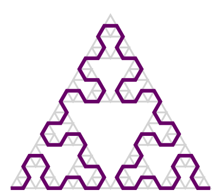
Illustrating Group Theory

A Coloring Book

Math is about more than just numbers. In this "book" the story of math is visual, told in shapes and patterns.



Group theory is a mathematical study with which we can explore symmetry.

ABOUT

This coloring book is both digital and on paper.

The paper copy is where the coloring is done - color through the concepts to explore symmetry and the beauty of math.

The digital copy brings the concepts and illustrations to life in interactive animations.

Print a copy: coloring-book.co/book.pdf
Digital copy: coloring-book.co

The illustrations in this book are drawn by algorithms that follow the symmetry rules of the groups each illustration represents. These algorithms use generative art techniques to add components of randomness - notice each illustration is unique.

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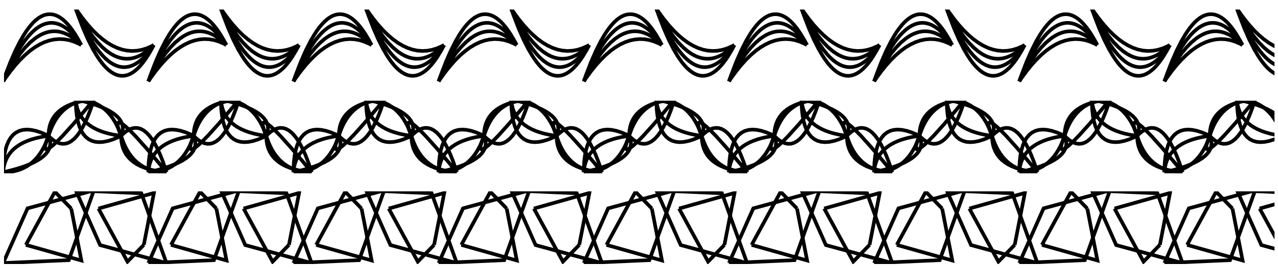
ROAD MAP

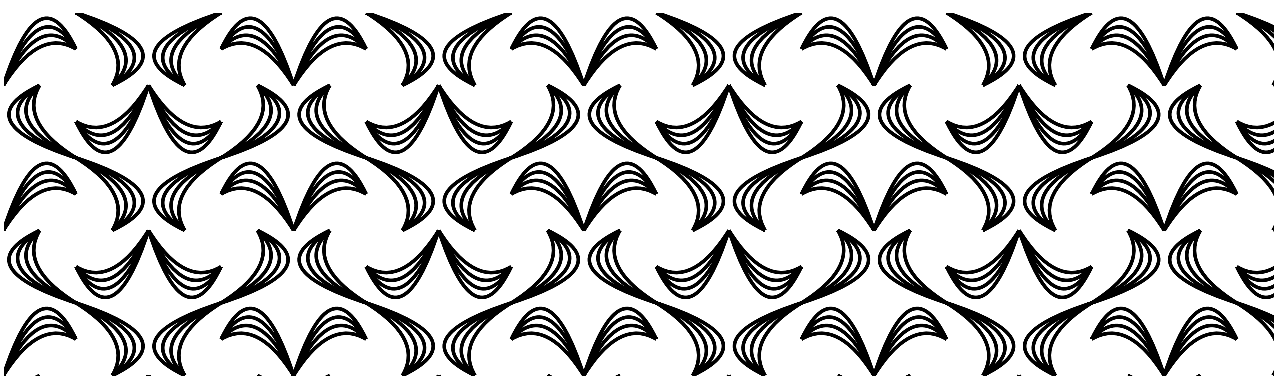
We'll color through the concepts of groups,

such as the FRIEZE PATTERNS

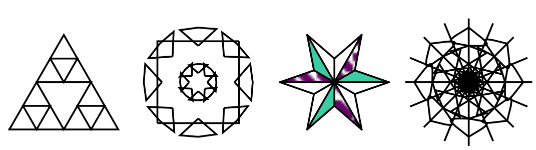


They start with a single shape that transforms and then repeats forever in opposite directions.

WALLPAPER PATTERNS repeat infinitely in even more directions.



But first we'll start with the basics of SHAPES & SYMMETRIES.



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SHAPES & SYMMETRIES

Symmetry presents itself in nature,

Landscape reflection in water

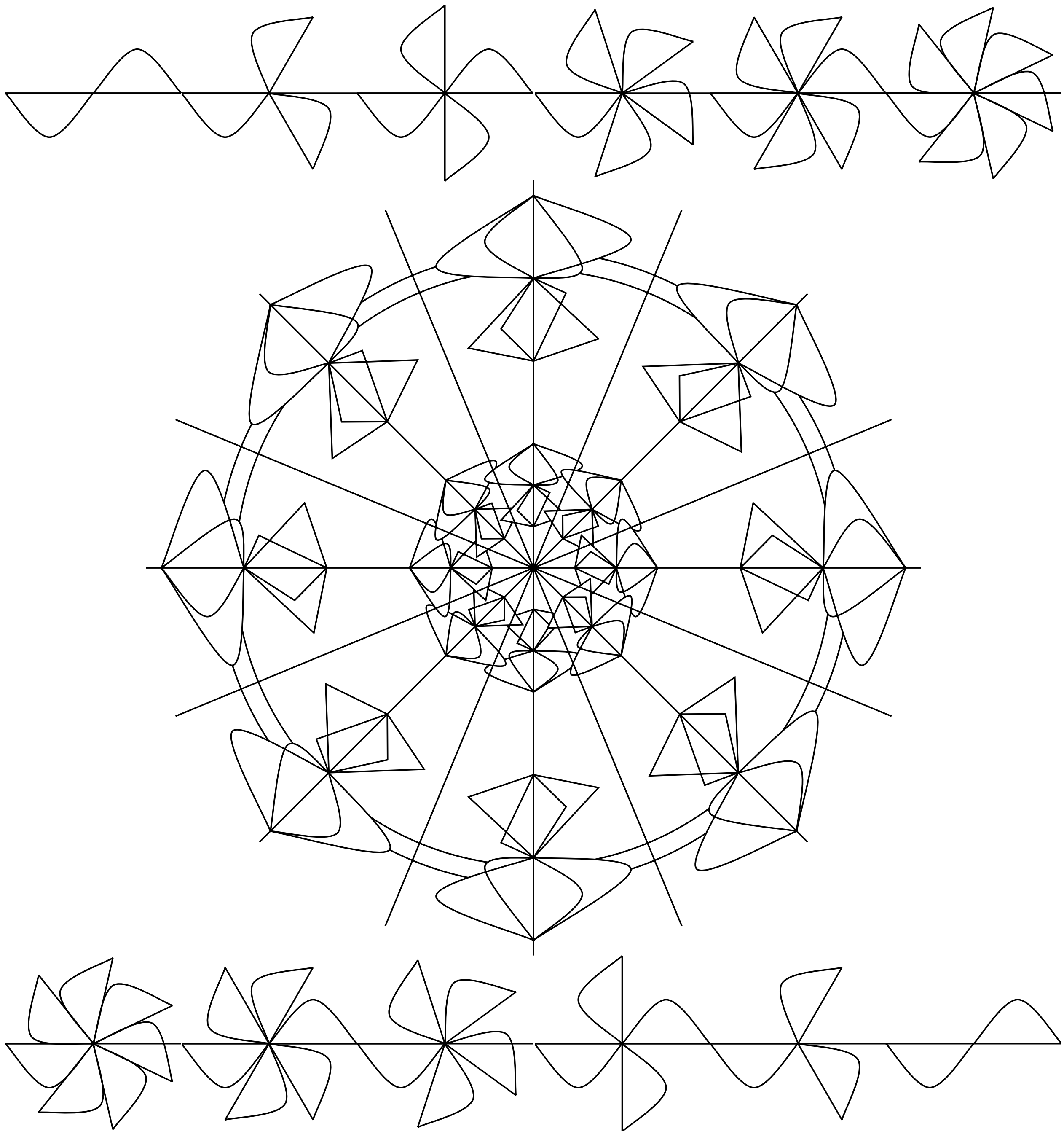
But often with imperfections.

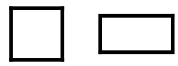
Moth

Acorn

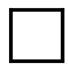
Starfish

Math creates a space where perfect symmetry can be considered.

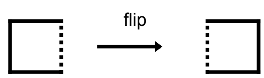




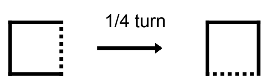
Some shapes have more symmetry than others.



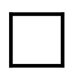
If while you blinked, a square was flipped,



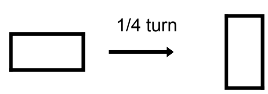
or turned a quarter of the way around,



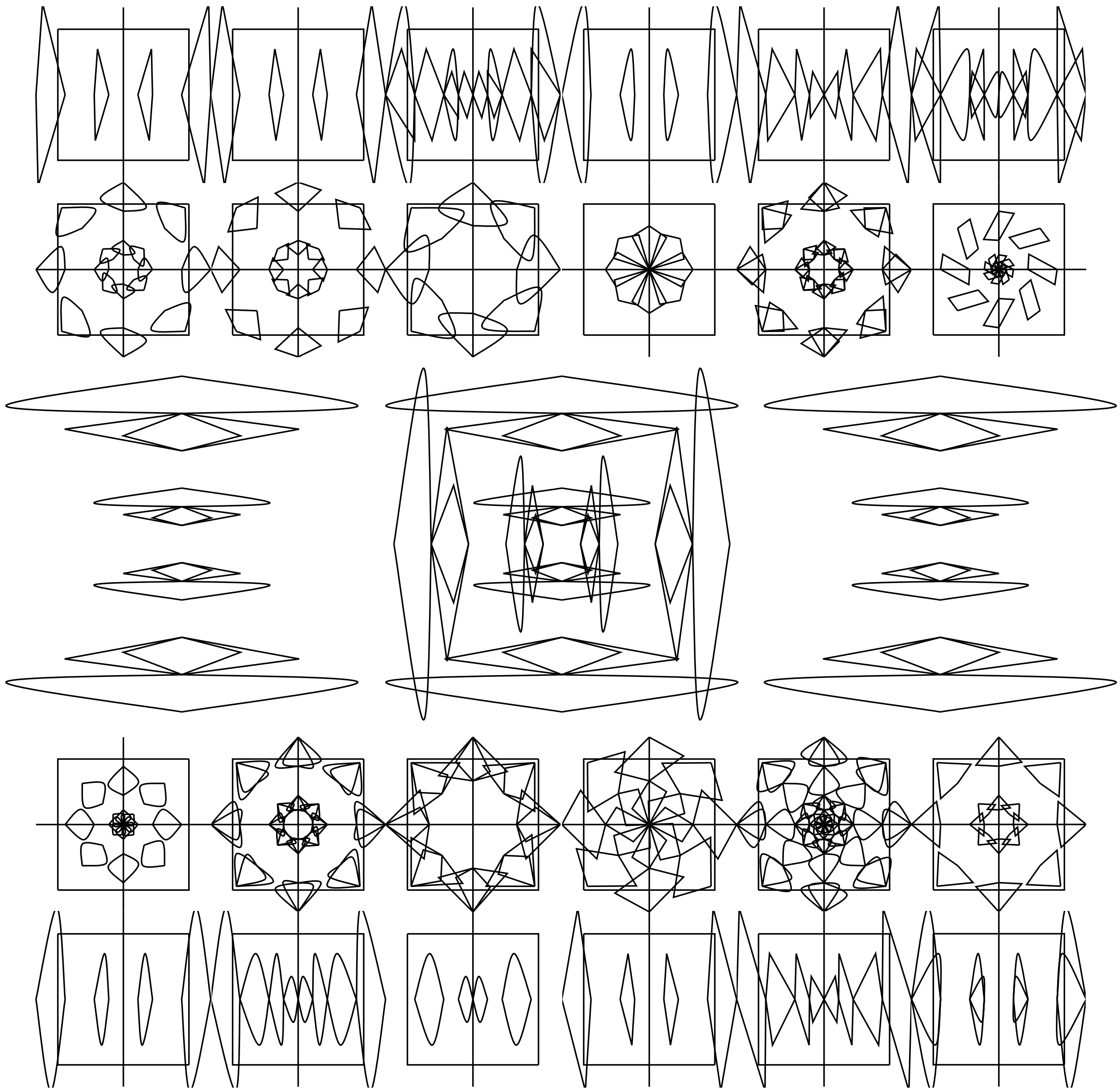
you would then still see the same square and not know.



This is not the case for a rectangle…

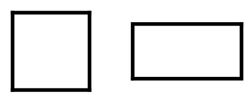


Challenge: Which of these shapes can be rotated by a $\frac{1}{4}$ *turn without changing in appearance?*

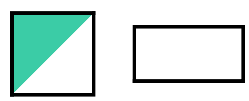


shapes with $\frac{1}{2}$ turns and shapes with $\frac{1}{4}$ turns

Our intuitive ideas of symmetry let us see that a square is "more symmetric" than a rectangle because it can be flipped and turned in more ways.

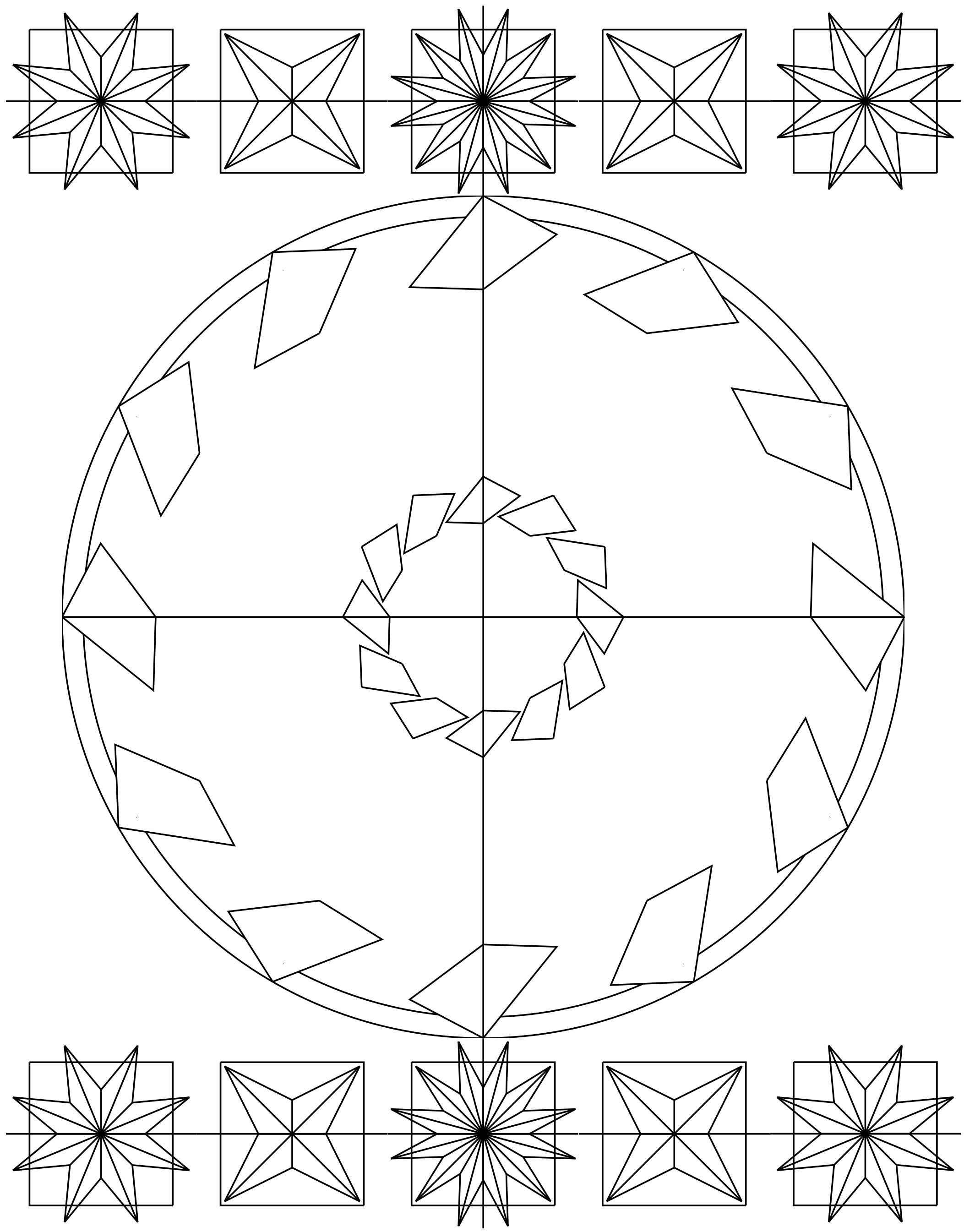


But this can change once color is added...



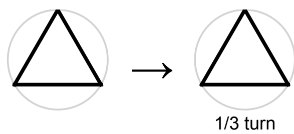
With color we will explore the world of perfect lines and symmetry.

Coloring Challenge: Can you color the shapes to make them "less symmetric"?

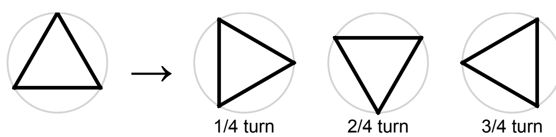


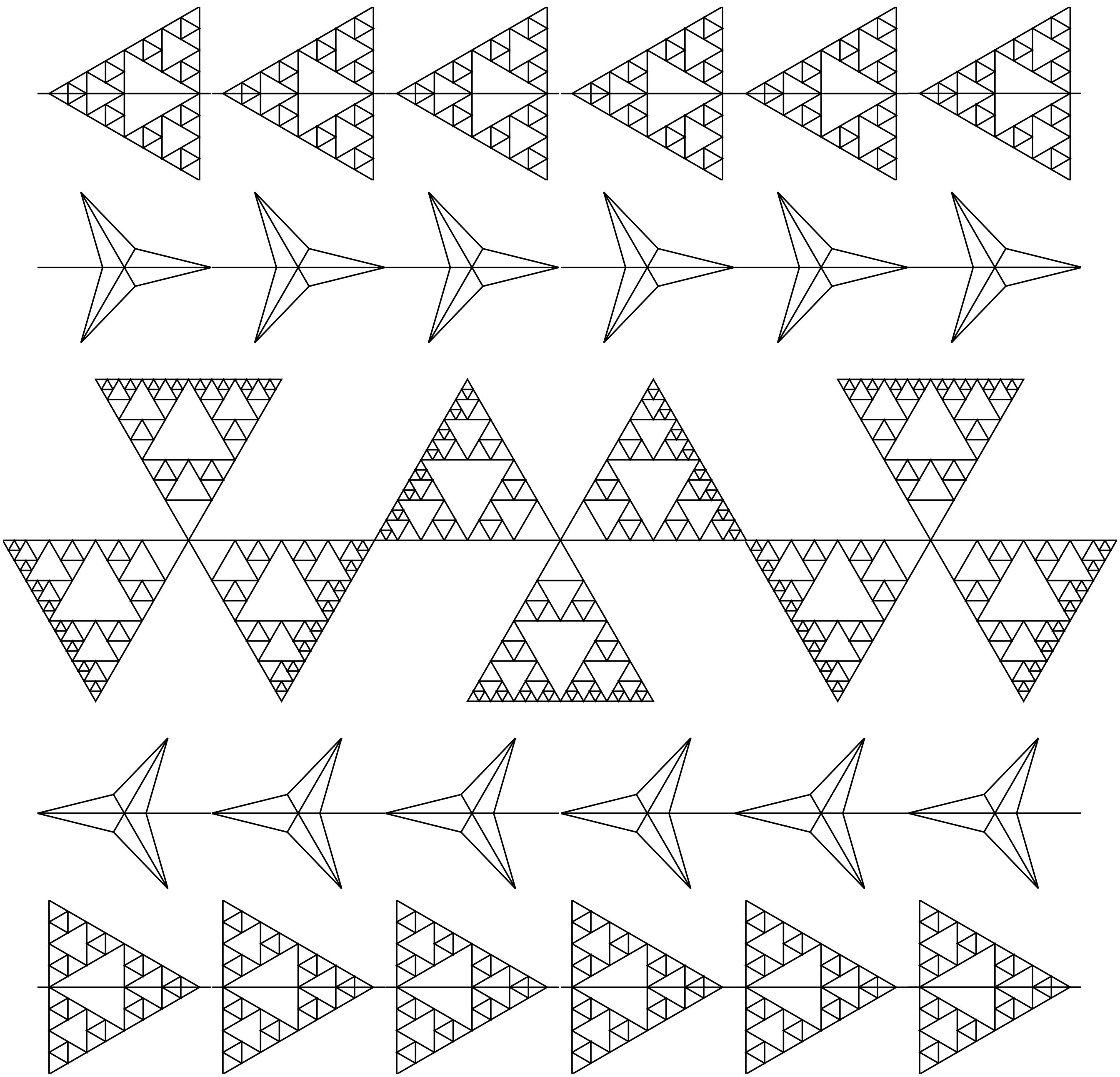
ROTATIONS

A regular triangle can be rotated $\frac{1}{3}$ of the way around a circle and appear unchanged.



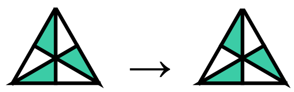
If the triangle is instead rotated by an arbitrary amount, like $\frac{1}{4}$ of the way around a circle, it will then appear changed, since it is oriented differently.





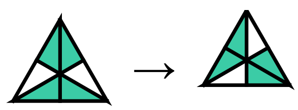
sierpinski triangles

We can find ways to color the triangle so that a $\frac{1}{3}$ turn still does not change it.

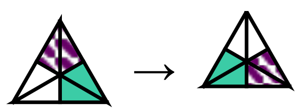


1/3 turn

While this will not work for other ways.

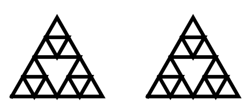


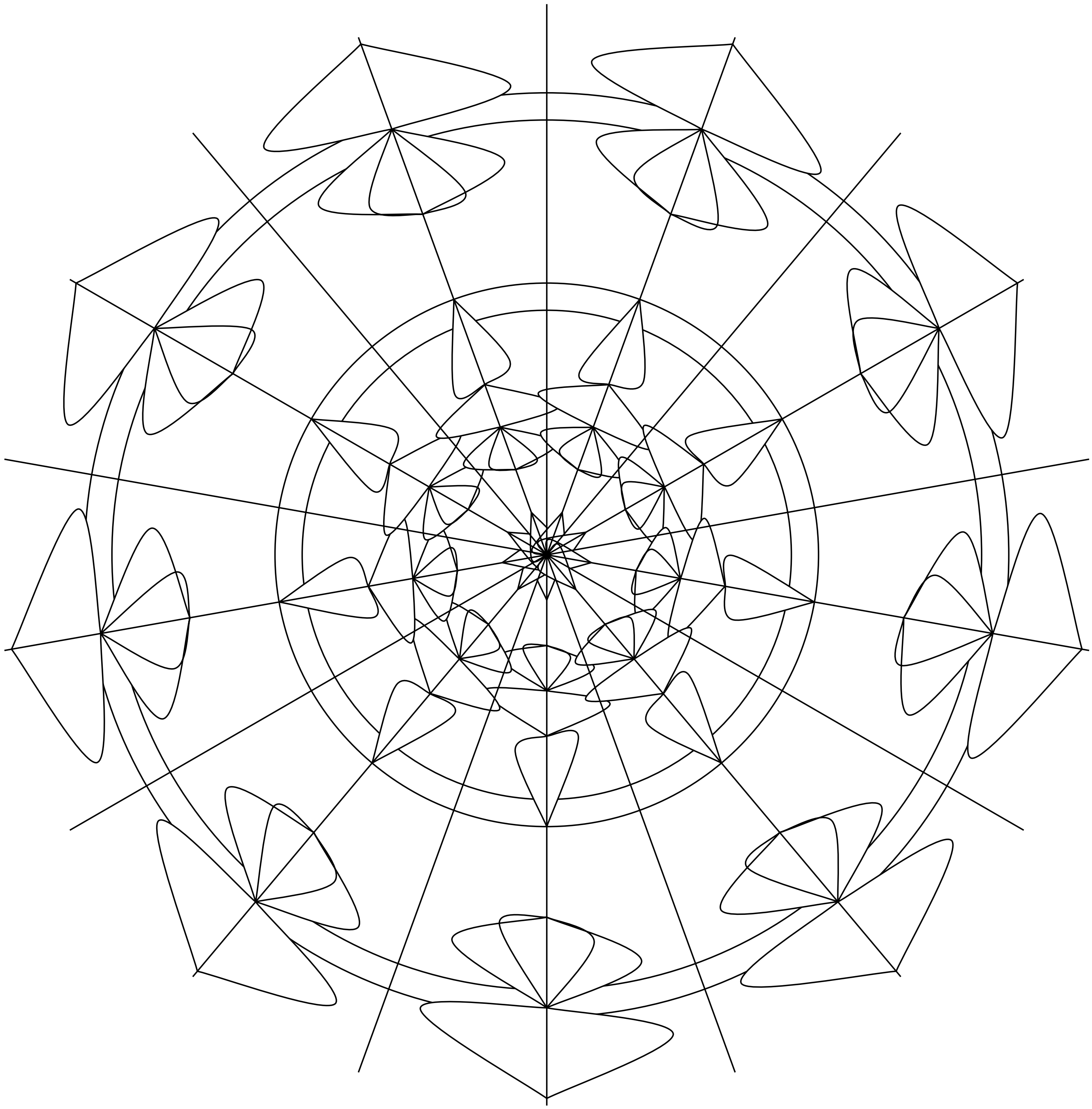
1/3 turn



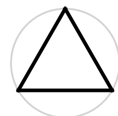
1/3 turn

Coloring Challenge: Can you color the shapes so that a $\frac{1}{3}$ *turn continues to leave their appearance unchanged?*



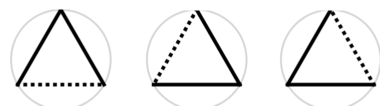


circular pattern of order 9

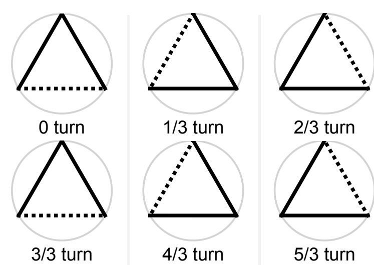


Our triangle can also rotate by more than a $\frac{1}{3}$ turn without changing.

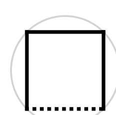
It can rotate by twice that much - $\frac{2}{3}$ of the way around the circle - or by 3 times that much, which is all the way around the circle.

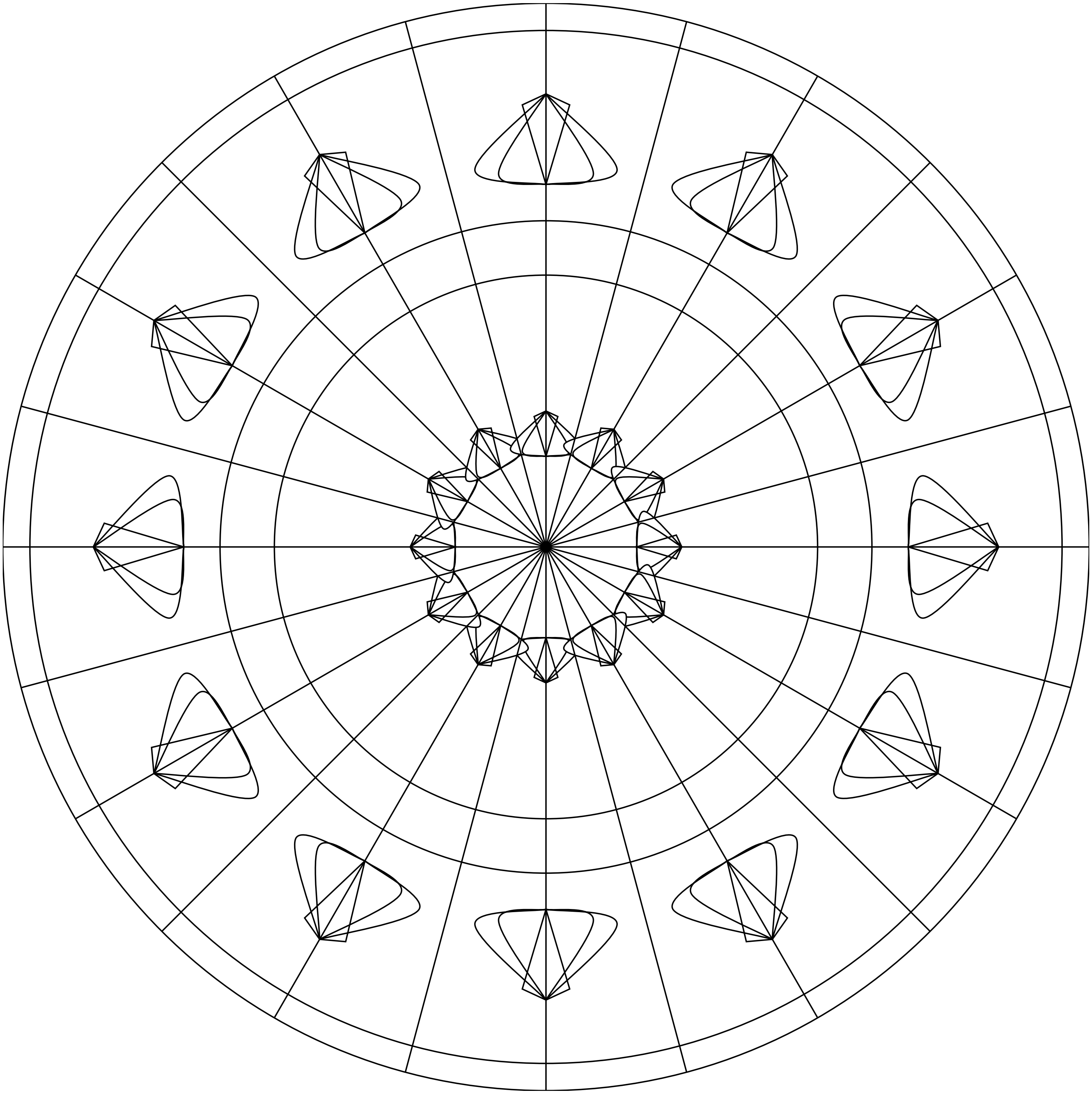


We can keep rotating - by 4 times that much, 5 times that much, 6 times... and keep going. The triangle seems to have an infinite number of rotations, but after 3 they become repetitive.



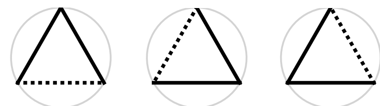
Challenge: How many ways can a square rotate without changing before the ways become repetitive?





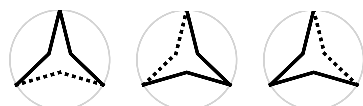
circular pattern of order 12

The triangle has only 3 unique rotations. We'll talk about rotations that are less than a full turn.

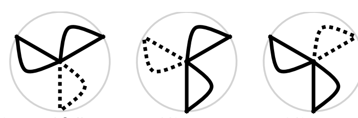


0 turn / full turn   1/3 turn   2/3 turn

Other shapes have these same 3 rotations. For this reason, we can say they all share the same symmetry group.

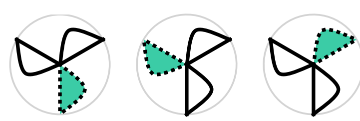


0 turn / full turn   1/3 turn   2/3 turn



0 turn / full turn   1/3 turn   2/3 turn

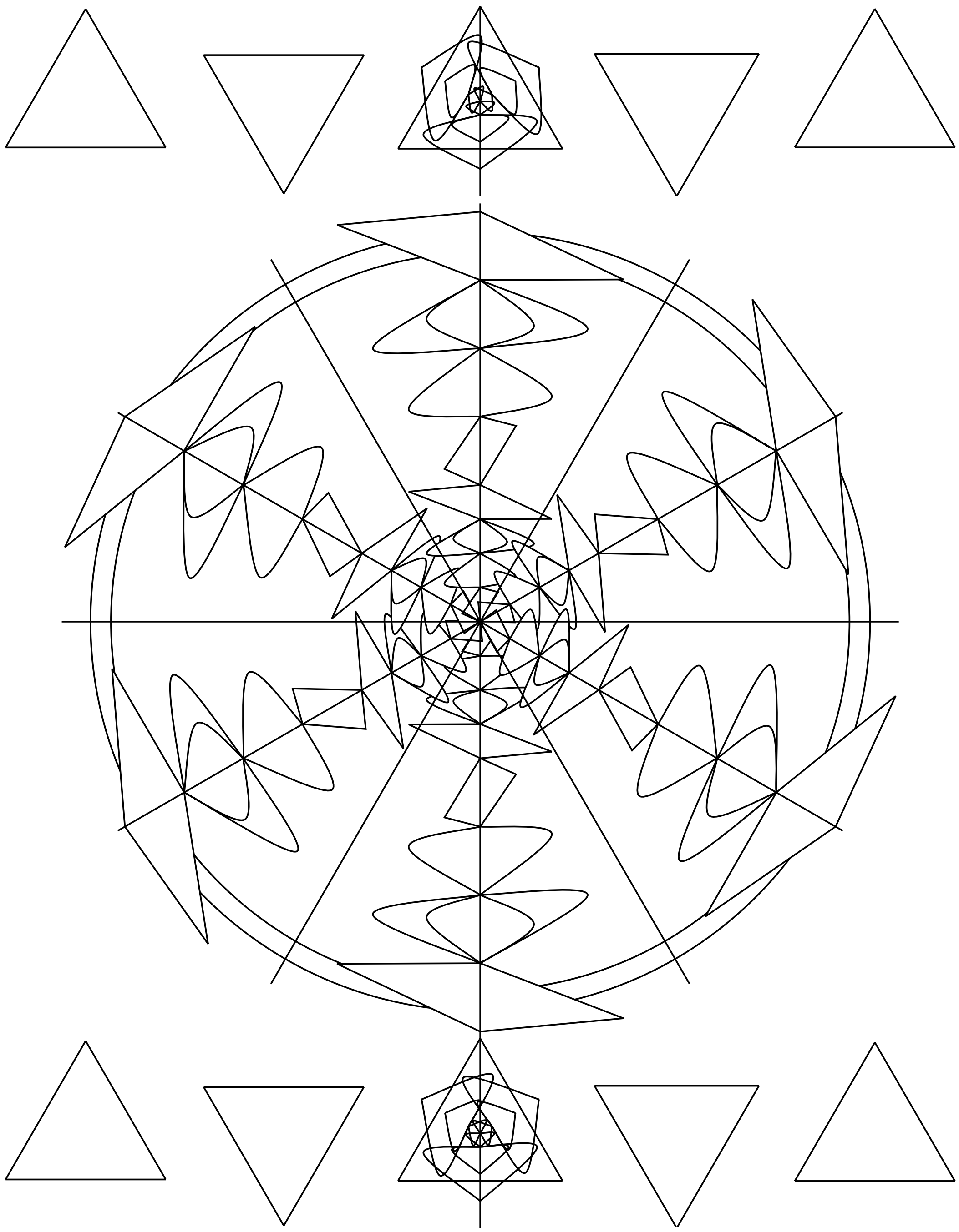
However, their rotational symmetry can be removed by adding color.



