

Participating media



15-468, 15-668, 15-868
Physics-based Rendering
Spring 2021, Lecture 17

Course announcements

- Programming assignment 4 posted, due Friday 4/9 at 23:59.
 - How many of you have looked at/started/finished it?
 - Any questions?
- Take-home quiz 6 due tonight.
- Take-home quiz 7 will be posted tonight.
- Details about final project proposals will be posted today.
 - Proposals due April 16th.
 - Extra office hours to discuss topics.
- Vote on Piazza for recitation hours.
- Suggest on Piazza topics for this week's reading group.

Graphics faculty candidate talk



- Speaker: **Mina Lukovic (MIT)**
- Title: **Transforming design and fabrication with computational discovery**
- Abstract: Recent advances in material science and computational fabrication provide promising opportunities for product design, mechanical and biomedical engineering, medical devices, robotics, architecture, art, and science. Engineered materials and personalized fabrication are revolutionizing manufacturing culture and having a significant impact on various scientific and industrial products. As new fabrication technologies emerge, effective computational tools are needed to fully exploit the potential of computational fabrication.

In this talk, I argue that computer science and mathematical models are essential for advancing and accelerating design practices and harnessing the potential of novel fabrication technologies. My aim is to transform the design workflow with computational tools and artificial intelligence and change “what?” and “how?” we can fabricate. I will discuss how the insights from differential geometry can help us understand existing materials and create new materials with specific performance. I will further demonstrate how grammars and deep learning can be combined for the autonomous discovery of terrain-optimized robots. Finally, I will show a data-efficient machine learning algorithm for optimal experiment design. Although different in methodologies, all these projects follow the same design pipeline and tackle two critical challenges: (i) providing tools for inverse design and (ii) accelerating design and fabrication with sophisticated algorithms.

Graphics lab meeting talk



- Speaker: **Pratul Srinivasan** (Google Research)
- Title: **Extending Neural Radiance Fields**
- Abstract: Neural volumetric scene representations such as Neural Radiance Fields (NeRF) have spurred exciting progress across many inverse rendering tasks. However, there is still a long way to go before NeRF-like representations can be used instead of more traditional representations in graphics pipelines. I'll be talking about our recent work to extend NeRF to support anti-aliasing during rendering, real-time rendering, and relighting.

Overview of today's lecture

- Wrap-up BRDFs.
- Participating media.
- Scattering material characterization.
- Volume rendering equation.
- Ray marching.
- Volumetric path tracing.
- Delta tracking.

Slide credits

Most of these slides were directly adapted from:

- Wojciech Jarosz (Dartmouth).

Fog



[Steve Lacey](#)

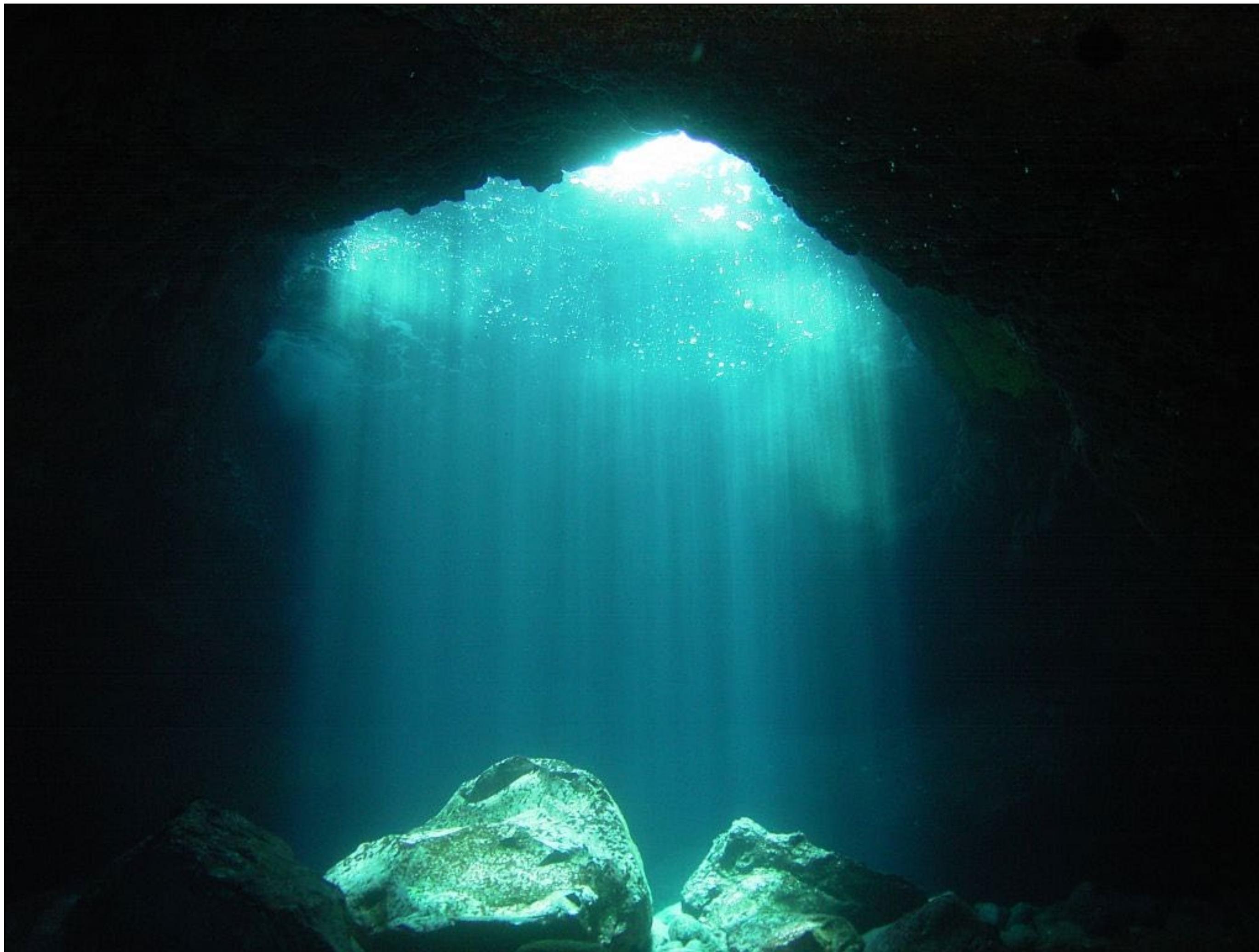
Clouds & Crepuscular rays



Fire



Underwater



Surface or Volume?



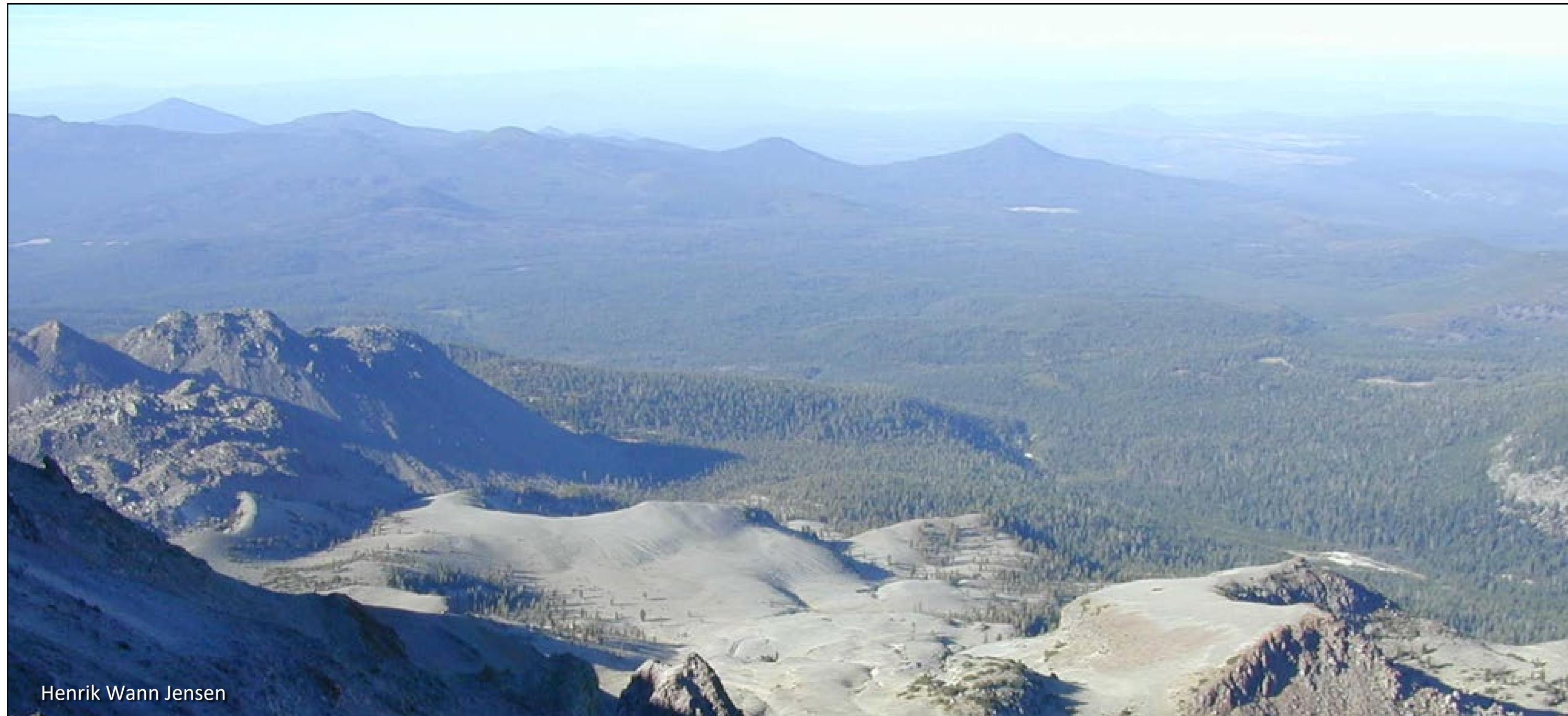
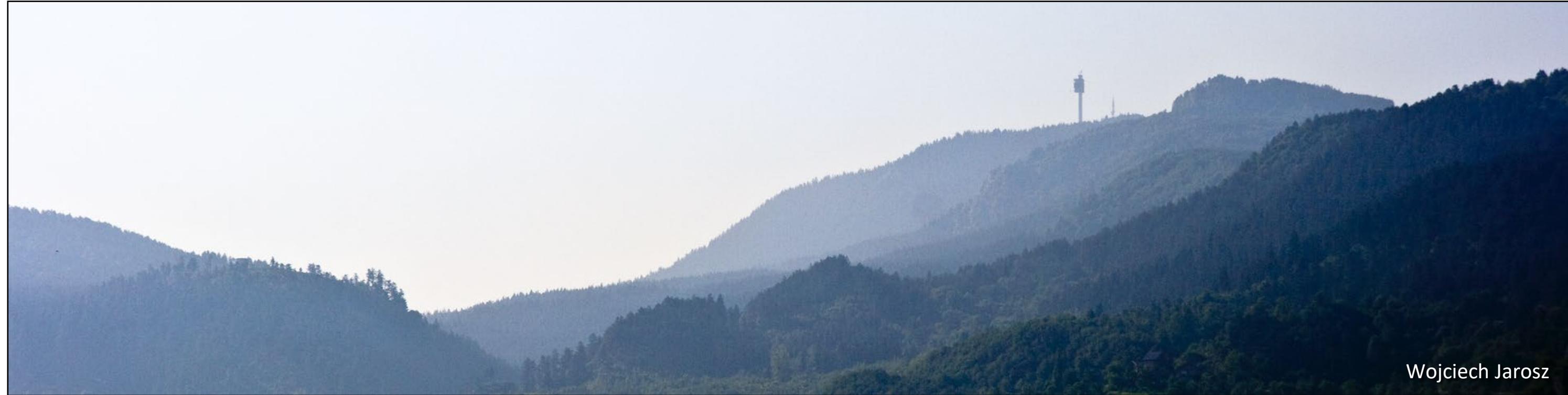
source: [Flickr](#)



Antelope Canyon, Az.



Aerial (Atmospheric) Perspective



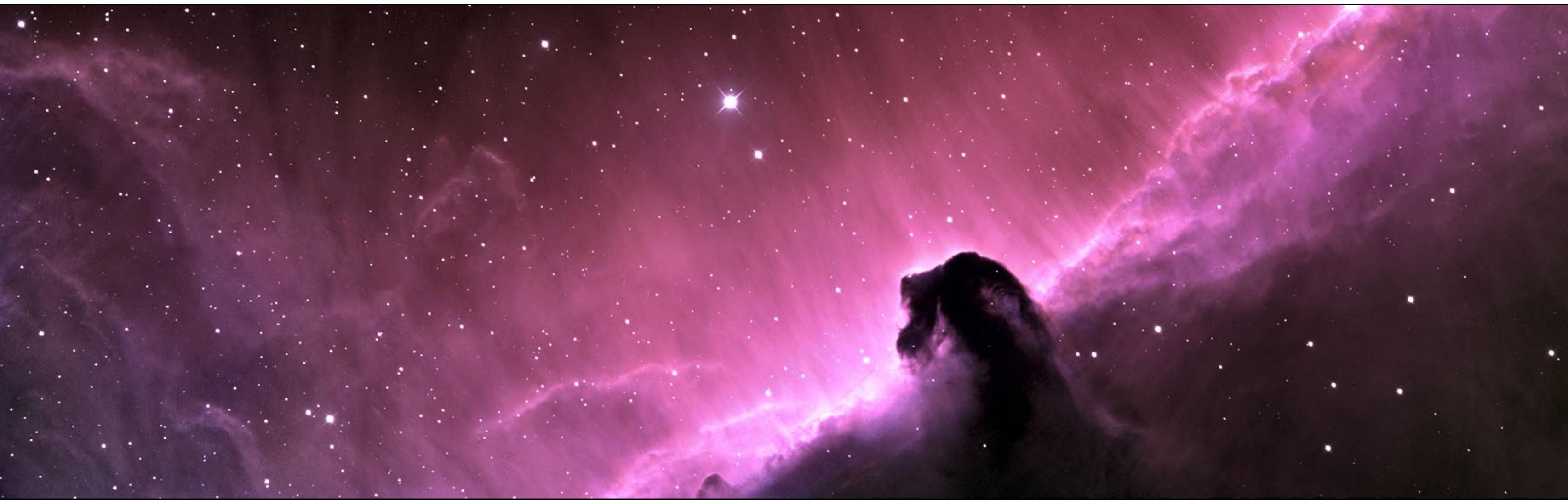
Leonardo da Vinci (1480)



Thus, if one is to be five times as distant, make it five times bluer.

—Treatise on Painting, Leonardo Da Vinci, pp 295, circa 1480.

Nebula



Emission



<http://wikipedia.org>

Absorption



<http://commons.wikimedia.org>

Scattering



<http://coclouds.com>

Defining Participating Media

Typically, we do not model particles of a medium explicitly
(wouldn't fit in memory, completely impractical to ray trace)

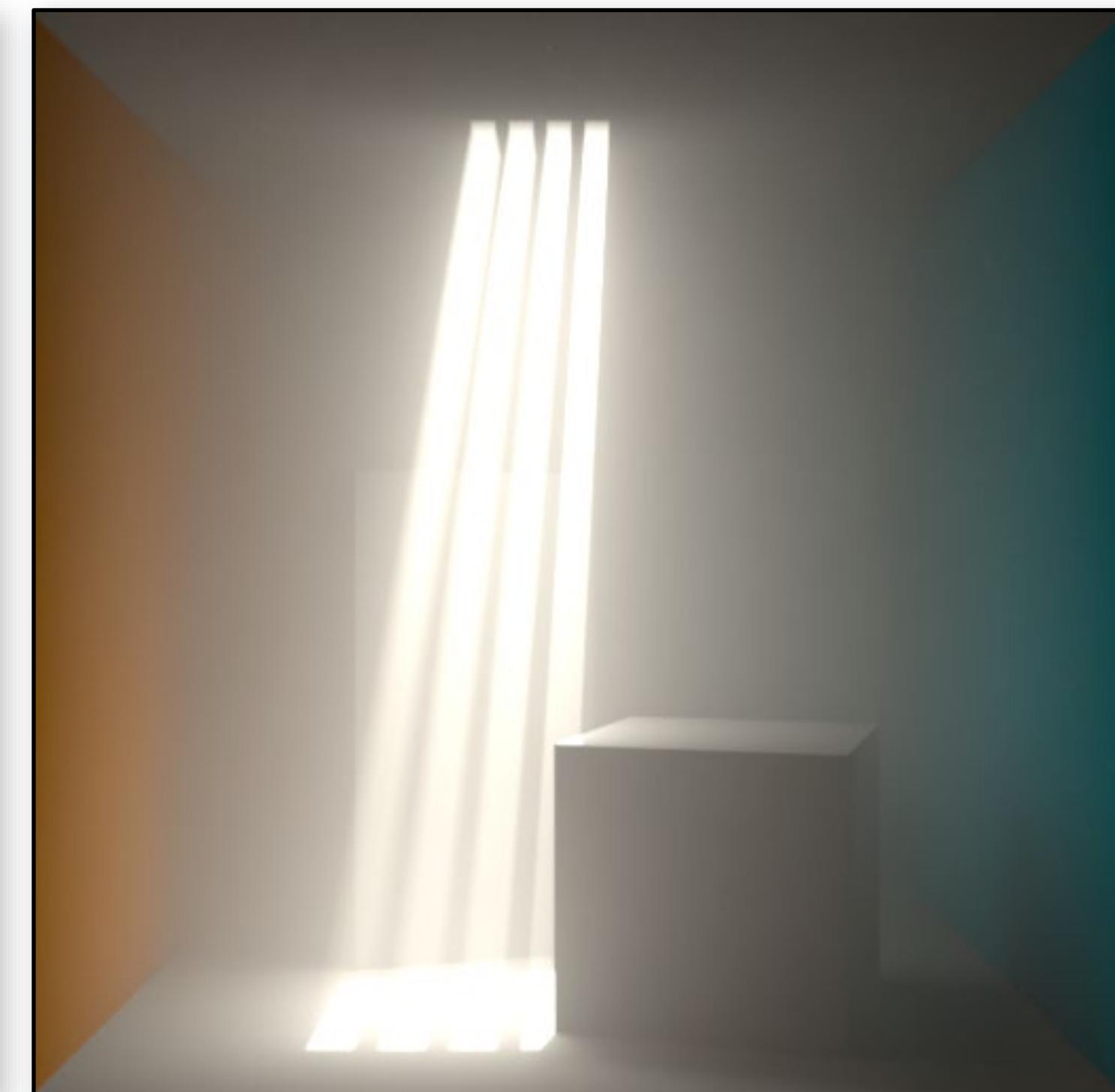
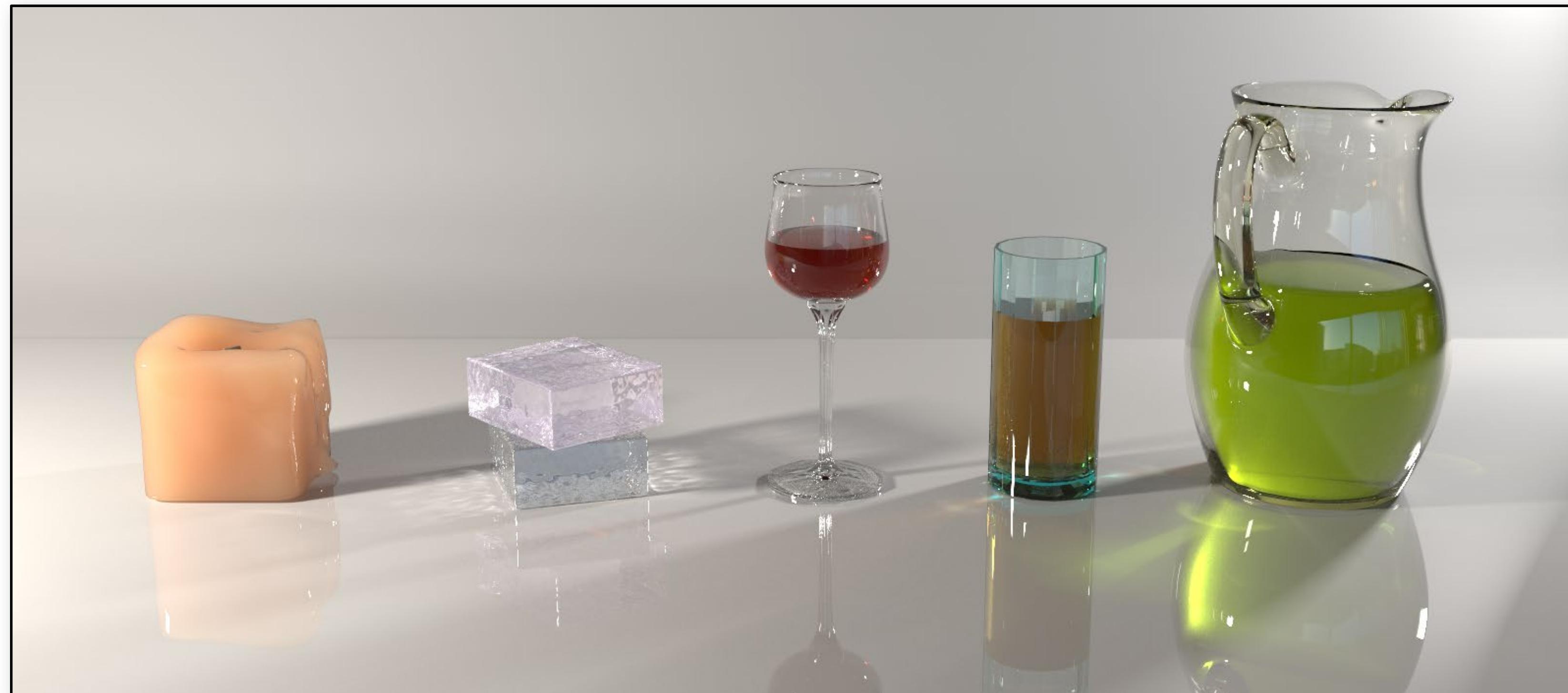
The properties are described statistically using various
coefficients and densities

- Conceptually similar idea as microfacet models

Defining Participating Media

Homogeneous:

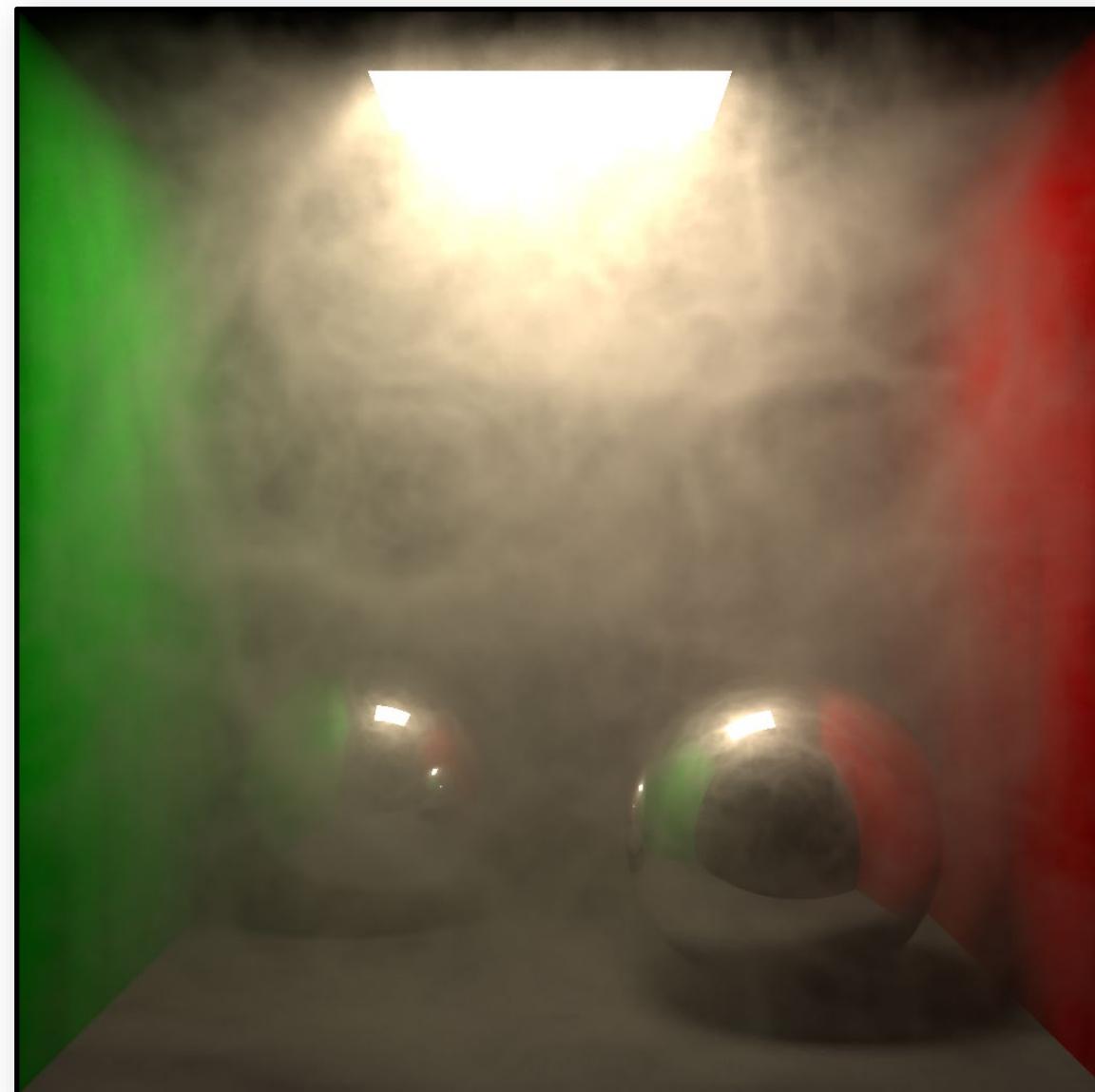
- Infinite or bounded by a surface or simple shape



Defining Participating Media

Heterogeneous (spatially varying coefficients):

- Procedurally, e.g., using a noise function
- Simulation + volume discretization, e.g., a voxel grid

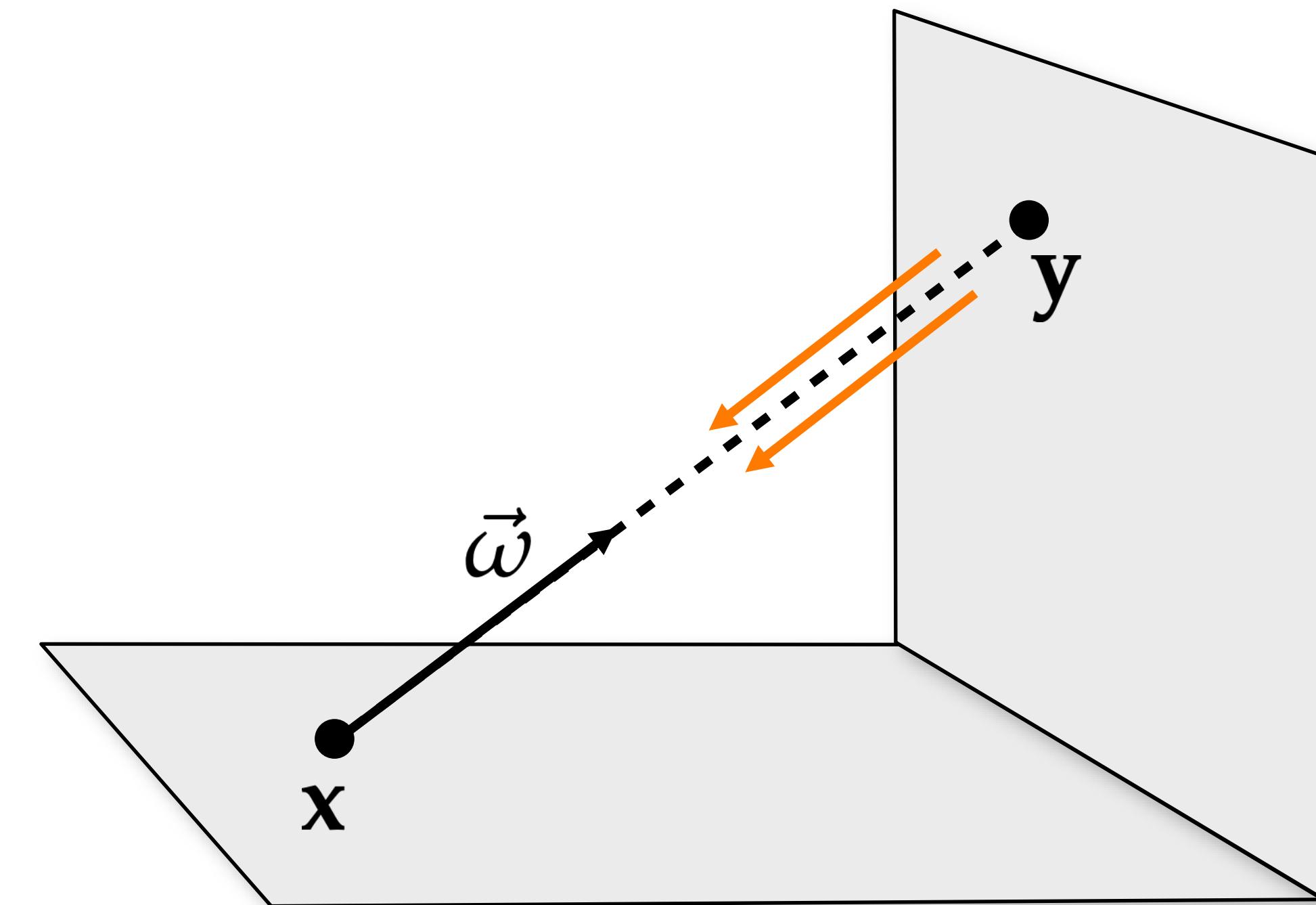


Radiance

The main quantity we are interested in for rendering is radiance

Previously: radiance *remains constant* along rays between surfaces

$$L_i(\mathbf{x}, \vec{\omega}) = L_o(\mathbf{y}, -\vec{\omega})$$
$$\mathbf{y} = r(\mathbf{x}, \vec{\omega})$$

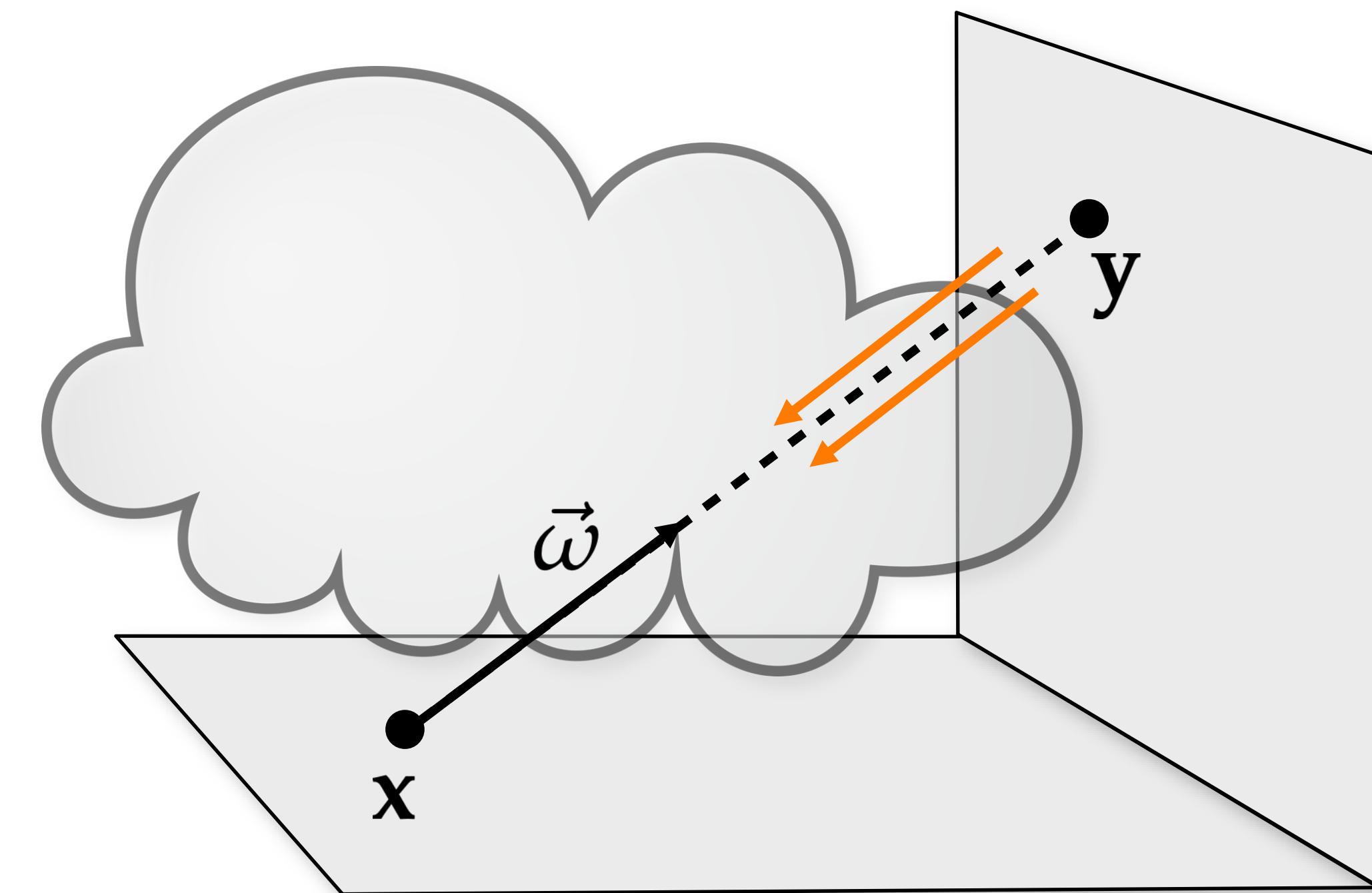


Radiance

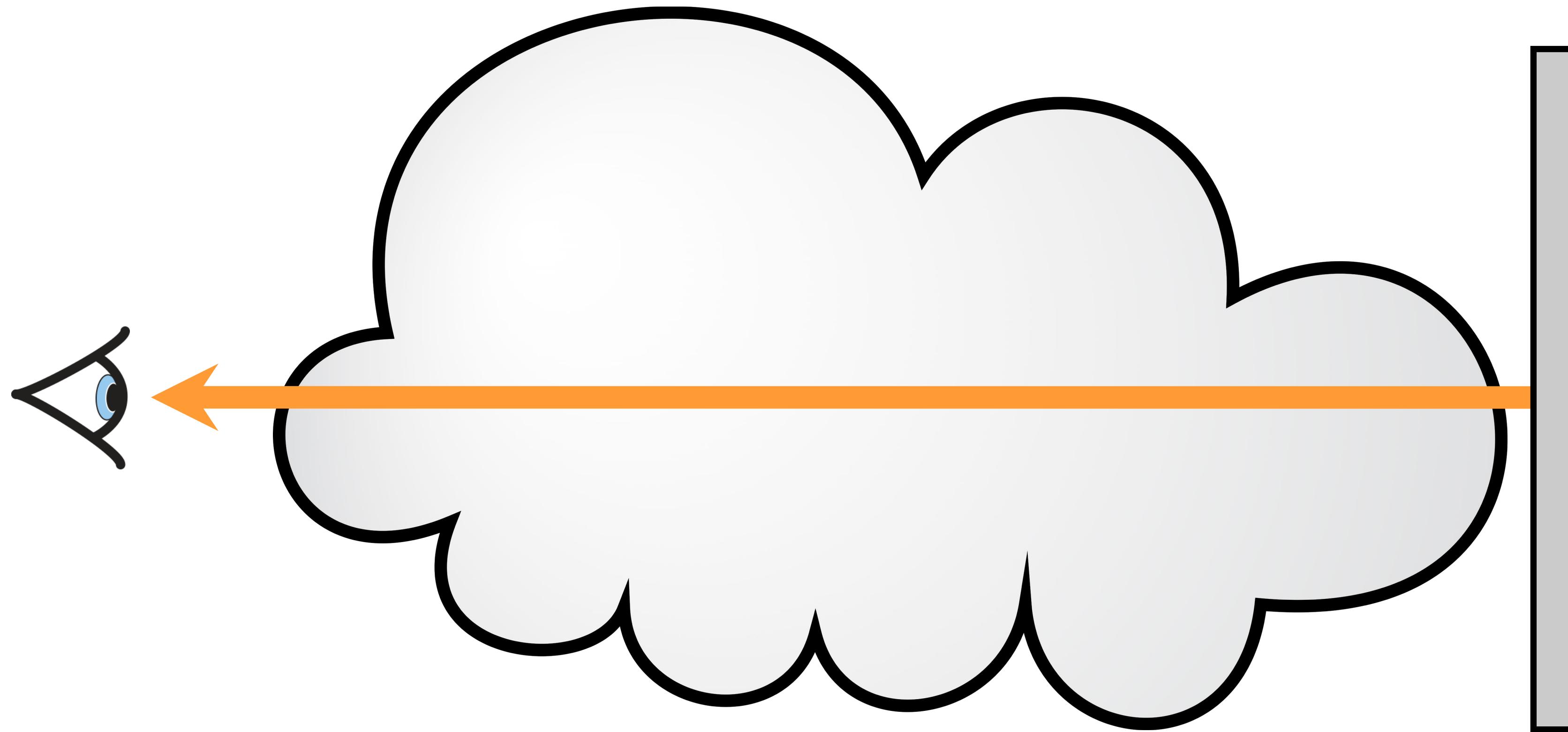
The main quantity we are interested in for rendering is radiance

Now: radiance may *change* along rays between surfaces

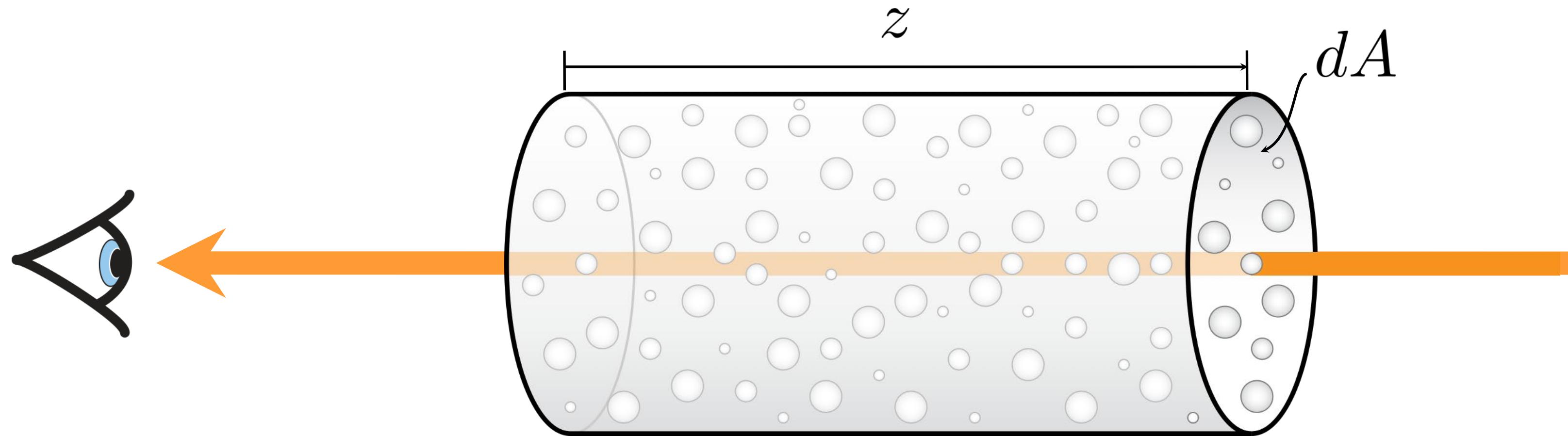
$$L_i(\mathbf{x}, \vec{\omega}) \neq L_o(\mathbf{y}, -\vec{\omega})$$
$$\mathbf{y} = r(\mathbf{x}, \vec{\omega})$$



Participating Media

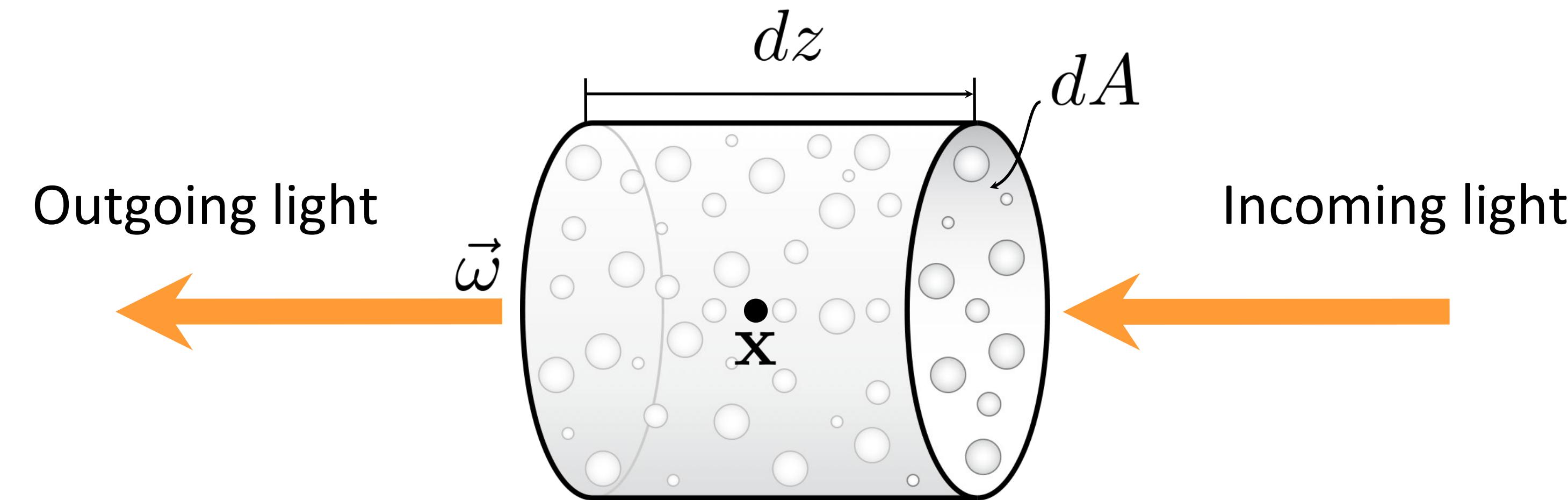


Differential Beam

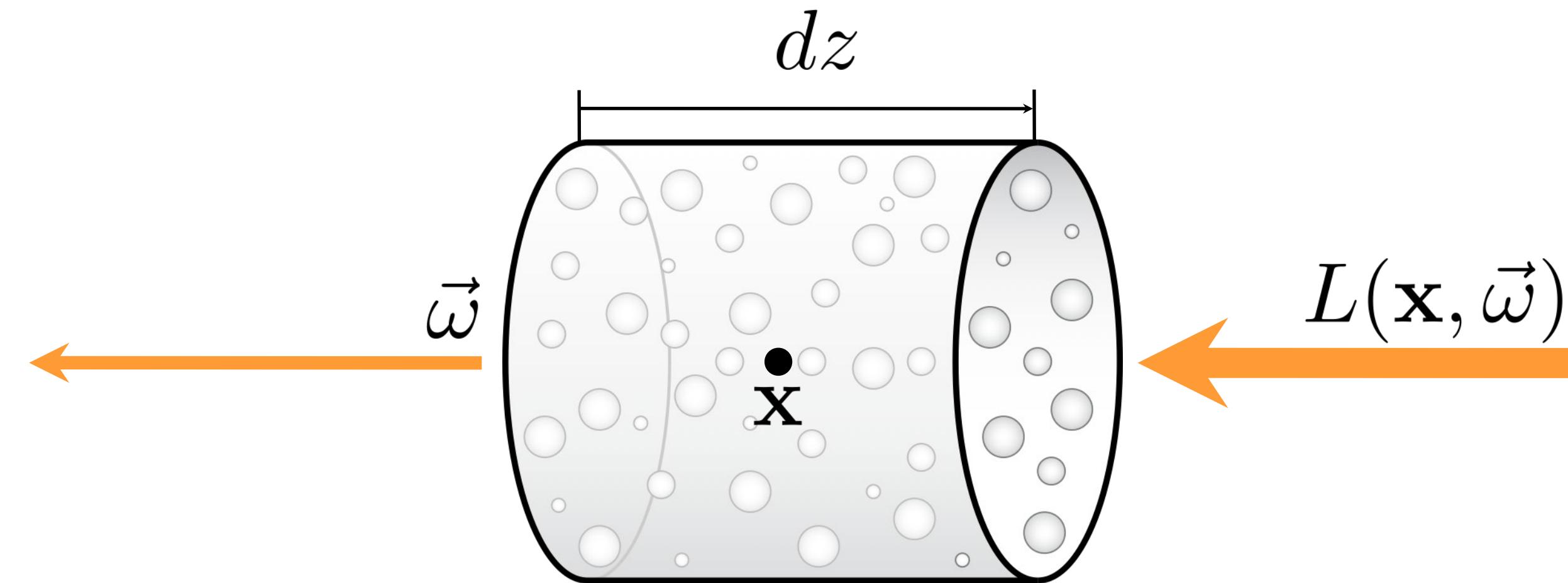


How much light is *lost/gained* along the differential beam
due to interactions of light with the medium?

