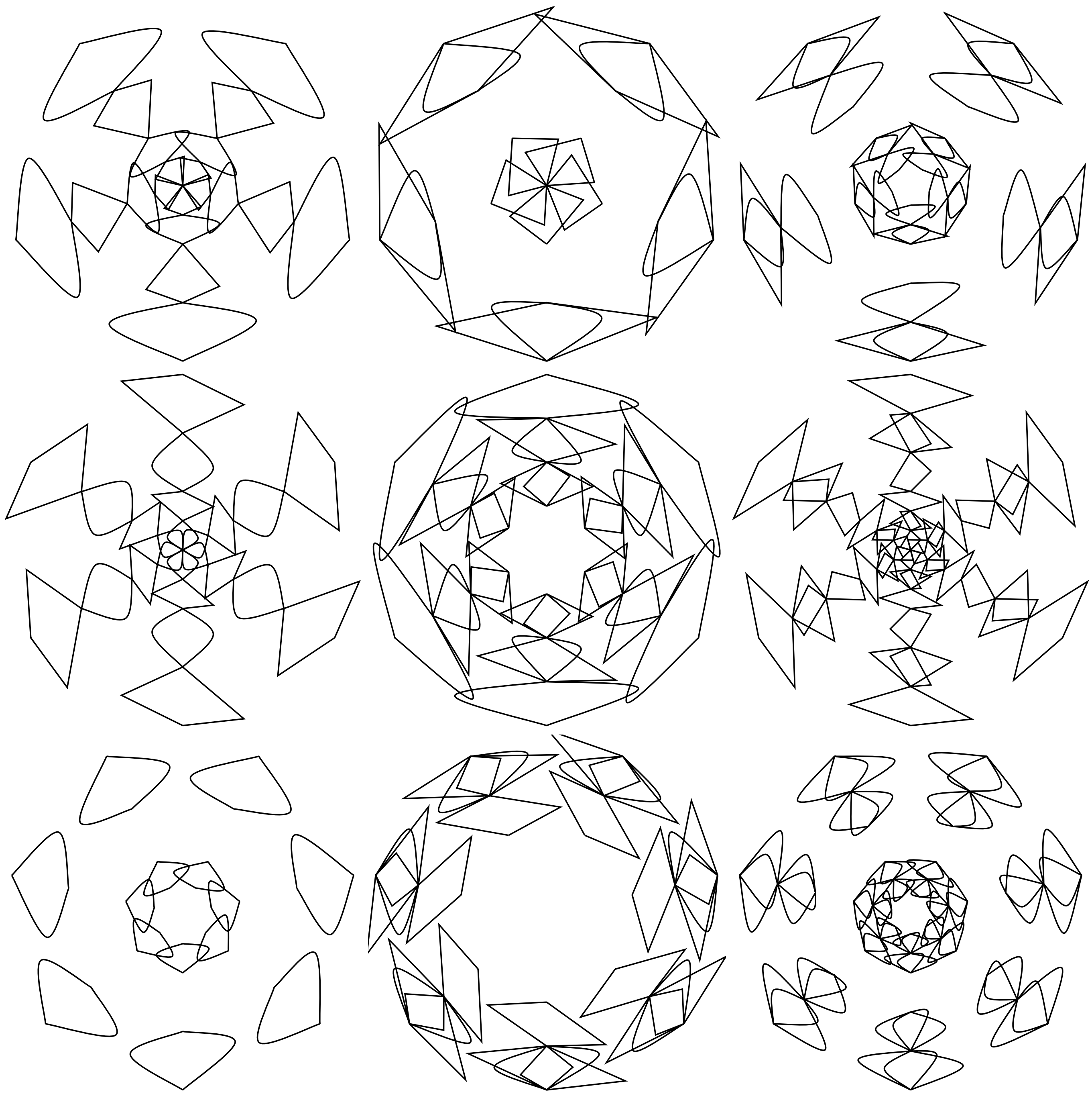
We can also see that a square has more rotational symmetry than a triangle, which in turn has more symmetry than a rectangle: A square has 4 unique rotations, while a triangle has 3, and a rectangle has only 2.



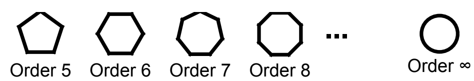


We don't need to stop at 4 rotations. We can find shapes with 5 rotations, 6 rotations, 7, 8, ... and keep going towards infinity.

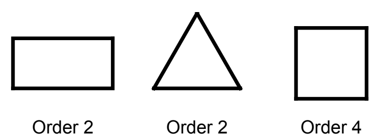




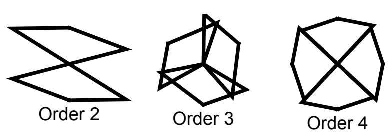
We need a better way to talk about this. We say a shape has rotational symmetry of **order n** if it has **n** unique rotations. This means the shape can be rotated $\frac{1}{n}$ of the way around the circle without changing its appearance.

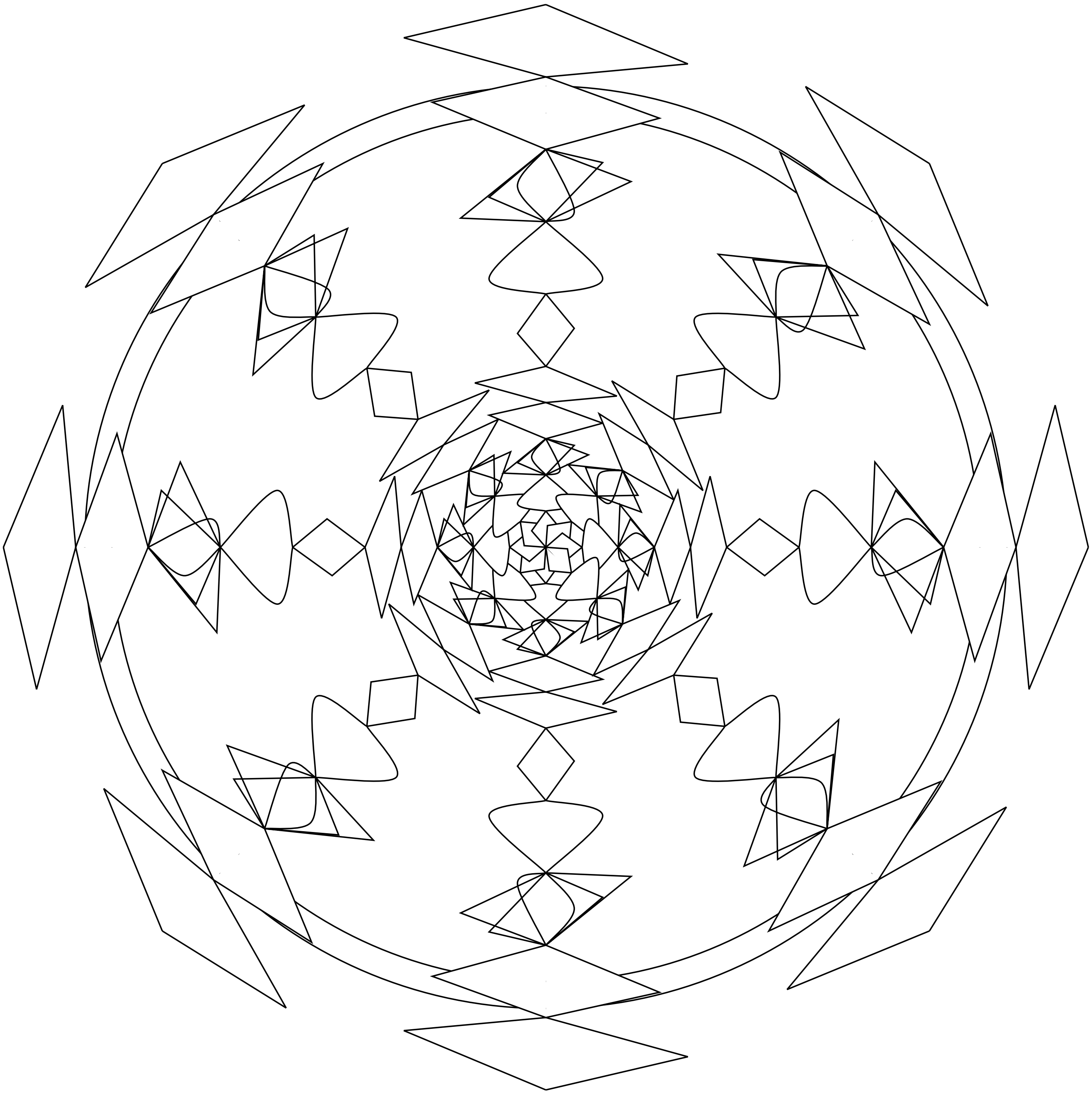


For example, a triangle can be rotated by $\frac{1}{3}$ of the way around the circle so a triangle has order 3. A square can be rotated $\frac{1}{4}$ of the way around the circle without changing, so it has order 4.



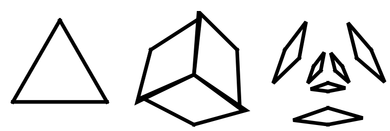
We can say the same for shapes that are not regular polygons.