0 turn / full turn   1/3 turn   2/3 turn

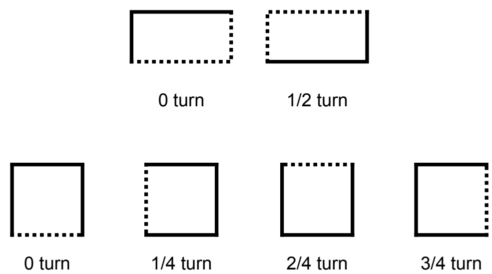
Now when our shape is rotated, its color shows it.

Coloring Challenge: Can you color the shapes to remove their rotational symmetry?



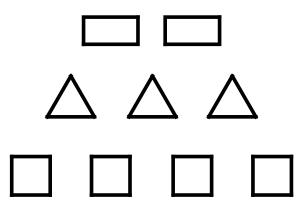
shapes with 3 rotations, and a shape with 6 rotations

Now that we can count rotations, we can be more precise when we say a square has more symmetry than a rectangle.



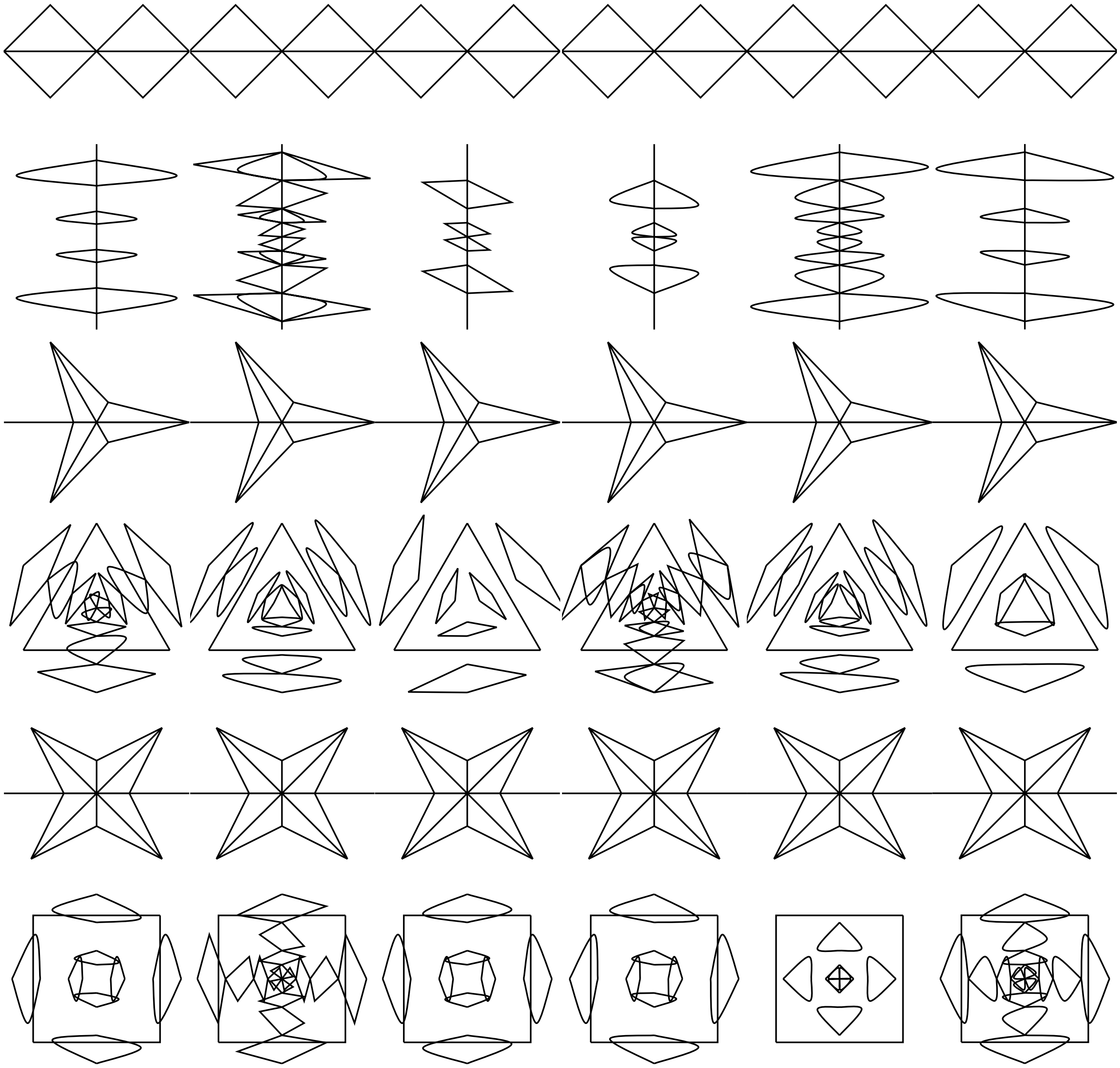
We can also see that a square has more rotational symmetry than a triangle, which in turn has more than a rectangle:

A rectangle has only 2 unique rotations, while a triangle has 3, and a square has 4.



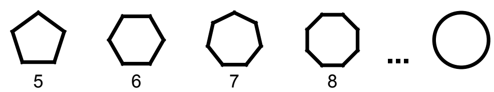
Challenge: Can you find all the shapes with 4 rotations?

Coloring challenge: Color the shapes with 4 rotations so that they have only 2 rotations.

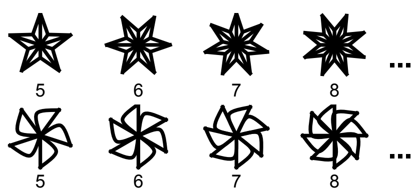


shapes with 2, 3, 4 rotations

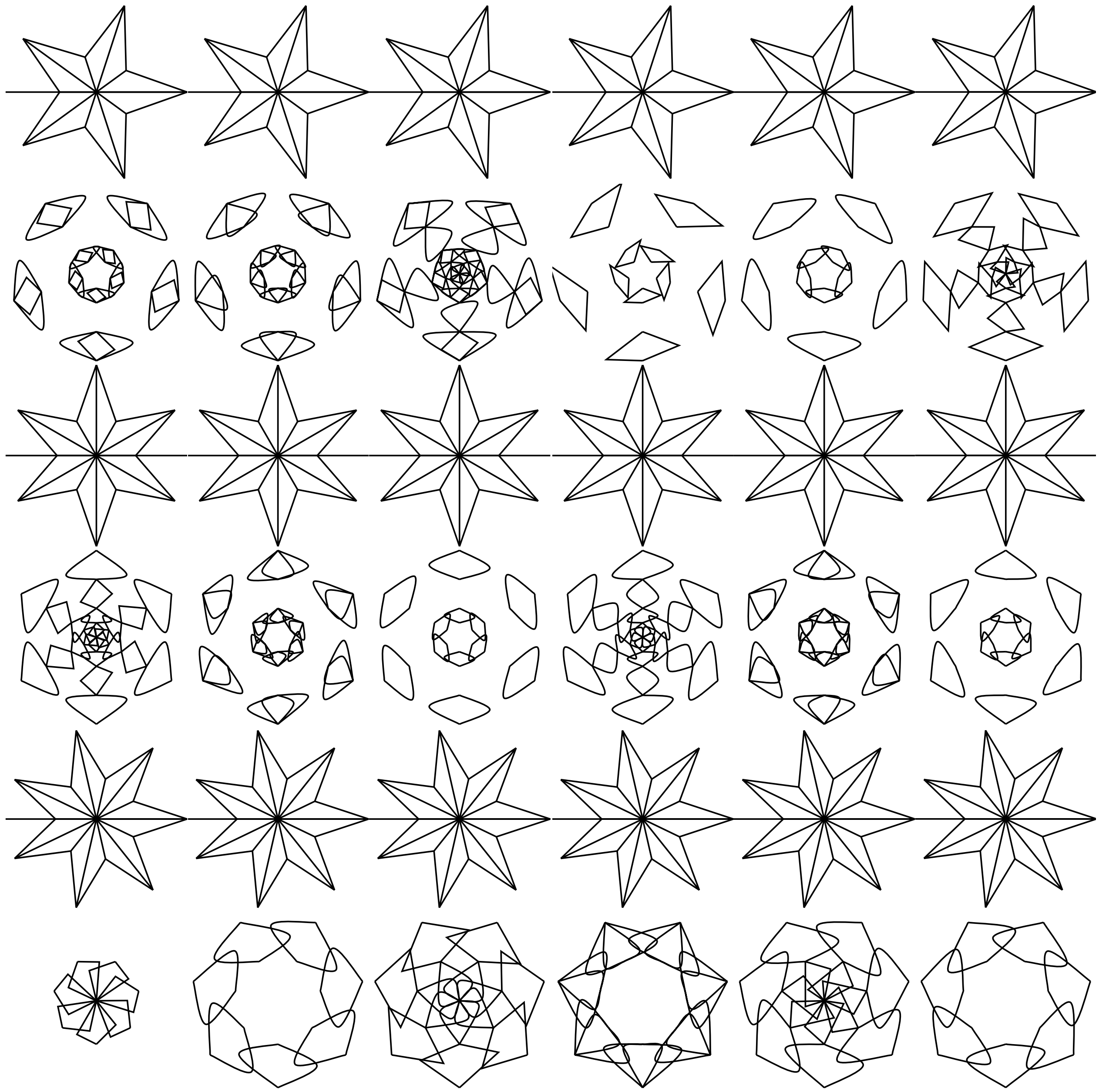
We don't need to stop at 4 rotations. We can find shapes with 5 rotations, 6 rotations, 7, 8, ... and keep going towards infinity.



These shapes don't need to be so simple.



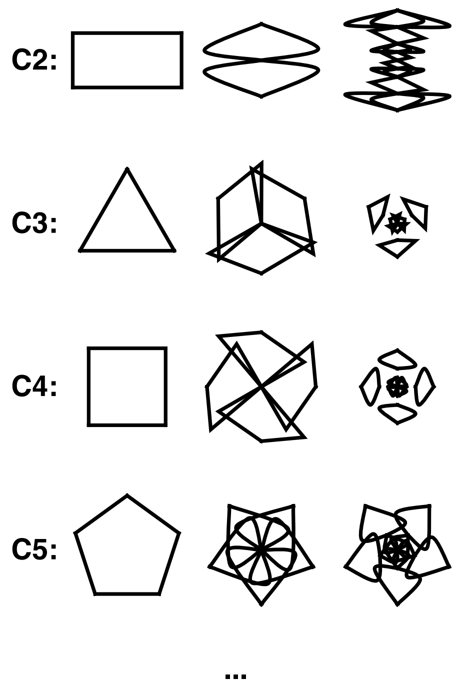
Challenge: Can you find all of the shapes with 7 rotations?



shapes with 5, 6, 7 rotations

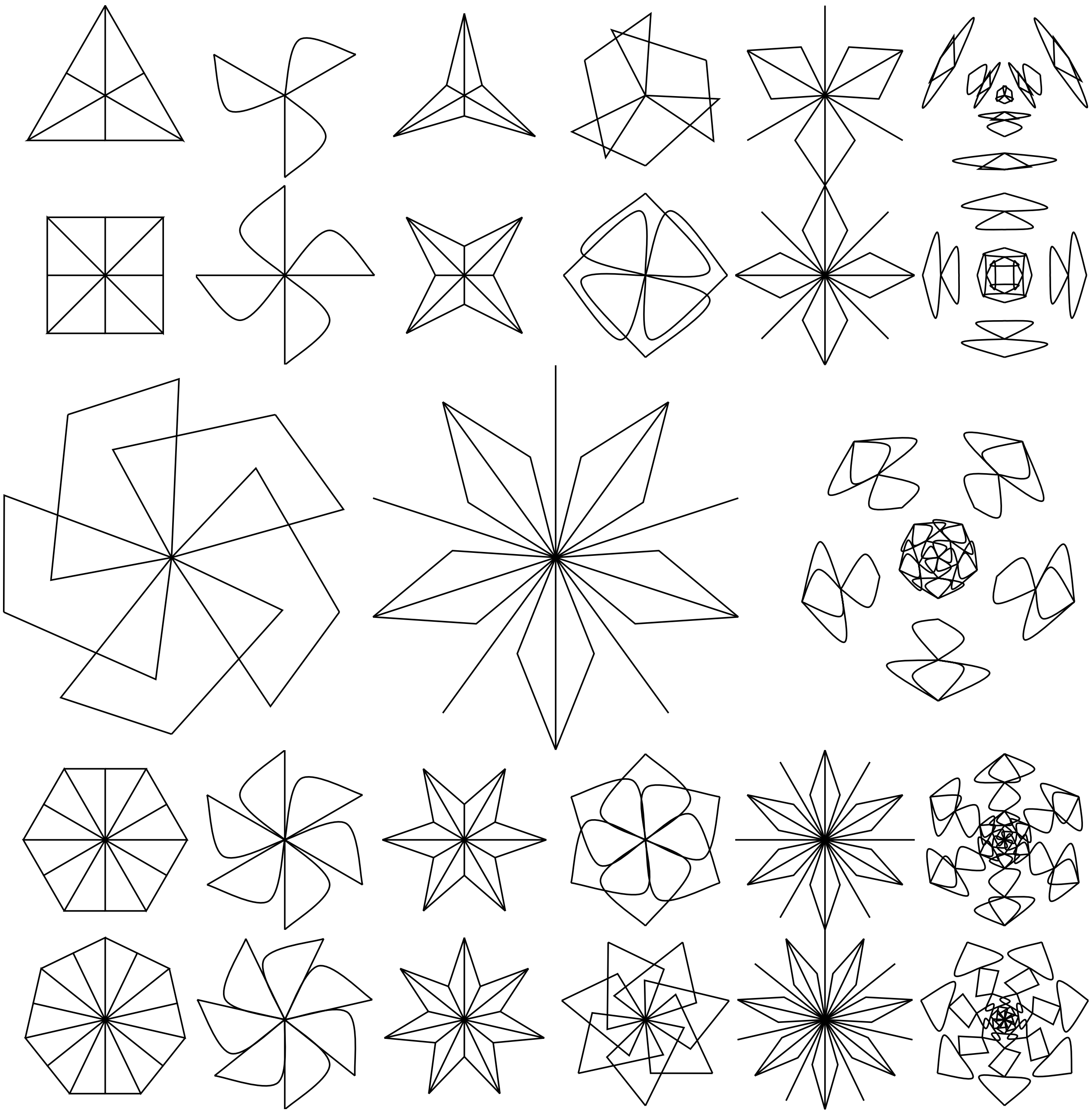
When shapes have the same number of rotations, they share a symmetry group.

We can call the group with 2 rotations C2, and call the the group with 3 rotations C3, call the group with 4 rotations C4, and so on...



These groups are called the cyclic groups.

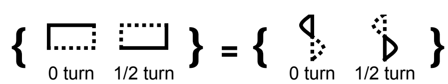
Challenge: Can you find all of the C5 and C6 shapes?



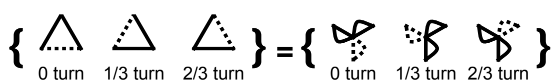
C3, C4, C5, C6, C7 shapes

Our shapes help us see our groups, but the members of the groups are the rotations, not the shapes.

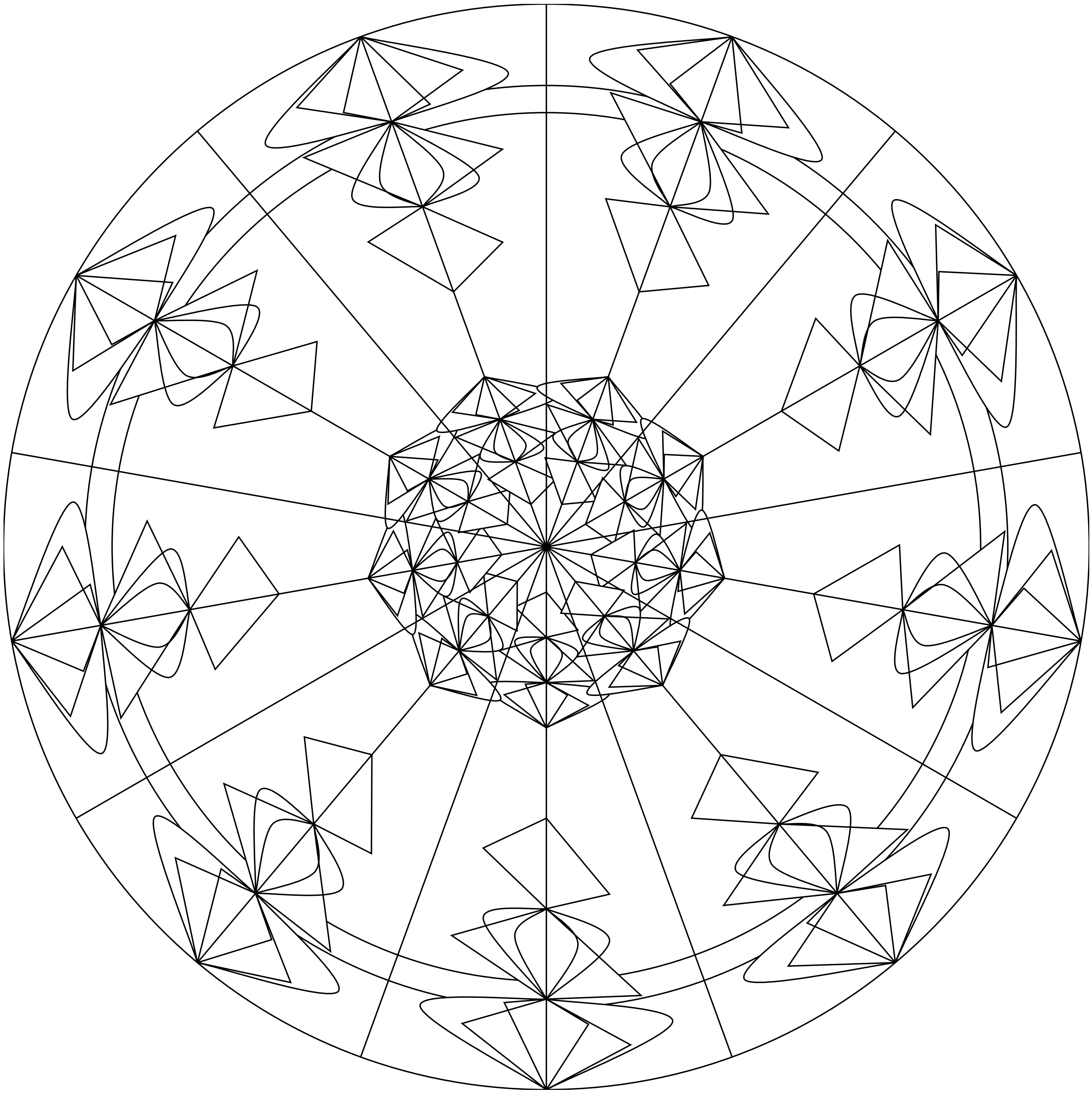
C2:



C3:

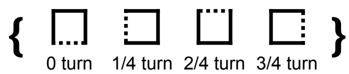


The rotations within each group are related to each other…



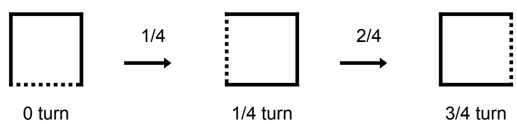
C9 shape (circular pattern)

C4:



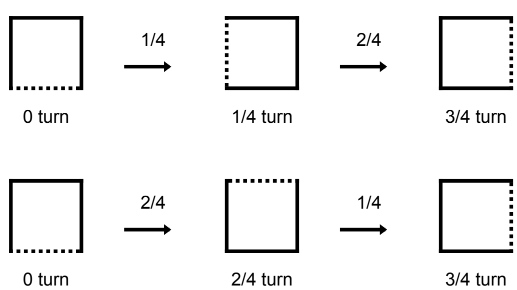
Another way to think about rotating a C4 shape by a $\frac{3}{4}$ turn is to rotate it by a $\frac{1}{4}$ turn and then rotate it again by a $\frac{2}{4}$ turn.

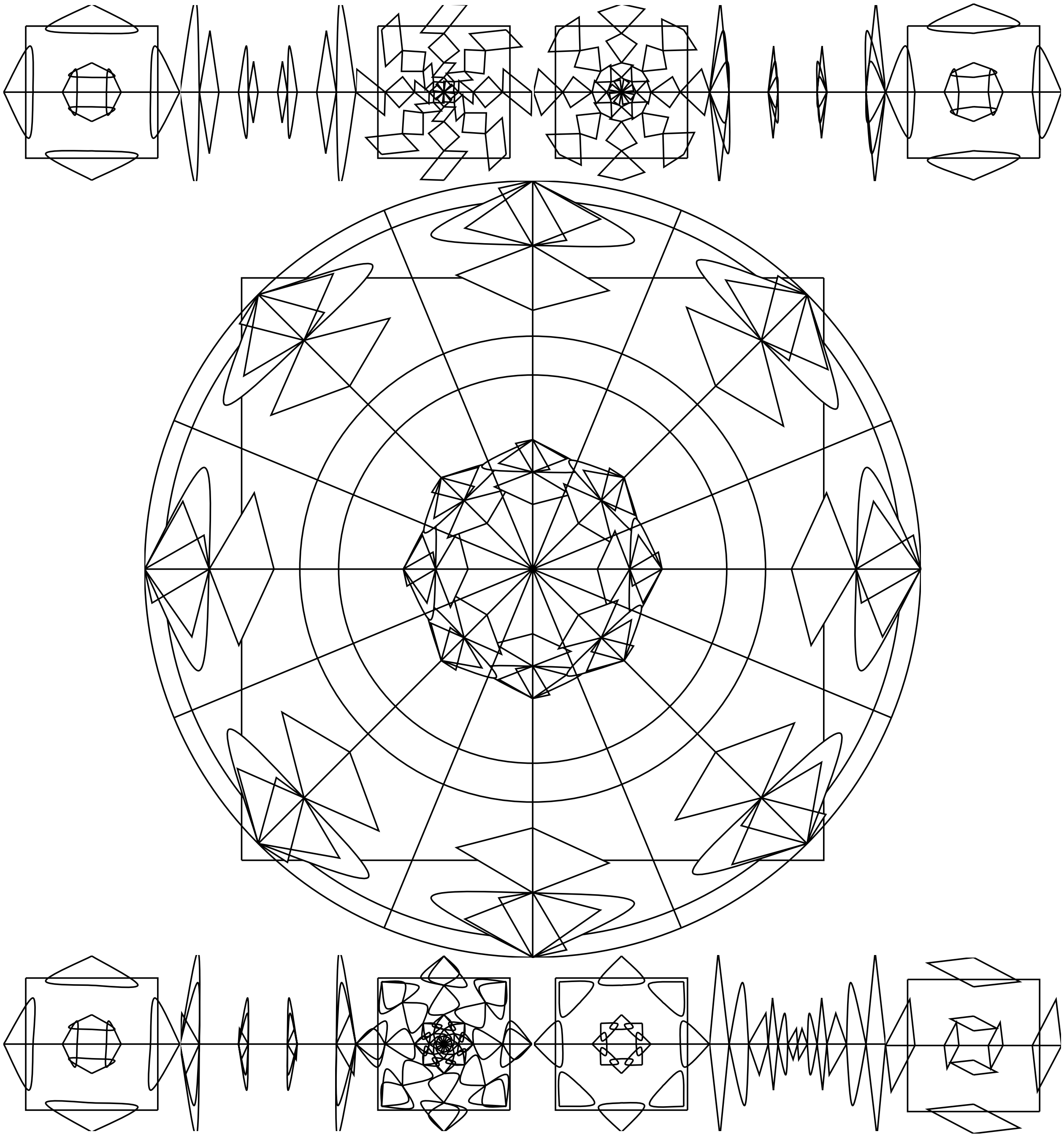
$$\textbf{C4:} \quad \tfrac{1}{4}\ \textbf{turn} * \tfrac{2}{4}\ \textbf{turn} \rightarrow \tfrac{3}{4}\ \textbf{turn}$$



Notice that the order in which these rotations are combined does not matter. The cyclic groups are <u>commutative</u>.

$$\textbf{C4:} \quad \tfrac{1}{4}\ \textbf{turn} * \tfrac{2}{4}\ \textbf{turn} = \tfrac{2}{4}\ \textbf{turn} * \tfrac{1}{4}\ \textbf{turn}$$





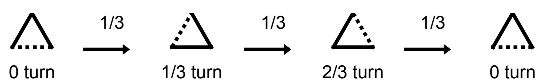
C2 and C4 shapes

Similarly, for our C3 group, a $\frac{2}{3}$ turn is the same as combining a $\frac{1}{3}$ turn with another $\frac{1}{3}$ turn.

$$\textbf{C3:}\qquad \tfrac{1}{3}\textbf{ turn} \ast \tfrac{1}{3}\textbf{ turn} \rightarrow \tfrac{2}{3}\textbf{ turn}$$

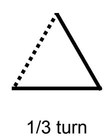
Adding another $\frac{1}{3}$ turn brings the shape back to its starting position - the 0 turn.

$$\textbf{C3:}\qquad \tfrac{1}{3}\textbf{ turn} \ast \tfrac{1}{3}\textbf{ turn} \ast \tfrac{1}{3}\textbf{ turn} \rightarrow \textbf{0 turn}$$

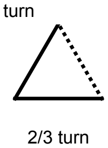


See, the $\frac{1}{3}$ turn can generate all of the rotations of C3 - it is a <u>generator</u> for our C3 group.

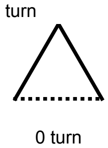
$$\tfrac{1}{3}\textbf{ turn} \longrightarrow$$

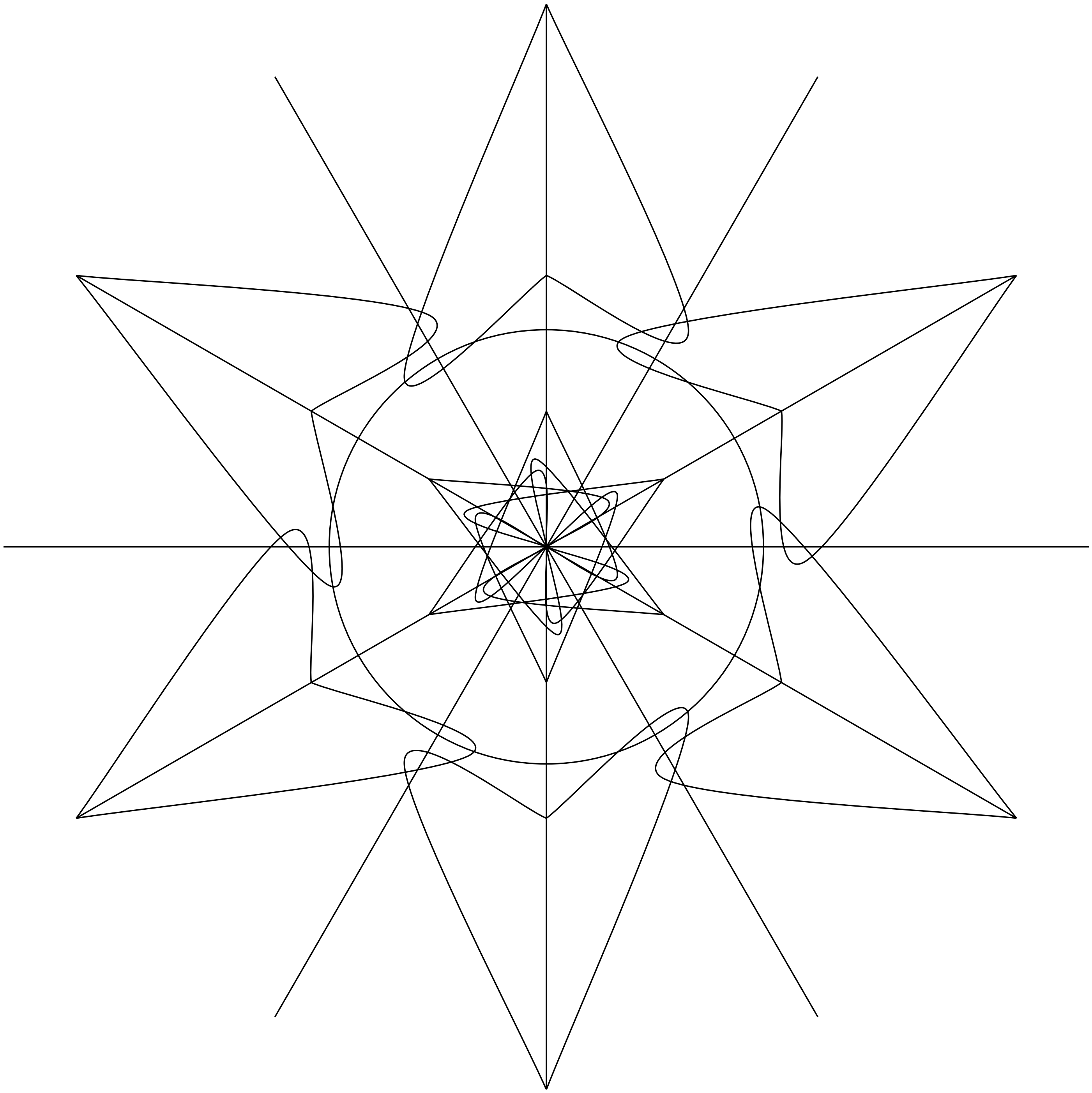


$$\tfrac{1}{3}\textbf{ turn} \ast \tfrac{1}{3}\textbf{ turn} \longrightarrow$$



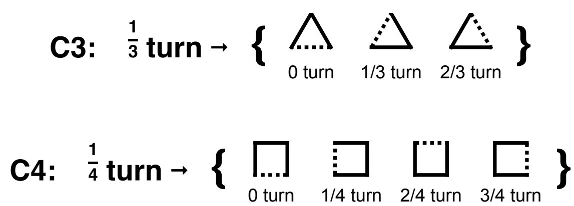
$$\tfrac{1}{3}\textbf{ turn} \ast \tfrac{1}{3}\textbf{ turn} \ast \tfrac{1}{3}\textbf{ turn} \longrightarrow$$



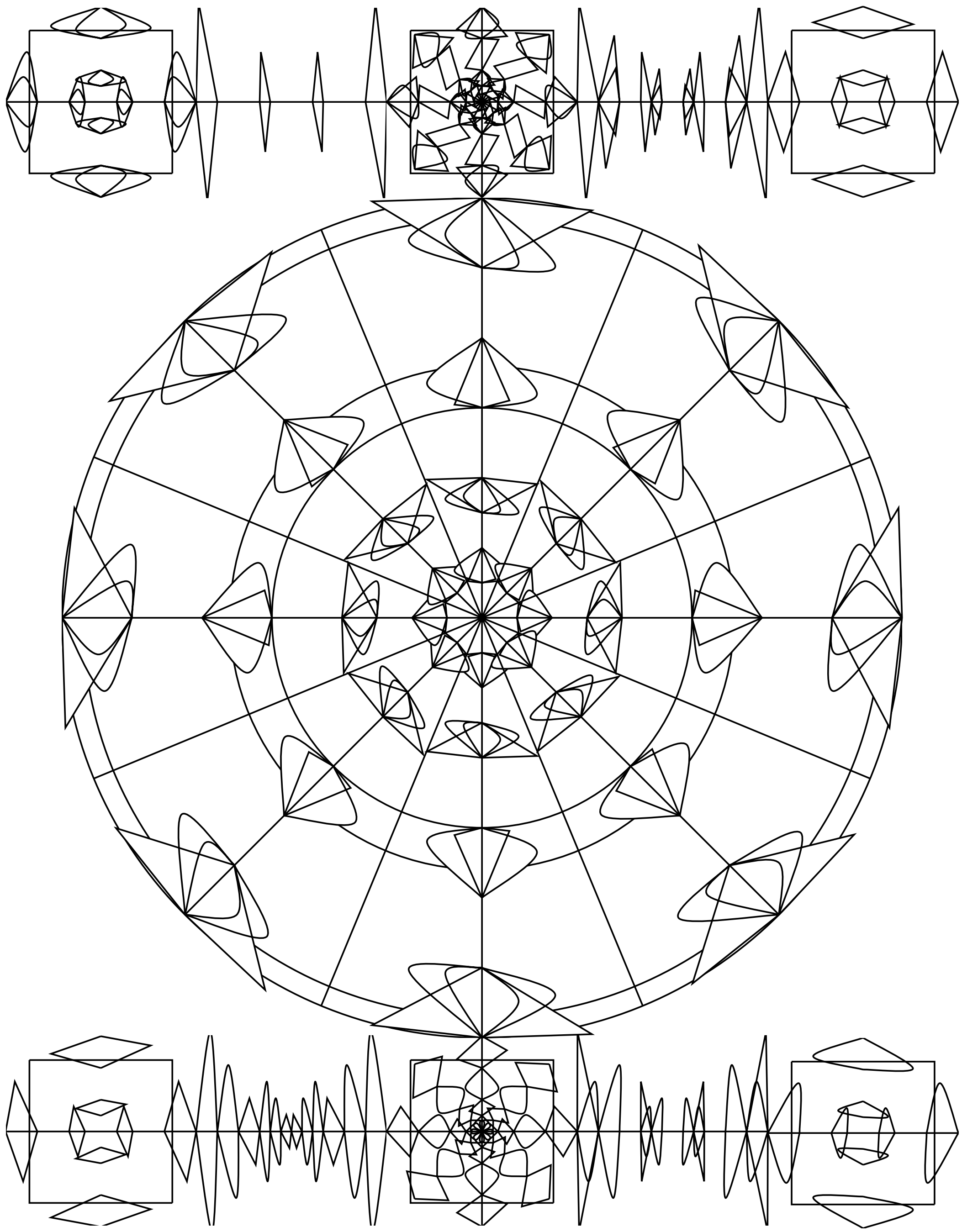


C6 design

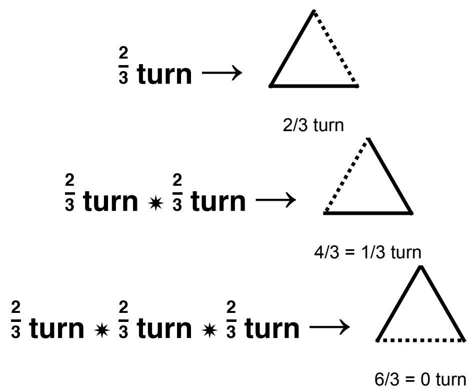
The $\frac{1}{3}$ turn is a generator for our C3 group, and similarly, the $\frac{1}{4}$ turn is a generator for our C4, because it can generate all of the rotations of our C4.