Differential Beam Segment



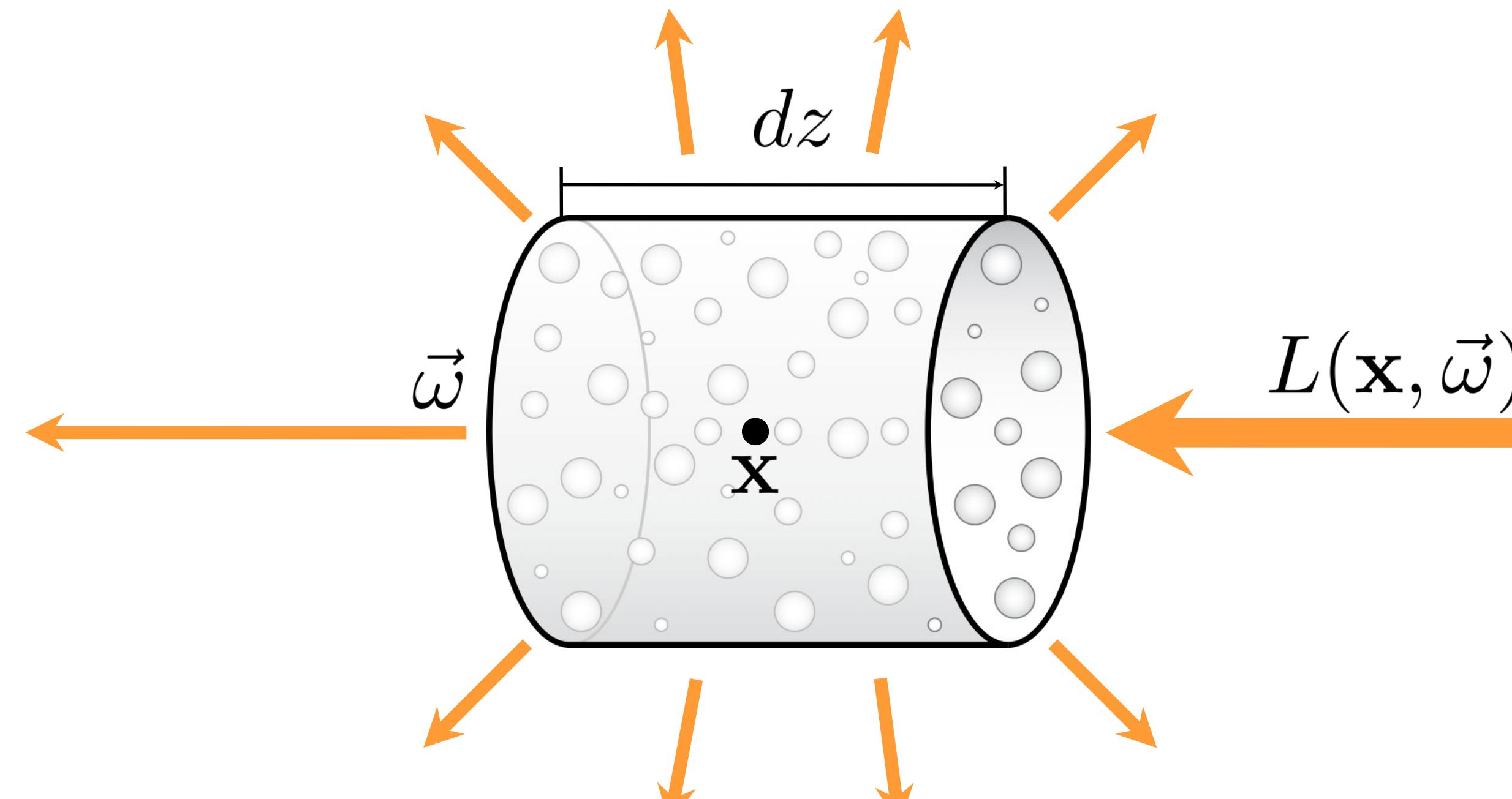
Absorption



$$dL(\mathbf{x}, \vec{\omega}) = -\sigma_a(\mathbf{x})L(\mathbf{x}, \vec{\omega})dz$$

$\sigma_a(\mathbf{x})$: absorption coefficient $[m^{-1}]$

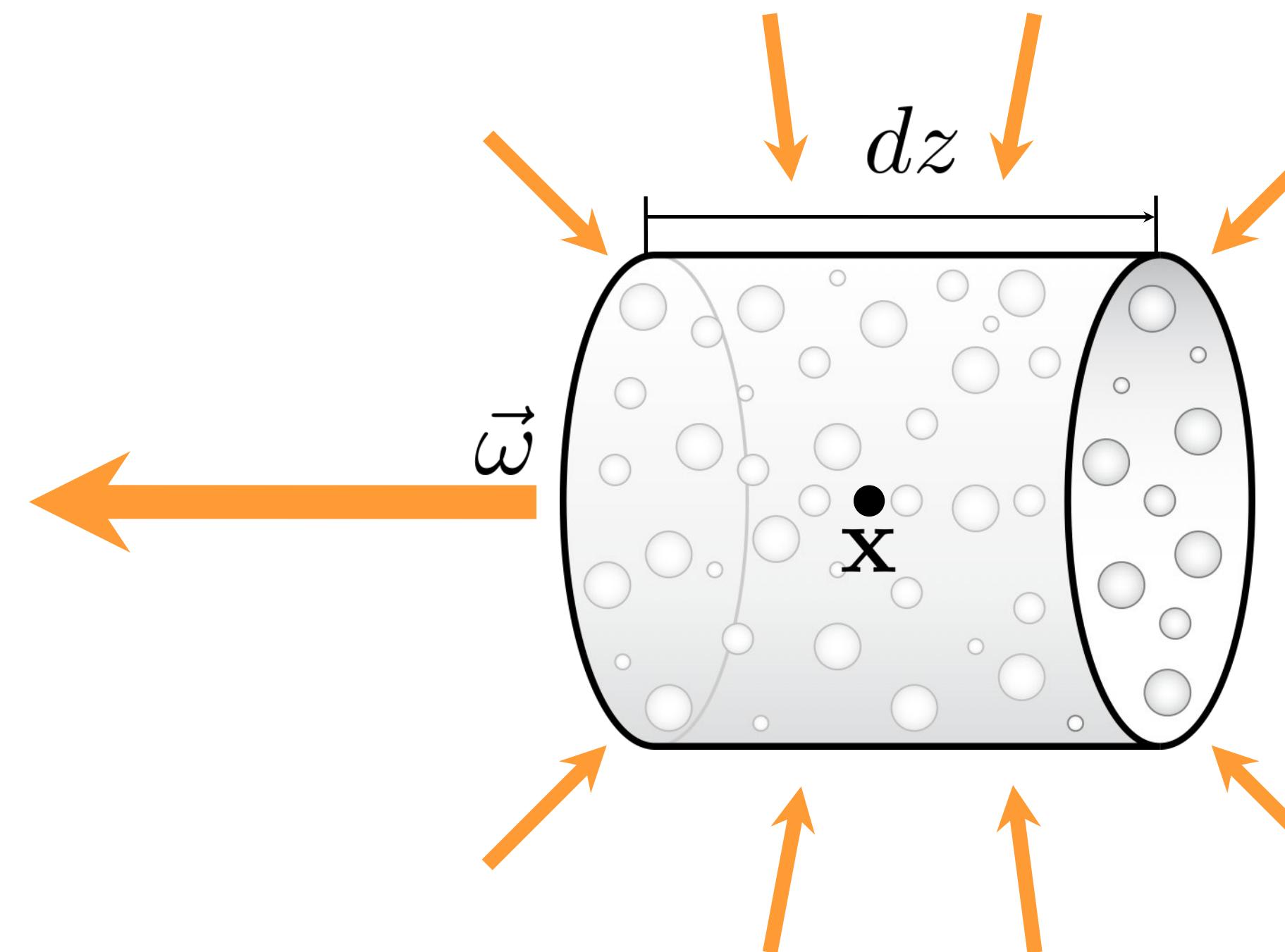
Out-scattering



$$dL(\mathbf{x}, \vec{\omega}) = -\sigma_s(\mathbf{x})L(\mathbf{x}, \vec{\omega})dz$$

$\sigma_s(\mathbf{x})$: scattering coefficient $[m^{-1}]$

In-scattering

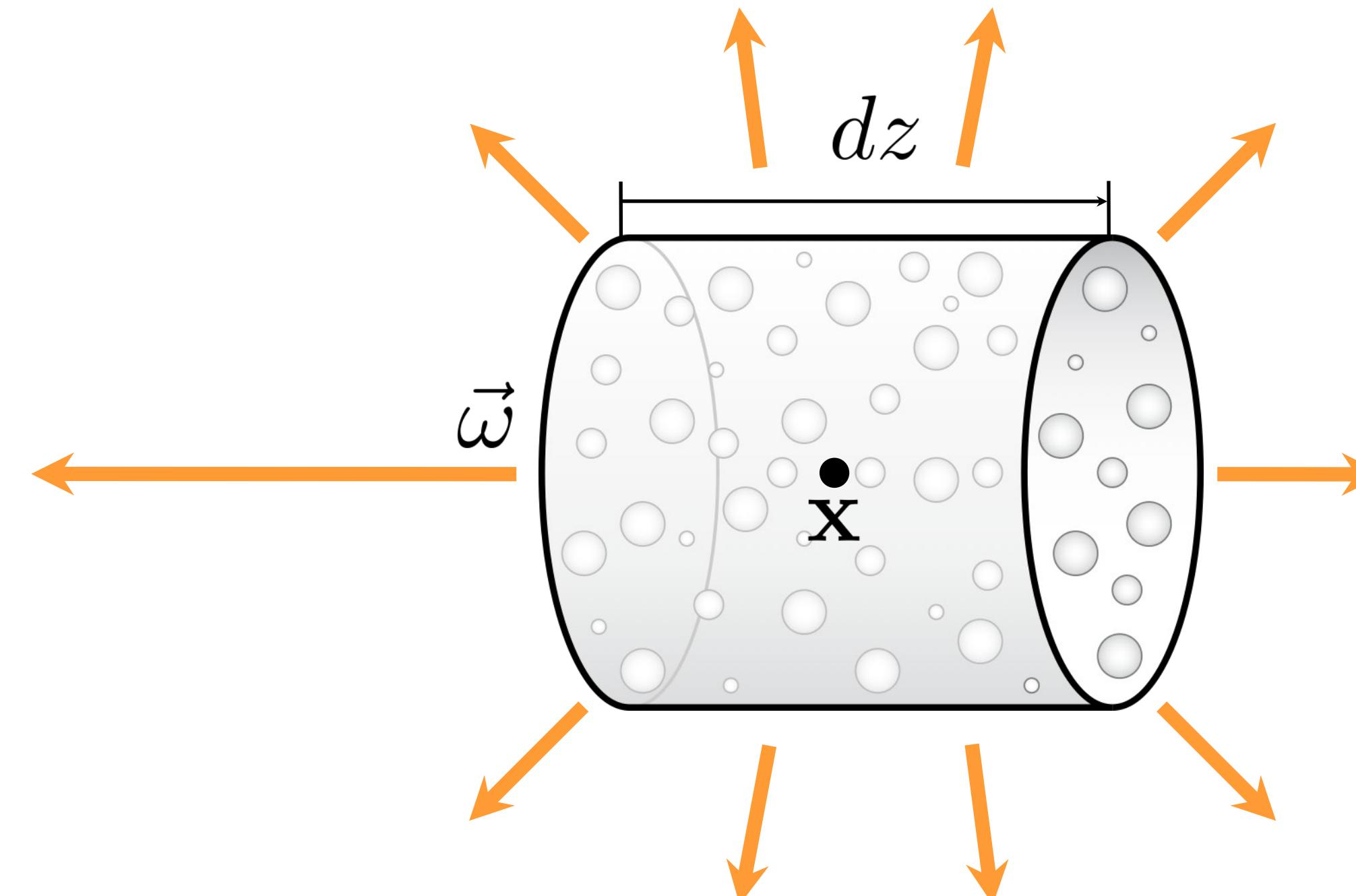


$$dL(\mathbf{x}, \vec{\omega}) = \sigma_s(\mathbf{x}) L_s(\mathbf{x}, \vec{\omega}) dz$$

$\sigma_s(\mathbf{x})$: scattering coefficient $[m^{-1}]$

$L_s(\mathbf{x}, \vec{\omega})$: in-scattered radiance

Emission

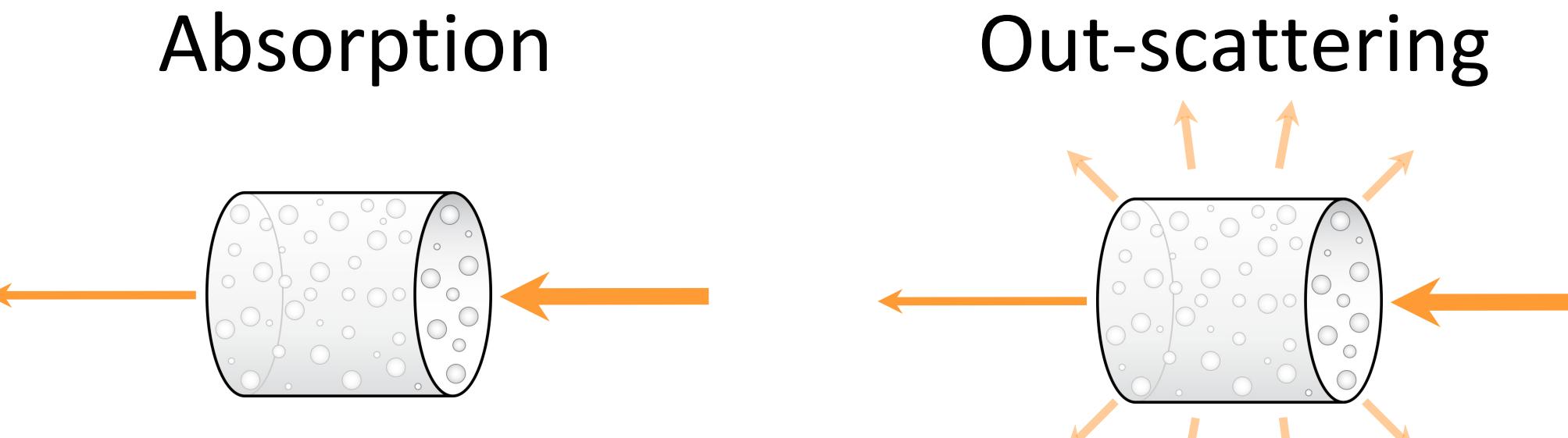


$$dL(\mathbf{x}, \vec{\omega}) = \sigma_a(\mathbf{x}) L_e(\mathbf{x}, \vec{\omega}) dz$$

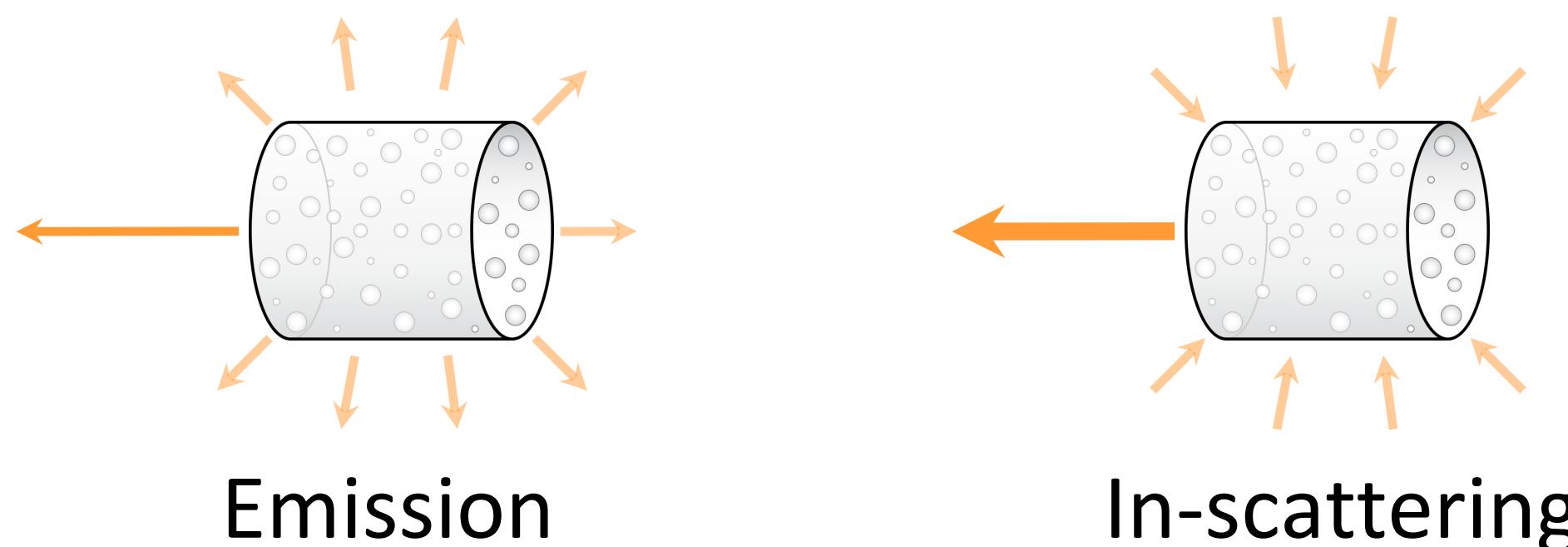
*Sometimes modeled without the absorption coefficient just by specifying a “source” term

$\sigma_a(\mathbf{x})$: absorption coefficient $[m^{-1}]$
 $L_e(\mathbf{x}, \vec{\omega})$: emitted radiance

Radiative Transfer Equation (RTE)



$$dL(\mathbf{x}, \vec{\omega}) = \boxed{-\sigma_a(\mathbf{x})L(\mathbf{x}, \vec{\omega})dz \quad -\sigma_s(\mathbf{x})L(\mathbf{x}, \vec{\omega})dz} \quad \text{Losses}$$
$$\boxed{+\sigma_a(\mathbf{x})L_e(\mathbf{x}, \vec{\omega})dz \quad +\sigma_s(\mathbf{x})L_s(\mathbf{x}, \vec{\omega})dz} \quad \text{Gains}$$

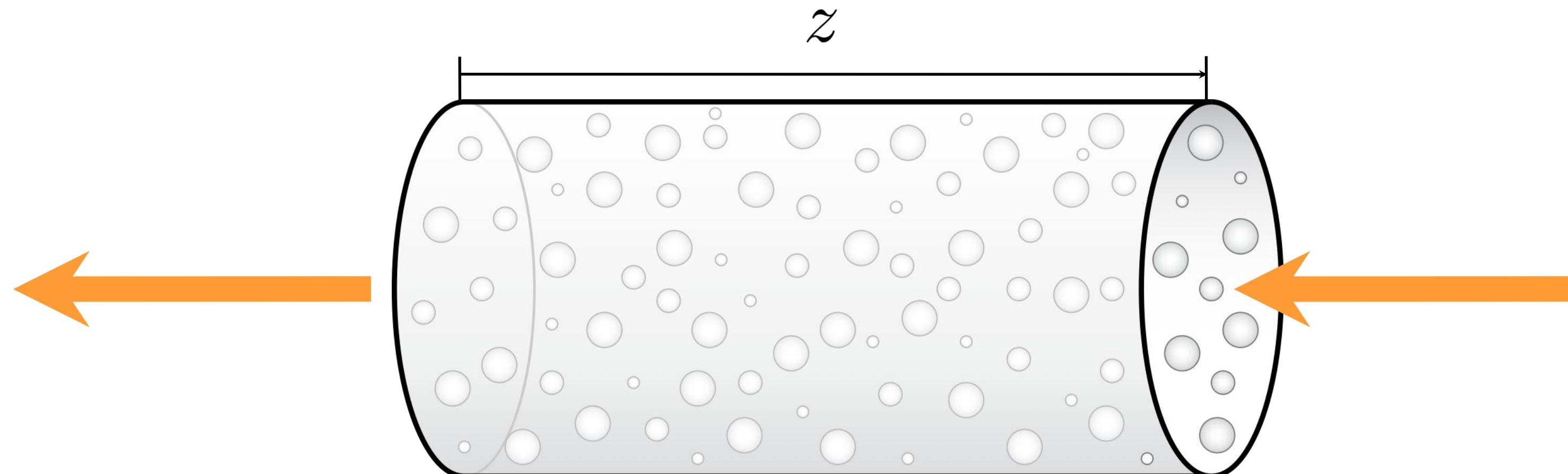


Losses (Extinction)

Absorption

$$\begin{aligned} dL(\mathbf{x}, \vec{\omega}) &= -\sigma_a(\mathbf{x})L(\mathbf{x}, \vec{\omega})dz - \sigma_s(\mathbf{x})L(\mathbf{x}, \vec{\omega})dz \\ &= -\sigma_t(\mathbf{x})L(\mathbf{x}, \vec{\omega})dz \end{aligned}$$

$\sigma_t(\mathbf{x})$: extinction coefficient $[m^{-1}]$
: total loss of light per unit distance



What about a beam with a finite length?

Extinction Along a Finite Beam

$$dL(\mathbf{x}, \vec{\omega}) = -\sigma_t(\mathbf{x})L(\mathbf{x}, \vec{\omega})dz \quad // \text{Assume constant } \sigma_t(\mathbf{x}), \text{ reorganize}$$

$$\frac{dL(\mathbf{x}, \vec{\omega})}{L(\mathbf{x}, \vec{\omega})} = -\sigma_t dz \quad // \text{Integrate along beam from 0 to } z$$

$$\ln(L_z) - \ln(L_0) = -\sigma_t z$$

$$\ln\left(\frac{L_z}{L_0}\right) = -\sigma_t z \quad // \text{Exponentiate}$$

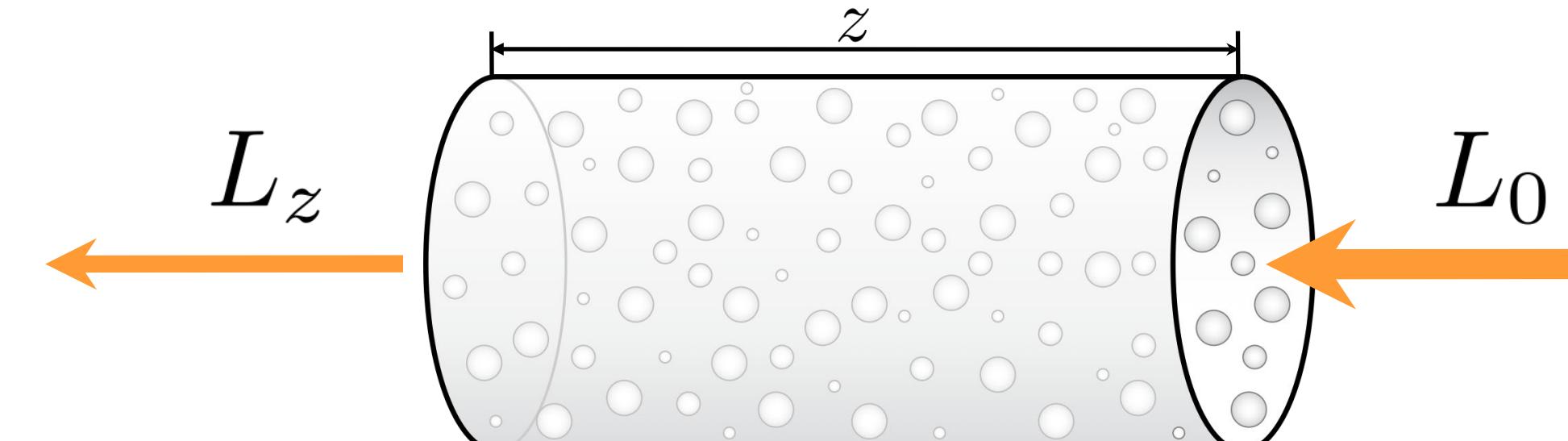
$$\frac{L_z}{L_0} = e^{-\sigma_t z}$$

Beer-Lambert Law

Expresses the remaining radiance after traveling a finite distance through a medium with constant extinction coefficient

The fraction is referred to as the *transmittance*

Think of this as fractional visibility between points


$$\frac{L_z}{L_0} = e^{-\sigma_t z}$$

Radiance at distance z

Radiance at the beginning of the beam

Transmittance

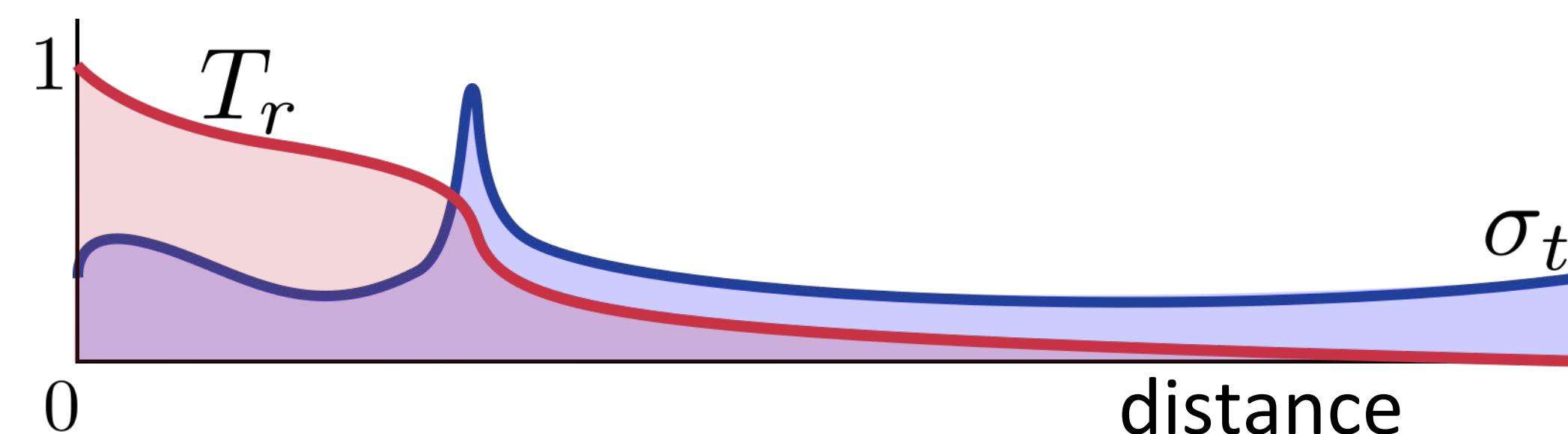
Homogeneous volume:

$$T_r(\mathbf{x}, \mathbf{y}) = e^{-\sigma_t \|\mathbf{x} - \mathbf{y}\|}$$

Heterogeneous volume (spatially varying σ_t):

$$T_r(\mathbf{x}, \mathbf{y}) = e^{-\int_0^{\|\mathbf{x} - \mathbf{y}\|} \sigma_t(t) dt}$$

Optical thickness



Transmittance

Homogeneous volume:

$$T_r(\mathbf{x}, \mathbf{y}) = e^{-\sigma_t \|\mathbf{x} - \mathbf{y}\|}$$

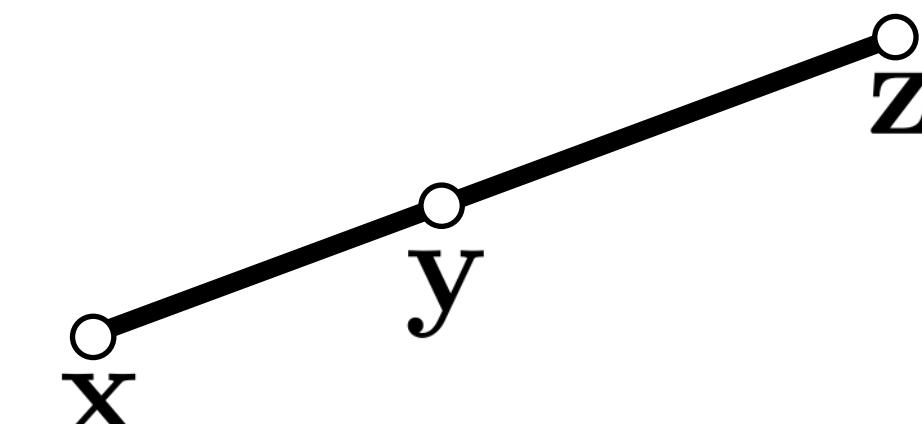
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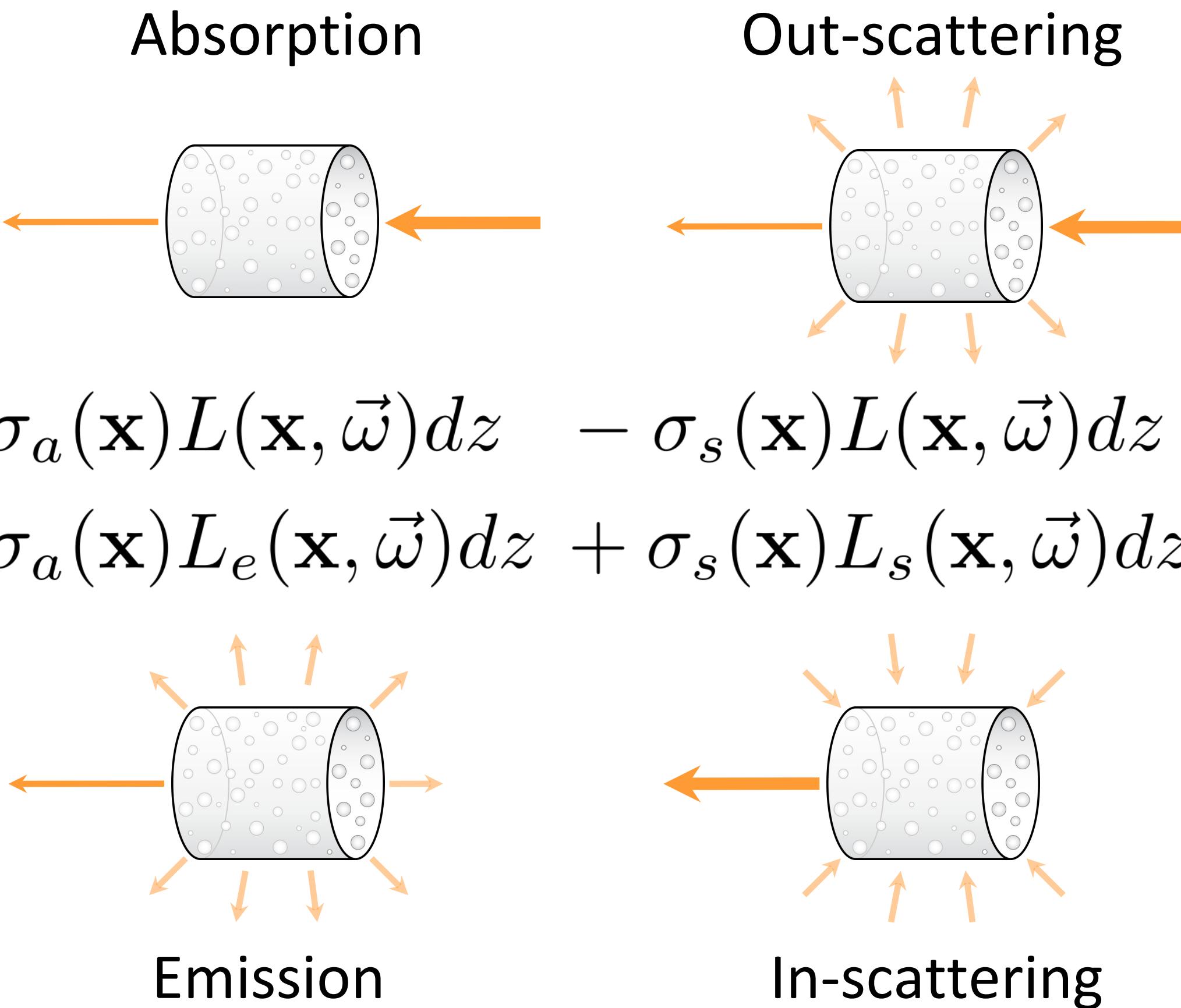
↑
Optical thickness

Transmittance is multiplicative:

$$T_r(\mathbf{x}, \mathbf{z}) = T_r(\mathbf{x}, \mathbf{y})T_r(\mathbf{y}, \mathbf{z})$$



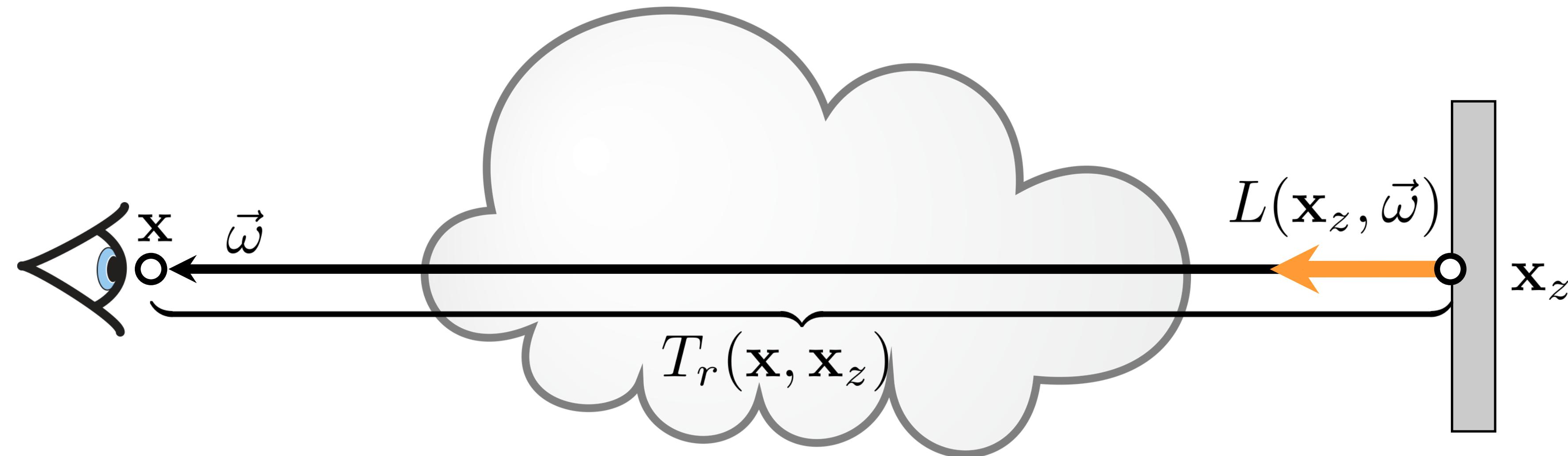
Radiative Transfer Equation (RTE)



Volume Rendering Equation

$$L(\mathbf{x}, \vec{\omega}) = T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega})$$

Reduced (background) surface radiance

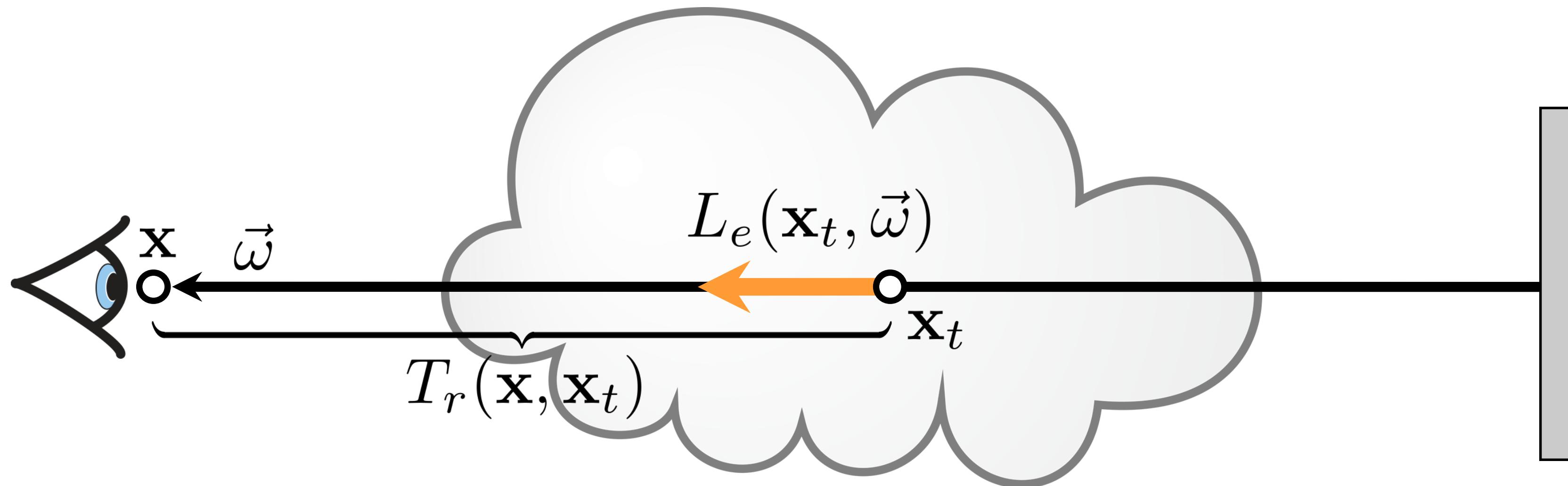


Volume Rendering Equation

$$L(\mathbf{x}, \vec{\omega}) = T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega})$$

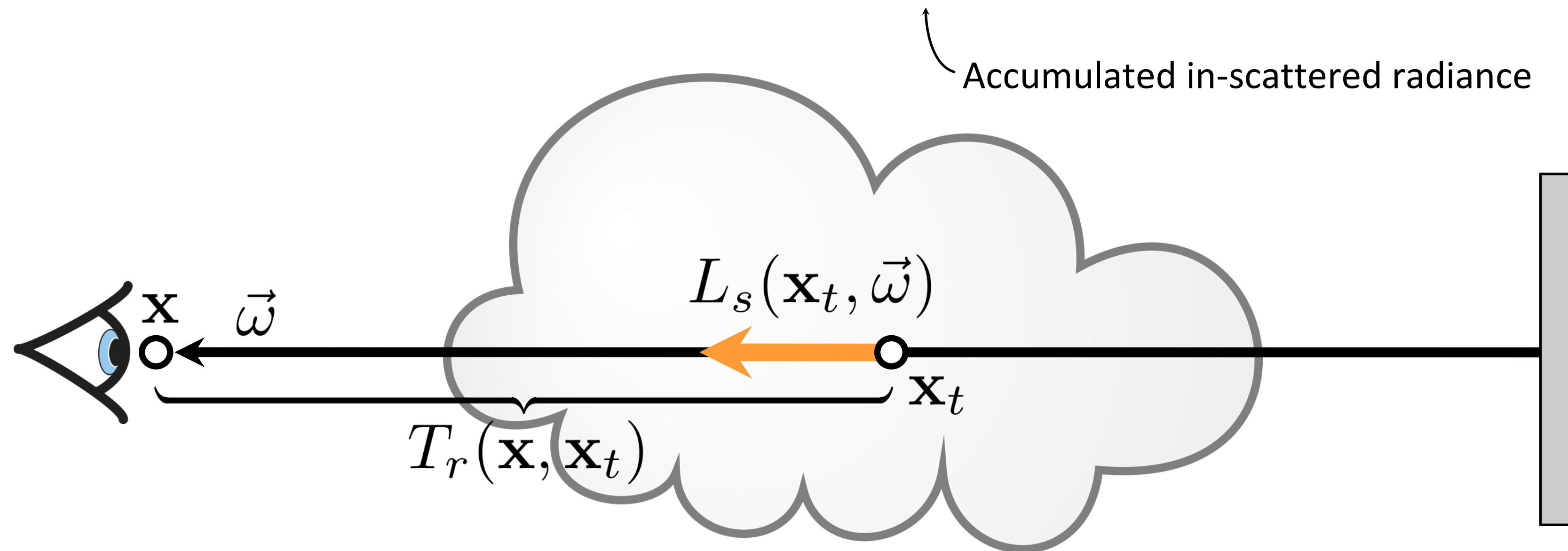
$$+ \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) dt$$

Accumulated emitted radiance



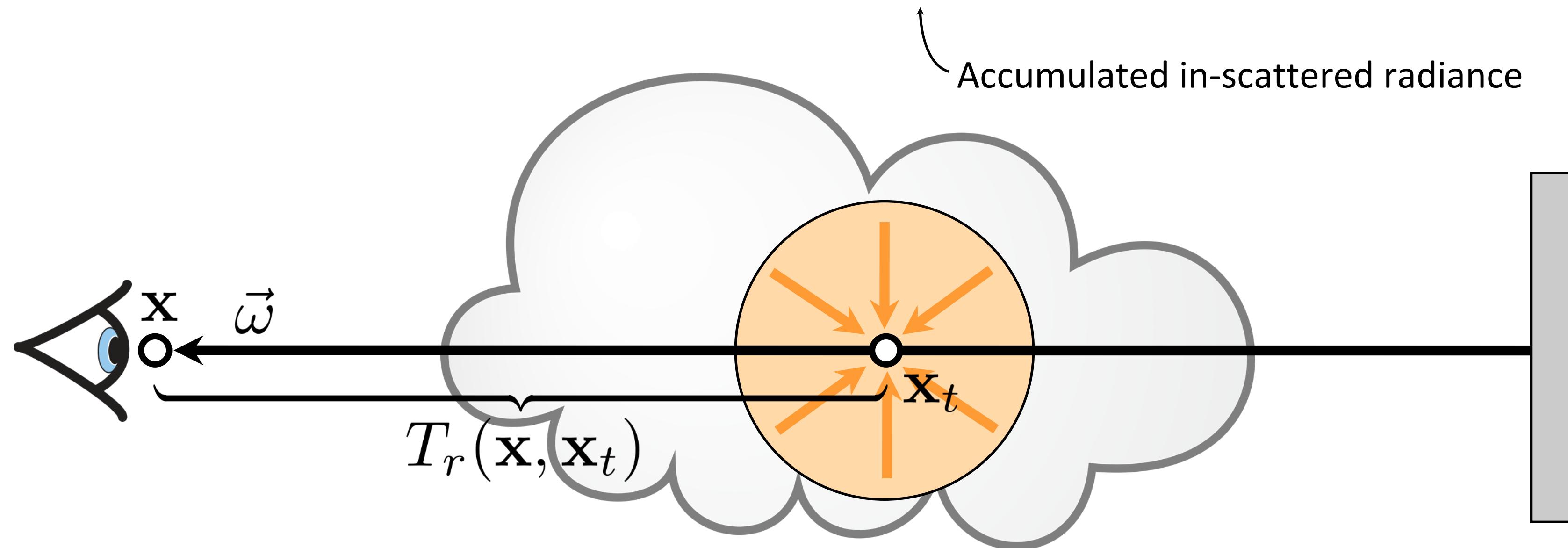
Volume Rendering Equation

$$L(\mathbf{x}, \vec{\omega}) = T_r(\mathbf{x}, \mathbf{x}_z)L(\mathbf{x}_z, \vec{\omega}) + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t)\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t, \vec{\omega})dt + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t)\sigma_s(\mathbf{x}_t)L_s(\mathbf{x}_t, \vec{\omega})dt$$



Volume Rendering Equation

$$L(\mathbf{x}, \vec{\omega}) = T_r(\mathbf{x}, \mathbf{x}_z)L(\mathbf{x}_z, \vec{\omega}) + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t)\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t, \vec{\omega})dt + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t)\sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega})L_i(\mathbf{x}_t, \vec{\omega}')d\vec{\omega}'dt$$



Volume Rendering Equation

$$\begin{aligned} L(\mathbf{x}, \vec{\omega}) = & T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega}) \\ & + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) dt \\ & + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) L_i(\mathbf{x}_t, \vec{\omega}') d\vec{\omega}' dt \end{aligned}$$

Scattering in Media

Phase Function f_p

Describes distribution of scattered light

Analog of BRDF but for scattering in media

Integrates to unity (unlike BRDF)

$$\int_{S^2} f_p(\mathbf{x}, \vec{\omega}', \vec{\omega}) d\vec{\omega}' = 1 \quad \text{Why do we have this property?}$$

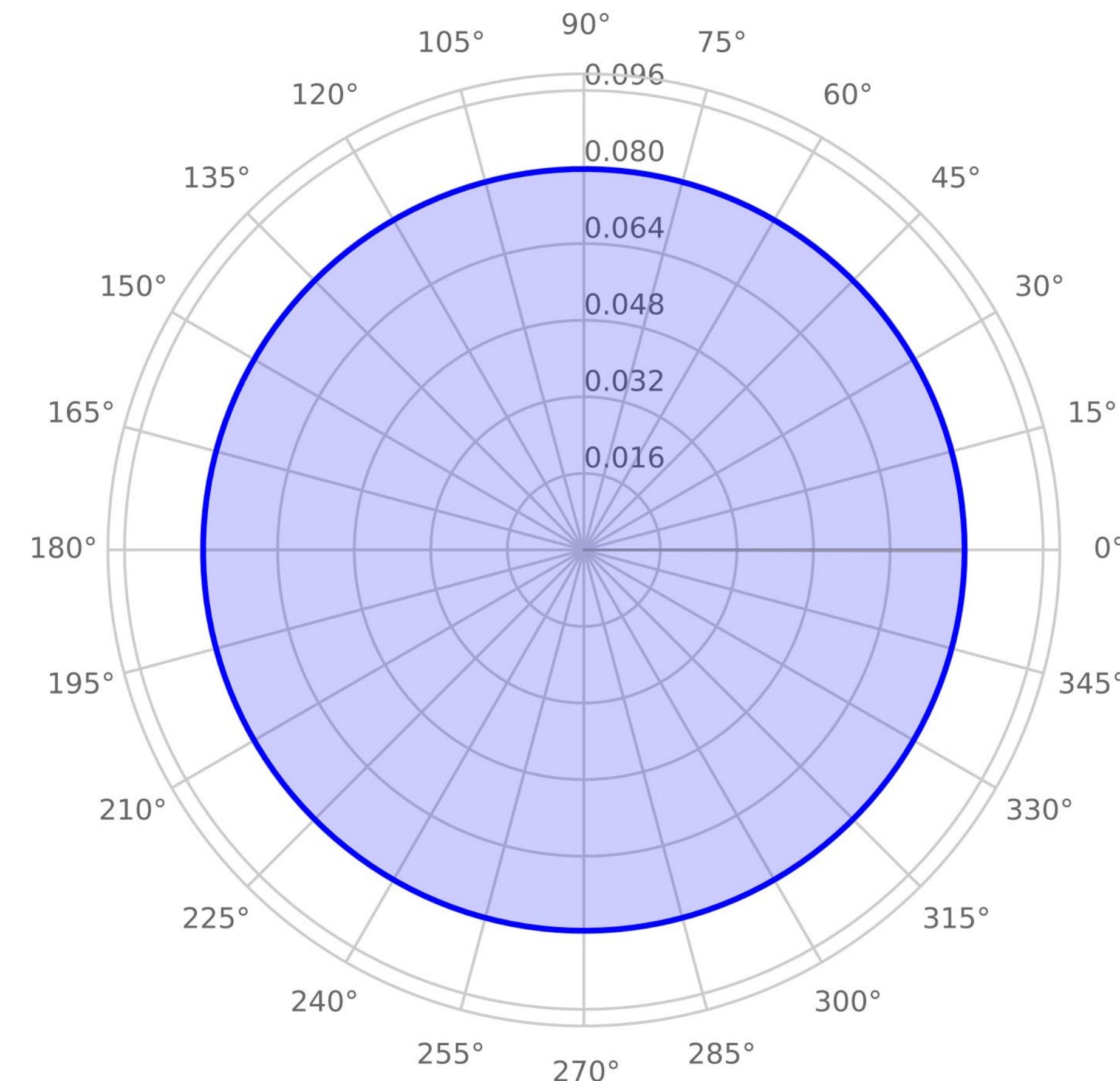
*We will use the same convention that phase function direction vectors always *point away* from the shading point \mathbf{x} . Many publications, however, use a different convention for phase functions, in which direction vectors “follow” the light, i.e. one direction points *towards* \mathbf{x} and the other *away* from \mathbf{x} . When reading papers, be sure to clarify the meaning of the vectors to avoid misinterpretation.

Isotropic Scattering

Uniform scattering, analogous to Lambertian BRDF

$$f_p(\vec{\omega}', \vec{\omega}) = \frac{1}{4\pi}$$

Where does this value come from?



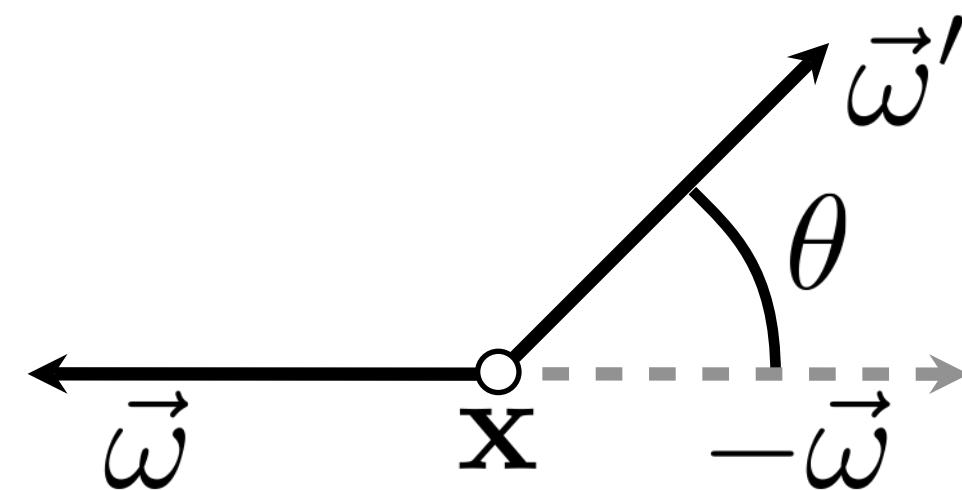
Anisotropic Scattering

Quantifying anisotropy (g , “average cosine”):

$$g = \int_{S^2} f_p(\mathbf{x}, \vec{\omega}', \vec{\omega}) \cos \theta d\vec{\omega}'$$

where:

$$\cos \theta = -\vec{\omega} \cdot \vec{\omega}'$$



$g = 0$: isotropic scattering (on average)

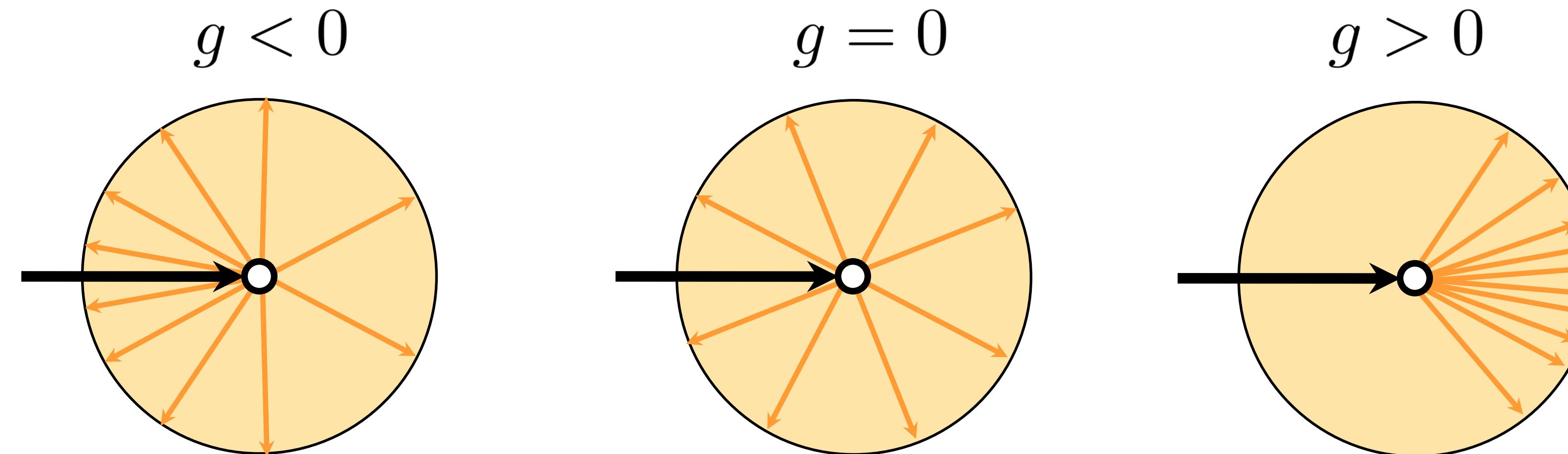
$g > 0$: forward scattering

$g < 0$: backward scattering

Henyey-Greenstein Phase Function

Anisotropic scattering

$$f_{p\text{HG}}(\theta) = \frac{1}{4\pi} \frac{1 - g^2}{(1 + g^2 - 2g \cos \theta)^{3/2}}$$

