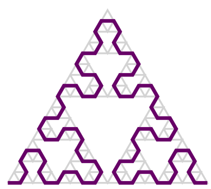
Color By Group Theory

Math is about more than just numbers. In this "book" the story of math is visual, told in shapes and patterns.



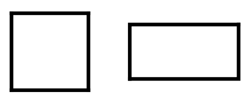
Group theory is a mathematical study with which we can explore symmetry.

This is a coloring book.

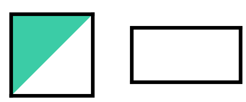
Table of Contents // Coloring Guide

Shapes & Symmetries

We have intuitive ideas of symmetry. Some shapes look "more symmetric" than others. For example, a square is "more symmetric" than a rectangle,

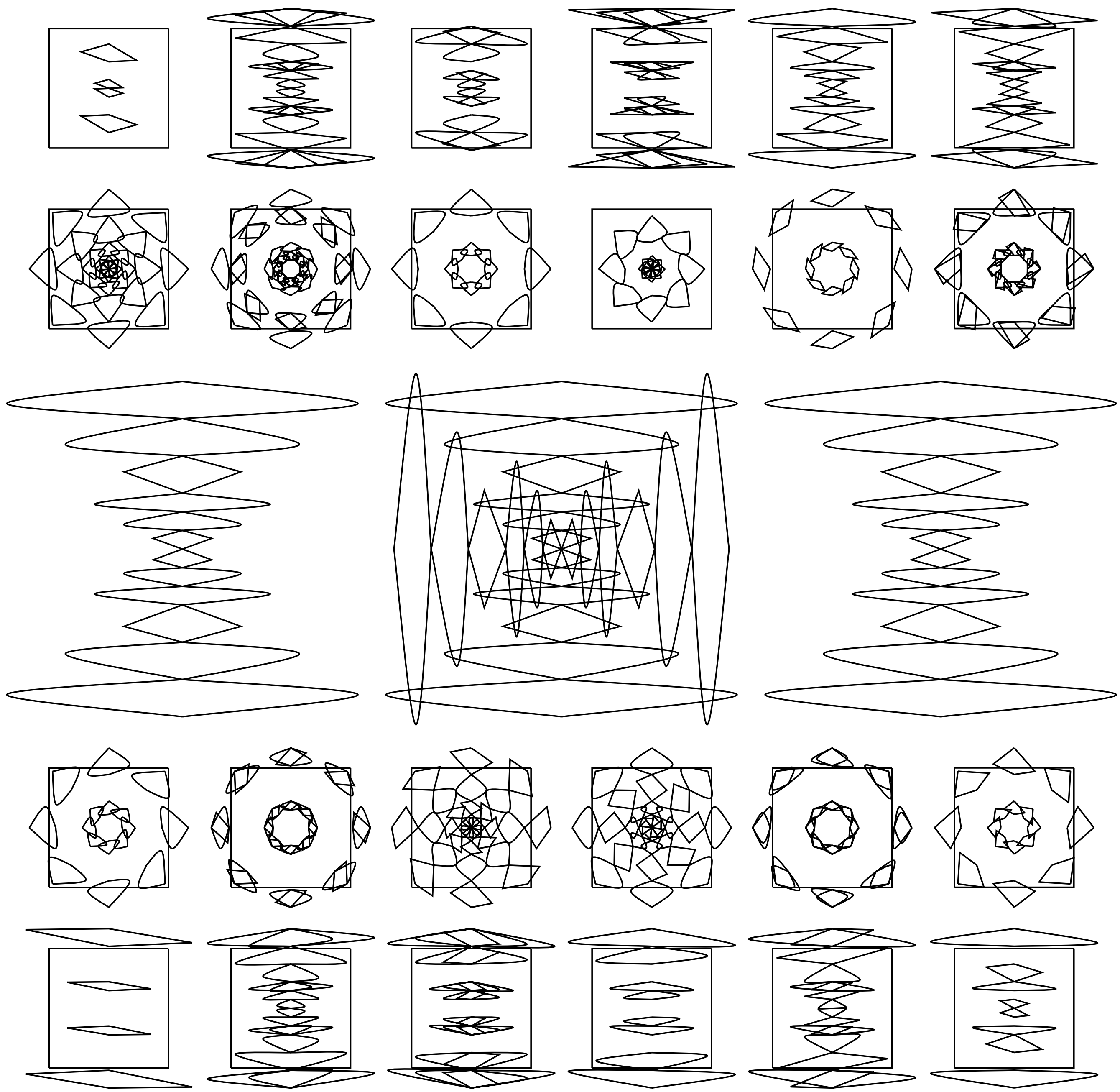


but this can change once color is added.



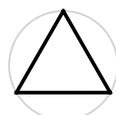
What does this mean? We can be more precise. We can even count how much symmetry a shape has.

Challenge: Can you color the shapes to give them the same "amount of symmetry"?

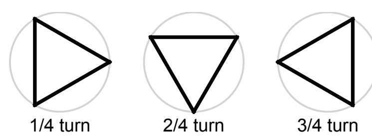


Rotations

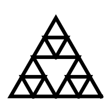
An equilateral triangle can be rotated $\frac{1}{3}$ of the way around a circle and appear unchanged.

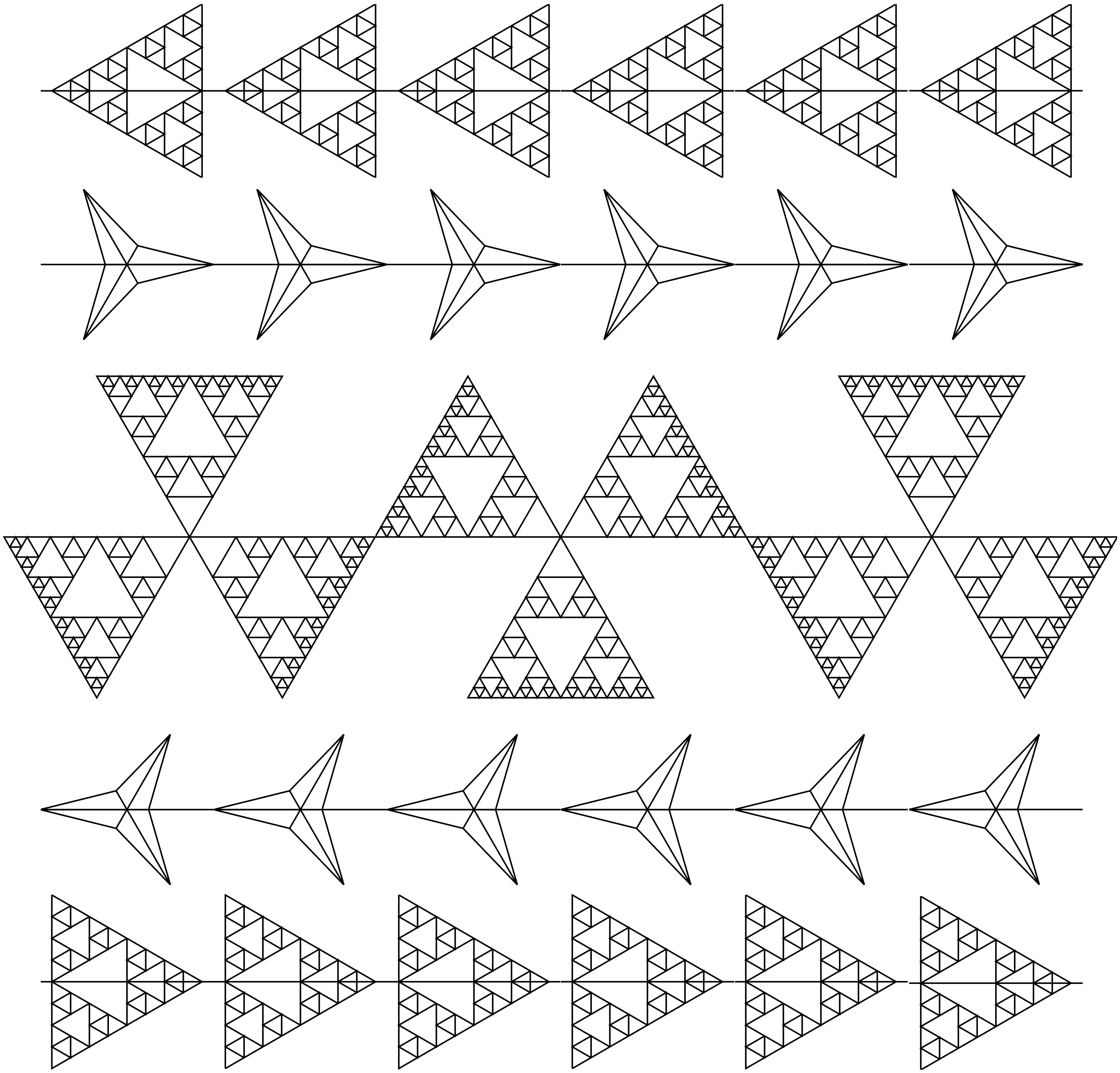


If the triangle were instead rotated by an arbitrary amount, like $\frac{1}{4}$ of the way around a circle, it would then appear changed, since it is oriented differently.

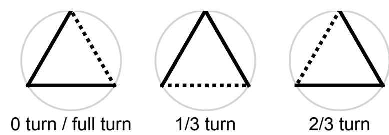


Challenge: Can you color the shapes so that a $\frac{1}{3}$ turn continues to leave their appearance unchanged?

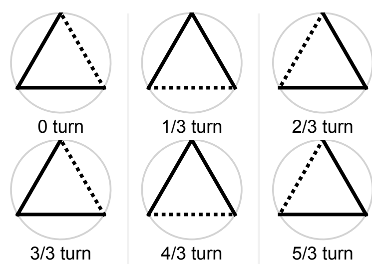


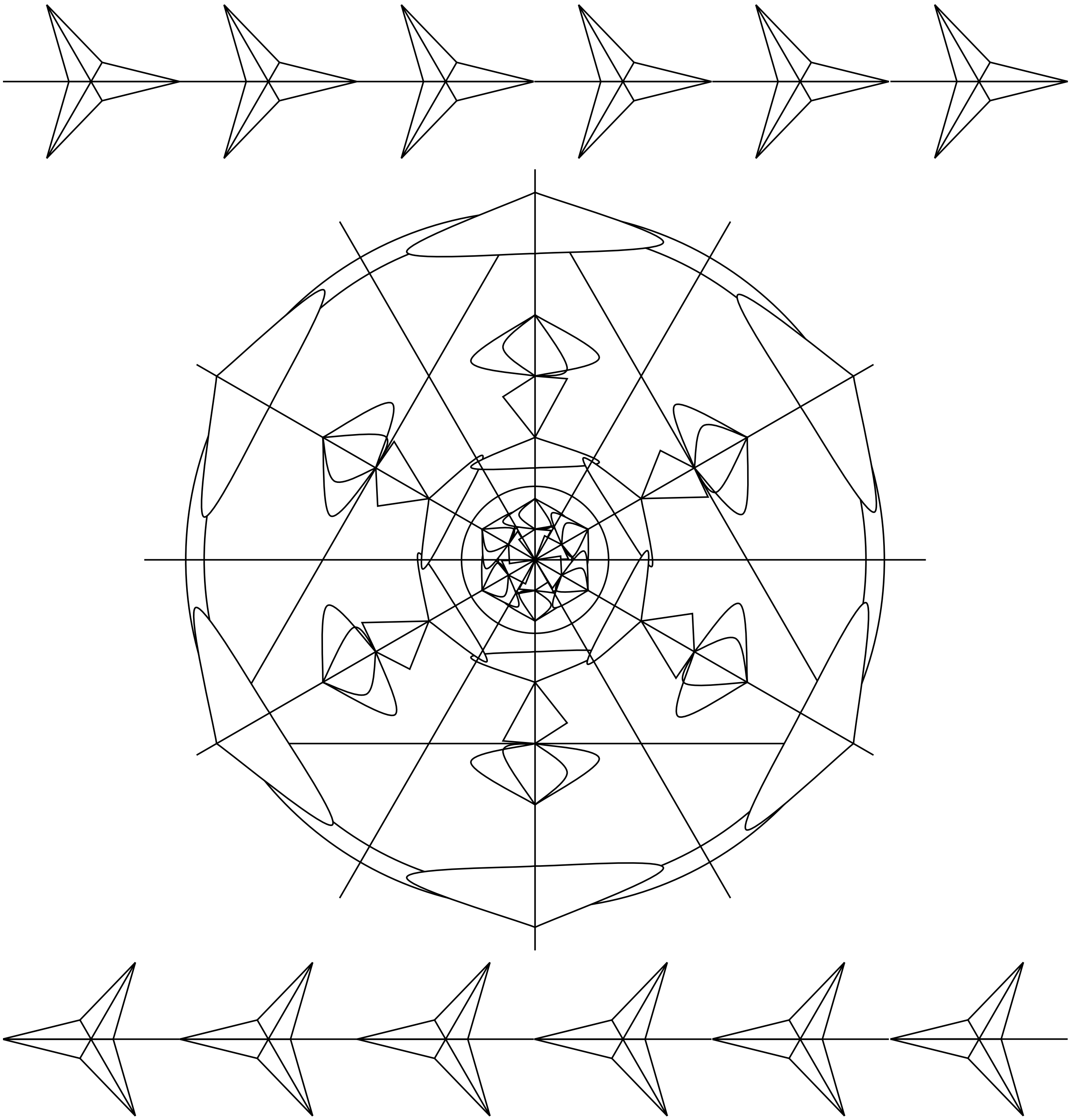


We can still rotate that triangle more than $\frac{1}{3}$ of the way around a circle without changing it. It can be rotated by twice that much - $\frac{2}{3}$ of the way around the circle - or by 3 times that much, which is all the way around the circle.

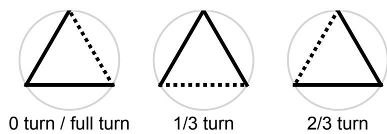


We can keep rotating - by 4 times that much, 5 times that much, 6 times... and keep going. The triangle seems to have an infinite number of rotations, but after rotating 3 times they become repetitive.

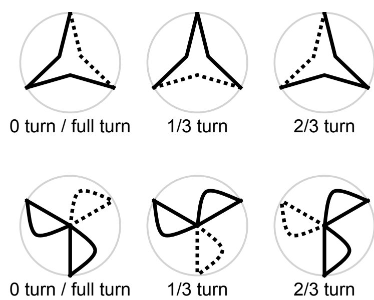


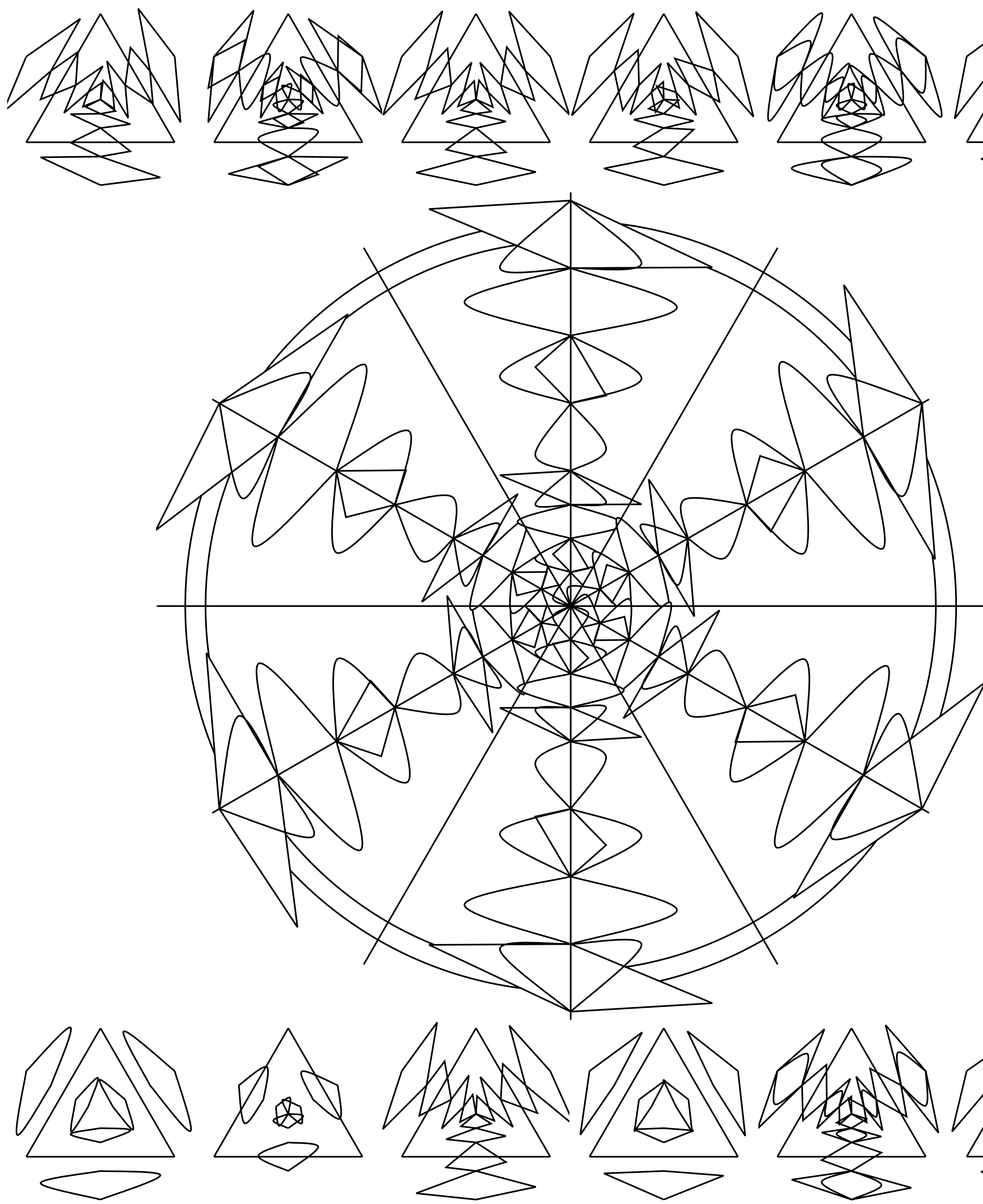


There are only 3 unique rotations for a triangle. We'll talk about them by referring to the rotations that are less than a full turn.