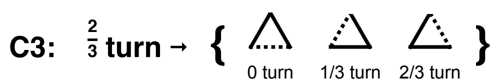
We could even choose different generators.



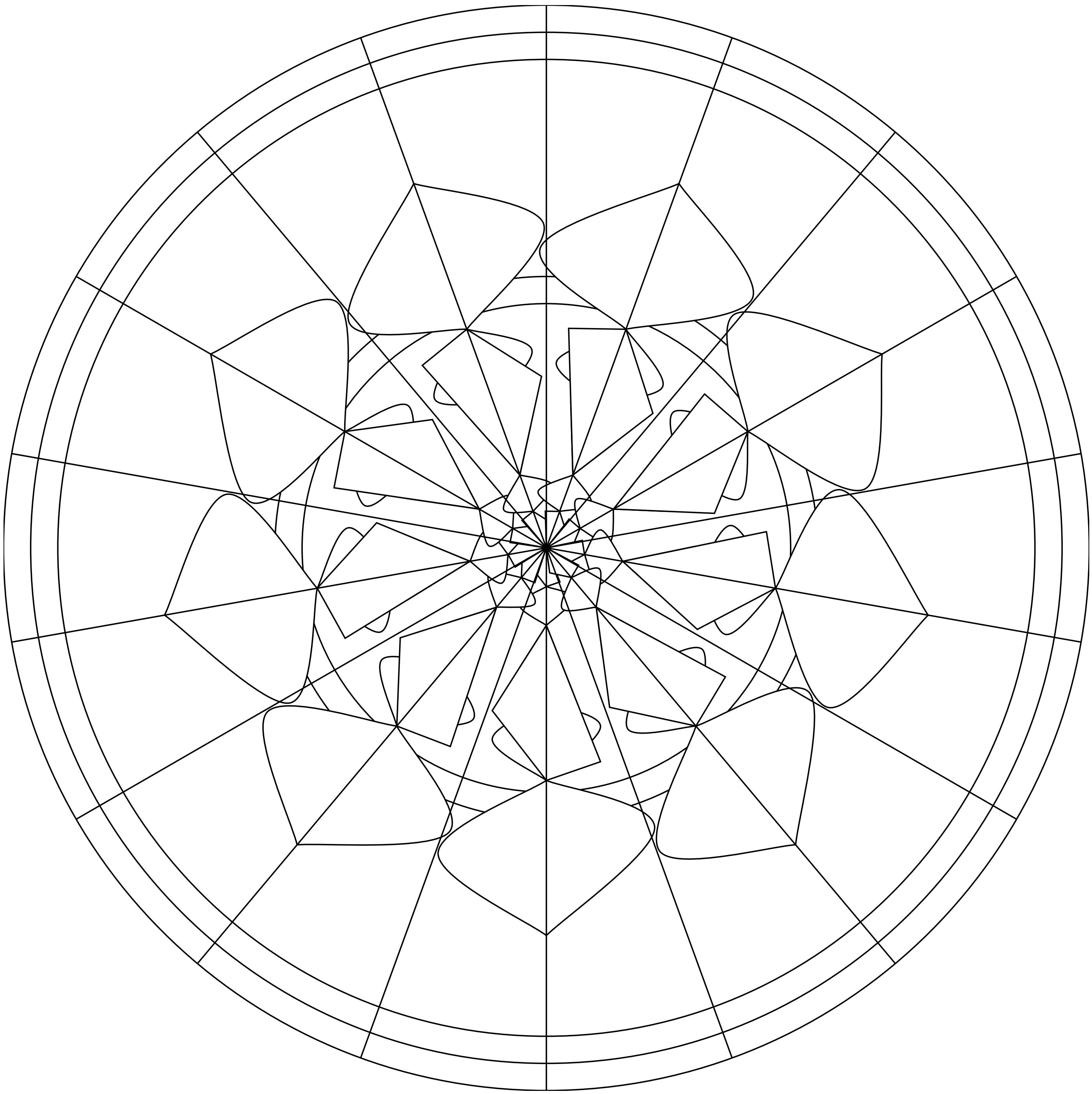
C2, C4, C8 shapes



We could have just as easily used a $\frac{2}{3}$ turn as our generator for C3 and ended up with the same result.

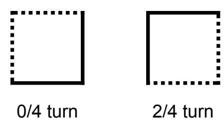


However, not all rotations are generators.

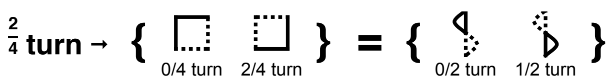


C9 shape (circular pattern)

A $\frac{2}{4}$ turn does not generate all of the rotations of our C4 group.



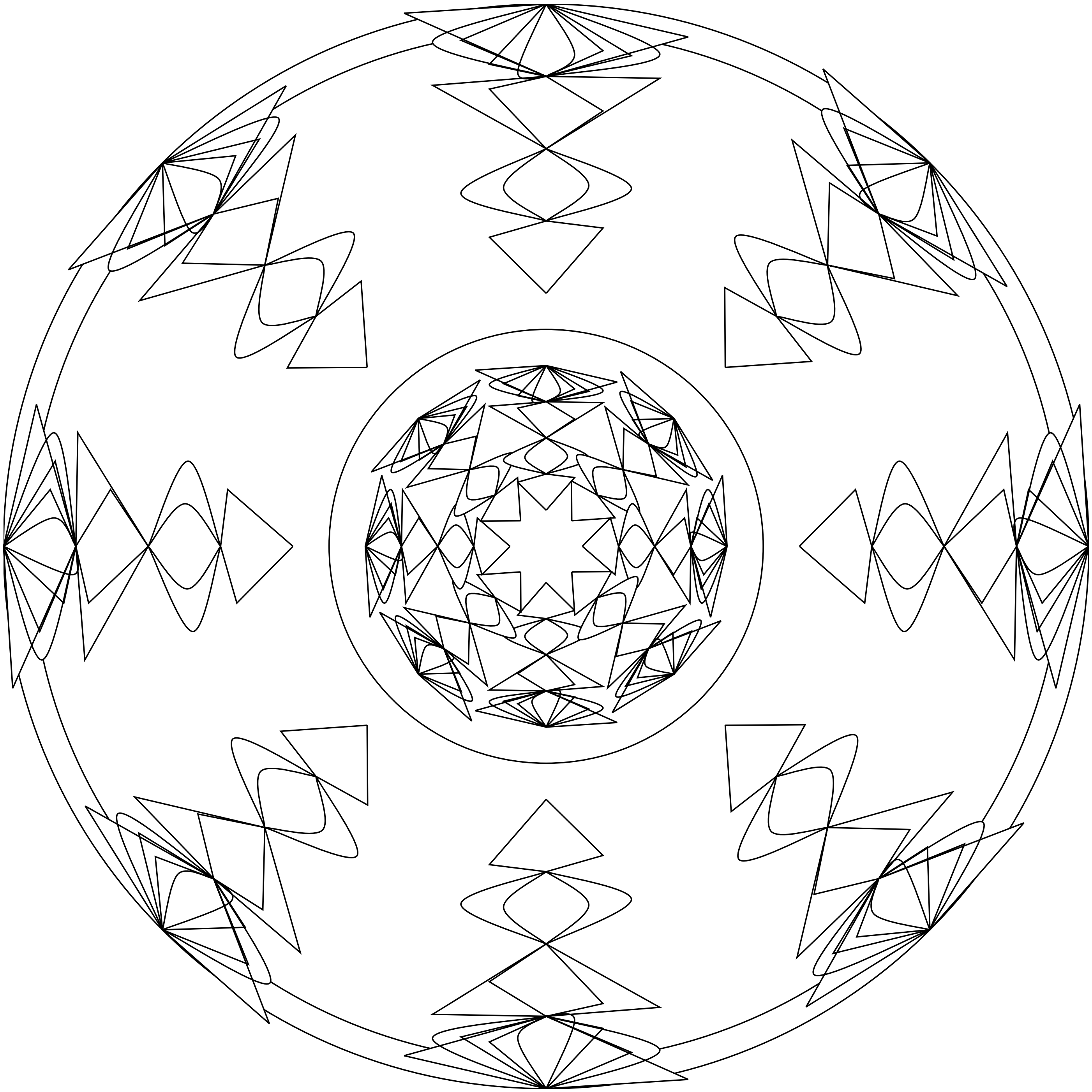
Instead a $\frac{2}{4}$ turn generates a smaller group -- our C2 group.



Another way to see this is with color...

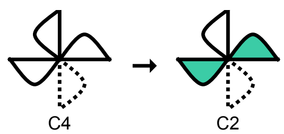
Challenge: Find all the generators for C4 and C8.

Challenge: Which rotations of C8 reduce C8 to C4 when used as generators?

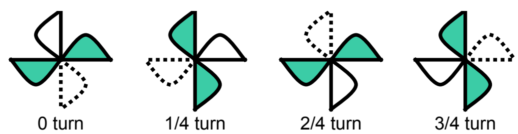


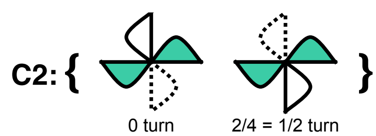
C8 shape (circular pattern)

We can transform a C4 shape into a C2 shape by coloring it.

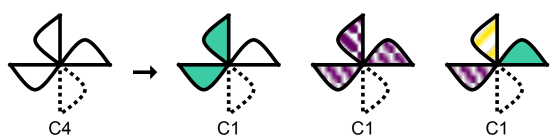


The only rotations that leave this colored shape unchanged are those of C2.

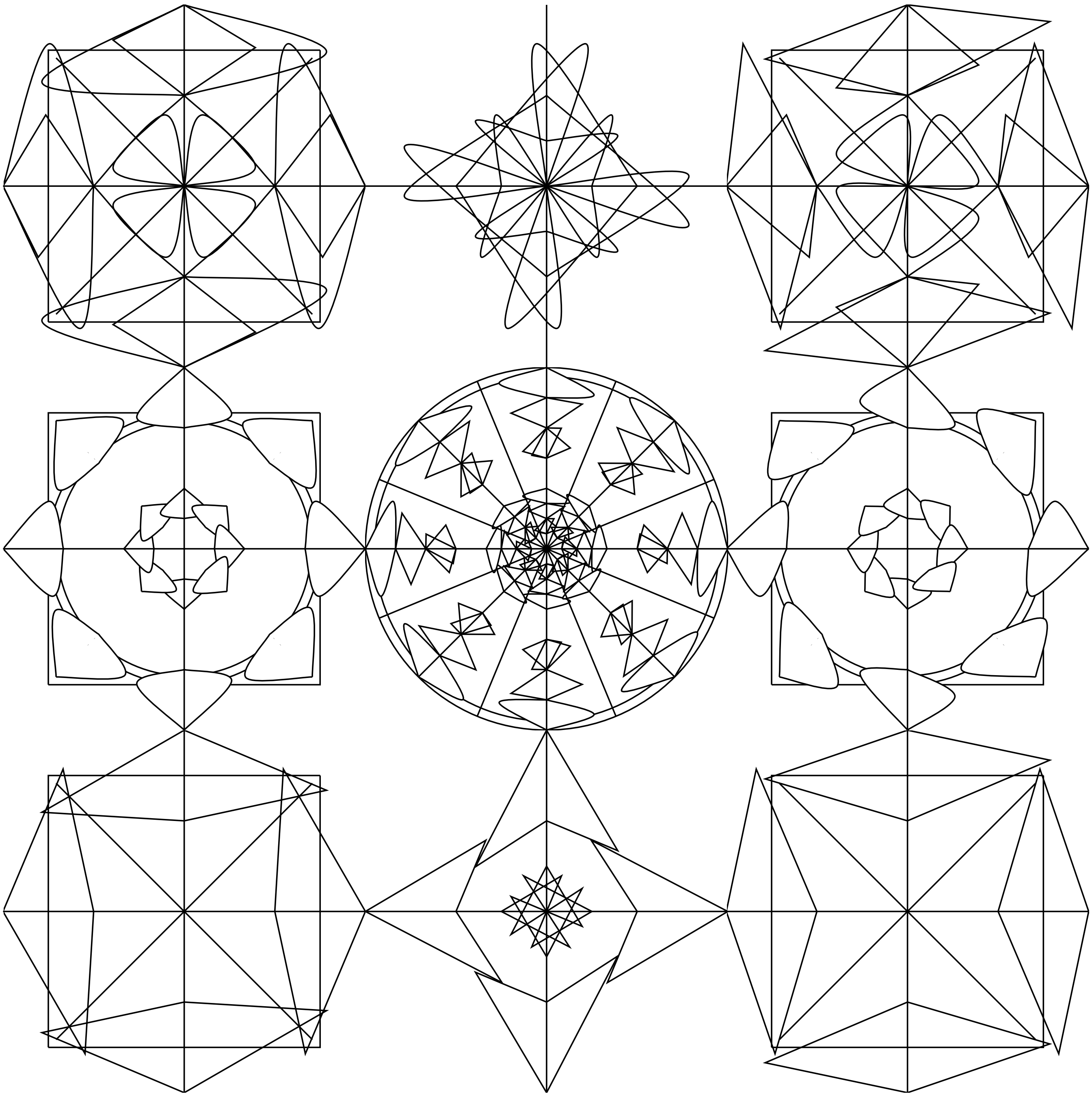




Not all colorings of our C4 shapes will transform them into C2 shapes.

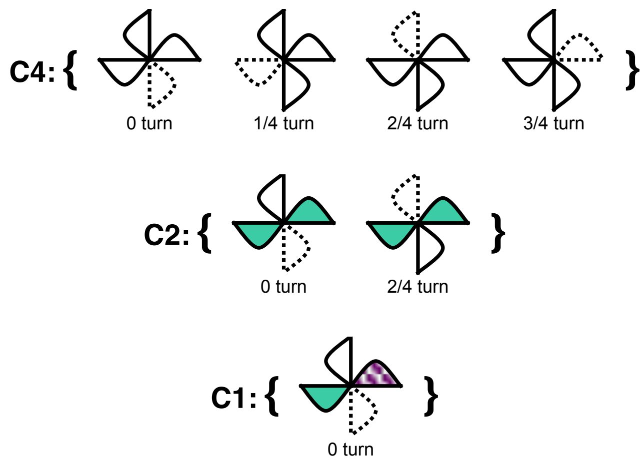


Coloring Challenge: Use color to transform the uncolored shapes into C2 shapes.



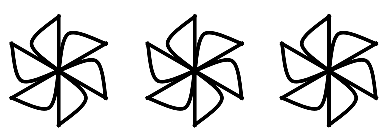
C4 and C8 shapes

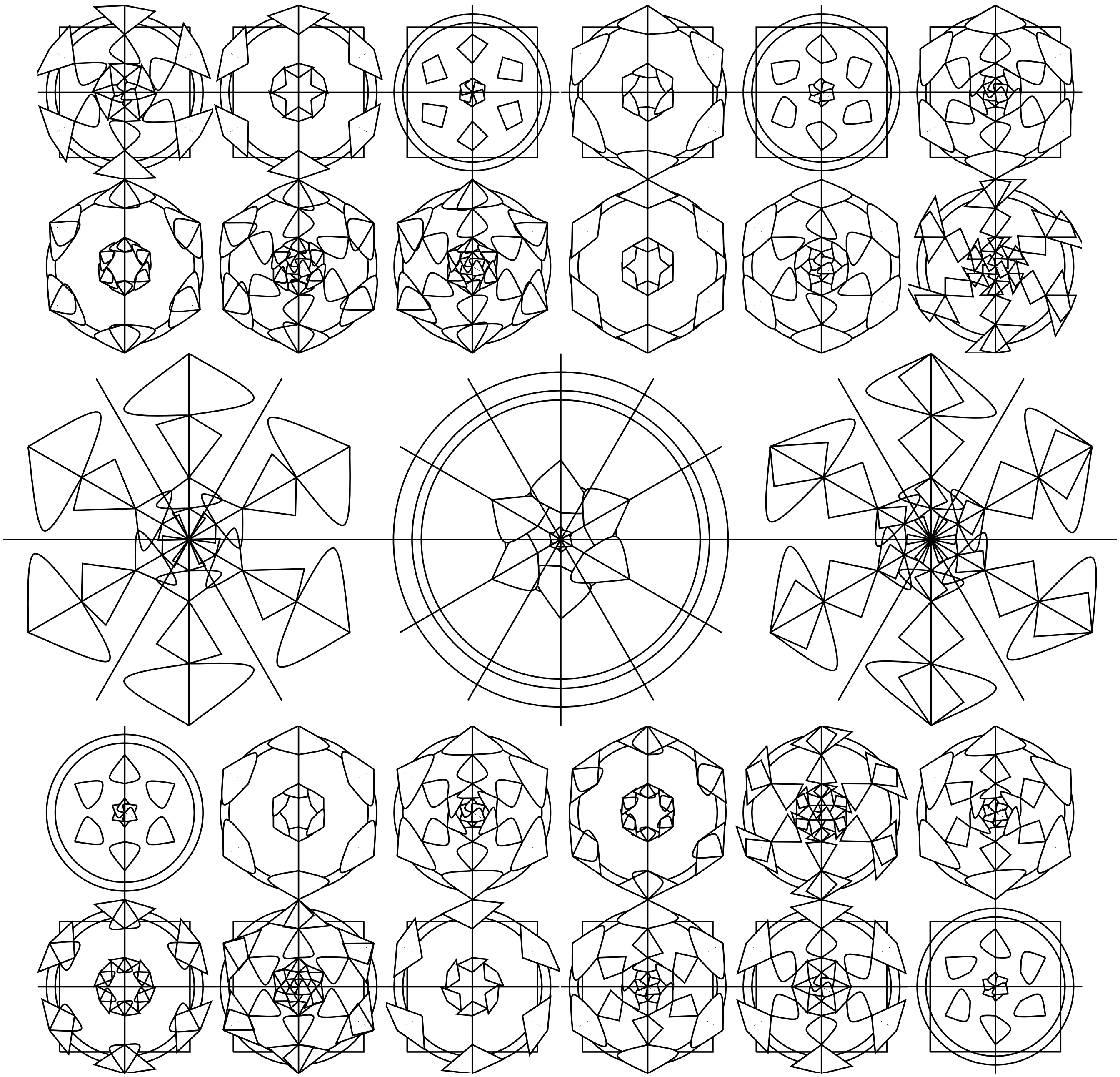
Color can reduce C4 shapes to C2 or C1 shapes because our C2 and C1 are subgroups of our C4 group. A subgroup is a group contained within a group.



Similarly, our C1, C2, and C3 groups are all subgroups of our C6 group.

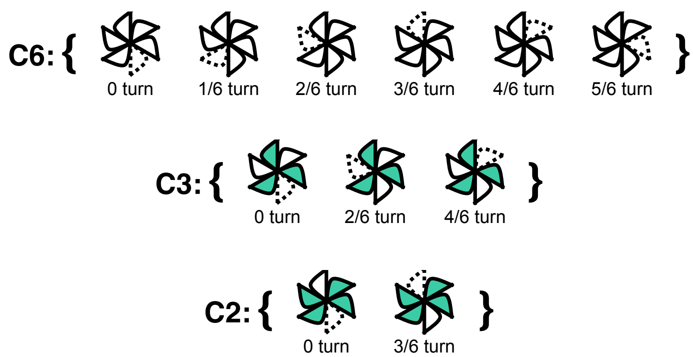
Coloring Challenge: Can you color the C6 shapes to reduce them to C1, C2, or C3 shapes?



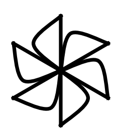


C2, C4, C6 shapes

Notice that a group has all of the rotations of its subgroups.

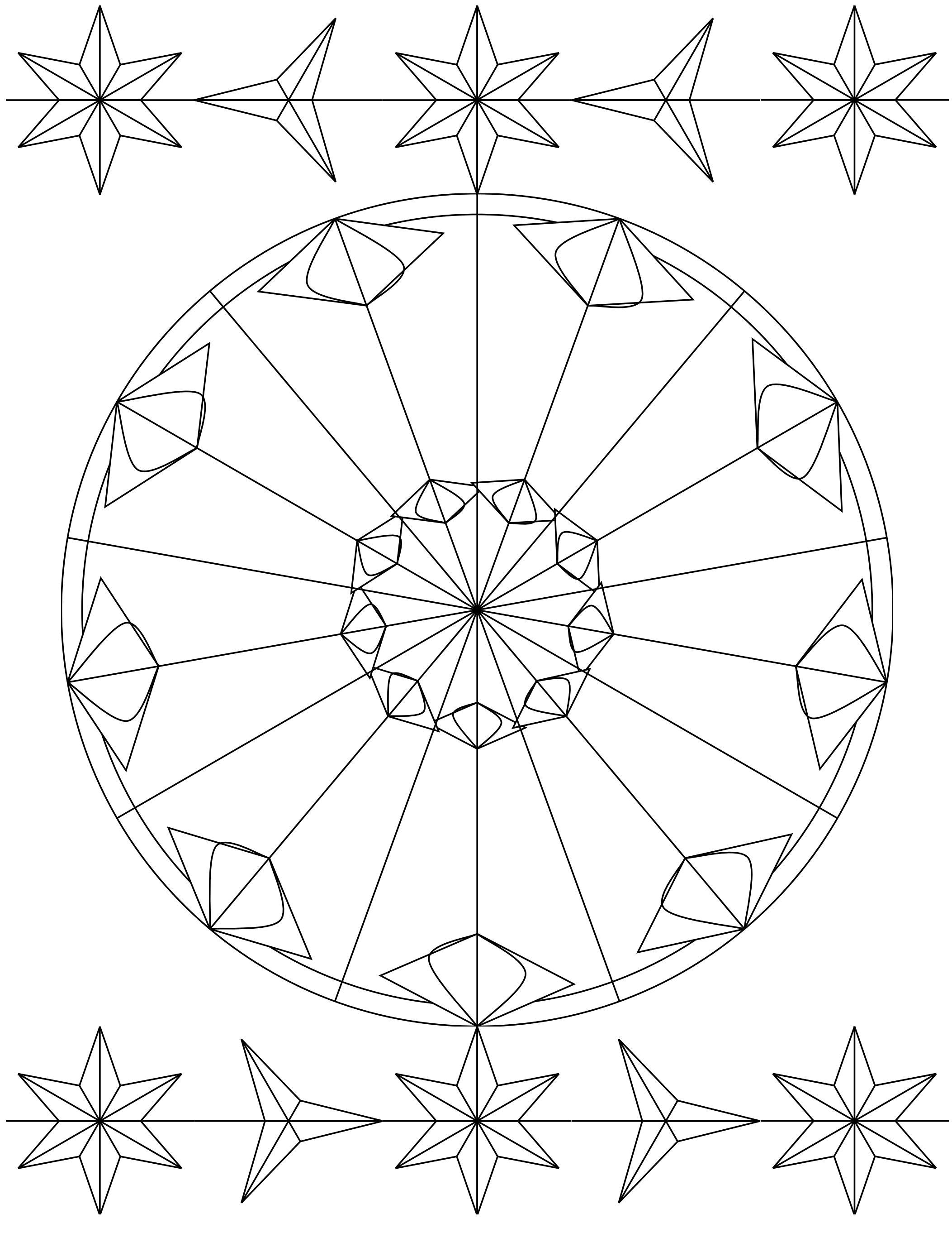


Try to color a C6 shape so that it has the rotations of C4.

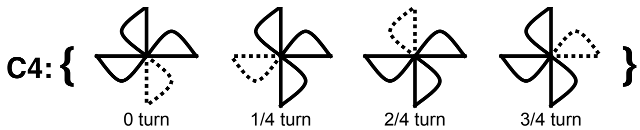


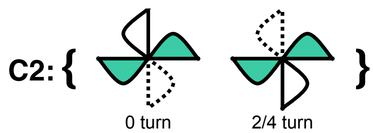
It can't be done. C4 is not a <u>subgroup</u> of C6.

There is more to it than that.

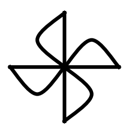


When we use color to reduce our shapes to belong to smaller groups, we give them a new set of rotations.



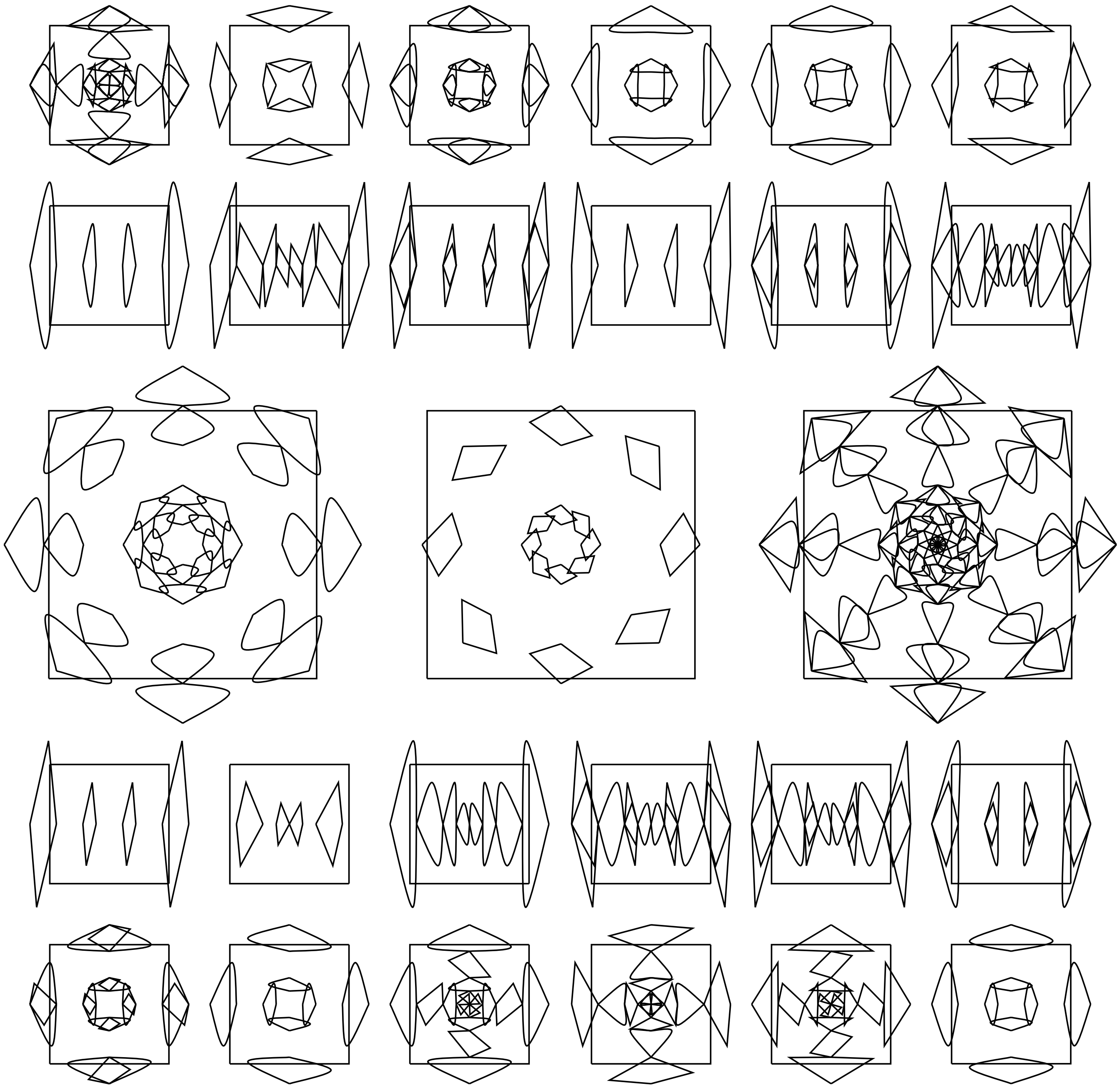


Not all sets of rotations are groups, and therefore cannot be <u>subgroups</u>. Try to color a shape in a way so that it has *only* a 0 turn and a $\frac{1}{4}$ turn.



It's impossible without also giving the shape a $\frac{2}{4}$ turn and a $\frac{3}{4}$ turn. That's because **{0 turn, $\frac{1}{4}$ turn }** is not a group, but **{0 turn, $\frac{1}{4}$ turn, $\frac{2}{4}$ turn, $\frac{3}{4}$ turn }** is.

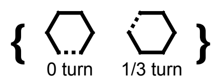
Why? This brings us back to combining rotations.



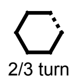
C2 and C4 shapes

In order for a set of rotations to be a group, any combination of rotations in the set must also be in the set. This rule is called <u>group closure</u>, and you can see it.

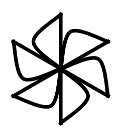
Take the set **{0 turn, $\frac{1}{3}$ turn}**.



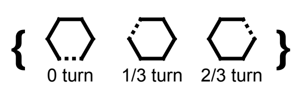
This is not a group because it's missing the $\frac{2}{3}$ turn, which is created by combining a $\frac{1}{3}$ with another $\frac{1}{3}$ turn.

$\frac{1}{3}$ turn $*$ $\frac{1}{3}$ turn → 

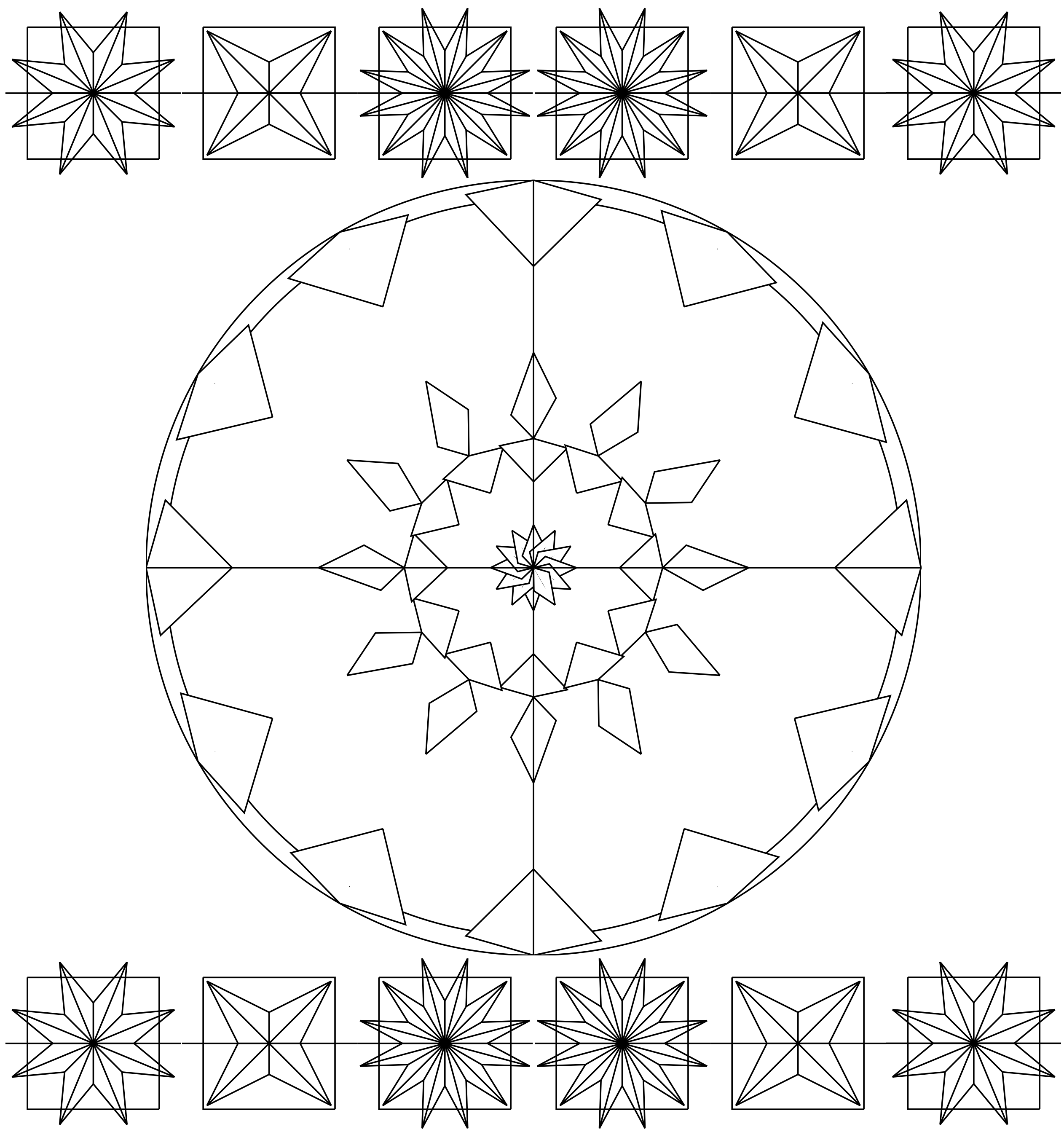
Can you color a shape to have the **{0 turn, $\frac{1}{3}$ turn}** rotations without a $\frac{2}{3}$ turn?