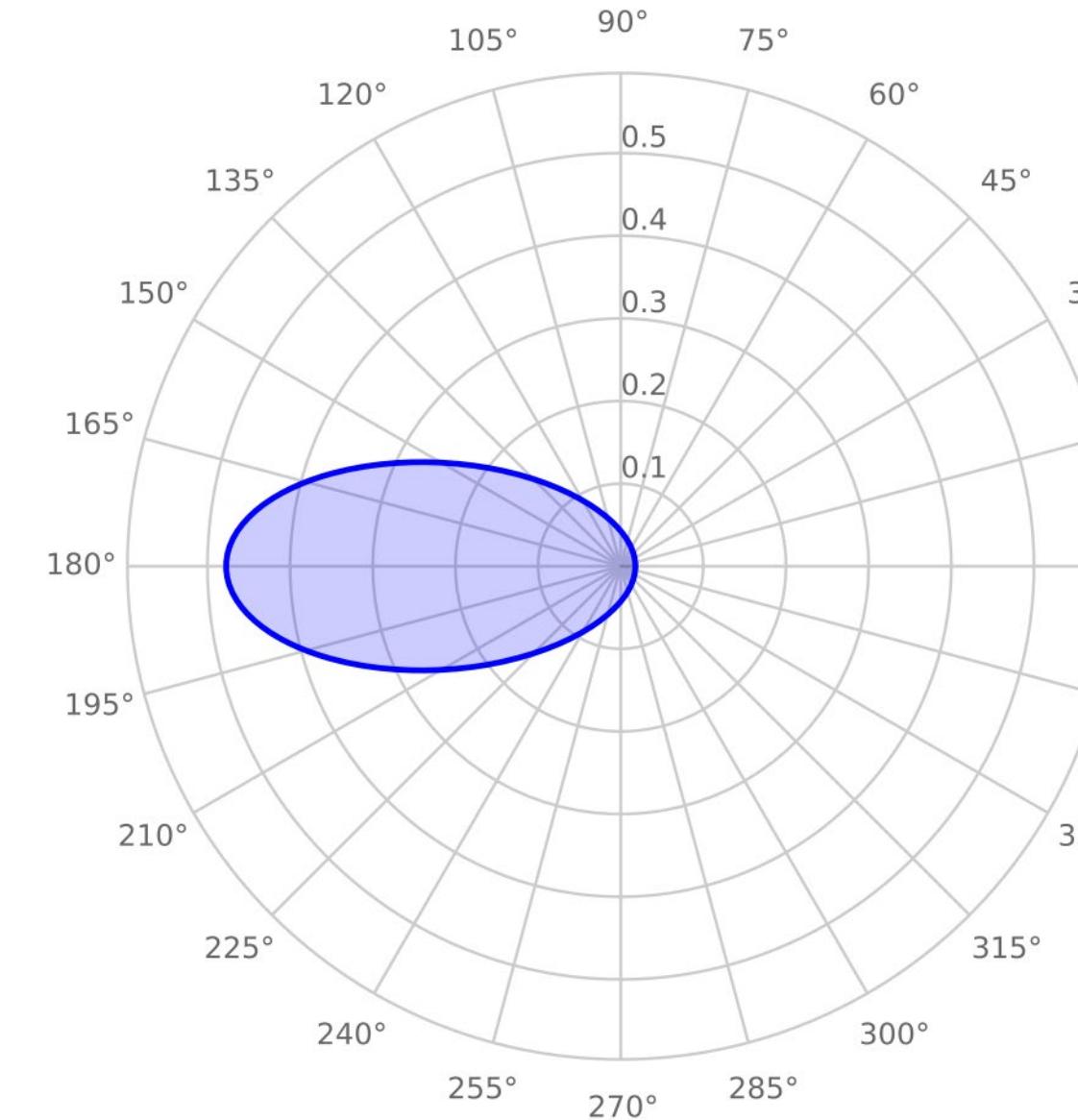


Henyey-Greenstein Phase Function

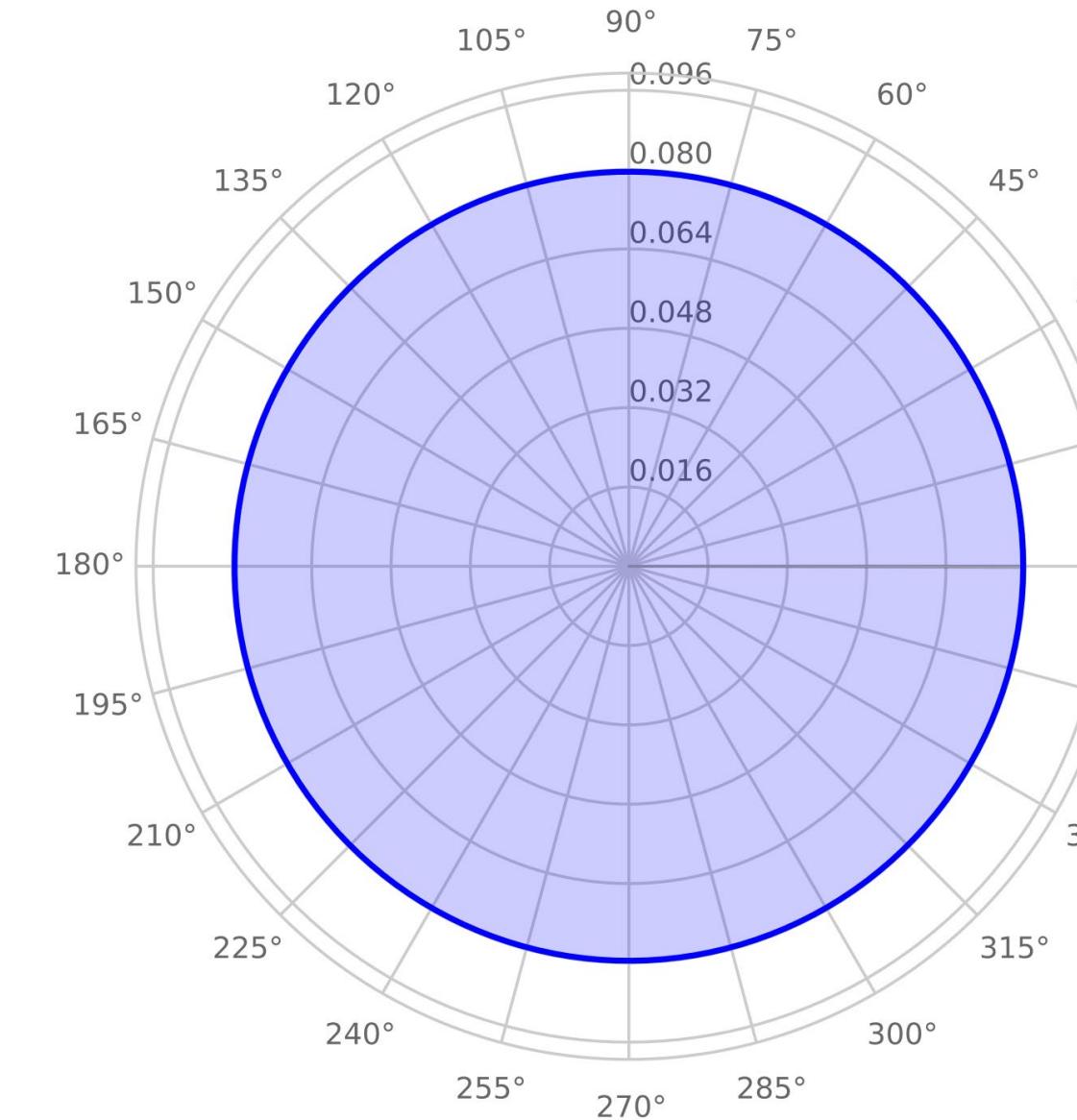
Anisotropic scattering

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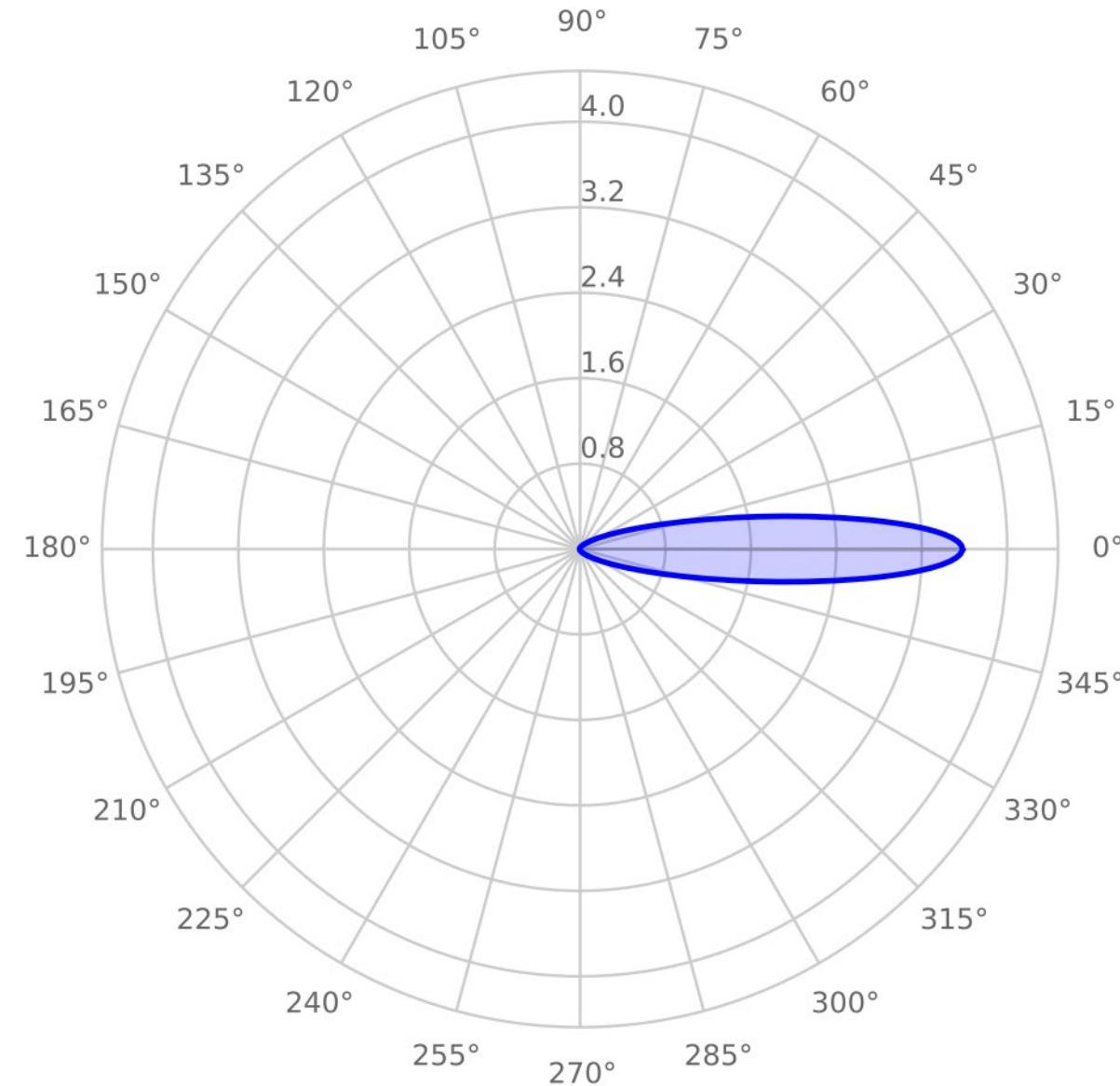
$$g = -0.5$$



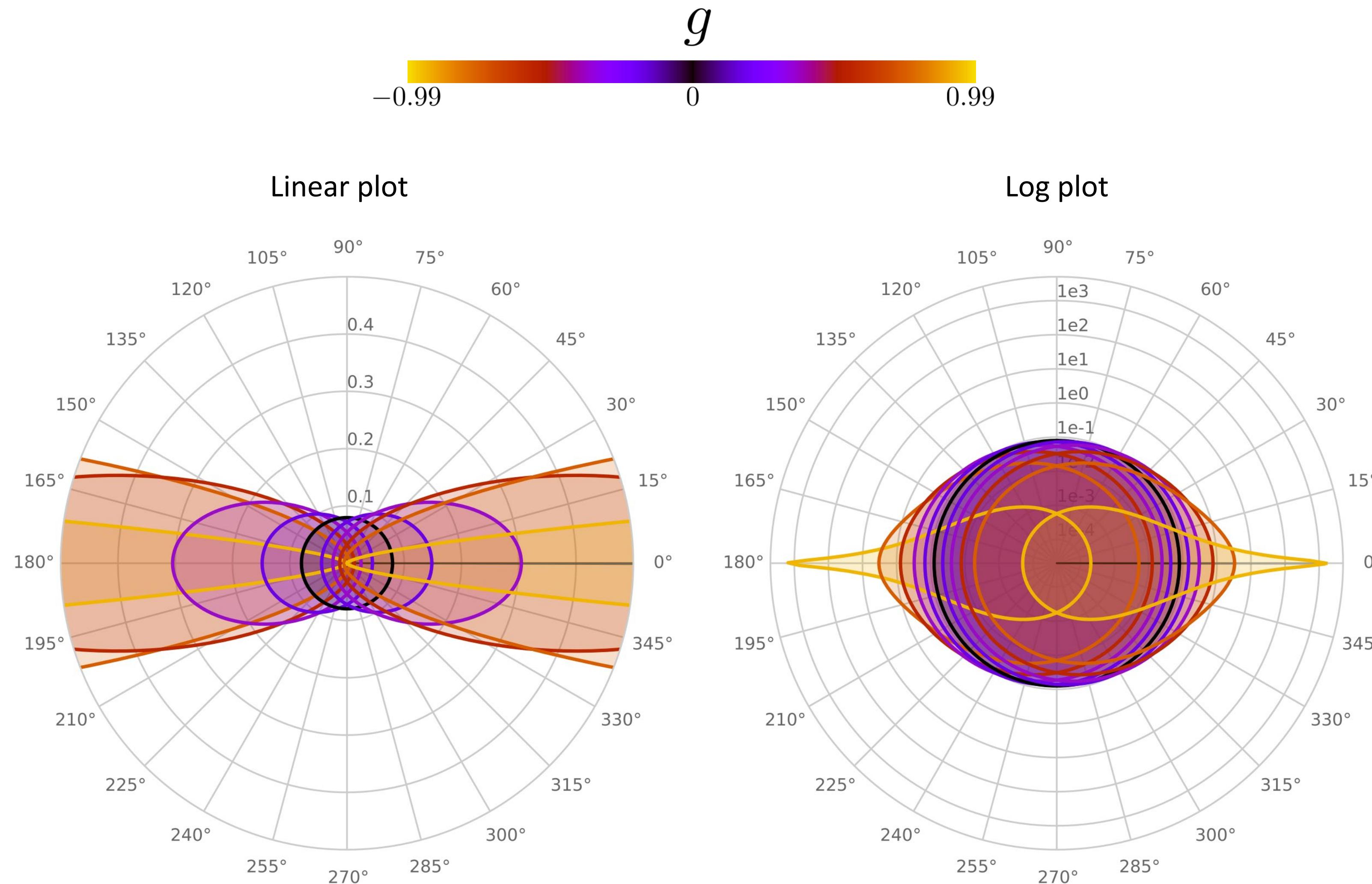
$$g = 0$$



$$g = 0.8$$



Henyey-Greenstein Phase Function



Henyey-Greenstein Phase Function

Empirical phase function

Introduced for intergalactic dust

Very popular in graphics and other fields

Schlick's Phase Function

Empirical phase function

Faster approximation of HG

$$f_{p\text{Schlick}}(\theta) = \frac{1}{4\pi} \frac{1 - k^2}{(1 - k \cos \theta)^2}$$

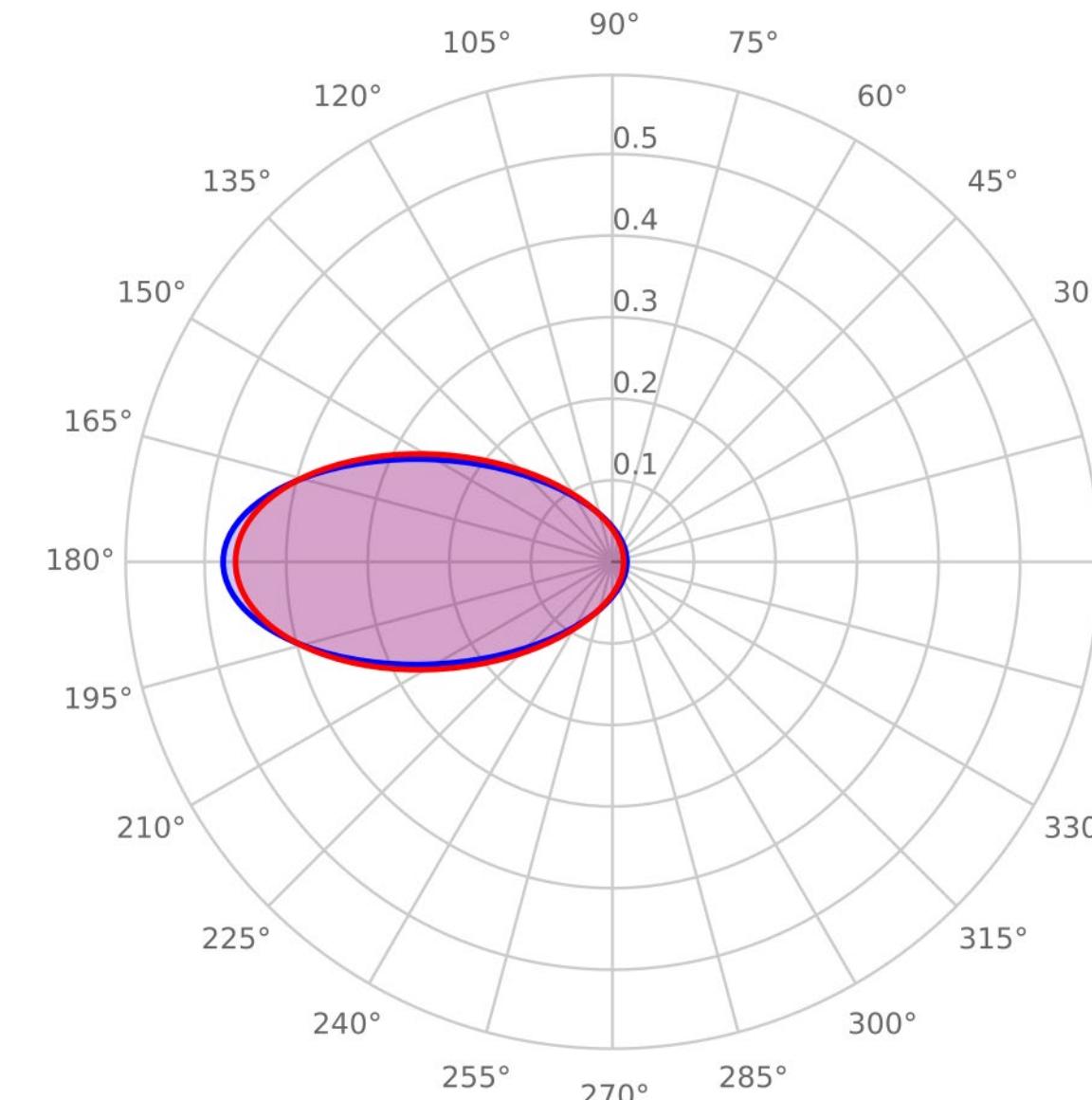
$$k = 1.55g - 0.55g^3$$

Schlick's Phase Function

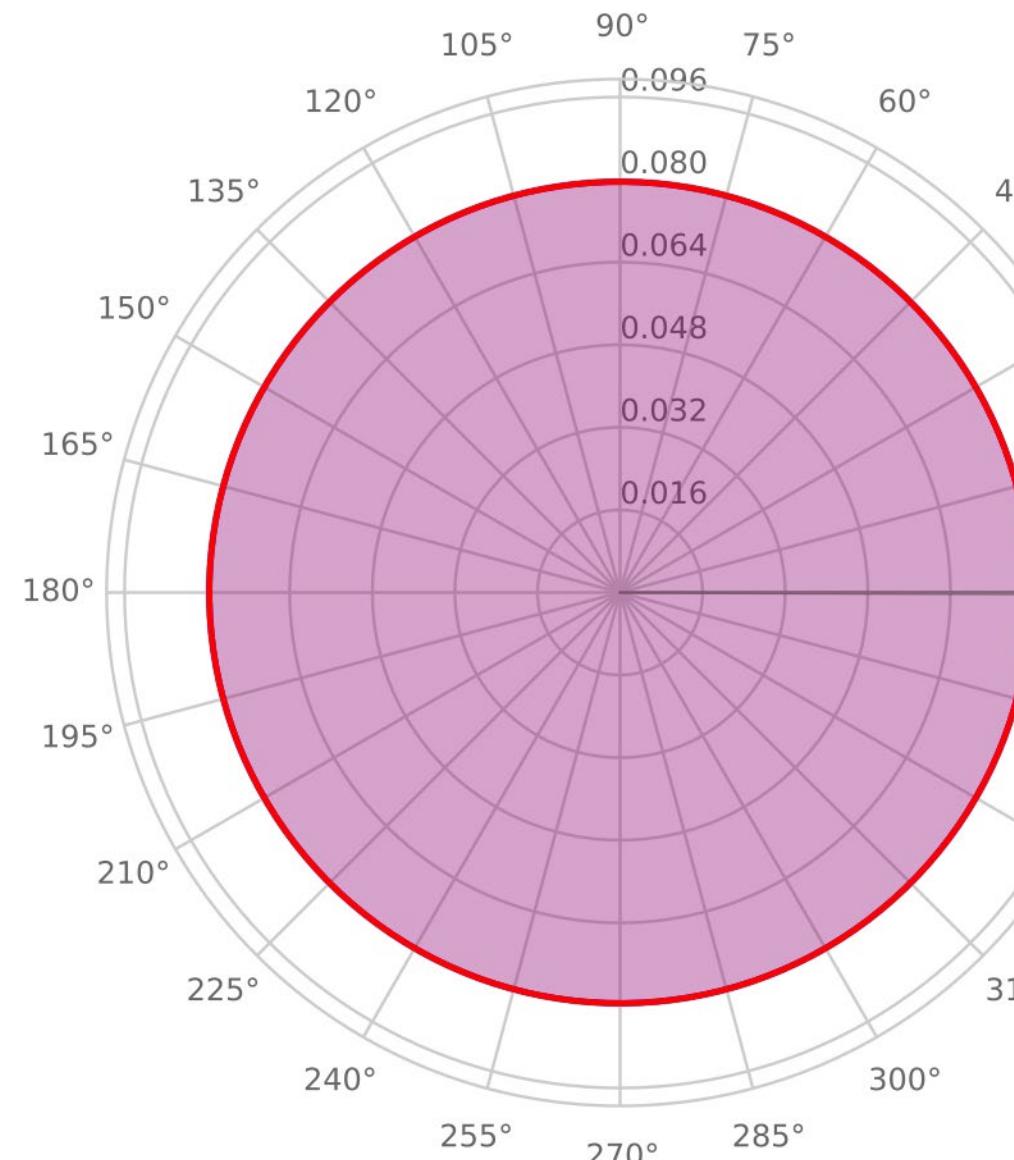
Empirical phase function

Faster approximation of HG

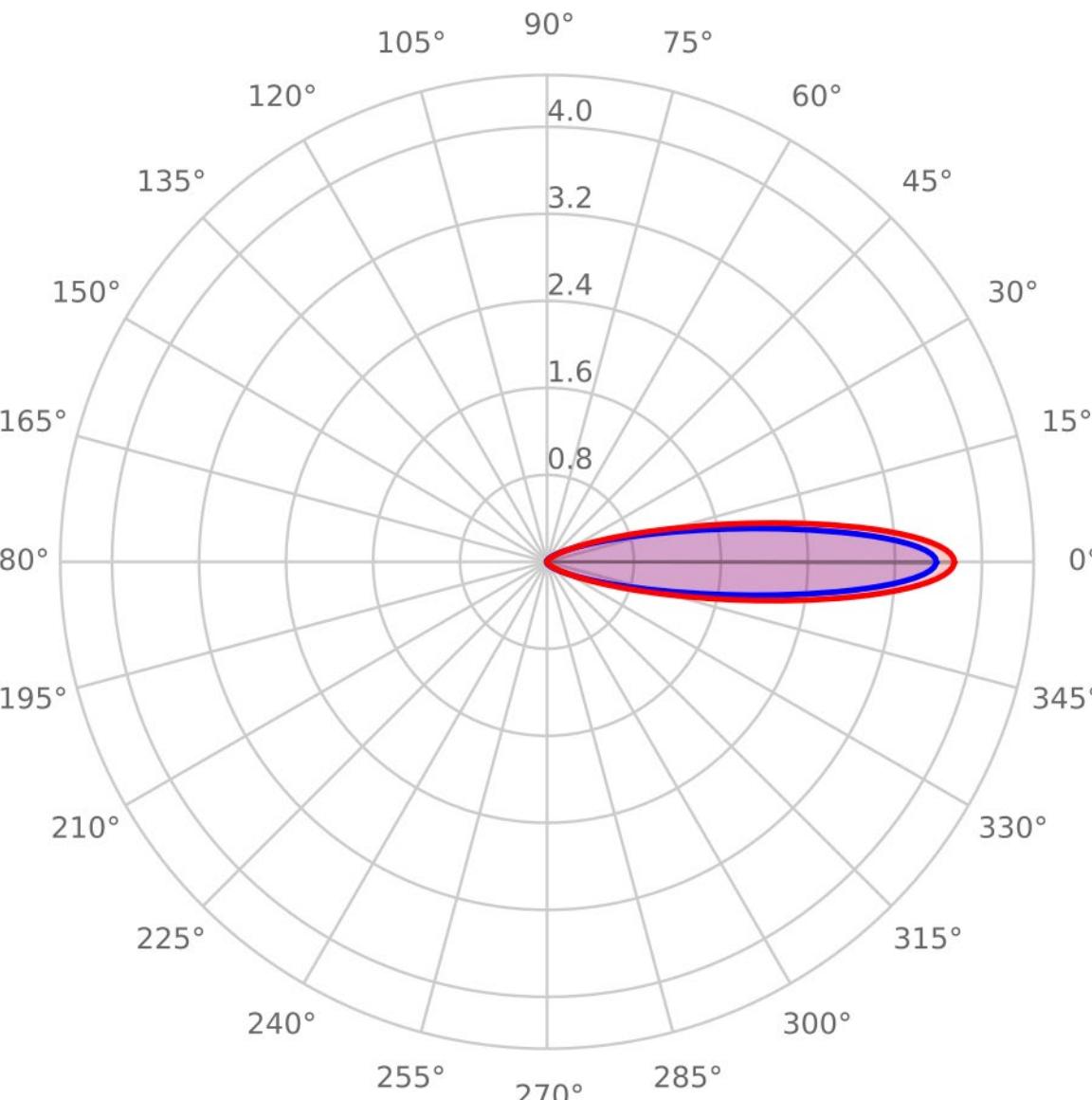
$$g = -0.5 \quad k = -0.706$$



$$g = 0 \quad k = 0$$



$$g = 0.8 \quad k = 0.96$$



— HG
— Schlick

Lorenz-Mie Scattering

If the diameter of scatterers is on the order of light wavelength, we cannot neglect the wave nature of light

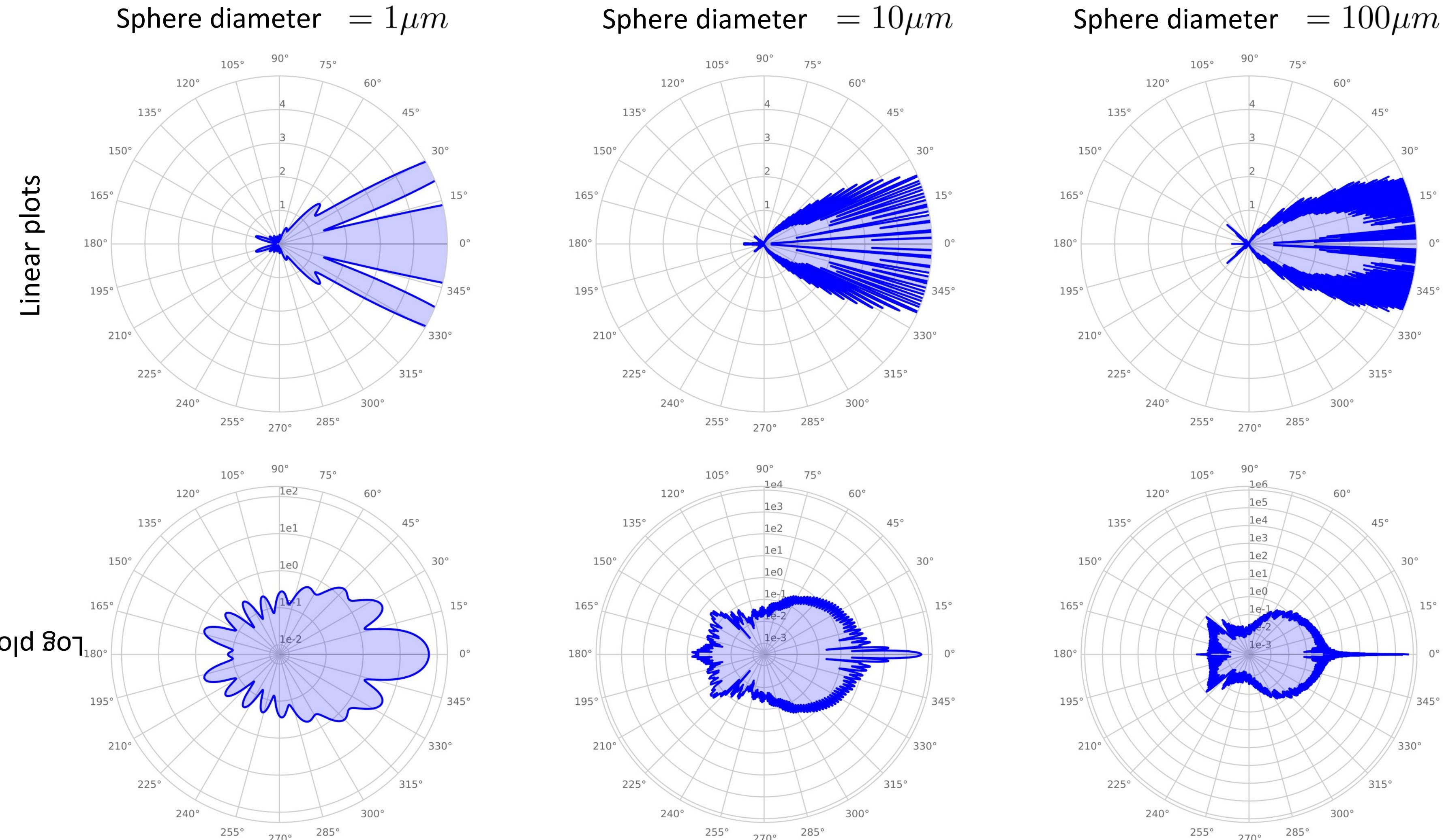
Solution to Maxwell's equations for scattering from any spherical dielectric particle

Explains many phenomena

Complicated:

- Solution is an infinite analytic series

Lorenz-Mie Phase Function



Data obtained from <http://www.philiplaven.com/mieplot.htm>

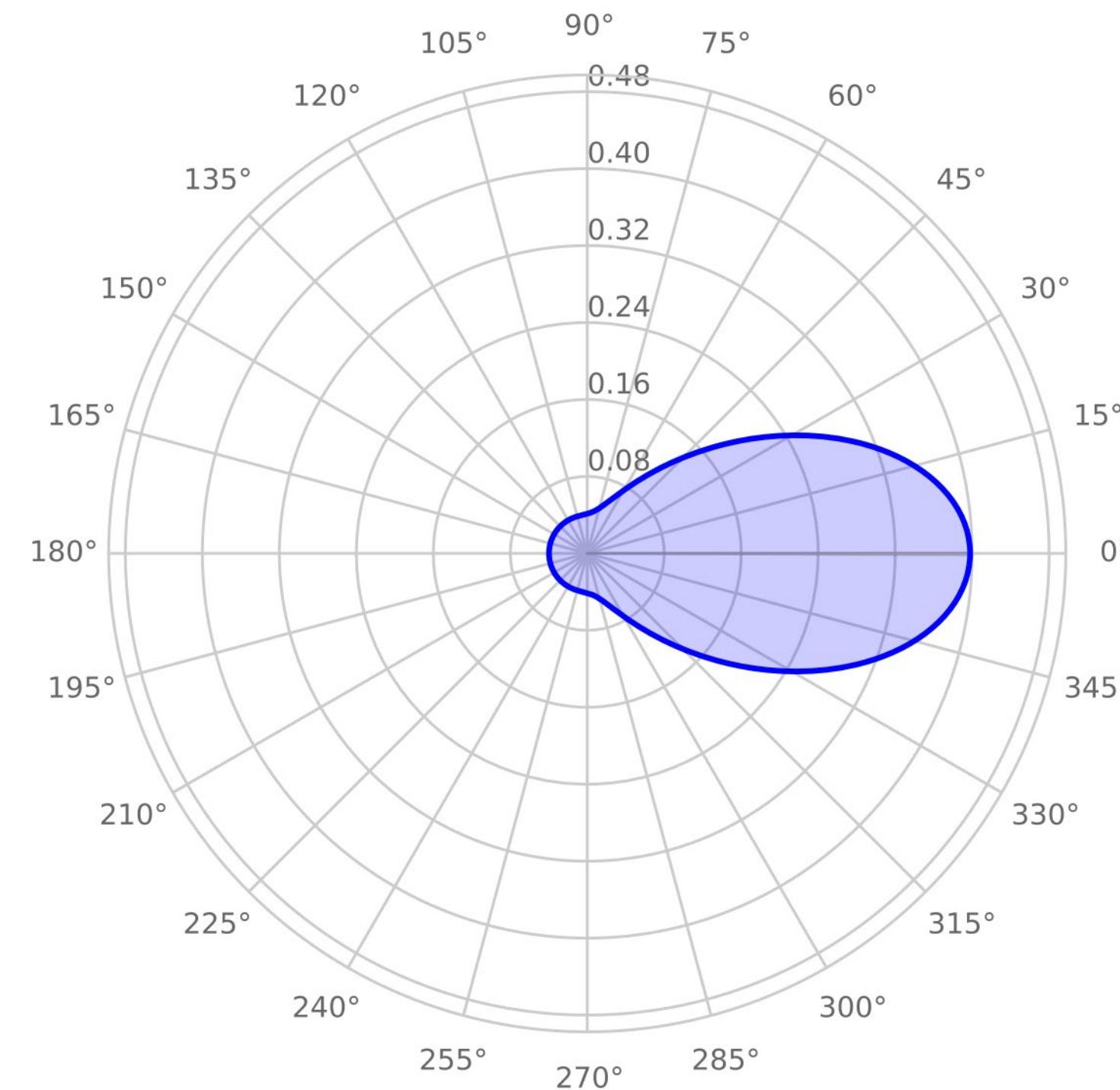
Rainbows



Lorenz-Mie Approximations

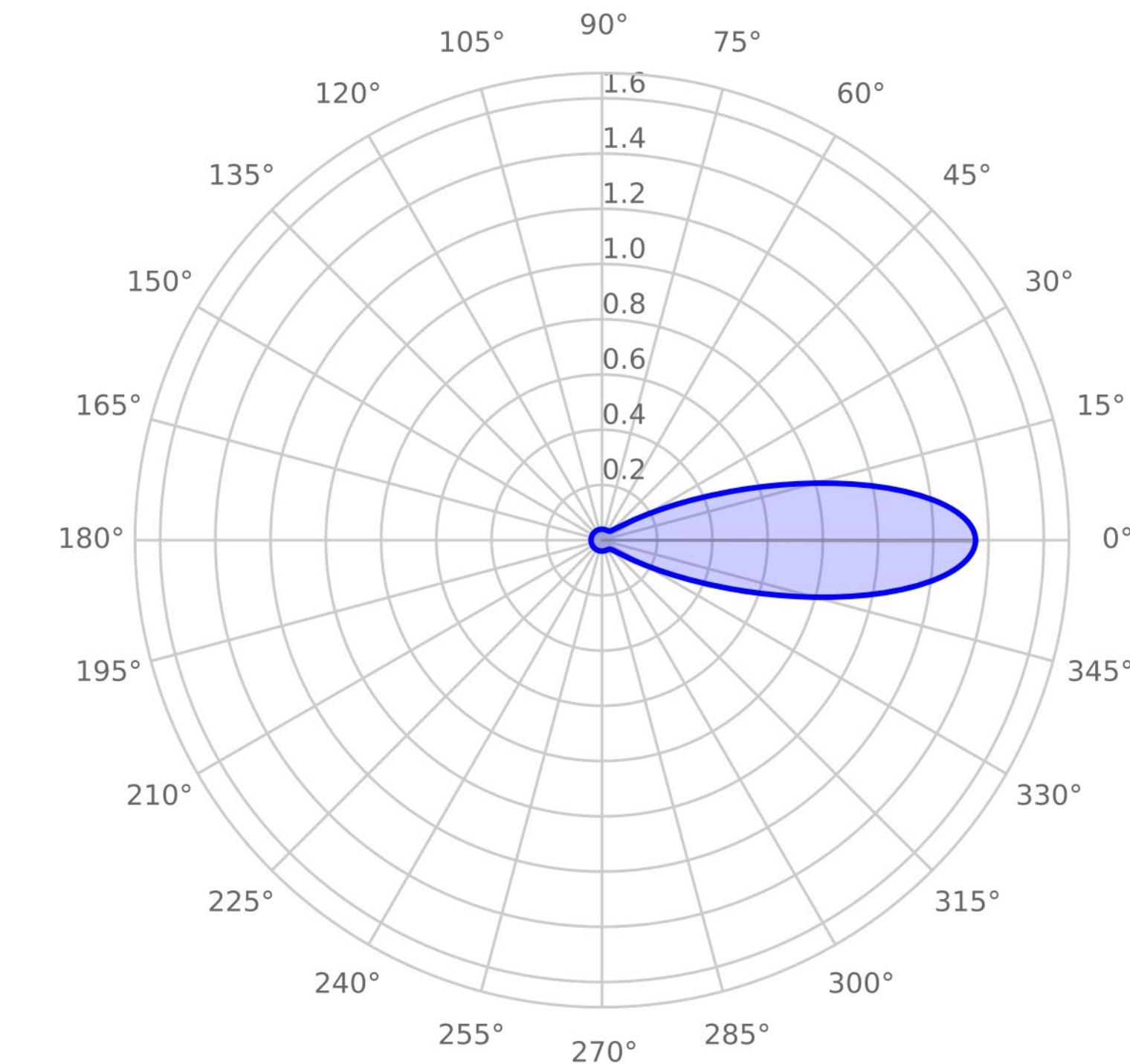
Hazy atmosphere

$$f_{p\text{ hazy}}(\theta) = \frac{1}{4\pi} \left(5 + \left(\frac{1 + \cos \theta}{2} \right)^8 \right)$$



Murky atmosphere

$$f_{p\text{ murky}}(\theta) = \frac{1}{4\pi} \left(17 + \left(\frac{1 + \cos \theta}{2} \right)^{32} \right)$$



Lorenz-Mie Approximations

Hazy atmosphere

$$f_{p \text{ hazy}}(\theta) = \frac{1}{4\pi} \left(5 + \left(\frac{1 + \cos \theta}{2} \right)^8 \right)$$



Murky atmosphere

$$f_{p \text{ murky}}(\theta) = \frac{1}{4\pi} \left(17 + \left(\frac{1 + \cos \theta}{2} \right)^{32} \right)$$



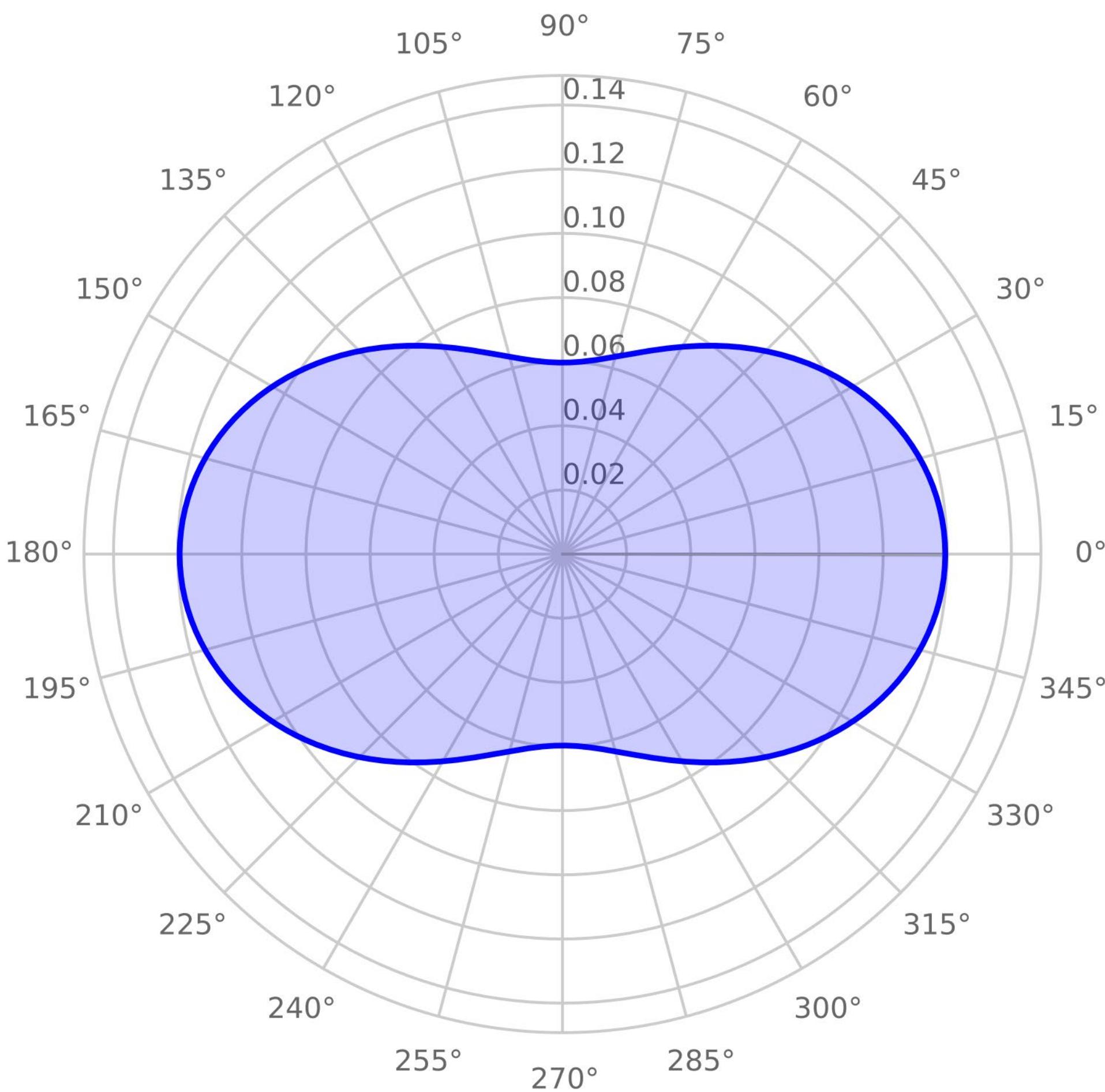
Rayleigh Scattering

Approximation of Lorenz-Mie for tiny scatterers that are typically smaller than 1/10th the wavelength of visible light

Used for atmospheric scattering, gasses, transparent solids

Highly wavelength dependent

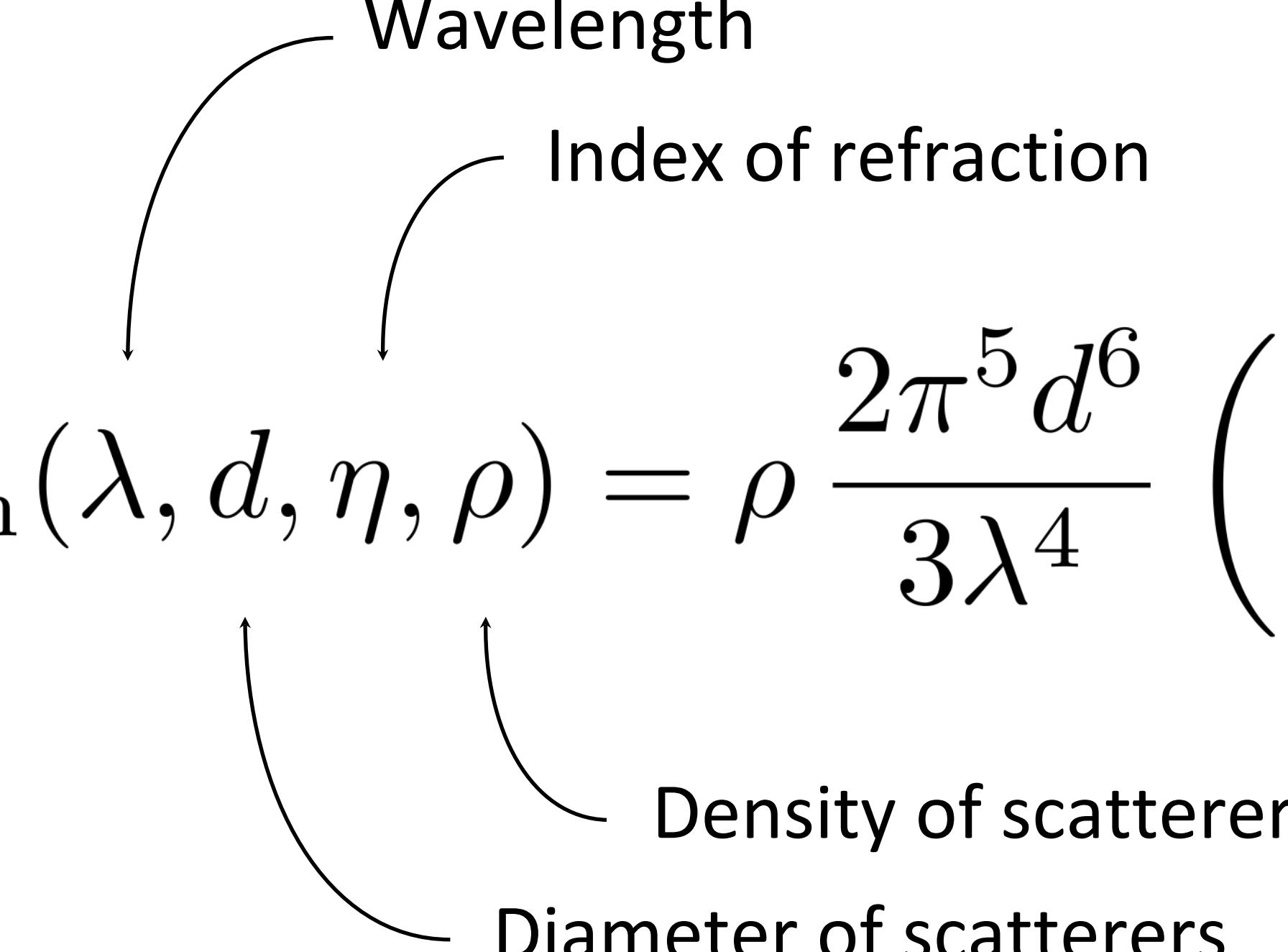
Rayleigh Phase Function



$$f_{p\text{Rayleigh}}(\theta) = \frac{3}{16\pi} (1 + \cos^2 \theta)$$

Scattering at right angles is half as likely as scattering forward or backward

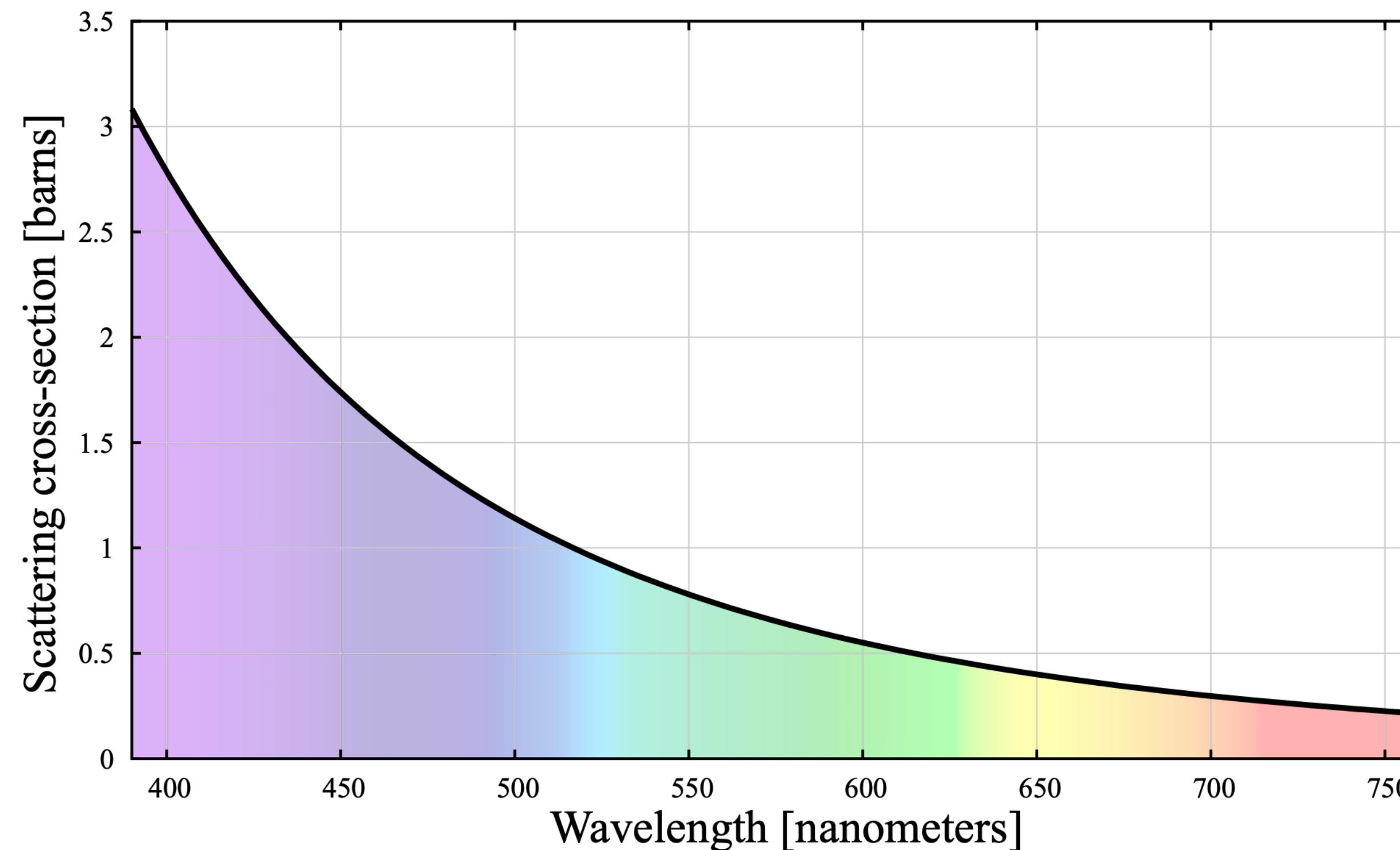
Rayleigh Scattering

$$\sigma_{s\text{Rayleigh}}(\lambda, d, \eta, \rho) = \rho \frac{2\pi^5 d^6}{3\lambda^4} \left(\frac{\eta^2 - 1}{\eta^2 + 2} \right)^2$$


The diagram illustrates the components of the Rayleigh scattering formula. It features a central equation for the scattering cross-section $\sigma_{s\text{Rayleigh}}$ as a function of wavelength λ , index of refraction η , density ρ , and diameter d . Four curly arrows originate from labels placed above and to the right of the equation and point to the respective variables: 'Wavelength' points to λ , 'Index of refraction' points to η , 'Density of scatterers' points to ρ , and 'Diameter of scatterers' points to d .