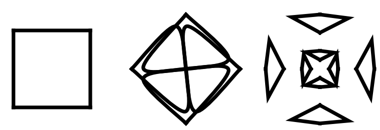


When shapes have the same rotational order, they share a cyclic group. We can use the name C3 to talk about the group of shapes that have 3 rotations. We can also use the name C4 for the shapes that have 4 rotations, C5 for the shapes that have 5 rotations, and so on.

C3:

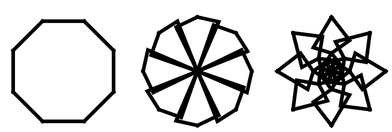


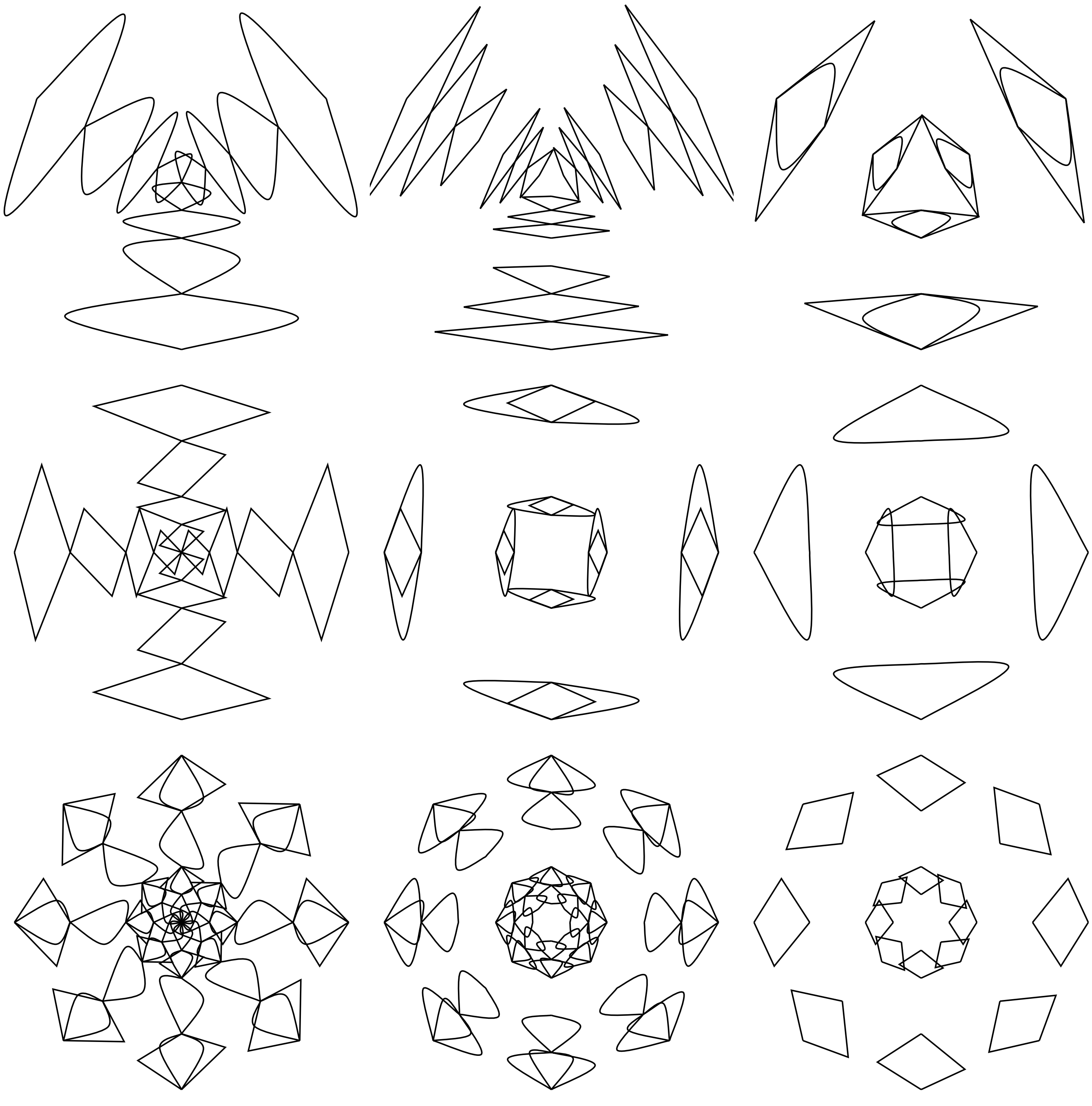
C4:



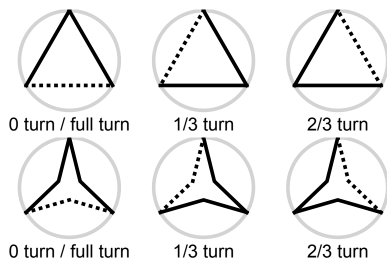
...

C8:

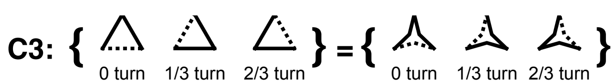




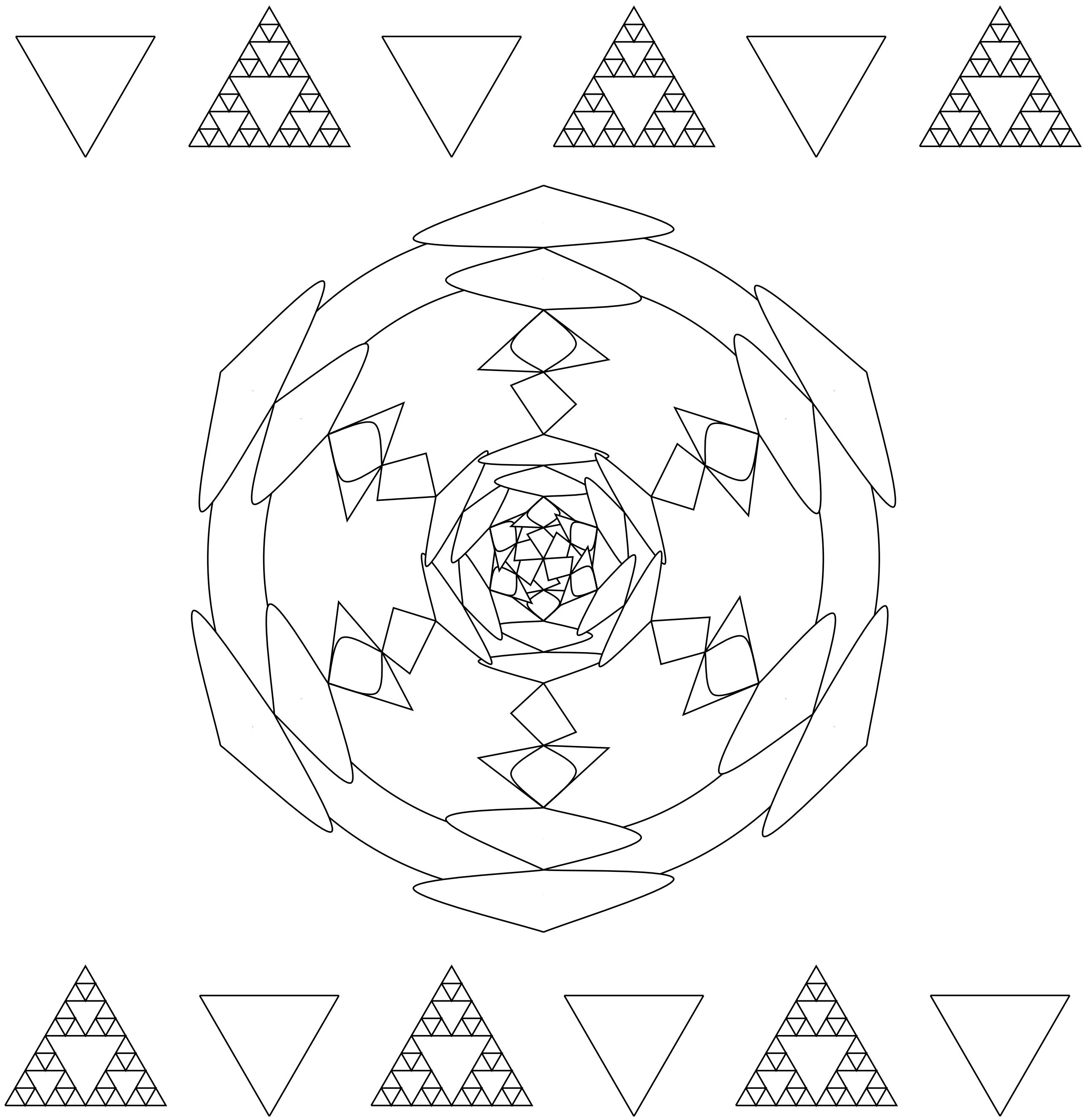
These shapes that share a <u>cyclic group</u> have the same group of rotations that leave their appearance unchanged. For example, the shapes of C3 all have the rotations **{0 turn, $\frac{1}{3}$ turn, $\frac{2}{3}$ turn }** and no more unique rotations other than those.



Our shapes help us see our groups, but the members of the groups are the rotations, not the shapes.

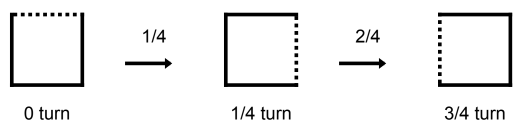


These rotations are related to each other.