No, but adding the $\frac{2}{3}$ turn to the set gives us our C3 group again.

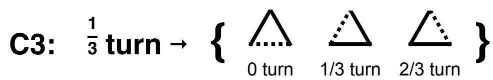


Coloring Challenge: Color the shapes to make them all C2 shapes.

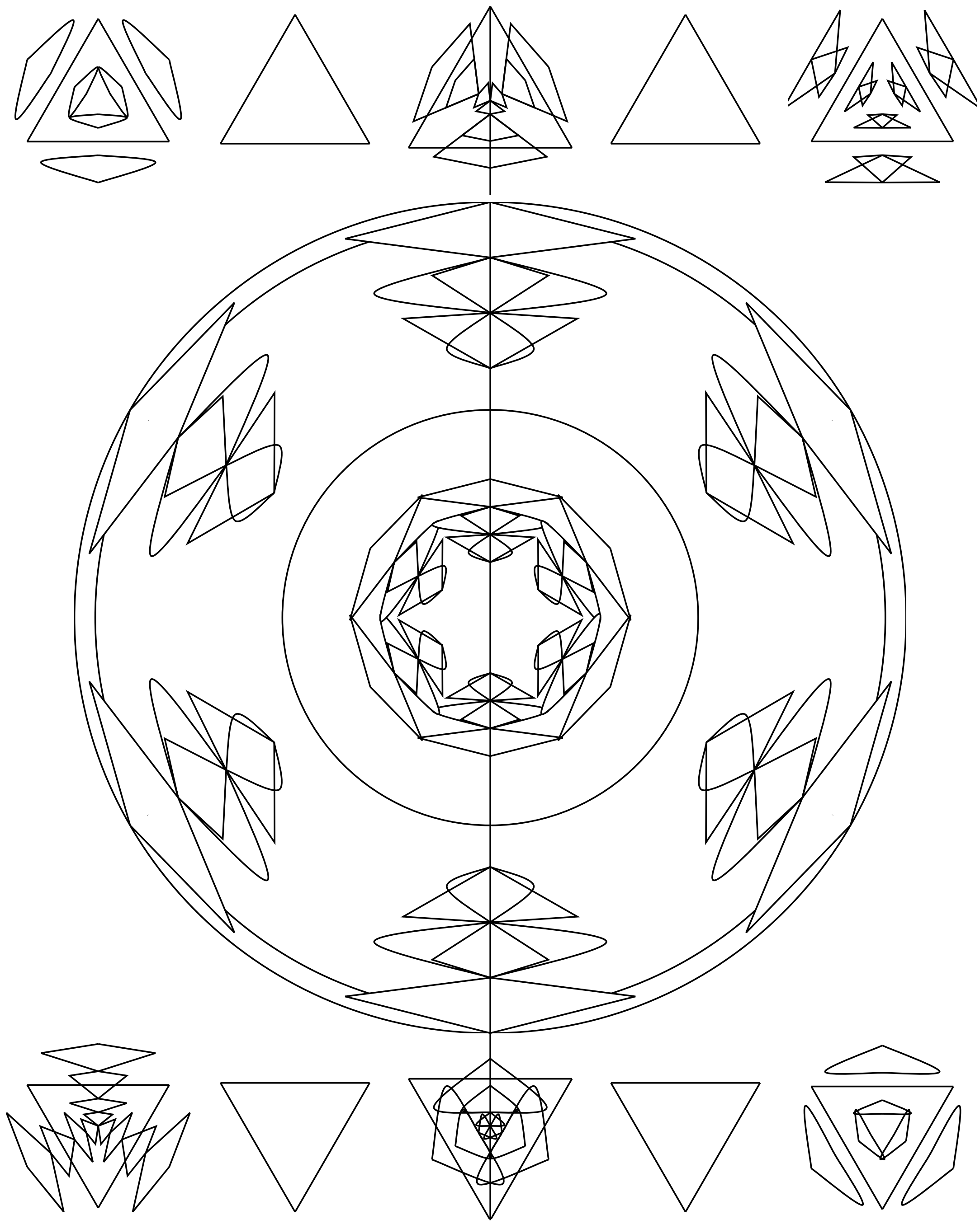


So far we've only been talking about groups of rotations.

These groups are cyclic. They can be created by combining just one rotation - the generator - multiple times with itself.



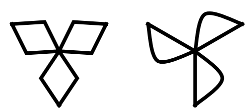
For our next groups, we have more generators, such as reflections.



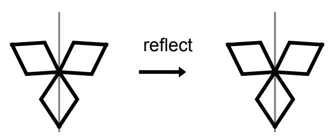
shapes with reflection

REFLECTIONS

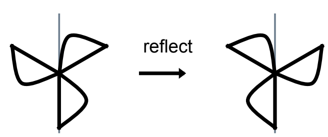
Even when two shapes have the same number of rotations, one can still have more symmetry than the other.



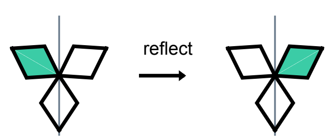
Some shapes have mirrors - they can reflect across internal, invisible lines without changing in appearance

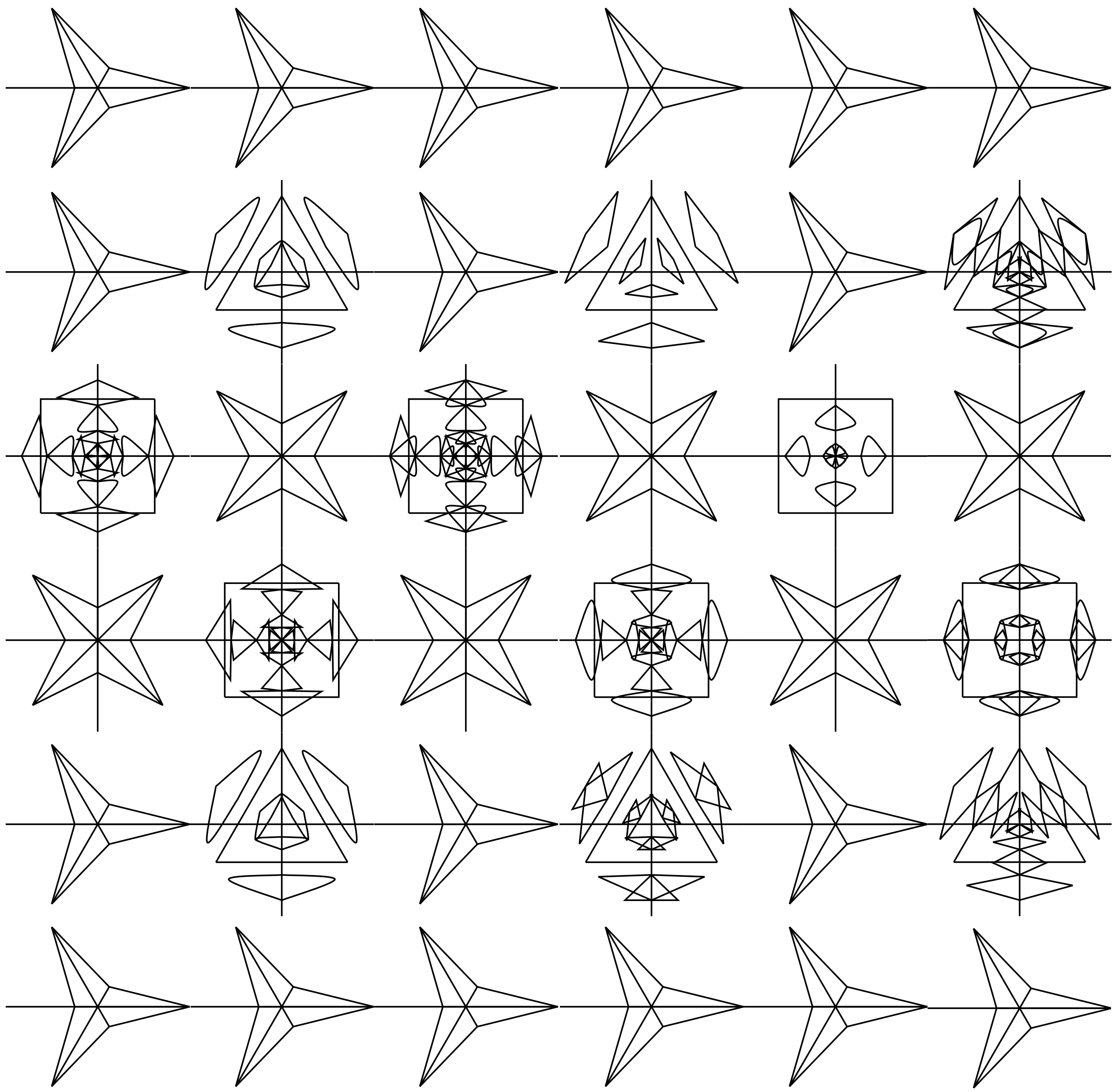


While others cannot.

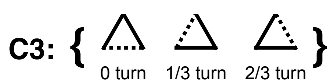


We'll see how these mirrors can be removed when color is added.

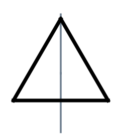




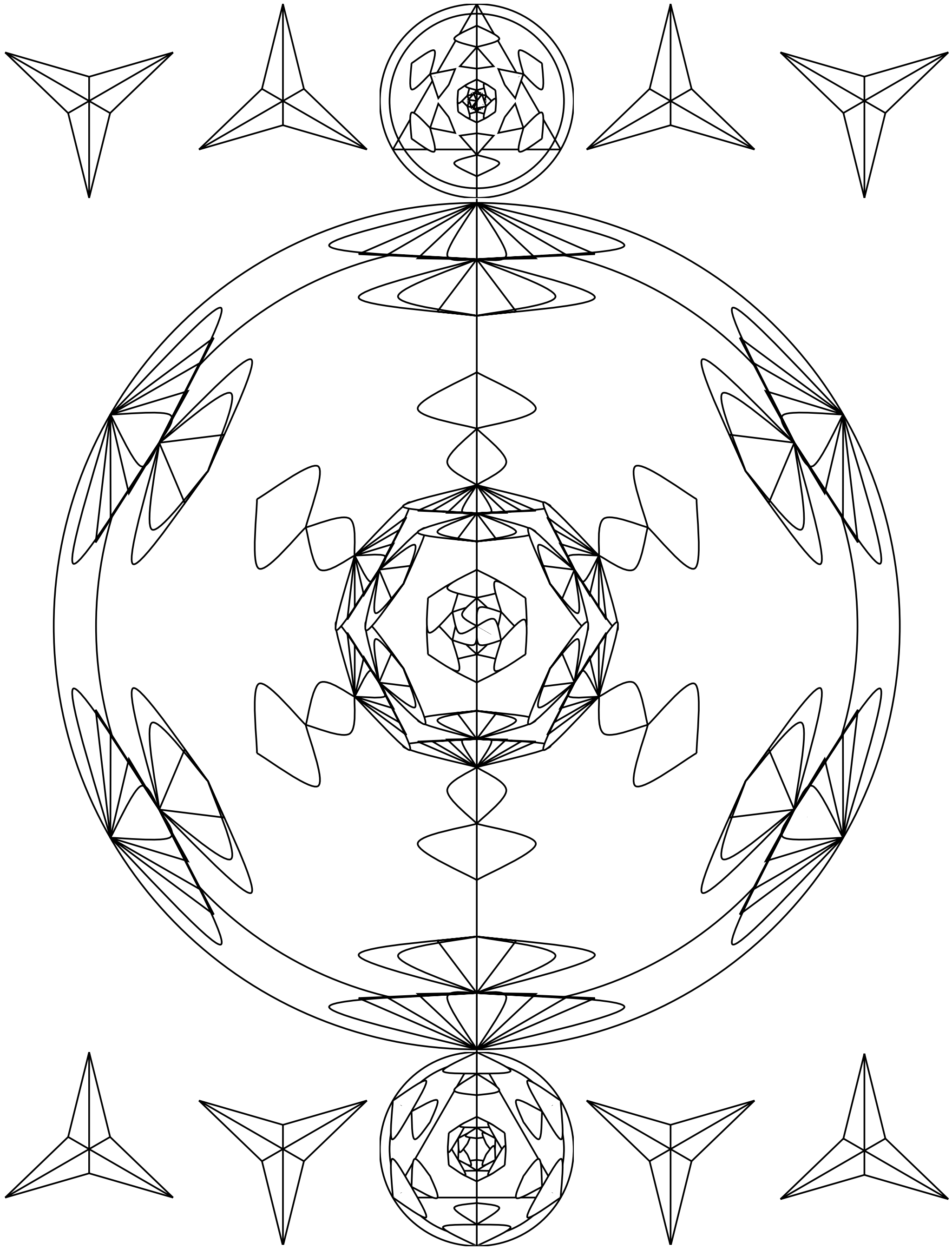
We saw that a single generator, the $\frac{1}{3}$ turn, could generate the entire group of rotations of a regular triangle, C3.



We can also reflect this triangle across a vertical mirror through its center.

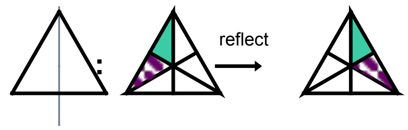


By coupling this mirror with a rotation, we can generate even larger groups.

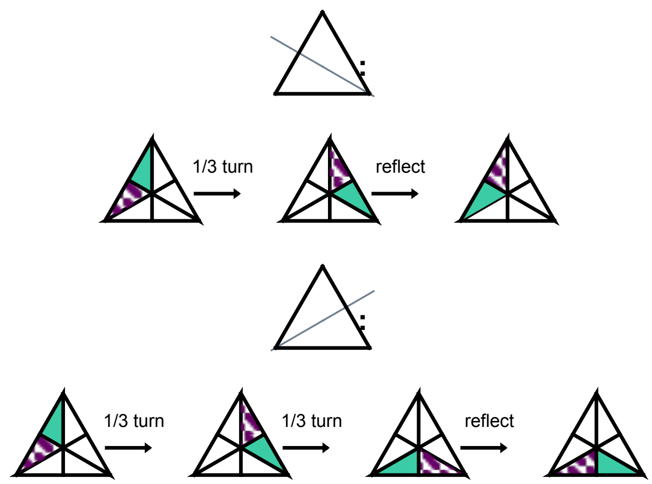


shapes with vertical mirrors

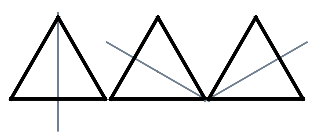
Reflections are easier to see with color.



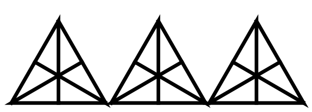
More mirror reflections can be generated by simply combining this vertical mirror with rotations.

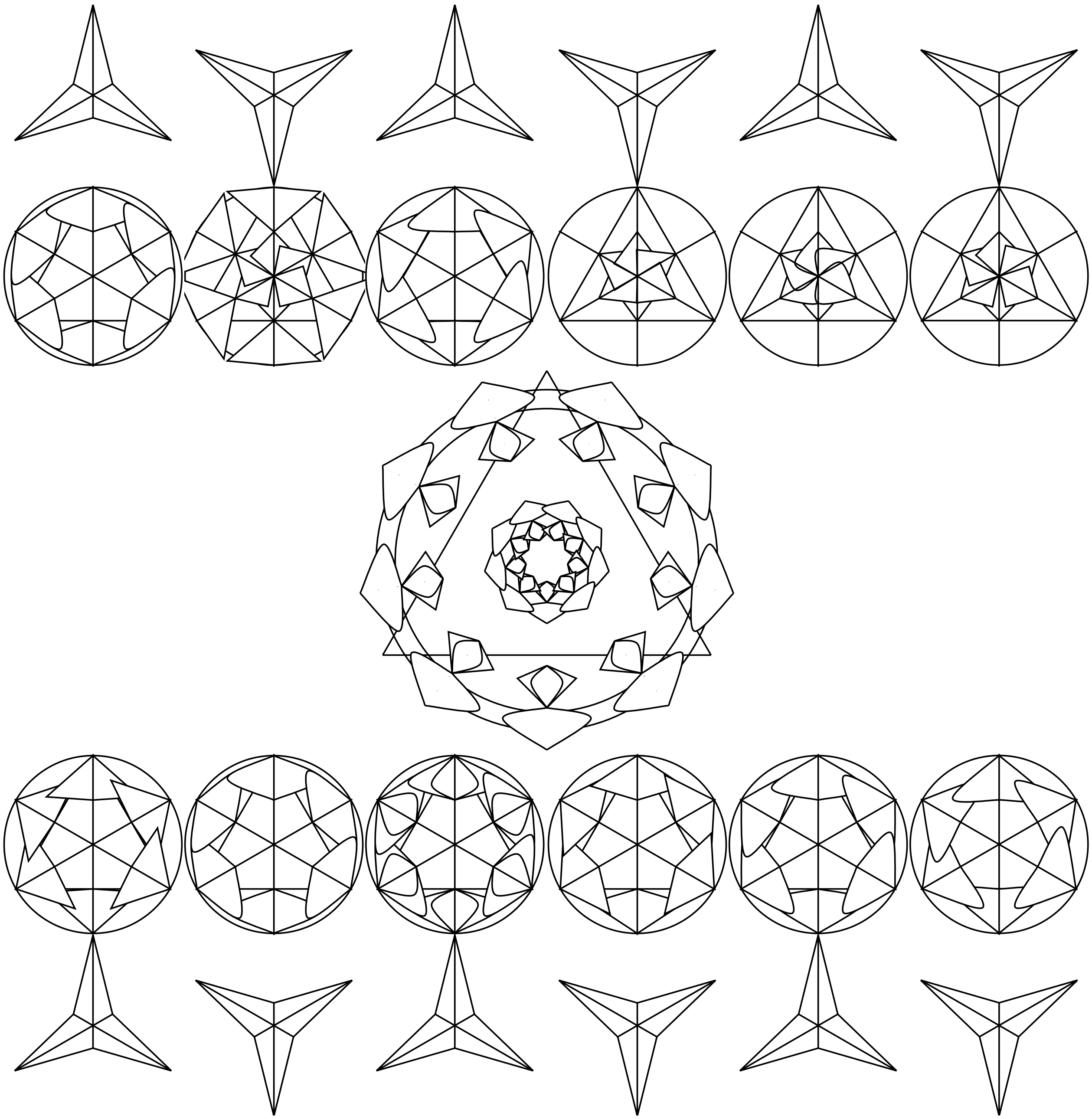


In total, a regular triangle has 3 unique mirrors.



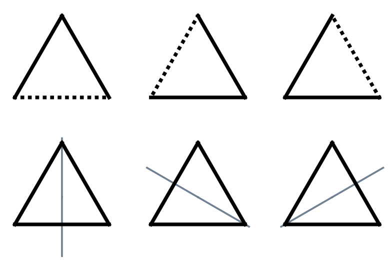
Coloring Challenge: Use color to show what happens to our triangle when it is reflected and then rotated. Is this different than rotating and then reflecting?



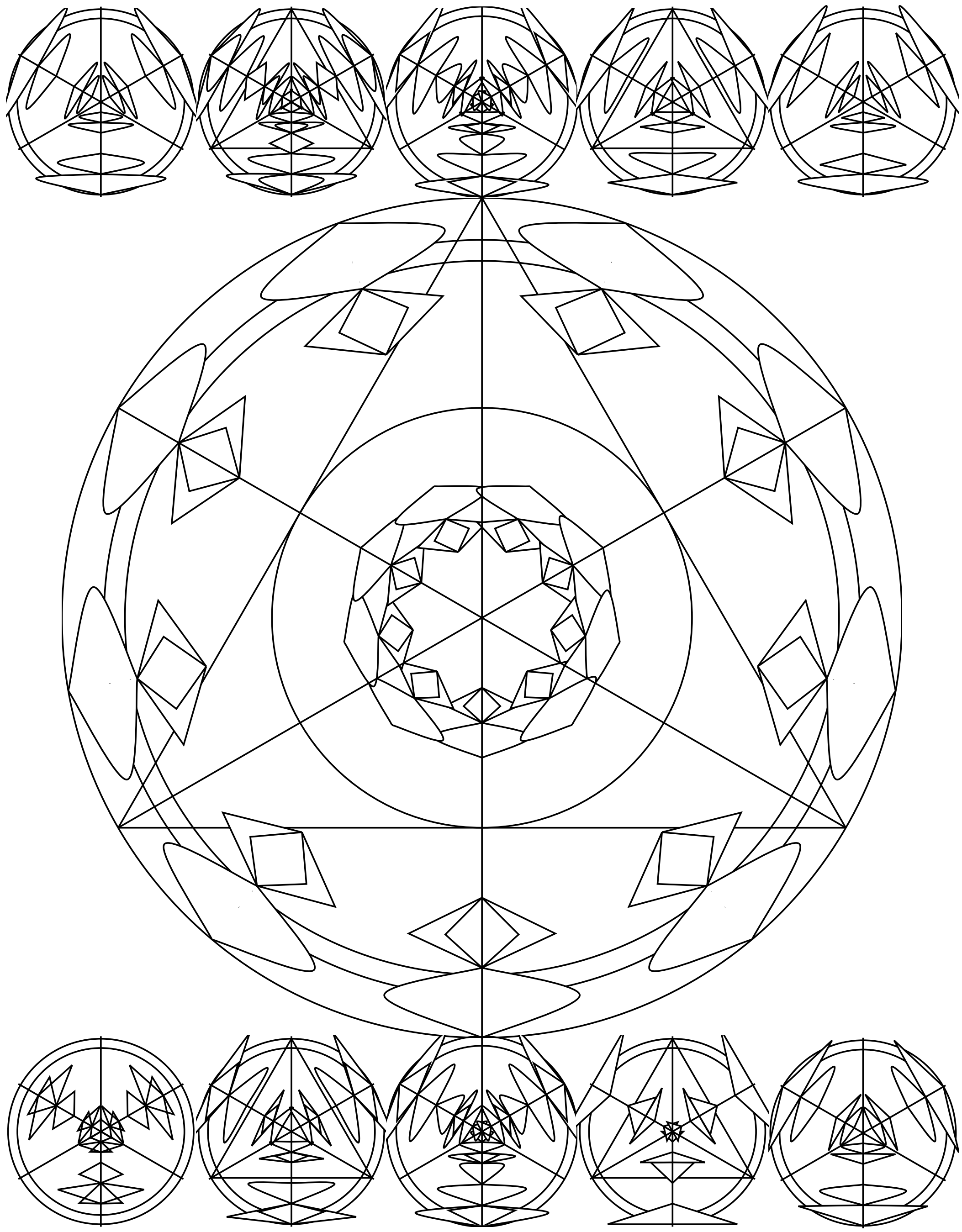


shapes with 3 rotations and 3 mirrors

With just a rotation and a mirror as generators, we generated a new, larger group of symmetries for a regular triangle.

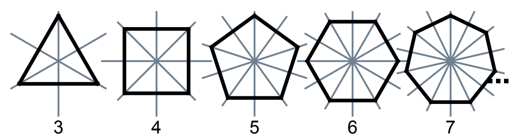


We can do the same for other shapes.

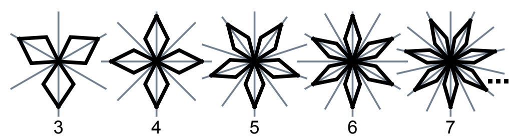


shapes with 3 rotations and 3 mirrors

Our triangle has 3 unique rotations and 3 unique reflections, a square has 4, and we can find shapes with 5, 6, 7, and keep going...

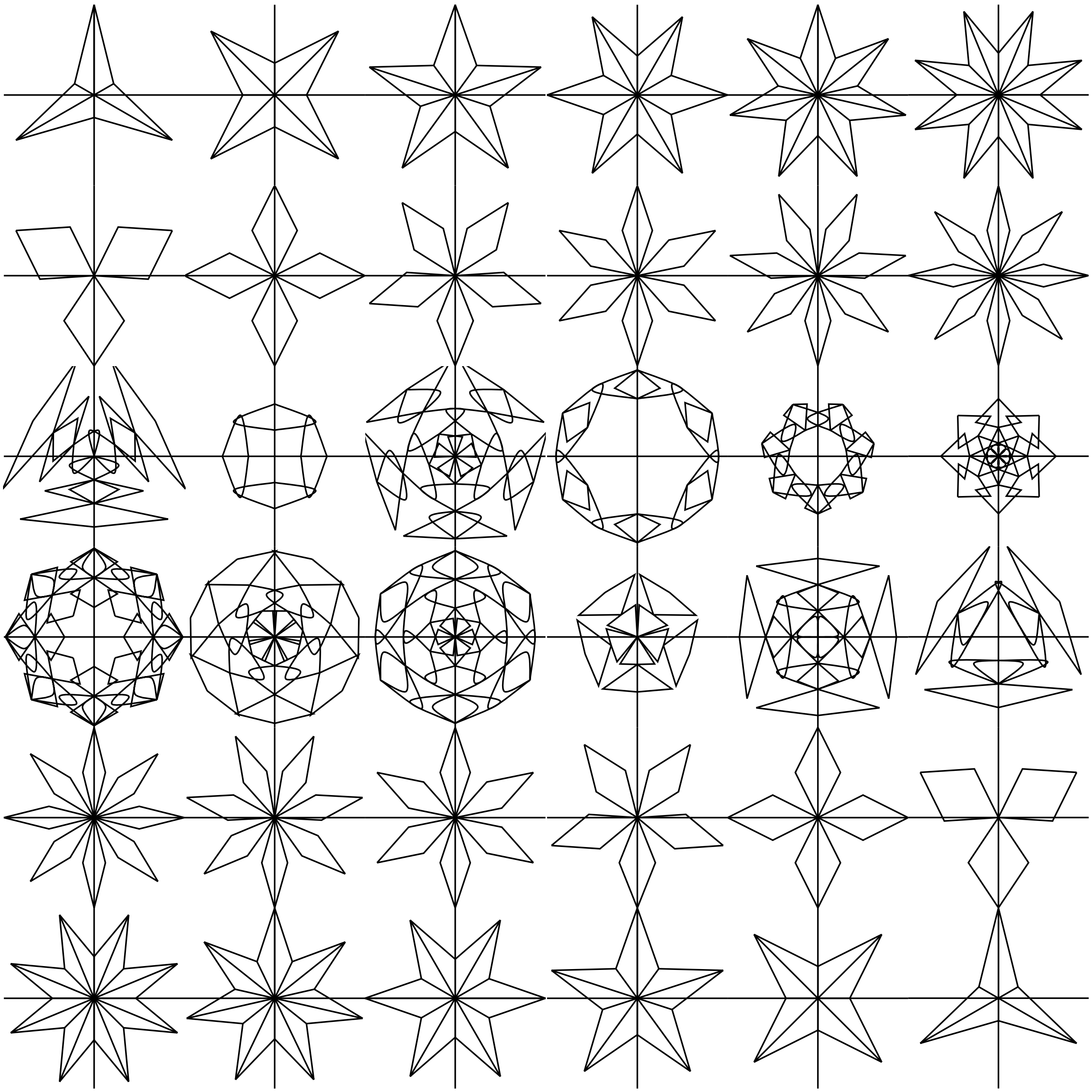


Shapes that are not regular polygons can have these same symmetries.



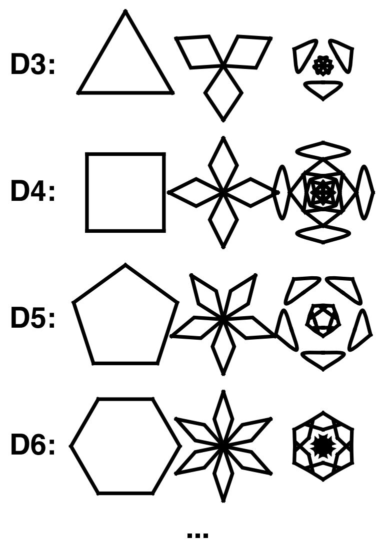
When shapes have the same set of symmetries, they share a symmetry <u>group</u>.

Challenge: Can you find the shapes that have the same rotations and reflections of a square?

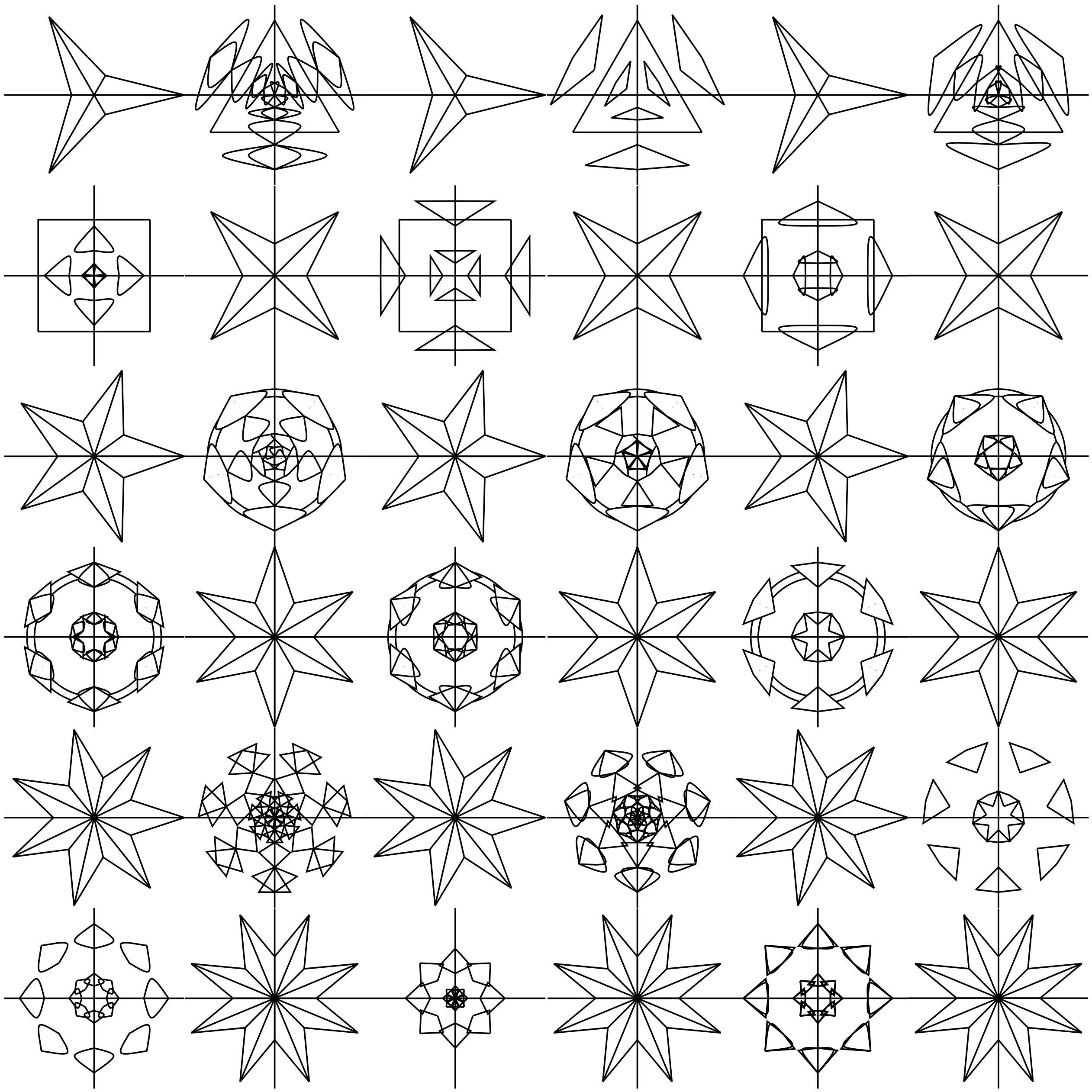


shapes with 3, 4, 5, 6, 7, 8 rotations and reflections

The symmetry group of a regular triangle and all the shapes that have its same symmetries, is called D3, while the symmetry group of a square is called D4, and so on...

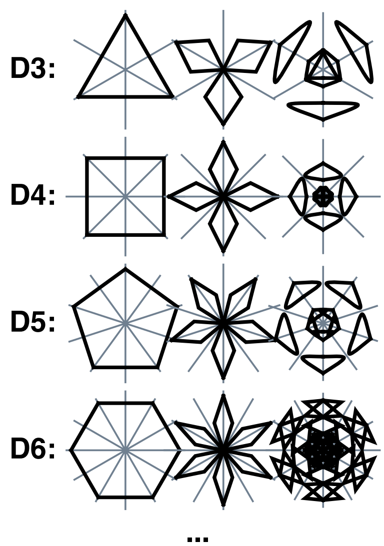


This series of groups is called the dihedral groups.