Rayleigh Scattering

$$\sigma_{s\text{Rayleigh}}(\lambda, d, \eta, \rho) = \rho \frac{2\pi^5 d^6}{3\lambda^4} \left(\frac{\eta^2 - 1}{\eta^2 + 2} \right)^2$$



Examples



Dana Stephenson/Getty Images

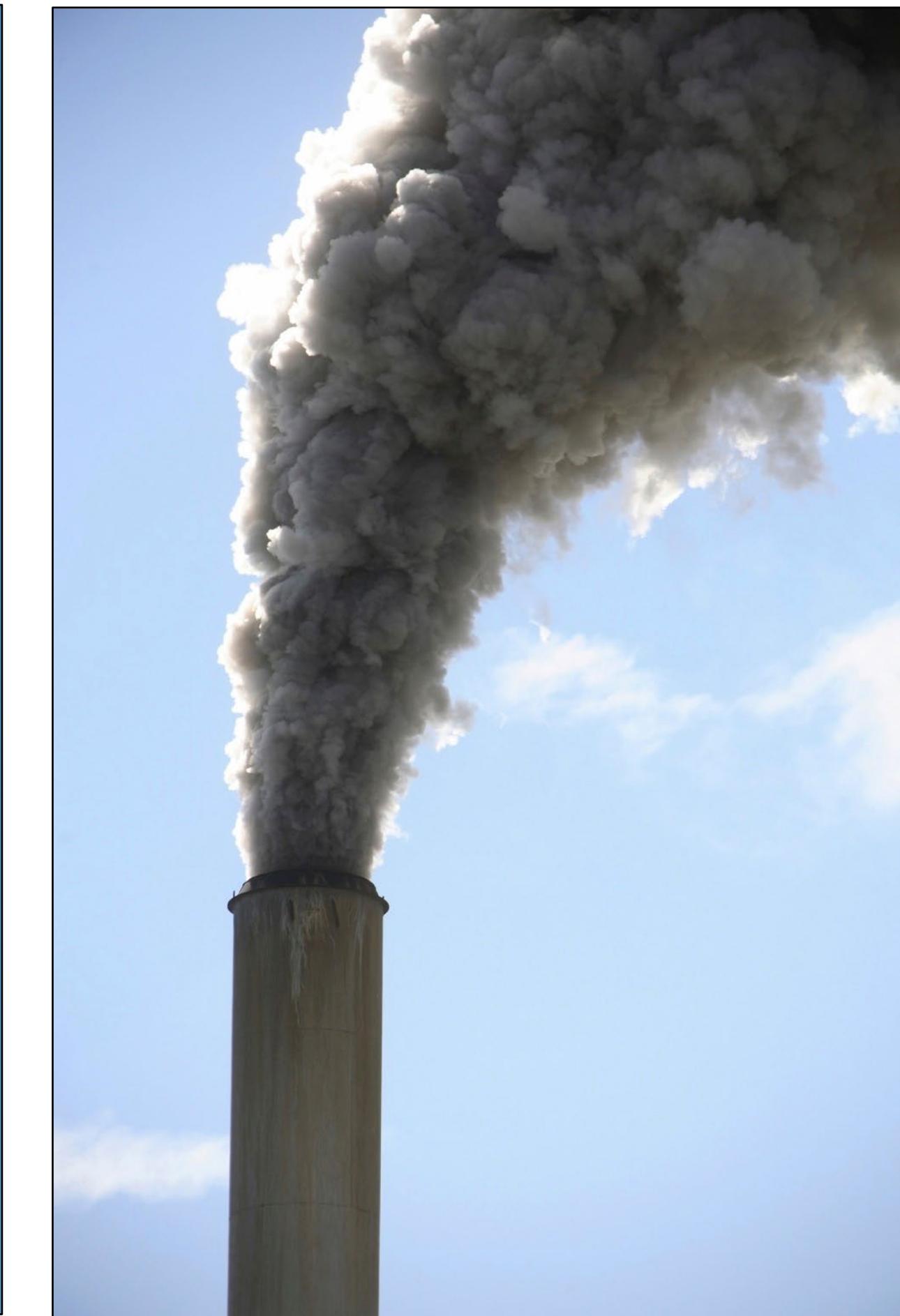
Examples

Steam



Forward scattering

Smoke



Backward scattering

Examples



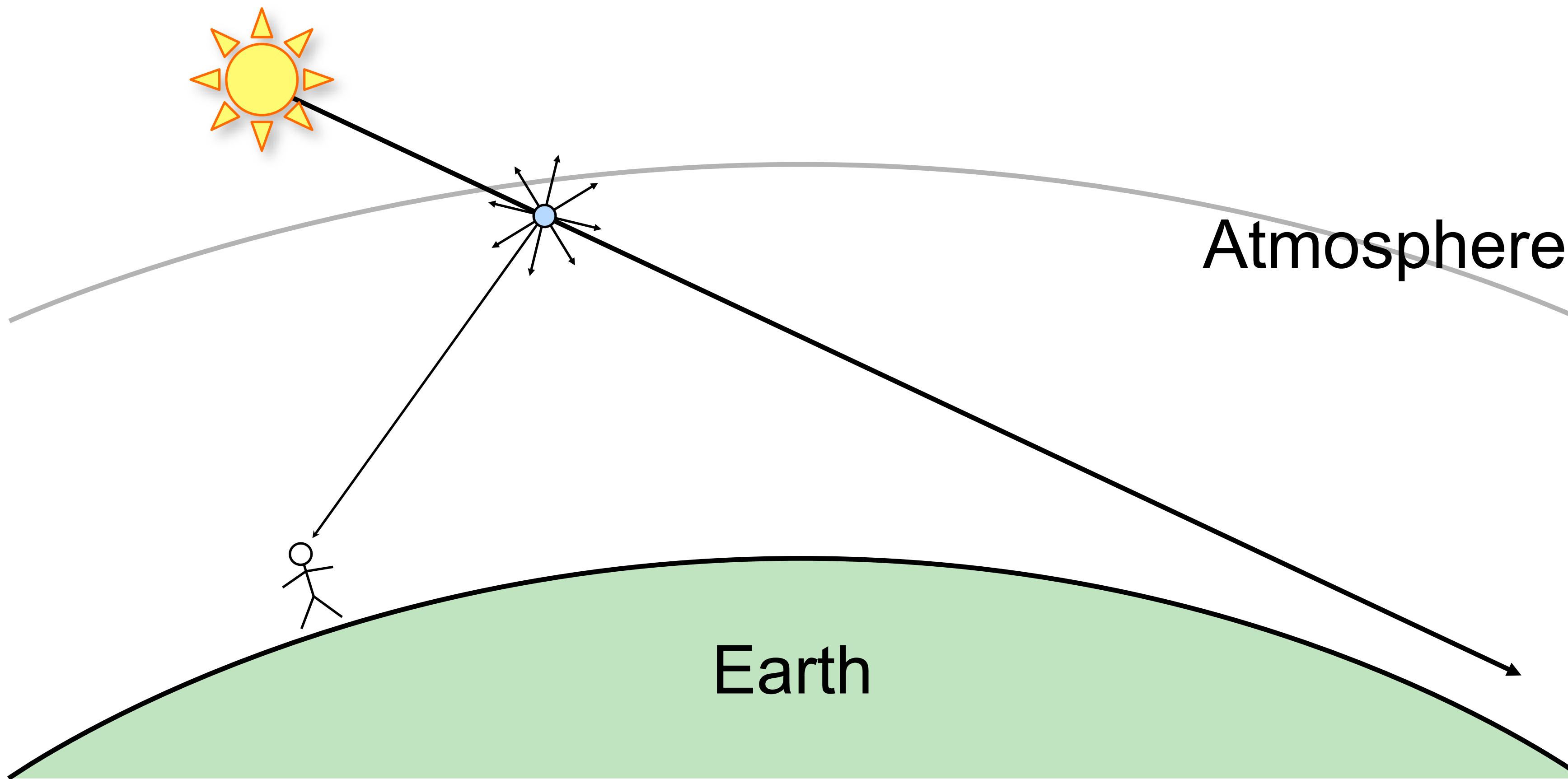
Isotropic scattering

Examples

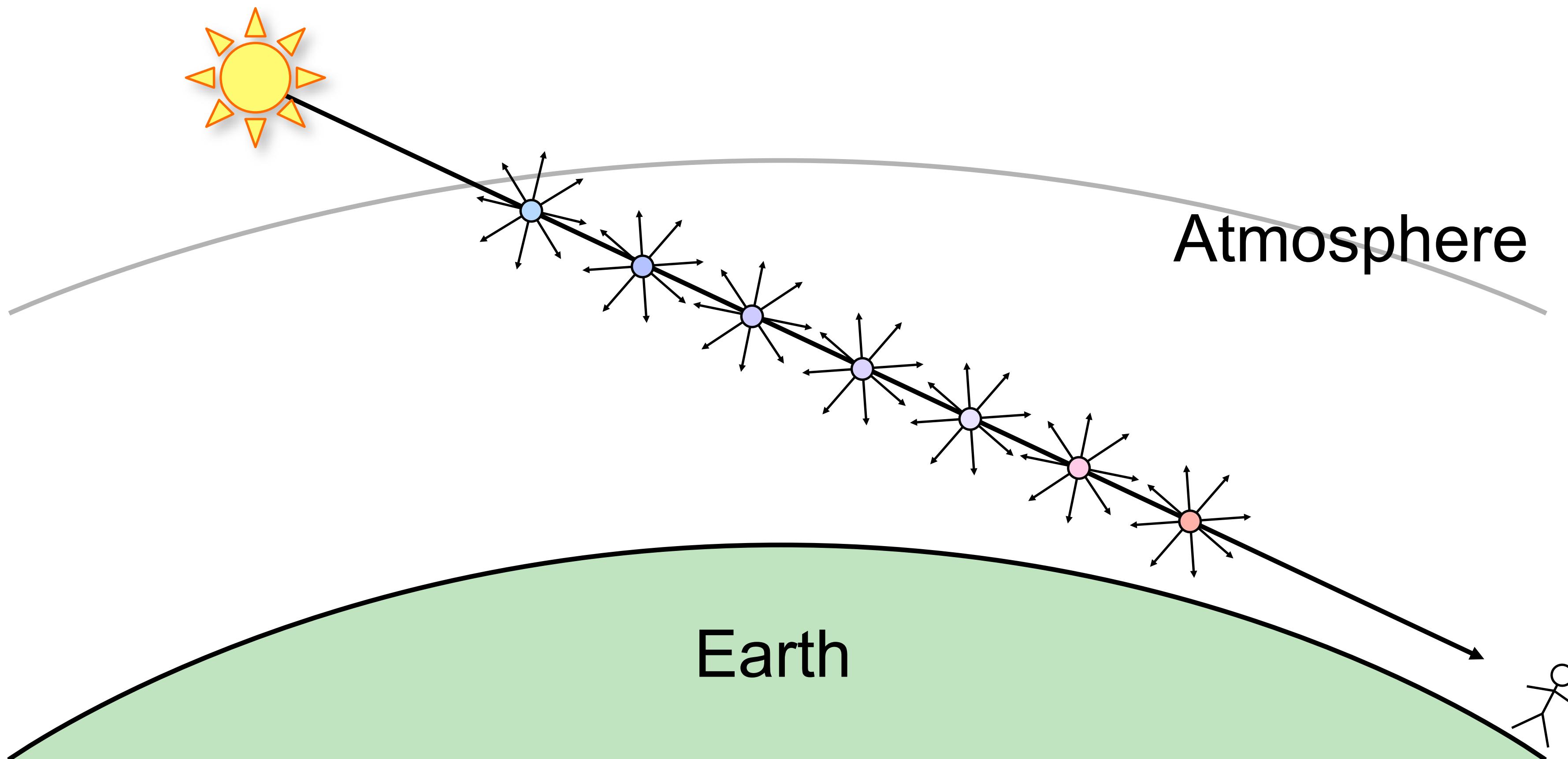


Forward scattering

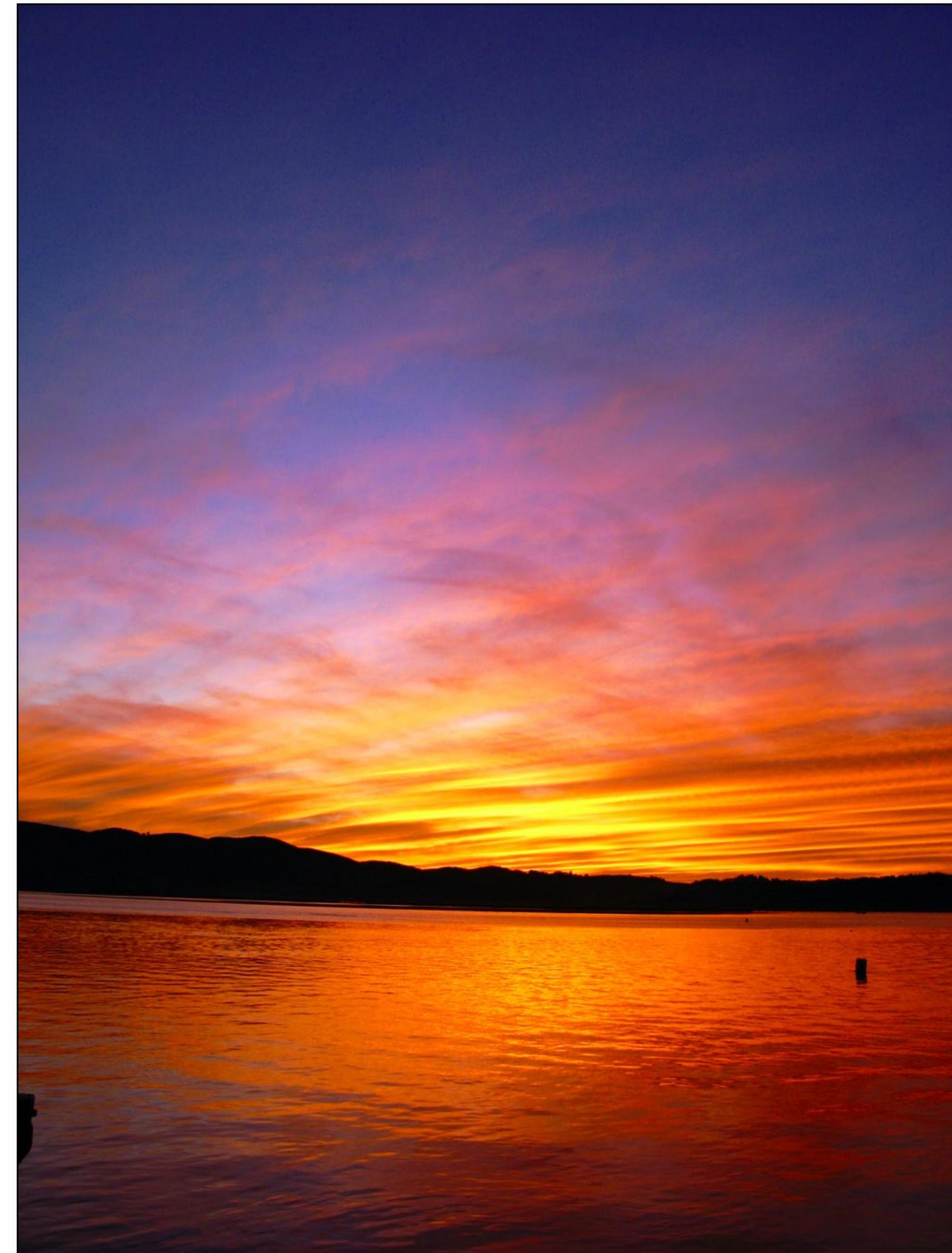
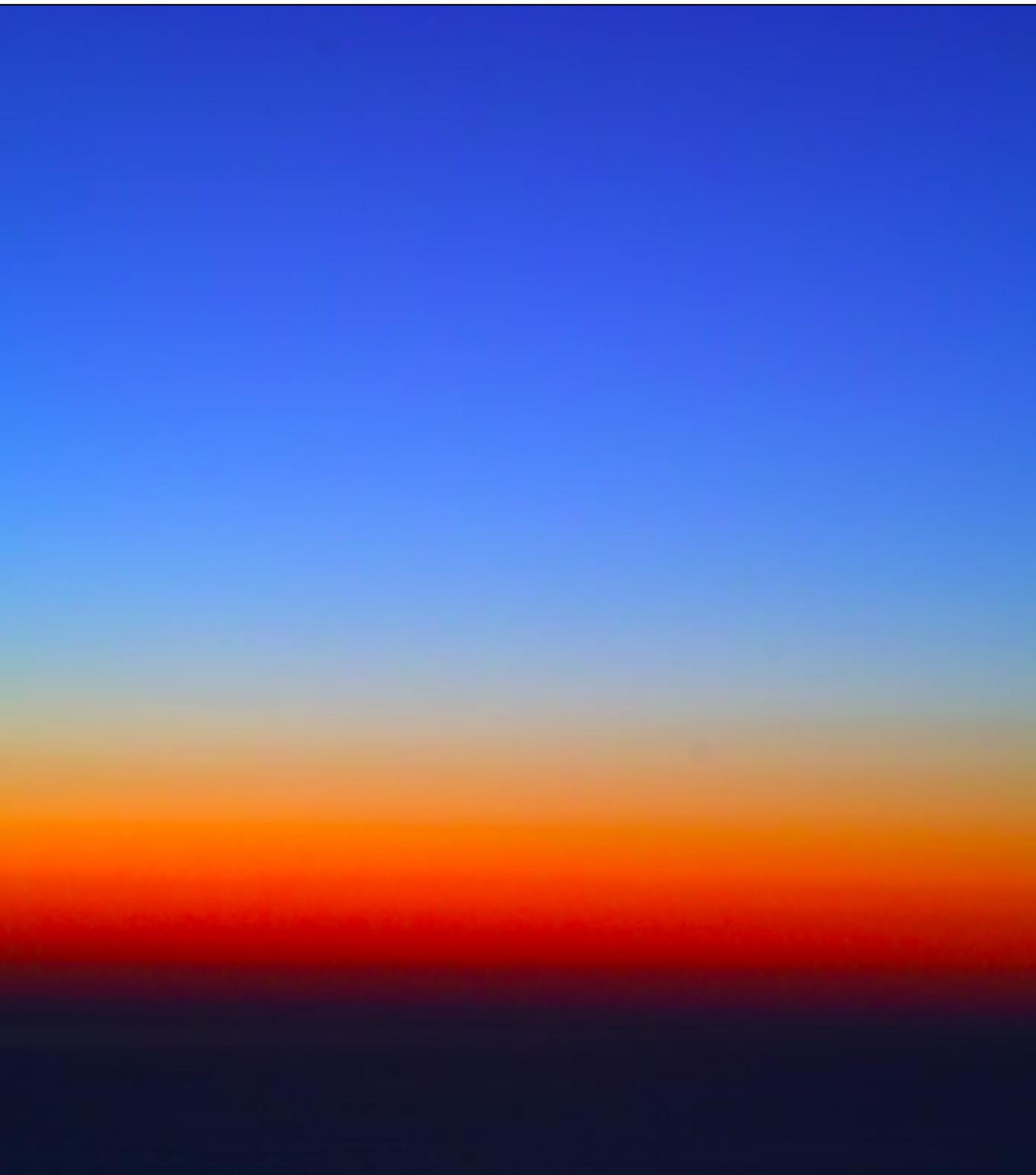
Why is the Sky Blue?



Why is the Sunset Red?



Rayleigh Scattering



Media Properties (Recap)

Given:

- Absorption coefficient	$\sigma_a(\mathbf{x})$	$[\text{m}^{-1}]$
- Scattering coefficient	$\sigma_s(\mathbf{x})$	$[\text{m}^{-1}]$
- Phase function	$f_p(\mathbf{x}, \vec{\omega}', \vec{\omega})$	$[\text{sr}^{-1}]$

Derived:

- Extinction coefficient	$\sigma_t(\mathbf{x}) = \sigma_a(\mathbf{x}) + \sigma_s(\mathbf{x})$	$[\text{m}^{-1}]$
- Albedo	$\alpha(\mathbf{x}) = \sigma_s(\mathbf{x}) / \sigma_t(\mathbf{x})$	[none]
- Mean-free path	$1 / \sigma_t(\mathbf{x})$	[m]
- Transmittance	$T_r(\mathbf{x}, \mathbf{y}) = e^{- \int_0^{\ \mathbf{x} - \mathbf{y}\ } \sigma_t(t) dt}$	[none]

Homogeneous Isotropic Medium

Given:

- Absorption coefficient	σ_a	$[\text{m}^{-1}]$
- Scattering coefficient	σ_s	$[\text{m}^{-1}]$
- Phase function	$\frac{1}{4\pi}$	$[\text{sr}^{-1}]$

Derived:

- Extinction coefficient	$\sigma_t = \sigma_a + \sigma_s$	$[\text{m}^{-1}]$
- Albedo	$\alpha = \sigma_s / \sigma_t$	[none]
- Mean-free path	$1 / \sigma_t$	[m]
- Transmittance	$T_r(\mathbf{x}, \mathbf{y}) = e^{-\sigma_t \ \mathbf{x} - \mathbf{y}\ }$	[none]

What is this?



source: [wikipedia](#)

Crepuscular Rays



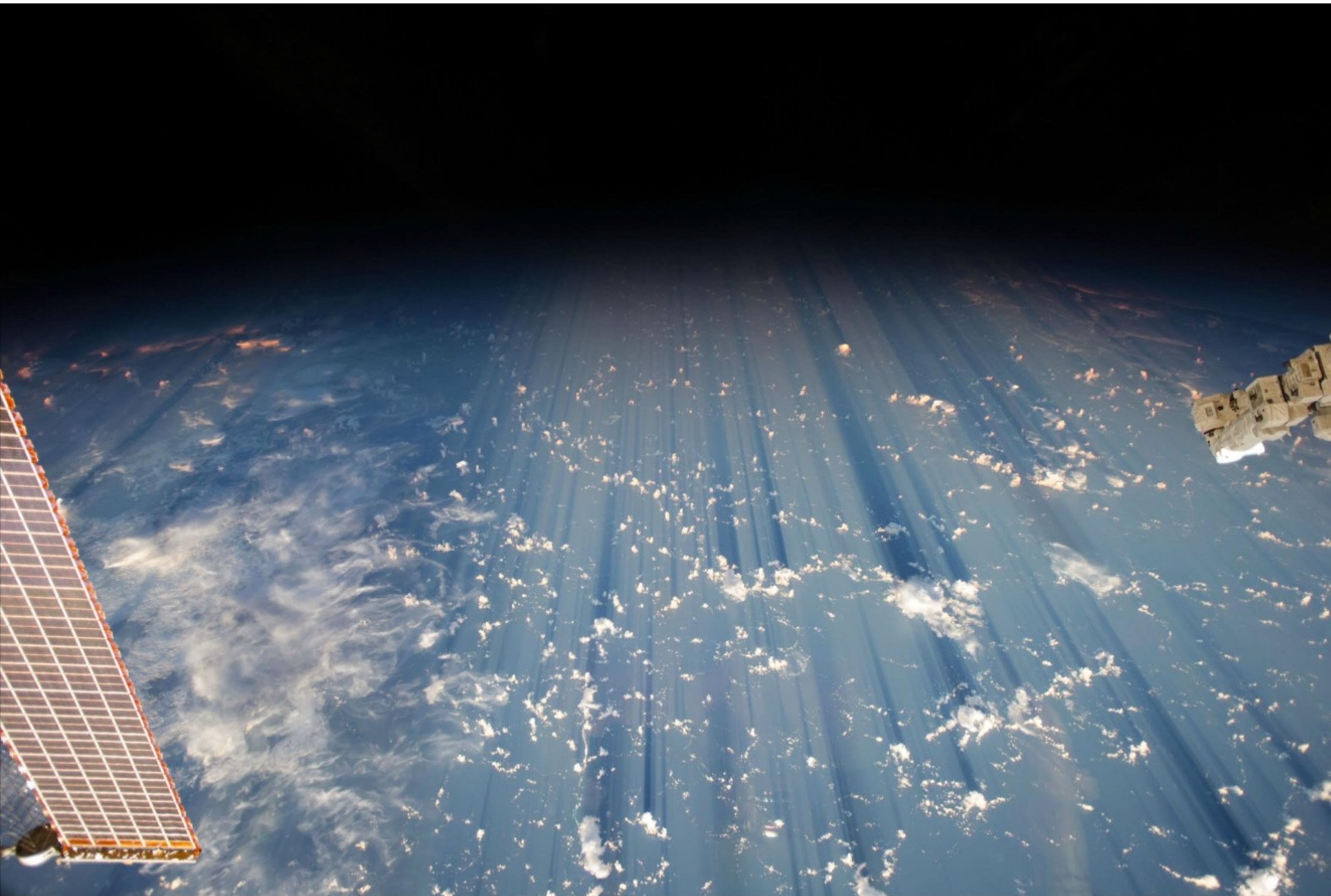
source: [wikipedia](#)

Anti-Crepuscular Rays



source: [wikipedia](#)

Crepuscular rays from space



Solving the Volume Rendering Equation

Complexity Progression

homogeneous vs. heterogeneous

scattering

- none
- fake ambient
- single
- multiple

Volume Rendering Equation

$$L(\mathbf{x}, \vec{\omega}) = T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega}) + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) dt + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) L_s(\mathbf{x}_t, \vec{\omega}) dt$$

Attenuated background radiance

Accumulated emitted radiance

Accumulated in-scattered radiance

Purely absorbing media

$$L(\mathbf{x}, \vec{\omega}) = T_r(\mathbf{x}, \mathbf{x}_z)L(\mathbf{x}_z, \vec{\omega})$$

Attenuated background radiance



Fog



<http://anordinarymom.wordpress.com>

Participating Media



$$L(\mathbf{x}, \vec{\omega}) = \int_0^s T_r(\mathbf{x} \leftrightarrow \mathbf{x}_t) \boxed{\sigma_s(\mathbf{x}_t)} L_i(\mathbf{x}_t, \vec{\omega}) dt + T_r(\mathbf{x} \leftrightarrow \mathbf{x}_s) L(\mathbf{x}_s, \vec{\omega})$$

$$L(\mathbf{x}, \vec{\omega}) = \boxed{\sigma_s} \int_0^s \boxed{T_r(\mathbf{x} \leftrightarrow \mathbf{x}_t)} L_i(\mathbf{x}_t, \vec{\omega}) dt + \boxed{T_r(\mathbf{x} \leftrightarrow \mathbf{x}_s)} L(\mathbf{x}_s, \vec{\omega})$$

$$L(\mathbf{x}, \vec{\omega}) = \sigma_s \int_0^s \boxed{e^{-t\sigma_t}} L_i(\mathbf{x}_t, \vec{\omega}) dt + \boxed{e^{-s\sigma_t}} L(\mathbf{x}_s, \vec{\omega})$$

Fog



$$L(\mathbf{x}, \vec{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t} L_i(\mathbf{x}_t, \vec{\omega}) dt + e^{-s\sigma_t} L(\mathbf{x}_s, \vec{\omega})$$

Homogeneous Ambient Media

Assume in-scattered radiance is an ambient constant:

$$L(\mathbf{x}, \vec{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t} [L_i(\mathbf{x}_t, \vec{\omega})] dt + e^{-s\sigma_t} L(\mathbf{x}_s, \vec{\omega})$$

Homogeneous Ambient Media

Assume in-scattered radiance is an ambient constant:

$$L(\mathbf{x}, \vec{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t} L_i(\mathbf{x}_t, \vec{\omega}) dt + e^{-s\sigma_t} L(\mathbf{x}_s, \vec{\omega})$$

$$L(\mathbf{x}, \vec{\omega}) = \sigma_s L_i \left[\int_0^s e^{-t\sigma_t} dt \right] + e^{-s\sigma_t} L(\mathbf{x}_s, \vec{\omega})$$

$$L(\mathbf{x}, \vec{\omega}) = \sigma_s L_i \left[\frac{1 - e^{-s\sigma_t}}{\sigma_t} \right] + e^{-s\sigma_t} L(\mathbf{x}_s, \vec{\omega})$$

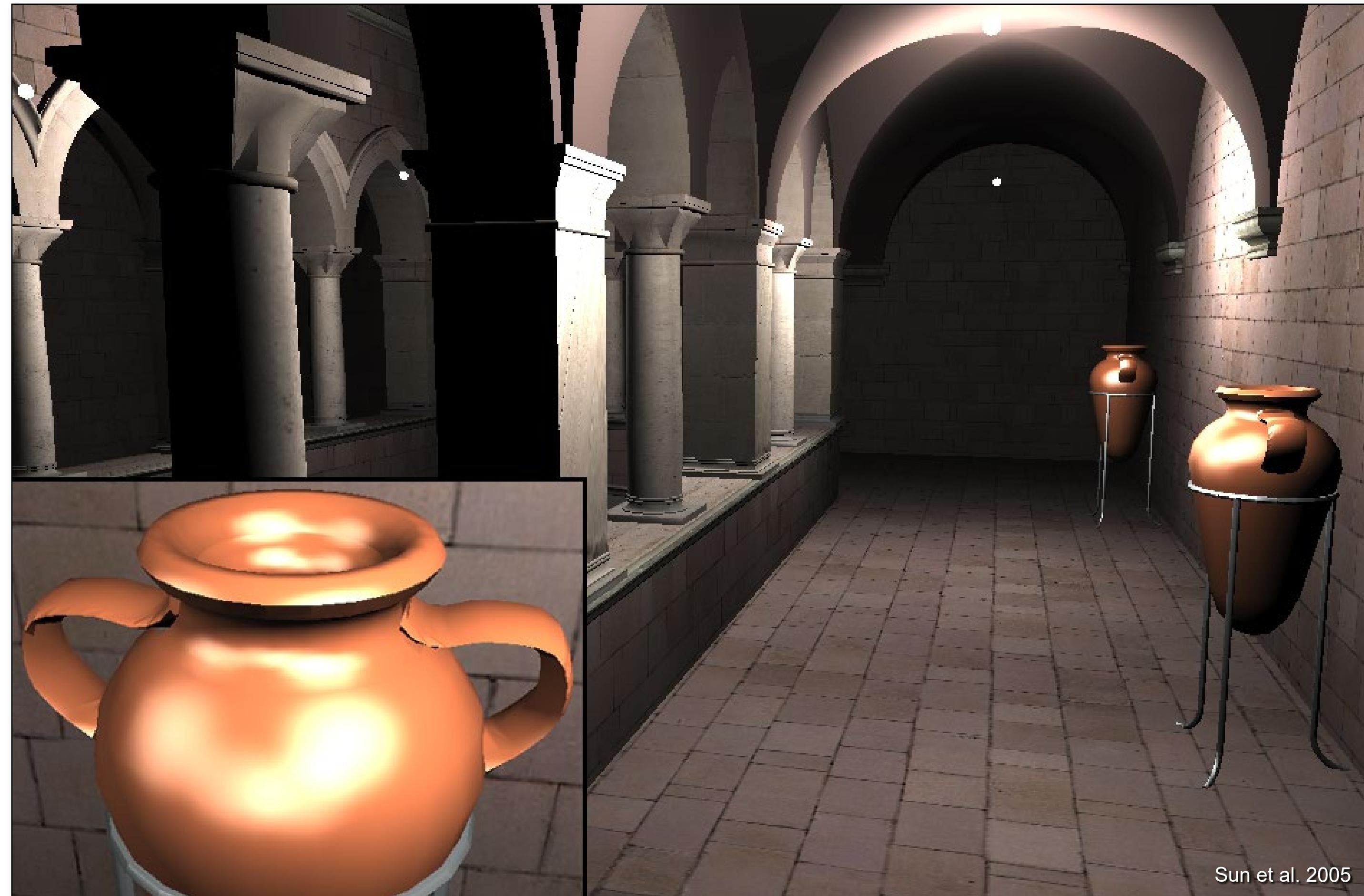
$$L(\mathbf{x}, \vec{\omega}) = \text{lerp} \left(\frac{\sigma_s}{\sigma_t} L_i, L(\mathbf{x}_s, \vec{\omega}), e^{-s\sigma_t} \right)$$

OpenGL Fog



Sun et al. 2005

OpenGL Clear Day



Sun et al. 2005

Fog









Volume Rendering Equation

$$\begin{aligned} L(\mathbf{x}, \vec{\omega}) = & T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega}) \\ & + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) dt \\ & + \boxed{\int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) L_s(\mathbf{x}_t, \vec{\omega}) dt} \end{aligned}$$

↑
Accumulated in-scattered radiance

In-scattered Radiance

$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) [L_s(\mathbf{x}_t, \vec{\omega})] dt$$

$$[L_s(\mathbf{x}_t, \vec{\omega})] = \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) [L_i(\mathbf{x}_t, \vec{\omega}')] d\vec{\omega}'$$

Single scattering

- L_i arrives directly from a light source (direct illum.)
i.e.:

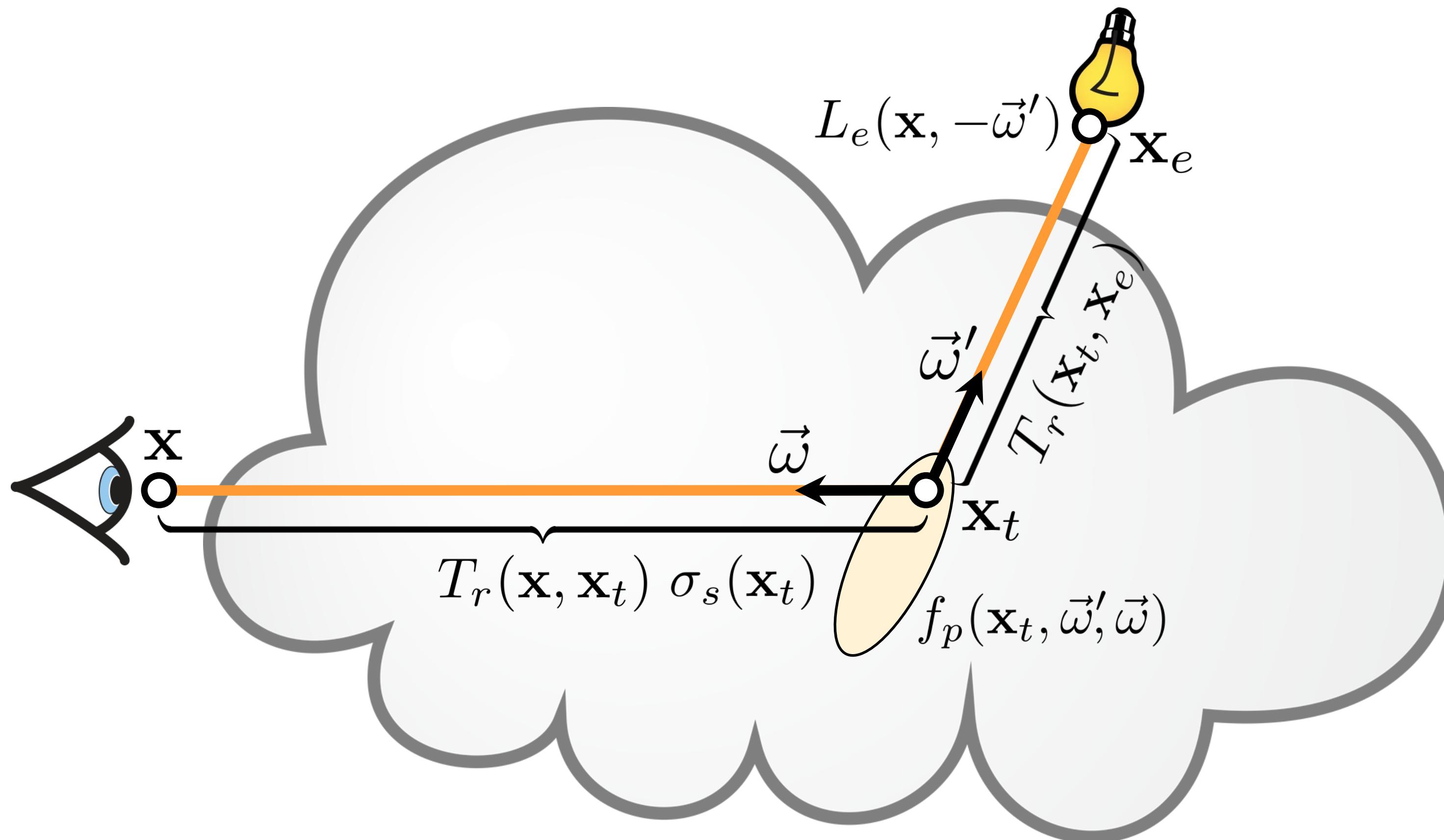
$$[L_i(\mathbf{x}, \vec{\omega})] = T_r(\mathbf{x}, r(\mathbf{x}, \vec{\omega})) L_e(r(\mathbf{x}, \vec{\omega}), -\vec{\omega})$$

Multiple scattering

- L_i arrives through multiple bounces (indirect illum.)

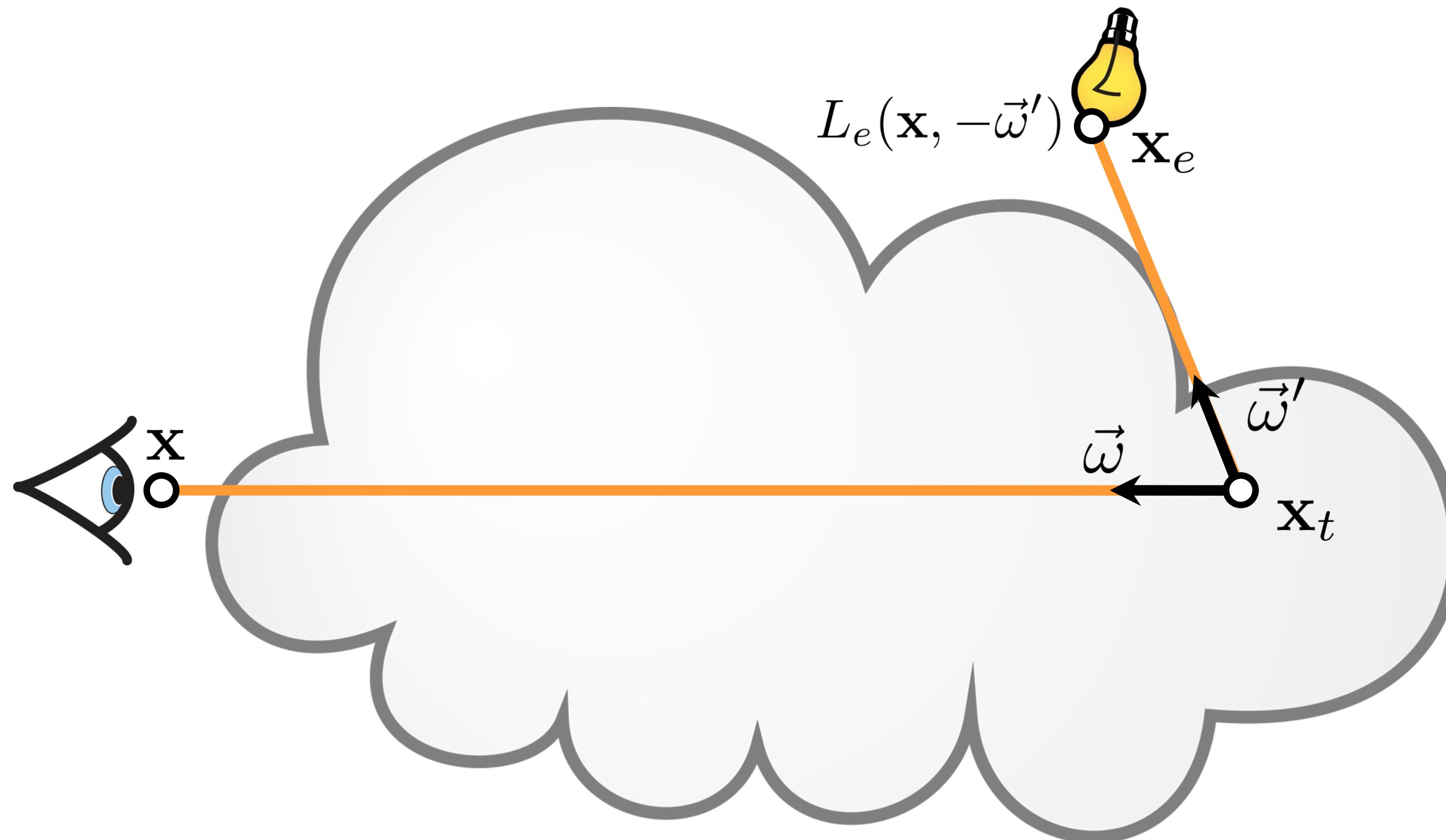
Single Scattering

$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) T_r(\mathbf{x}_t, \mathbf{x}_e) L_e(\mathbf{x}_e, -\vec{\omega}') d\vec{\omega}' dt$$



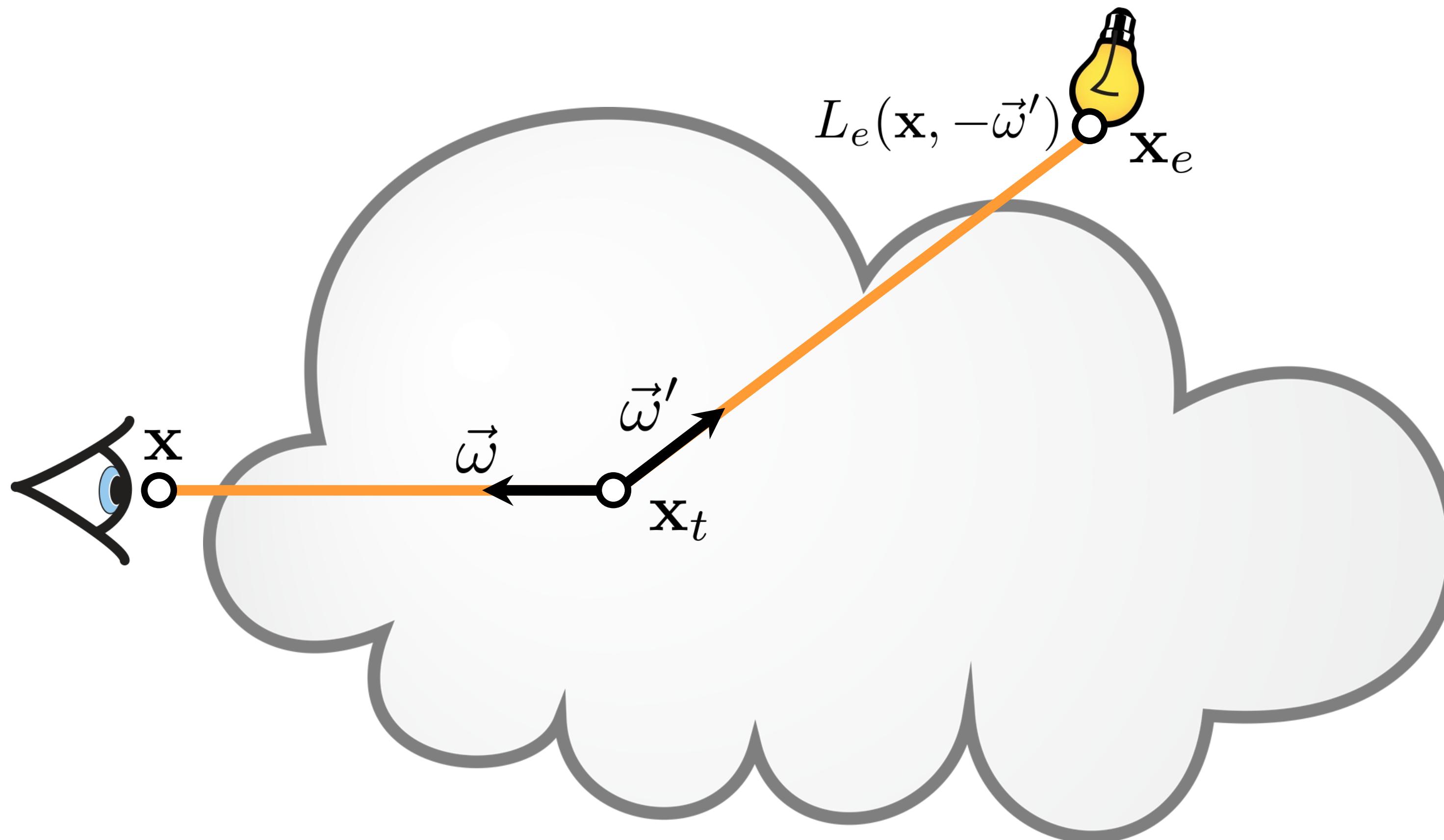
Single Scattering

$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) T_r(\mathbf{x}_t, \mathbf{x}_e) L_e(\mathbf{x}_e, -\vec{\omega}') d\vec{\omega}' dt$$



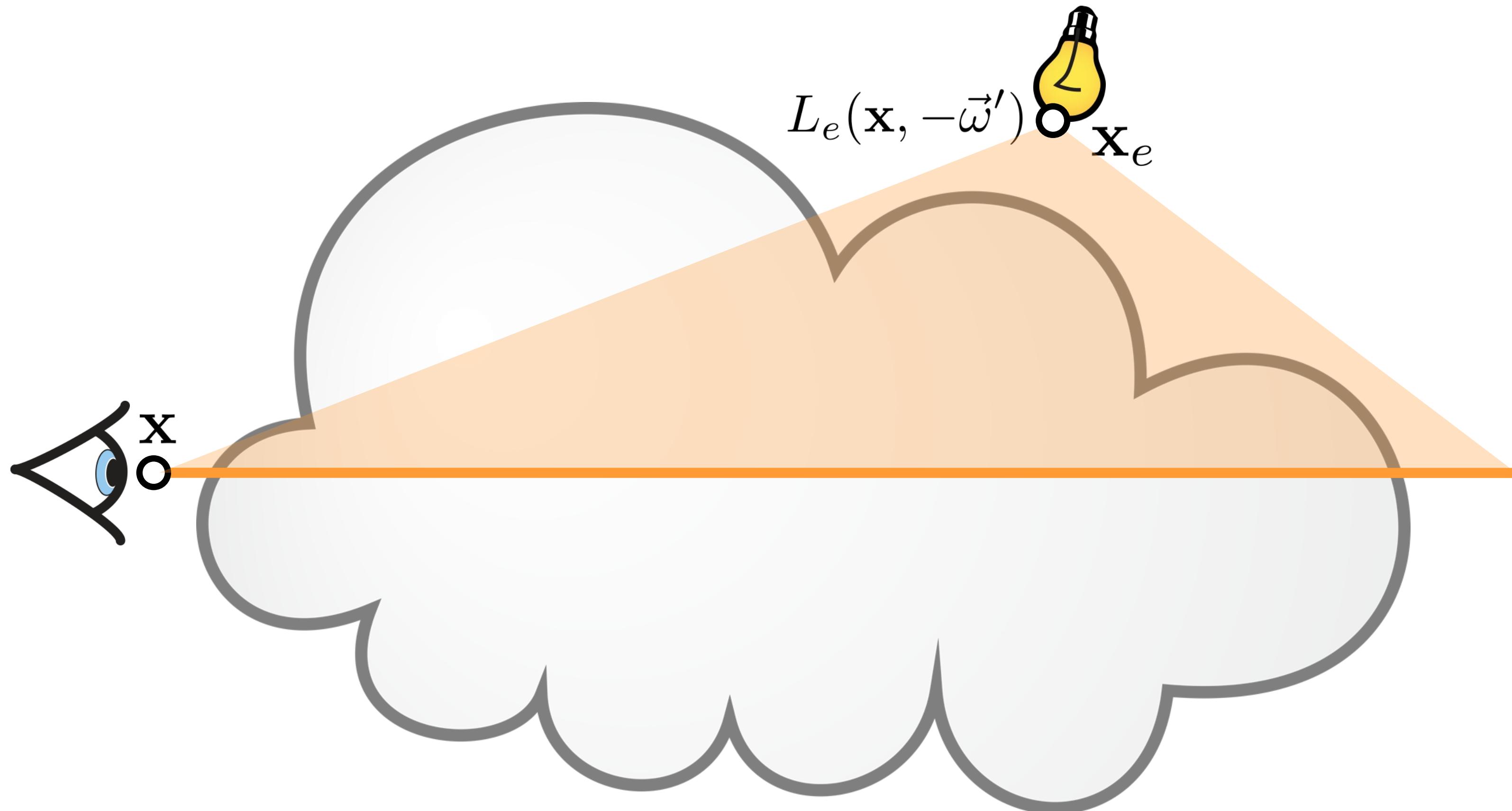
Single Scattering

$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) T_r(\mathbf{x}_t, \mathbf{x}_e) L_e(\mathbf{x}_e, -\vec{\omega}') d\vec{\omega}' dt$$



Single Scattering

$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) T_r(\mathbf{x}_t, \mathbf{x}_e) L_e(\mathbf{x}_e, -\vec{\omega}') d\vec{\omega}' dt$$



Single Scattering

$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) T_r(\mathbf{x}_t, \mathbf{x}_e) L_e(\mathbf{x}_e, -\vec{\omega}') d\vec{\omega}' dt$$

(Semi-)analytic solutions:

- Sun et al. [2005]
- Pegoraro et al. [2009, 2010]

Numerical solutions:

- Ray-marching
- Equiangular sampling

Analytic Single Scattering

$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) T_r(\mathbf{x}_t, \mathbf{x}_e) L_e(\mathbf{x}_e, -\vec{\omega}') d\vec{\omega}' dt$$

Assumptions:

- Homogeneous medium
- Point or spot light
- Relatively simple phase function
- No occlusion

$$L(\mathbf{x}, \vec{\omega}) = \frac{\Phi}{4\pi} \frac{1}{4\pi} \sigma_s \int_0^z e^{-\sigma_t \|\mathbf{x}, \mathbf{x}_t\|} \frac{e^{-\sigma_t \|\mathbf{x}_t, \mathbf{x}_p\|}}{\|\mathbf{x}_t, \mathbf{x}_p\|^2} dt$$

OpenGL Fog



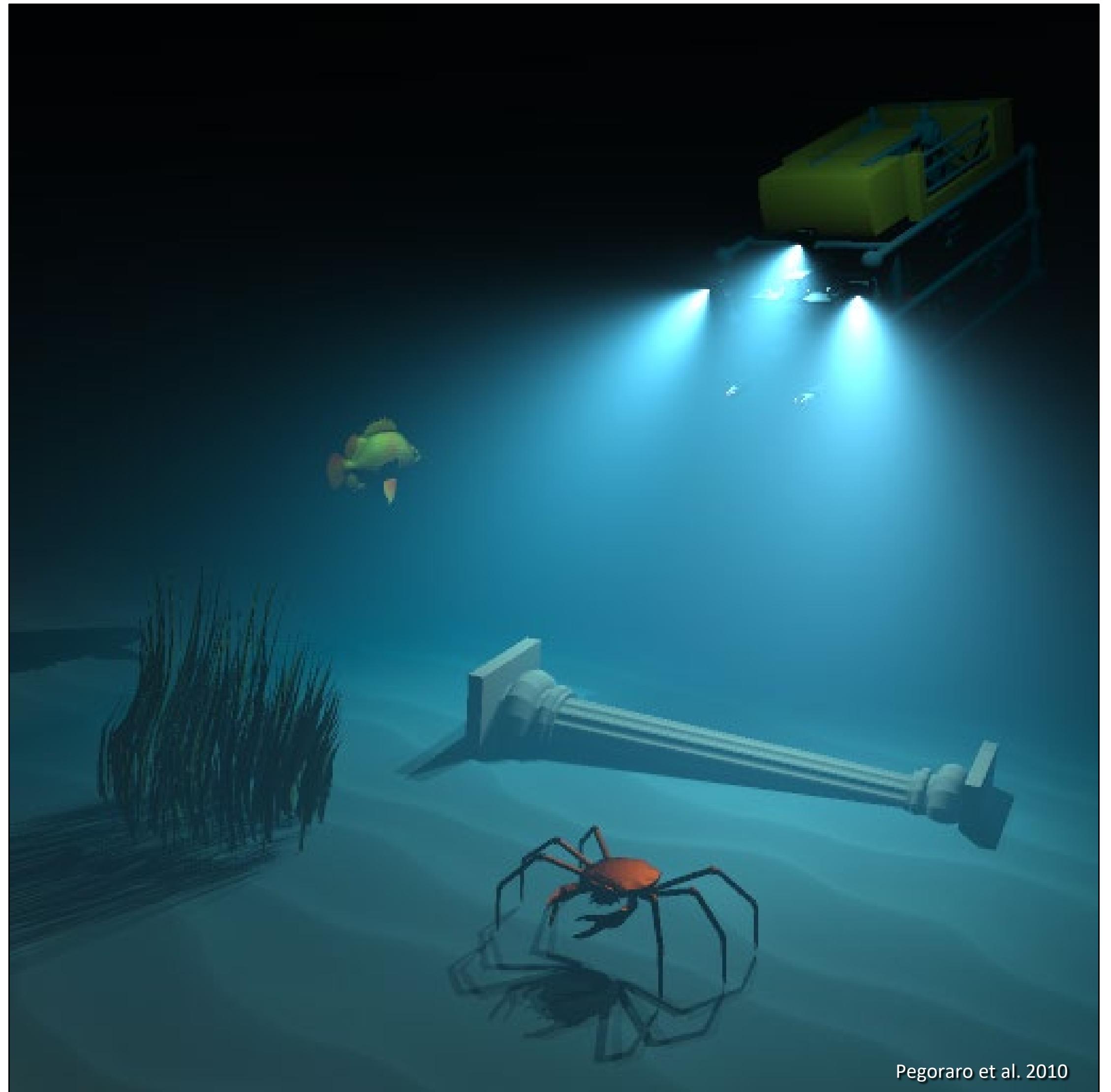
Sun et al. 2005

Analytic Single Scattering



Sun et al. 2005

Analytic Single Scattering





Andreas Levers



Andreas Levers

Analytic Single Scattering

$$L_m(x_a, x_b, \vec{\omega}) = \frac{\kappa_s}{h} e^{\kappa_t(x_a - x_b)} 2 \sum_{n=0}^{N-1} c(n) \sum_{k=0}^{2n} d(n, k) \int_{v_a}^{v_b} \frac{e^{-Hv}}{(v^2 + 1)^{n+1}} v^k dv$$

$$\begin{aligned} \int \frac{e^{av}}{(v^2 + 1)^m} v^n dv &= \frac{1}{2^{m-1}} \sum_{l=0}^{m-1} \frac{1}{2^l} \binom{m-1+l}{m-1} \left(\sum_{k=0}^{\min\{m-1-l, n\}} \binom{n}{k} \left(\frac{a^{m-1-l-k}}{(m-1-l-k)!} E(a, v, m-n-l+k) \right. \right. \\ &\quad \left. \left. - e^{av} \sum_{j=1}^{m-1-l-k} \frac{(j-1)!}{(m-1-l-k)!} \frac{a^{m-1-l-k-j}}{(v^2 + 1)^j} \sum_{\substack{i=0 \\ i \equiv (m-n-l+k-j) \pmod{2}}}^{\leq j} (-1)^{\frac{m-n-l+k-j+i}{2}} \binom{j}{i} v^i \right) \right) \\ &\quad + \frac{e^{av}}{a} \sum_{k=0}^{\leq n-m+l} \binom{n}{k} \sum_{j=0}^{n-m+l-k} \frac{(n-m+l-k)!}{j!} \frac{1}{(-a)^{n-m+l-k-j}} \sum_{\substack{i=0 \\ i \equiv (-m+l+k-j) \pmod{2}}}^{\leq j} (-1)^{\frac{-m+l+k-j+i}{2}} \binom{j}{i} v^i \end{aligned}$$

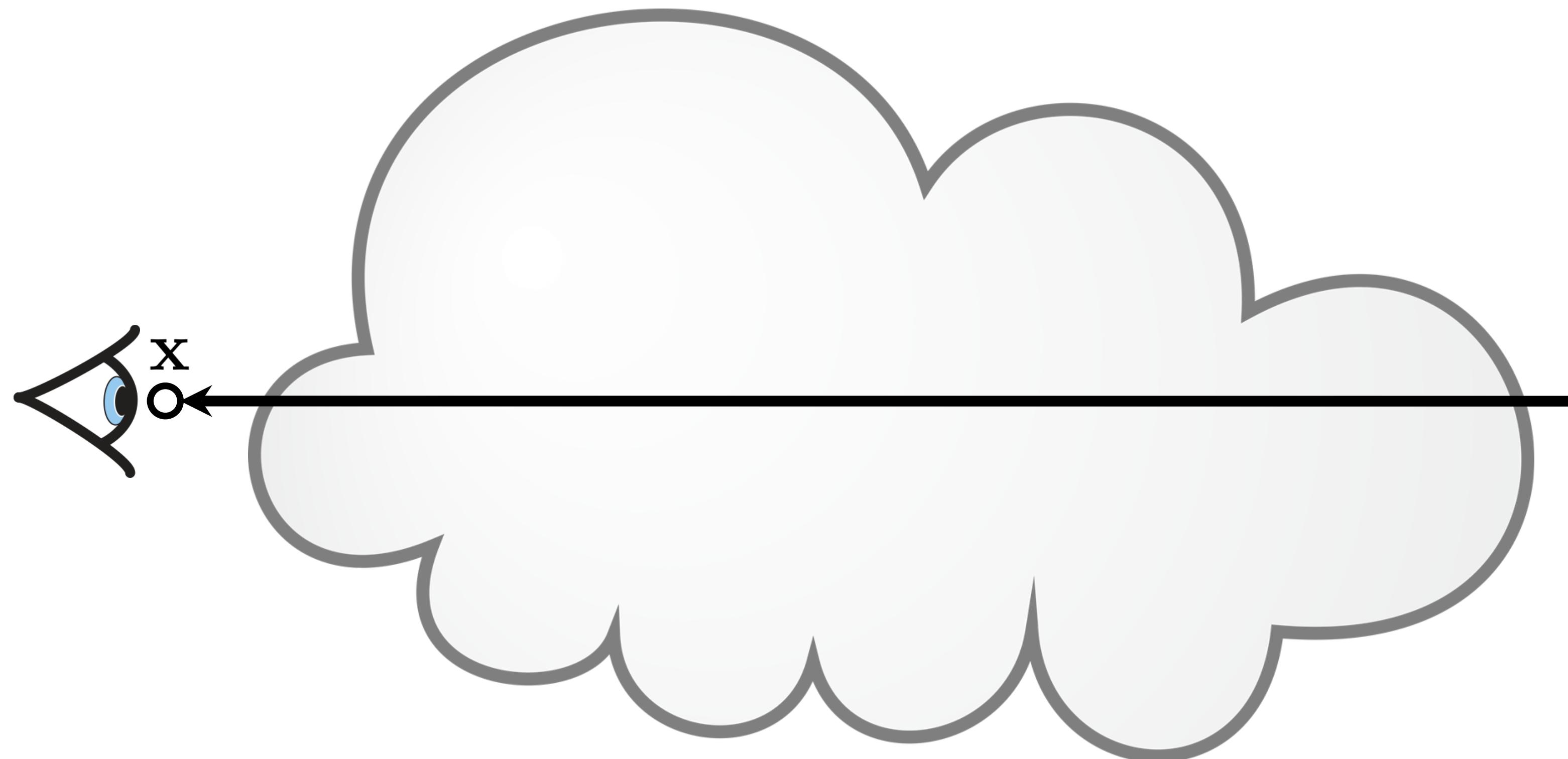
No shadows, implementation nightmare, computationally intensive...