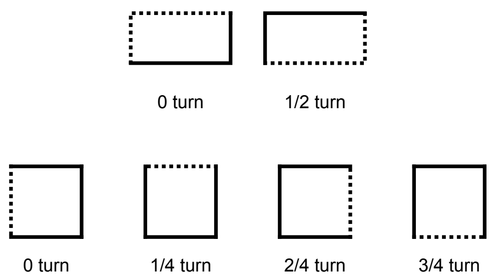


Other shapes, not just equilateral triangles, have the same 3 rotations. For this reason, we can say they all share the same group (C3).

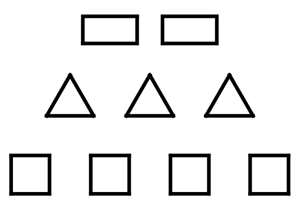


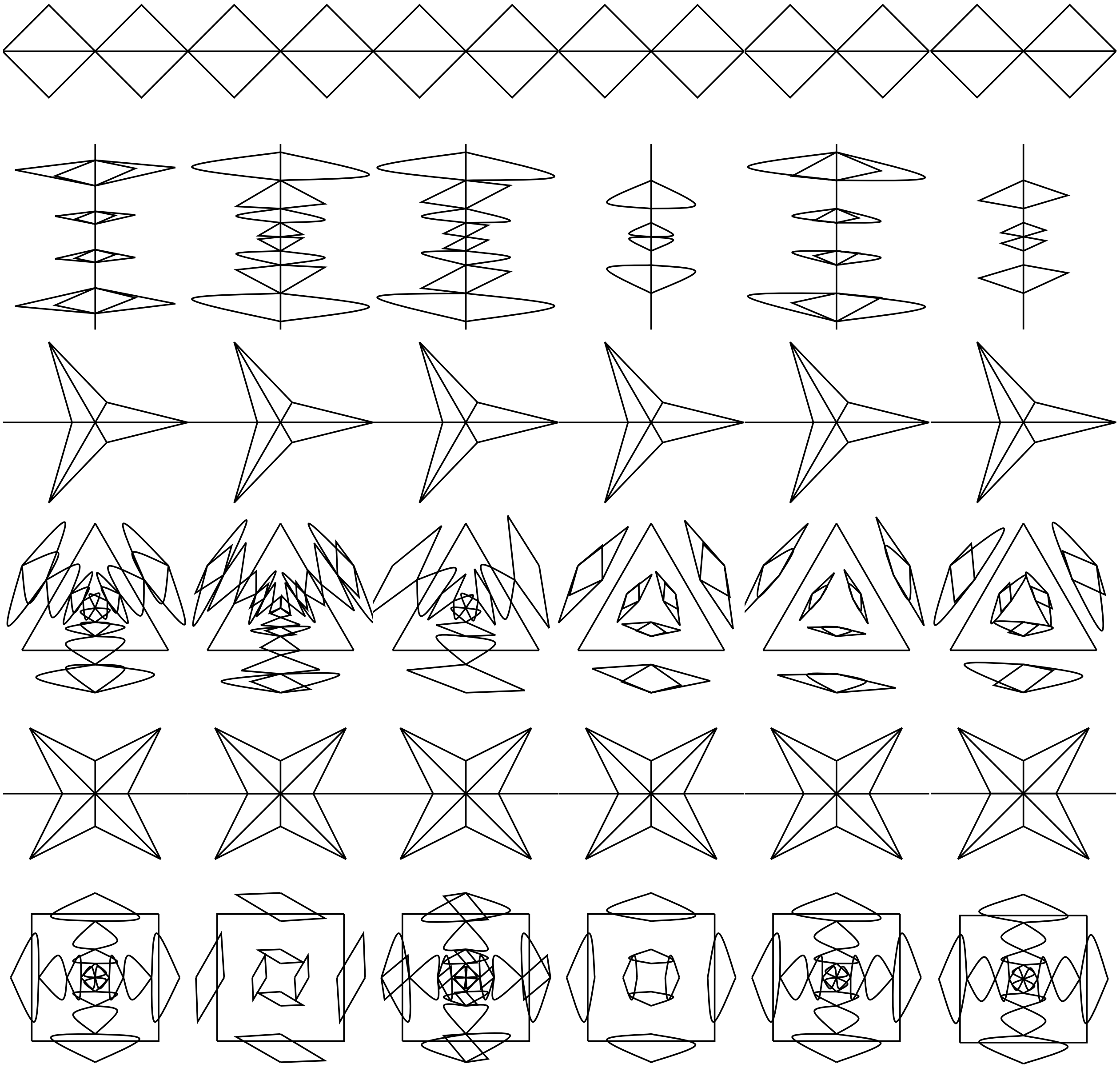


Now that we can count rotations, we can be more precise when we say a square has more symmetry than a rectangle.

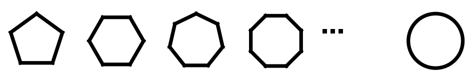


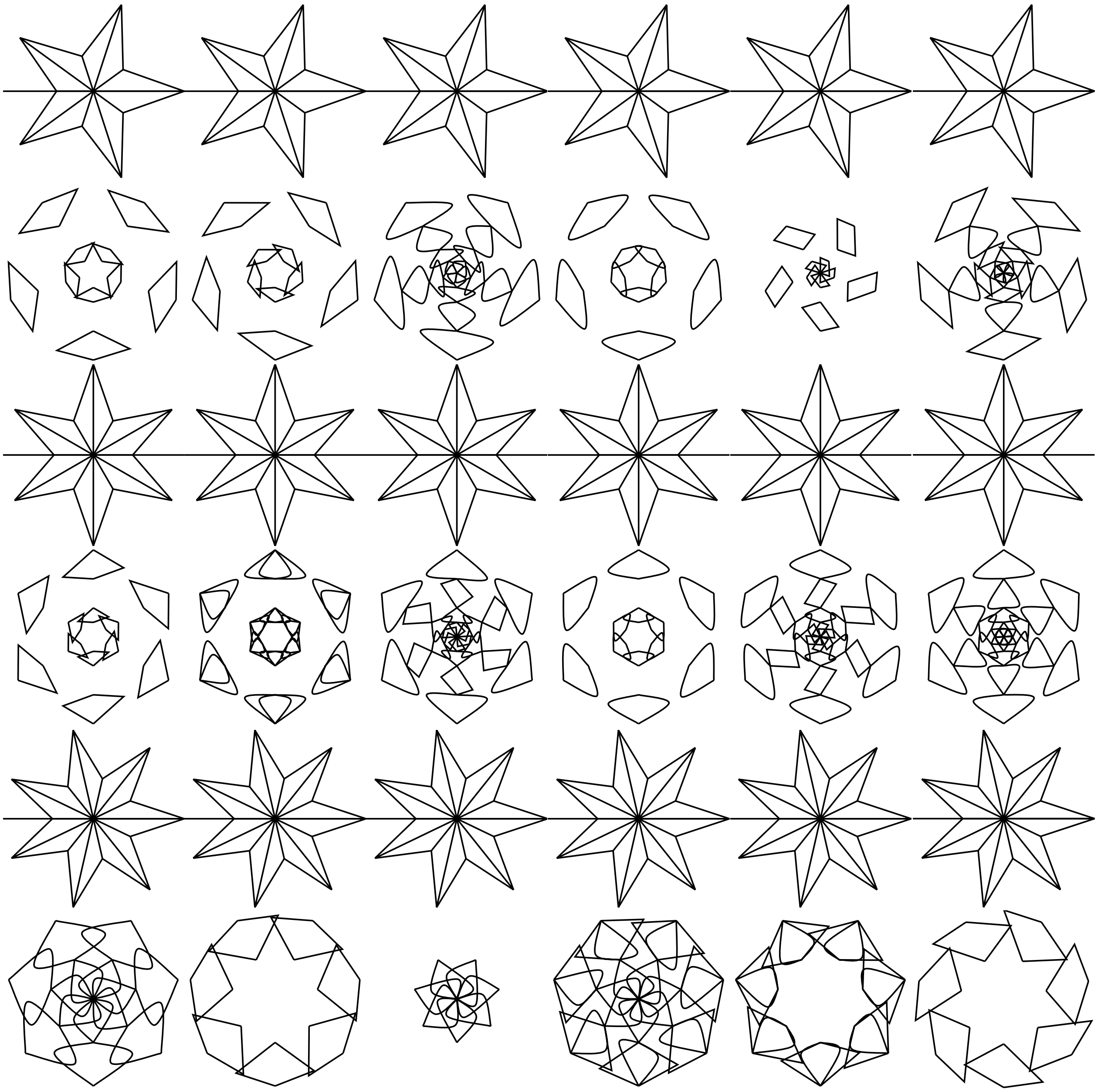
We can also see that a square has more rotational symmetry than a triangle, which in turn has more symmetry than a rectangle: A square has 4 unique rotations, while a triangle has 3, and a rectangle has only 2.



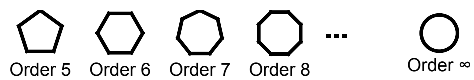


We don't need to stop at 4 rotations. We can find shapes with 5 rotations, 6 rotations, 7, 8, ... and keep going towards infinity.

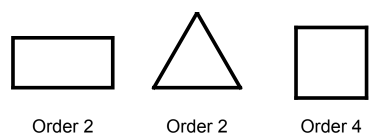




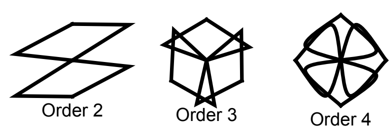
We need a better way to talk about this. We say a shape has rotational symmetry of **order n** if it has **n** unique rotations. This means the shape can be rotated $\frac{1}{n}$ of the way around the circle without changing its appearance.

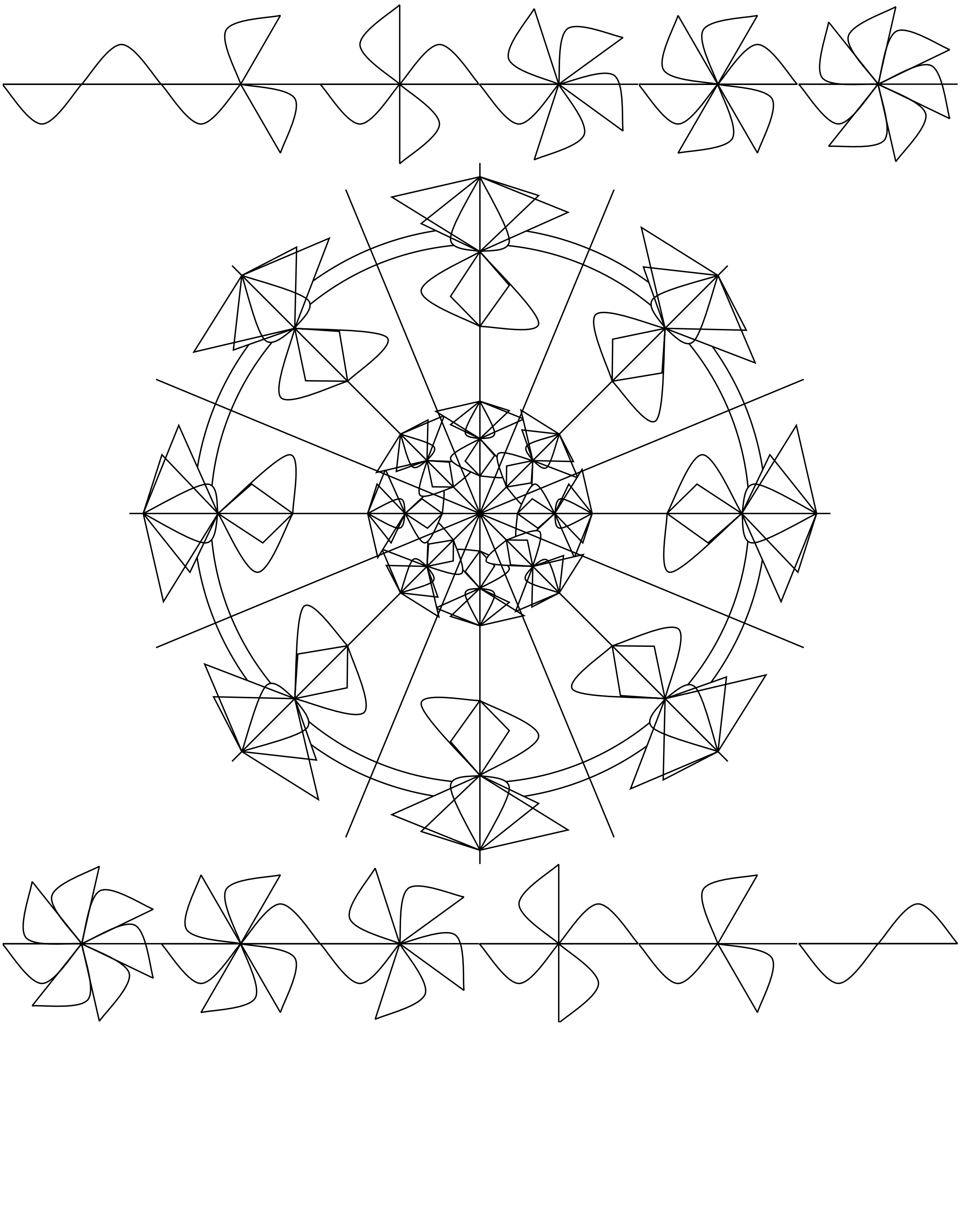


For example, a triangle can be rotated by $\frac{1}{3}$ of the way around the circle so a triangle has order 3. A square can be rotated $\frac{1}{4}$ of the way around the circle without changing, so it has order 4.

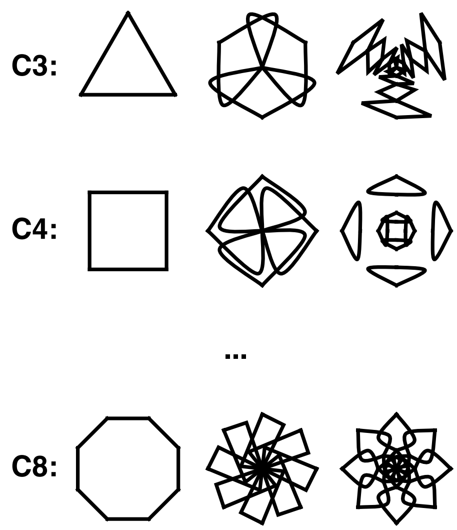


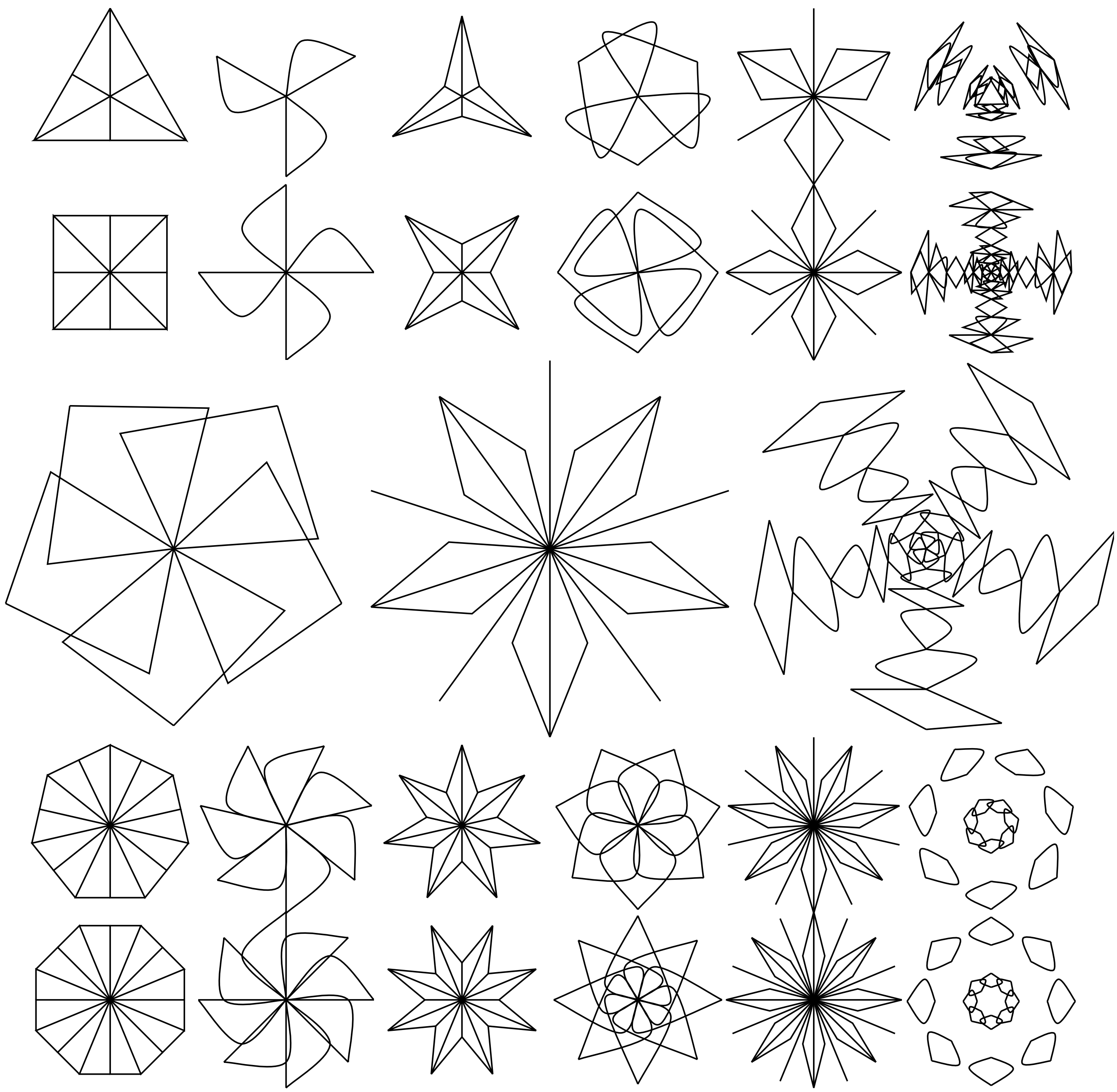
We can say the same for shapes that are not regular polygons.