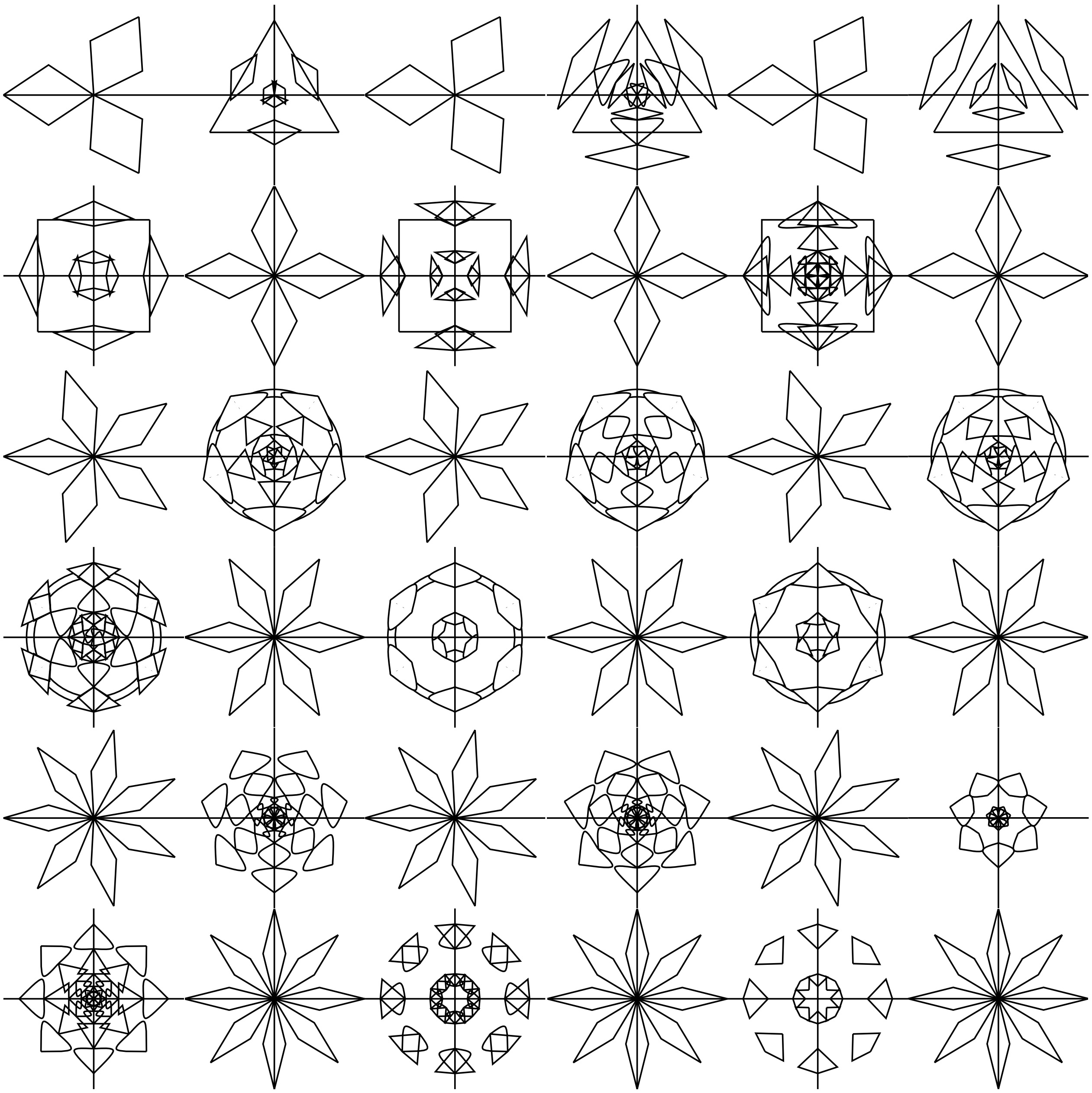


D3, D4, D5, D6, D7, D8 shapes

These shapes that share a symmetry group may look different, but when viewed through the lens of group theory, they look the same. Only their symmetries - the rotations and reflections that leave them unchanged - are seen.

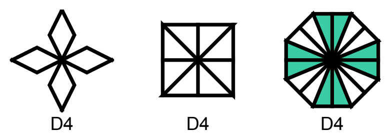


Challenge: Which dihedral group does each uncolored shape belong to?

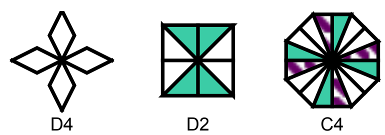


D3, D4, D5, D6, D7, D8 shapes

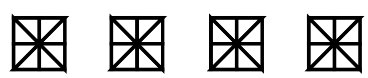
By looking for rotations and reflections, we can see when shapes share a symmetry group.

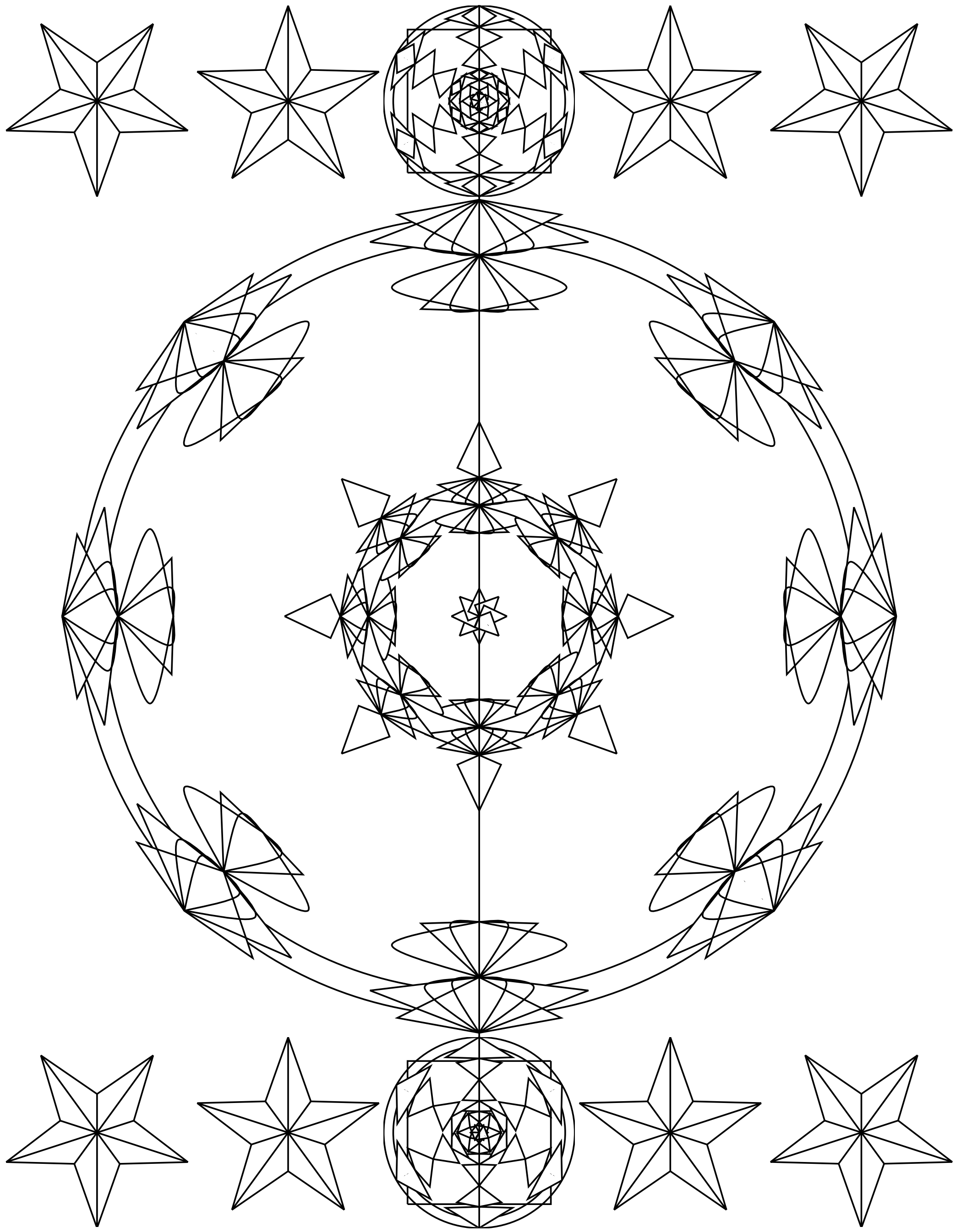


And when they do not.

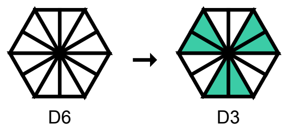


Coloring Challenge: Can you color the shapes so that none of them share a symmetry group?

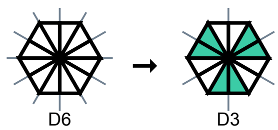




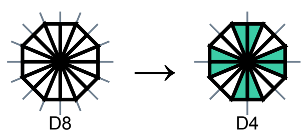
Again, color can reduce the amount of symmetry a shape has.



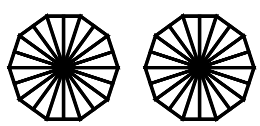
For example, a D6 shape has 6 mirrors and 6 rotations, but color can transform it into a shape with only 3 mirrors and 3 rotations - a D3 shape.

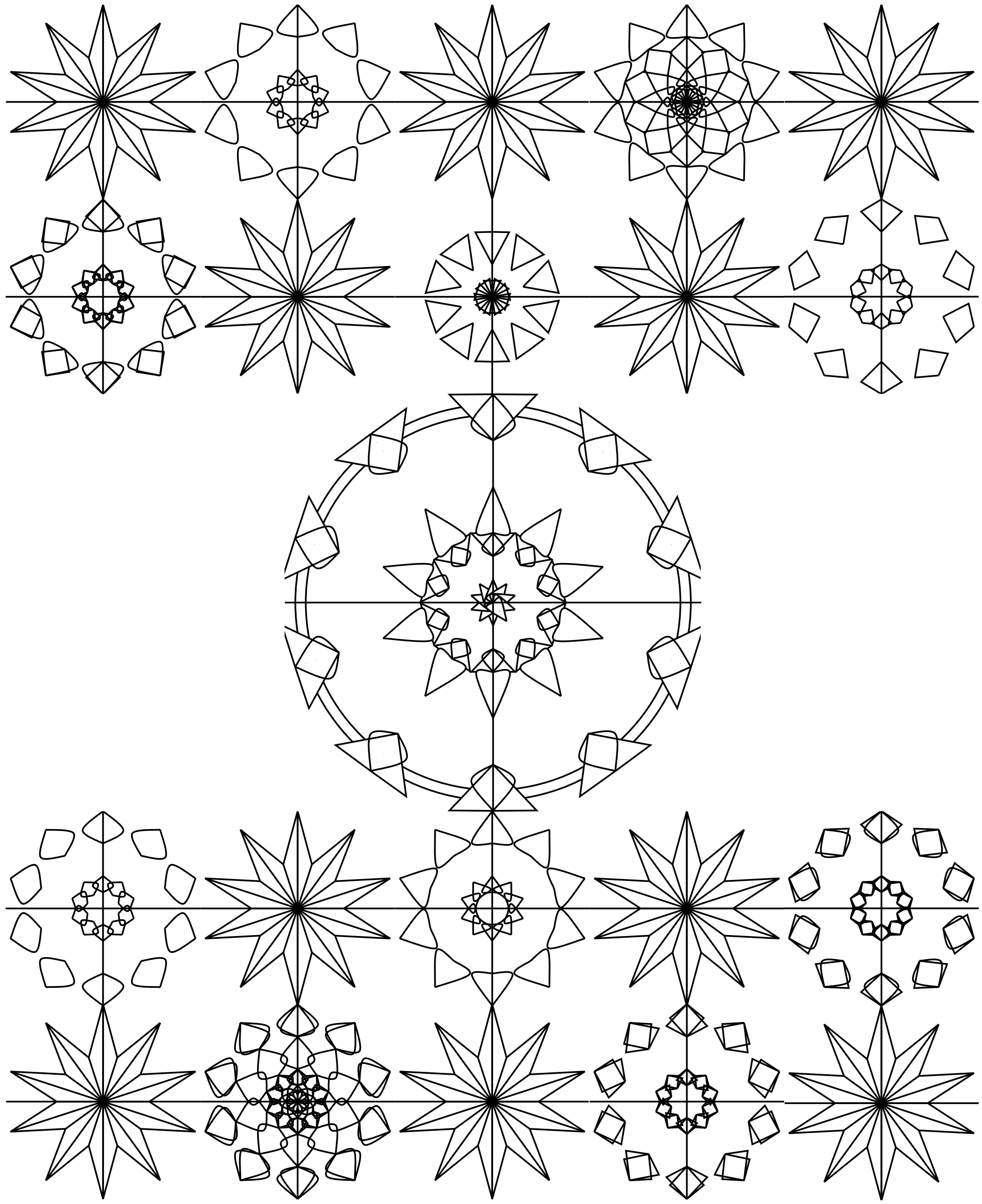


Color can reduce a D6 shape to a D3 shape because D3 is a subgroup of D6. Similarly, D4 is a subgroup of D8.

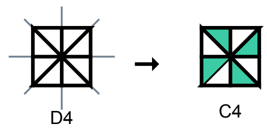


Coloring Challenge: Can you use color to reduce the D10 shapes to D5 shapes?

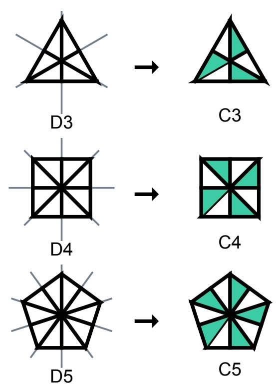




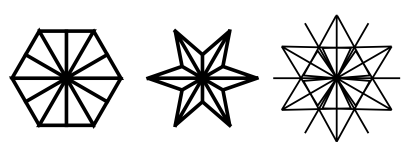
What happens when color is added to remove only mirrors and not rotations?

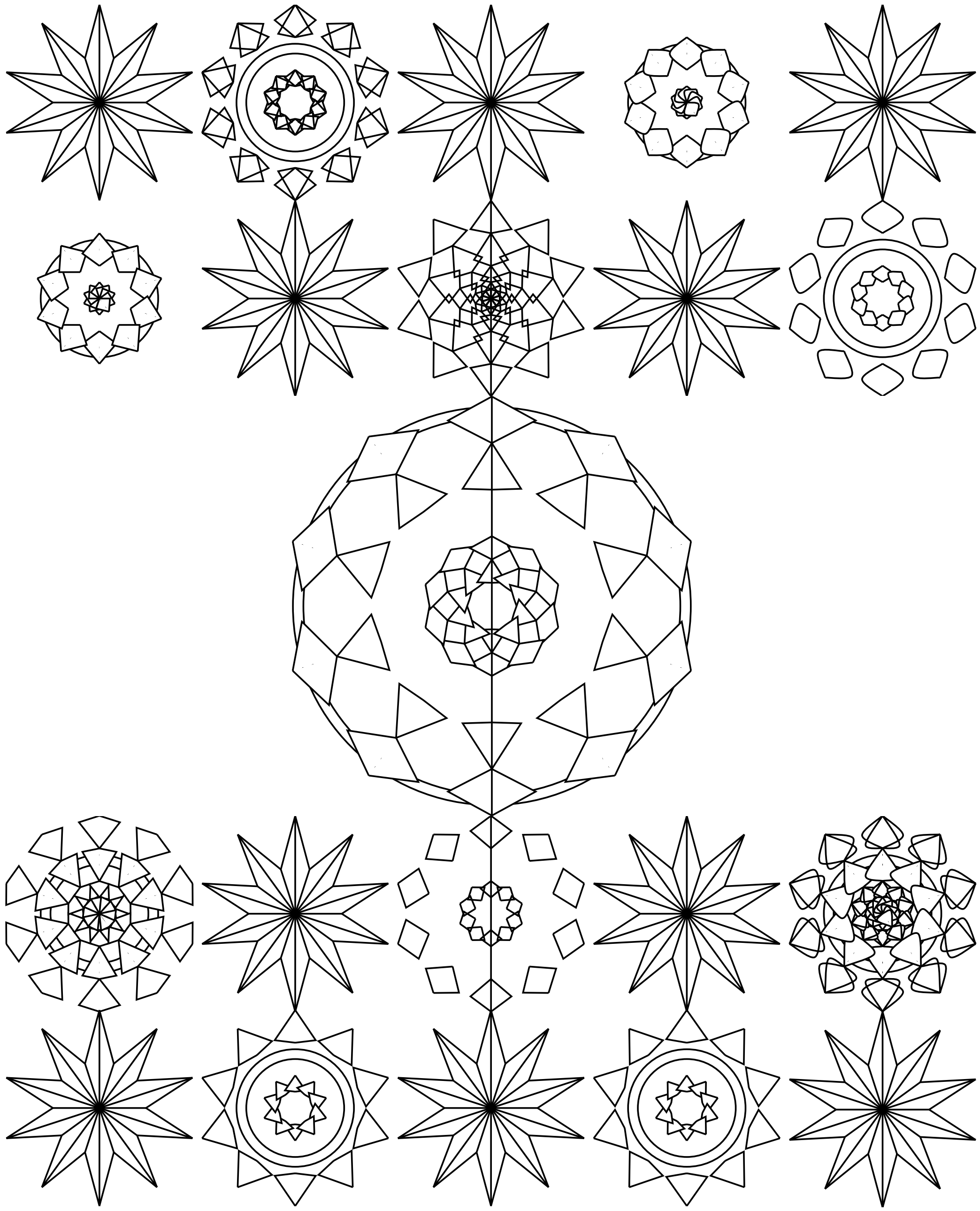


The dihedral groups have mirror reflections, while the cyclic groups do not. When these mirrors are removed, we can see the cyclic groups are subgroups of the dihedral groups.

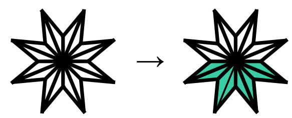


Coloring Challenge: Color the D6 shapes to transform them into C6 shapes.

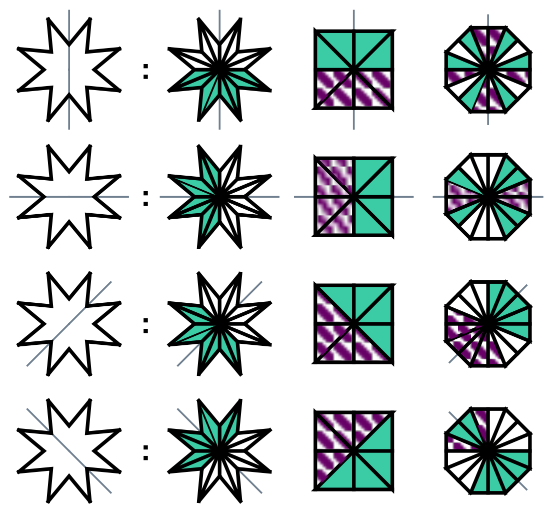




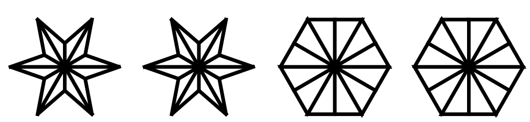
Color can also take away a shape's rotations.

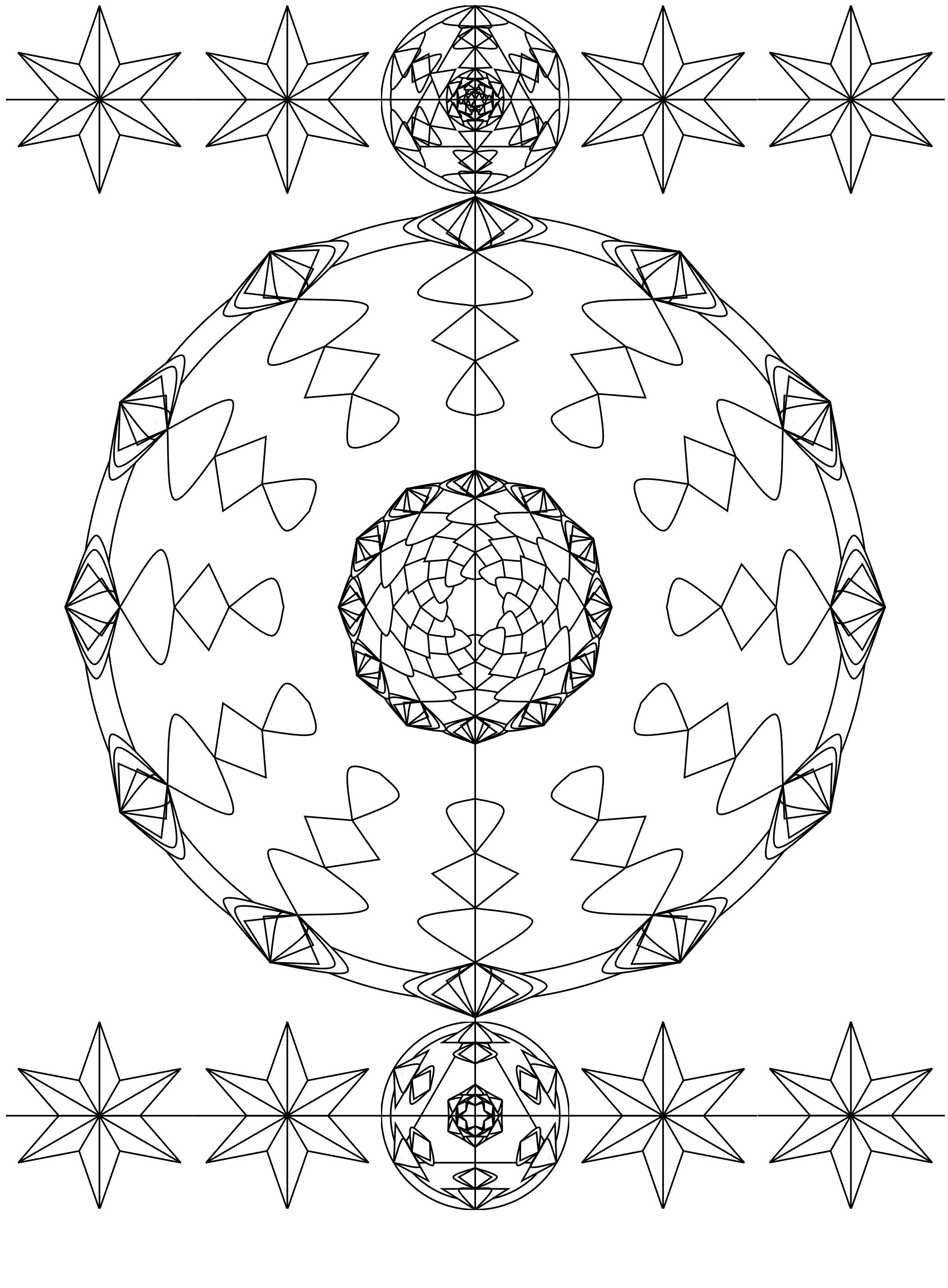


Coloring in this way leads to finding subgroups with only mirror reflections.

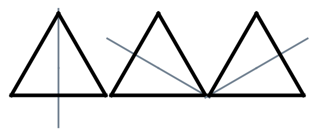


Coloring Challenge: Use color to reduce D6 shapes to D3 shapes. Then add more color to remove their rotations so that they only have reflection.

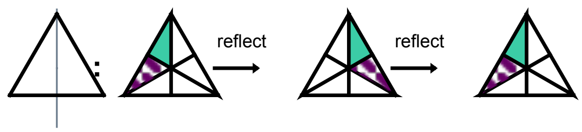




There's something about these mirrors that you may have already noticed.

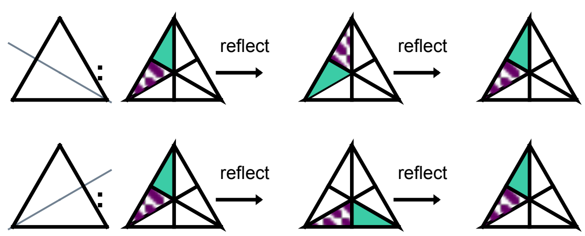


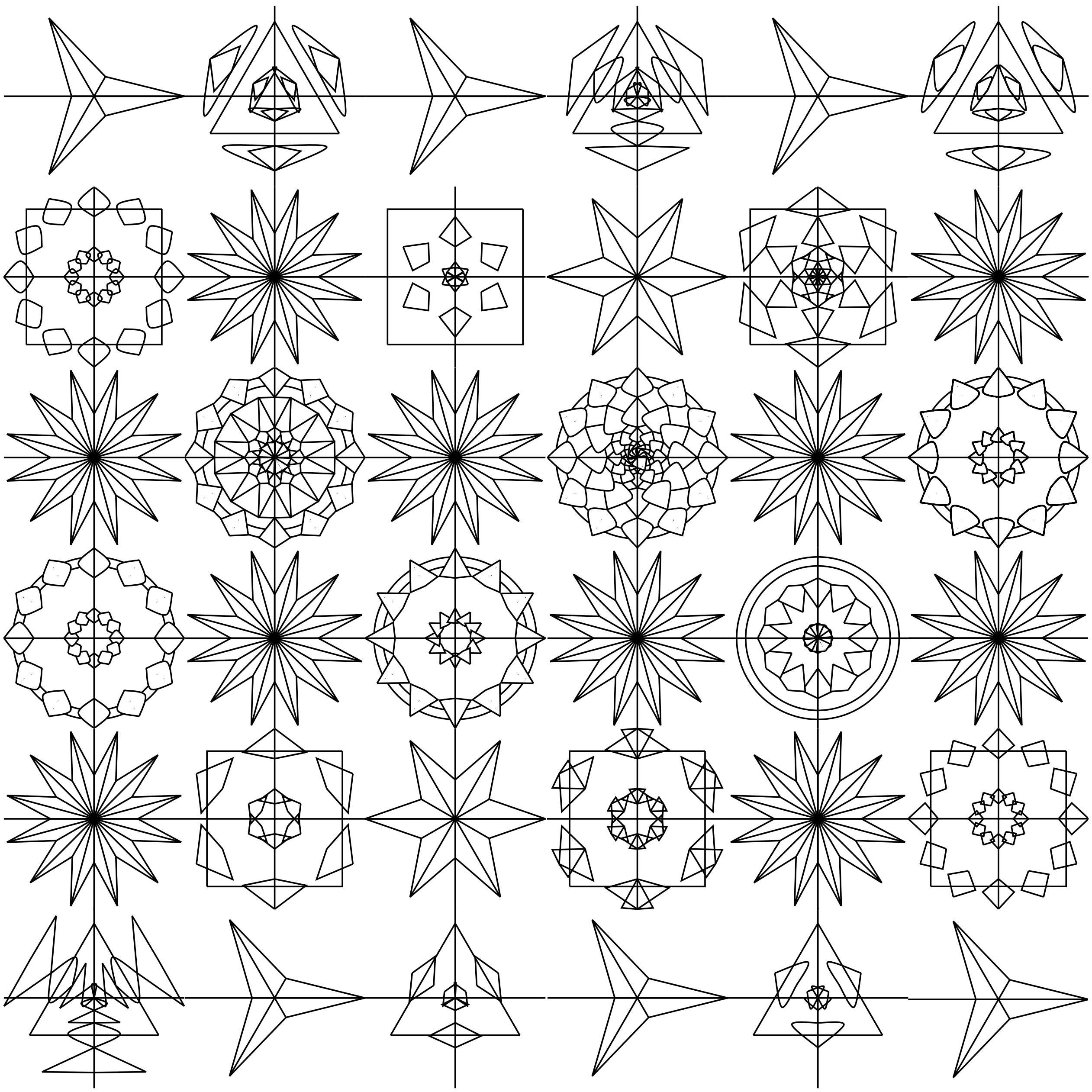
Reflecting a shape across the same mirror twice in a row is the same as not reflecting it at all.



The second reflection reverses the work of the first reflection.

The same can be said for the other mirrors we found.

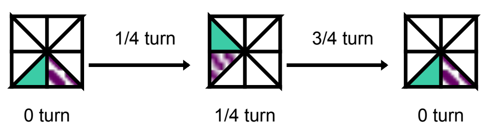




You may have also noticed that our rotations can be reversed as well.

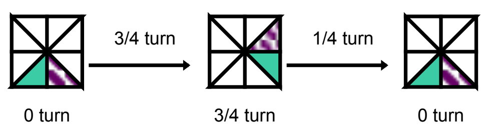
When a square is rotated by a $\frac{1}{4}$ turn, rotating again by a $\frac{3}{4}$ turn brings it back to the position it started in. The result is the same as a 0 turn.

$\frac{1}{4}$ turn $\ast$ $\frac{3}{4}$ turn = 0 turn

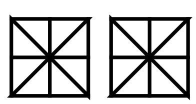


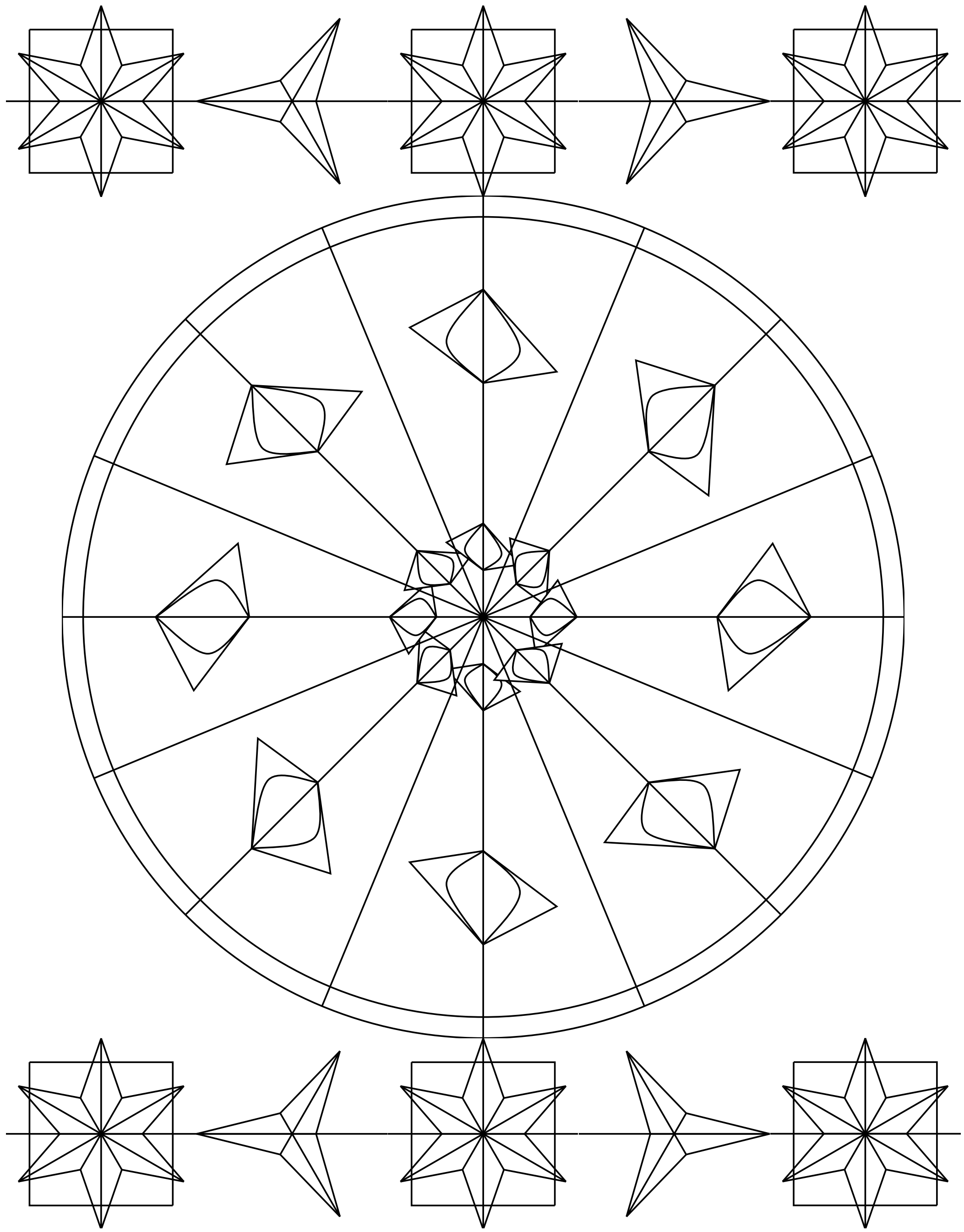
The same can be said the other way around.

$\frac{3}{4}$ turn $\ast$ $\frac{1}{4}$ turn = 0 turn



Challenge: Which rotation in C4 is the reverse of the $\frac{2}{4}$ turn?



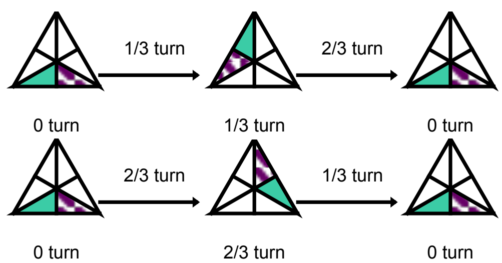


When one transformation, like a $\frac{1}{4}$ turn, reverses the work of another transformation, it's called an <u>inverse</u>.