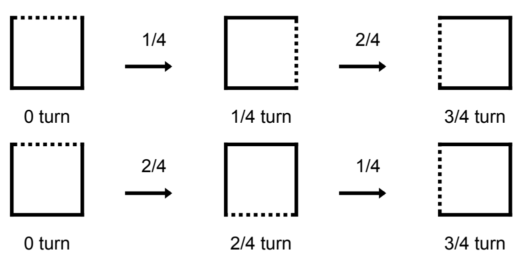
Another way to think about rotating a C4 shape by a $\frac{3}{4}$ turn is to rotate it by a $\frac{1}{4}$ turn and then rotate it again by a $\frac{2}{4}$ turn.

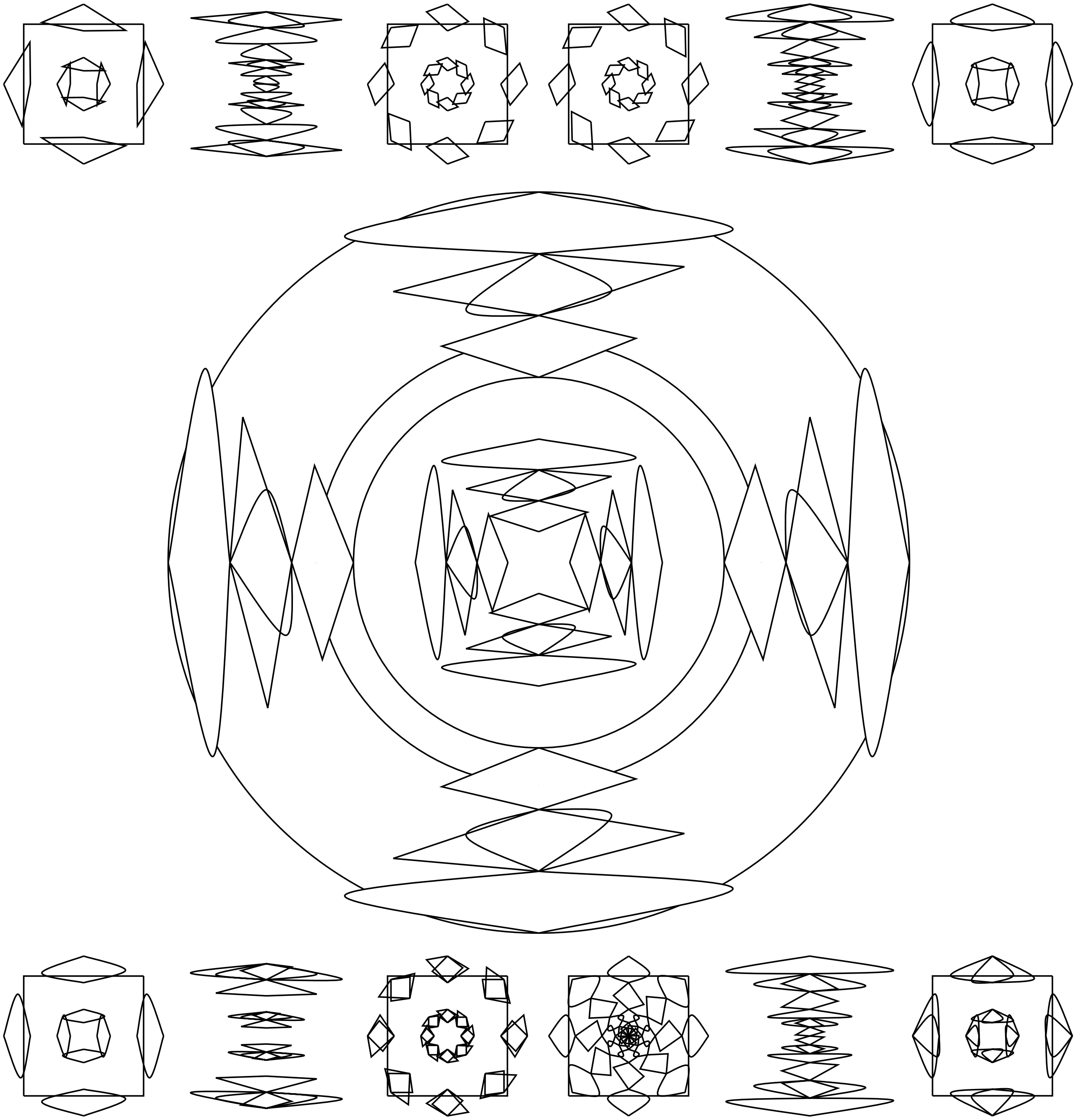
$$\textbf{C4:} \quad \tfrac{1}{4}\ \textbf{turn} \ast \tfrac{2}{4}\ \textbf{turn} \rightarrow \tfrac{3}{4}\ \textbf{turn}$$



Notice that the order in which these rotations are applied does not matter. The Cn groups are <u>commutative</u>.

$$\textbf{C4:} \quad \tfrac{1}{4}\ \textbf{turn} \ast \tfrac{2}{4}\ \textbf{turn} = \tfrac{2}{4}\ \textbf{turn} \ast \tfrac{1}{4}\ \textbf{turn}$$



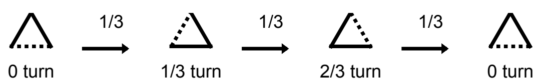


Similarly, for our C3 group, a $\frac{2}{3}$ turn is the same as combining a $\frac{1}{3}$ turn with another $\frac{1}{3}$ turn.

$$\textbf{C3:} \qquad \tfrac{1}{3}\ \textbf{turn} \ast \tfrac{1}{3}\ \textbf{turn} \rightarrow \tfrac{2}{3}\ \textbf{turn}$$

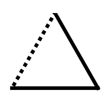
Rotating again by $\frac{1}{3}$ brings the shape back to its starting position - the 0 turn.

$$\textbf{C3:} \qquad \tfrac{1}{3}\ \textbf{turn} \ast \tfrac{1}{3}\ \textbf{turn} \ast \tfrac{1}{3}\ \textbf{turn} \rightarrow \textbf{0 turn}$$



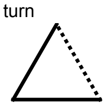
See, the $\frac{1}{3}$ turn can generate all of the rotations of C3 - it is a <u>generator</u> for our C3 group.