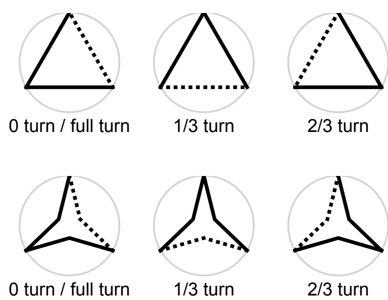


When shapes have the same rotational order, they share a cyclic group. We can use the name C3 to talk about the group of shapes that have 3 rotations. We can also use the name C4 for the shapes that have 4 rotations, C5 for the shapes that have 5 rotations, and so on.

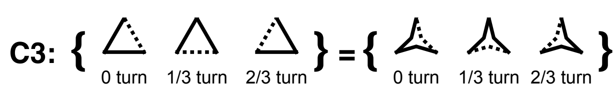




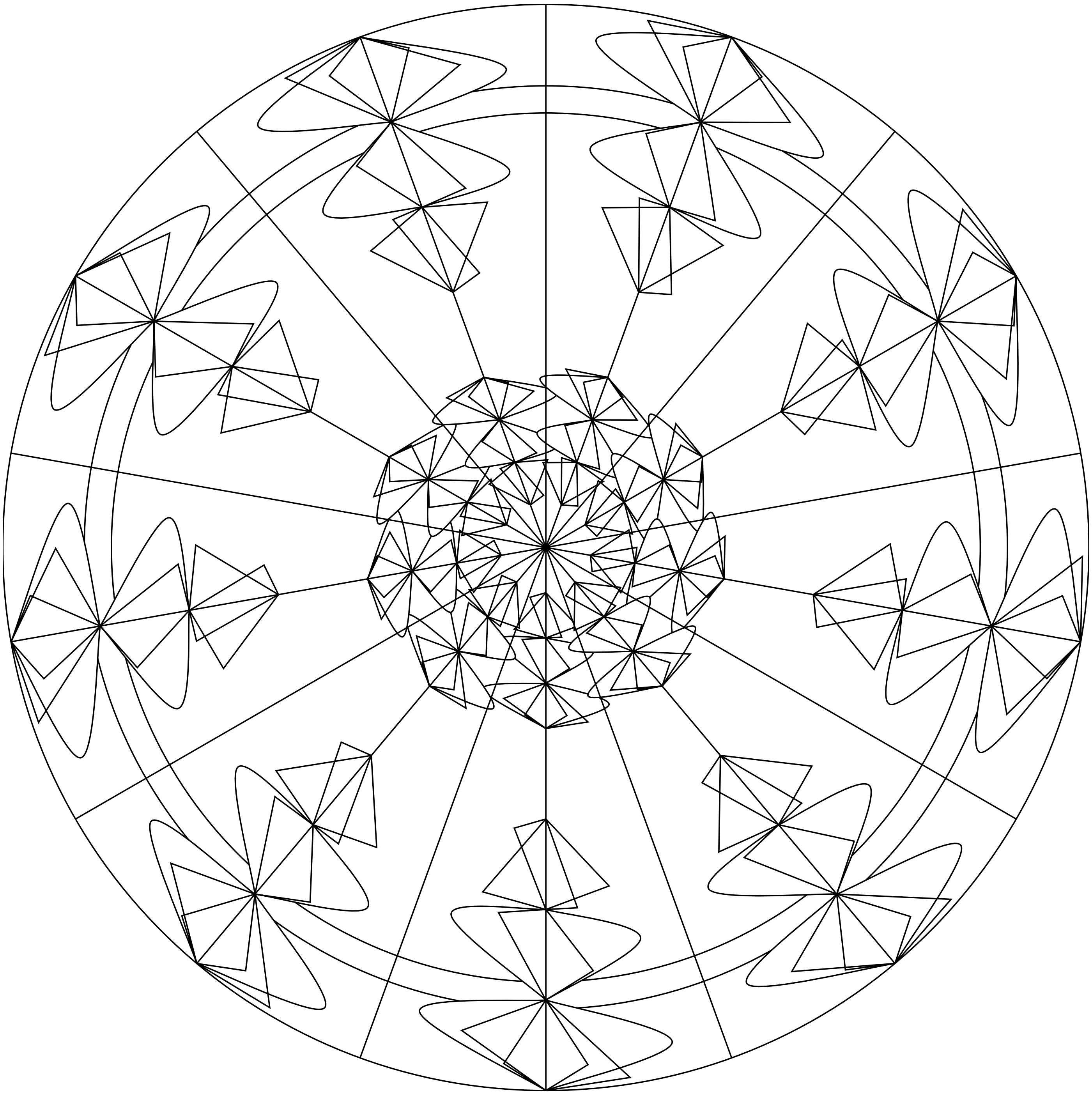
These shapes that share a cyclic group have the same group of rotations that leave their appearance unchanged. For example, the shapes of C3 all have the rotations **{0 turn, $\frac{1}{3}$ turn, $\frac{2}{3}$ turn }** and no more unique rotations other than those.



Our shapes help us see our groups, but the members of the groups are the rotations, not the shapes.

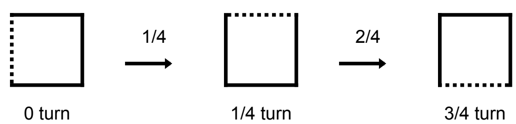


These rotations are related to each other.



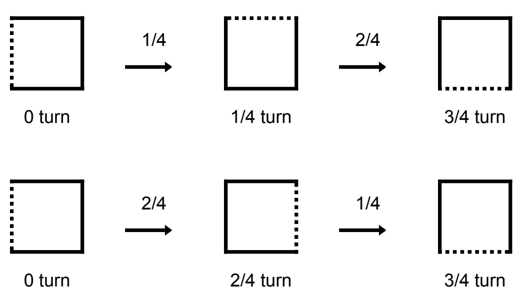
Another way to think about rotating a C4 shape by a $\frac{3}{4}$ turn is to rotate it by a $\frac{1}{4}$ turn and then rotate it again by a $\frac{2}{4}$ turn.

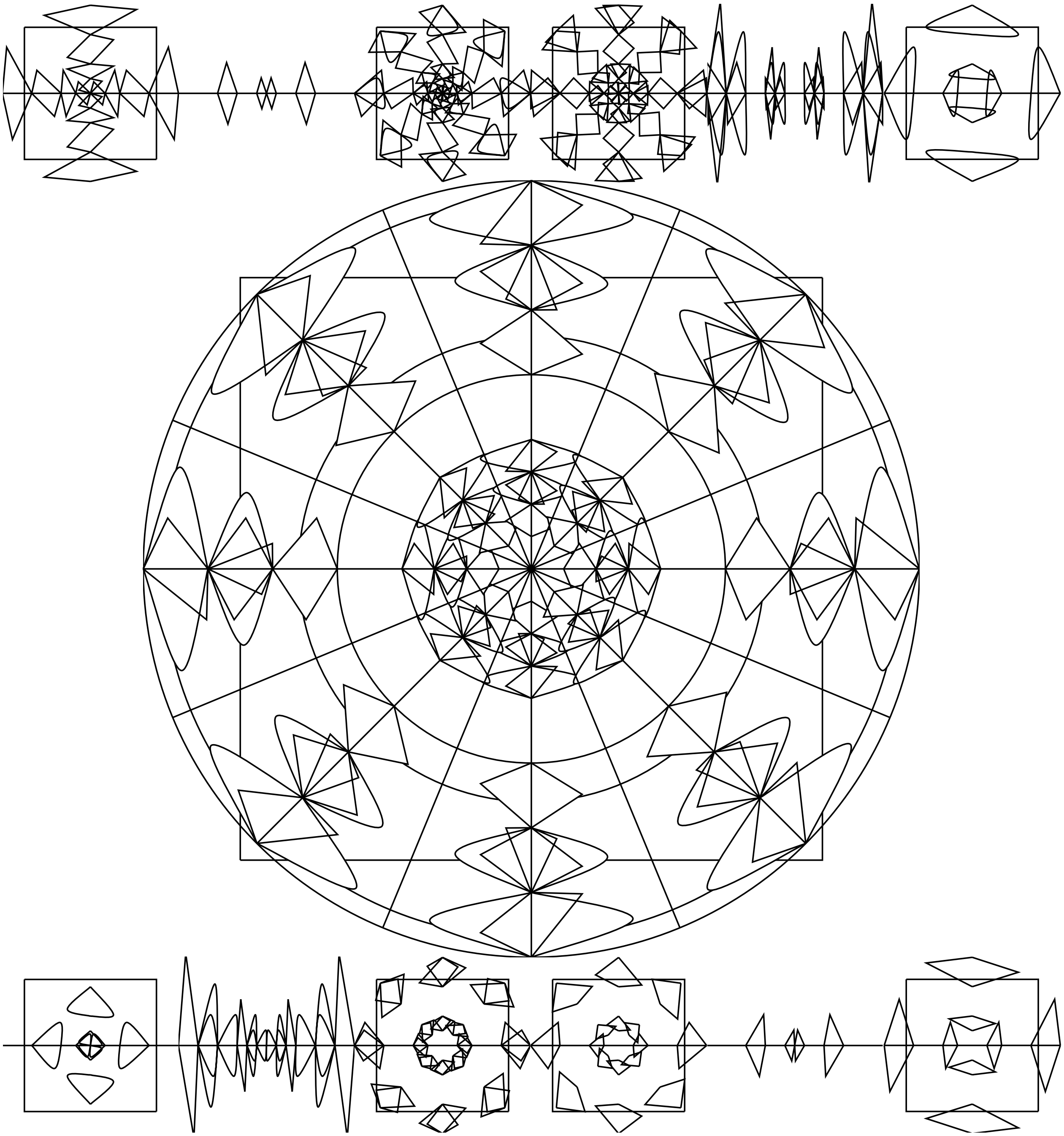
$$\textbf{C4:} \quad \tfrac{1}{4}\text{ turn} * \tfrac{2}{4}\text{ turn} \rightarrow \tfrac{3}{4}\text{ turn}$$



Notice that the order in which these rotations are applied does not matter. The Cn groups are <u>commutative</u>.

$$\textbf{C4:} \quad \tfrac{1}{4}\text{ turn} * \tfrac{2}{4}\text{ turn} = \tfrac{2}{4}\text{ turn} * \tfrac{1}{4}\text{ turn}$$



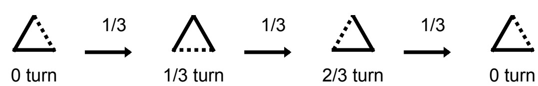


Similarly, for our C3 group, a $\frac{2}{3}$ turn is the same as combining a $\frac{1}{3}$ turn with another $\frac{1}{3}$ turn.

C3: $\quad \frac{1}{3}$ **turn** $\ast$ $\frac{1}{3}$ **turn** $\rightarrow$ $\frac{2}{3}$ **turn**

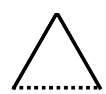
Adding another $\frac{1}{3}$ turn brings the shape back to its starting position - the 0 turn.

C3: $\quad \frac{1}{3}$ **turn** $\ast$ $\frac{1}{3}$ **turn** $\ast$ $\frac{1}{3}$ **turn** $\rightarrow$ **0 turn**



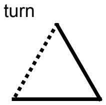
See, the $\frac{1}{3}$ turn can generate all of the rotations of C3 - it is a generator for our C3 group.

$\frac{1}{3}$ **turn** $\longrightarrow$



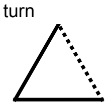
1/3 turn

$\frac{1}{3}$ **turn** $\ast$ $\frac{1}{3}$ **turn** $\longrightarrow$

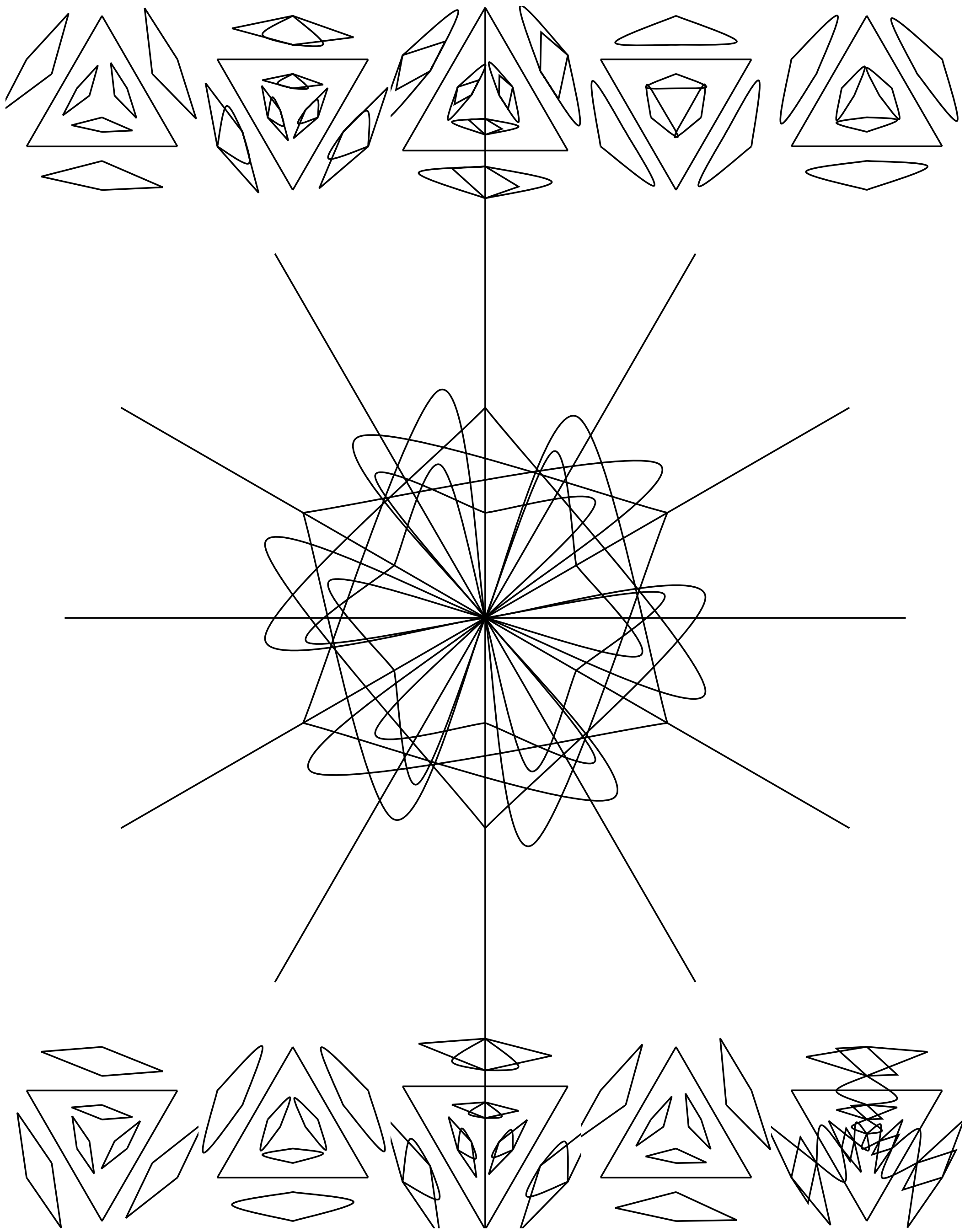


2/3 turn

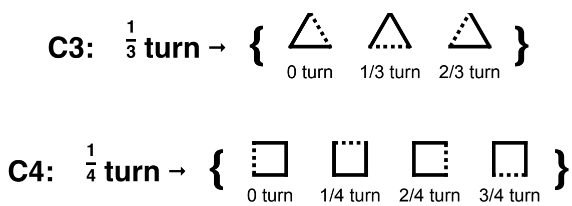
$\frac{1}{3}$ **turn** $\ast$ $\frac{1}{3}$ **turn** $\ast$ $\frac{1}{3}$ **turn** $\longrightarrow$



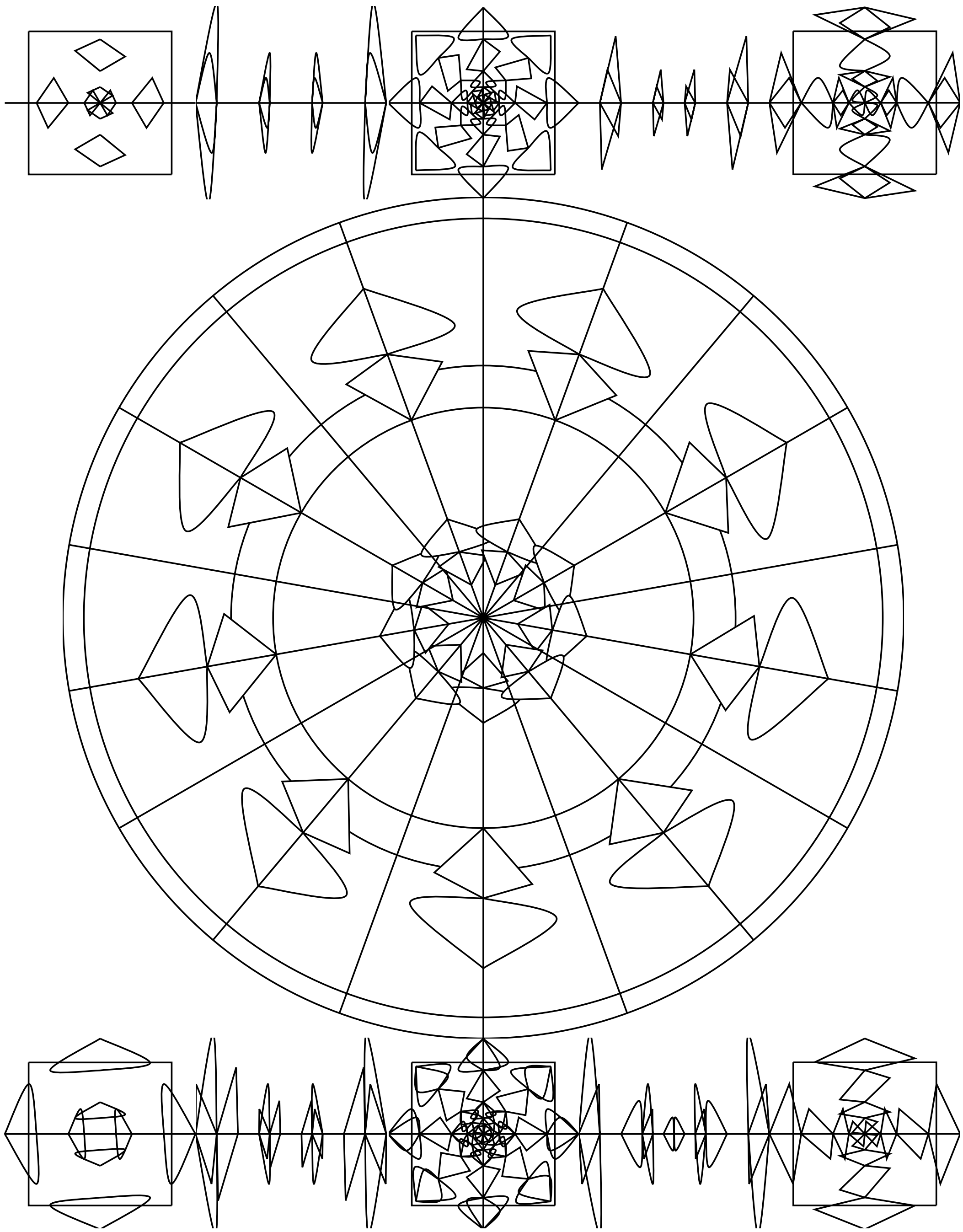
0 turn

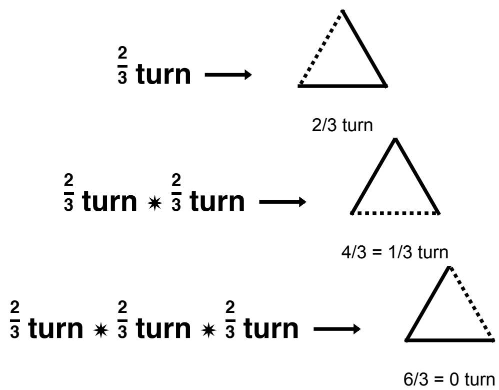


The $\frac{1}{3}$ turn is a generator for our C3 group, and similarly, the $\frac{1}{4}$ turn is a generator for our C4, because it can generate all of the rotations of our C4.