Let's try brute force!

Ray-Marching

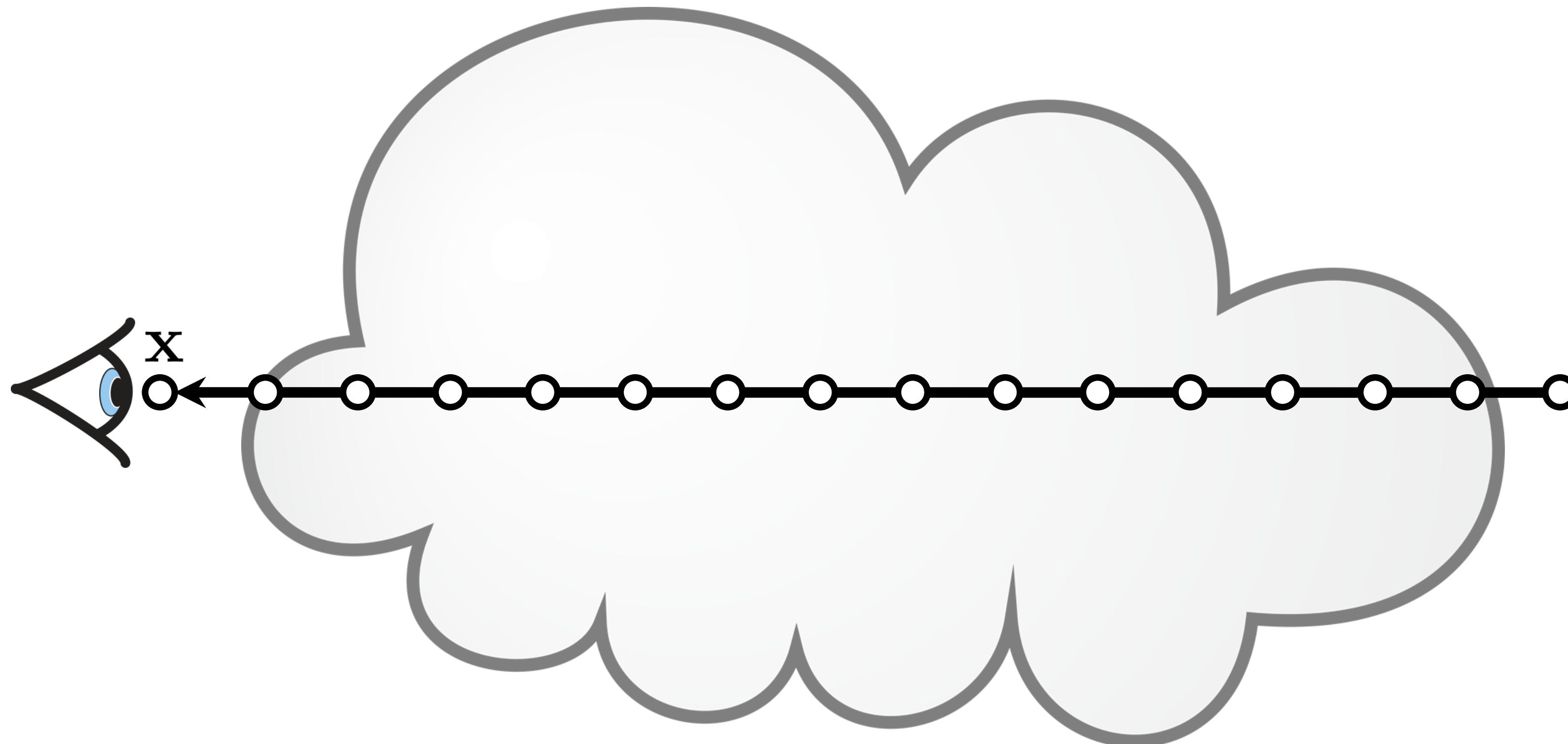
$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) L_s(\mathbf{x}_t, \vec{\omega}) dt$$

↓
Approximate with Riemann sum



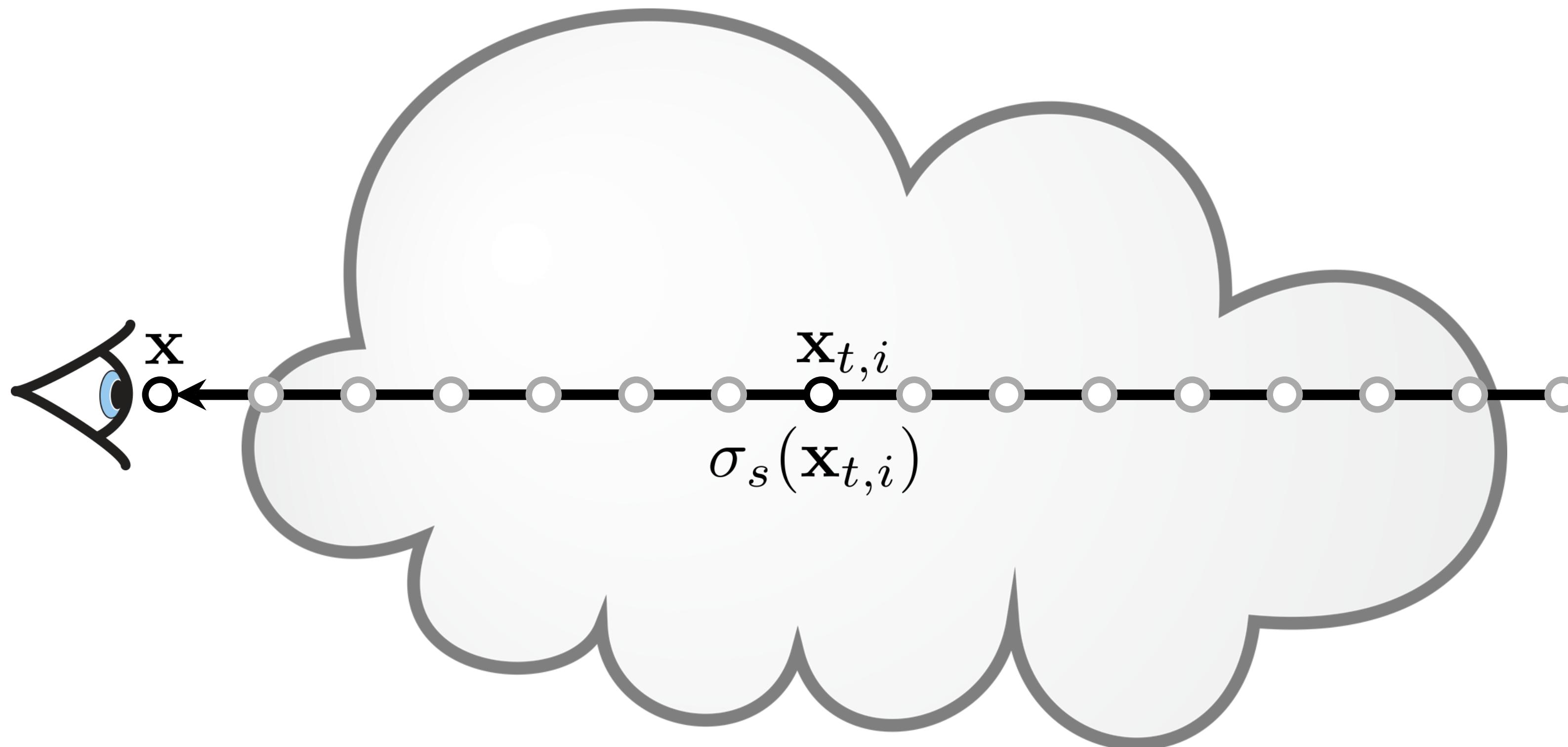
Ray-Marching

$$L(\mathbf{x}, \vec{\omega}) \approx \sum_{i=1}^N T_r(\mathbf{x}, \mathbf{x}_{t,i}) \sigma_s(\mathbf{x}_{t,i}) L_s(\mathbf{x}_{t,i}, \vec{\omega}) \Delta t$$



Ray-Marching

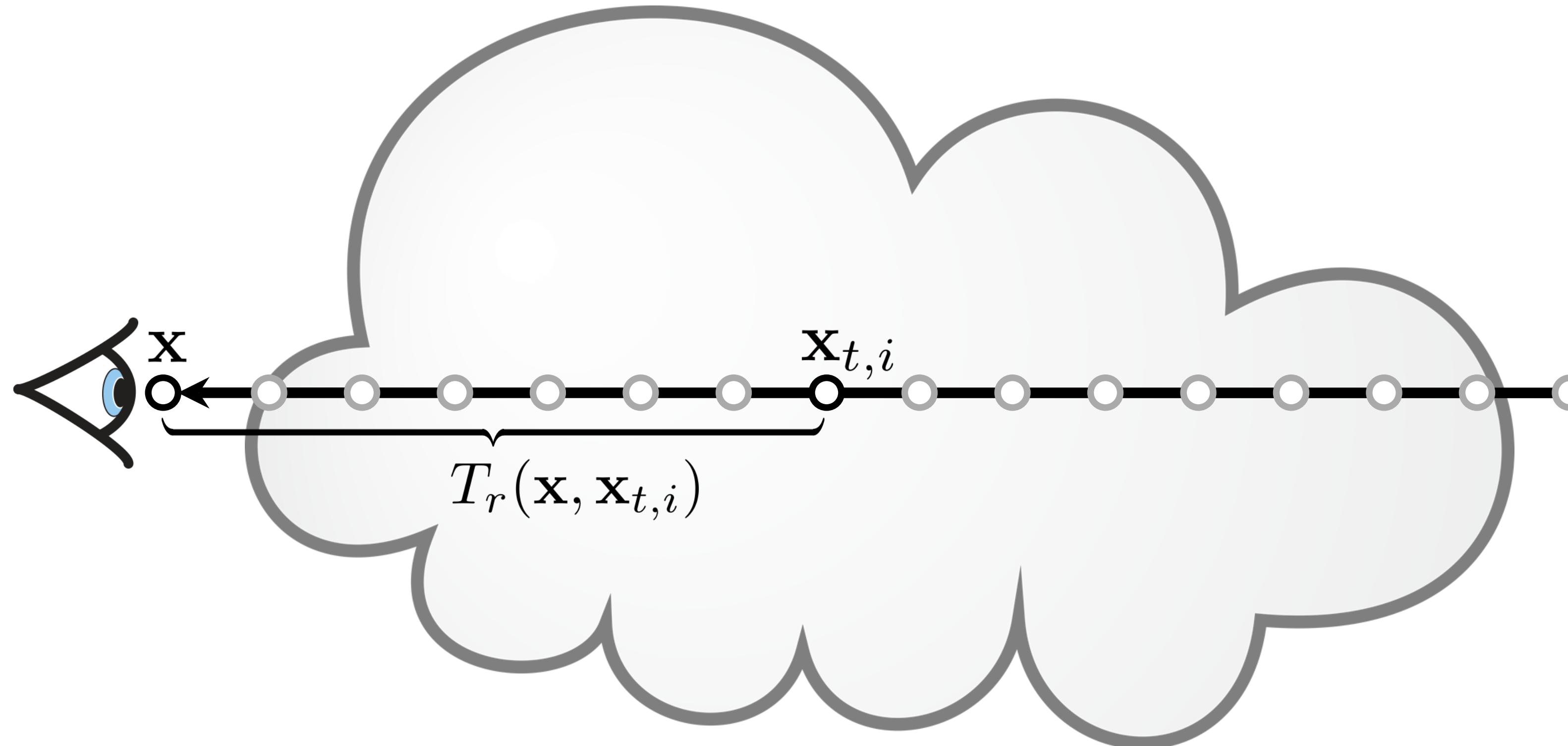
$$L(\mathbf{x}, \vec{\omega}) \approx \sum_{i=1}^N T_r(\mathbf{x}, \mathbf{x}_{t,i}) \sigma_s(\mathbf{x}_{t,i}) L_s(\mathbf{x}_{t,i}, \vec{\omega}) \Delta t$$



Ray-Marching

$$L(\mathbf{x}, \vec{\omega}) \approx \sum_{i=1}^N T_r(\mathbf{x}, \mathbf{x}_{t,i}) \sigma_s(\mathbf{x}_{t,i}) L_s(\mathbf{x}_{t,i}, \vec{\omega}) \Delta t$$

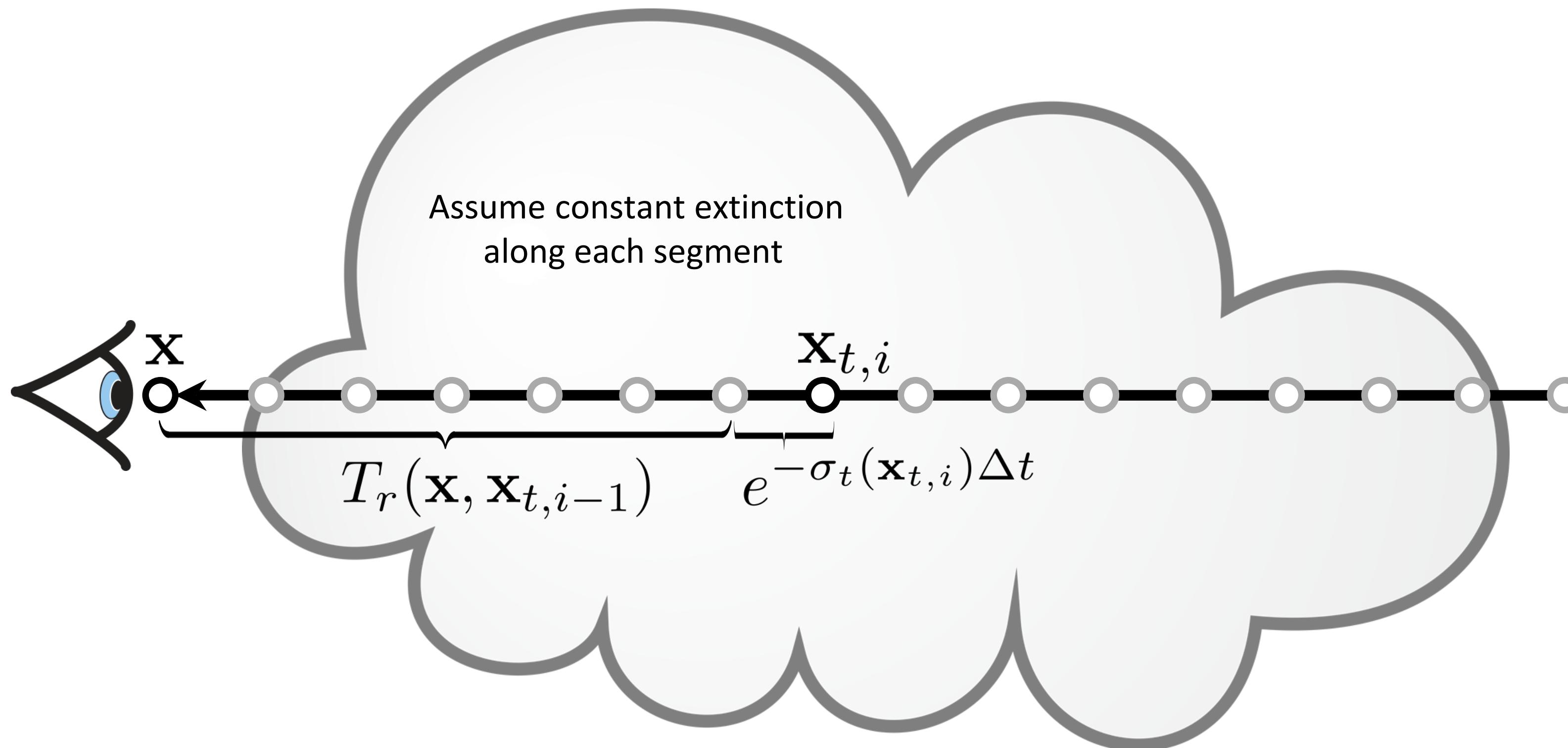
Homogeneous volume: $T_r(\mathbf{x}, \mathbf{x}_{t,i}) = e^{-\sigma_t \|\mathbf{x}, \mathbf{x}_{t,i}\|}$



Ray-Marching

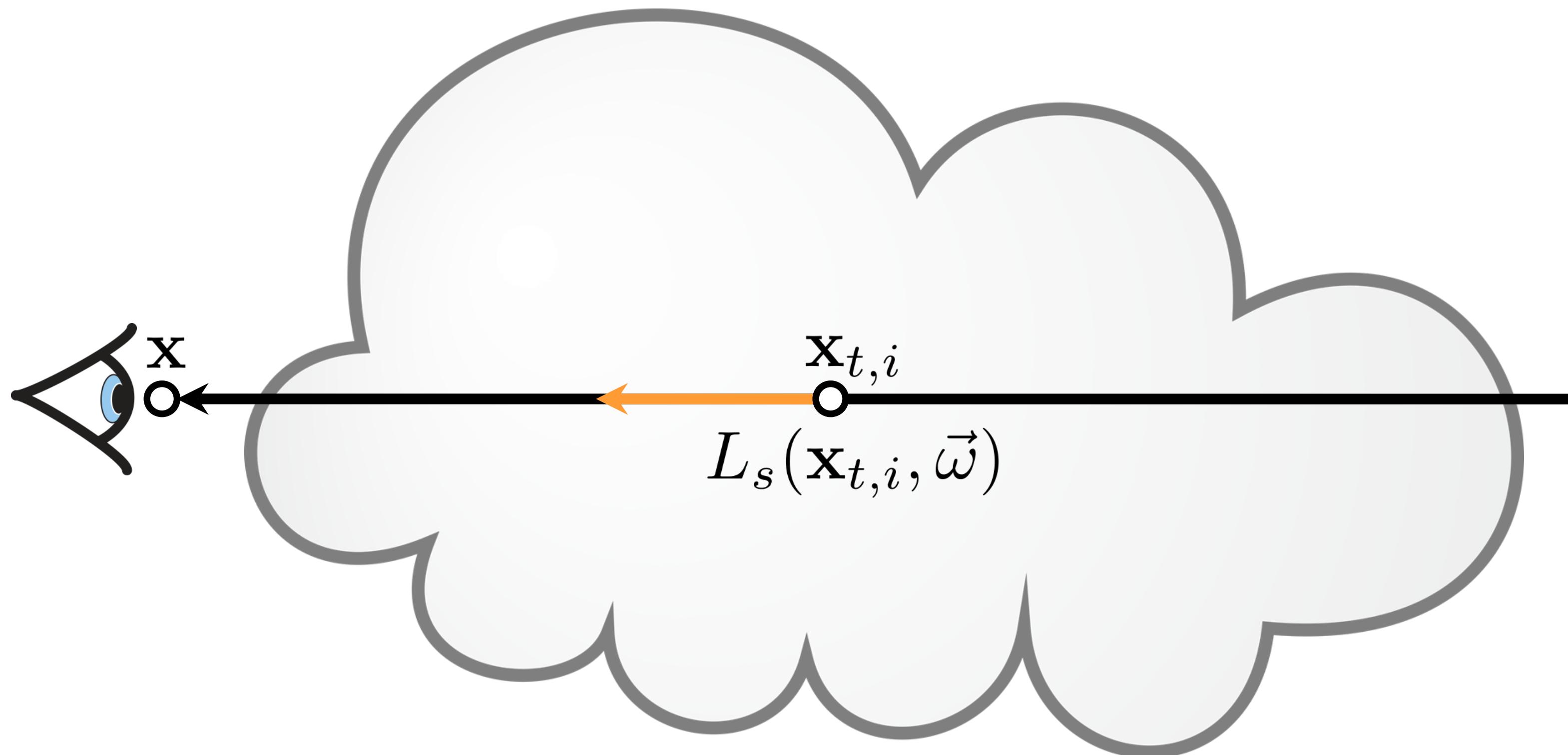
$$L(\mathbf{x}, \vec{\omega}) \approx \sum_{i=1}^N T_r(\mathbf{x}, \mathbf{x}_{t,i}) \sigma_s(\mathbf{x}_{t,i}) L_s(\mathbf{x}_{t,i}, \vec{\omega}) \Delta t$$

Heterogeneous volume: $T_r(\mathbf{x}, \mathbf{x}_{t,i}) = T_r(\mathbf{x}, \mathbf{x}_{t,i-1}) e^{-\sigma_t(\mathbf{x}_{t,i}) \Delta t}$



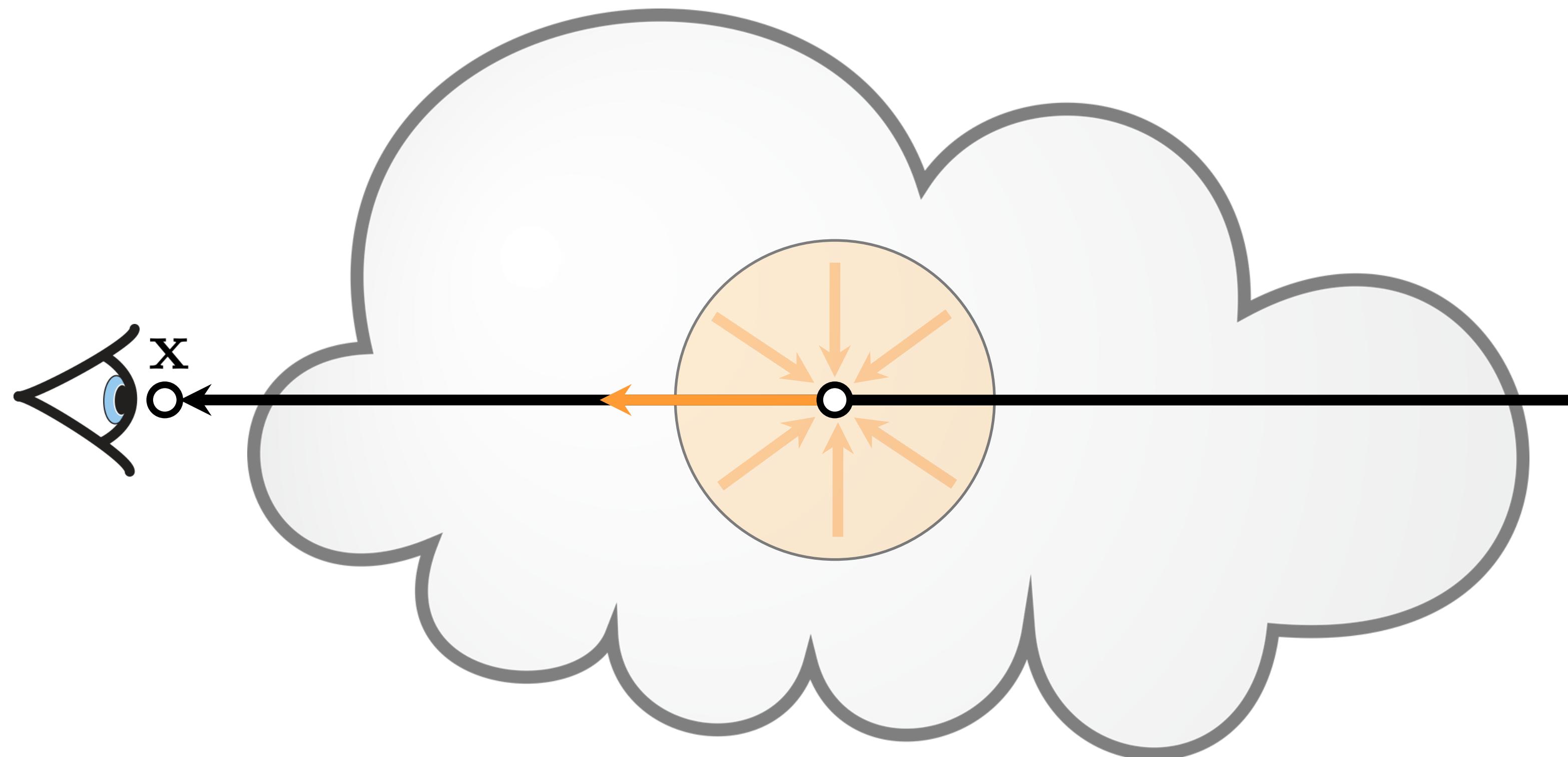
Ray-Marching

$$L(\mathbf{x}, \vec{\omega}) \approx \sum_{i=1}^N T_r(\mathbf{x}, \mathbf{x}_{t,i}) \sigma_s(\mathbf{x}_{t,i}) L_s(\mathbf{x}_{t,i}, \vec{\omega}) \Delta t$$



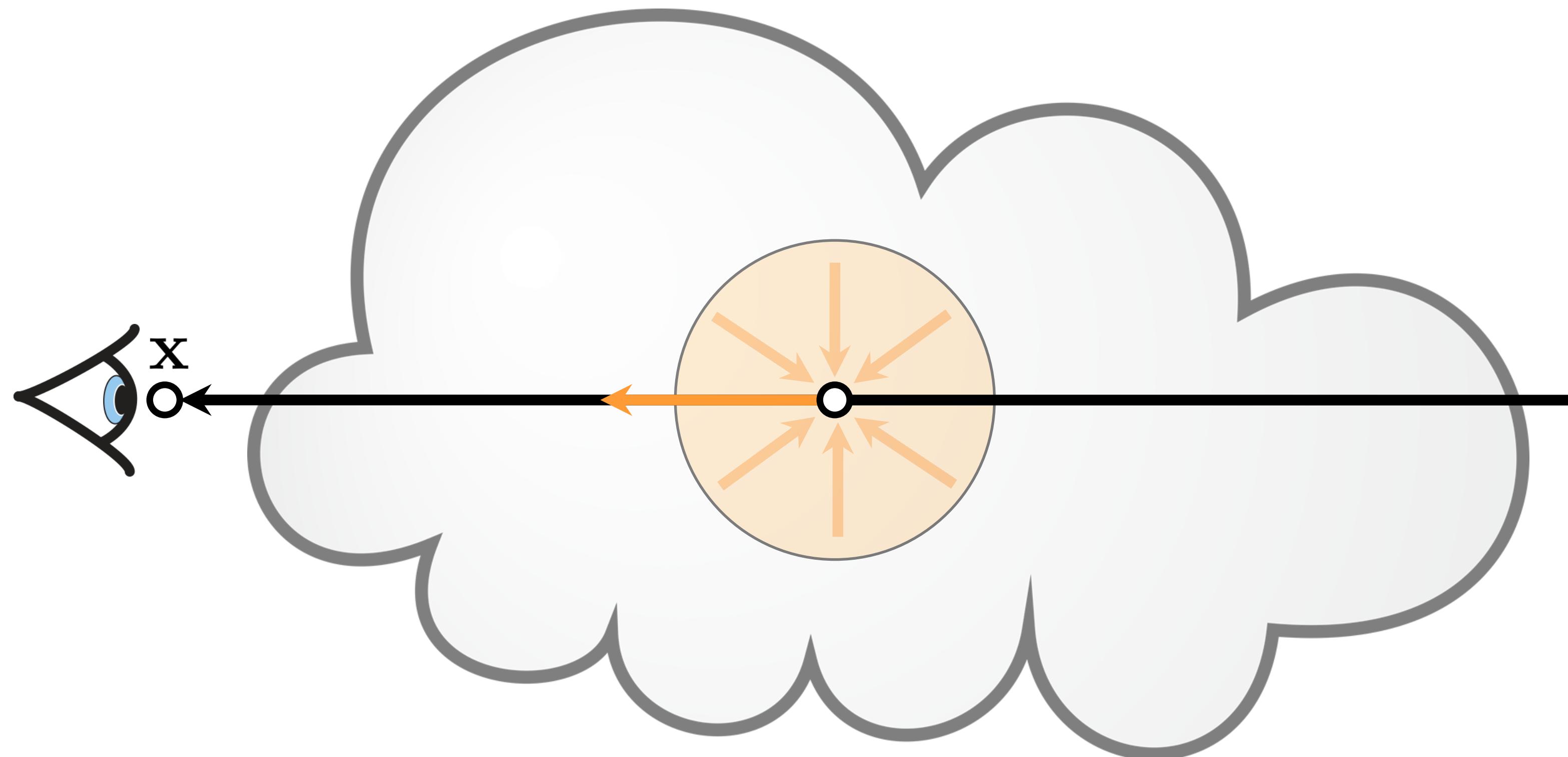
Ray-Marching

$$L_s(\mathbf{x}_t, \vec{\omega}) = \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) L_i(\mathbf{x}_t, \vec{\omega}') d\vec{\omega}'$$



Ray-Marching

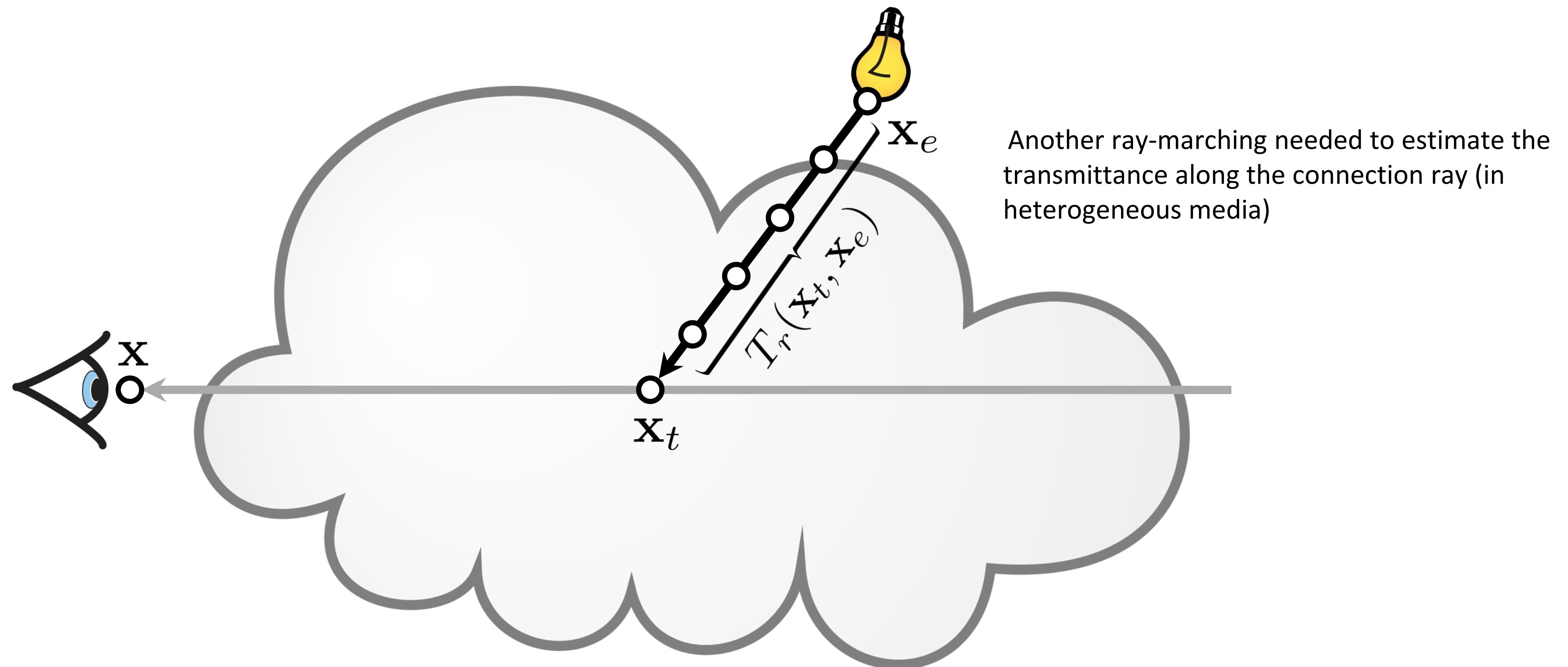
$$L_s(\mathbf{x}_t, \vec{\omega}) \approx \frac{1}{M} \sum_{j=0}^M \frac{f_p(\mathbf{x}_t, \vec{\omega}'_j, \vec{\omega}) L_i(\mathbf{x}_t, \vec{\omega}'_j)}{p(\vec{\omega}'_j)}$$



Ray-Marching

Single scattering:

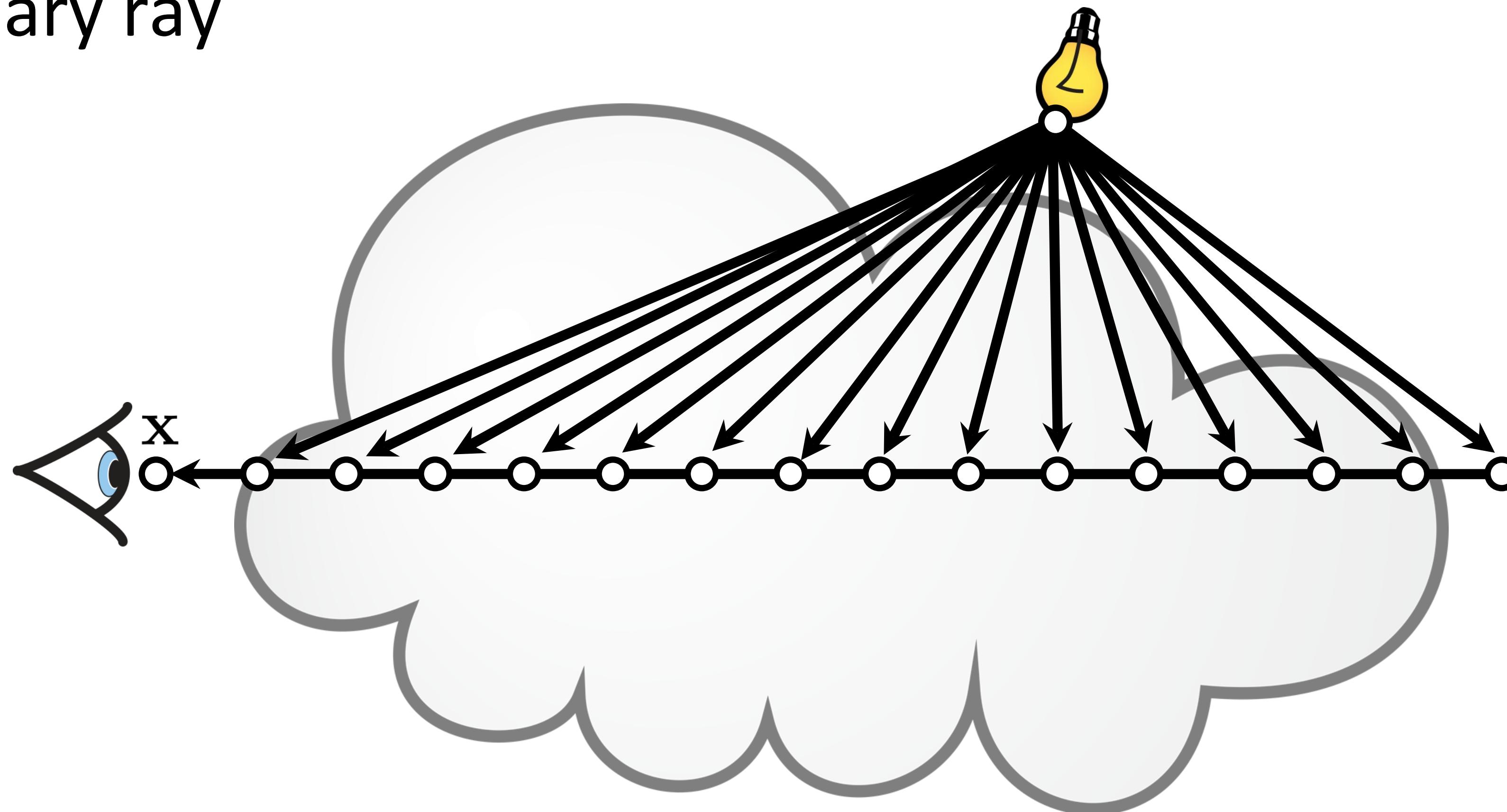
$$L_i(\mathbf{x}_t, \vec{\omega}) = T_r(\mathbf{x}_t, \mathbf{x}_e) L_e(\mathbf{x}_e, -\vec{\omega})$$



Ray-Marching in Heterogeneous Media

Marching towards the light source

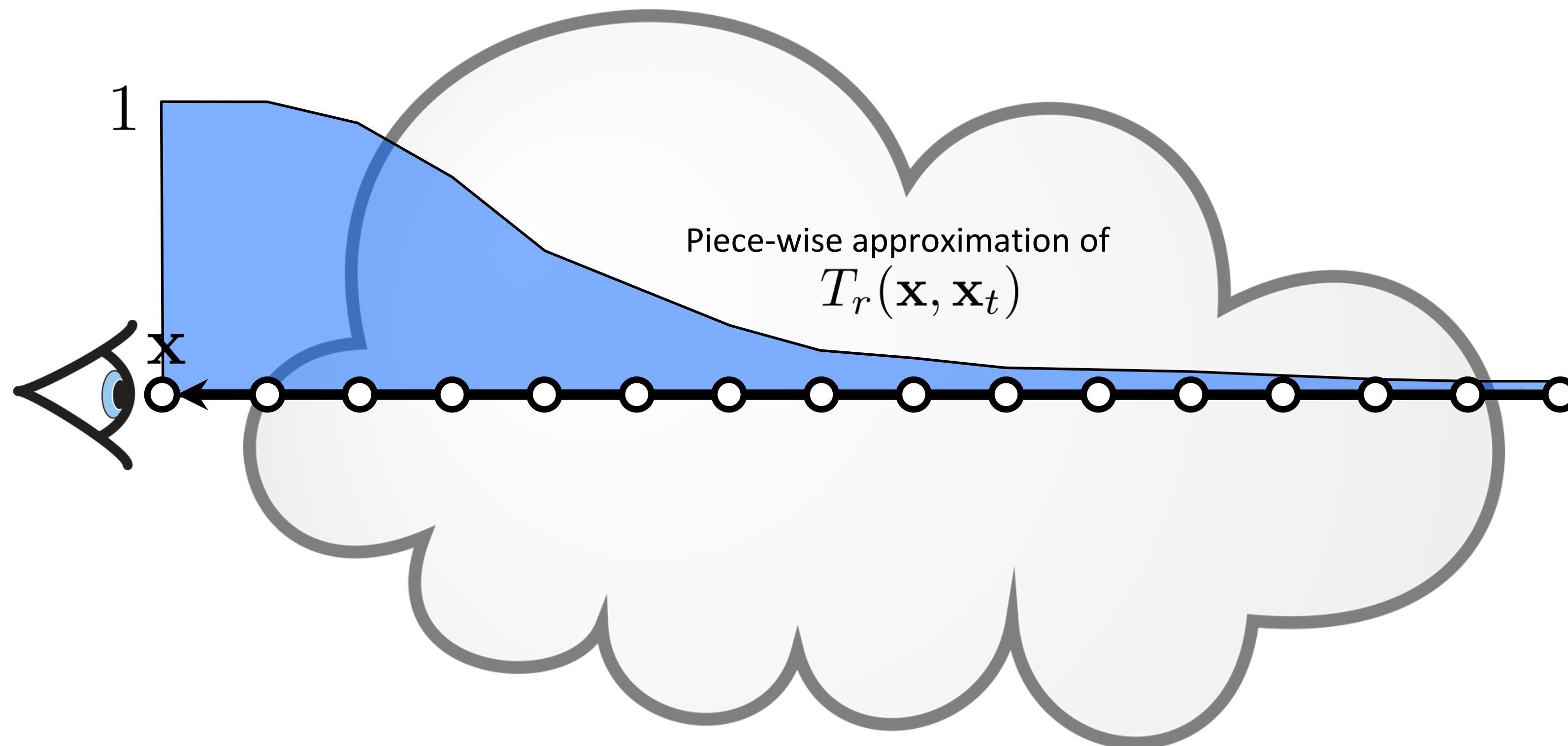
- Connections are expensive, many, and uniformly distributed along the primary ray



Decoupled Transmittance and In-scattering

1. Ray-march and cache transmittance

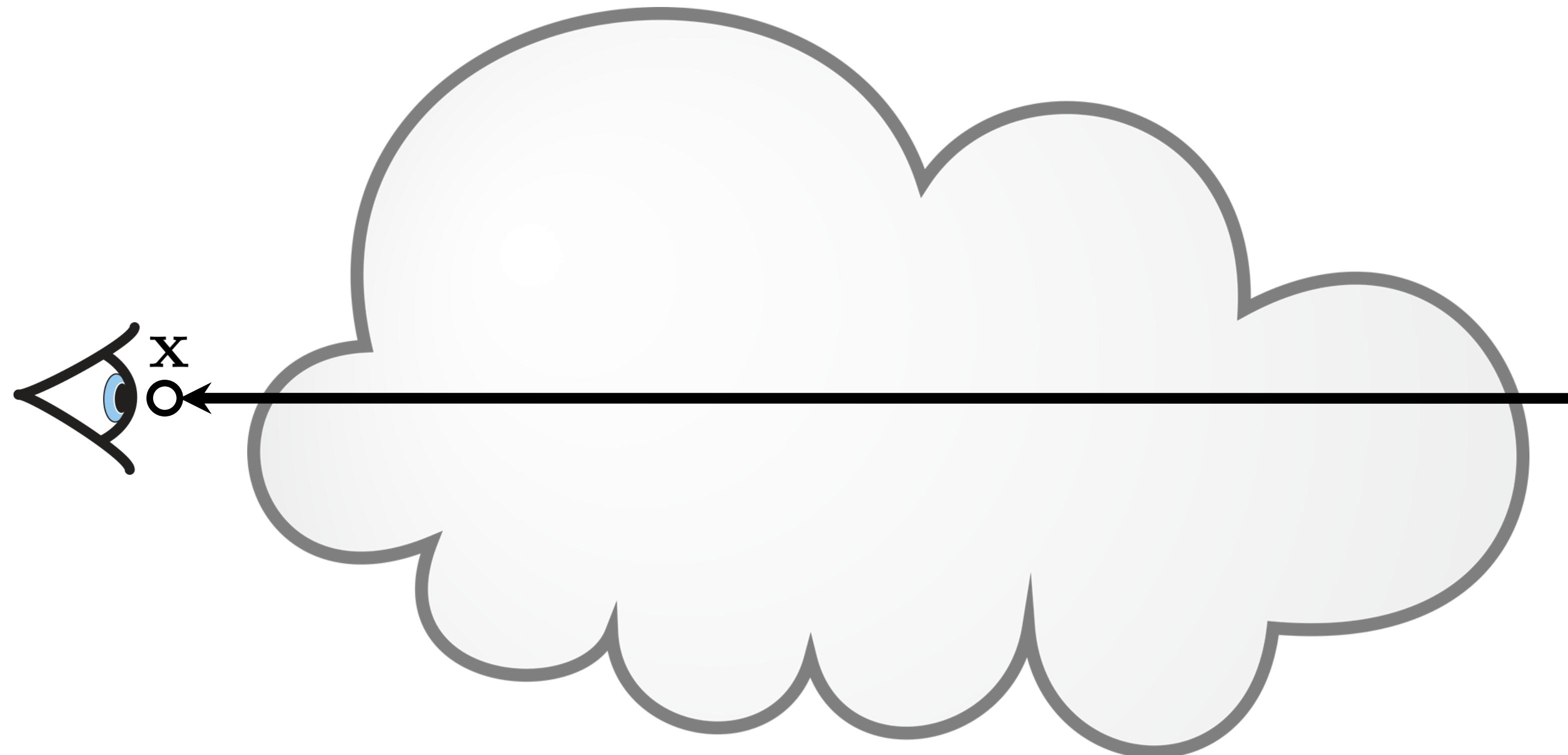
- Choose step-size w.r.t. frequency content to accurately capture variations



Decoupled Transmittance and In-scattering

2. Estimate in-scattering using MC integration

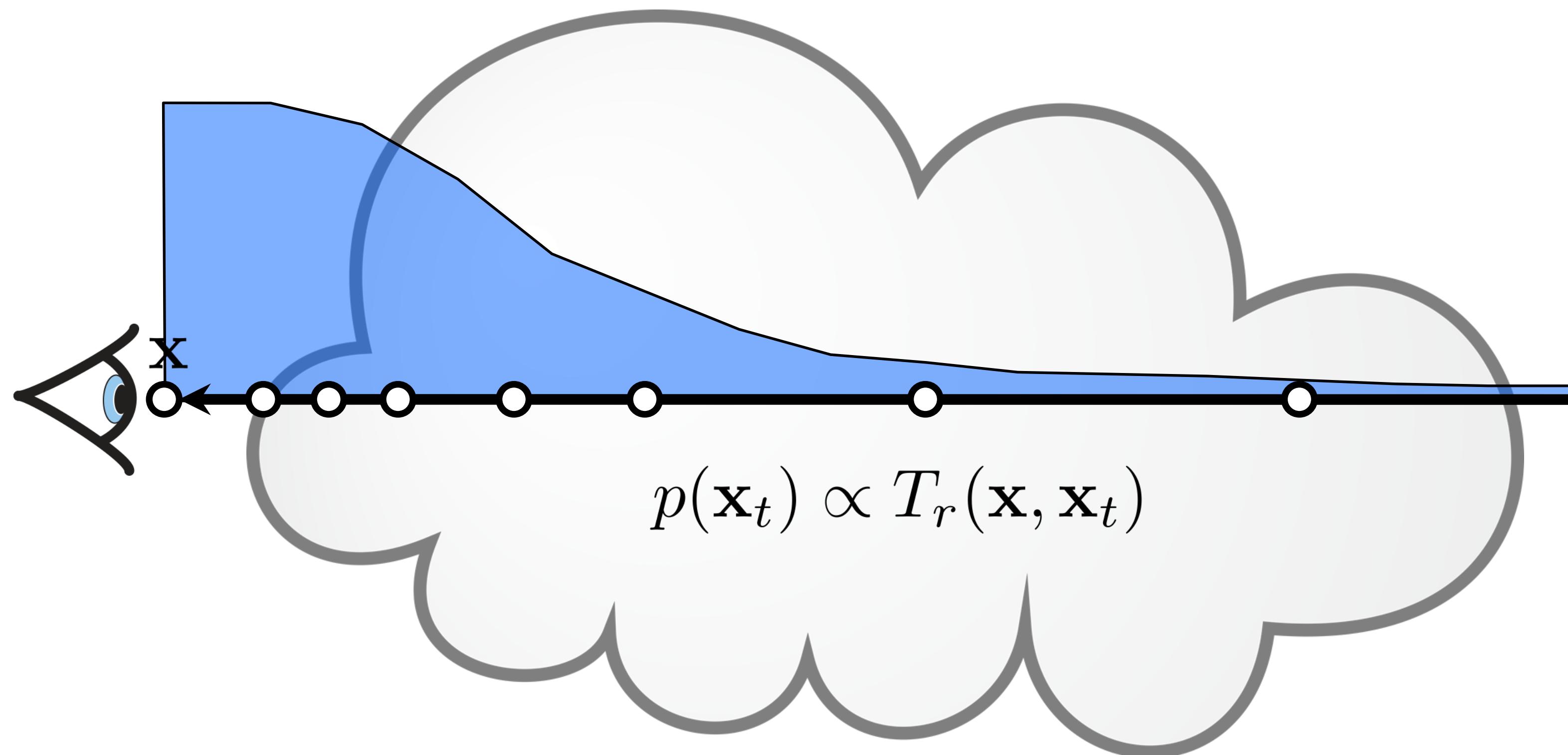
- Distribute samples \propto (part of) the integrand