$$\tfrac{1}{3}\ \textbf{turn} \longrightarrow$$



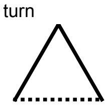
1/3 turn

$$\tfrac{1}{3}\ \textbf{turn} \ast \tfrac{1}{3}\ \textbf{turn} \longrightarrow$$

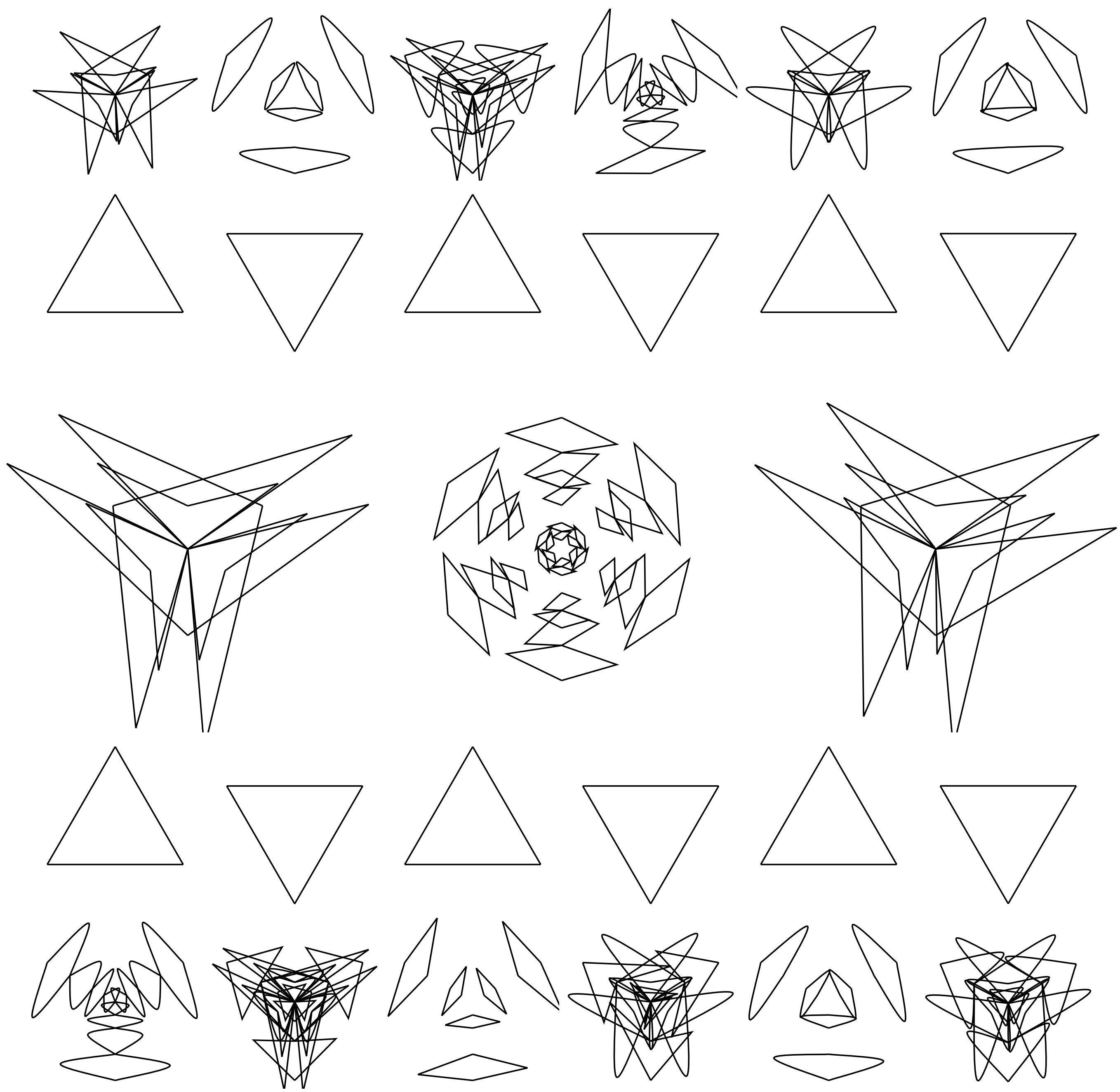


2/3 turn

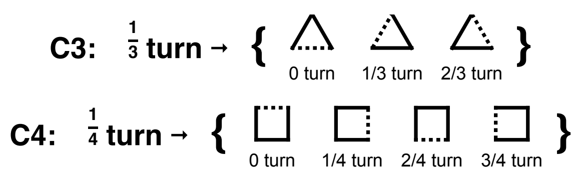
$$\tfrac{1}{3}\ \textbf{turn} \ast \tfrac{1}{3}\ \textbf{turn} \ast \tfrac{1}{3}\ \textbf{turn} \longrightarrow$$



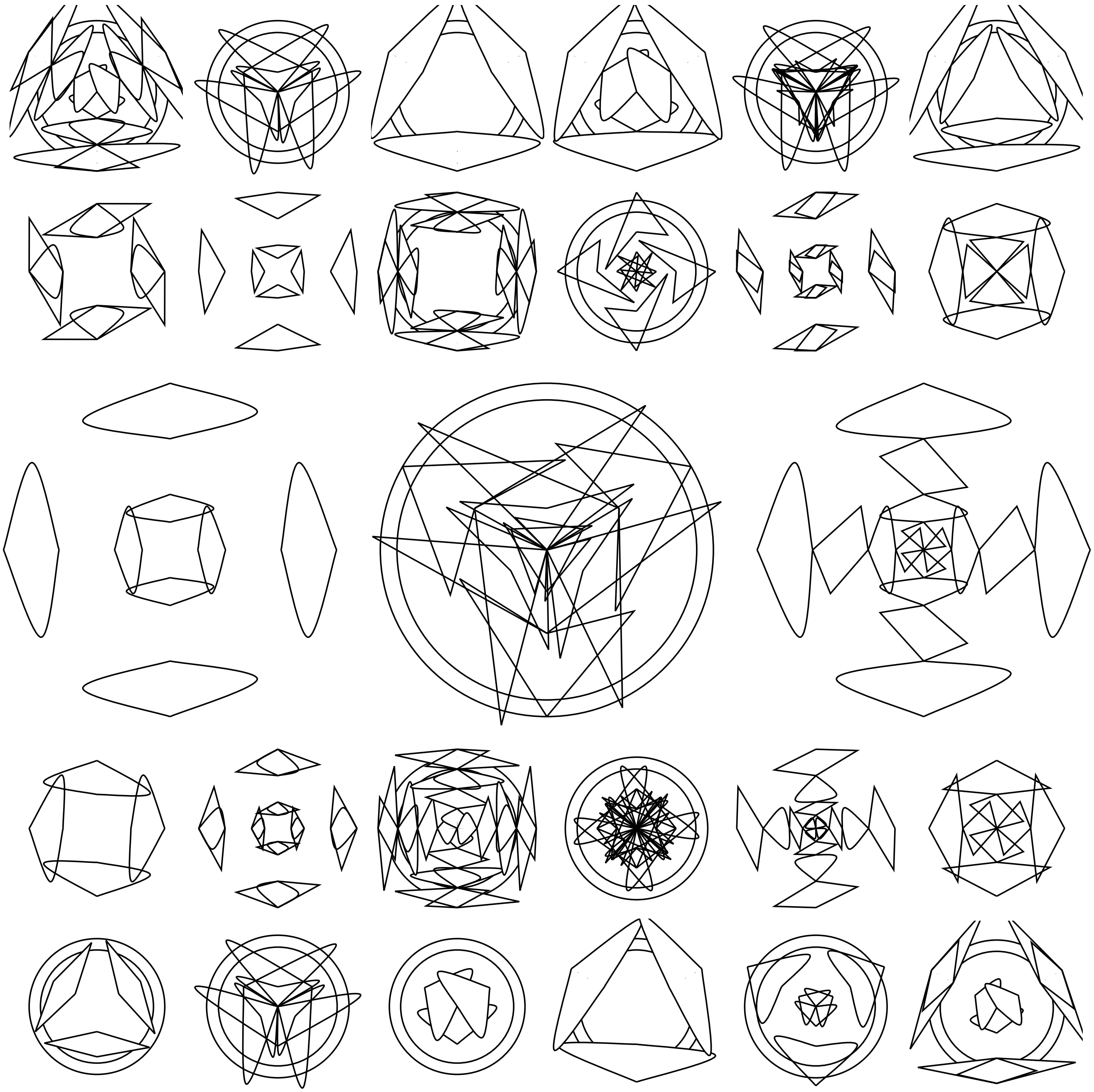
0 turn

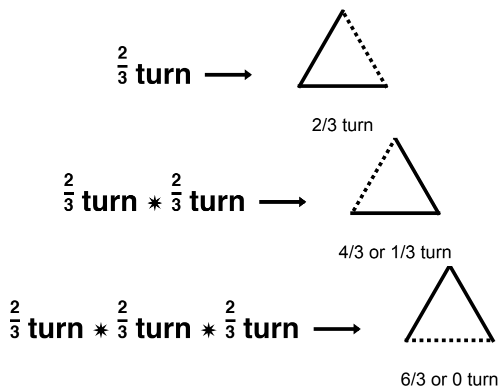


The $\frac{1}{3}$ turn is a generator for our C3 group, and similarly, the $\frac{1}{4}$ turn is a generator for our C4, because it can generate all of the rotations of our C4.

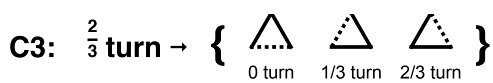


We could even choose different generators.