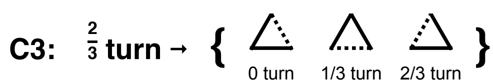


We could even choose different generators.

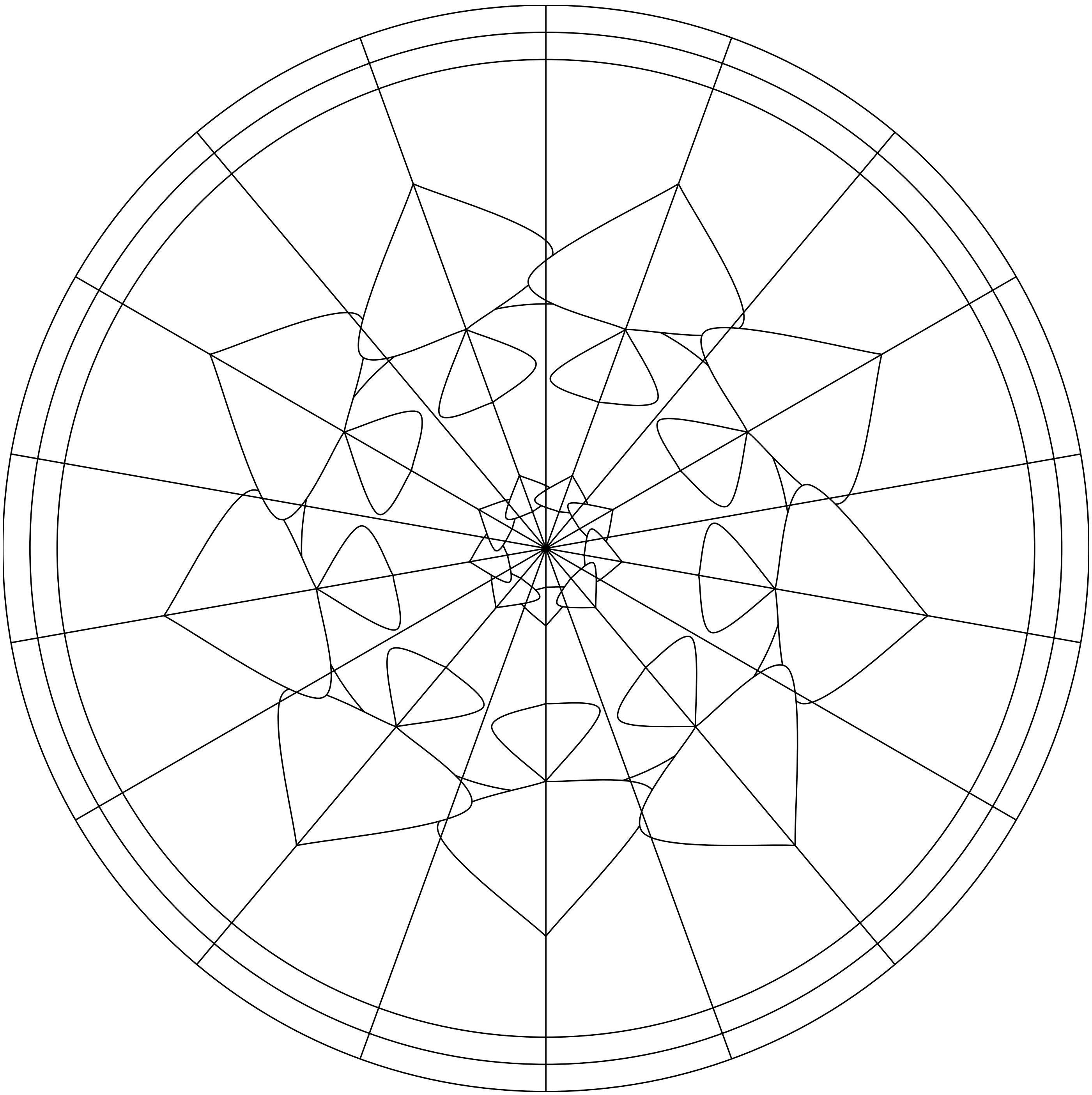




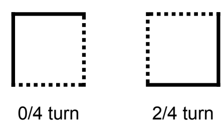
We could have just as easily used a $\frac{2}{3}$ turn as our generator for C3 and ended up with the same result.



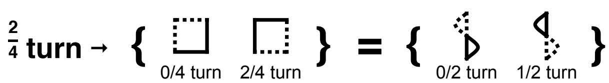
However, not all rotations are generators.



A $\frac{2}{4}$ turn does not generate all of the rotations of our C4 group.

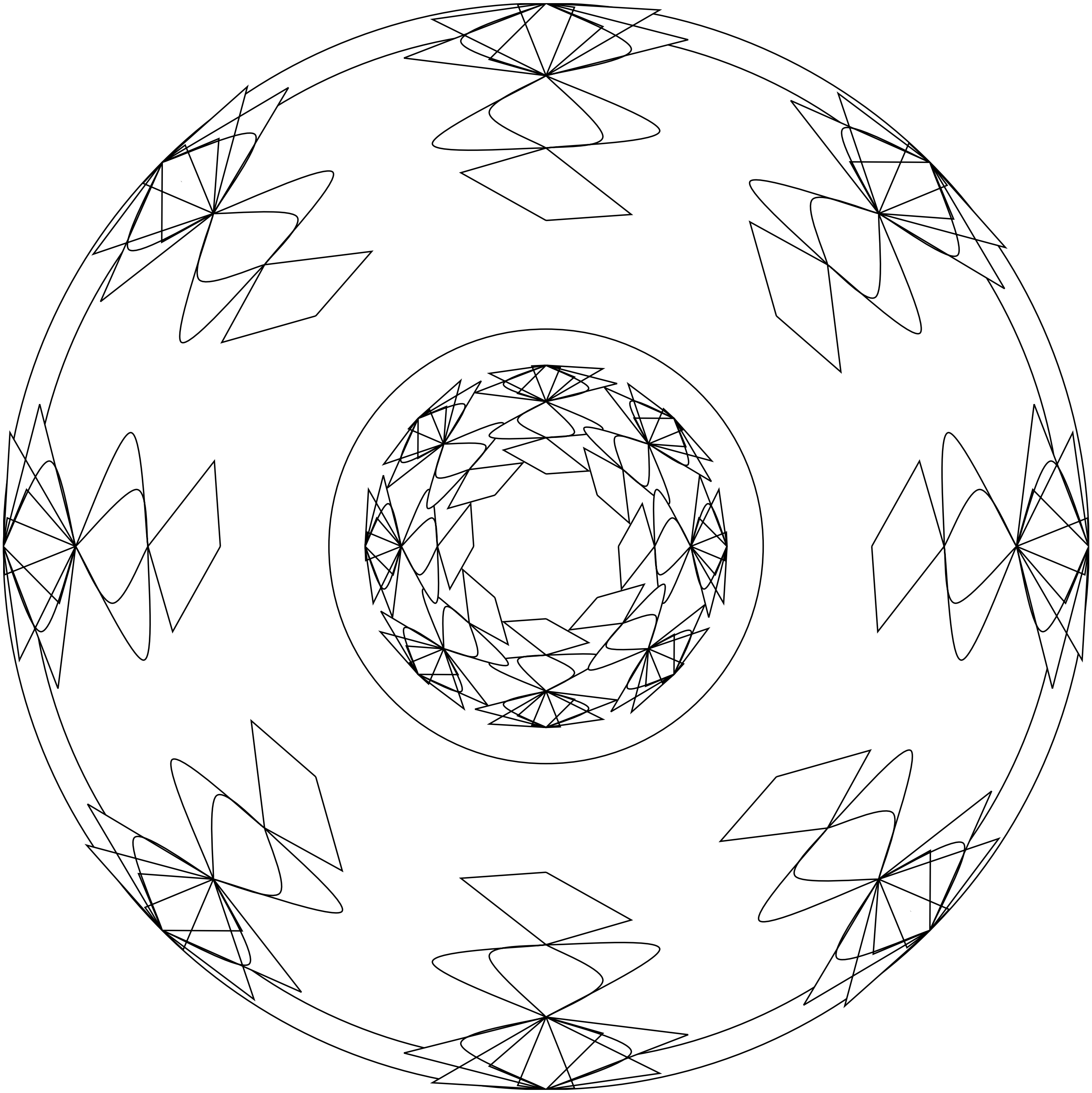


Instead a $\frac{2}{4}$ turn generates a smaller group -- our C2 group.

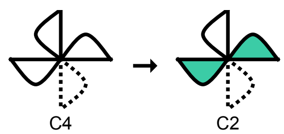


Challenge: Find all the generators for C4 and C8.

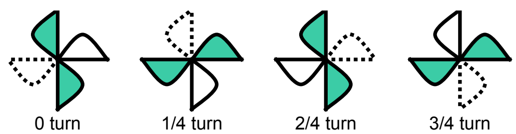
Challenge: Which rotations of C8 reduce C8 to C4 when used as generators?

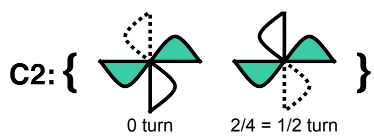


We can transform a C4 shape into a C2 shape by coloring it.

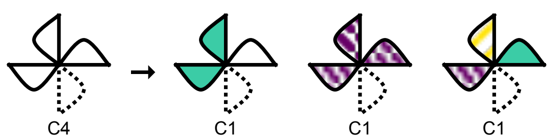


The only rotations that leave this colored shape unchanged are those of C2.

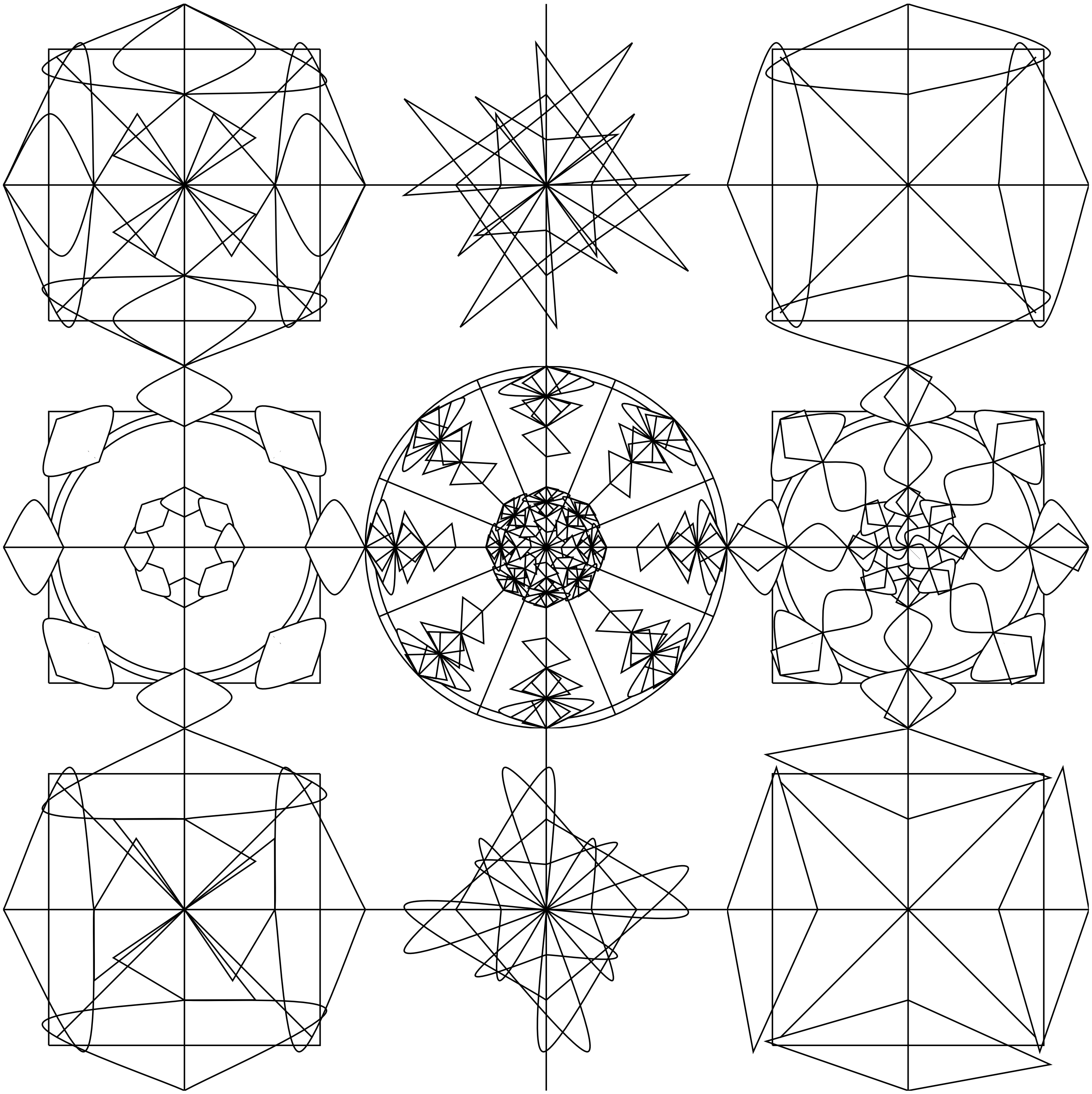




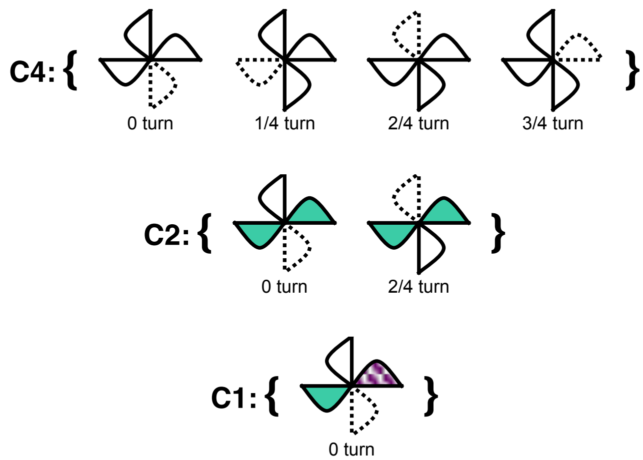
Not all colorings of our C4 shapes will transform them into C2 shapes.



Challenge: Color the C4 shapes to reduce them to C2 shapes.