Decoupled Transmittance and In-scattering

2. Estimate in-scattering using MC integration

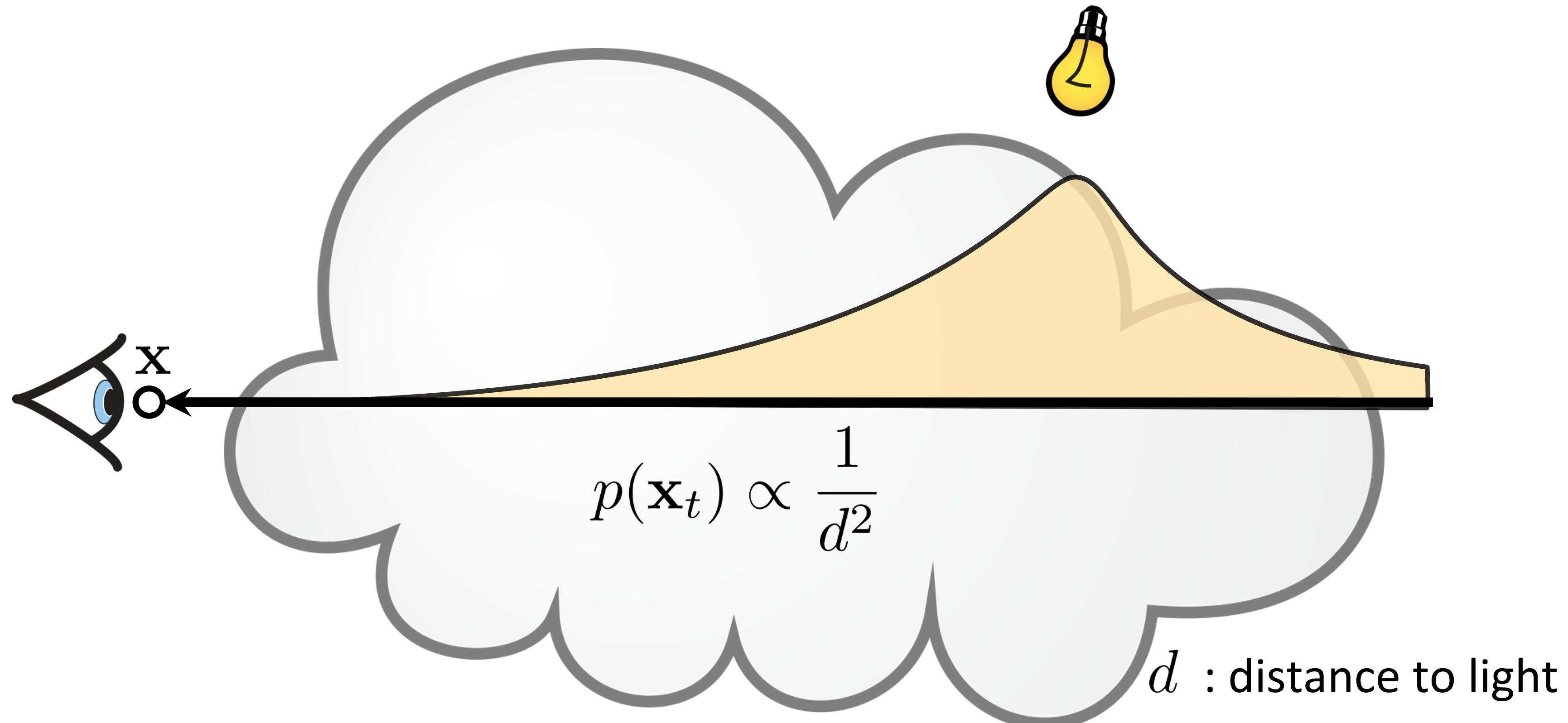
- Distribute samples \propto (part of) the integrand



Decoupled Transmittance and In-scattering

2. Estimate in-scattering using MC integration

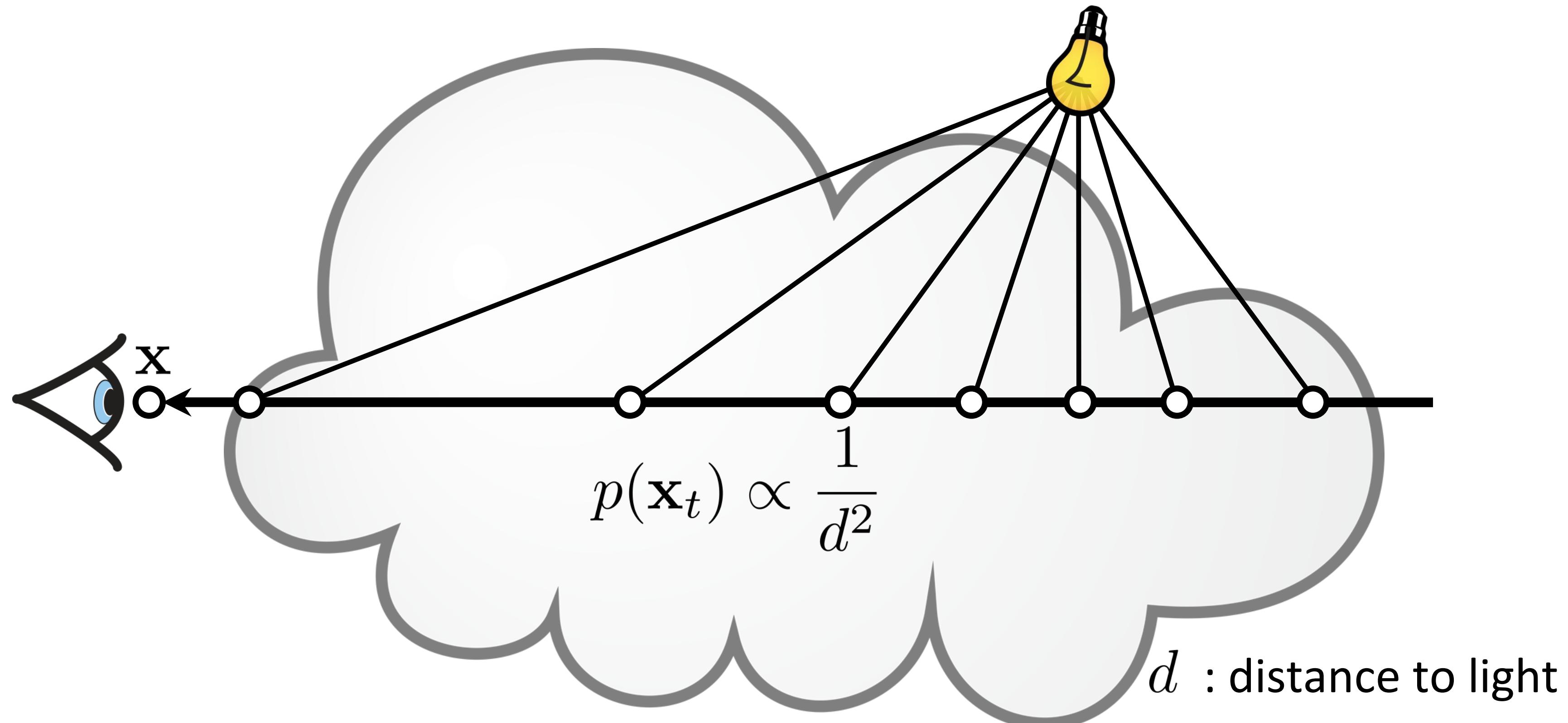
- Distribute samples \propto (part of) the integrand



Decoupled Transmittance and In-scattering

2. Estimate in-scattering using MC integration

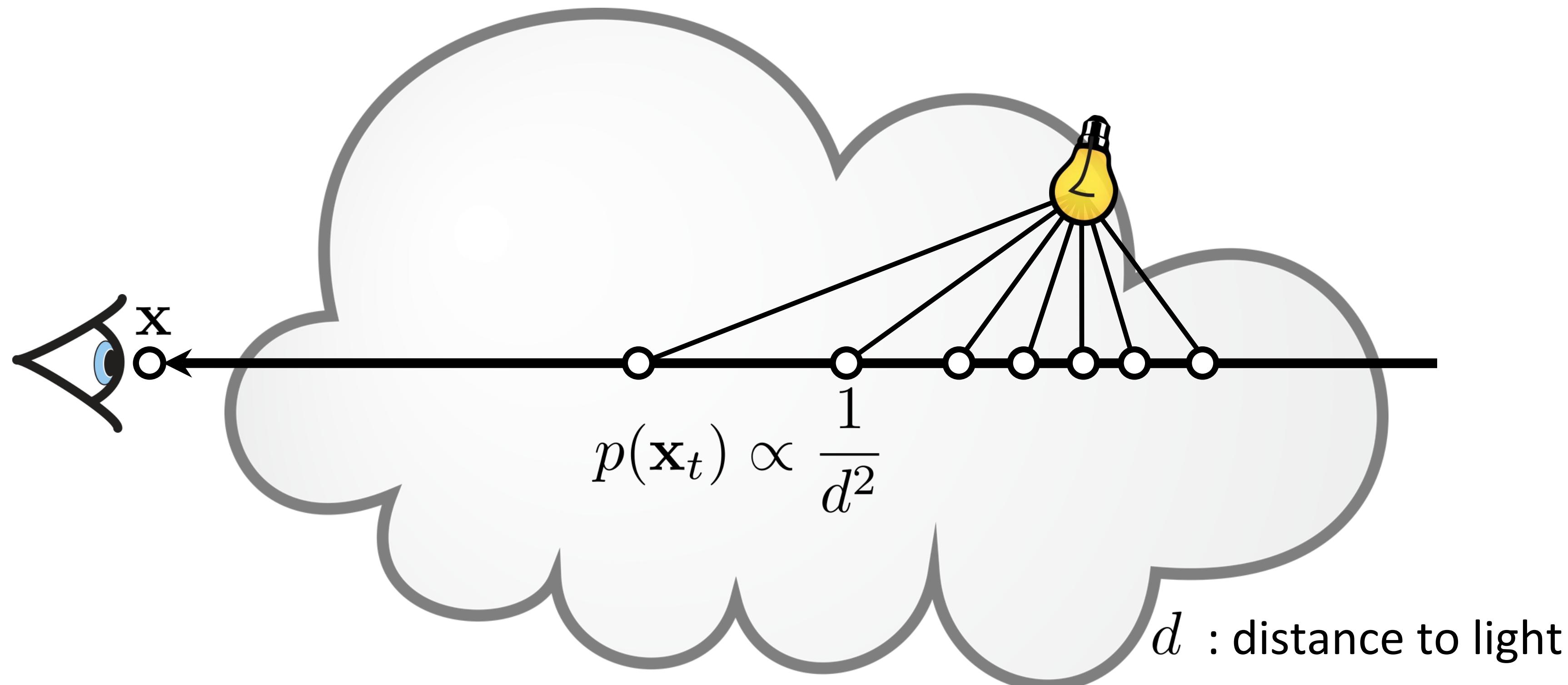
- Distribute samples \propto (part of) the integrand



Decoupled Transmittance and In-scattering

2. Estimate in-scattering using MC integration

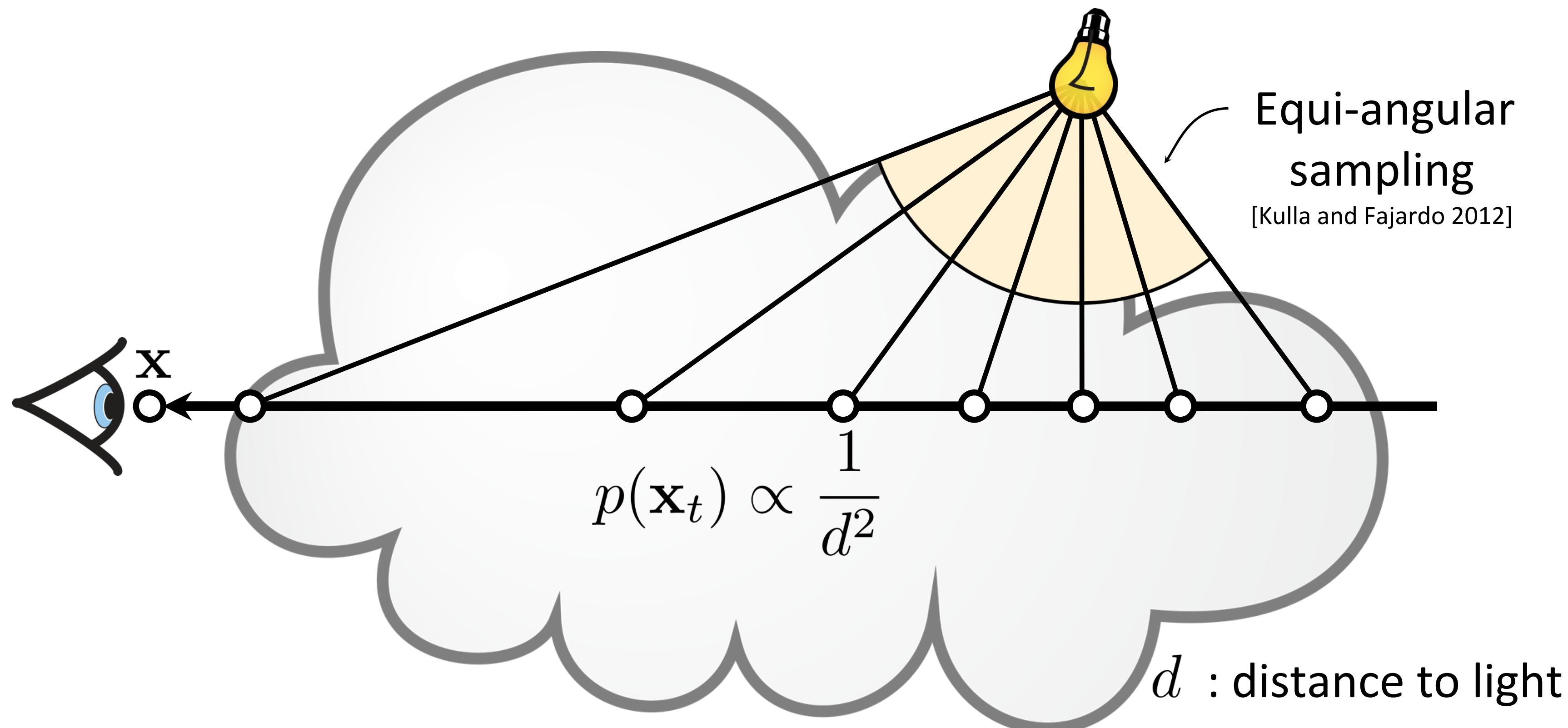
- Distribute samples \propto (part of) the integrand



Decoupled Transmittance and In-scattering

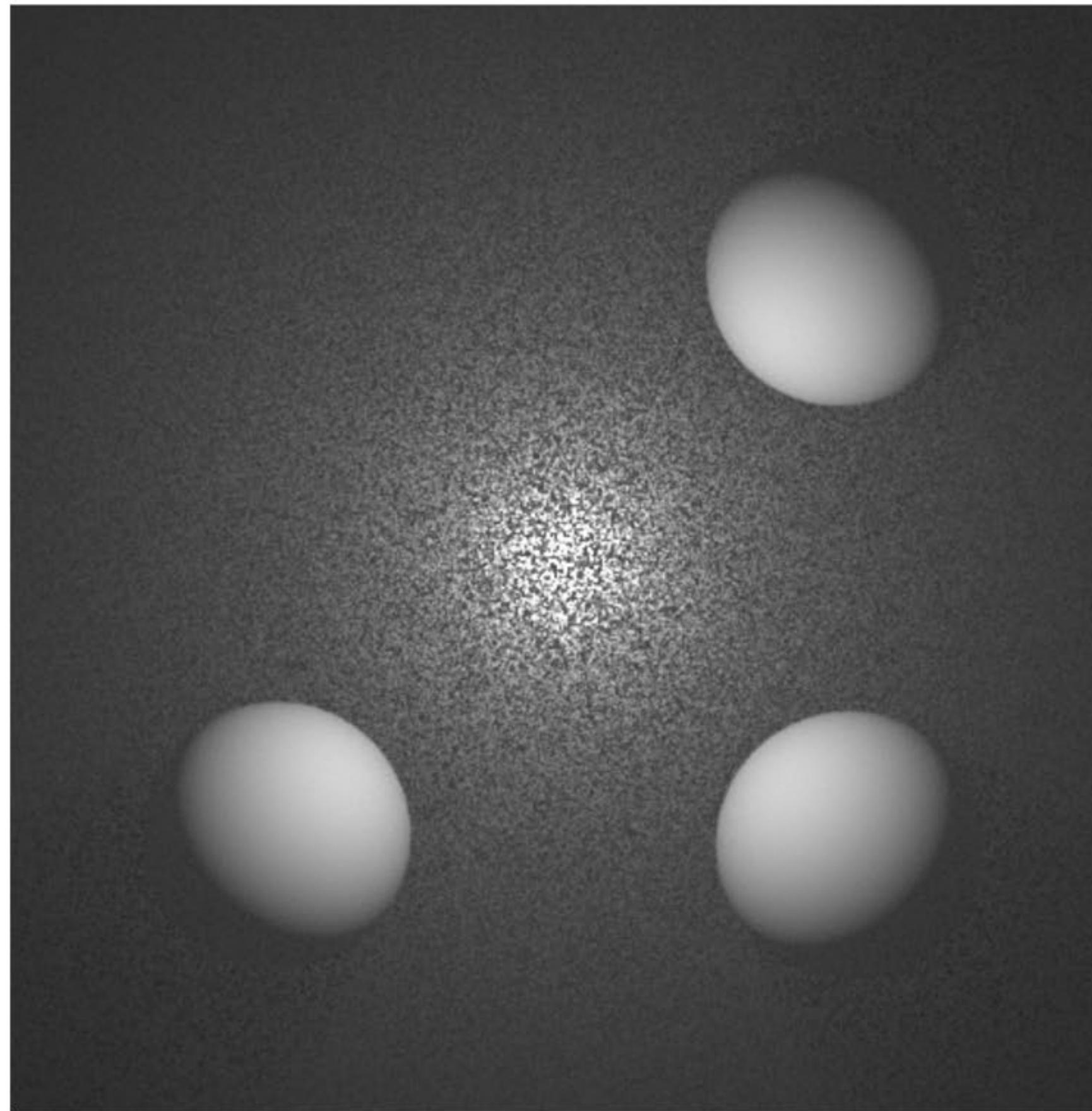
2. Estimate in-scattering using MC integration

- Distribute samples \propto (part of) the integrand

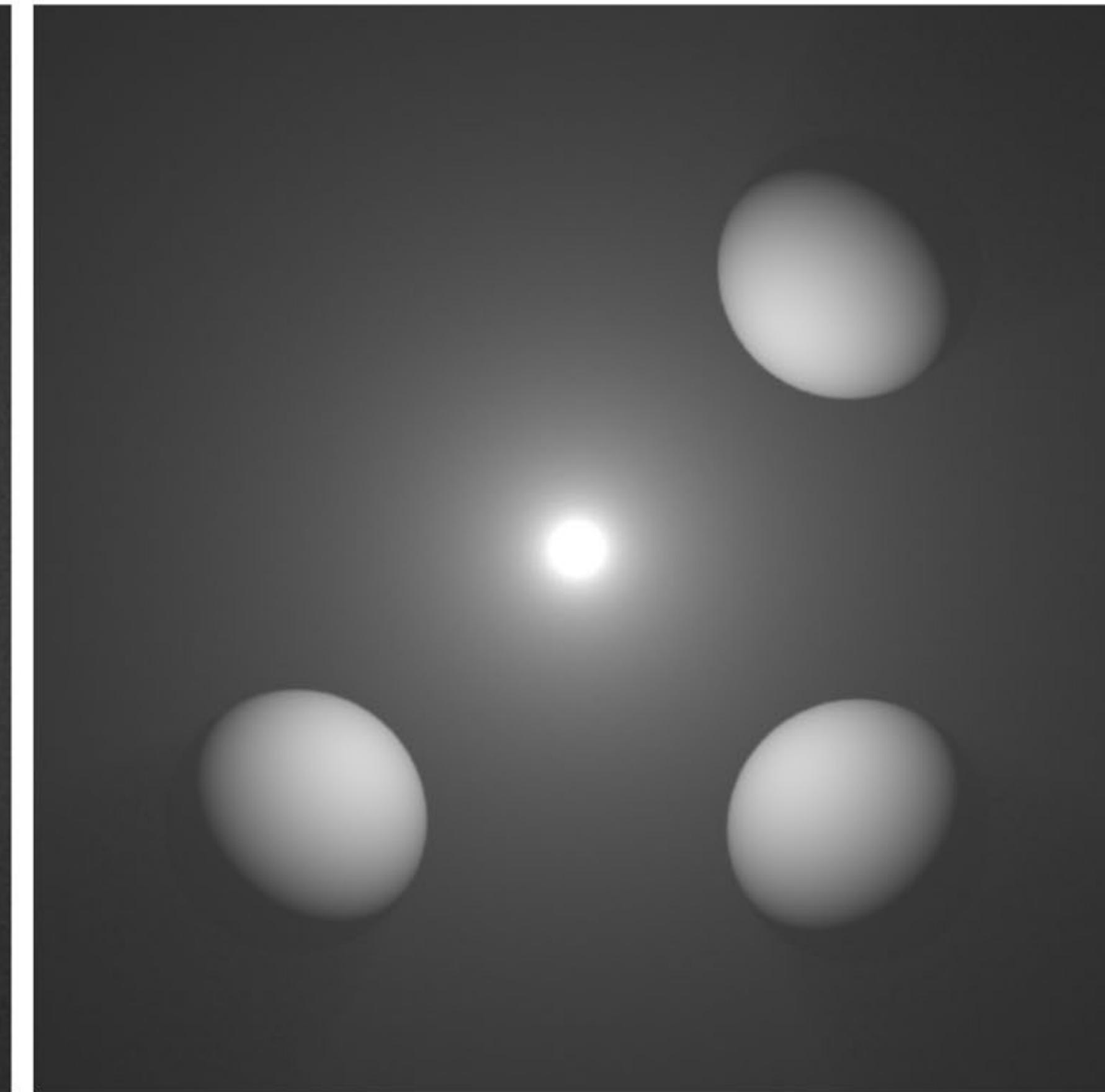


Decoupled Transmittance and In-scattering

Ray-marching



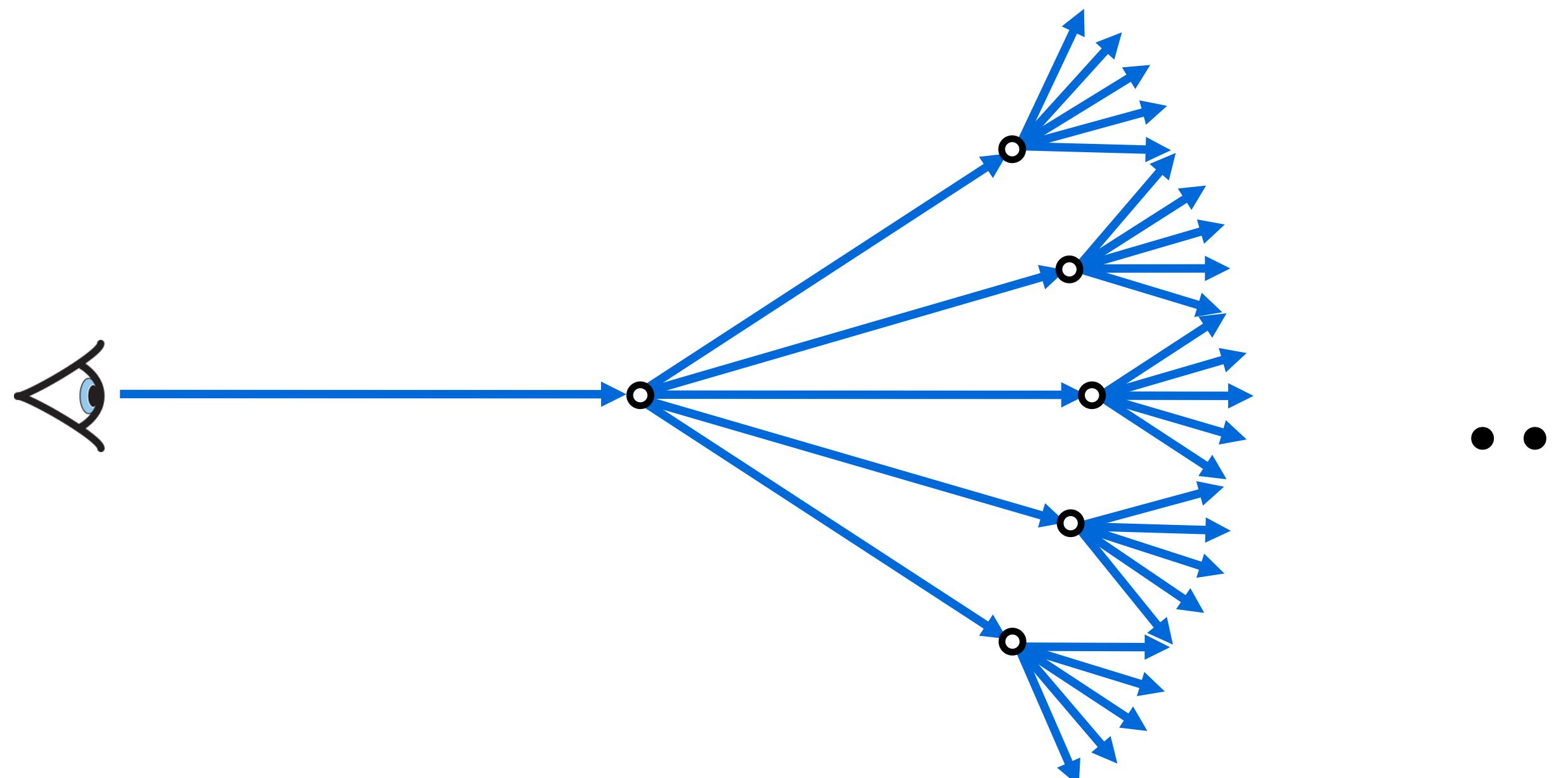
Equiangular sampling



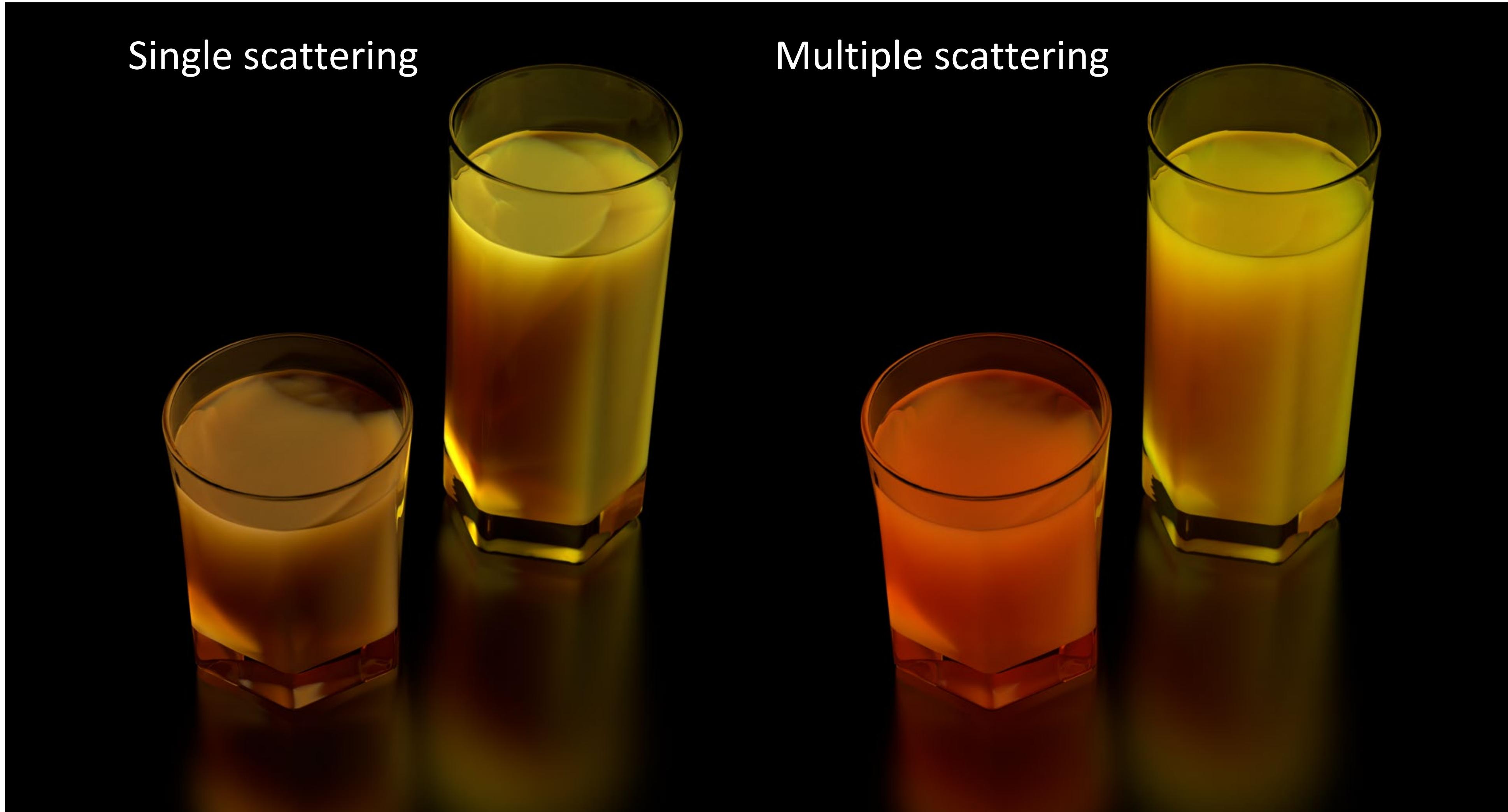
Multiple Bounces

Same concept as in recursive Monte Carlo ray tracing, but taking into account volumetric scattering

Exponential growth:



Visual Break



Volumetric Path Tracing

Volumetric Path Tracing

Motivation:

- Same as with standard path tracing: avoid the exponential growth

Paths can:

- Reflect/refract off surfaces
- Scatter inside a volume

Volume Rendering Equation

$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) dt + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) L_s(\mathbf{x}_t, \vec{\omega}) dt + T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega})$$

Accumulated emitted radiance

Attenuated background radiance

Accumulated in-scattered radiance

Volume Rendering Equation

$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \left[\sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) + \sigma_s(\mathbf{x}_t) L_s(\mathbf{x}_t, \vec{\omega}) \right] dt + T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega})$$

Accumulated emitted + in-scattered radiance

Attenuated background radiance

Volume Rendering Equation

$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \left[\sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) + \sigma_s(\mathbf{x}_t) L_s(\mathbf{x}_t, \vec{\omega}) \right] dt + T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega})$$

1-Sample Monte Carlo Estimator

$$\begin{aligned}\langle L(\mathbf{x}, \vec{\omega}) \rangle &= \frac{T_r(\mathbf{x}, \mathbf{x}_t)}{p(t)} \left[\sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) + \sigma_s(\mathbf{x}_t) L_s(\mathbf{x}_t, \vec{\omega}) \right] \\ &+ \frac{T_r(\mathbf{x}, \mathbf{x}_z)}{P(z)} L(\mathbf{x}_z, \vec{\omega})\end{aligned}$$

$p(t)$ - probability *density* of distance t

$P(z)$ - *probability* of exceeding distance z

1-Sample Monte Carlo Estimator

$$\begin{aligned}\langle L(\mathbf{x}, \vec{\omega}) \rangle &= \frac{T_r(\mathbf{x}, \mathbf{x}_t)}{p(t)} \left[\sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) + \sigma_s(\mathbf{x}_t) \frac{f_p(\vec{\omega}, \vec{\omega}_i) L(\mathbf{x}_t, \vec{\omega}_i)}{p(\vec{\omega}_i)} \right] \\ &+ \frac{T_r(\mathbf{x}, \mathbf{x}_z)}{P(z)} L(\mathbf{x}_z, \vec{\omega})\end{aligned}$$

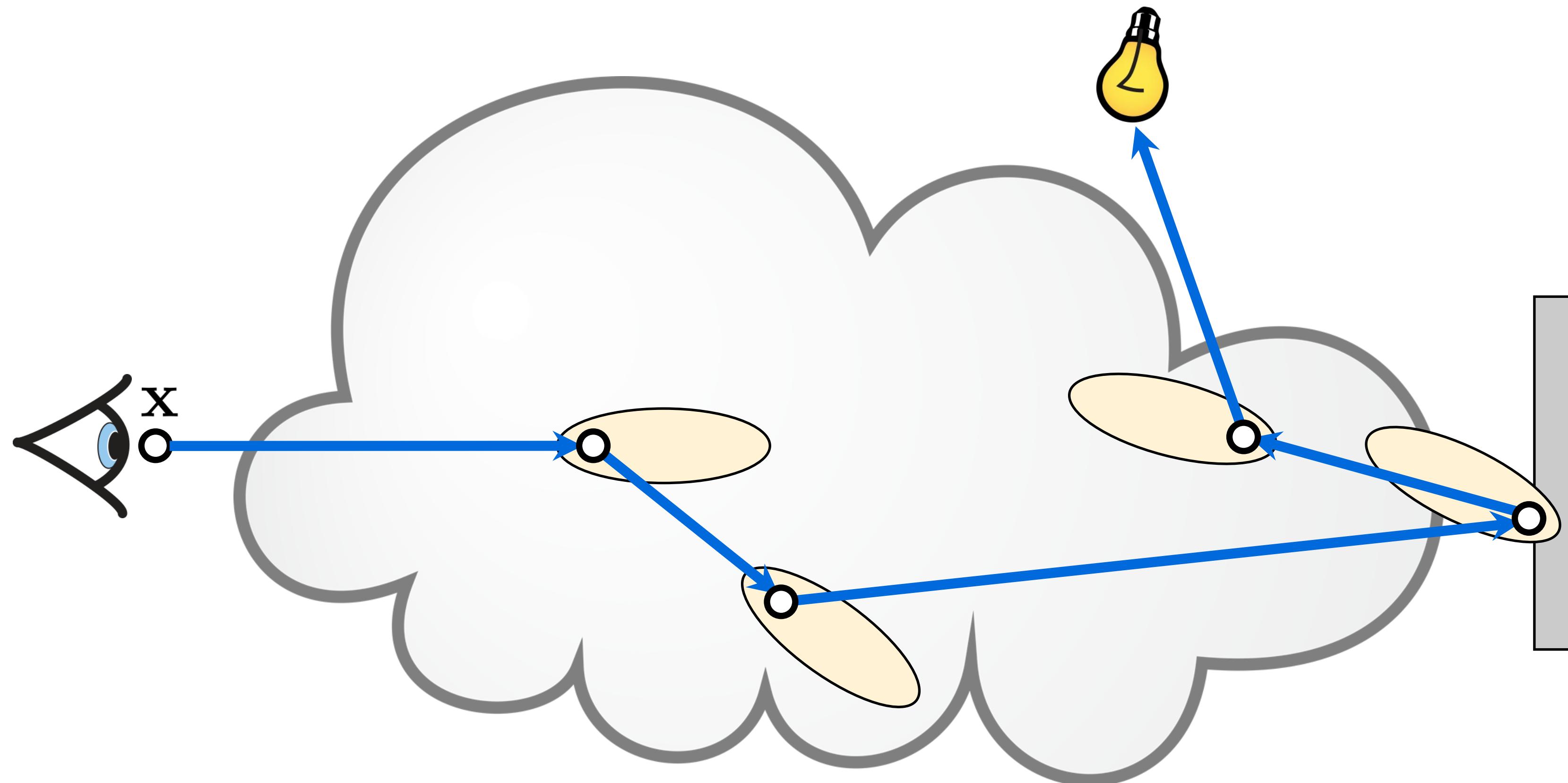
$p(t)$ - probability *density* of distance t

$P(z)$ - *probability* of exceeding distance z

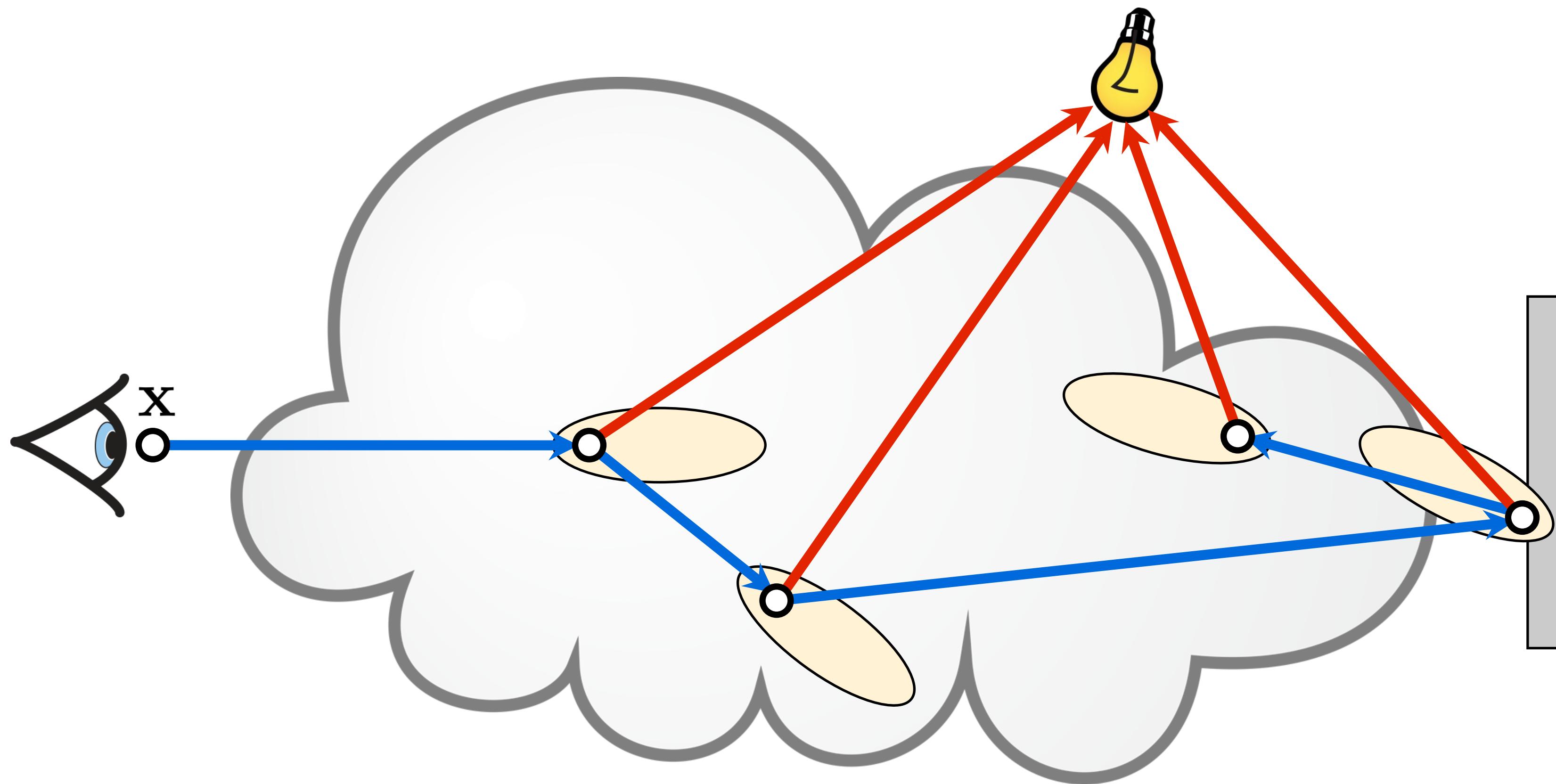
$p(\vec{\omega}_i)$ - probability *density* of direction $\vec{\omega}_i$

Volumetric Path Tracing

1. Sample distance to next interaction
2. Scatter in the volume or bounce off a surface



Volumetric Path Tracing with NEE



Sampling the Phase Function

Isotropic:

- Uniform sphere sampling

Henyey-Greenstein:

- Using the inversion method we can derive

$$\cos \theta = \frac{1}{2g} \left(1 + g^2 - \left(\frac{1 - g^2}{1 - g + 2g\xi_1} \right)^2 \right)$$

$$\phi = 2\pi\xi_2$$

- PDF is the value of the HG phase function

Free-path Sampling

Free-path (or free-flight distance):

- Distance to the next interaction within the medium
- Dense media (e.g. milk): short mean-free path
- Thin media (e.g. atmosphere): long mean-free path

Ideally, we want to sample proportional to (part of) integrand,
e.g. transmittance:

$$p(\mathbf{x}_t | (\mathbf{x}, \vec{\omega})) \propto T_r(\mathbf{x}, \mathbf{x}_t)$$

$p(t) \propto T_r(t)$

simplified notation for brevity

Free-path Sampling

Homogeneous media:

$$T_r(t) = e^{-\sigma_t t}$$

- PDF: $p(t) \propto e^{-\sigma_t t}$

$$p(t) = \frac{e^{-\sigma_t t}}{\int_0^\infty e^{-\sigma_t s} ds} = \sigma_t e^{-\sigma_t t}$$

- CDF: $P(t) = \int_0^t \sigma_t e^{-\sigma_t s} ds = 1 - e^{-\sigma_t t}$

- Inverted CDF:

$$P^{-1}(\xi) = -\frac{\ln(1 - \xi)}{\sigma_t}$$

Free-path Sampling

Homogeneous media:

$$T_r(t) = e^{-\sigma_t t}$$

Recipe:

- Generate random number

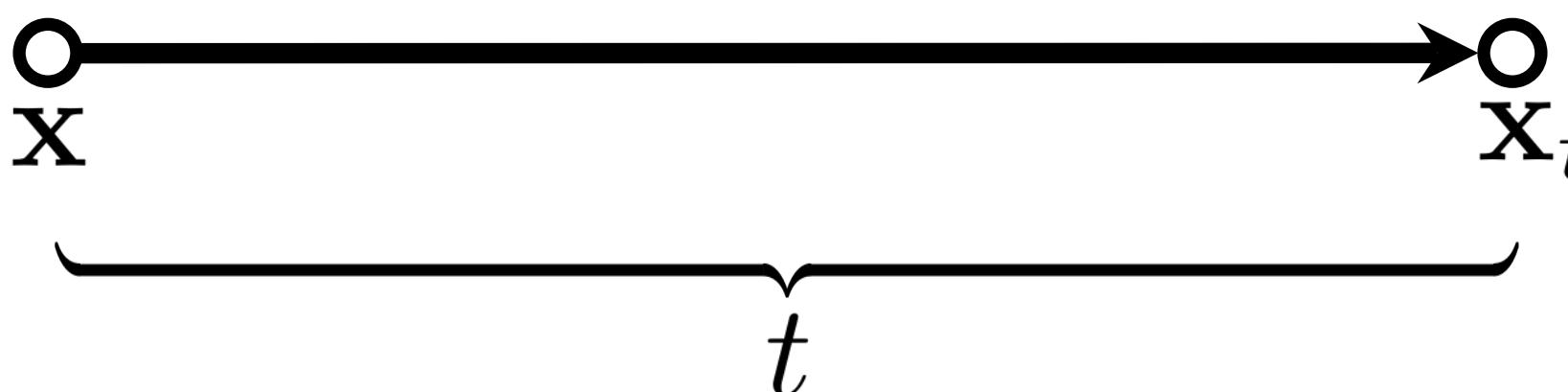
$$\xi$$

- Sample distance

$$t = -\frac{\ln(1 - \xi)}{\sigma_t}$$

- Compute PDF

$$p(t) = \sigma_t e^{-\sigma_t t}$$



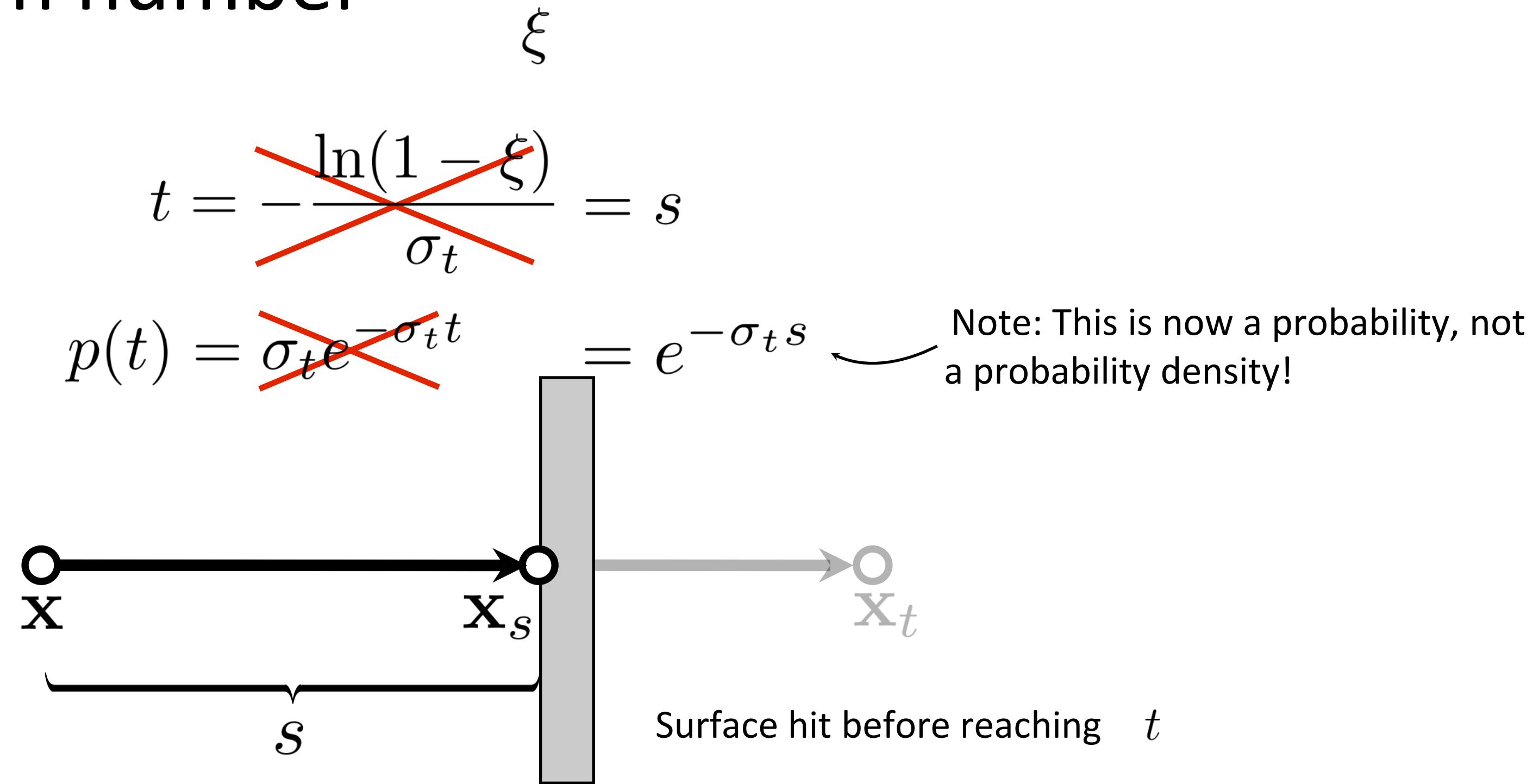
Free-path Sampling

Homogeneous media:

$$T_r(t) = e^{-\sigma_t t}$$

Recipe:

- Generate random number
- Sample distance
- Compute PDF



Volumetric PT for Homogeneous Volumes

```
Color vPT(x,  $\omega$ )
tmax = nearestSurface(x,  $\omega$ )
t = -log(1 - randf()) /  $\sigma_t$  // Sample free path
if t < tmax: // Volume interaction
    x += t *  $\omega$ 
    pdf_t =  $\sigma_t$  * exp(- $\sigma_t$  * t)
    ( $\omega'$ , pdf_omega') = samplePF( $\omega$ )
    return Tr(t) / pdf_t * ( $\sigma_a$  * Le(x,  $\omega$ ) +  $\sigma_s$  * PF( $\omega$ ,  $\omega'$ ) * vPT(x,  $\omega'$ ) / pdf_omega')
else: // Surface interaction
    x += tmax *  $\omega$ 
    Pr_tmax = exp(- $\sigma_t$  * tmax)
    ( $\omega'$ , pdf_omega') = sampleBRDF(n,  $\omega$ )
    return Tr(tmax) / Pr_tmax * (Le(x,  $\omega$ ) + BRDF( $\omega$ ,  $\omega'$ ) * vPT(x,  $\omega'$ ) / pdf_omega')
```

$$\langle L(\mathbf{x}, \vec{\omega}) \rangle = \frac{T_r(\mathbf{x}, \mathbf{x}_t)}{p(t)} \left[\sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) + \sigma_s(\mathbf{x}_t) \frac{f_p(\vec{\omega}, \vec{\omega}_i) L(\mathbf{x}_t, \vec{\omega}_i)}{p(\vec{\omega}_i)} \right] + \frac{T_r(\mathbf{x}, \mathbf{x}_z)}{P(z)} L(\mathbf{x}_z, \vec{\omega})$$

Volumetric PT for Homogeneous Volumes

```
Color vPT( $\mathbf{x}$ ,  $\omega$ )
tmax = nearestSurface( $\mathbf{x}$ ,  $\omega$ )
t = -log(1 - randf()) /  $\sigma_t$  // Sample free path
if t < tmax: // Volume interaction
     $\mathbf{x}$  += t *  $\omega$ 
    pdf_t =  $\sigma_t$  * exp(- $\sigma_t$  * t)
    ( $\omega'$ , pdf_omega') = samplePF( $\omega$ )
    // Note: transmittance and PF cancel out with PDFs except for a constant factor 1/ $\sigma_t$ 
    return Tr(t) / pdf_t * ( $\sigma_a$  * L_e( $\mathbf{x}$ ,  $\omega$ ) +  $\sigma_s$  * PF( $\omega$ ,  $\omega'$ ) * vPT( $\mathbf{x}$ ,  $\omega'$ ) / pdf_omega')
else: // Surface interaction
     $\mathbf{x}$  += tmax *  $\omega$ 
    Pr_tmax = exp(- $\sigma_t$  * tmax)
    ( $\omega'$ , pdf_omega') = sampleBRDF( $\mathbf{n}$ ,  $\omega$ )
    // Note: transmittance and prob of sampling the distance cancel out
    return Tr(tmax) / Pr_tmax * (L_e( $\mathbf{x}$ ,  $\omega$ ) + BRDF( $\omega$ ,  $\omega'$ ) * vPT( $\mathbf{x}$ ,  $\omega'$ ) / pdf_omega')
```

$$\langle L(\mathbf{x}, \vec{\omega}) \rangle = \frac{T_r(\mathbf{x}, \mathbf{x}_t)}{p(t)} \left[\sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) + \sigma_s(\mathbf{x}_t) \frac{f_p(\vec{\omega}, \vec{\omega}_i) L(\mathbf{x}_t, \vec{\omega}_i)}{p(\vec{\omega}_i)} \right] + \frac{T_r(\mathbf{x}, \mathbf{x}_z)}{P(z)} L(\mathbf{x}_z, \vec{\omega})$$

Volumetric PT for Homogeneous Volumes

```
Color vPT(x,  $\omega$ )
tmax = nearestSurface(x,  $\omega$ )
t = -log(1 - randf()) /  $\sigma_t$  // Sample free path
if t < tmax: // Volume interaction
    x += t *  $\omega$ 
    pdf_t =  $\sigma_t$  * exp(- $\sigma_t$  * t)
    ( $\omega'$ , pdf_omega') = samplePF( $\omega$ )
    // Note: transmittance and PF cancel out with PDFs except for a constant factor 1/ $\sigma_t$ 
    return  $\sigma_a/\sigma_t$  *  $L_e(\mathbf{x}, \omega)$  +  $\sigma_s/\sigma_t$  * vPT(x,  $\omega'$ )
else: // Surface interaction
    x += tmax *  $\omega$ 
    Pr_tmax = exp(- $\sigma_t$  * tmax)
    ( $\omega'$ , pdf_omega') = sampleBRDF(n,  $\omega$ )
    // Note: transmittance and prob of sampling the distance cancel out
    return  $L_e(\mathbf{x}, \omega)$  + BRDF( $\omega$ ,  $\omega'$ ) * vPT(x,  $\omega'$ ) / pdf_omega'
```

$$\langle L(\mathbf{x}, \vec{\omega}) \rangle = \frac{T_r(\mathbf{x}, \mathbf{x}_t)}{p(t)} \left[\sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) + \sigma_s(\mathbf{x}_t) \frac{f_p(\vec{\omega}, \vec{\omega}_i) L(\mathbf{x}_t, \vec{\omega}_i)}{p(\vec{\omega}_i)} \right] + \frac{T_r(\mathbf{x}, \mathbf{x}_z)}{P(z)} L(\mathbf{x}_z, \vec{\omega})$$

What about heterogeneous media?