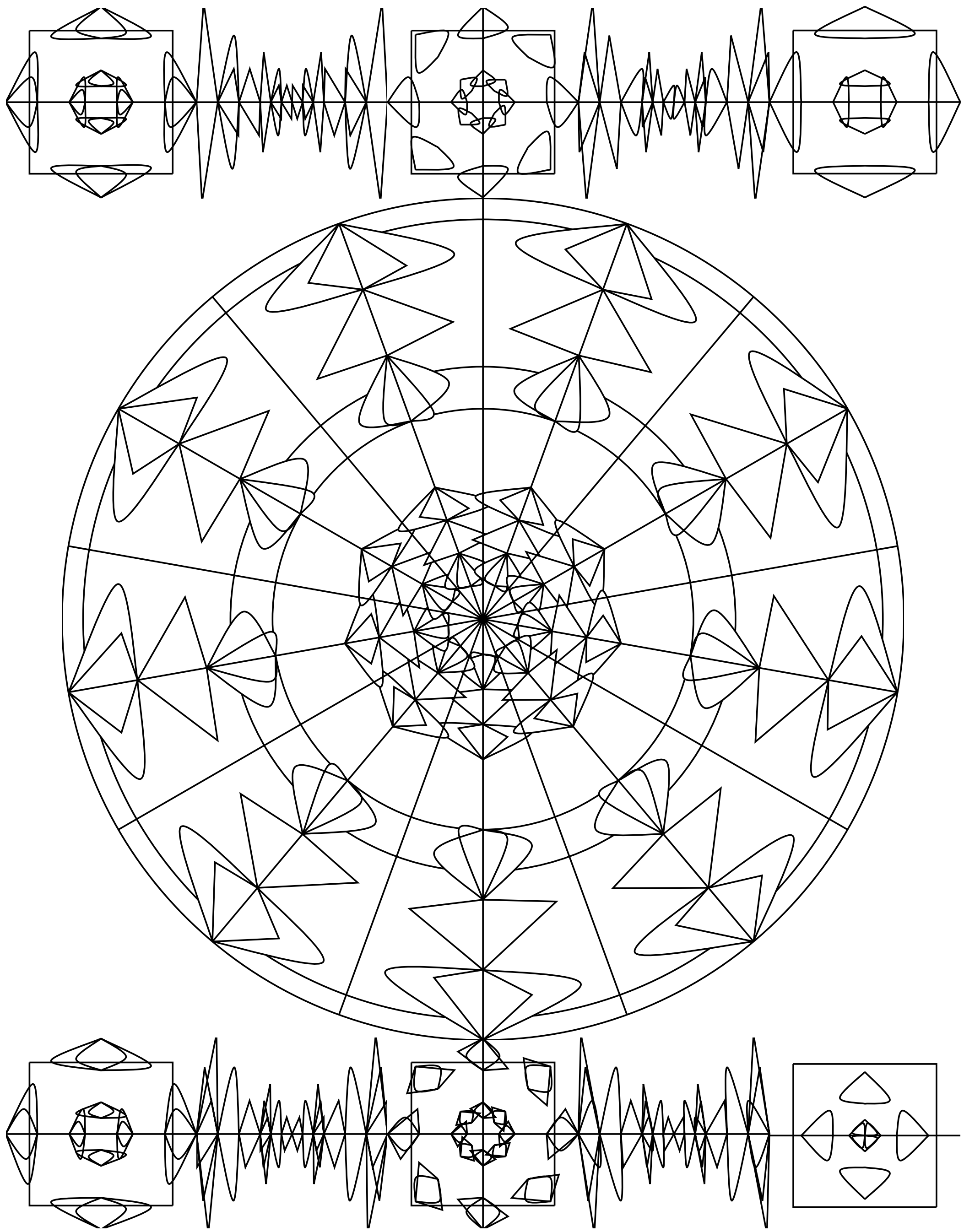
The $\frac{1}{4}$ turn is the <u>inverse</u> of the $\frac{3}{4}$ turn in C4, and vise-versa.

Similarly, the $\frac{1}{3}$ turn and $\frac{2}{3}$ turn are <u>inverses</u> in C3.

C3: $\frac{1}{3}$ turn ✷ $\frac{2}{3}$ turn = 0 turn = $\frac{2}{3}$ turn ✷ $\frac{1}{3}$ turn



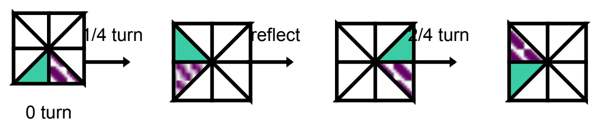
Challenge: What is the inverse of a vertical reflection? What is the inverse of any reflection?



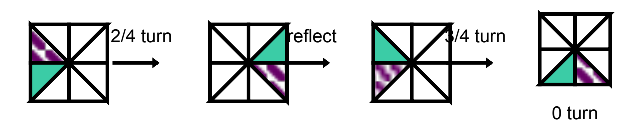
D2, D4, D9 shapes

All of the transformations in our cyclic and dihedral groups have inverses.

Even when a shape undergoes a combination of reflections and rotations,

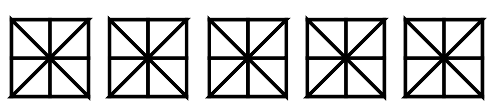


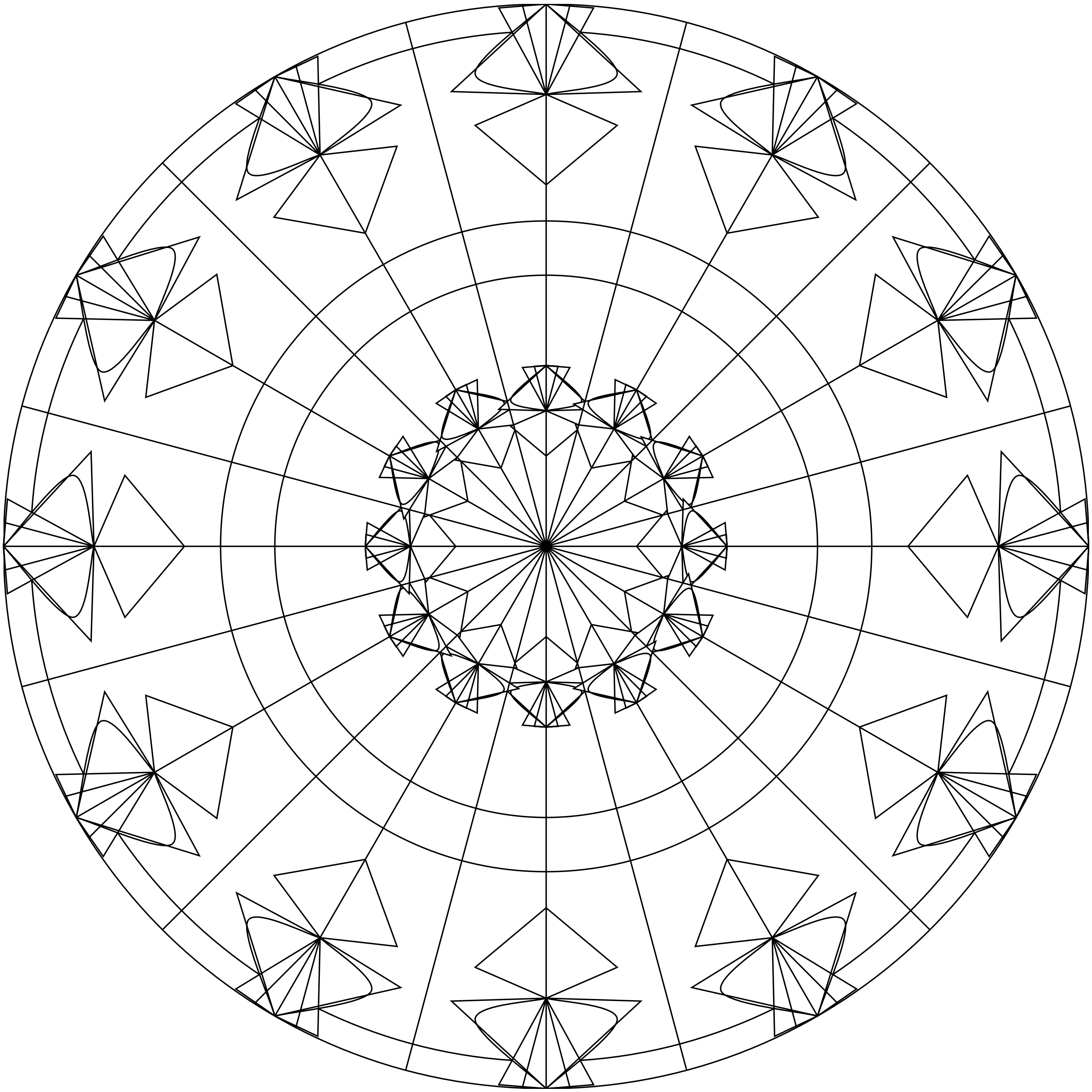
The transformations can be reversed and the shape can end back in the position it started.



This is a rule in group theory: Any member of a group has an inverse that is also in the group. And remember, the members of our groups are the reflections and rotations that transform our shapes.

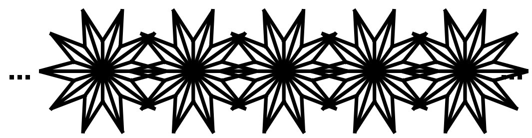
Coloring Challenge: Color the squares to show the result of rotating by a $\frac{3}{4}$ turn and then reflecting across a vertical mirror. Then find the combination of transformations that brings the square back to its starting position.



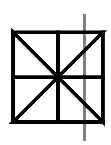


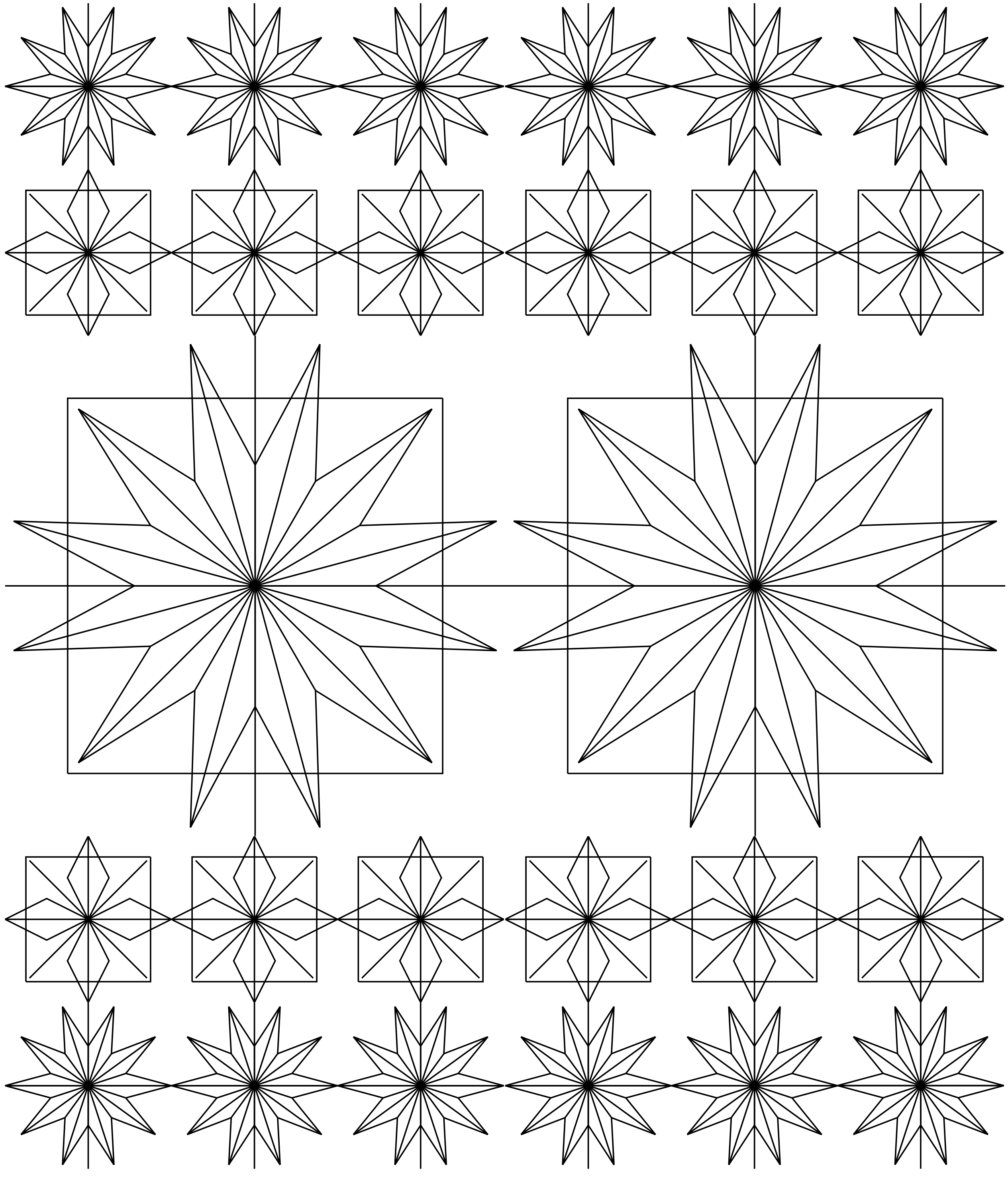
D12 (circular pattern)

There are bigger groups to see and more types of symmetry to talk about. There are even transformations that take our illustrations beyond shapes and generate patterns that repeat forever.



Challenge: What would happen if you reflected a shape across a mirror that sat next to the shape rather than through its center?

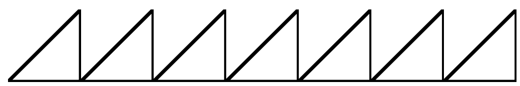




patterns of repeated shapes with mirror reflection

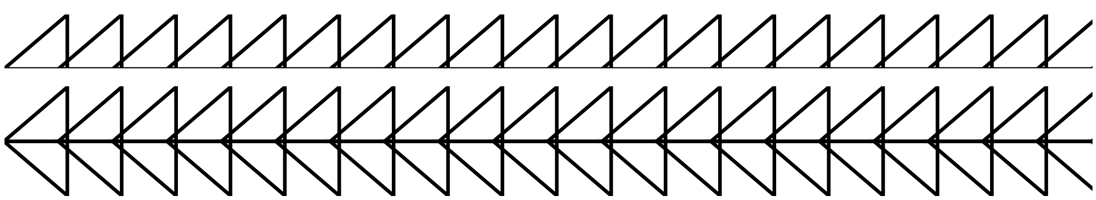
FRIEZE GROUPS

The **Frieze Groups** can be seen in patterns that repeat infinitely in opposite directions.

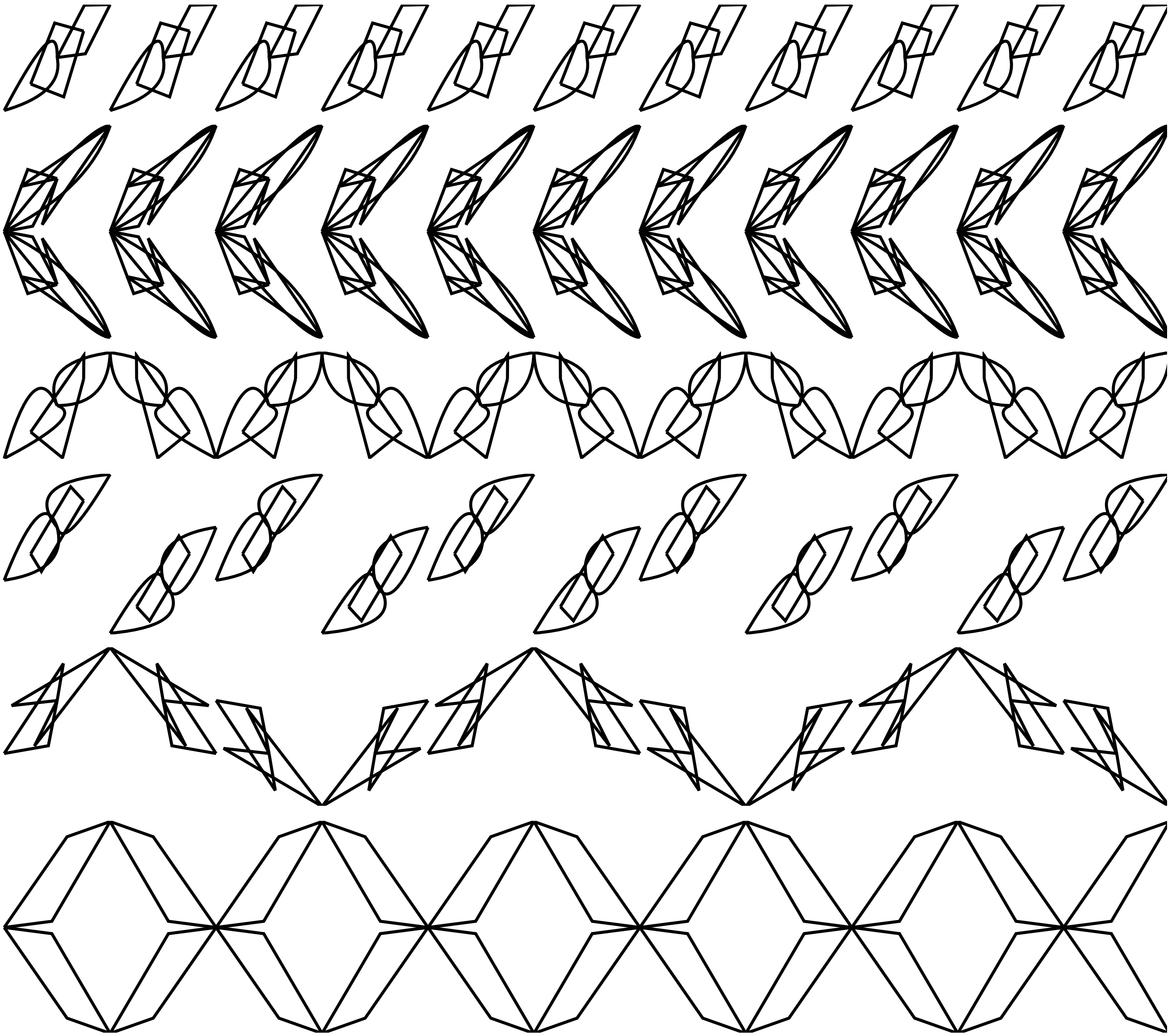
...  ...

A page cannot do patterns justice. It cuts them off when really they continue repeating forever...

Challenge: Can you see how the following patterns repeat across the page?

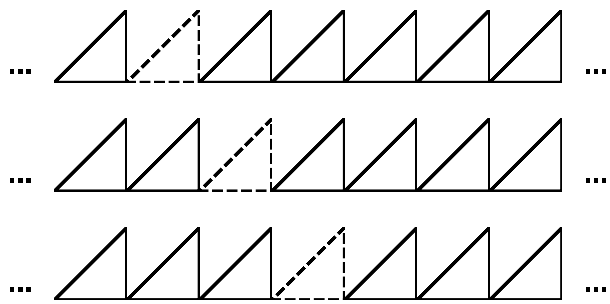


Coloring Challenge: Color the patterns in a way that maintains their repetition.



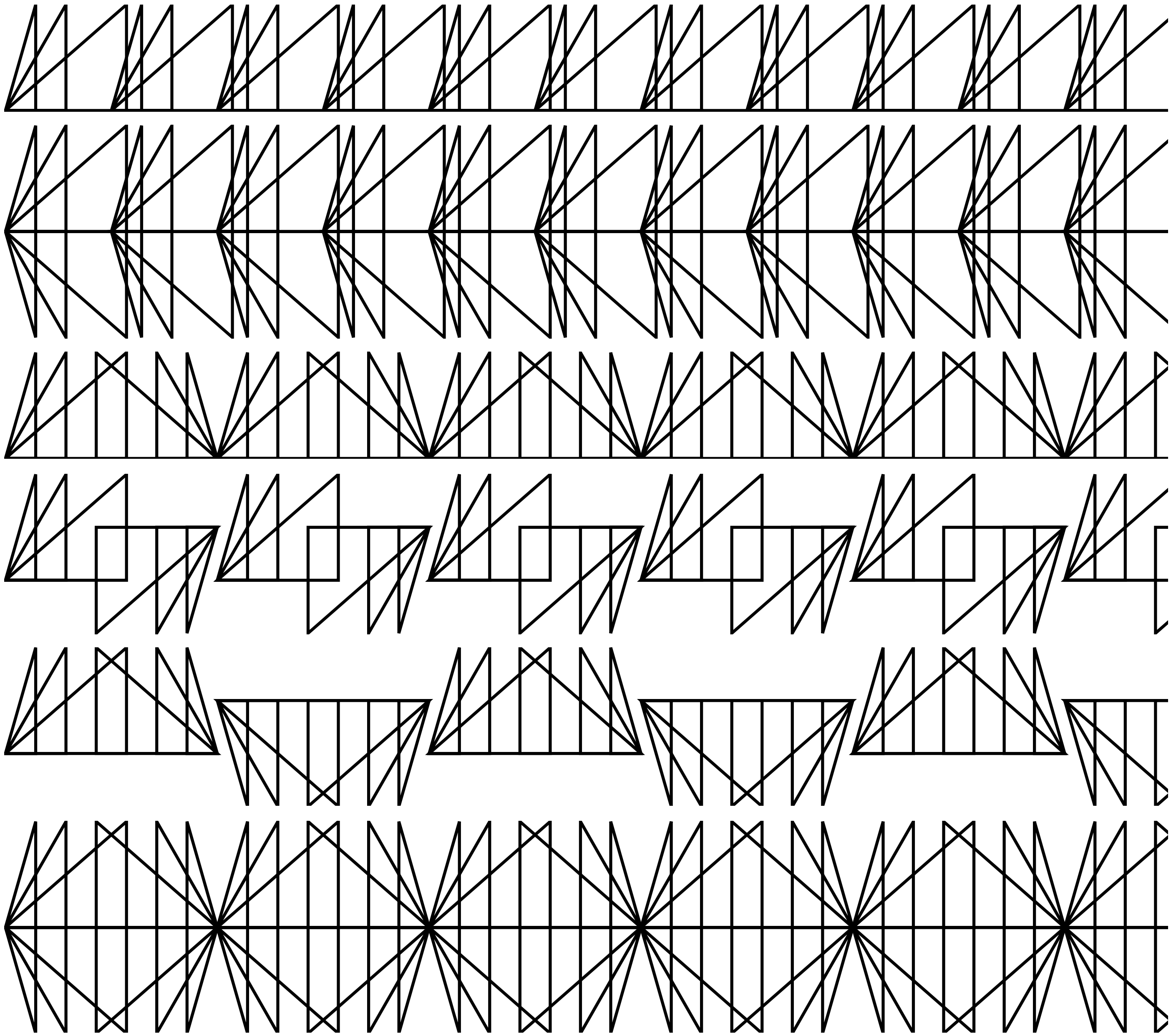
frieze patterns

This repetition introduces another type of symmetry - these patterns can shift over and still be the same pattern.



This shift is called **translation**.

Coloring Challenge: Color the patterns to make them less repetitive.



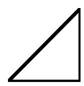
different frieze patterns with the same fundamental domain

p1

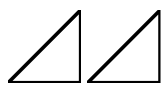
Translation

The simplest patterns can be generated by **translation** alone.

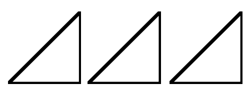
See, they can start with a single piece,

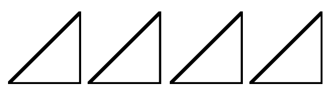


that gets copied, and then shifted over,



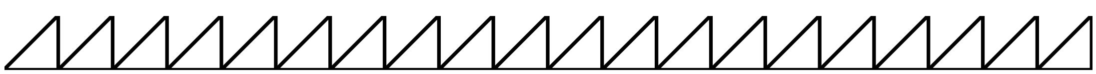
again and again...

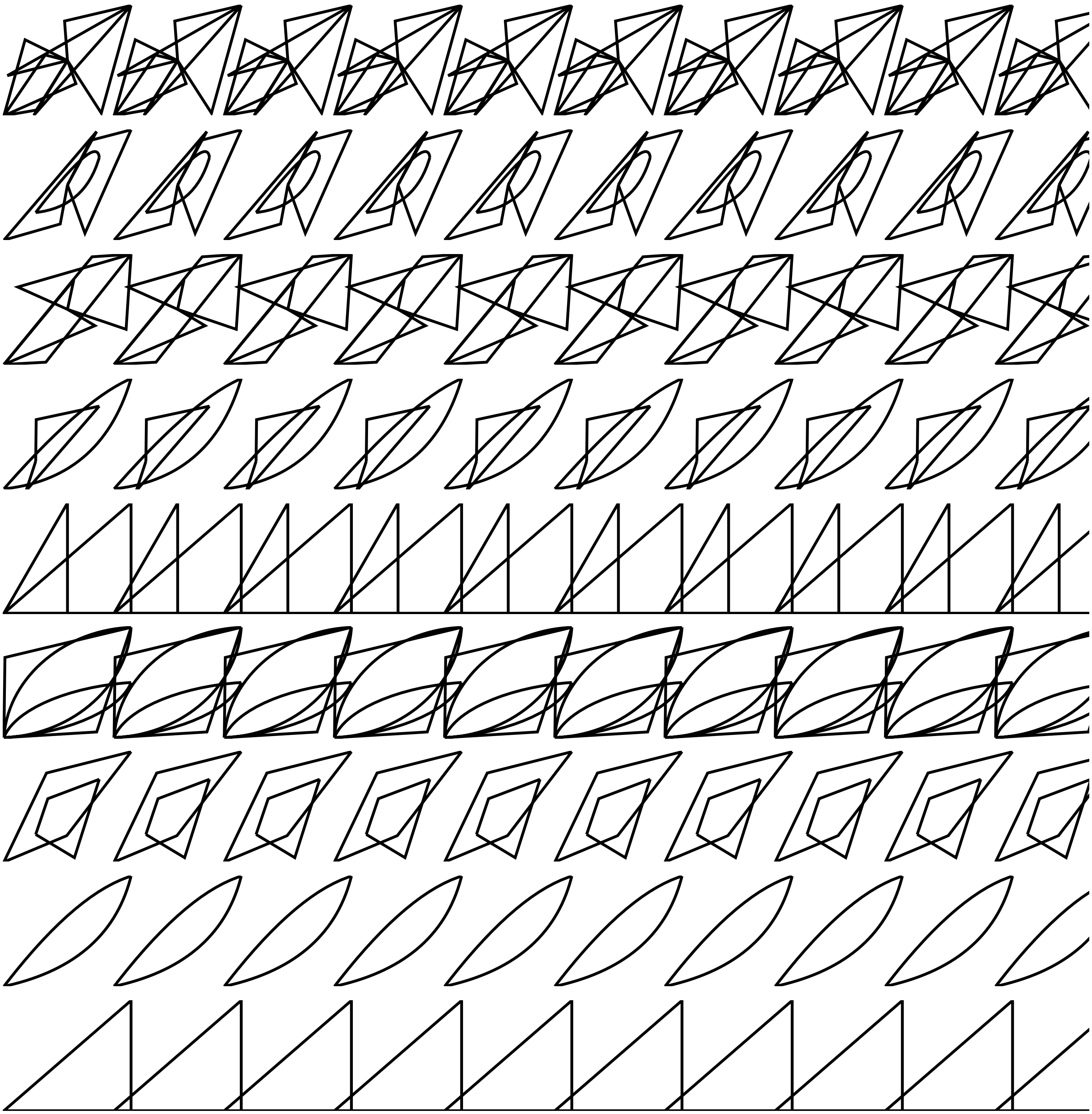




...

The following patterns may look different, but since they all have the same single generator of translation, they have the same symmetry - they belong to the same frieze group.





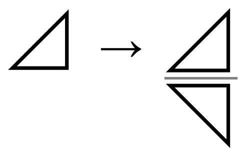
p1 frieze patterns: {translation}

p11m

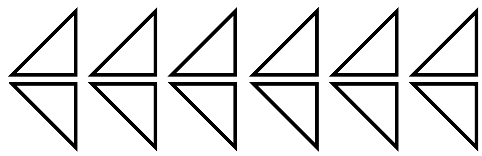
Horizontal Reflection & Translation

Our patterns can have more symmetry than just translation.

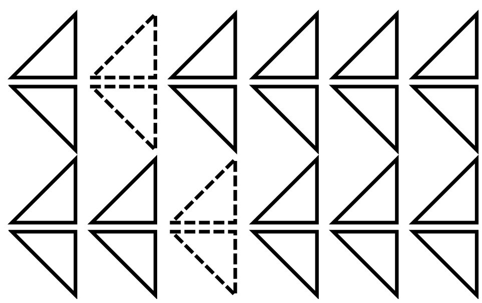
Reflecting a piece across a horizontal mirror before translating it,



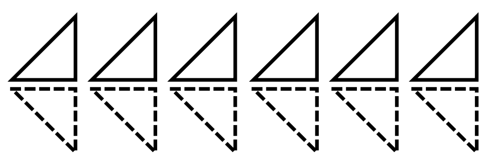
generates a new pattern, with more symmetry than the one before.



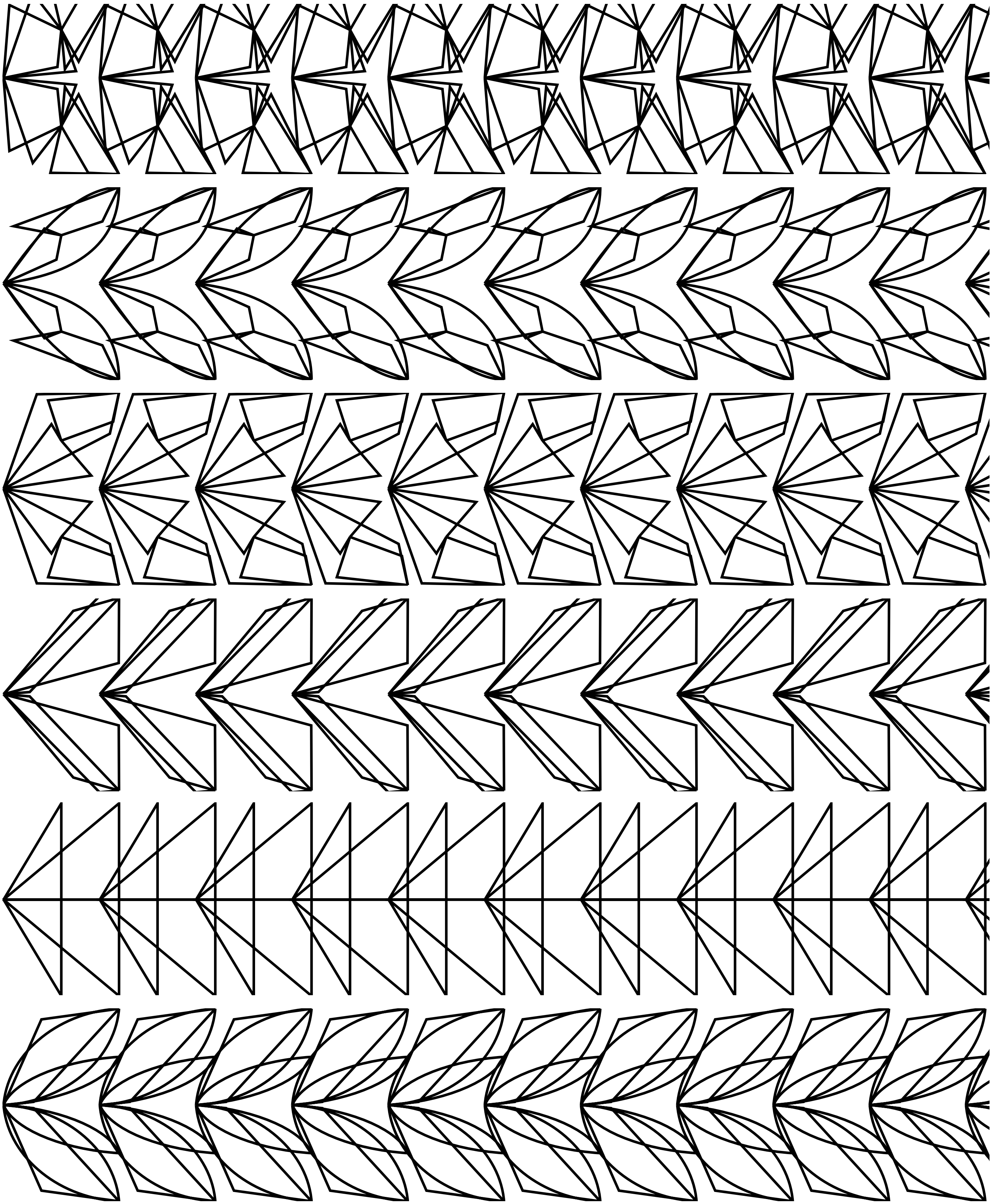
It still has **translation** - it can still shift over without changing -



but it also has a **horizontal mirror:** The whole pattern can reflect across the same mirror that transformed our first piece, and appear unchanged.



Challenge: Can you find the mirrors in the following patterns?

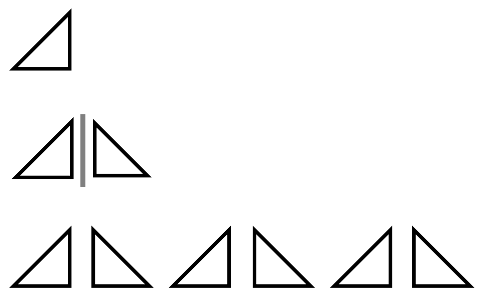


p11m frieze patterns: {horizontal reflection, translation}

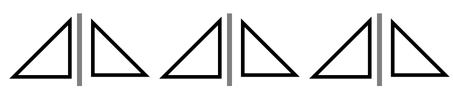
p1m1

Vertical Reflection & Translation

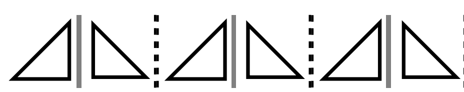
Patterns can have **vertical mirrors** as well.