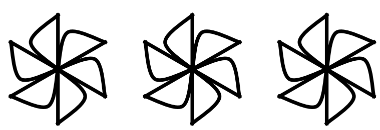


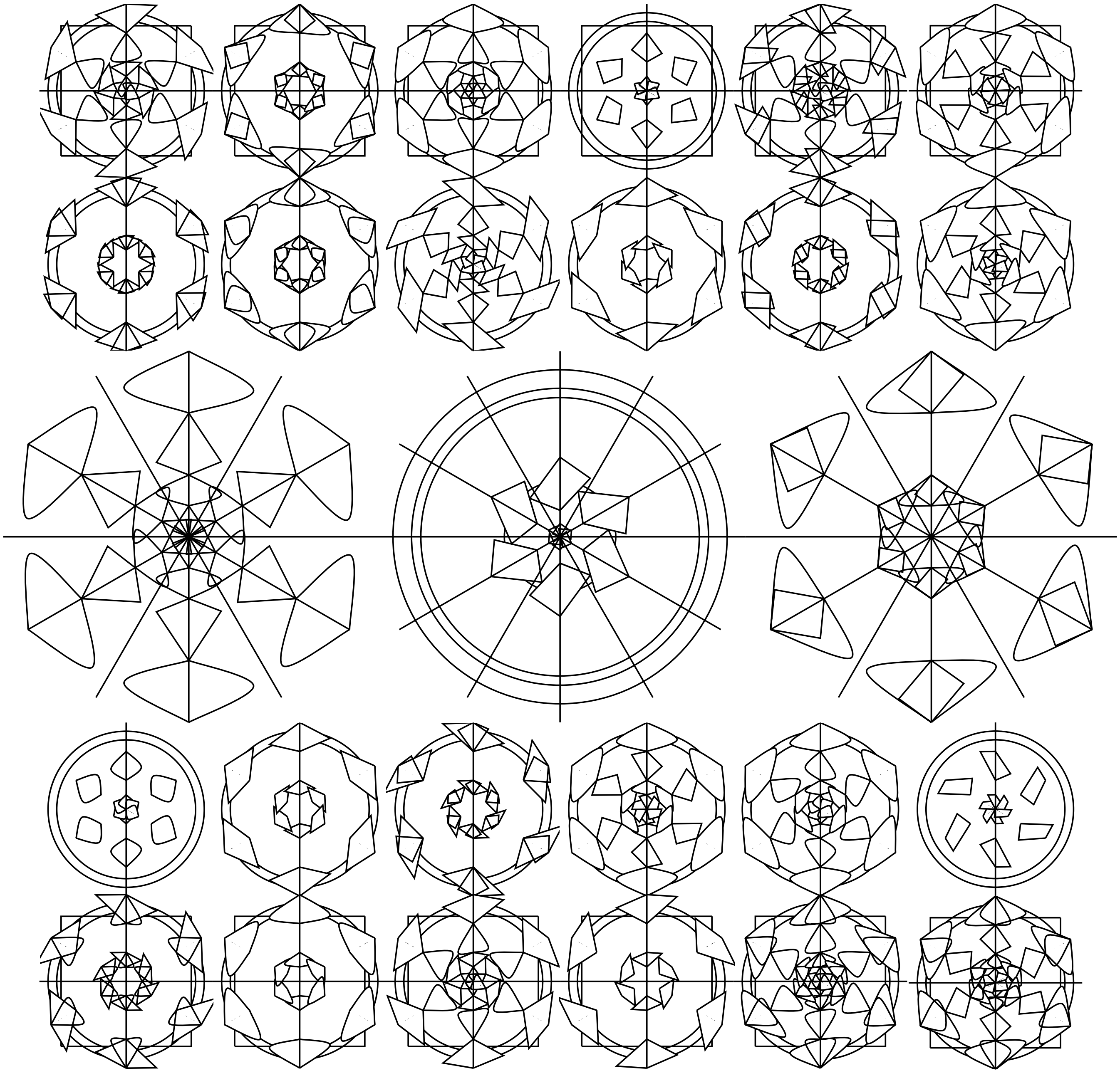
Color can reduce C4 shapes to C2 or C1 shapes because our C2 and C1 are <u>subgroups</u> of our C4 group. A <u>subgroup</u> is a group contained within a group.



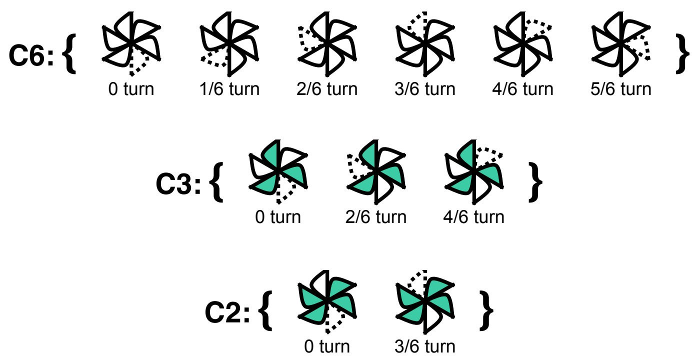
Similarly, our C1, C2, and C3 groups are all <u>subgroups</u> of our C6 group.

Challenge: Can you color the C6 shapes to reduce them to C1, C2, or C3 shapes?

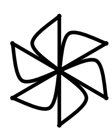




Notice that a group has all of the rotations of its subgroups.

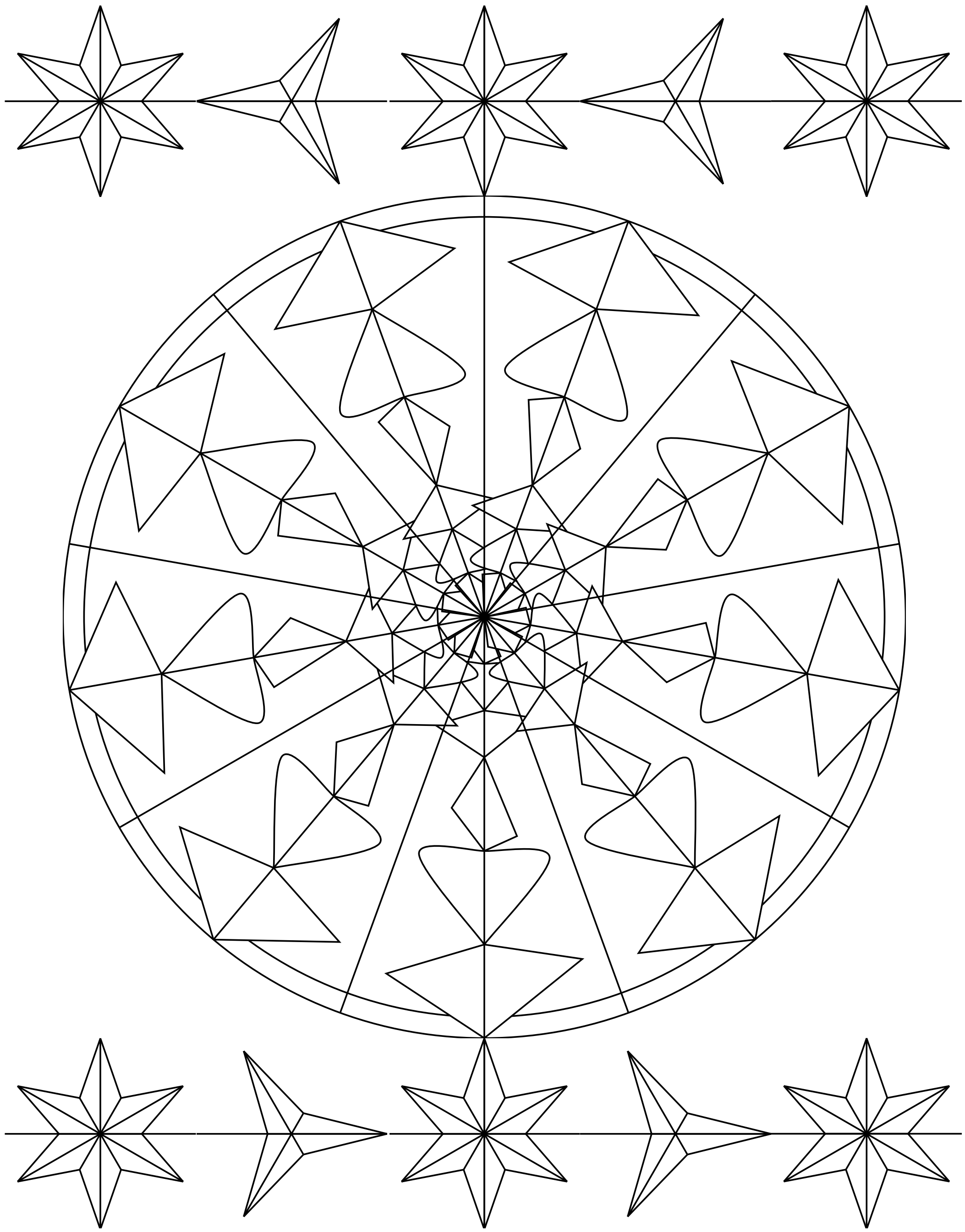


Try to color a C6 shape so that it has the rotations of C4.

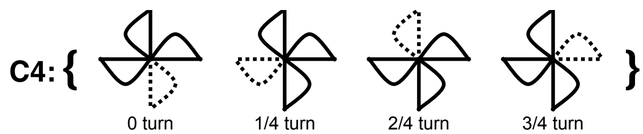


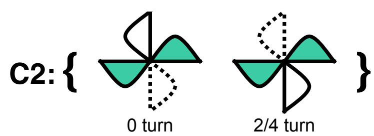
It can't be done. C4 is not a <u>subgroup</u> of C6.

There is more to it than that.

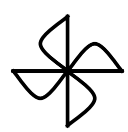


When we use color to reduce our shapes to belong to smaller groups, we give them a new set of rotations.



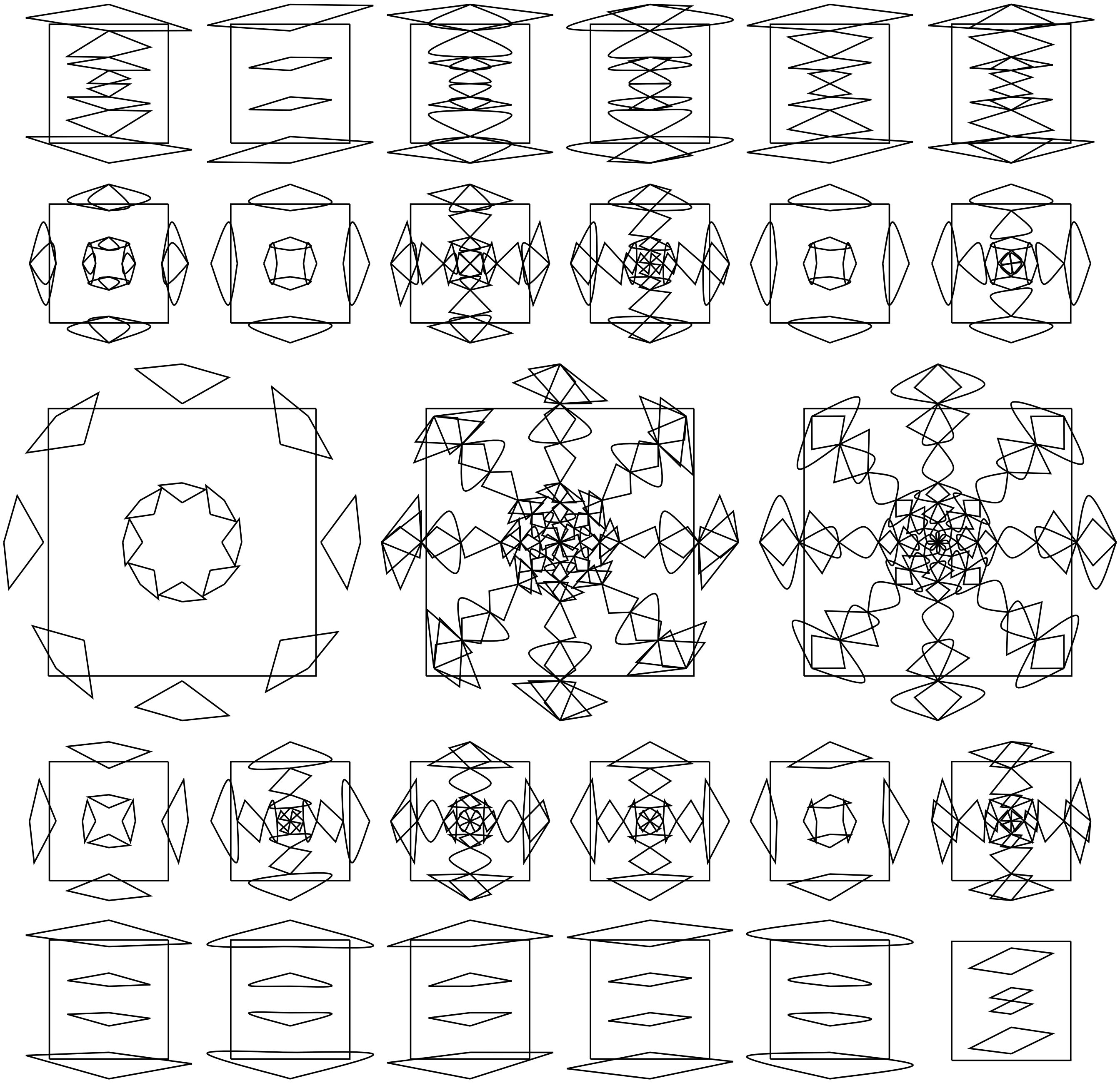


Not all sets of rotations are groups, and therefore cannot be subgroups. Try to color a shape in a way so that it has *only* a 0 turn and a $\frac{1}{4}$ turn.



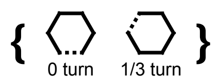
It's impossible without also giving the shape a $\frac{2}{4}$ turn and a $\frac{3}{4}$ turn. That's because **{0 turn, $\frac{1}{4}$ turn }** is not a group, but **{0 turn, $\frac{1}{4}$ turn, $\frac{2}{4}$ turn, $\frac{3}{4}$ turn }** is.

Why? This brings us back to combining rotations.

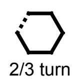


In order for a set of rotations to be a group, any combination of rotations in the set must also be in the set. This rule is called group closure, and you can see it.

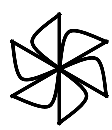
Take the set **{0 turn, $\frac{1}{3}$ turn}**.



This is not a group because it's missing the $\frac{2}{3}$ turn, which is created by combining a $\frac{1}{3}$ with another $\frac{1}{3}$ turn.

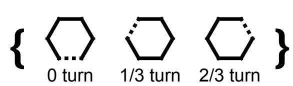
$\frac{1}{3}$ turn $*$ **$\frac{1}{3}$ turn** $\rightarrow$ 

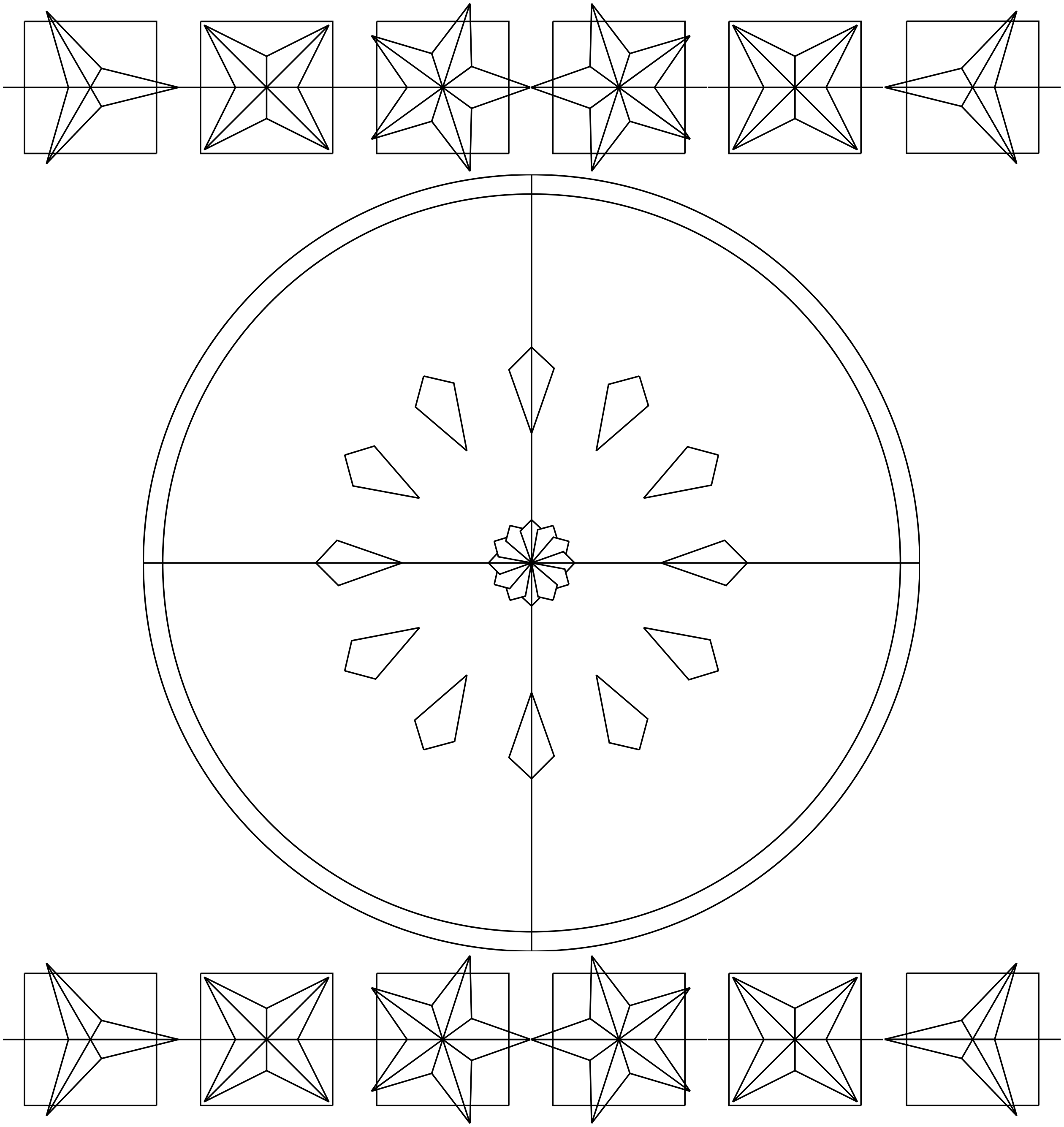
Try to color a shape to have the **{0 turn, $\frac{1}{3}$ turn}** rotations but not a **$\frac{2}{3}$ turn**.



It cannot be done.

Adding the **$\frac{2}{3}$ turn** to the set gives us our C3 group again.