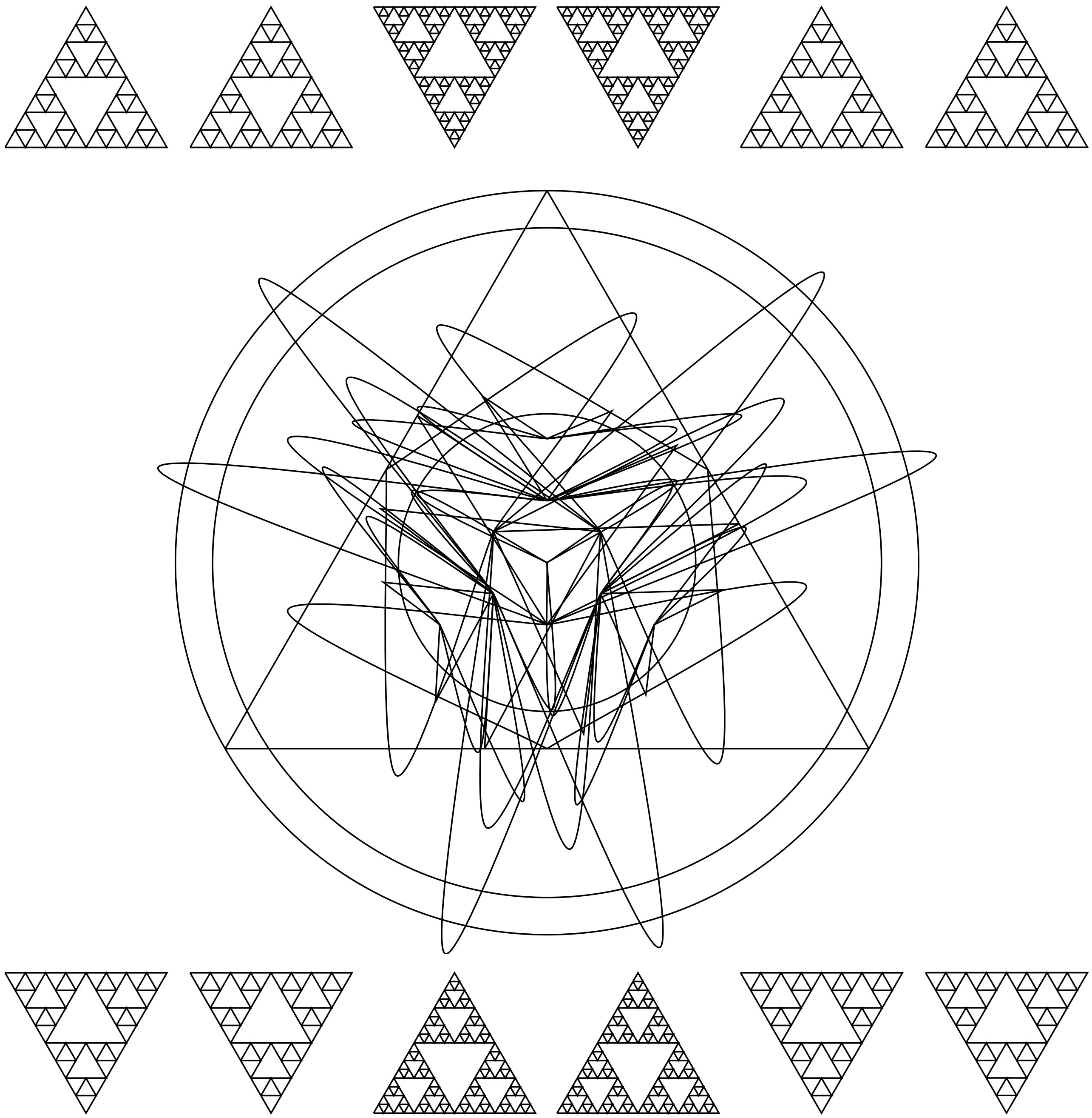




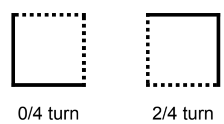
We could have just as easily used a $\frac{2}{3}$ turn as our generator for C3 and ended up with the same result.



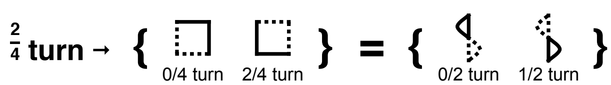
However, not all rotations are generators.



A $\frac{2}{4}$ turn does not generate all of the rotations of our C4 group.

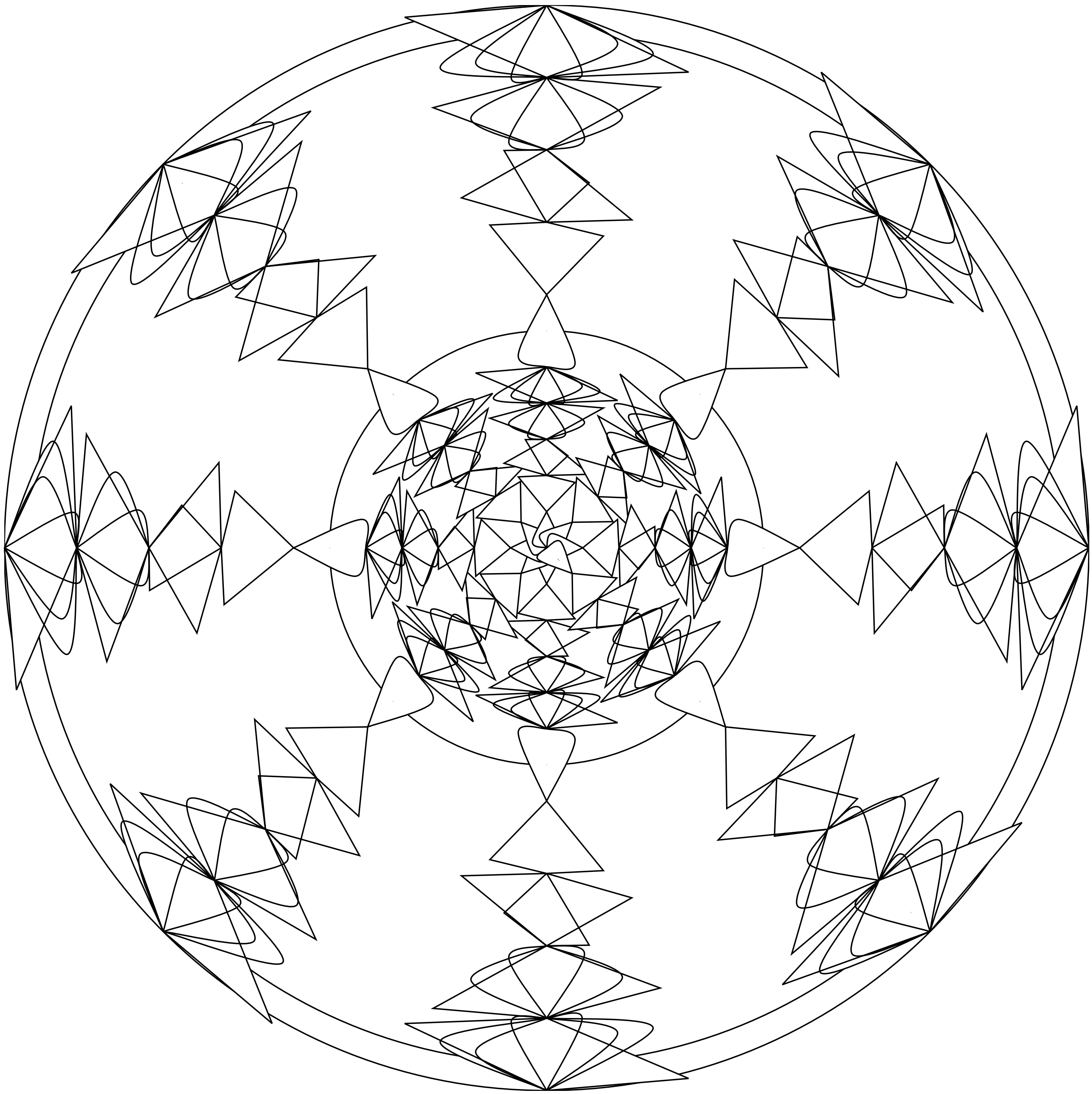


Instead a $\frac{2}{4}$ turn generates a smaller group -- our C2 group.

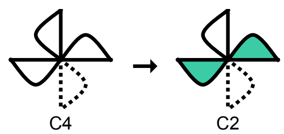


Challenge: Find all the generators for C4 and C8.

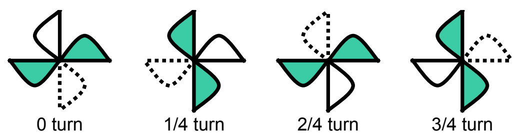
Challenge: Which rotations of C8 reduce C8 to C4 when used as generators?



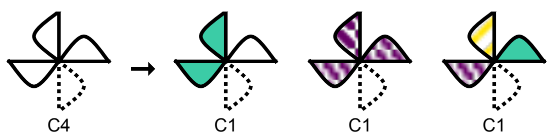
We can transform a C4 shape into a C2 shape by coloring it.



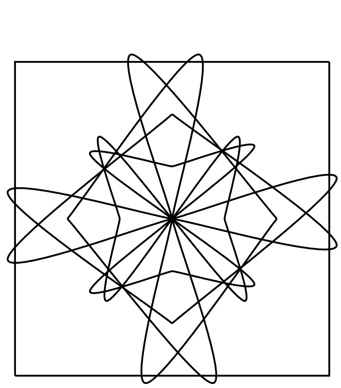
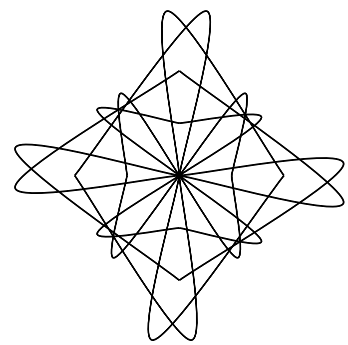
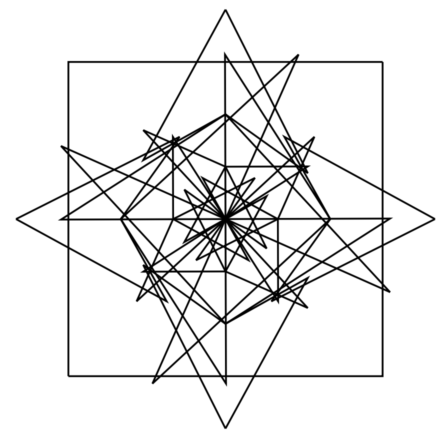
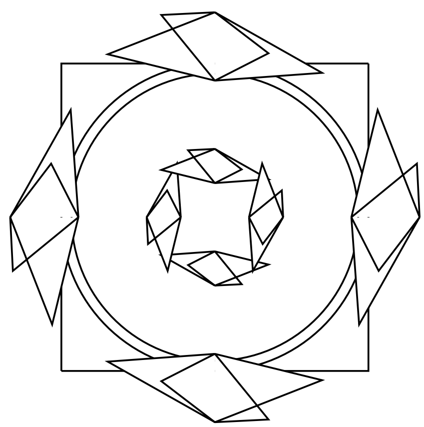
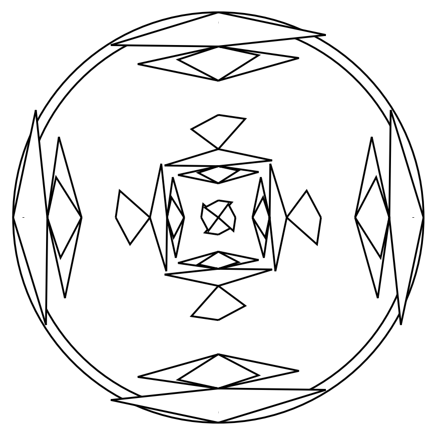
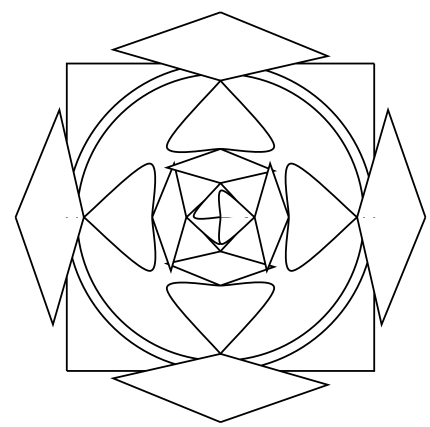
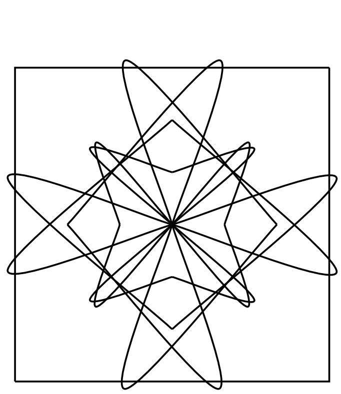
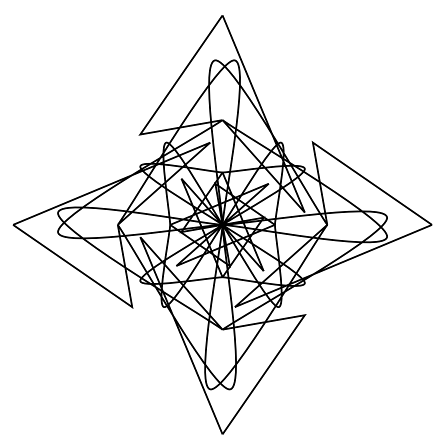
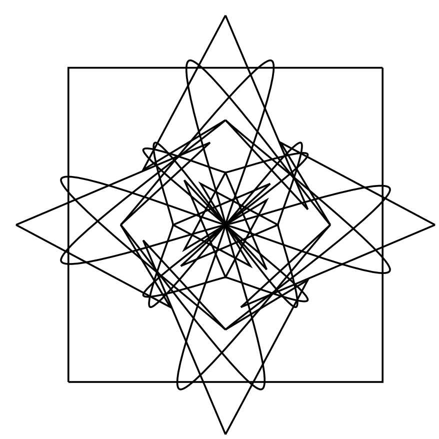
The only rotations that leave this colored shape unchanged are those of C2.