These mirrors shift over with each repeated translation, so once a pattern has one vertical mirror, it has an infinite number of vertical mirrors.



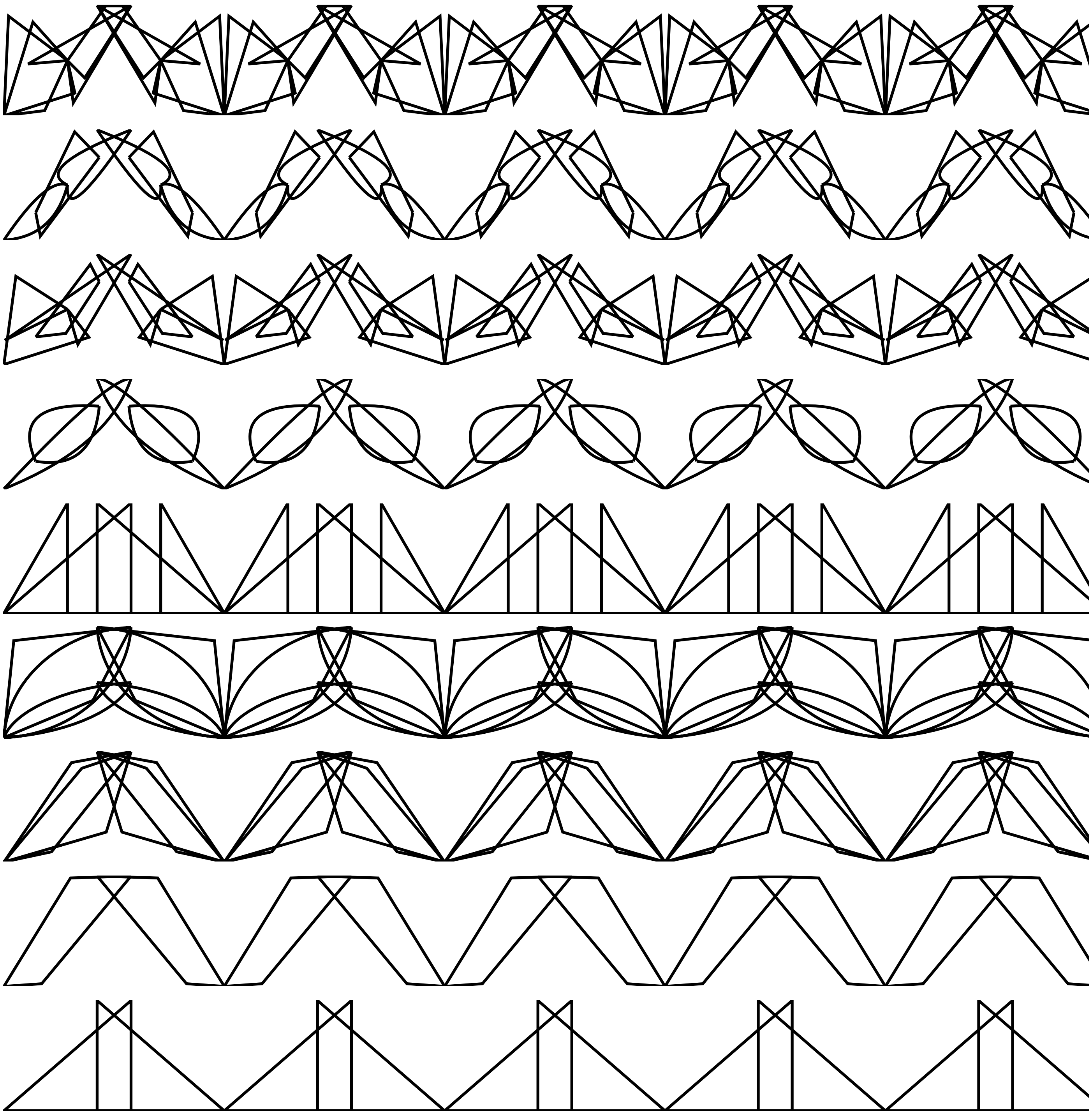
Twice that many, really.



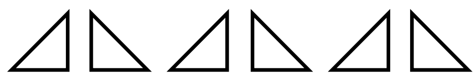
Even though we start with a vertical mirror on one side of each piece, as the pattern repeats, another different vertical mirror shows itself.

Challenge: Can you find the mirrors in the following patterns?

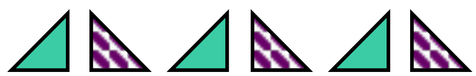
Coloring Challenge: Can you color the patterns to remove the mirrors?



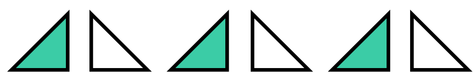
p1m1 frieze patterns: {vertical reflection, translation}



All of these mirrors can be removed with color.



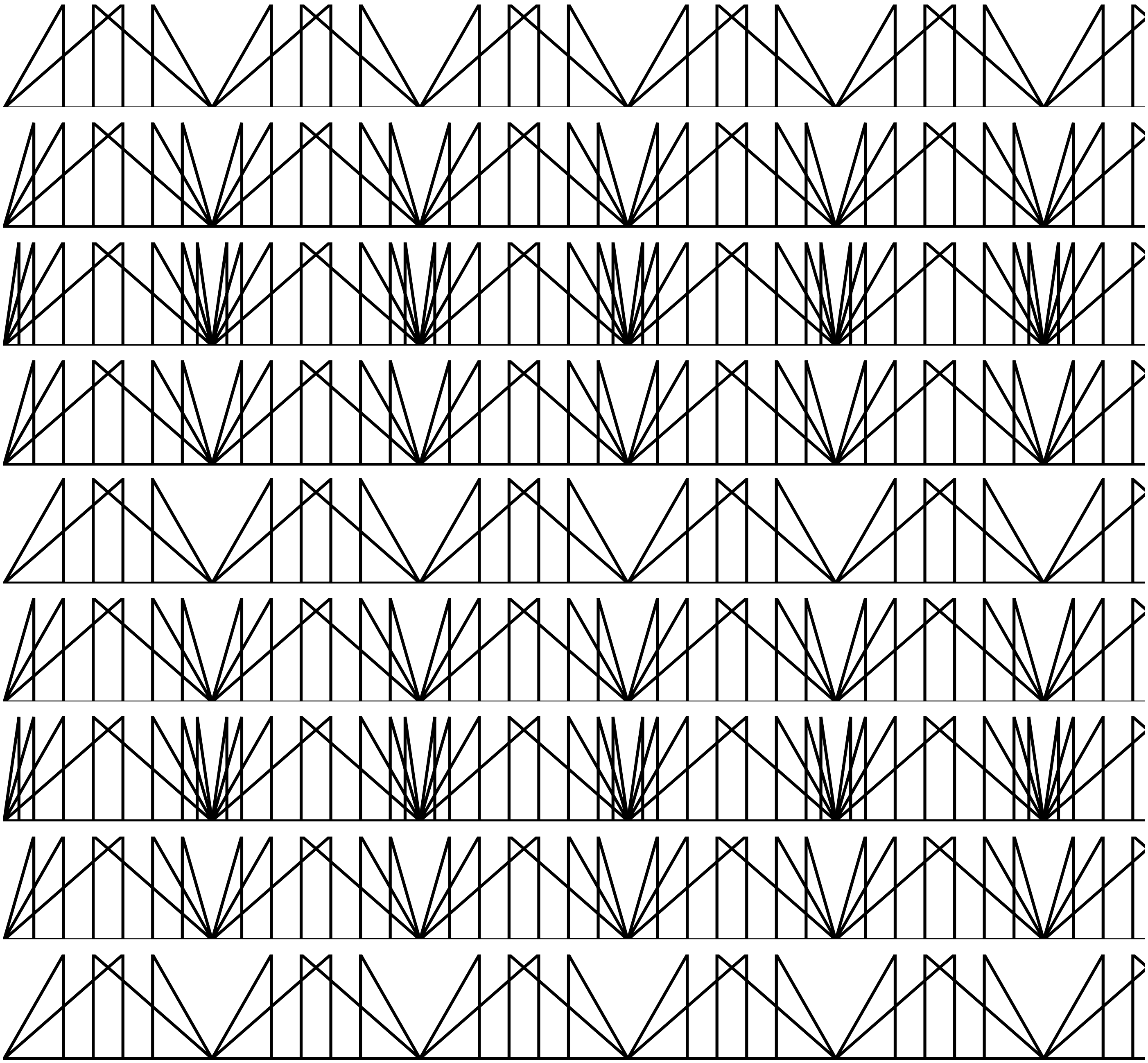
With color, we can reduce the patterns so that translation is their only symmetry.



Why can we do this?

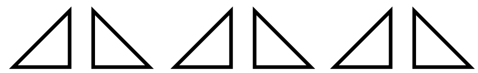
This brings us back to subgroups.

Coloring Challenge: Color the patterns to remove any mirrors so that translation is their only symmetry.

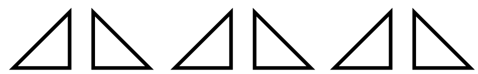


p1m1 frieze patterns: {vertical reflection, translation}

Our patterns with vertical reflection belong to a symmetry group with translation and vertical reflection.

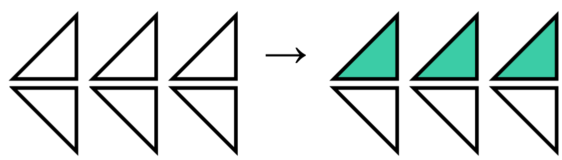
{vertical reflection, translation}: 

Naturally, the group with only translation is a subgroup.

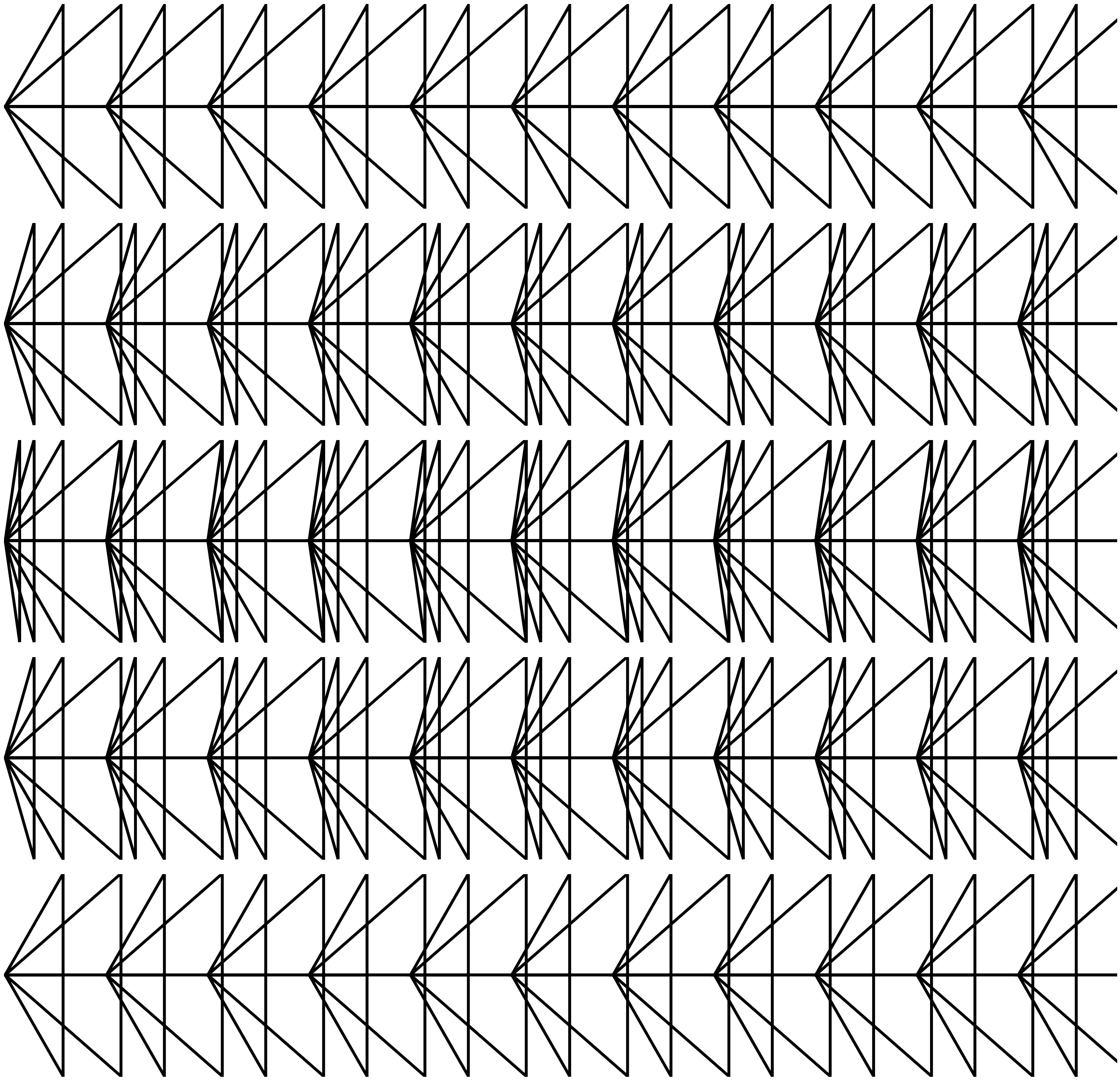
{vertical reflection, translation}: 

{translation}: 

The same goes for our patterns with horizontal reflection. Color can remove their mirrors as well, and reduce them to patterns with only translation.



Coloring Challenge: Can you find all of the mirrors in these patterns and use color to remove them?



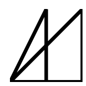
p11m frieze patterns: {horizontal reflection, translation}

<u>p2</u>

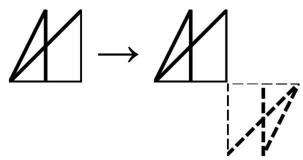
Rotation & Translation

Frieze patterns can also have ½ turns.

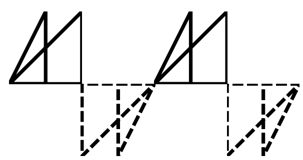
They can be generated by a single piece,

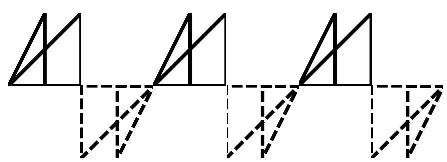


that rotates by a ½ turn around a point,

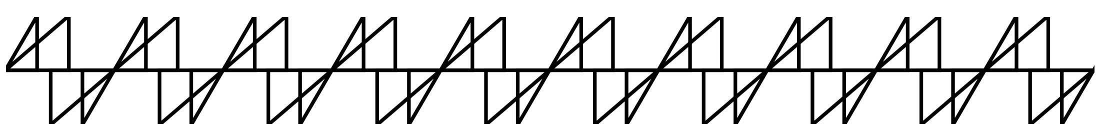


before translating.

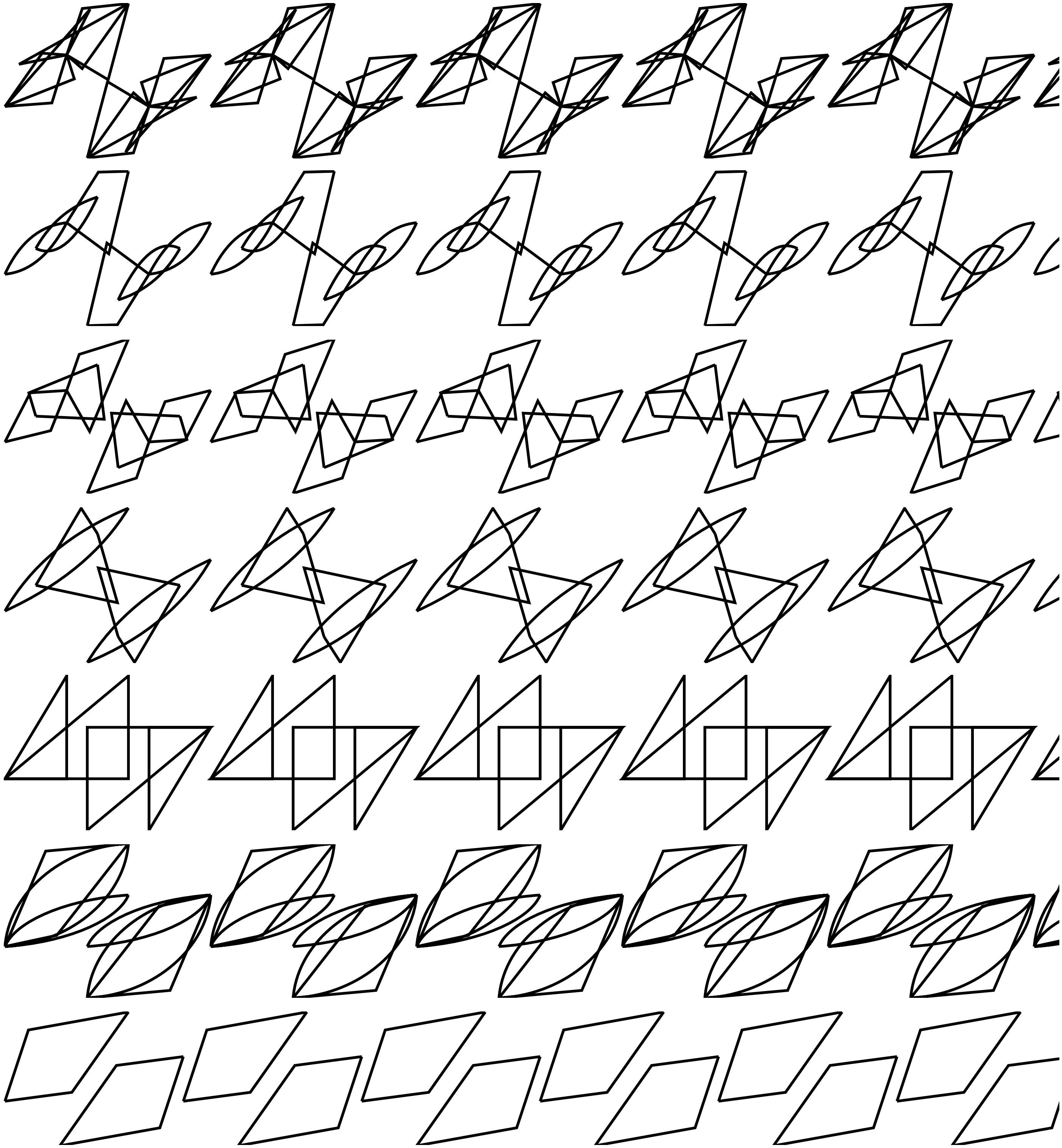




...



Challenge: Can you find the ½ turns in the following patterns?

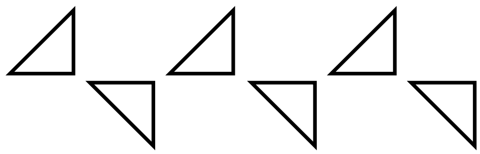


p2 frieze patterns: {½ turns, translation}

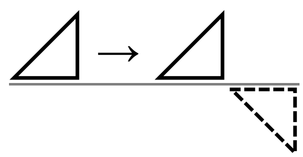
p11g

Glide Reflection & Translation

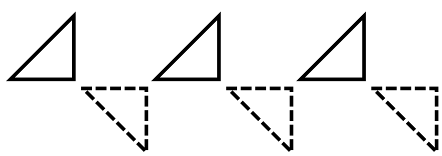
There is another type of symmetry called **glide reflection**.



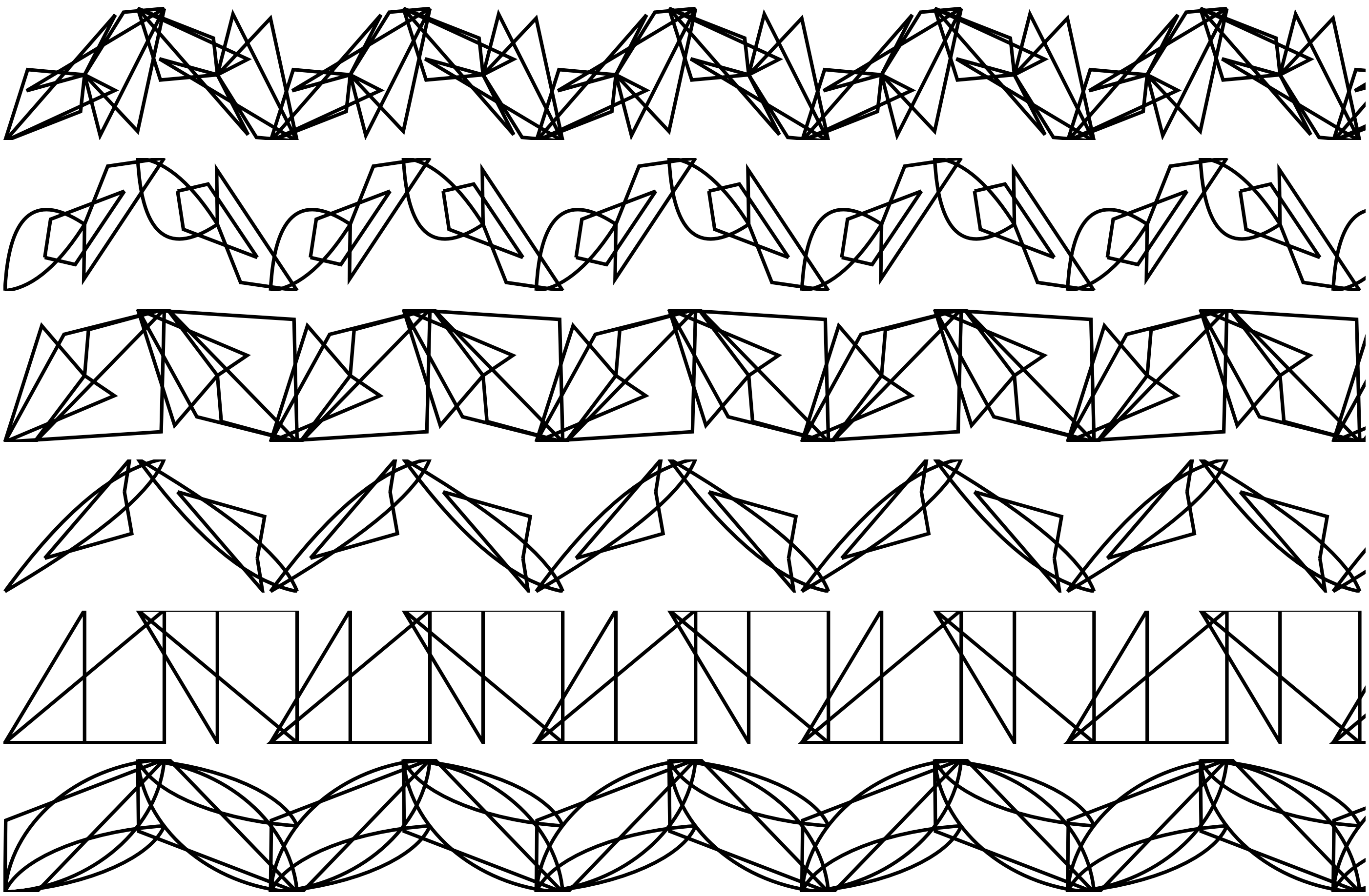
A glide reflection is a transformation that reflects across a mirror line at the same time as translating along it.



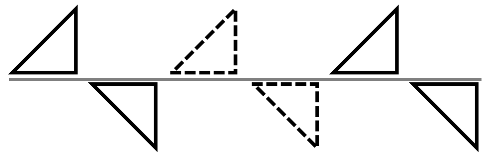
By continuing to translate or glide, a pattern with glide reflection is generated.



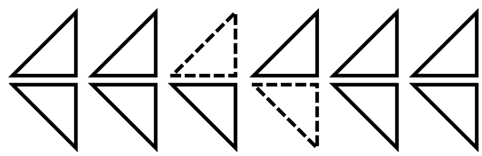
Challenge: Can you find the glide reflections in the following patterns?



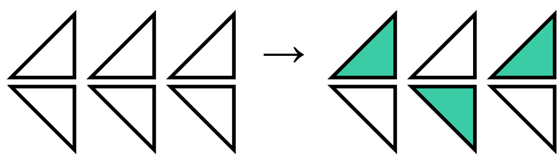
p11g frieze patterns: {glide reflection, translation}



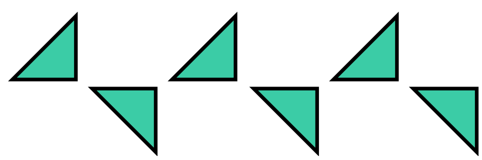
Glide reflections show themselves in other patterns as well. The patterns we generated with horizontal reflection have glide reflection too,



and color can reduce them

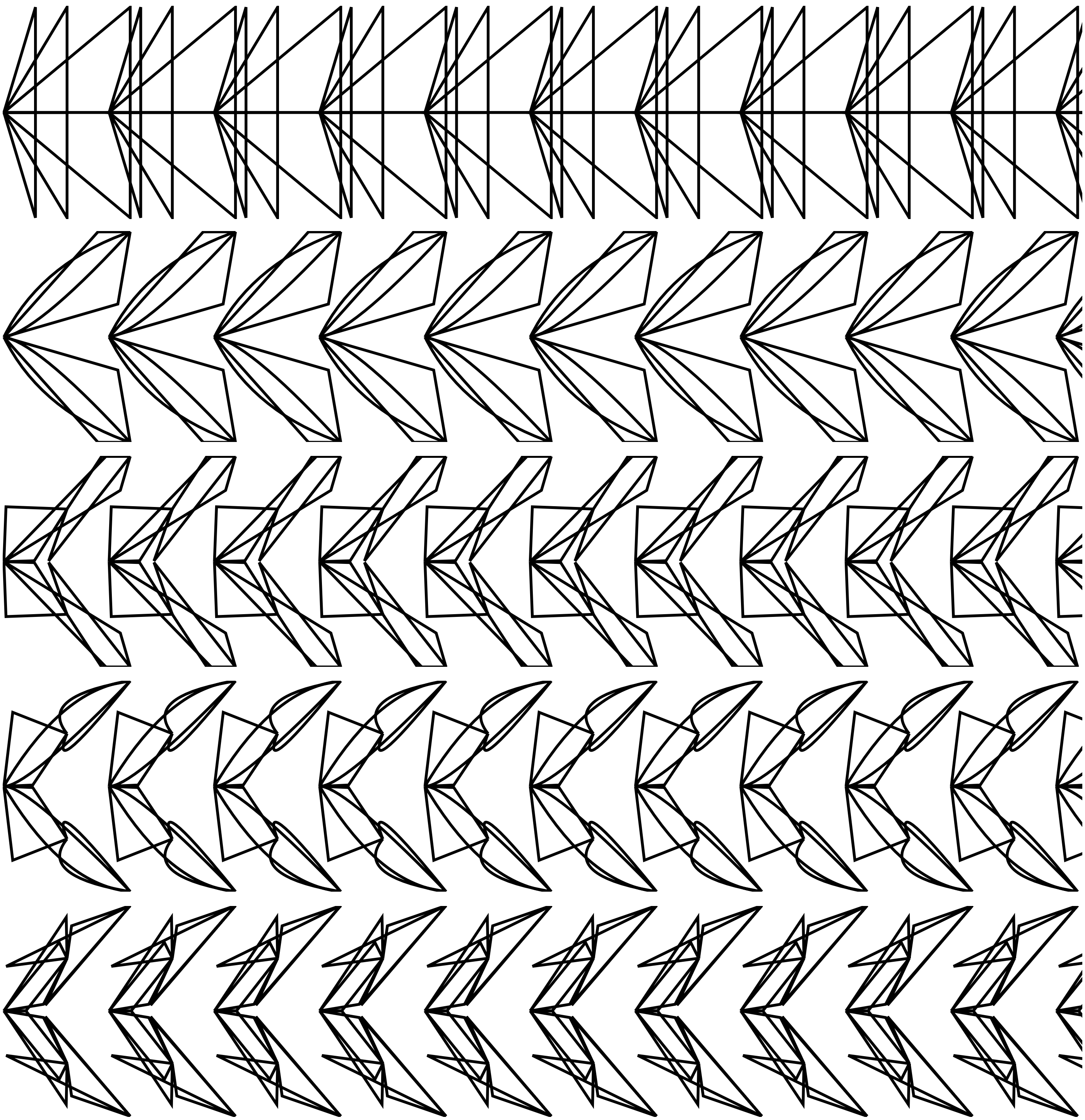


to patterns with glide reflection only.



Coloring Challenge: Use color to reduce the patterns with horizontal reflection so that they only have glide reflection.

Coloring Challenge: Can you add more color to then remove the glide reflections?



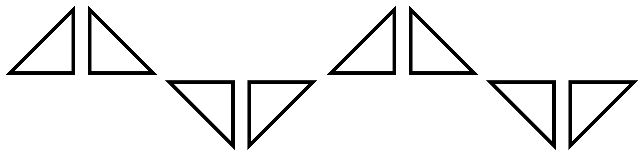
p11m frieze patterns: {horizontal reflection, translation}

p2mg

Glide Reflection & Vertical Reflection & Rotation & Translation

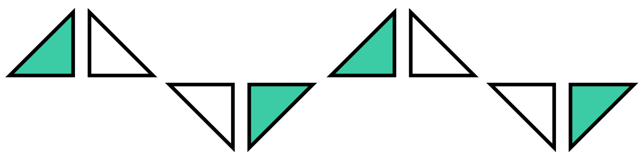
Combining more of these symmetries yields more groups of patterns. Take the group

{glide reflection, vertical reflection, rotation, translation}.

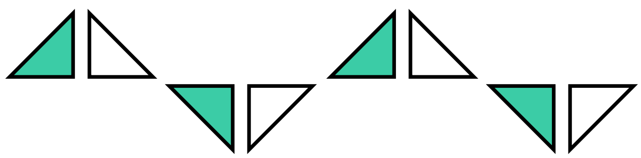


Color can again reduce this pattern to have the same symmetry as the simpler patterns we already colored.

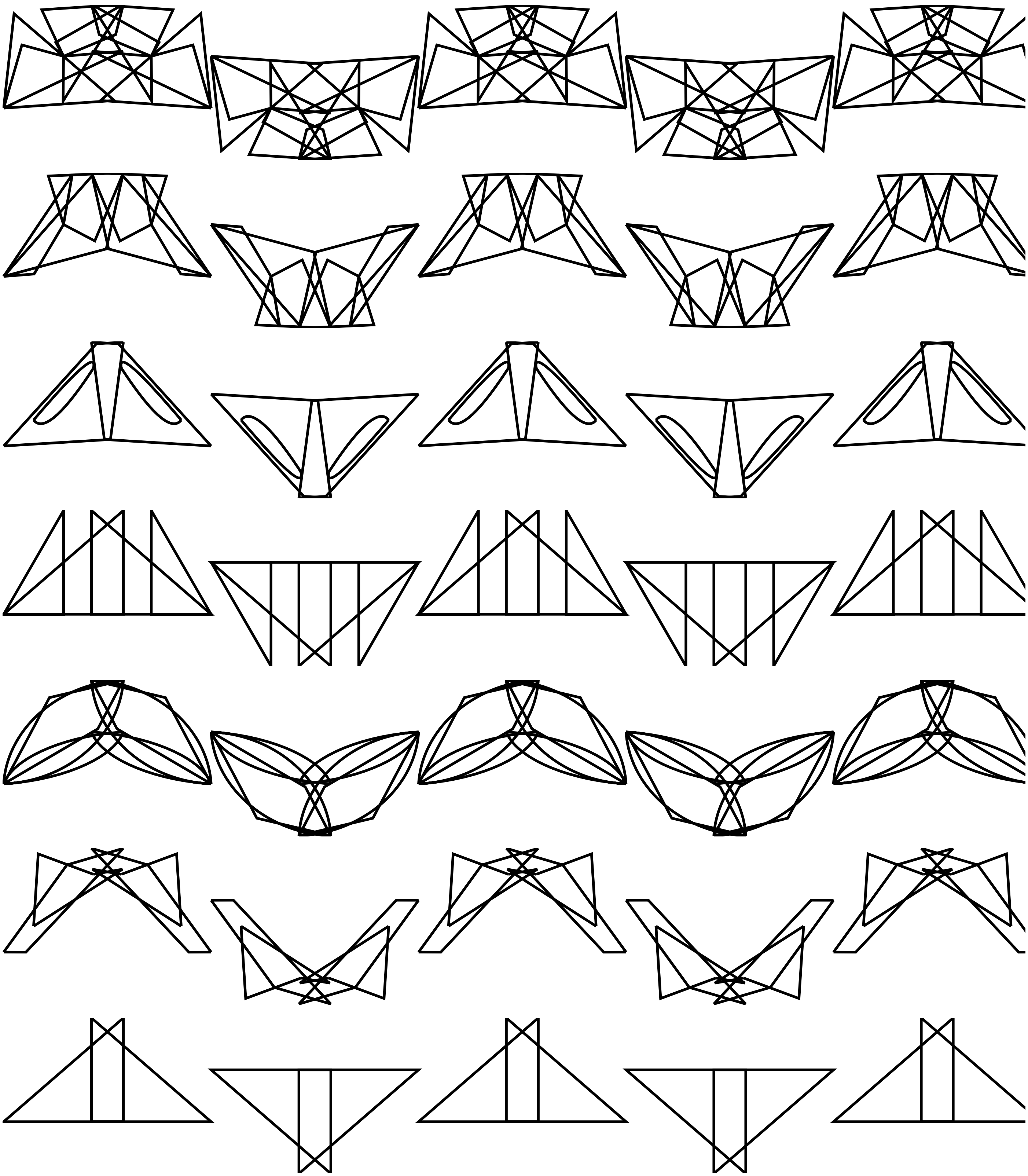
{rotation, translation}:



{glide reflection, translation}:



Coloring Challenge: Use color to reduce the amount of symmetry in the patterns so that they only have vertical reflection and translation.