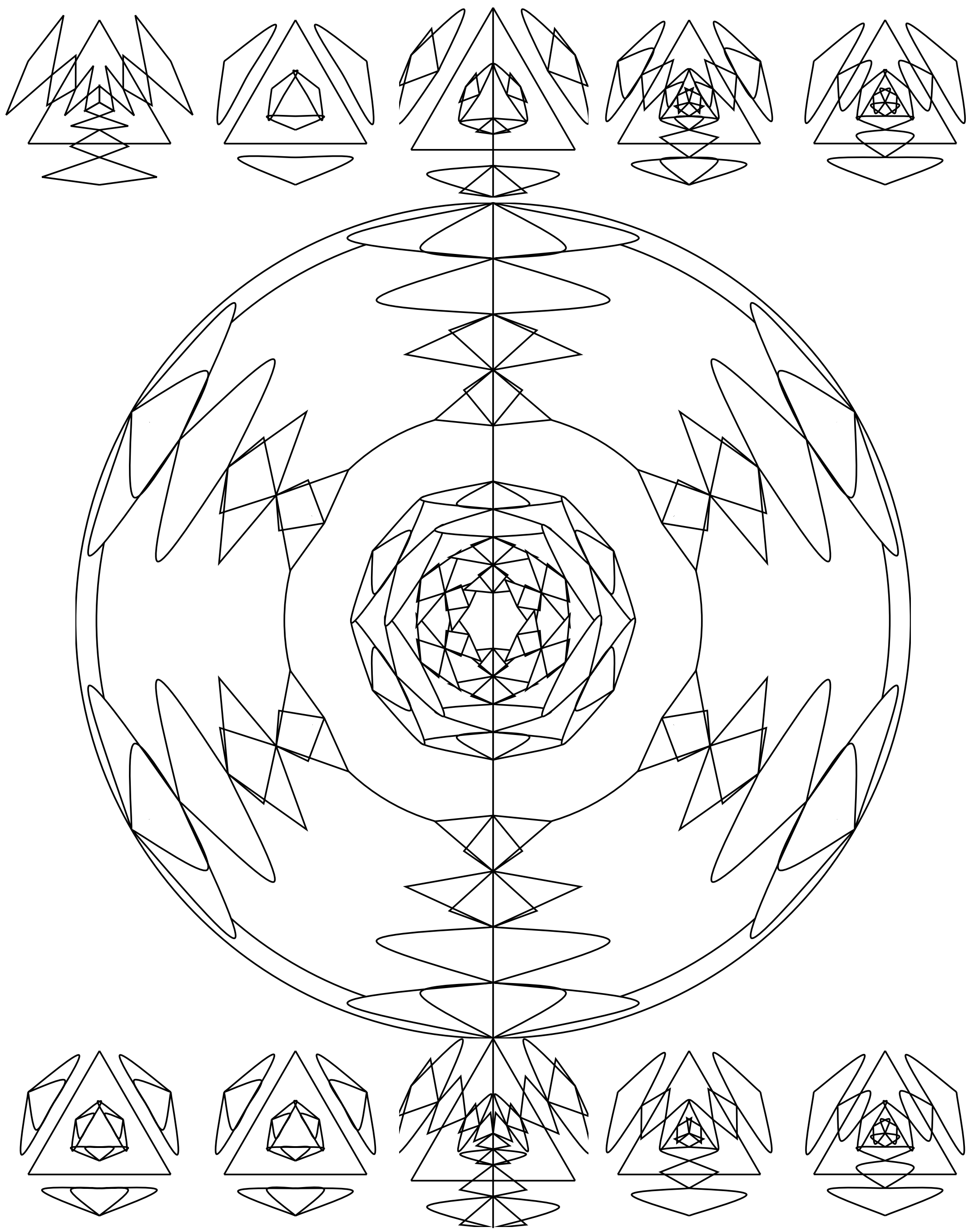


So far we've only been talking about groups of rotations.

These groups are cyclic. They can be created by combining just one rotation - the generator - multiple times with itself.

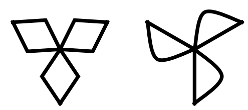
C3: $\frac{1}{3}$ **turn** →

For our next groups, we have more generators, such as reflections.

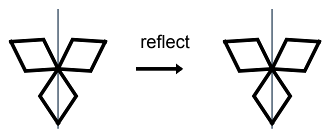


Reflections

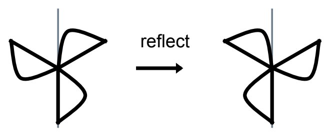
Even when two shapes have the same number of rotations, one can still have more symmetry than the other.



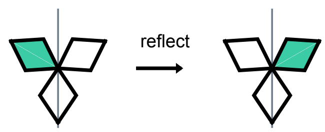
Some shapes have mirrors - they can reflect across internal, invisible lines without changing in appearance

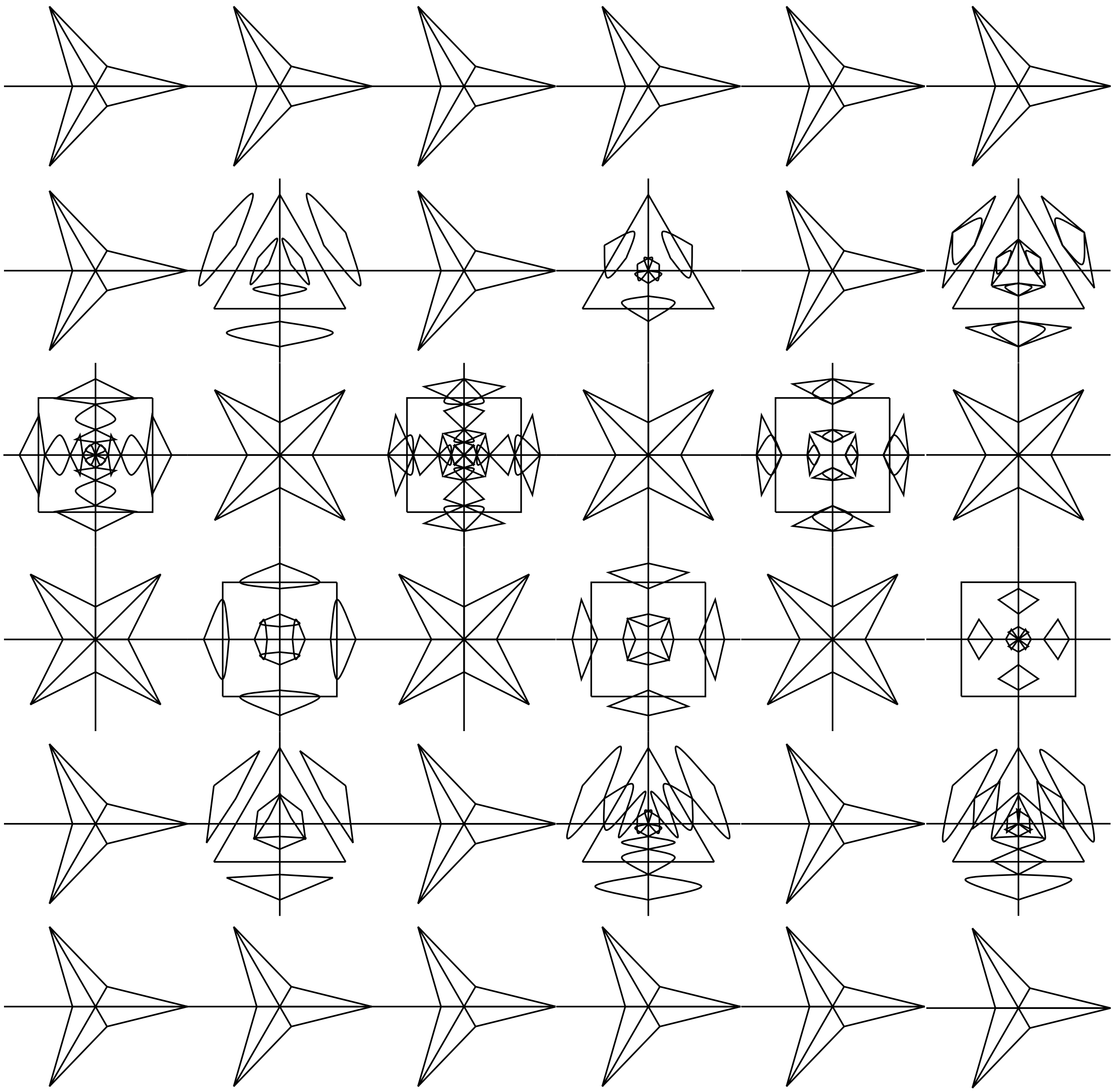


While others cannot.



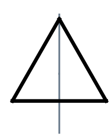
We'll see how these mirrors can be removed when color is added.



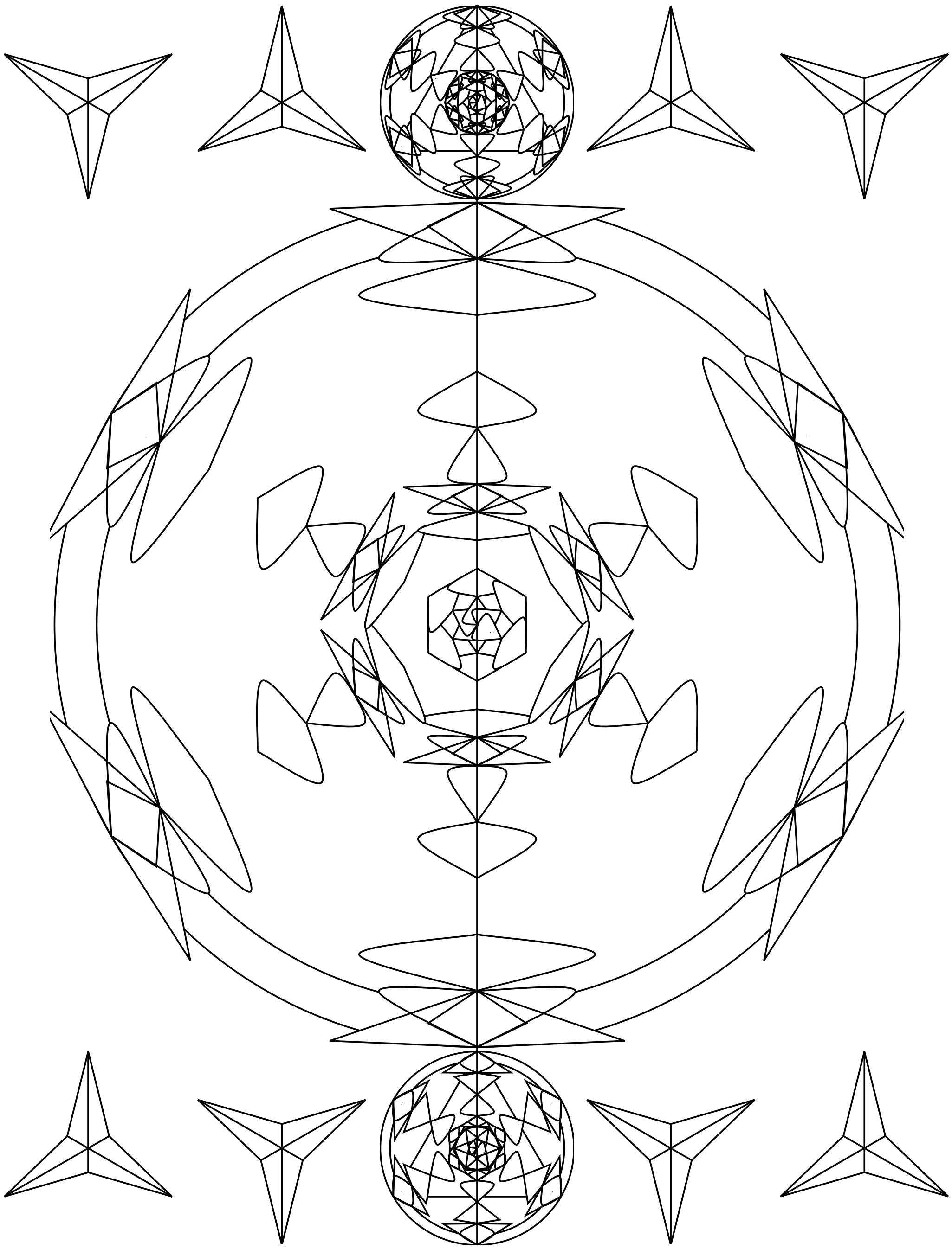


We saw that a single generator, the $\frac{1}{3}$ turn, could generate the entire group of rotations of a regular triangle, C3.

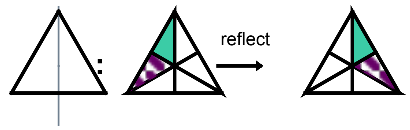
We can also reflect this triangle across a vertical mirror through its center.



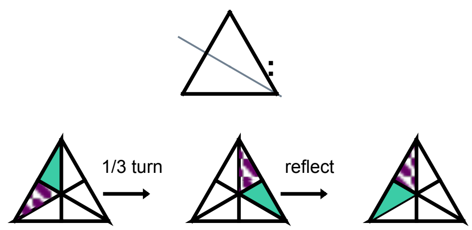
By coupling this mirror with a rotation, we can generate even larger groups.

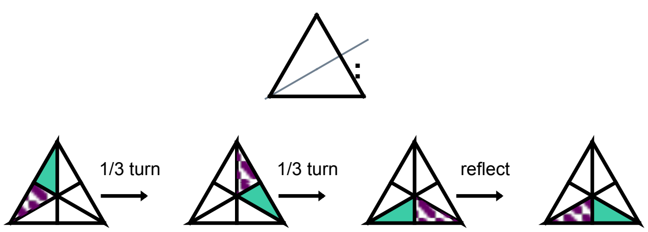


Reflections are easier to see with color.

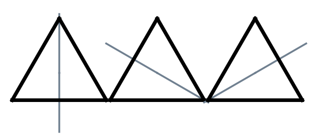


More mirror reflections can be generated by simply combining this vertical mirror with rotations.

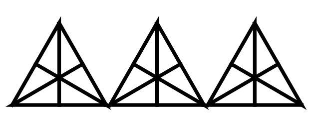


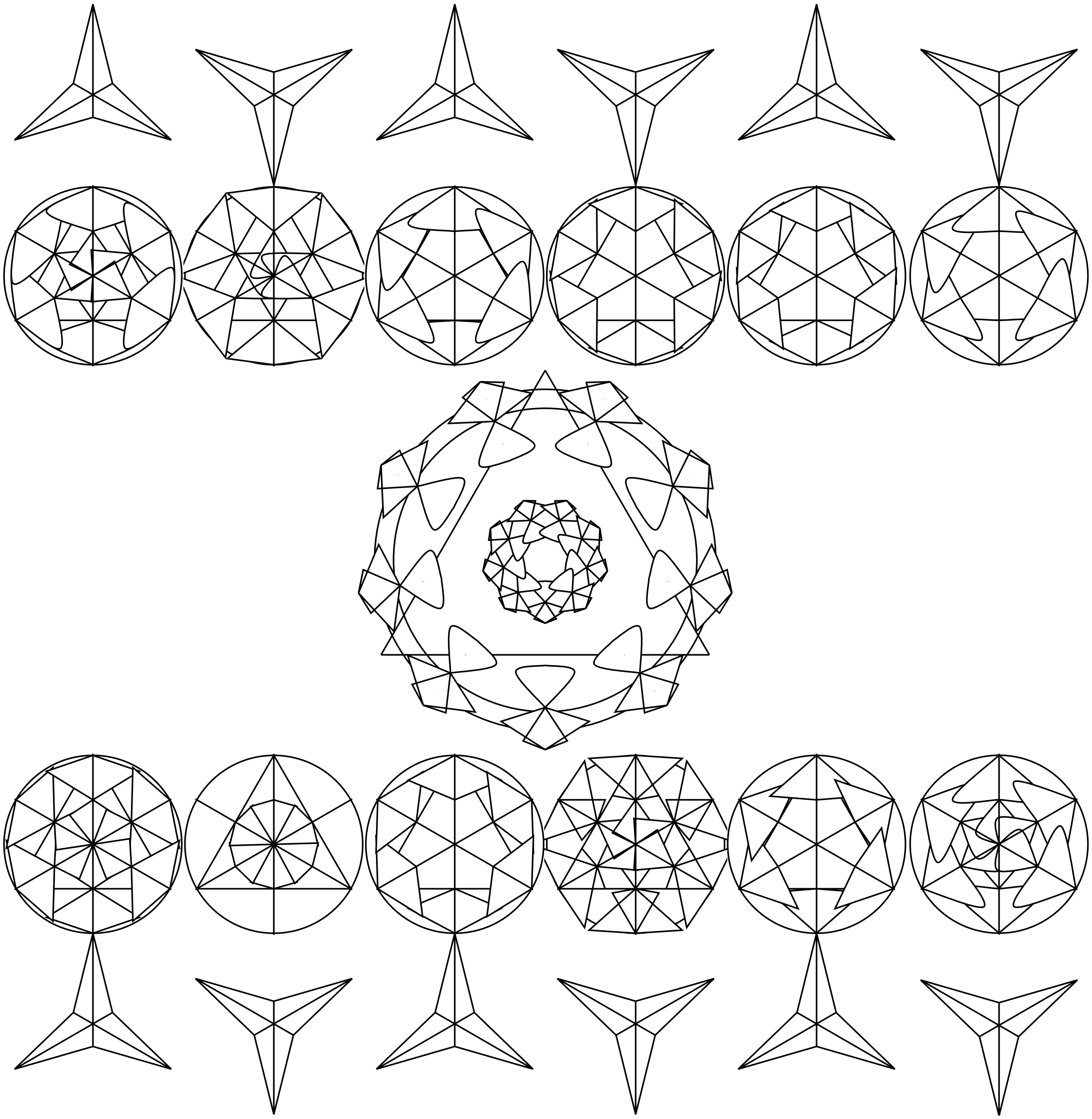


In total, a regular triangle has 3 unique mirrors.

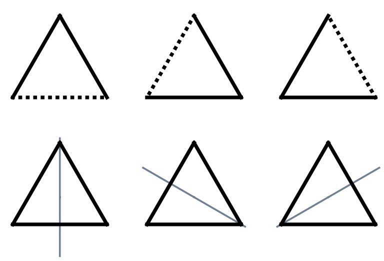


Challenge: Use color to show what happens to our triangle when it is reflected and then rotated. Is this different than rotating and then reflecting?