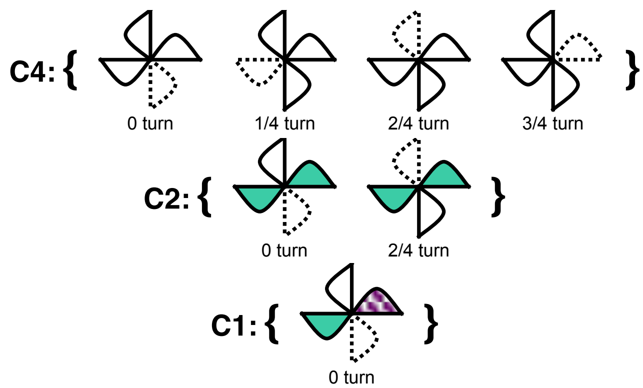
Not all colorings of our C4 shapes will transform them into C2 shapes.



Try coloring the C4 shapes to reduce them to C2 shapes.

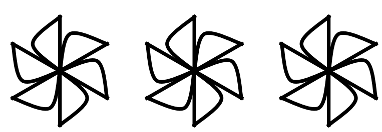










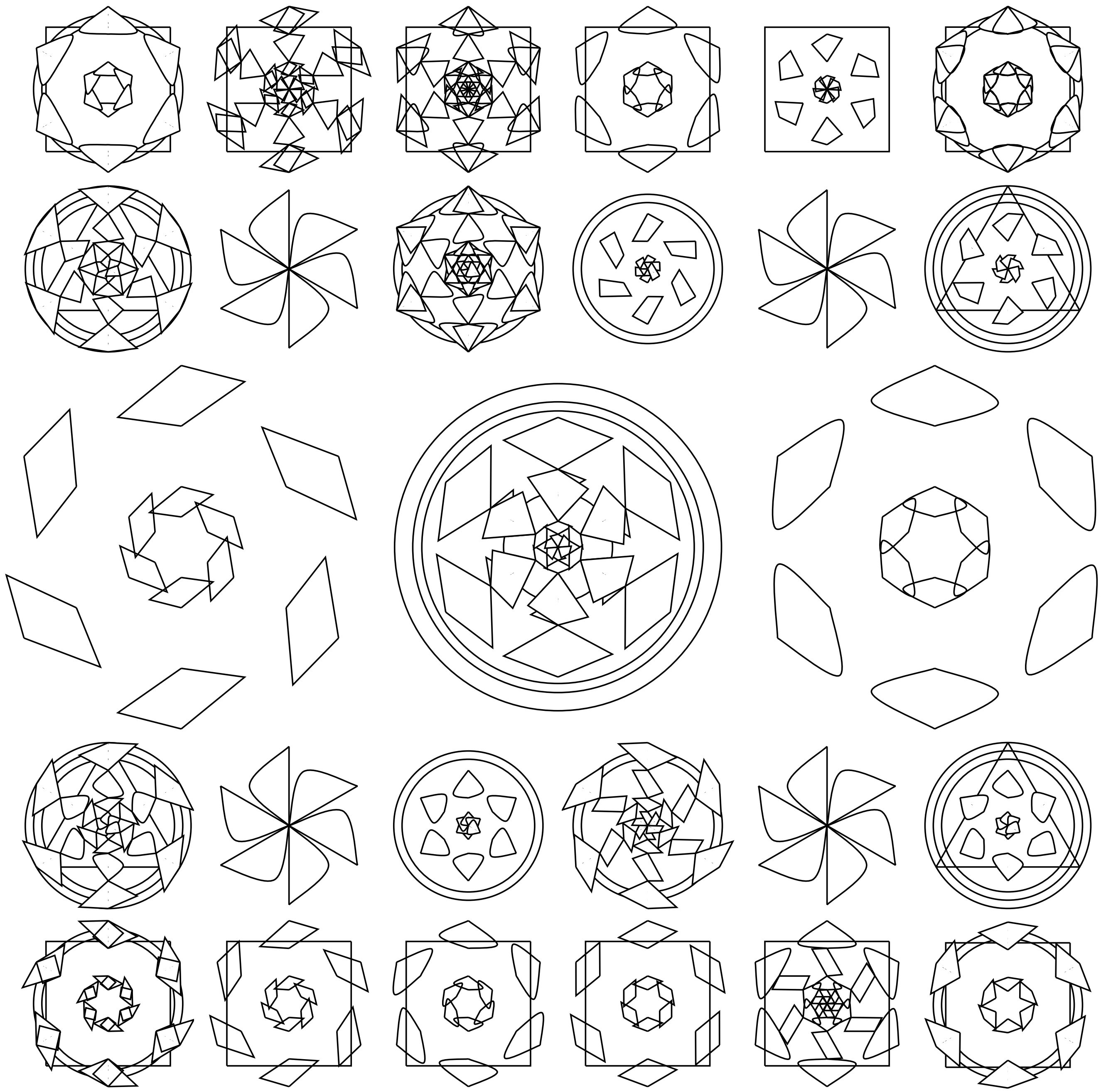
Color can reduce C4 shapes to C2 or C1 shapes because our C2 and C1 are <u>subgroups</u> of our C4 group. A <u>subgroup</u> is a group contained within a group.



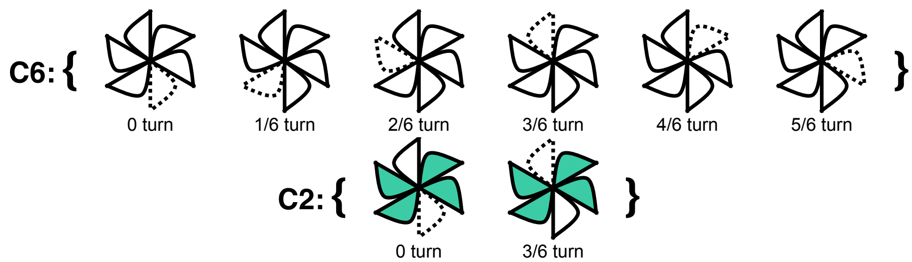
Similarly, our C1, C2, and C3 groups are all <u>subgroups</u> of our C6 group.

Challenge: Can you color the C6 shapes to reduce them to C1, C2, or C3 shapes?

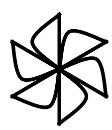




Notice that a group has all of the rotations of its subgroups.