Free-path Sampling

Heterogeneous media: $T_r(t) = e^{\int_0^t -\sigma_t(s)ds}$

- Closed-form solutions exist only for simple media
 - e.g. linearly or exponentially varying extinction
- Other solutions:
 - Regular tracking (3D DDA)
 - Ray marching
 - Delta tracking

Free-path Sampling

How to sample the flight distance to the next interaction?

Random variable representing flight distance

$$T(t) = e^{- \int_0^t \sigma_t(s) ds} = \boxed{P(X > t) \quad P(X \leq t) = F(t)}$$

CDF

Partition of unity

$$F(t) = 1 - T(t)$$

Recipe for generating samples

Free-path Sampling

Cumulative distribution function (**CDF**)

$$F(t) = 1 - T(t) = 1 - e^{-\tau(t)}$$

Probability density function (**PDF**)

$$p(t) = \frac{dF(t)}{dt} = \frac{d}{dt} \left(1 - e^{-\tau(t)} \right) = \sigma_t(t) e^{-\tau(t)}$$

Inverted cumulative distr. function (**CDF⁻¹**)

$$\xi = 1 - e^{-\tau(t)}$$

Solve for t

$$\int_0^t \sigma_t(s) \, ds = -\ln(1 - \xi)$$

Approaches for finding t:

- 1) **ANALYTIC** (closed-form CDF⁻¹)
- 2) **SEMI-ANALYTIC** (regular tracking)
- 3) **APPROXIMATE** (ray marching)

Regular Tracking (Semi-Analytic)

For piecewise-simple (e.g. piecewise-constant), summation replaces integration

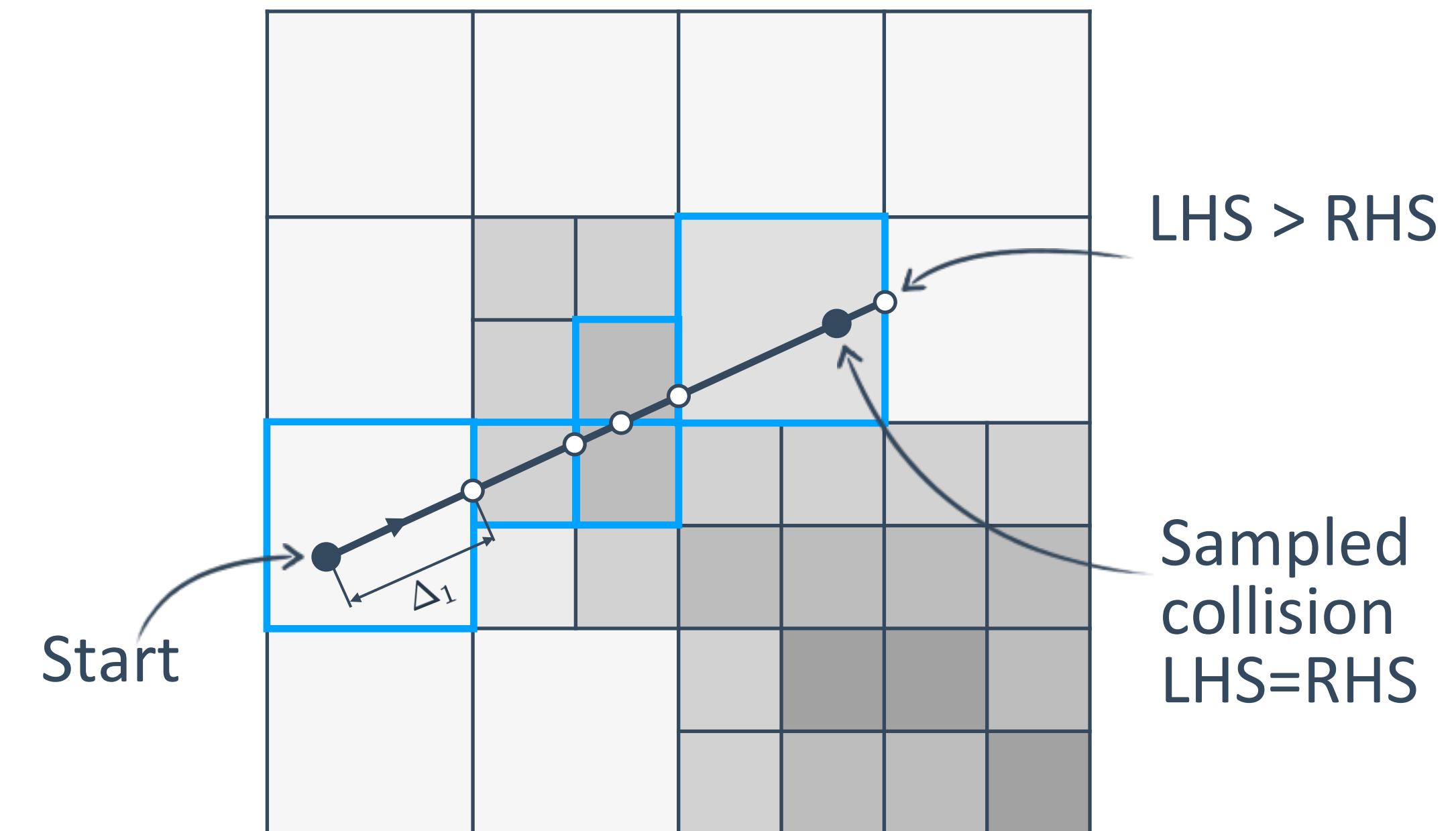
$$\int_0^t \sigma_t(s) \, ds = -\ln(1 - \xi)$$

$$\sum_{i=1}^k \sigma_{t,i} \Delta_i = -\ln(1 - \xi)$$

Regular tracking:

- 1) Draw a random number ξ
- 2) While $LHS < RHS$
move to the next intersection
- 3) Find the exact location
in the last segment analytically

(Hierarchical) voxel grid



Ray Marching

Find the collision distance approximately

$$\int_0^t \sigma_t(s) \, ds = -\ln(1 - \xi)$$

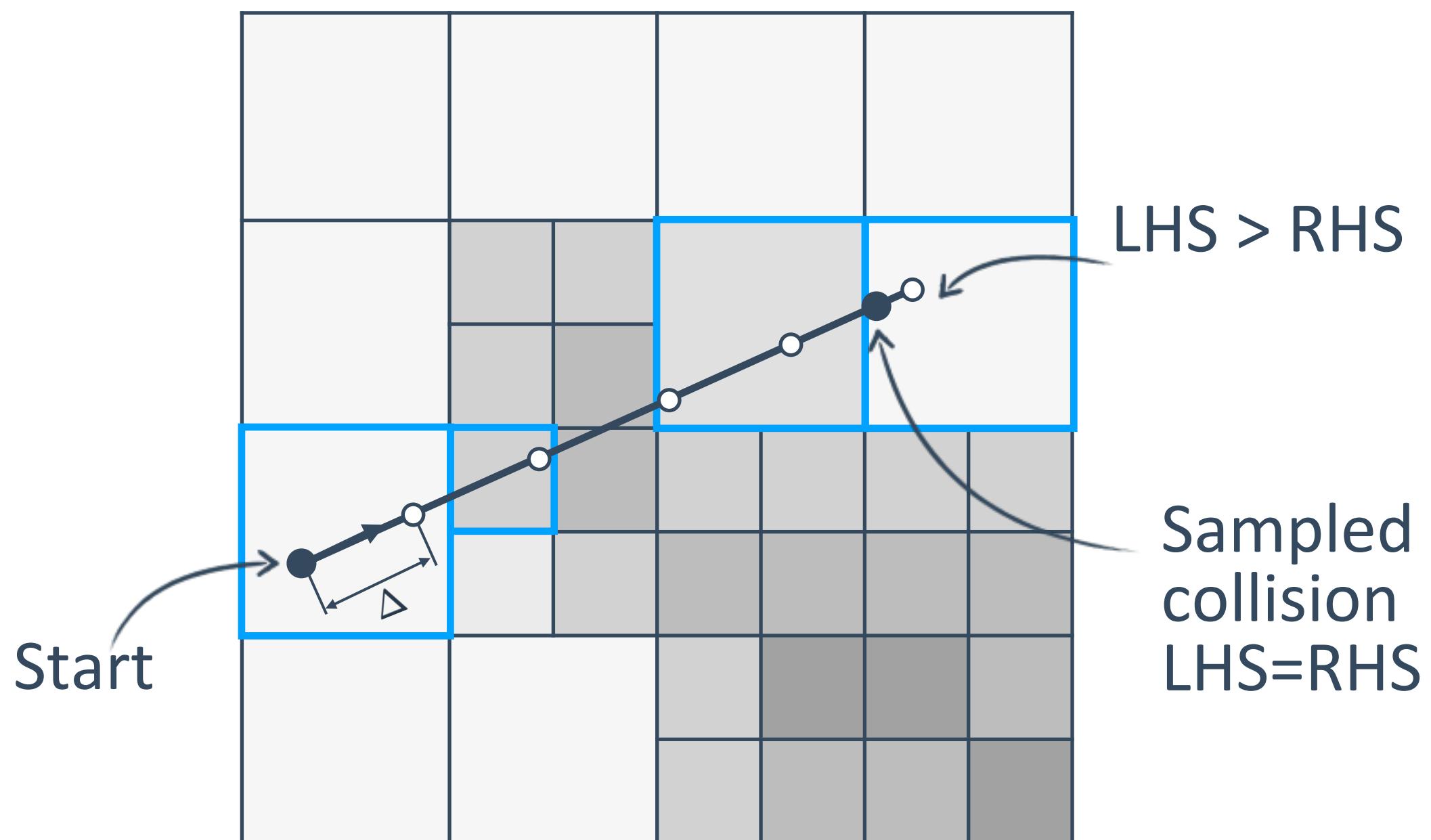
$$\sum_{i=1}^k \sigma_{t,i} \Delta = -\ln(1 - \xi)$$

Constant step

Ray marching:

- 1) Draw a random number ξ
- 2) While $\text{LHS} < \text{RHS}$
make a (fixed-size) step
- 3) Find the exact location
in the last segment analytically

(Hierarchical) voxel grid



Ray Marching

Find the collision distance approximately

$$\int_0^t \sigma_t(s) \, ds = -\ln(1 - \xi)$$

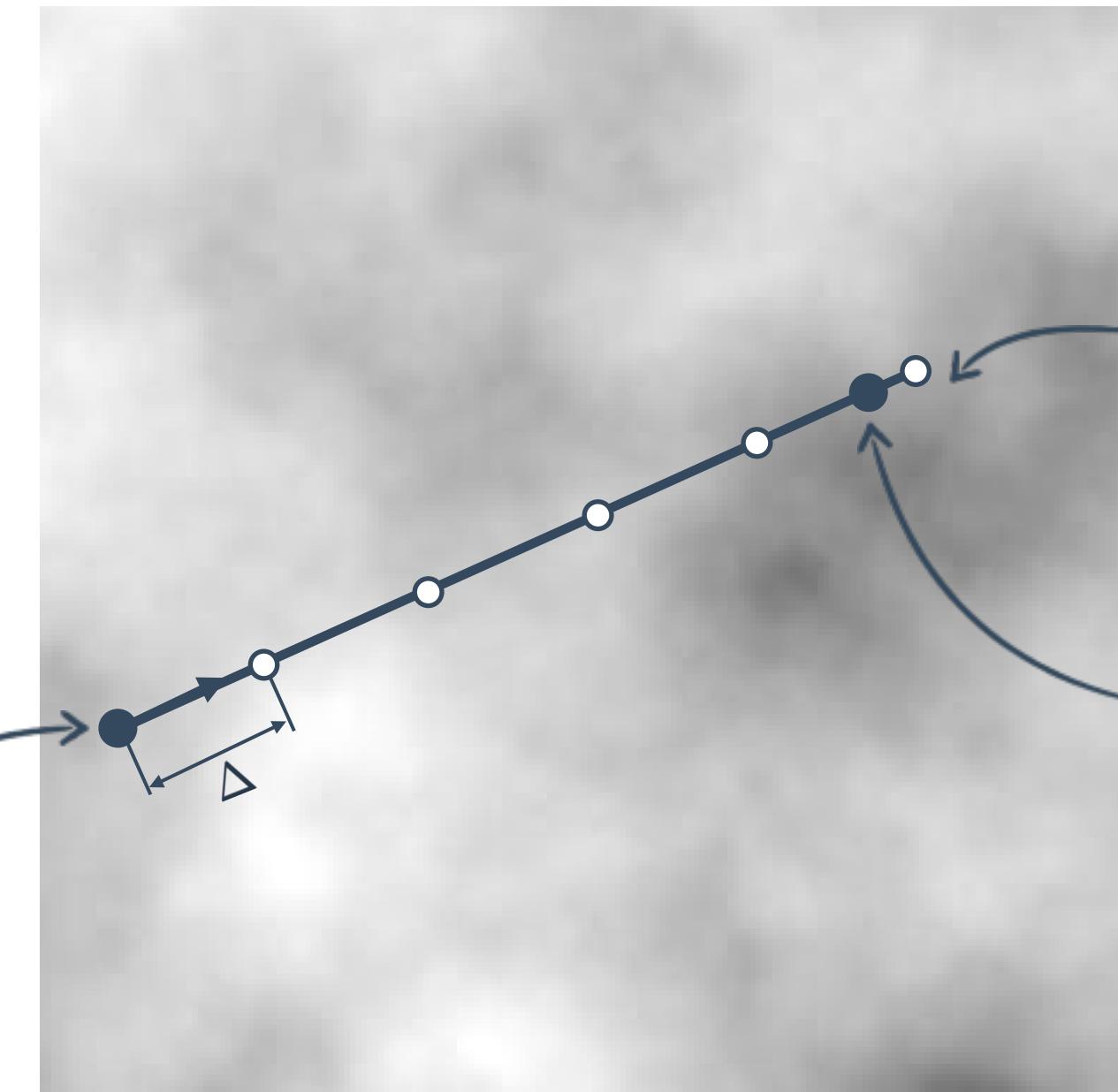
$$\sum_{i=1}^k \sigma_{t,i} \Delta = -\ln(1 - \xi)$$

Constant step

Ray marching:

- 1) Draw a random number ξ
- 2) While $\text{LHS} < \text{RHS}$
make a (fixed-size) step
- 3) Find the exact location
in the last segment analytically

General volume



$\text{LHS} > \text{RHS}$

Sampled
collision
 $\text{LHS} = \text{RHS}$

Ray Marching

Find the collision distance approximately

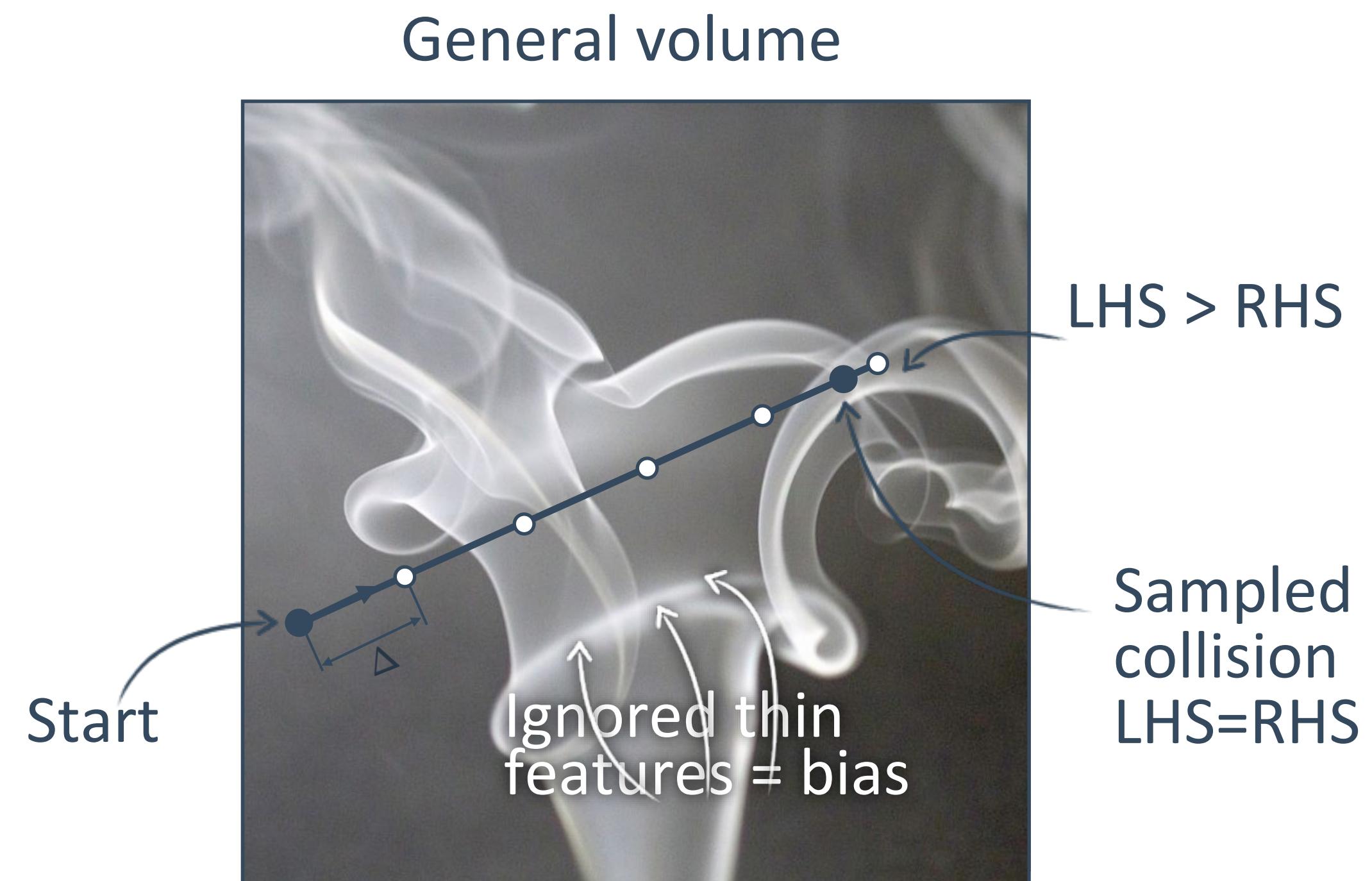
$$\int_0^t \sigma_t(s) \, ds = -\ln(1 - \xi)$$

$$\sum_{i=1}^k \sigma_{t,i} \Delta = -\ln(1 - \xi)$$

Constant step

Ray marching:

- 1) Draw a random number ξ
- 2) While $\text{LHS} < \text{RHS}$
make a (fixed-size) step
- 3) Find the exact location
in the last segment analytically



Free-path Sampling

ANALYTIC CDF-¹

- ▶ Efficient & simple, limited to few volumes
- ▶ Simple volumes (e.g. homogeneous)
- ▶ Unbiased

REGULAR TRACKING

- ▶ Iterative, inefficient if free paths cross many boundaries
- ▶ Piecewise-simple volumes
- ▶ Unbiased

RAY MARCHING

- ▶ Iterative, inaccurate (or inefficient) for media with high frequencies
- ▶ Any volume
- ▶ Biased

Common approach: sample optical thickness, find corresponding distance

Delta Tracking

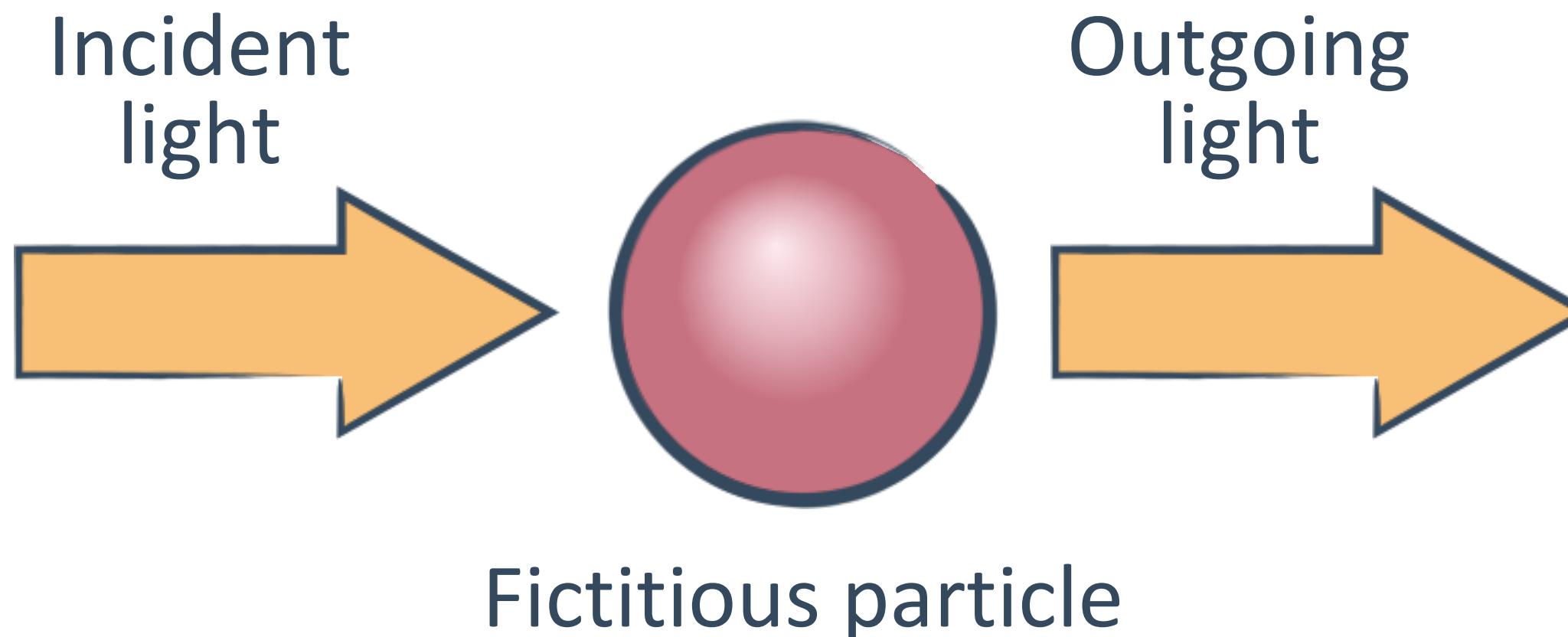
(a.k.a. Woodcock tracking, pseudo scattering, hole tracking, null-collision method,...)

Delta tracking idea

Add **FICTITIOUS MATTER** to homogenize medium

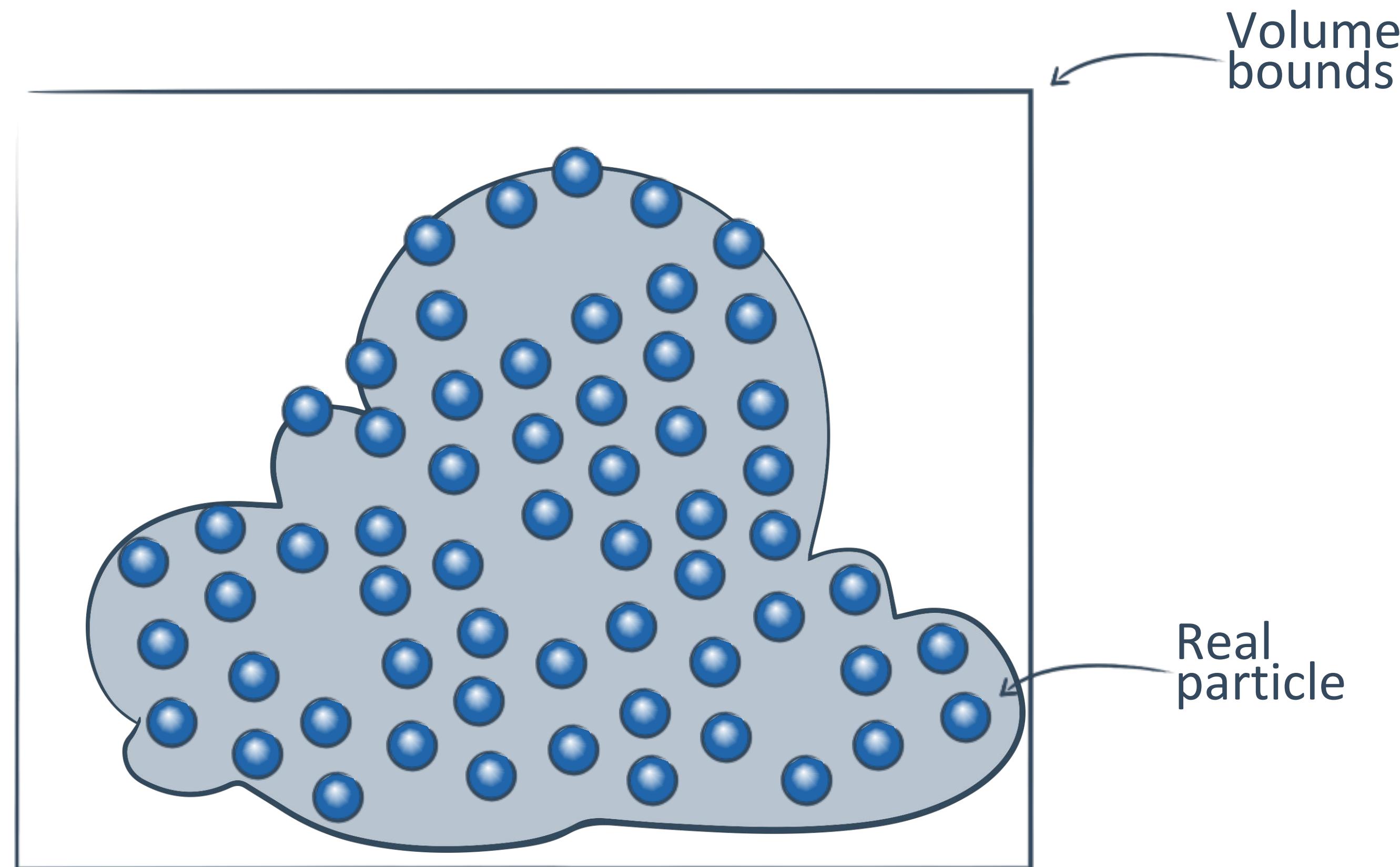
- albedo: $\alpha(\mathbf{x}) = 1$

- phase function: $f_p(\omega, \omega') = \delta(\omega - \omega')$

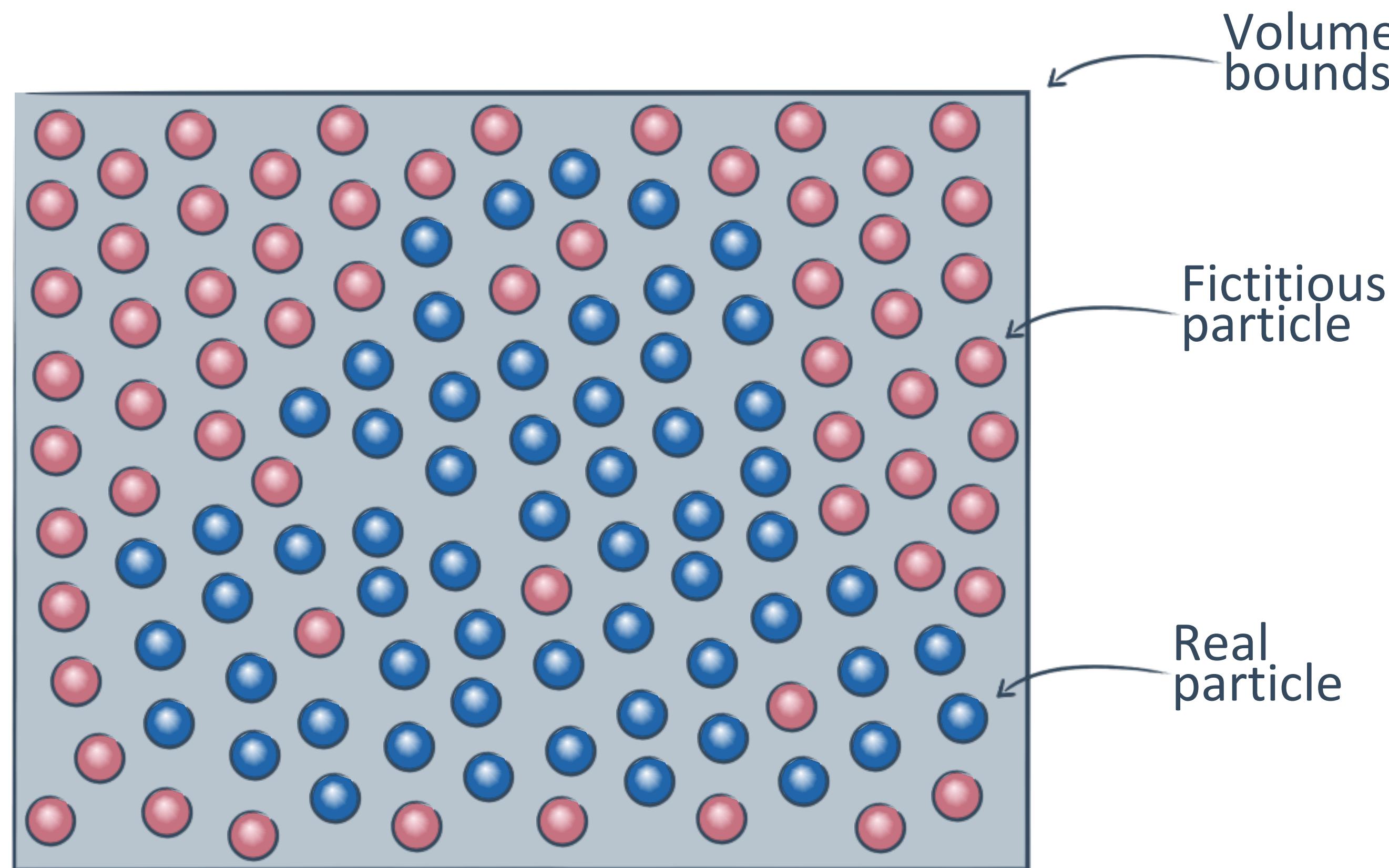


Presence of fictitious matter
does not impact light transport

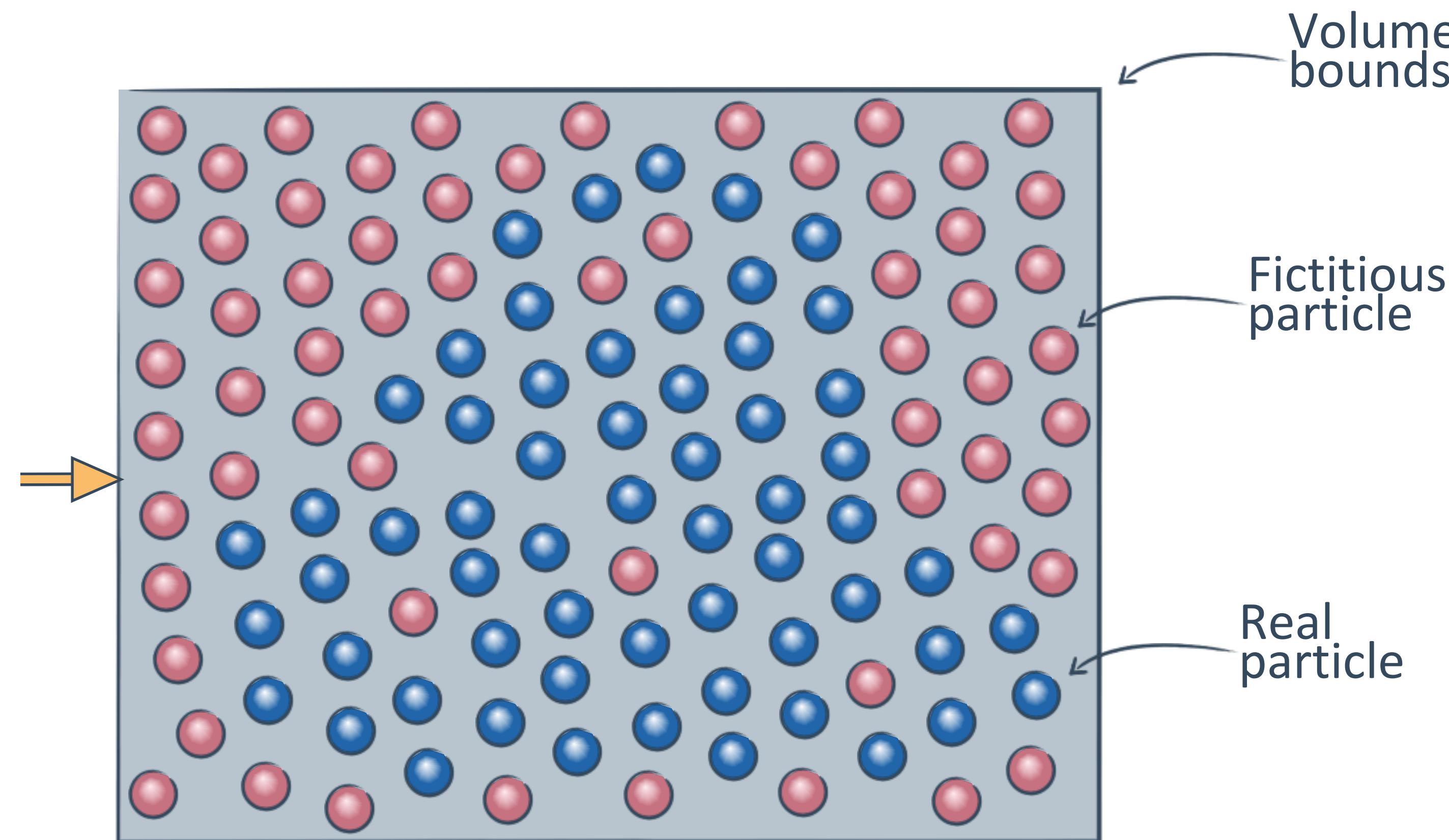
Homogenization



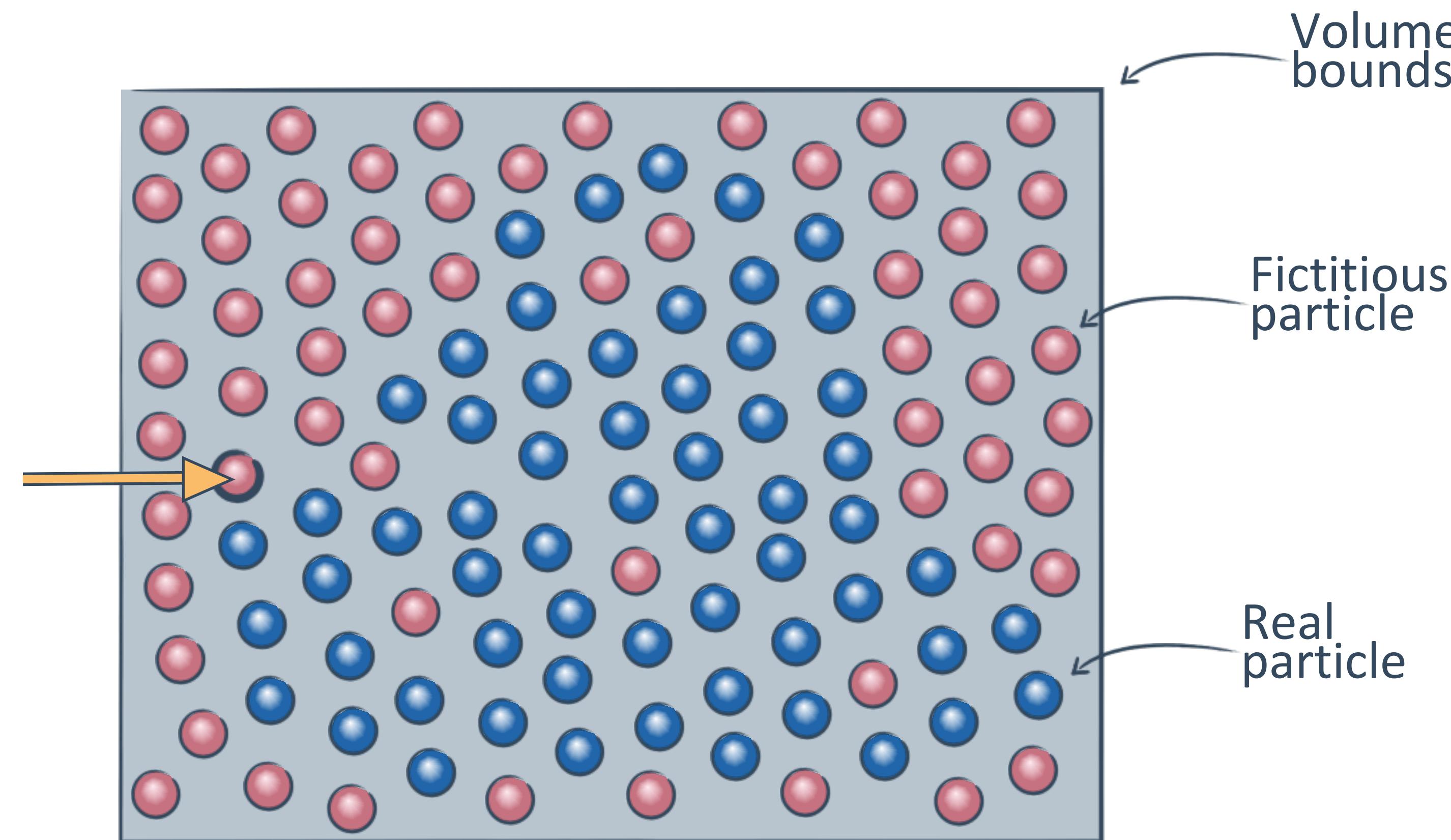
Homogenization



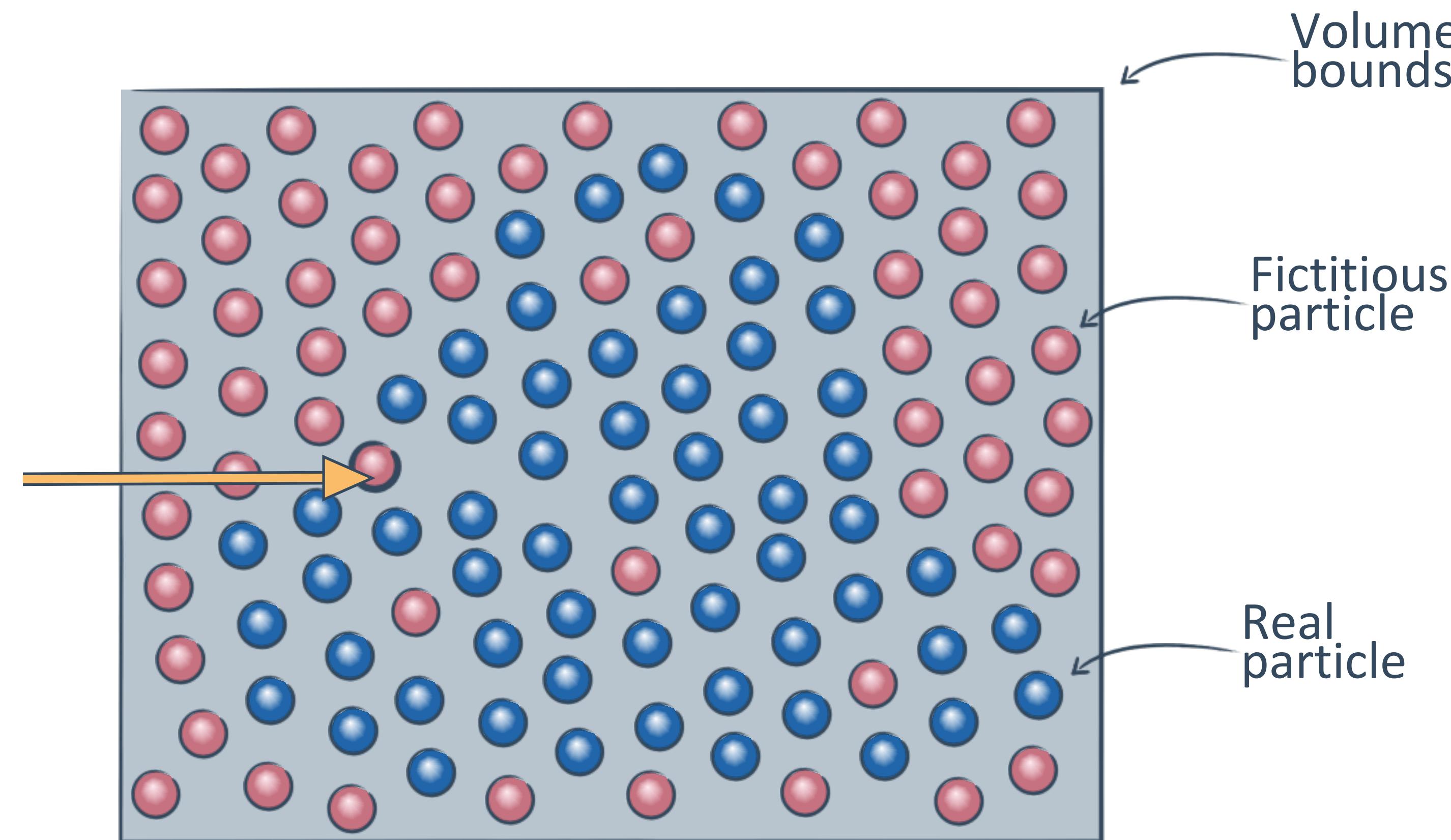
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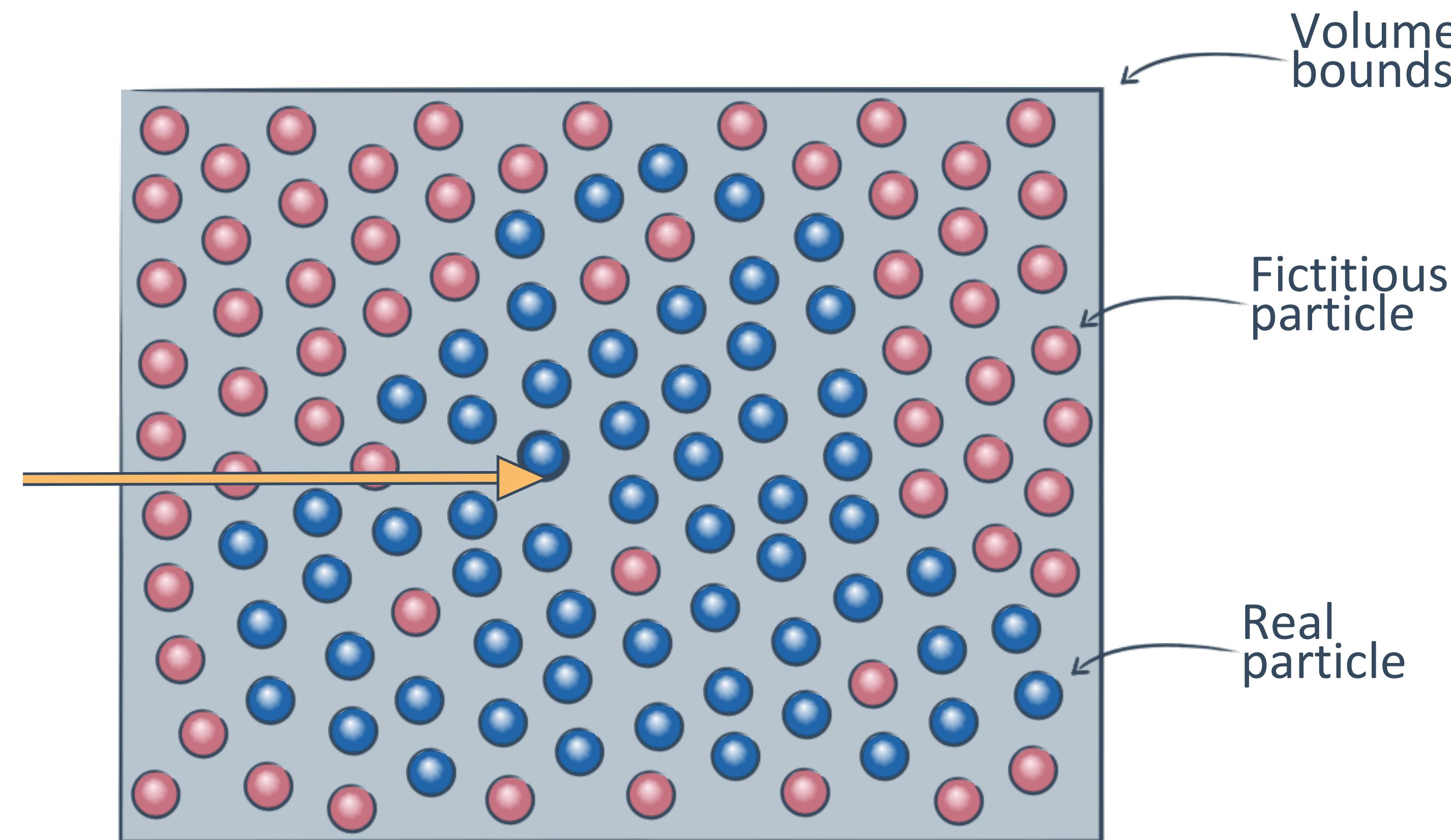
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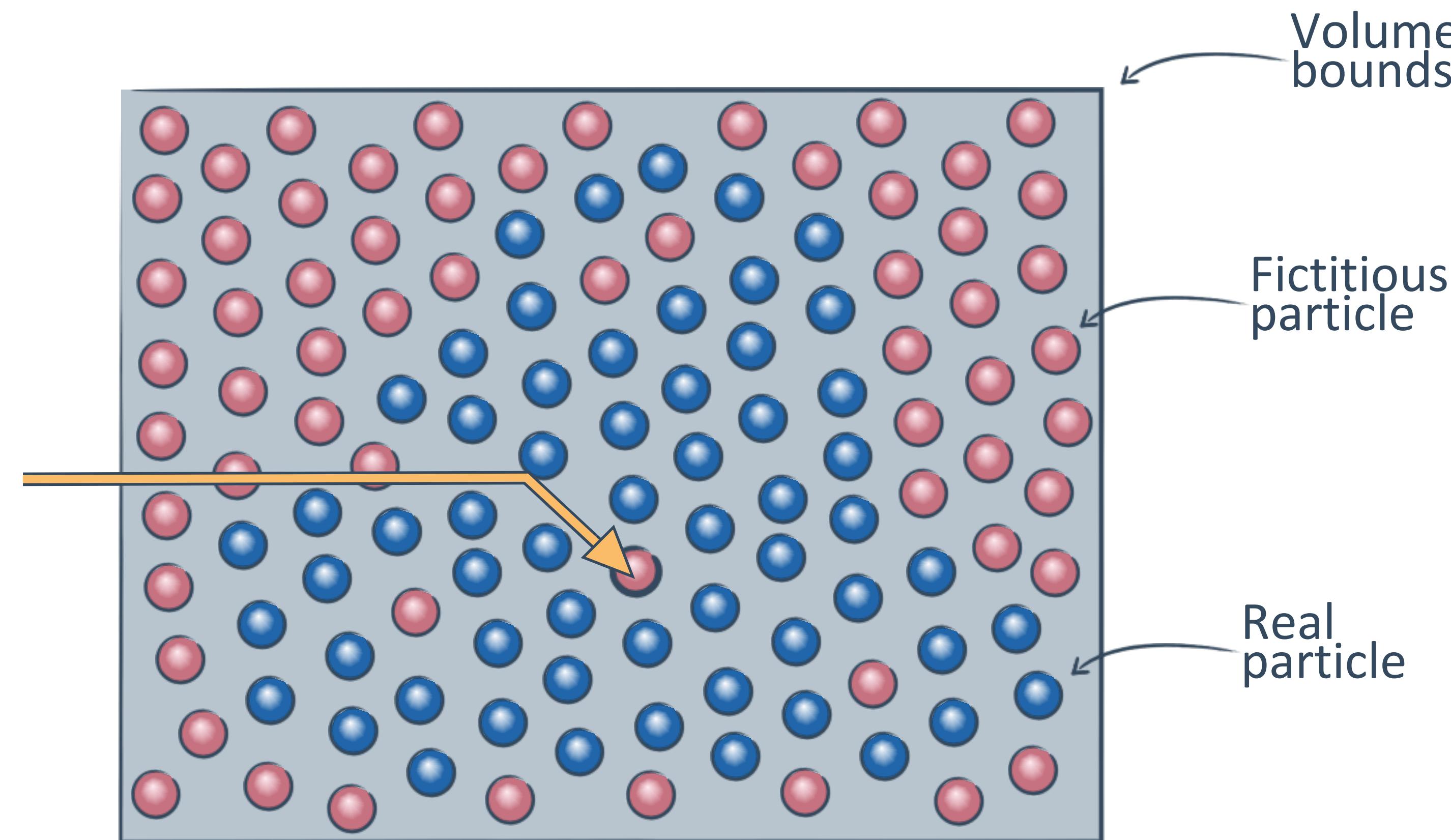
Homogenization