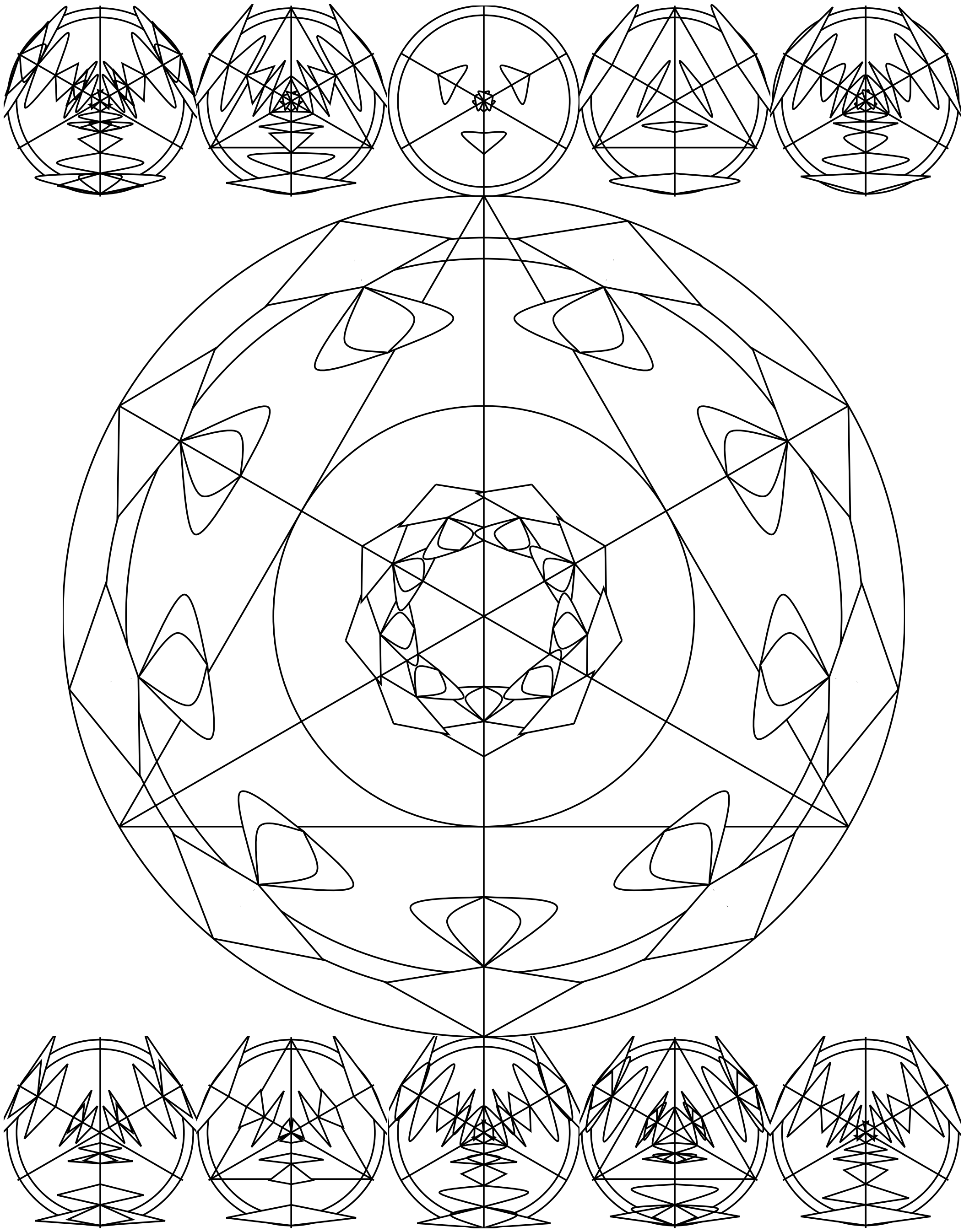




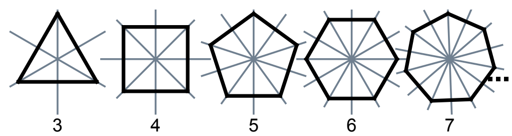
With just a rotation and a mirror as generators, we generated a new, larger group of symmetries for a regular triangle.



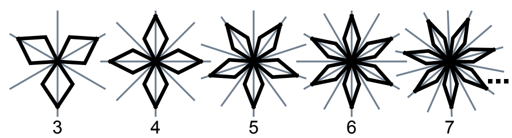
We can do the same for other shapes.



Our triangle has 3 unique rotations and 3 unique reflections, a square has 4, and we can find shapes with 5, 6, 7, and keep going...

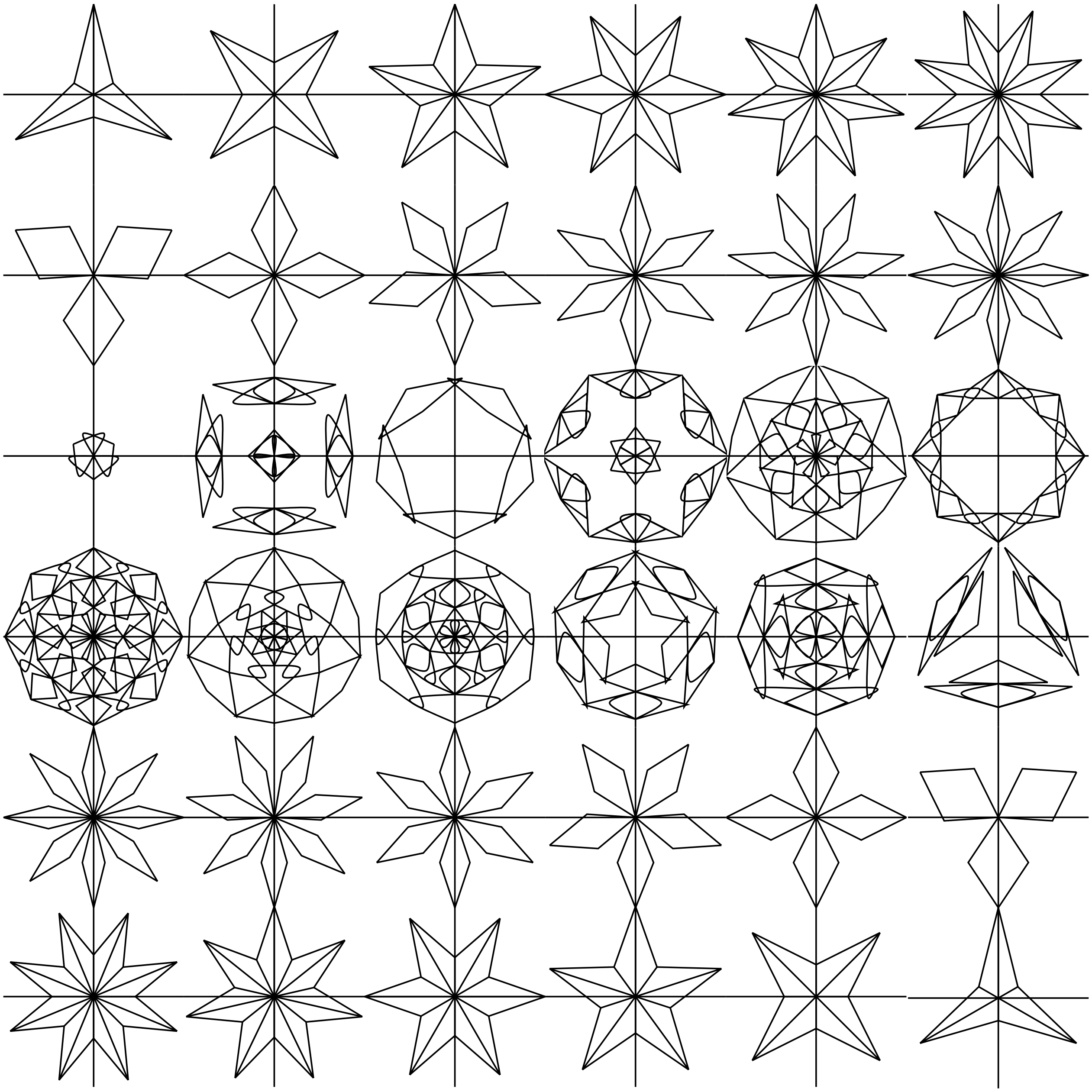


Shapes that are not regular polygons have these same symmetries.

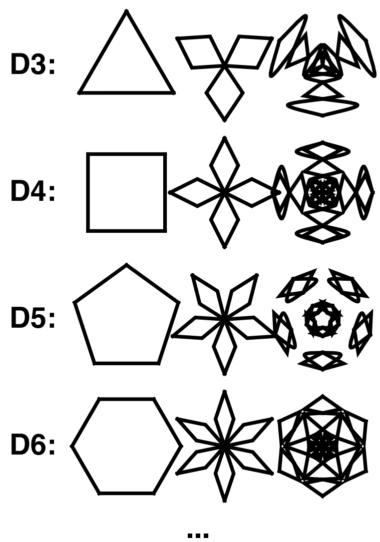


When shapes have the same set of symmetries, they share a symmetry group.

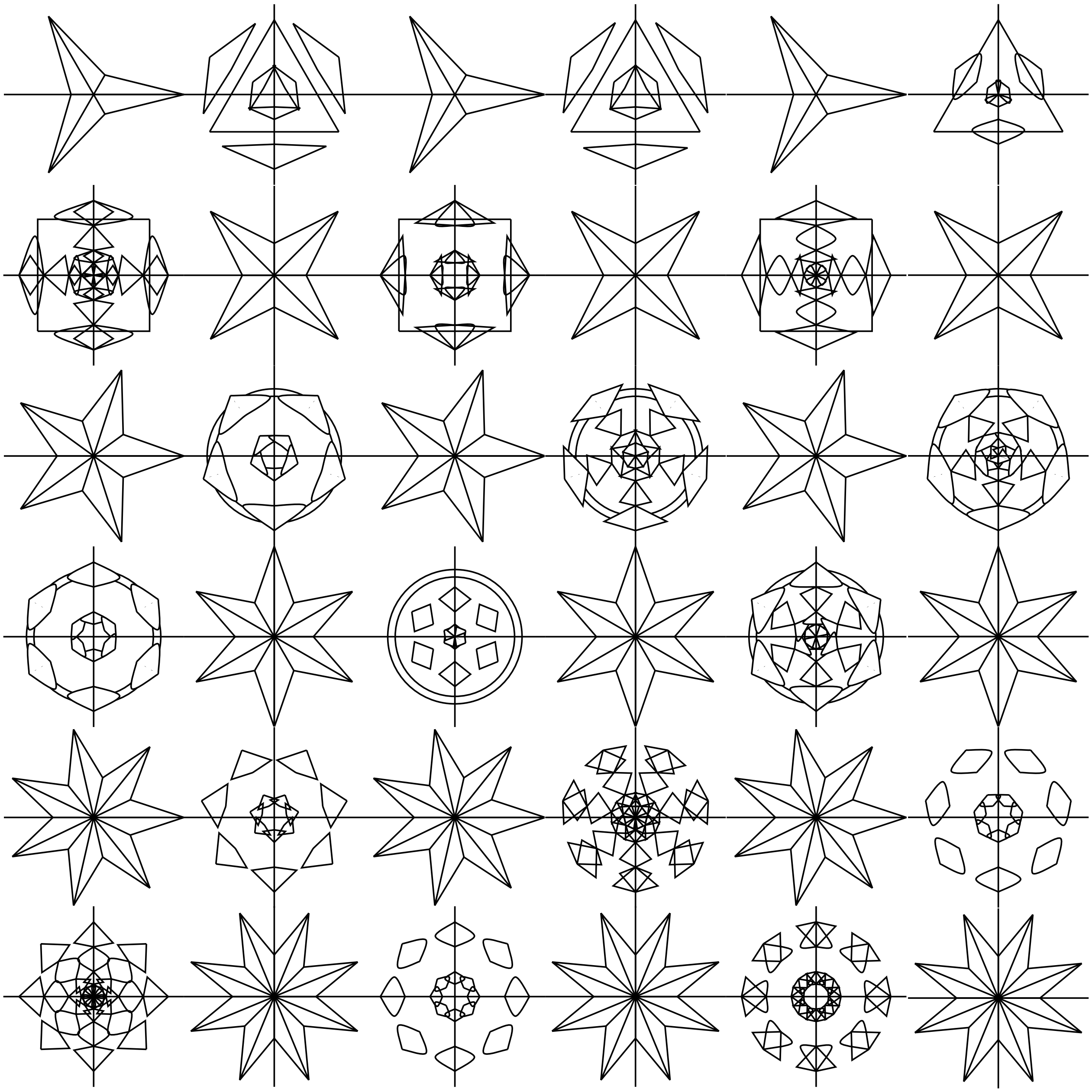
Challenge: Can you find the shapes that have the same rotations and reflections of a square?



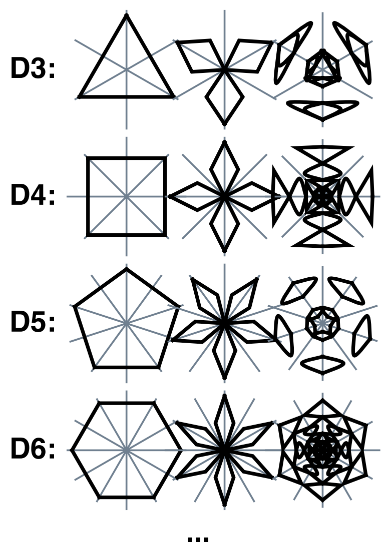
The symmetry group of a regular triangle and all the shapes that have its same symmetries, is called D3, while the symmetry group of a square is called D4, and so on...



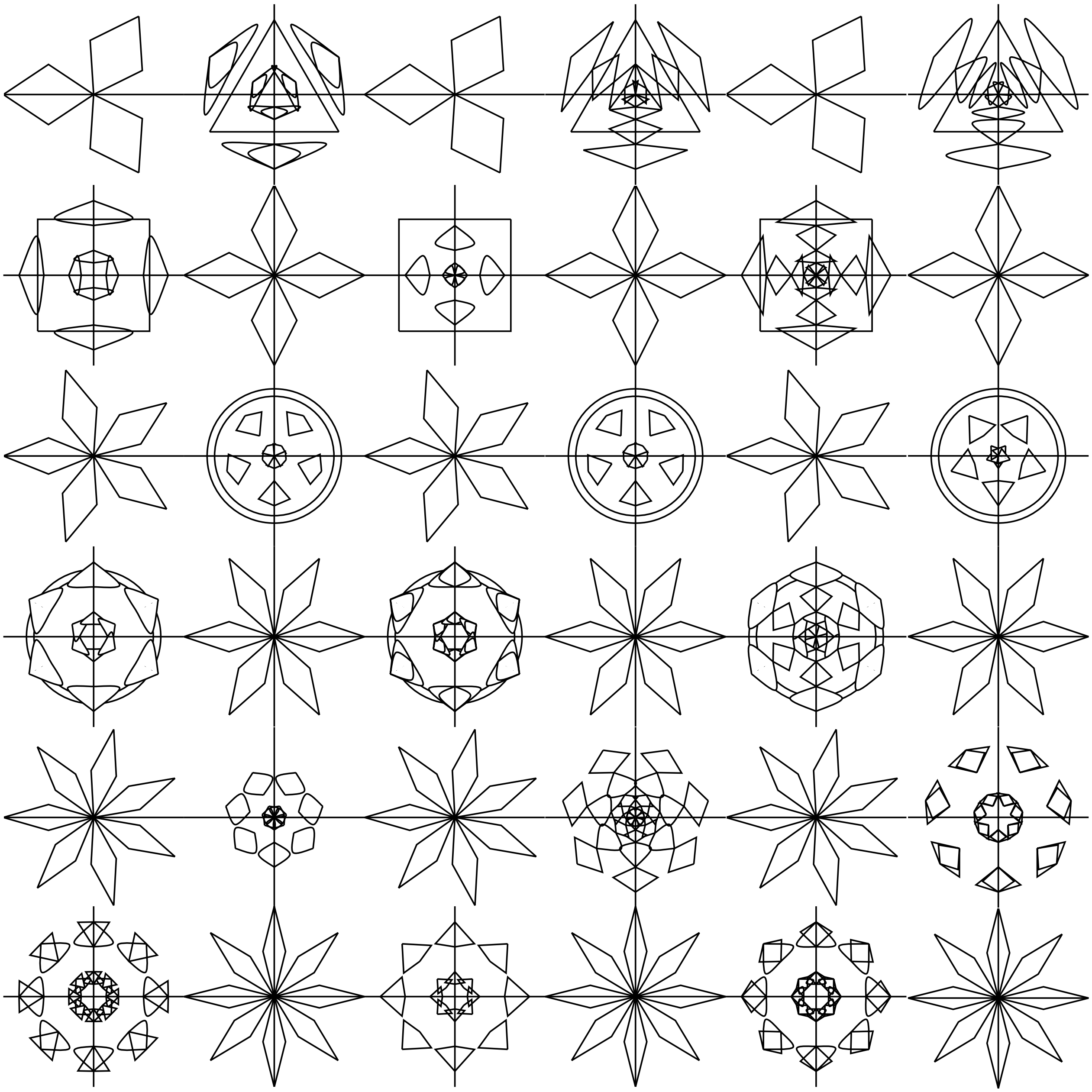
This series of groups is called the dihedral groups.



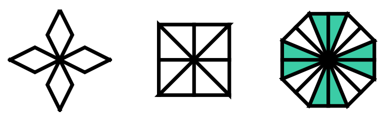
These shapes that share a symmetry group may look different, but when viewed through the lens of group theory, they look the same. Only their symmetries - the rotations and reflections that leave them unchanged - are seen.



Challenge: Which dihedral group does each uncolored shape belong to?

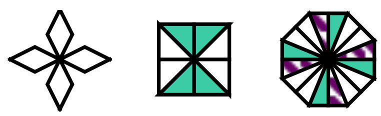


By looking for rotations and reflections, we can see when shapes share a symmetry group.