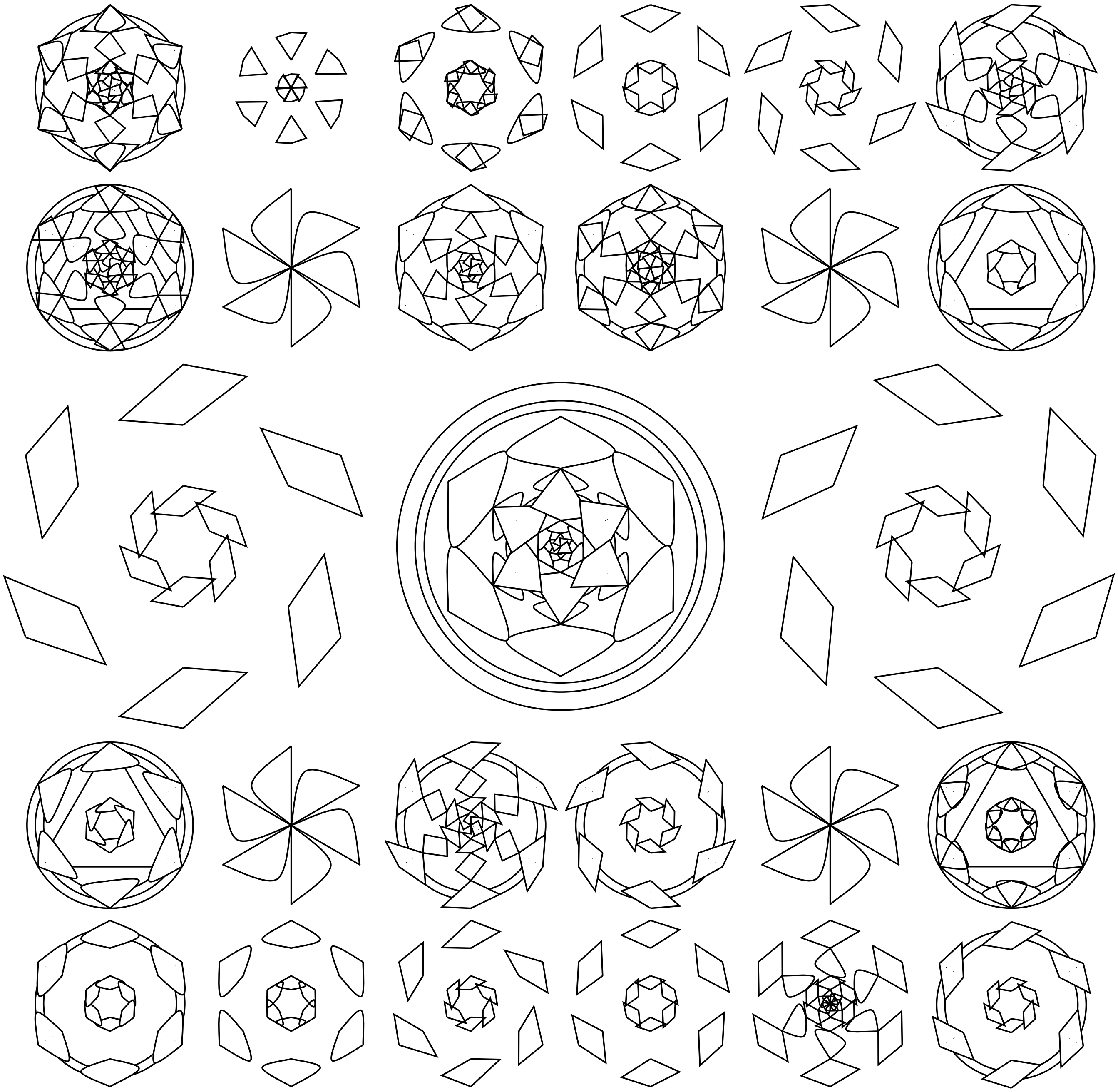


Try to color a C6 shape so that it has the rotations of C4.

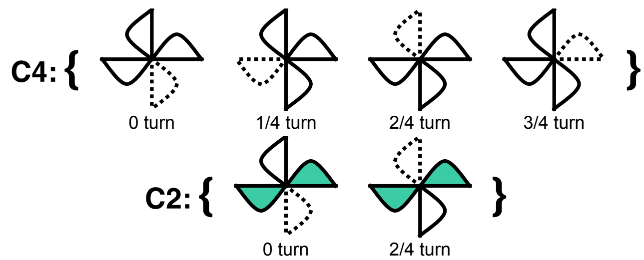


It can't be done. C4 is not a <u>subgroup</u> of C6.

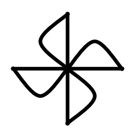
There is more to it than that.



When we use color to reduce our shapes to belong to smaller groups, we give them a new set of rotations.

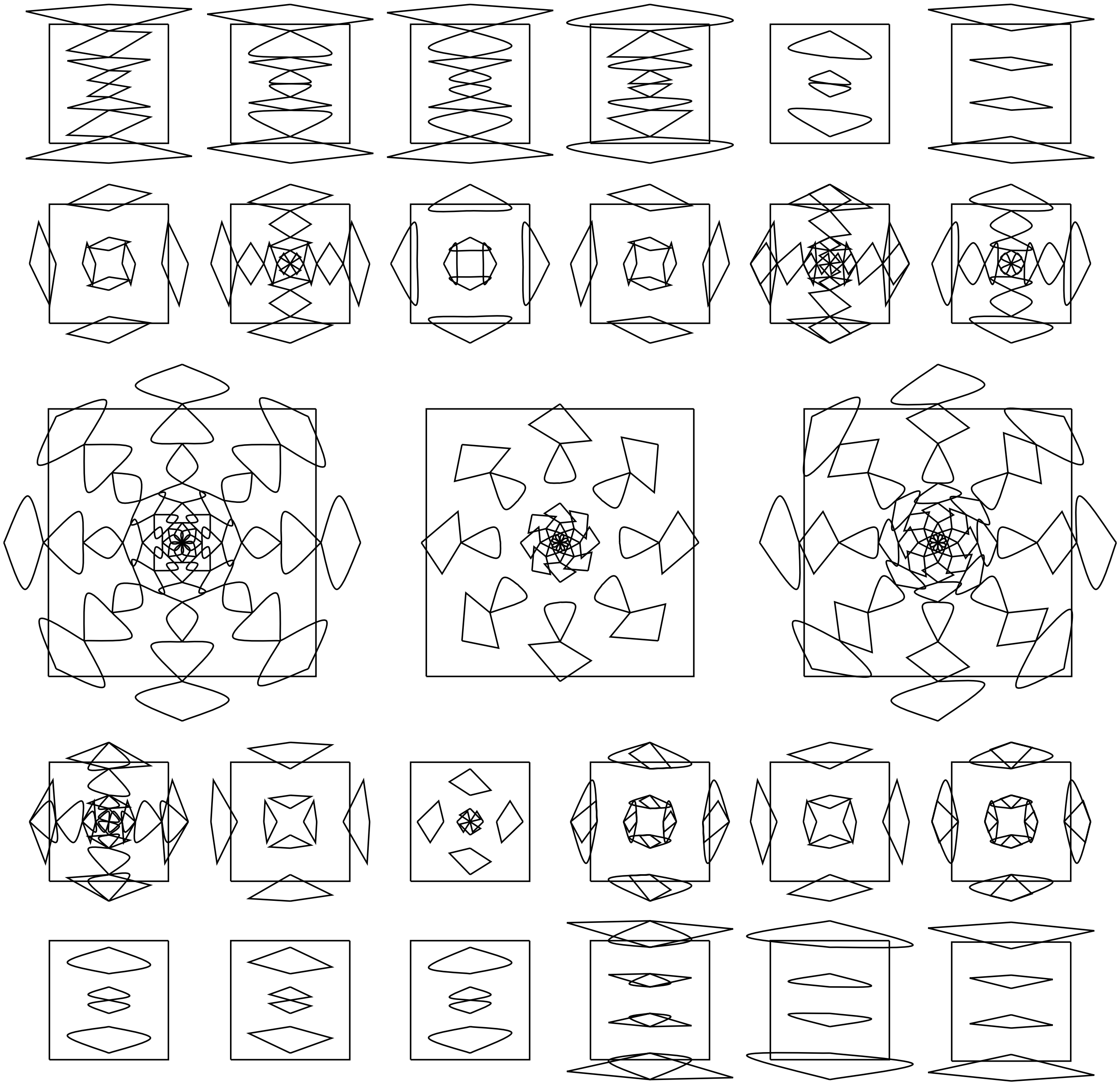


Not all sets of rotations are groups, and therefore cannot be subgroups. Try to color a shape in a way so that it has *only* a 0 turn and a $\frac{1}{4}$ turn.



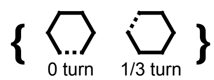
It's impossible without also giving the shape a $\frac{2}{4}$ turn and a $\frac{3}{4}$ turn: **{0 turn, $\frac{1}{4}$ turn }** is not a group, but **{0 turn, $\frac{1}{4}$ turn, $\frac{2}{4}$ turn, $\frac{3}{4}$ turn }** is.

Why? This brings us back to combining rotations.



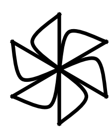
In order for a set of rotations to be a group, any combination of rotations in the set must also be in the set. This rule is called <u>group closure</u>, and you can see it.

Take the set **{0 turn, $\frac{1}{3}$ turn}**.



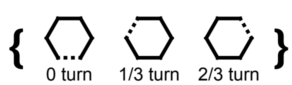
This is not a group because it's missing the $\frac{2}{3}$ turn, which is created by combining a $\frac{1}{3}$ with another $\frac{1}{3}$ turn.