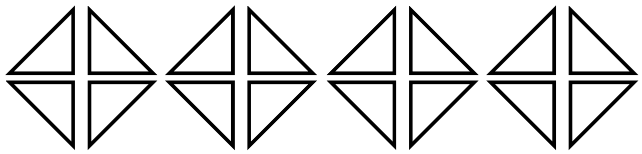
p2mg frieze patterns: {glide reflection, horizontal reflection, translation}

p2mm

Horizontal Reflection & Vertical Reflection & Rotation & Glide Reflection & Translation

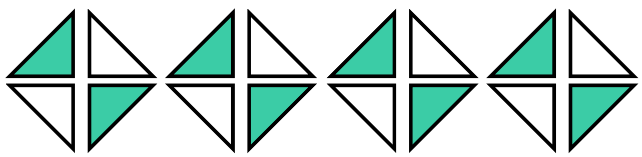
The group that has all of our symmetries,

{horizontal reflection, vertical reflection, rotation, glide reflection, translation},

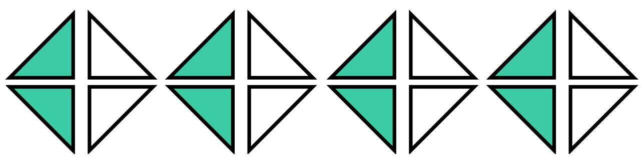


can be reduced to each of the patterns we have already seen.

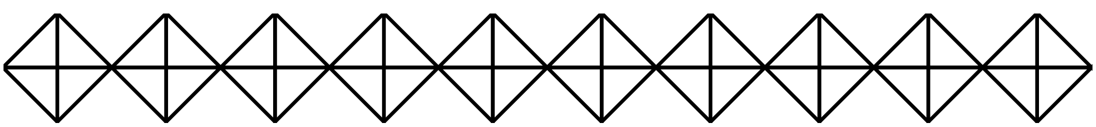
{rotation, translation}:



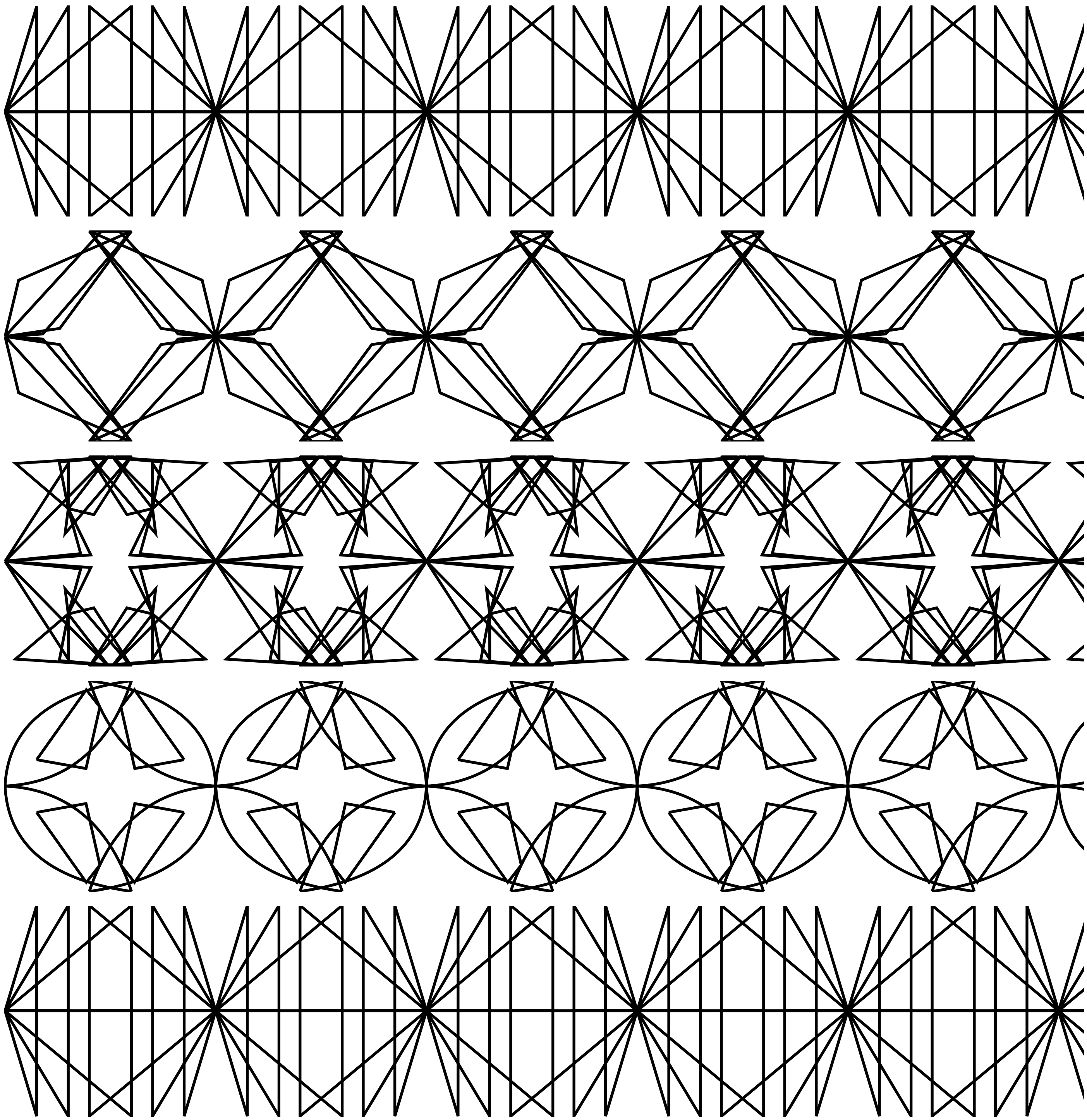
{horizontal reflection, translation}:



You can find the rest.



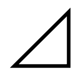
Coloring Challenge: Use color to transform the patterns of the group {horizontal reflection, vertical reflection, rotation, glide reflection, translation} *into each of the simpler pattern groups we have seen.*



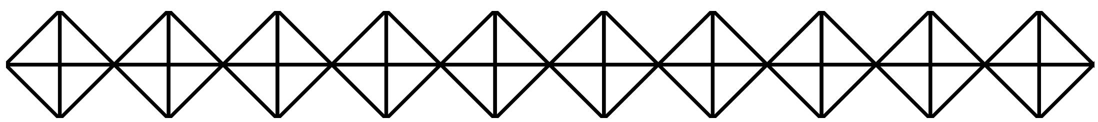
p2mm frieze patterns: [horizontal reflection, vertical reflection, glide reflection, $\frac{1}{2}$ turns, translation]

We have now colored all 7 Frieze Groups. There are no other groups of patterns that repeat forever in one direction.

Surprised? Then try to generate more by again starting with a single piece.

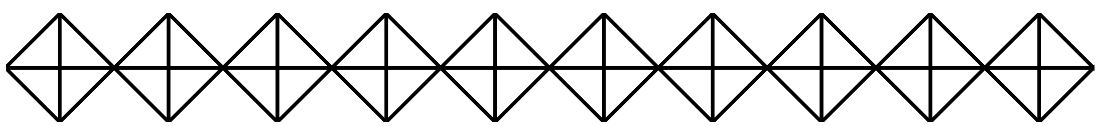


Or use color to reduce a pattern to one with combinations of symmetries that we did not yet see, like **{horizontal reflection, vertical reflection, translation}**.

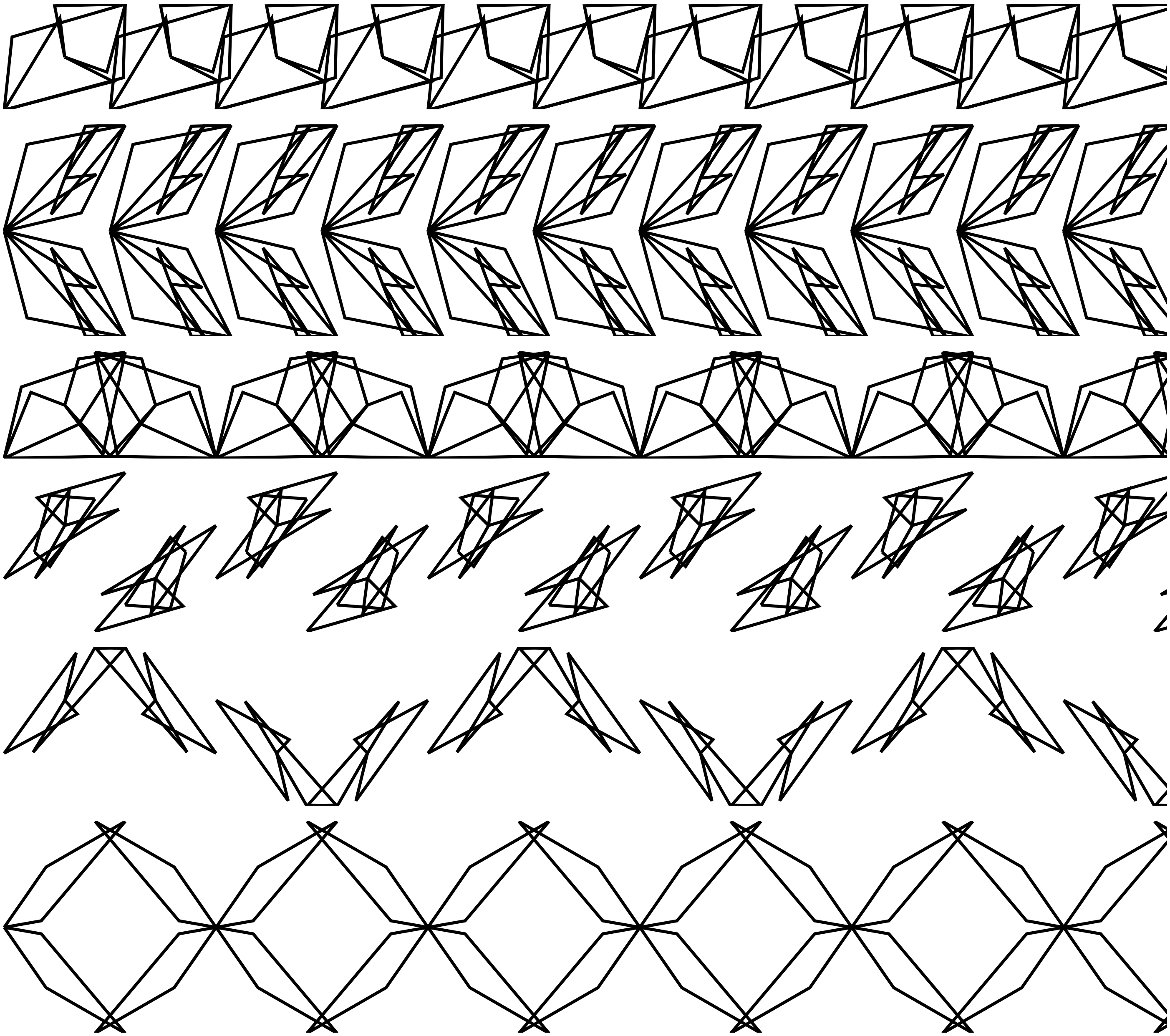


You will have to give up.

Combining **horizontal reflection** and **vertical reflection** brings about **glide reflection** and **rotation**. This is just one example of how combining symmetries brings us back to patterns we already have.



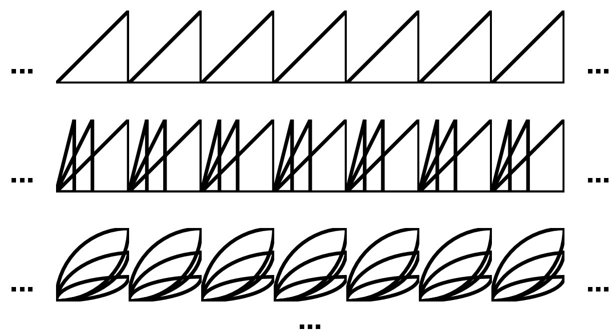
Challenge: What happens when you combine glide reflection with rotation?



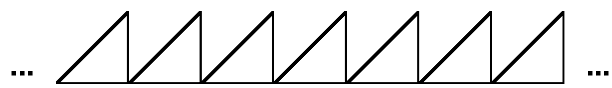
There may only be 7 Frieze Groups, but each has an infinite number of patterns.

Even our simplest group can appear in an endless number of ways.

{translation}:

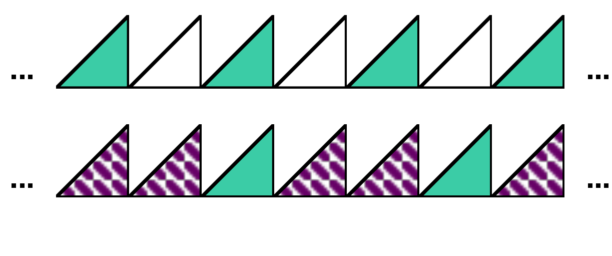


Yet when we focus on just one of these patterns,



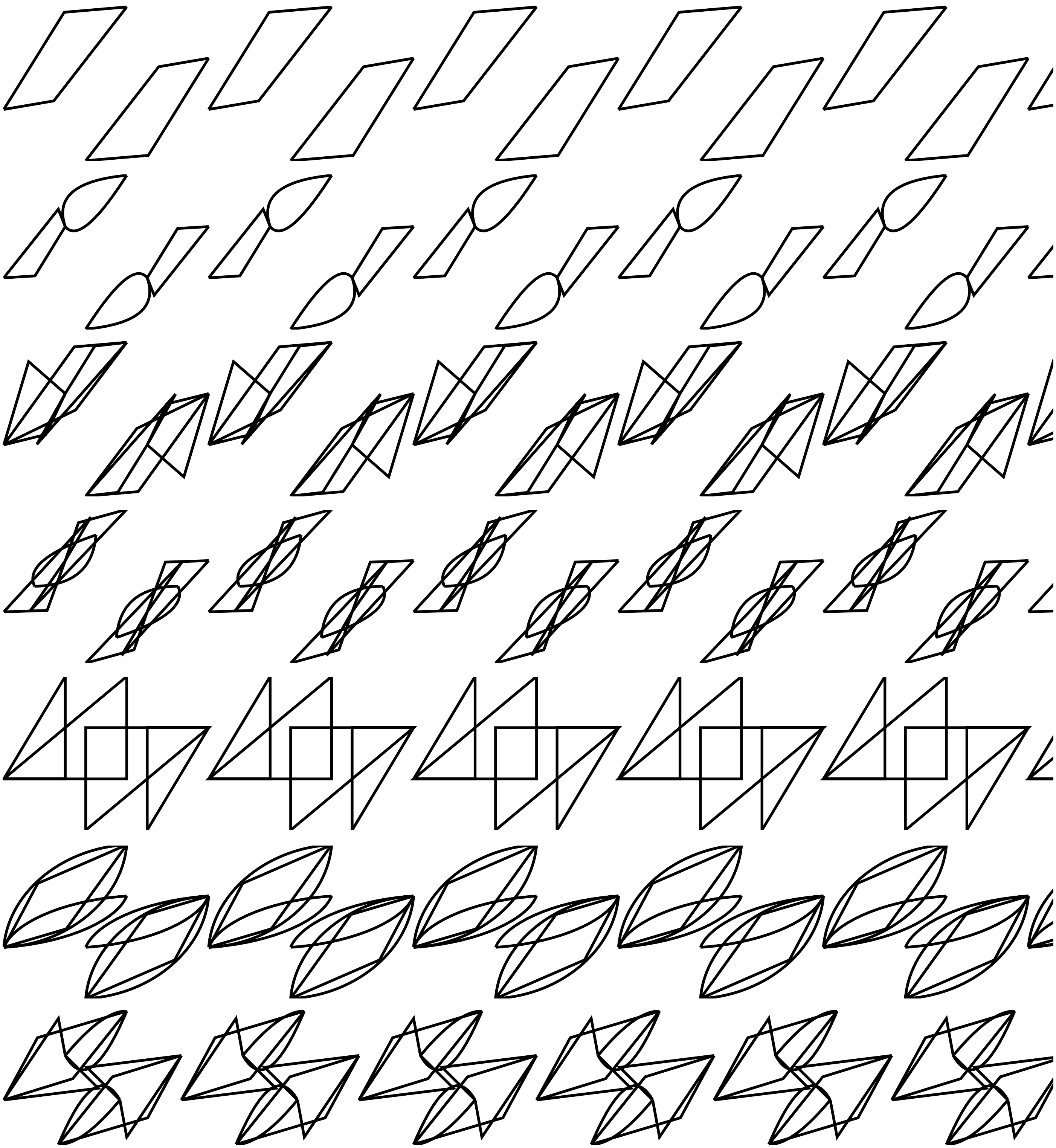
color can transform it into infinitely more that still belong to the same group.

{translation}:

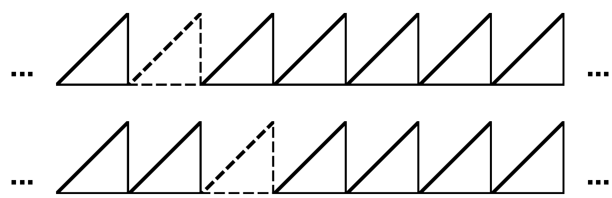


We can see why...

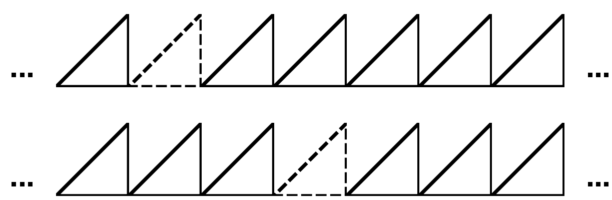
Coloring Challenge: Color the patterns in a way so that their symmetry group does not change.



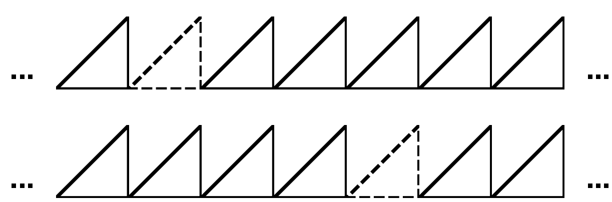
A **translation** shifts our patterns over by a certain distance without changing them,



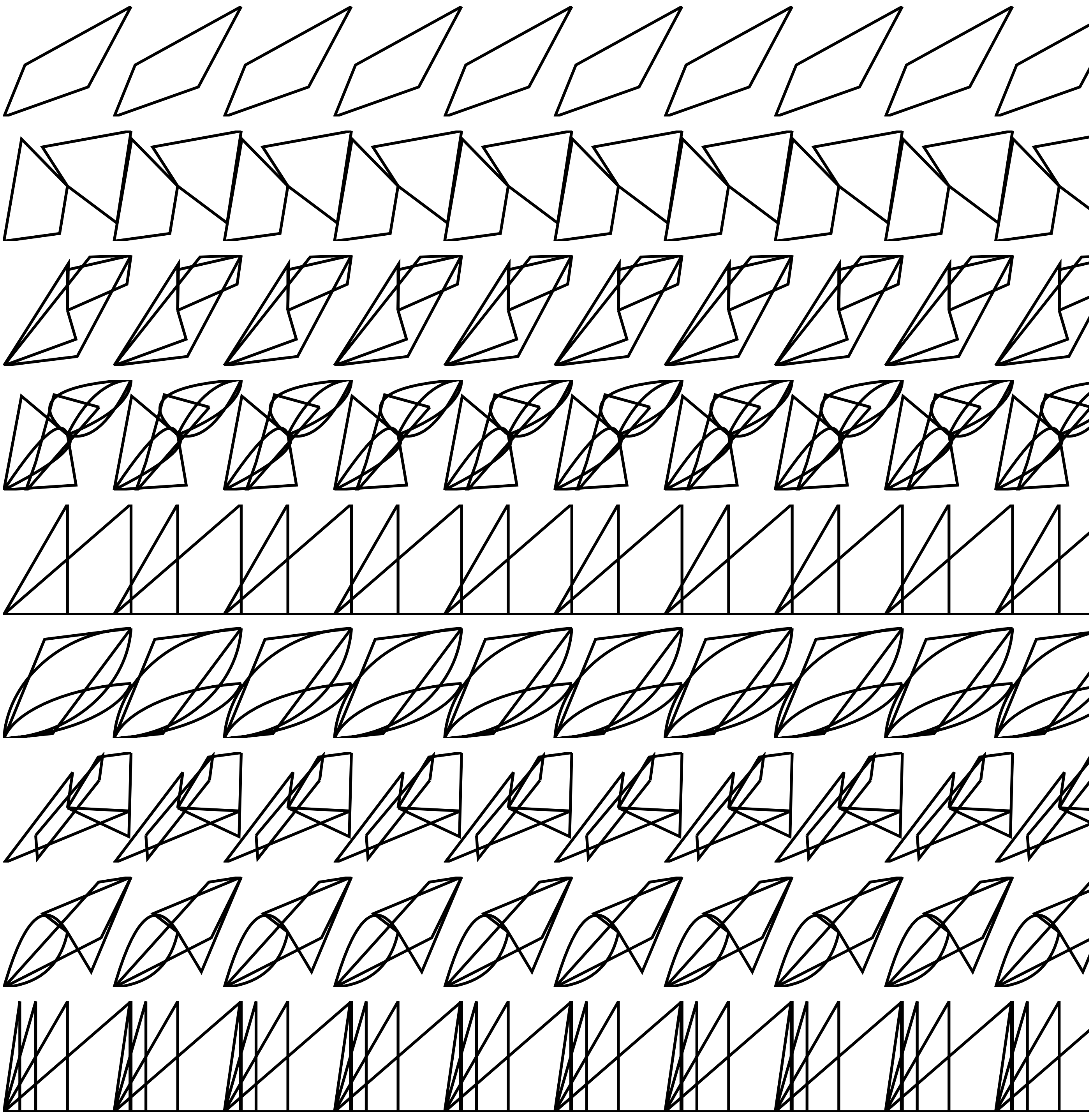
but shifting by twice that distance is also a translation.



So is 3 times that distance,

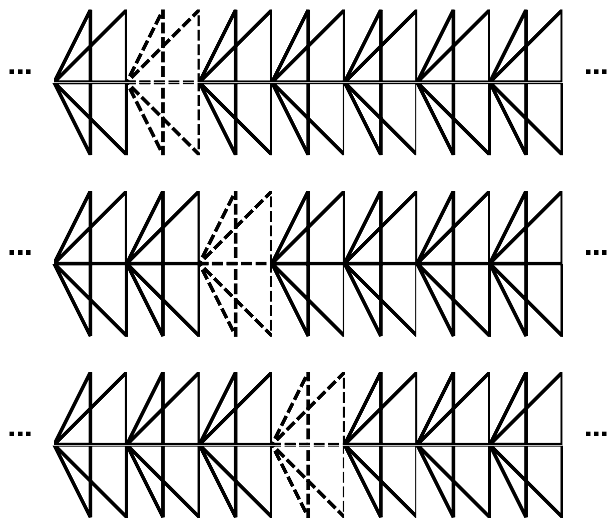


and so on...

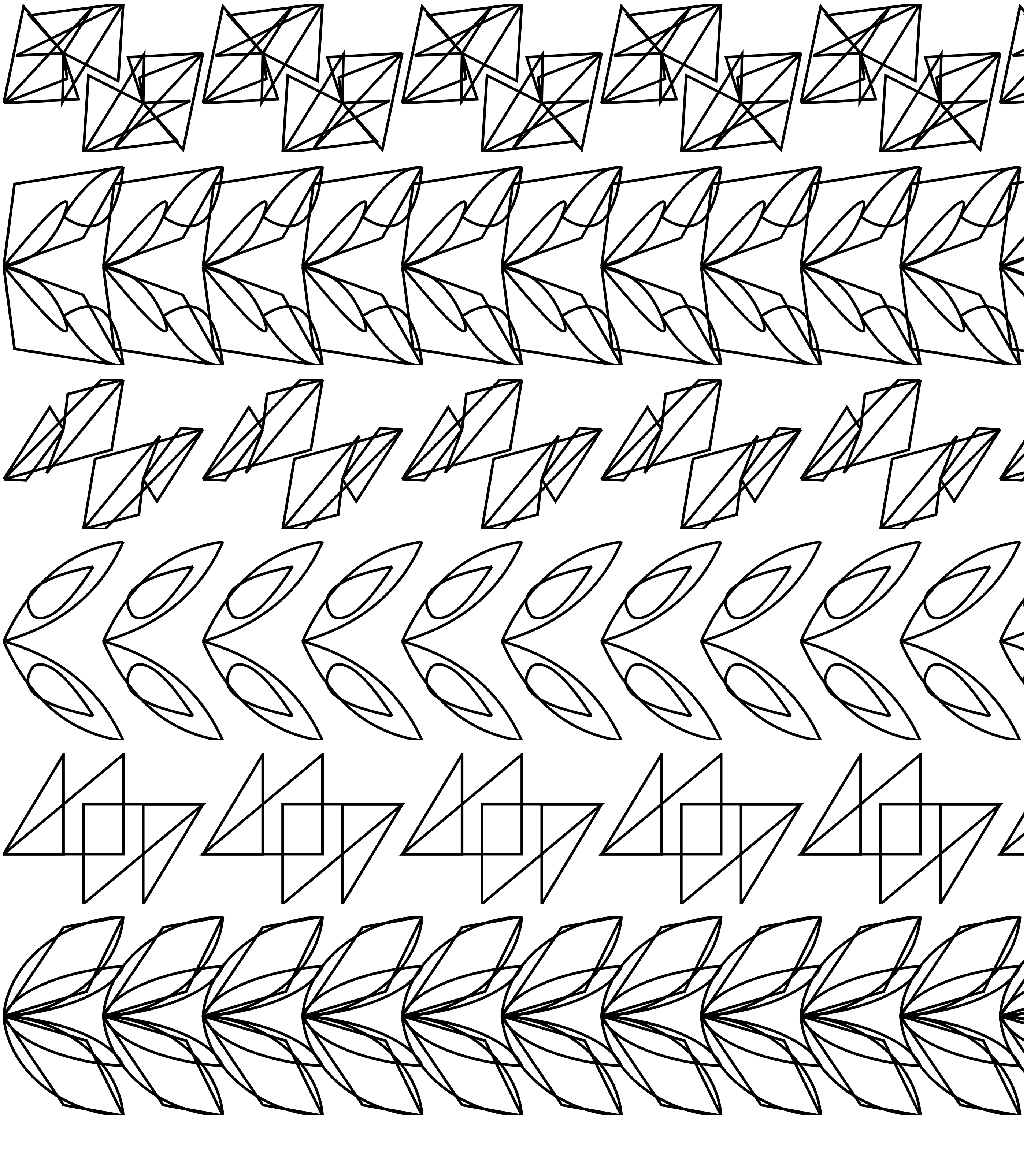


We can see this in more complex patterns as well.

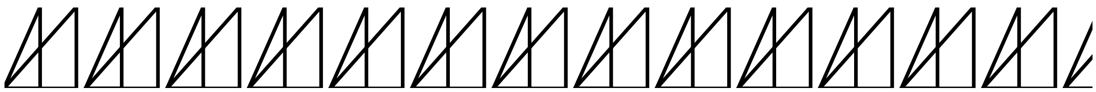
{horizontal reflection, translation}:



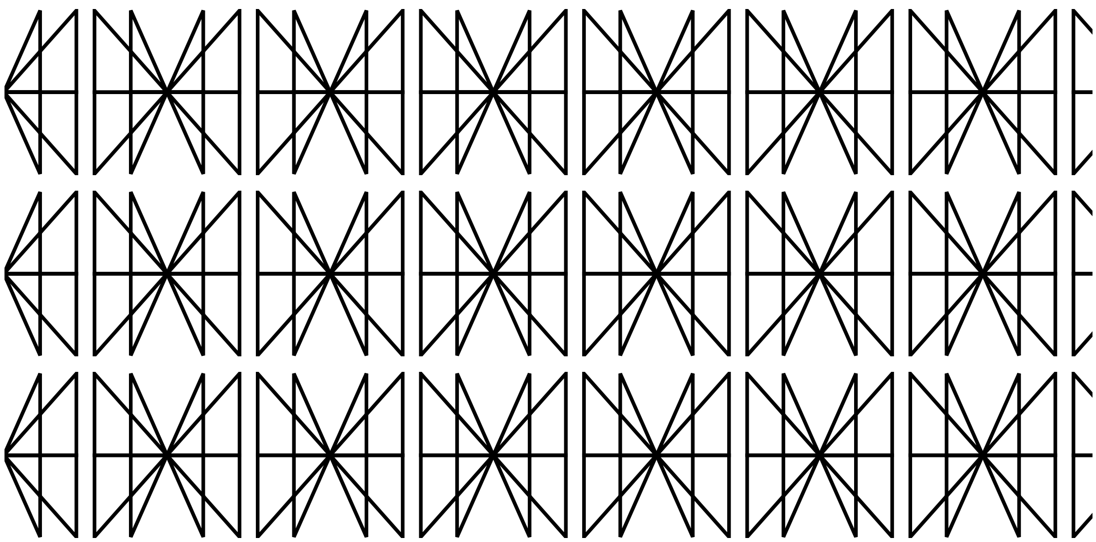
Coloring challenge: Color the patterns in a way so that the distance between their repetitions is larger than it was without color.



The 7 Frieze Groups may yield an infinite number of patterns, but this infinity is small compared to the number of patterns of the Wallpaper Groups.

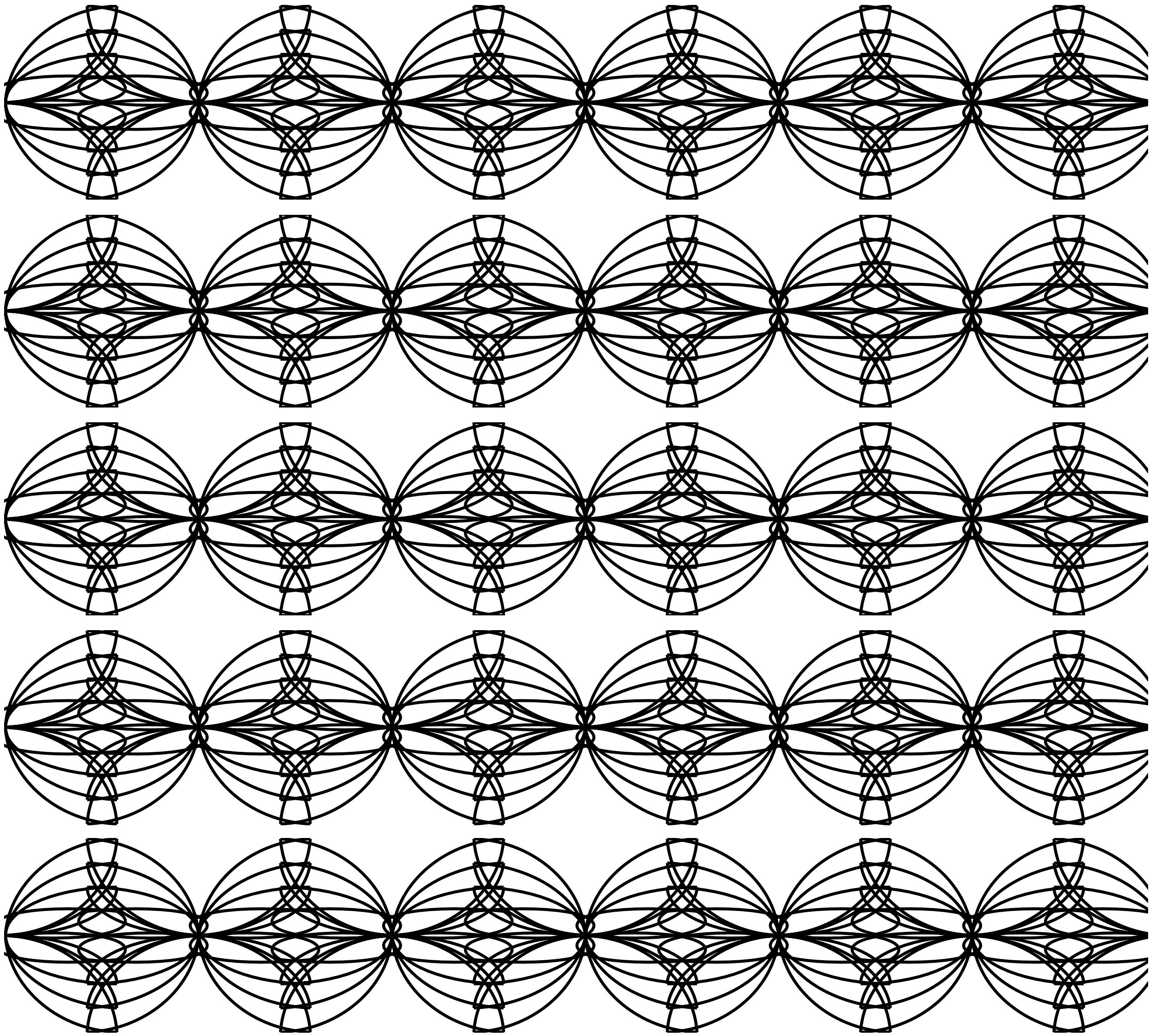


Frieze patterns are limited to repetition in one direction, but Wallpaper patterns do not have that limit.



When that limit is removed for the Wallpaper Groups, the number of possible patterns and amount of symmetry within them grows beyond what we have colored.

Challenge: Can you find all of the directions in which the pattern repeats?



More is coming soon.