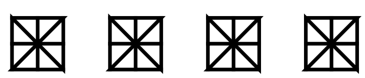
D4 D4 D4

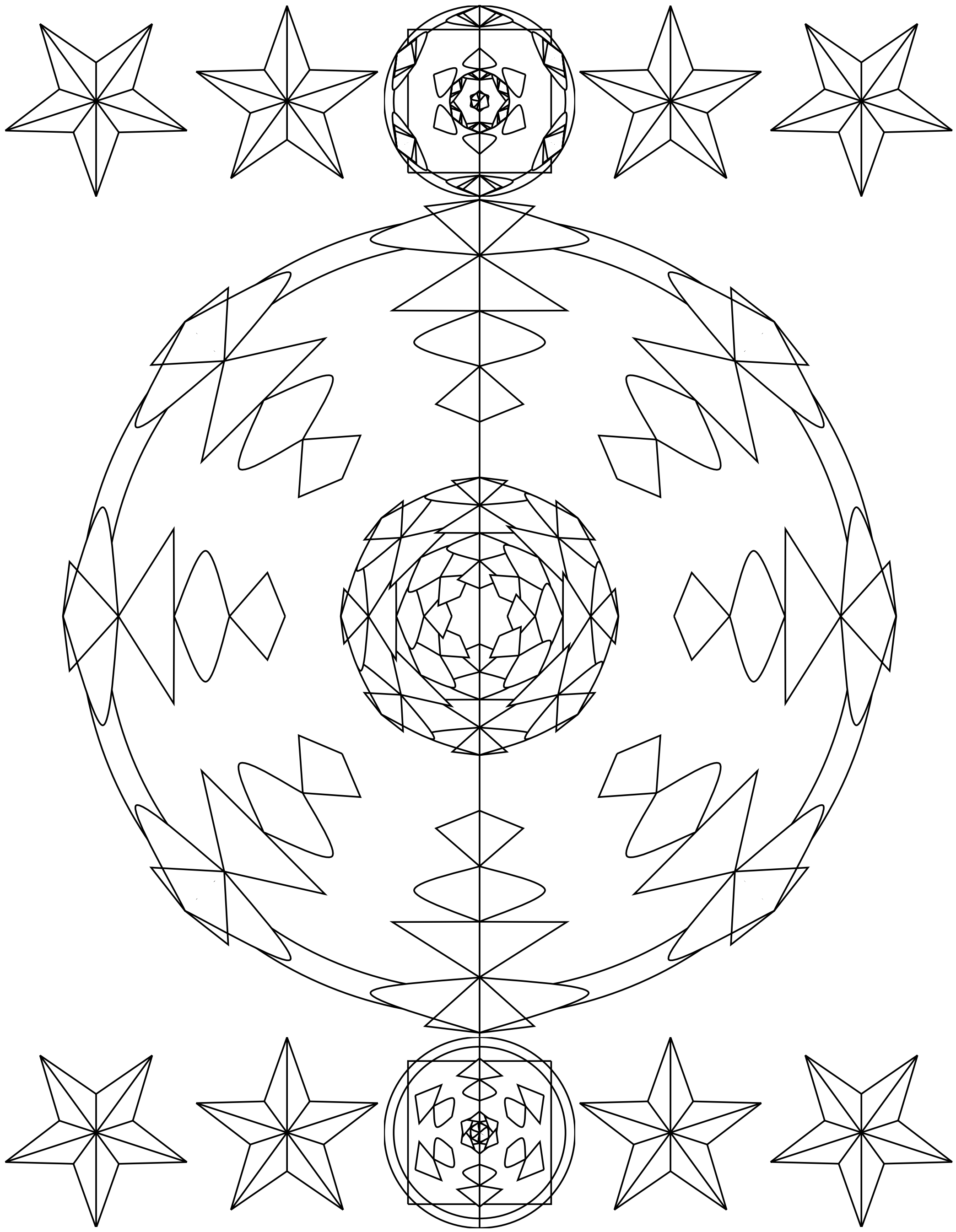
And when they do not.



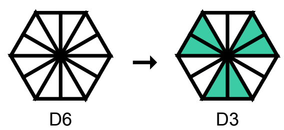
D4 D2 C4

Challenge: Can you color the shapes so that none of them share a symmetry group?

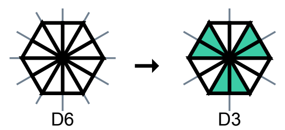




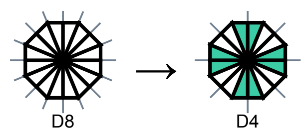
Again, color can reduce the amount of symmetry a shape has.



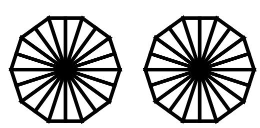
For example, a D6 shape has 6 mirrors and 6 rotations, but color can transform it into a shape with only 3 mirrors and 3 rotations - a D3 shape.

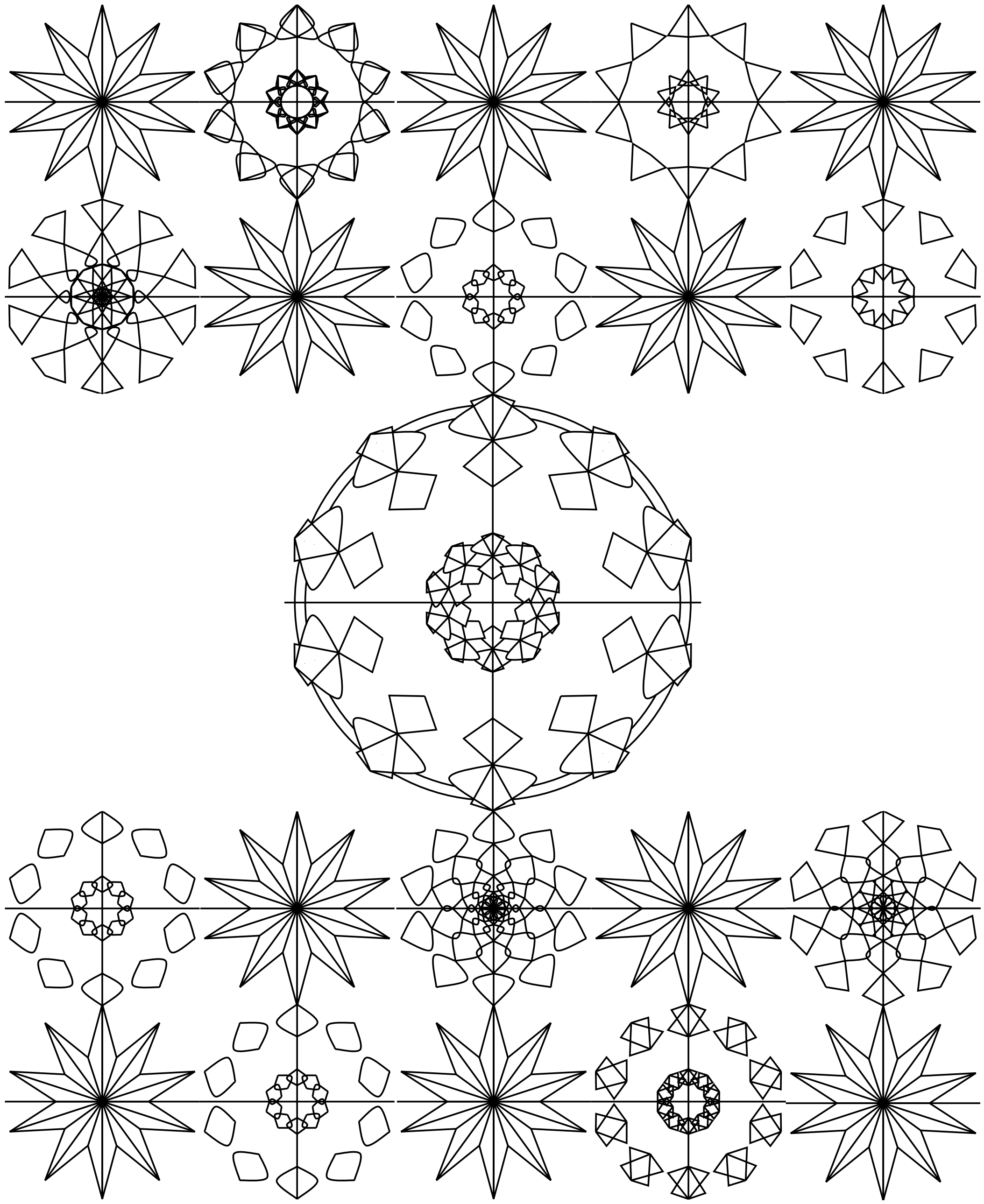


Color can reduce a D6 shape to a D3 shape because D3 is a <u>subgroup</u> of D6. Similarly, D4 is a <u>subgroup</u> of D8.

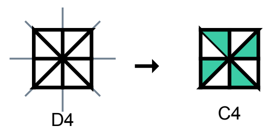


Challenge: Can you use color to reduce the D10 shapes to D5 shapes?

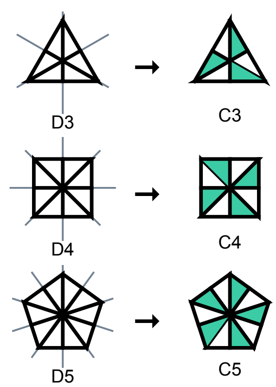




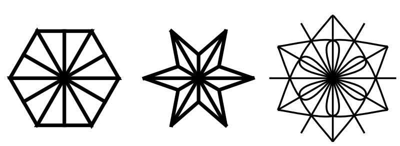
What happens when color is added to remove only mirrors and not rotations?

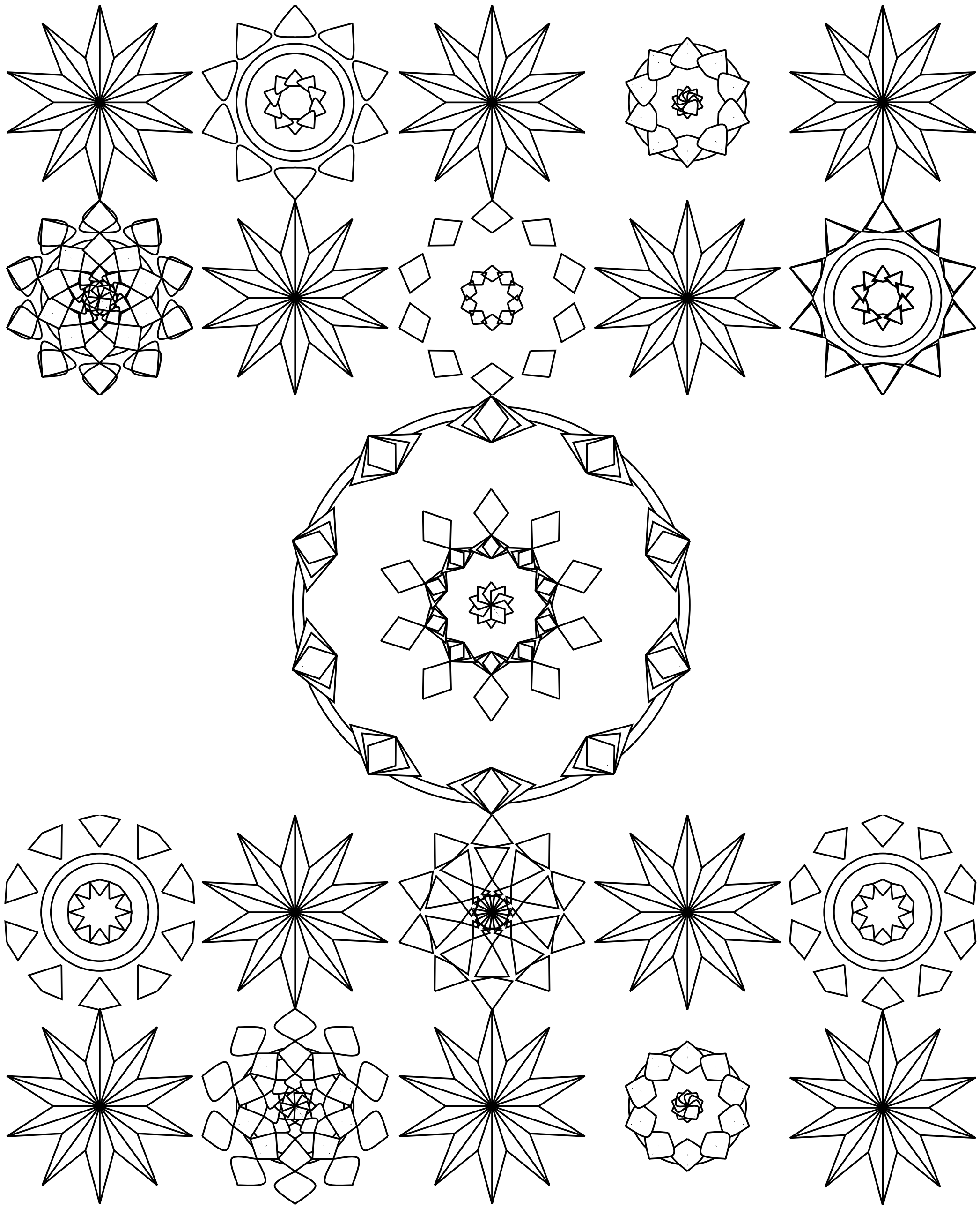


The dihedral groups have mirror reflections, while the cyclic groups do not. When these mirrors are removed, we can see the cyclic groups are subgroups of the dihedral groups.

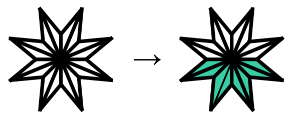


Challenge: Color the D6 shapes to transform them into C6 shapes.

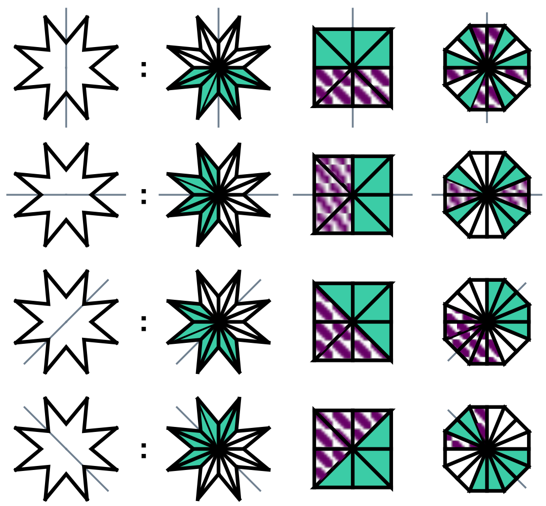




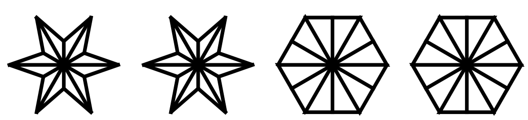
Color can also take away a shape's rotations.

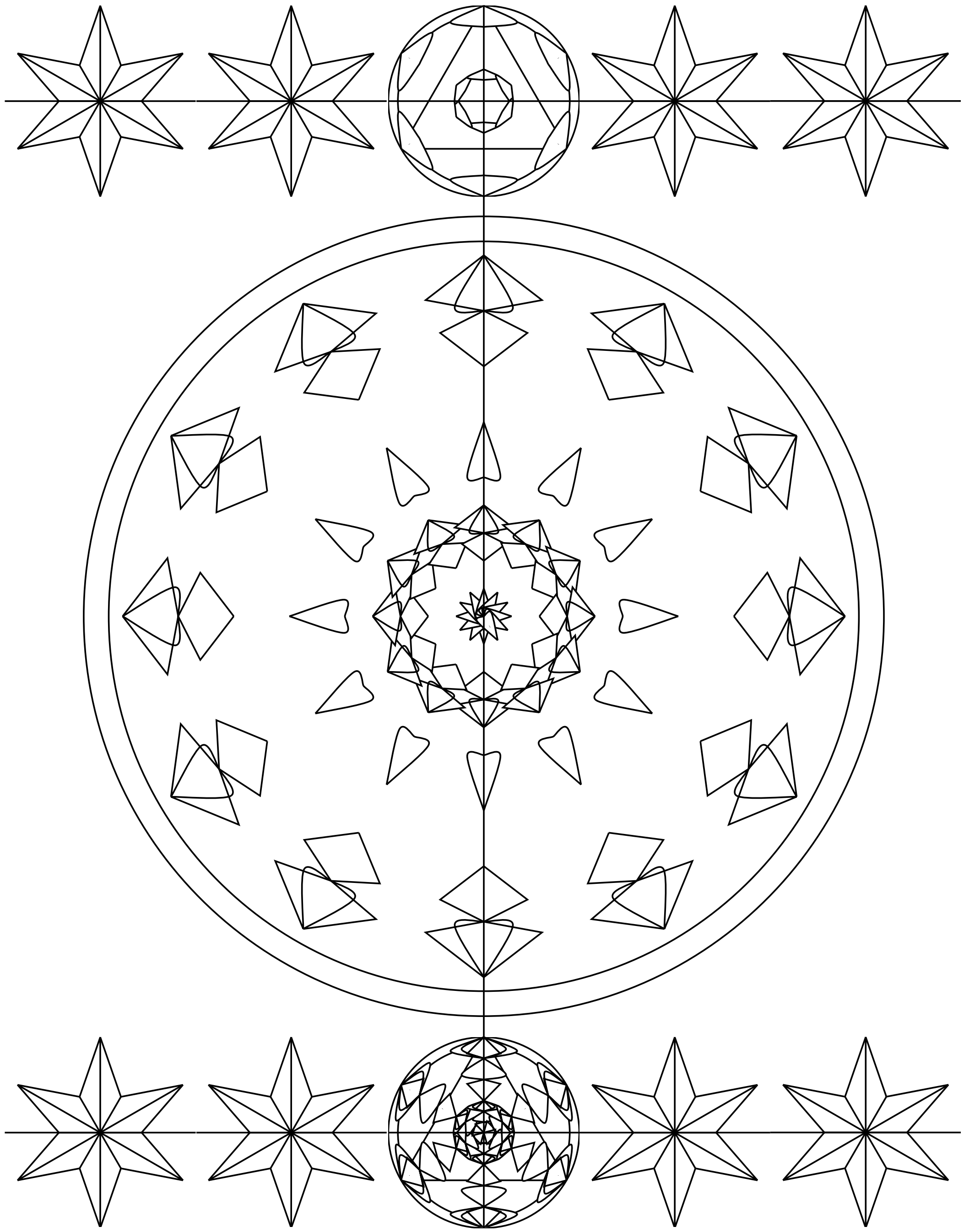


Coloring in this way leads to finding subgroups with only mirror reflections.



Challenge: Use color to reduce D6 shapes to D3 shapes. Then add more color to remove their rotations so that they only have reflection.





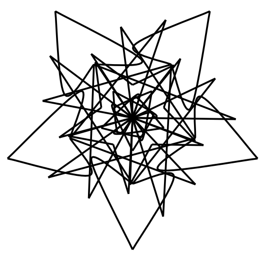
More is coming soon.

About

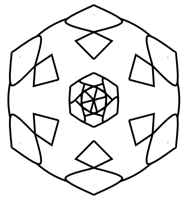
This is a coloring book that is both digital and on paper.

By coloring on paper, readers can explore symmetry and the beauty of math. Follow along on the digital copy to bring the concepts and illustrations to life in interactive animations.

Printed copy: *Coming soon*
Digital copy: coloring-book.co



The illustrations in this book are designed by algorithms that follow the symmetry rules of the groups that the illustrations belong to. These algorithms also use generative art techniques to add components of randomness - notice the illustrations never repeat.



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