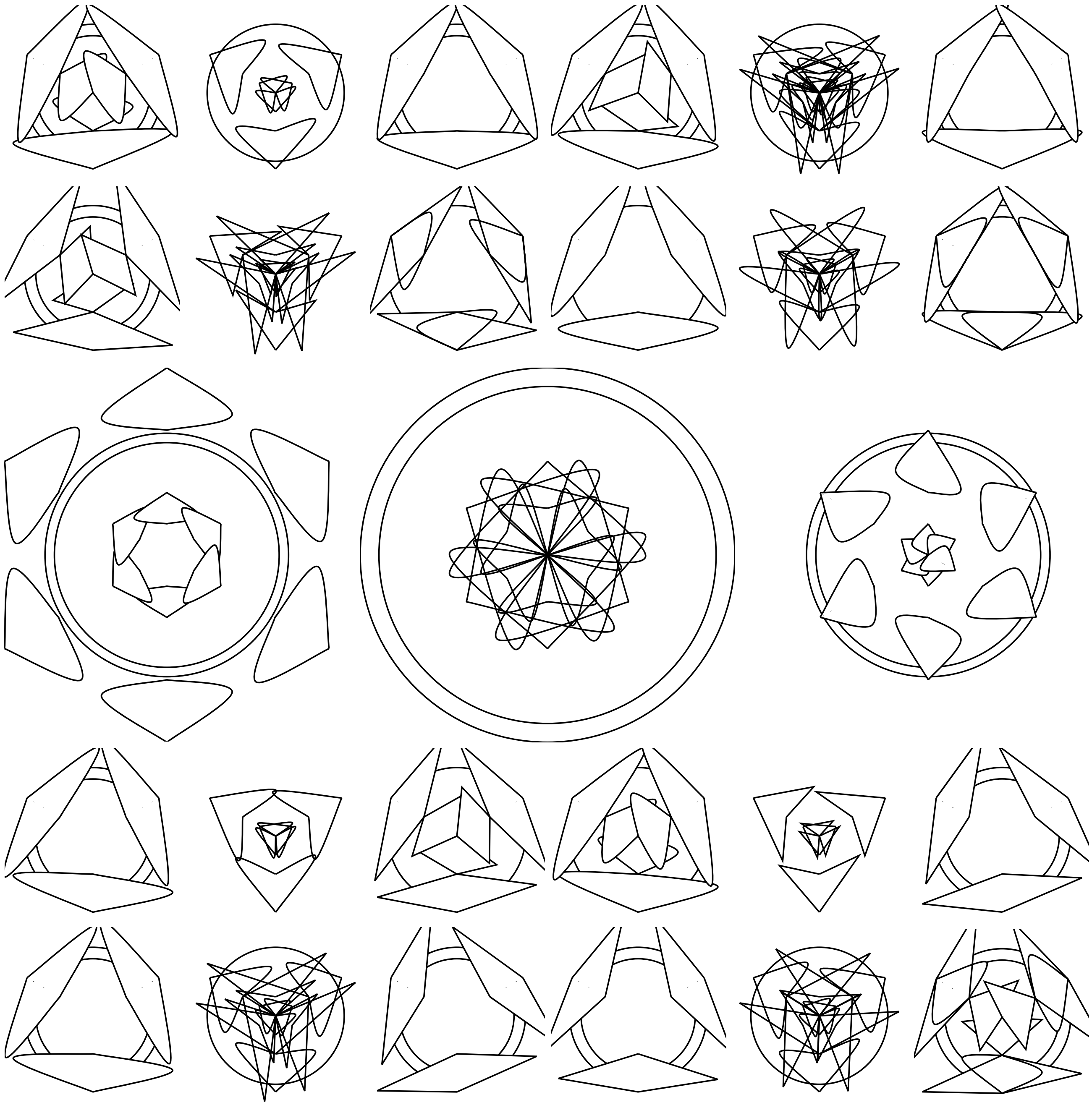
Try to color a shape to have the **{0 turn, $\frac{1}{3}$ turn}** rotations but not a $\frac{2}{3}$ **turn**.



It cannot be done.

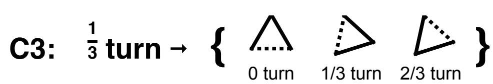
Adding the $\frac{2}{3}$ **turn** to the set gives us our C3 group again.



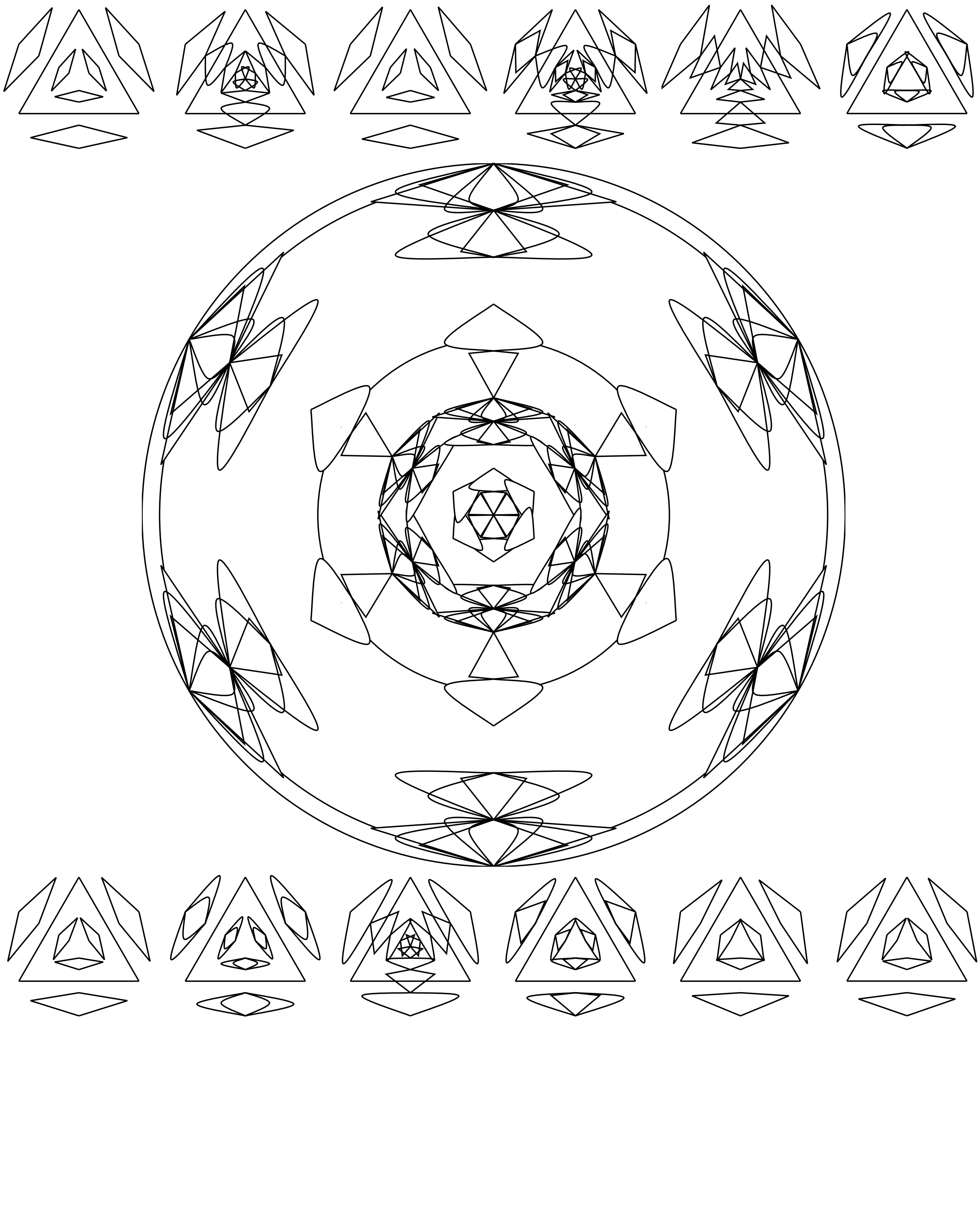


So far we've only been talking about groups of rotations.

These groups are cyclic. They can be created by combining just one rotation - the generator - multiple times with itself.



For our next groups, we have more generators, such as reflections.



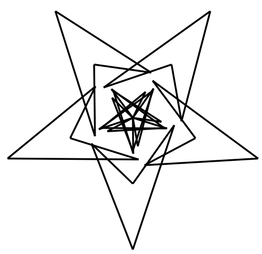
More is coming soon.

About

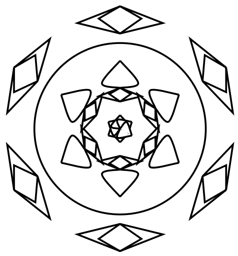
This is a coloring book that is both digital and on paper.

By coloring on paper, readers can explore symmetry and the beauty of math. Follow along on the digital copy to bring the concepts and illustrations to life in interactive animations.

Printed copy: *Coming soon*
Digital copy: coloring-book.co



The illustrations in this book are designed by algorithms that follow the symmetry rules of the groups that the illustrations belong to. These algorithms also use generative art techniques to add components of randomness - notice the illustrations never repeat.



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