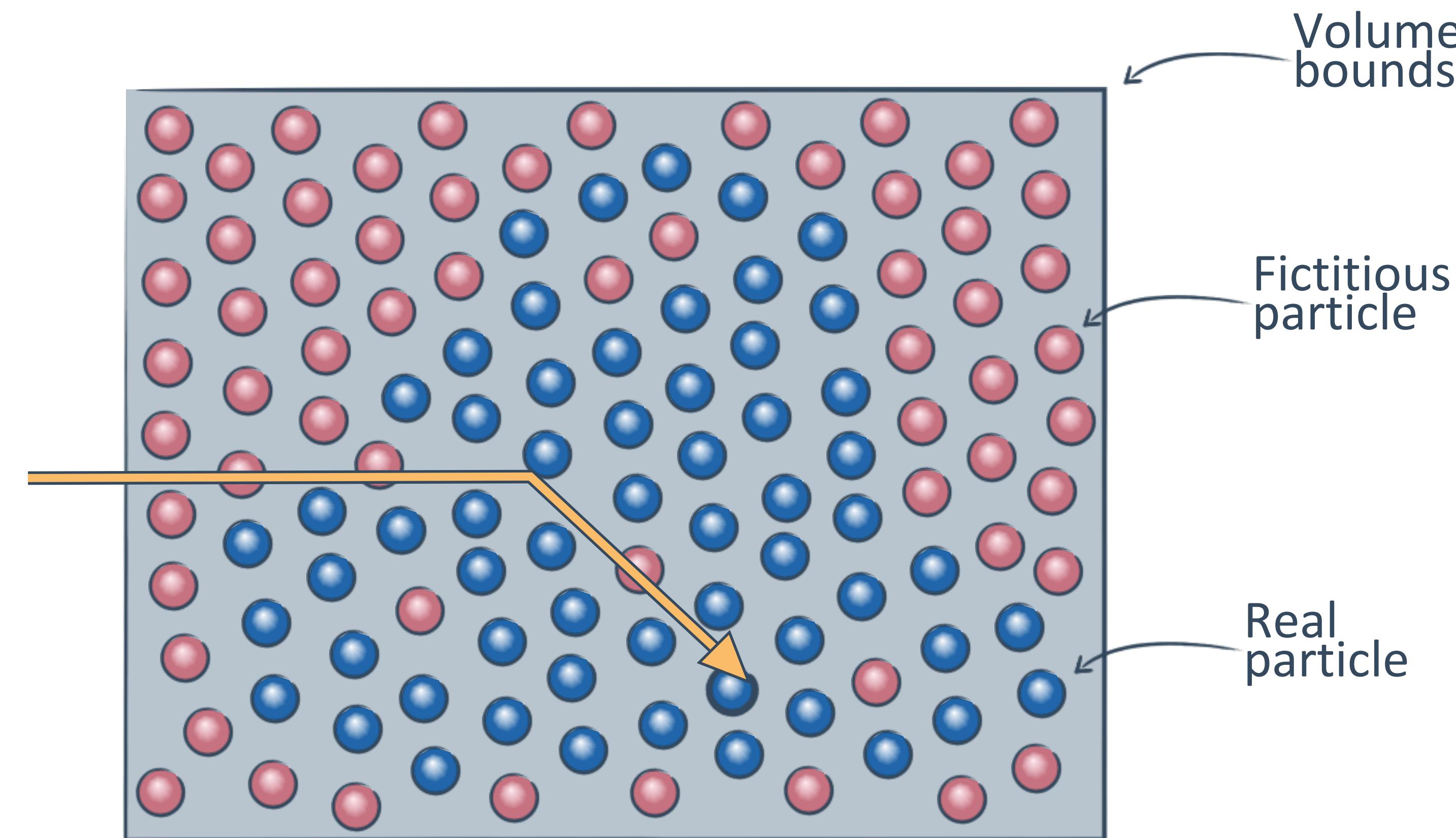
Homogenization



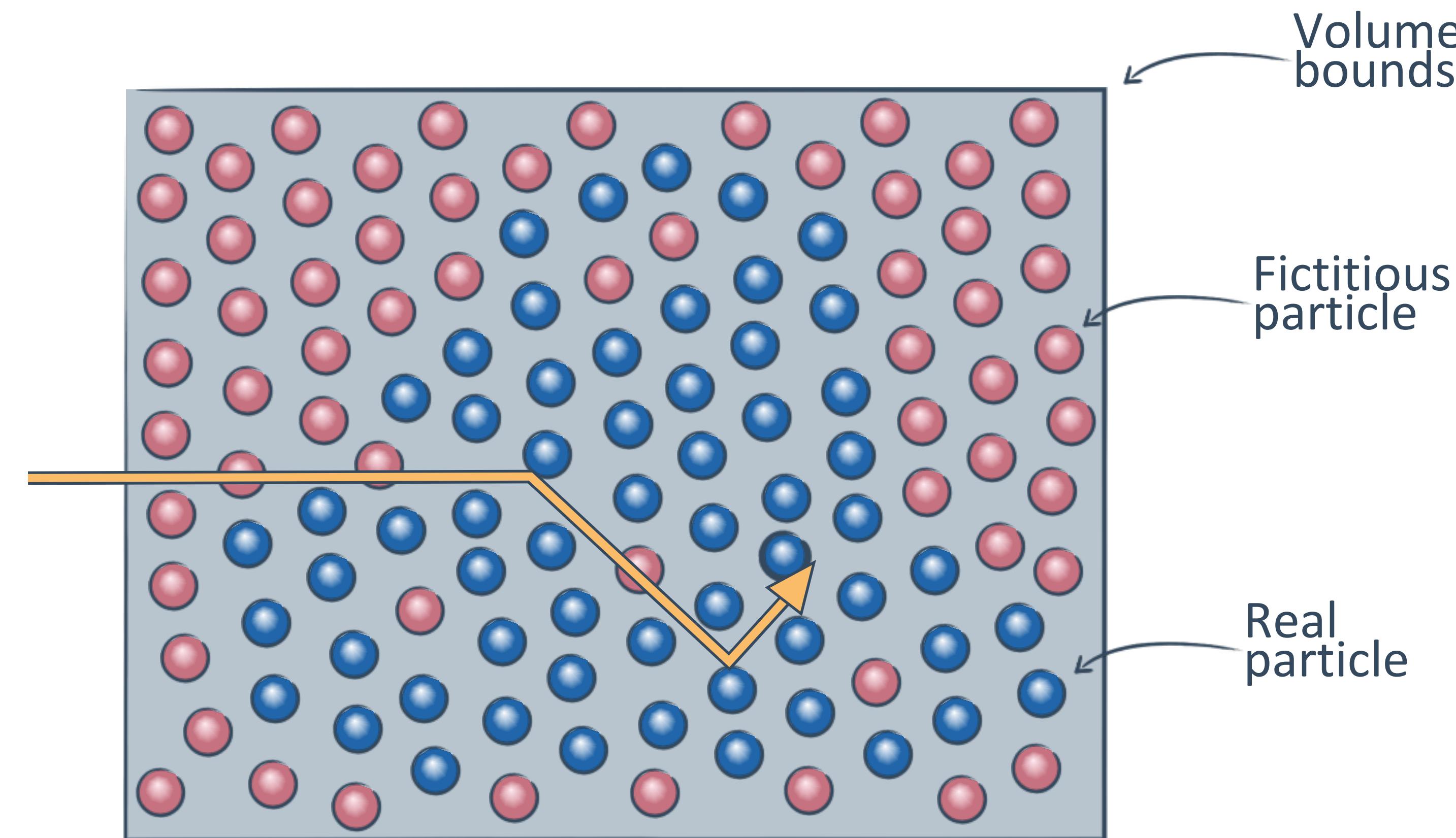
Homogenization



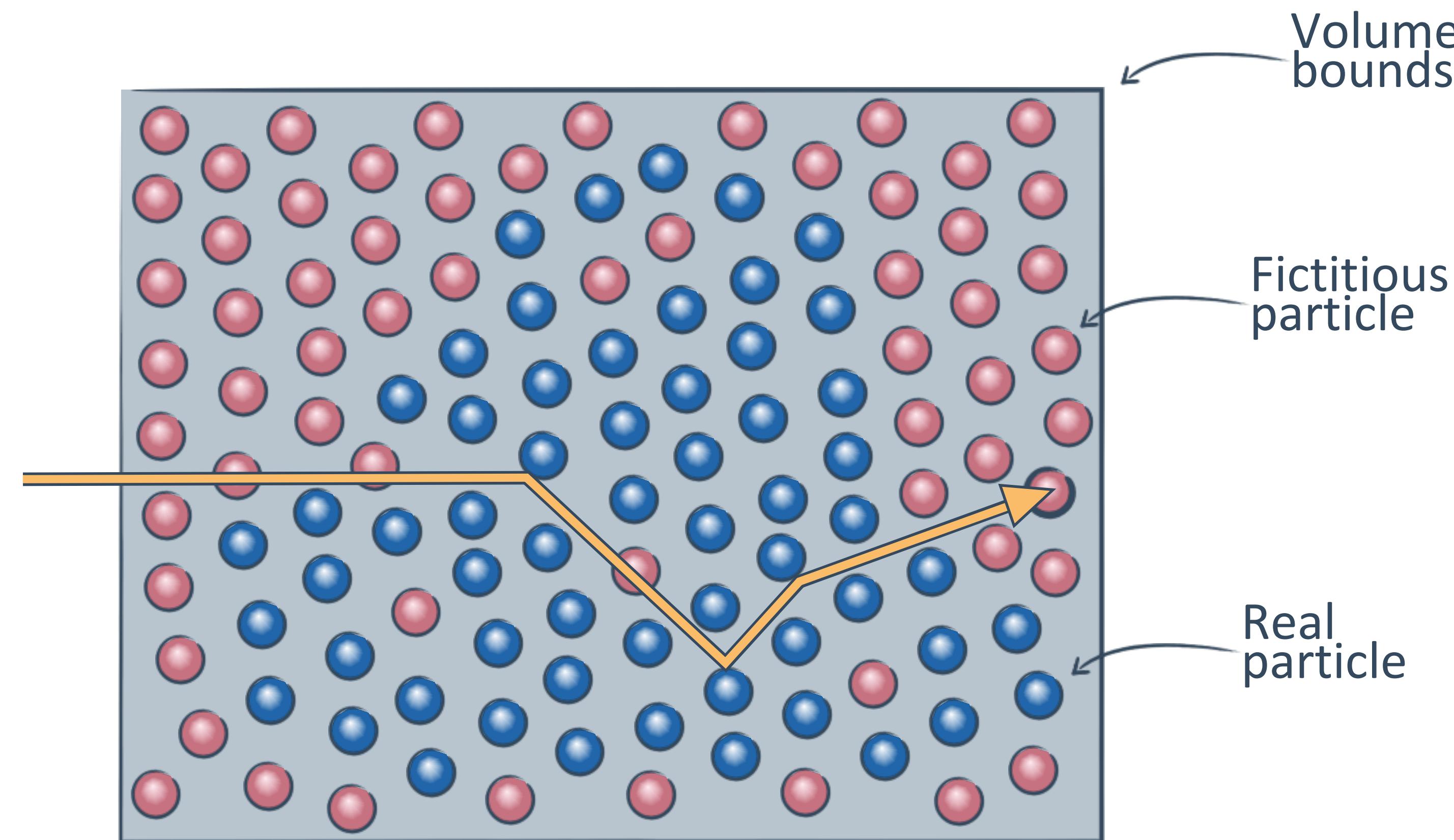
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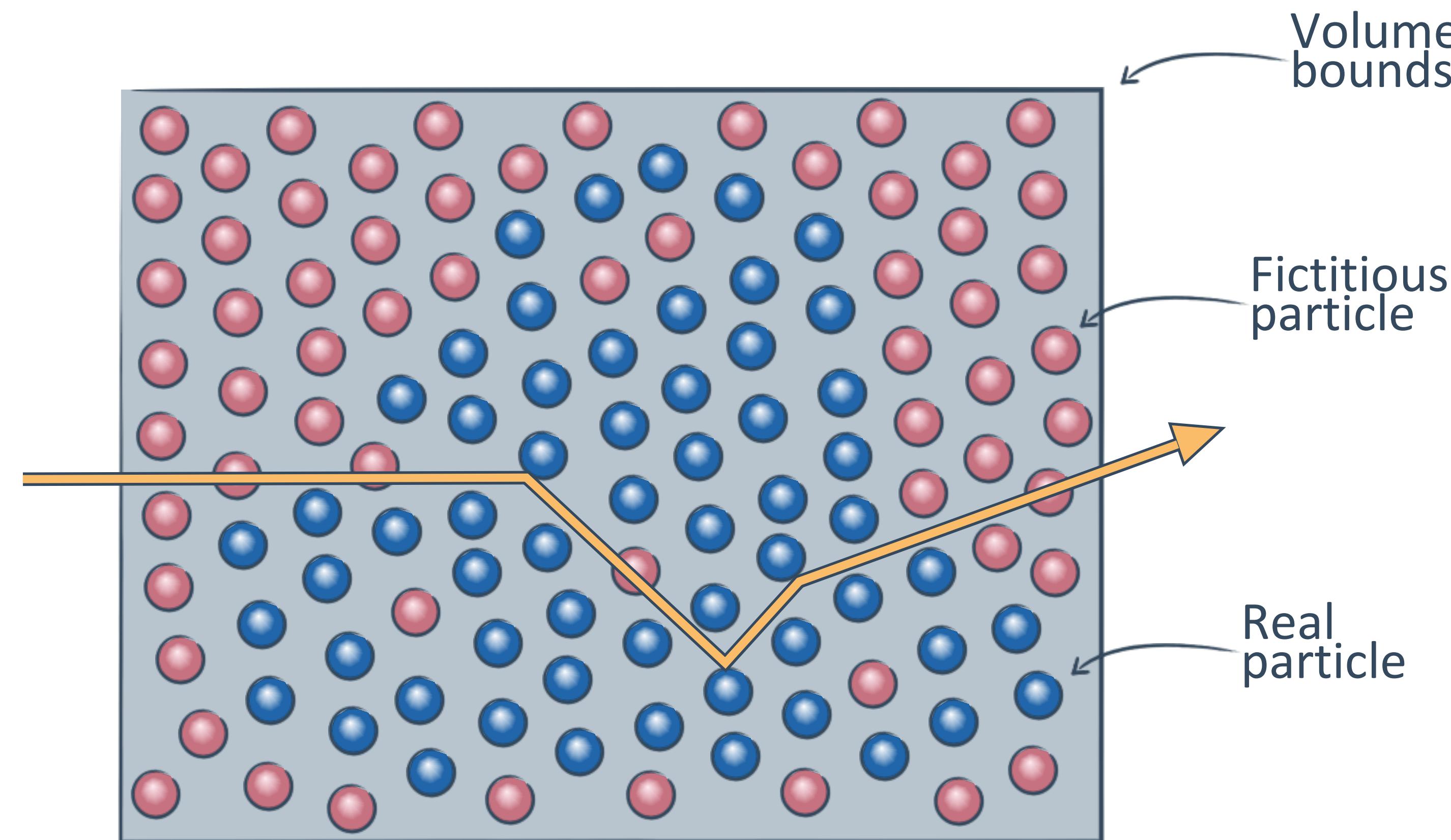
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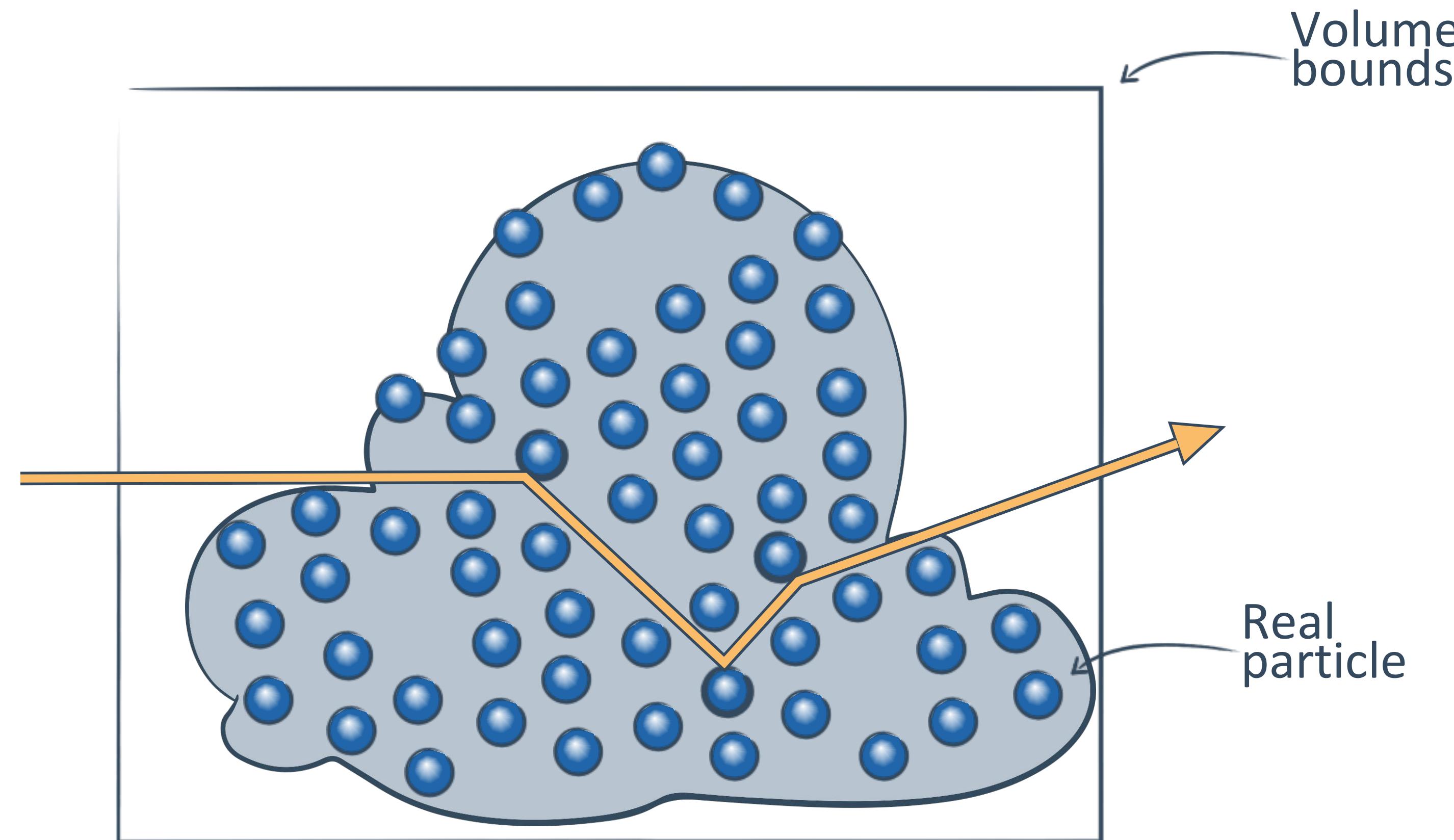
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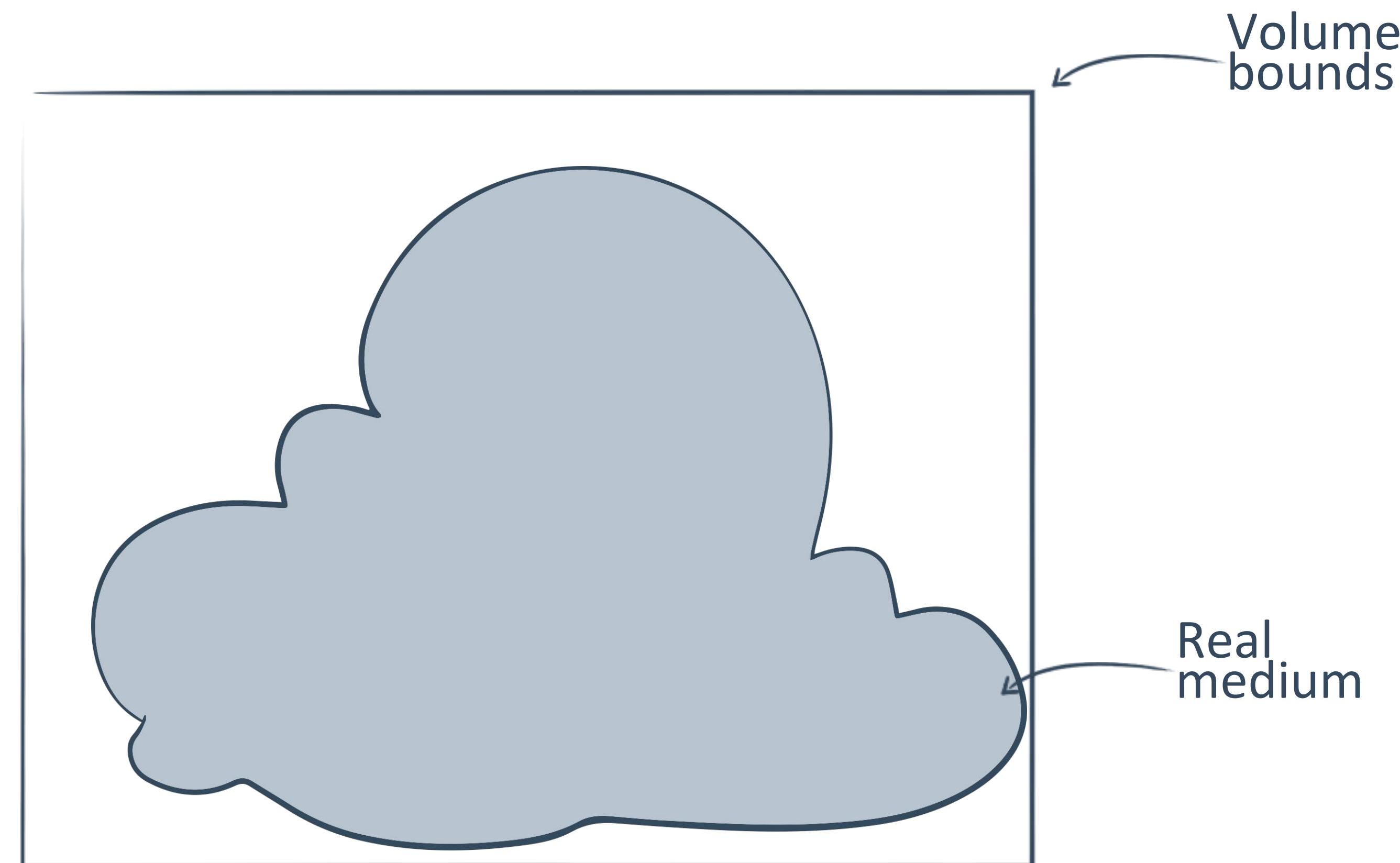
Homogenization



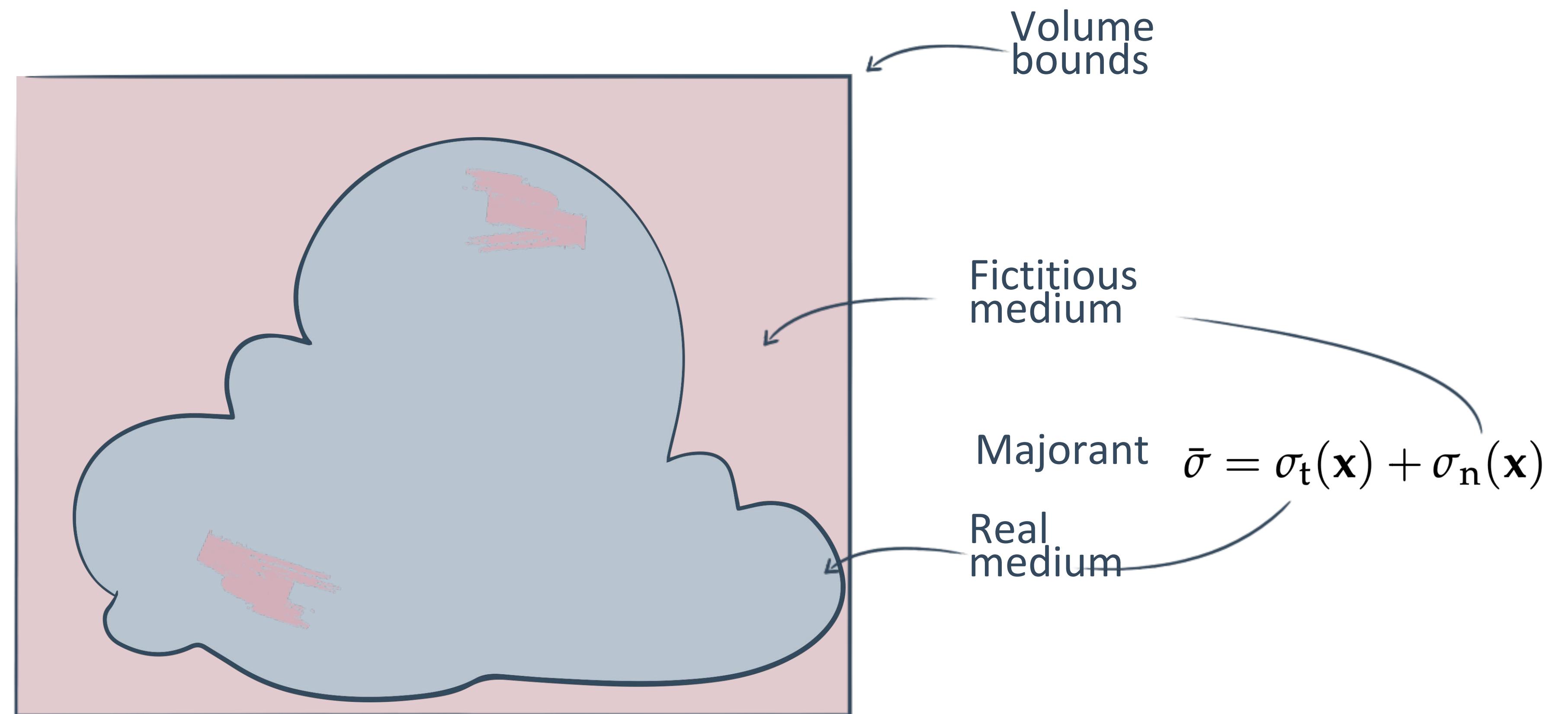
Homogenization



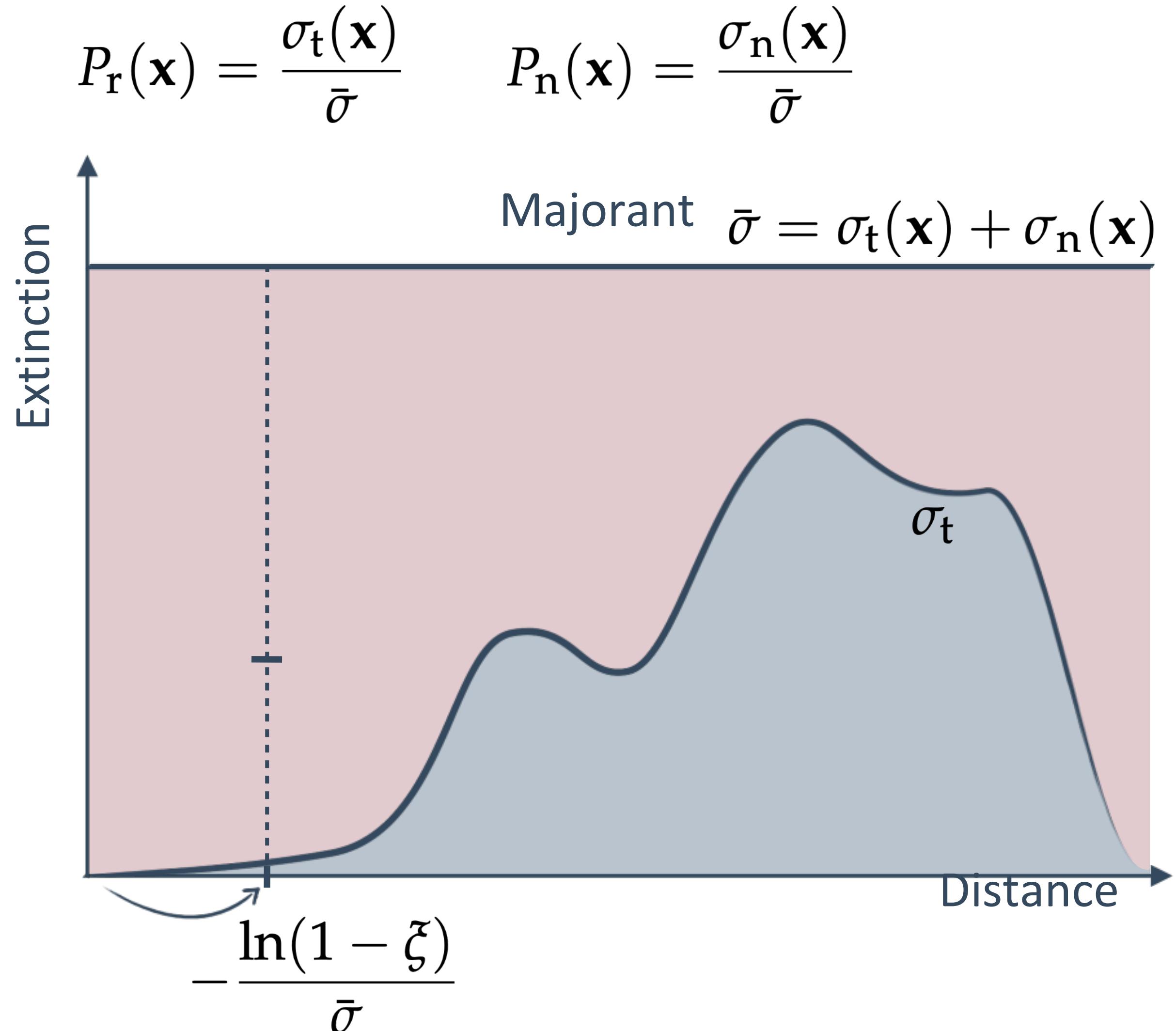
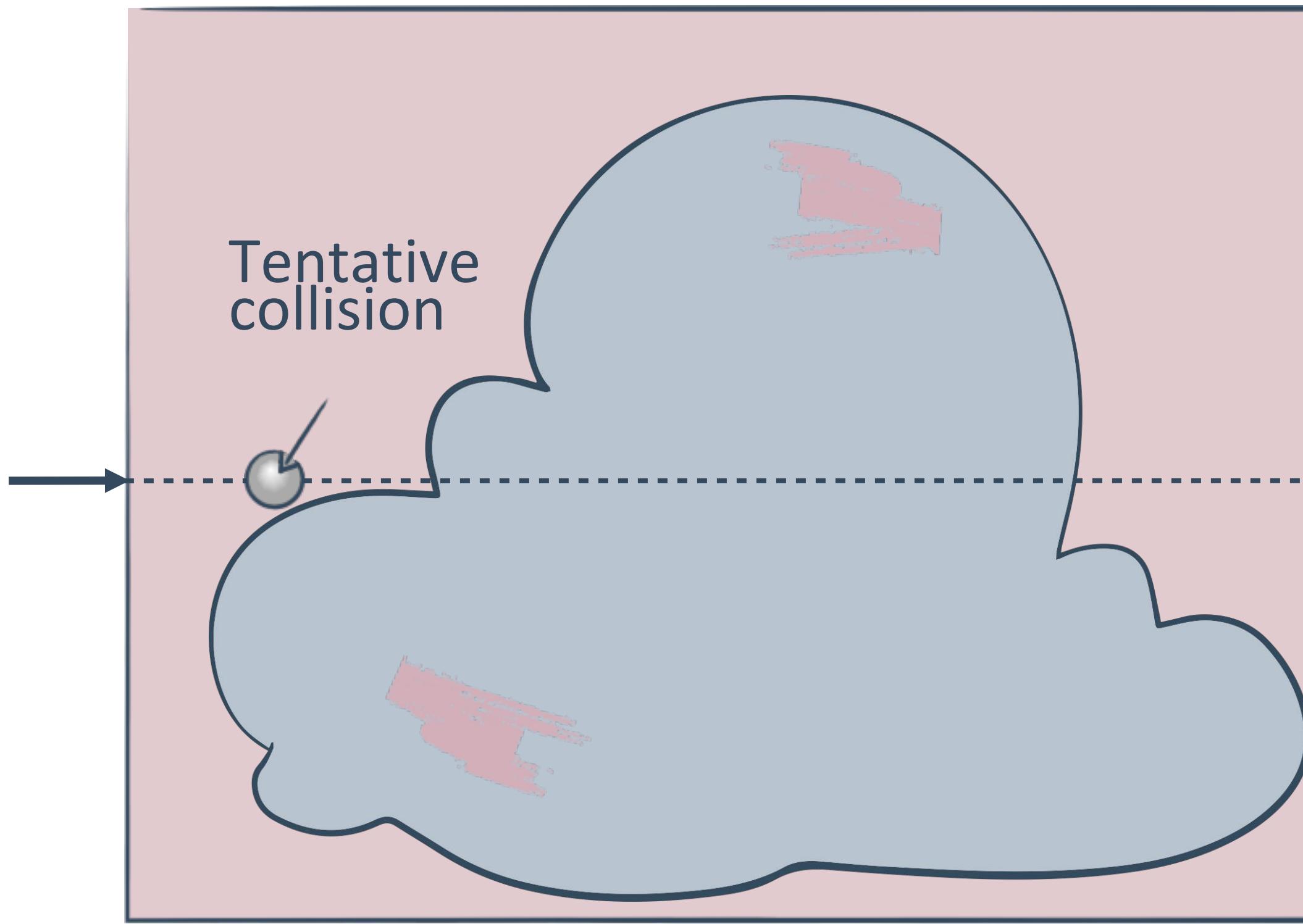
Stochastic Sampling



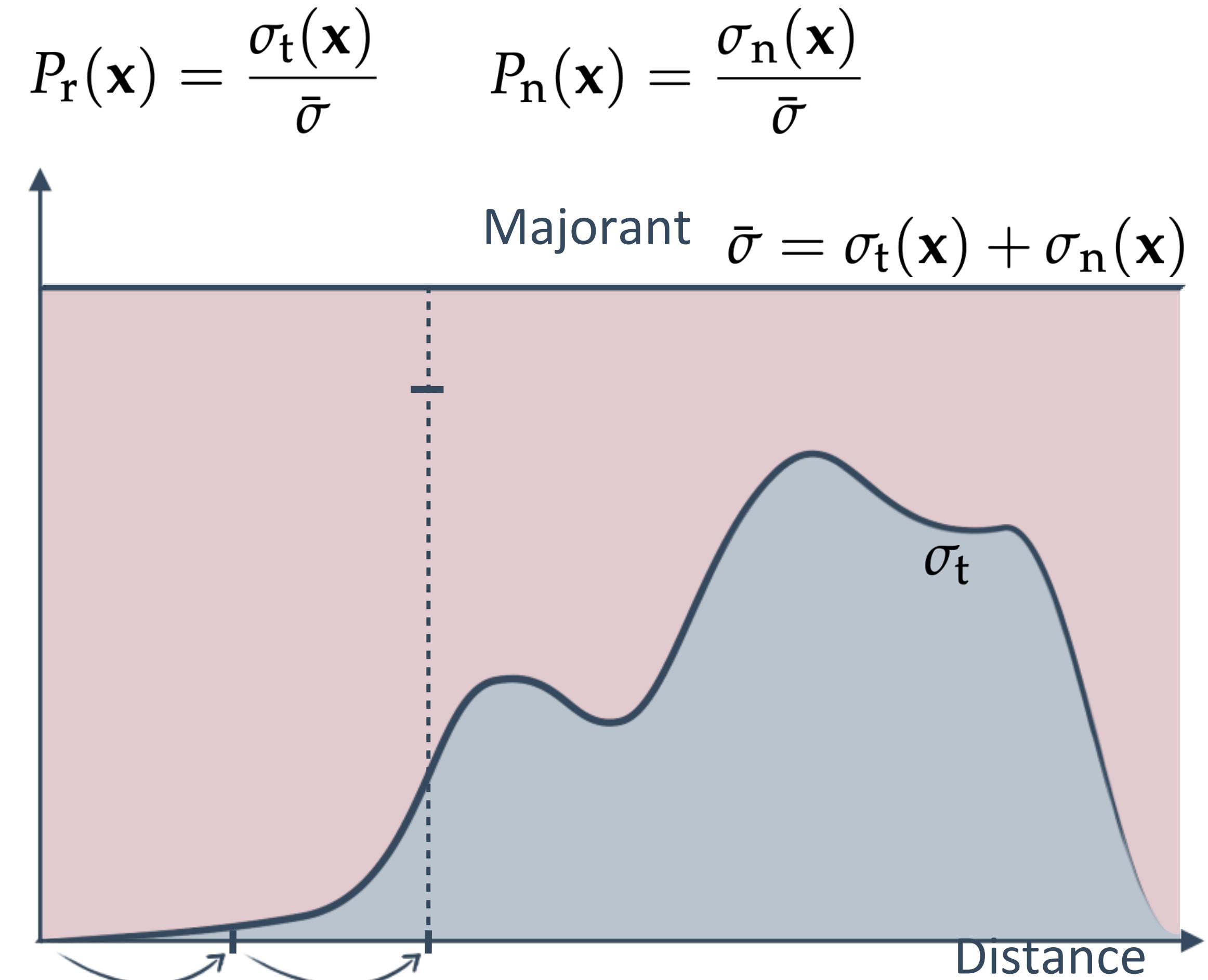
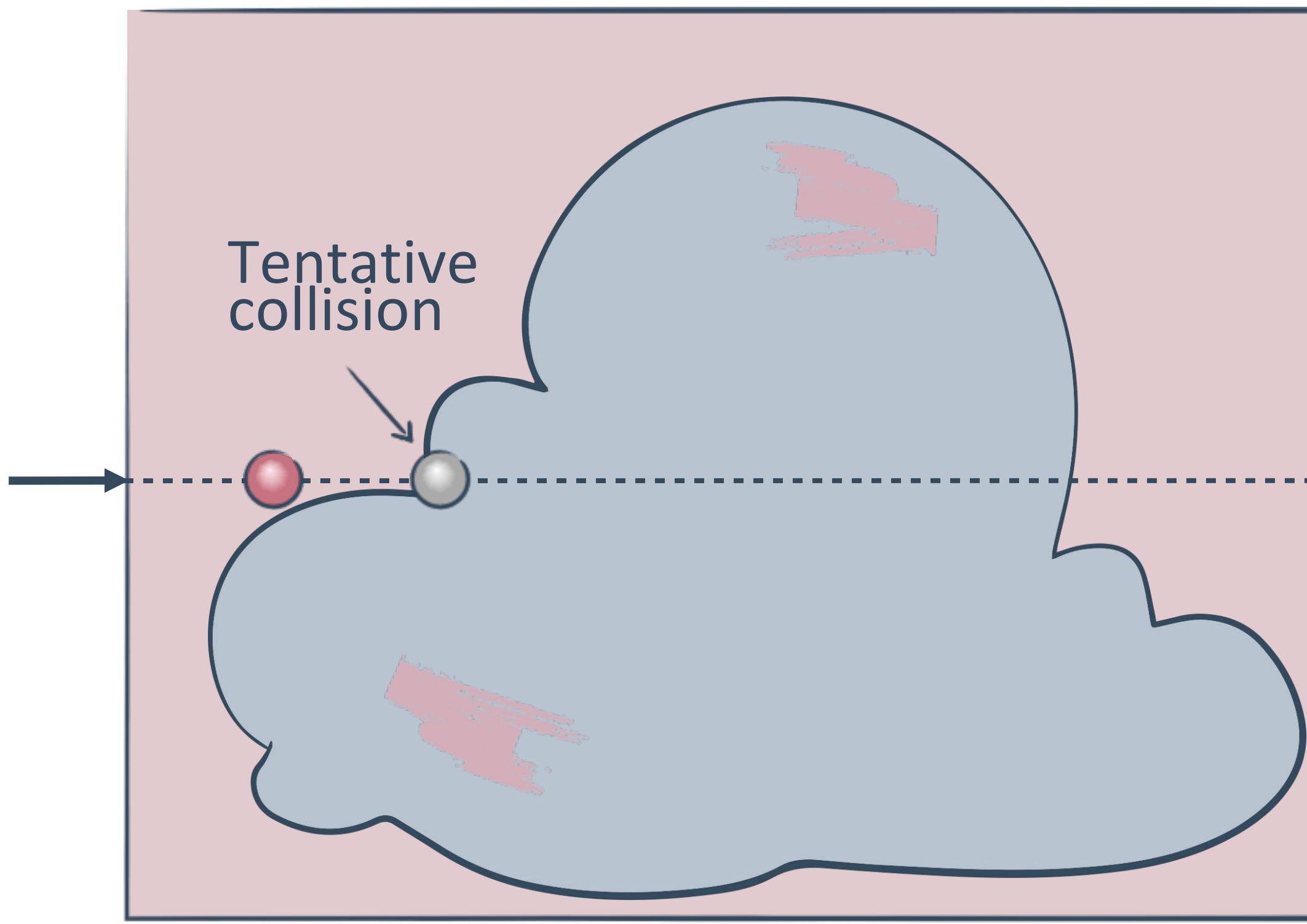
Stochastic Sampling



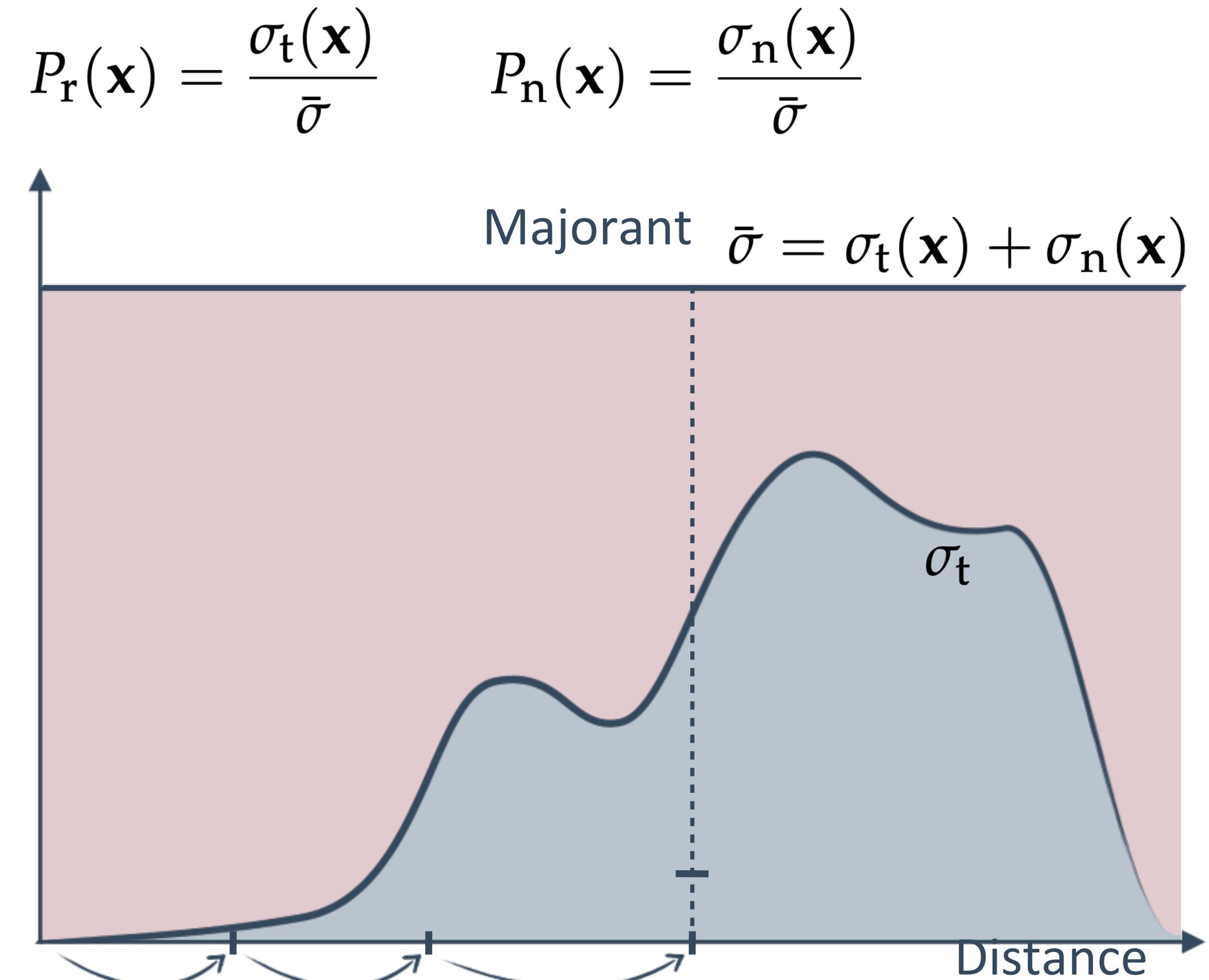
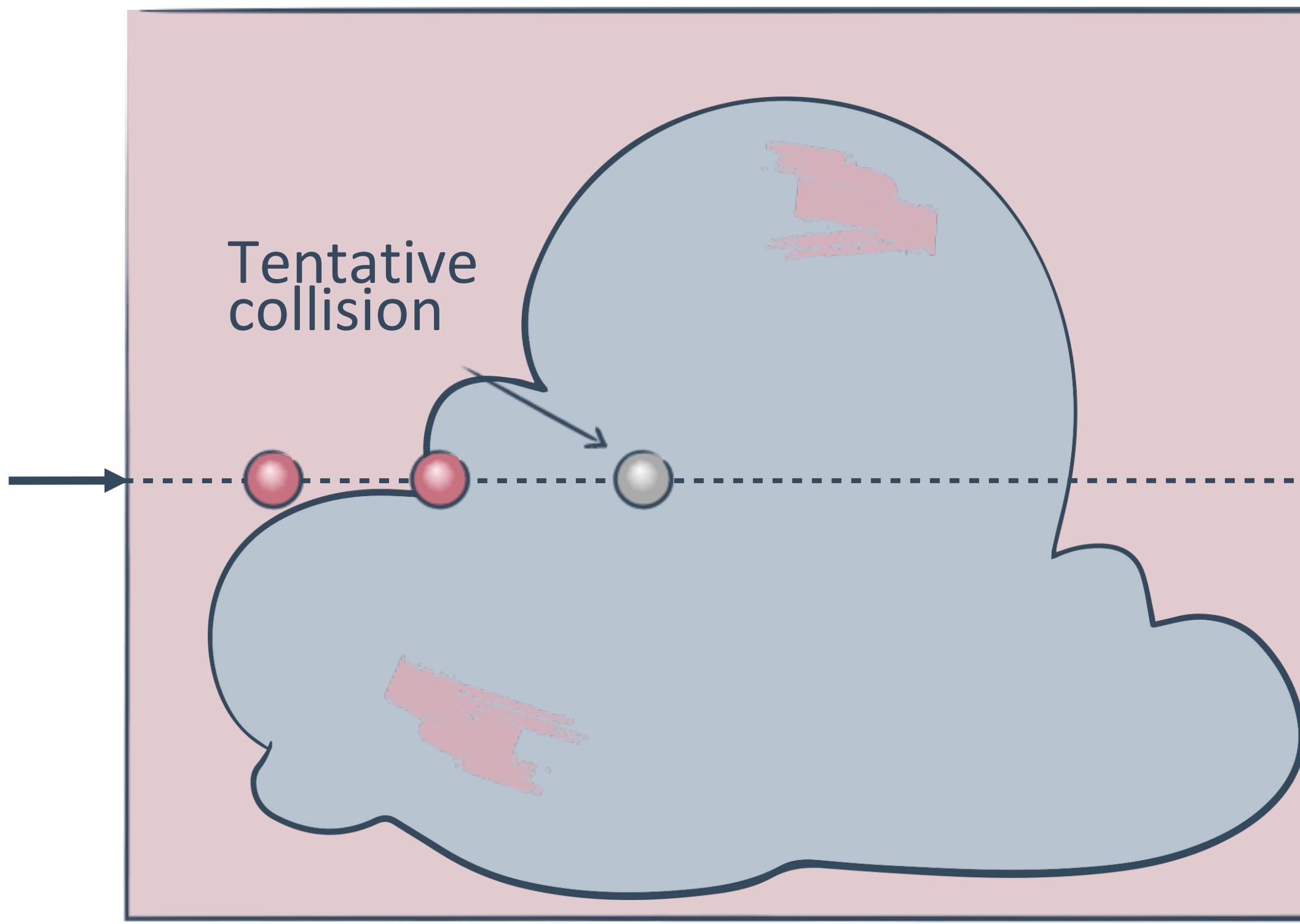
Stochastic Sampling



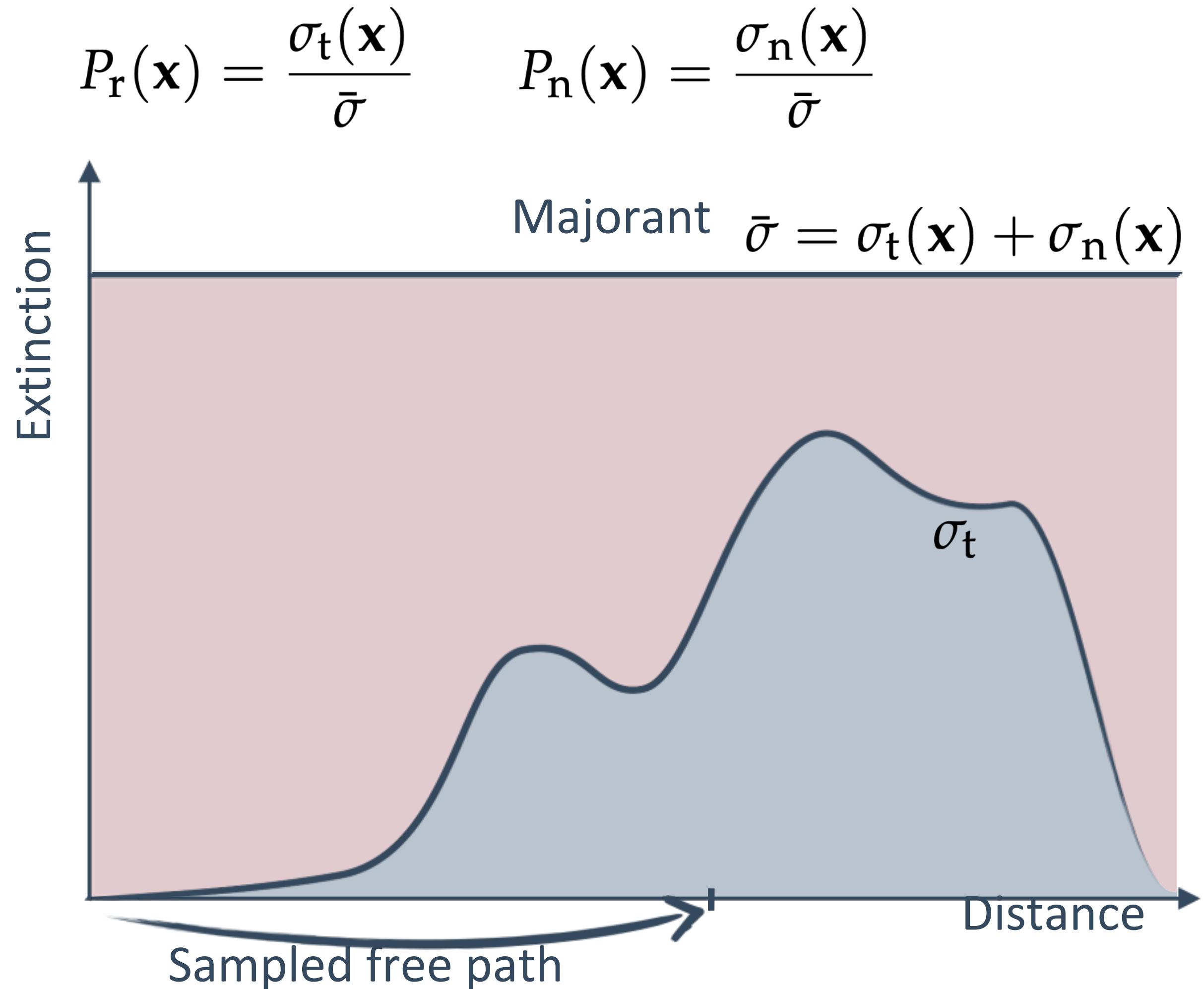
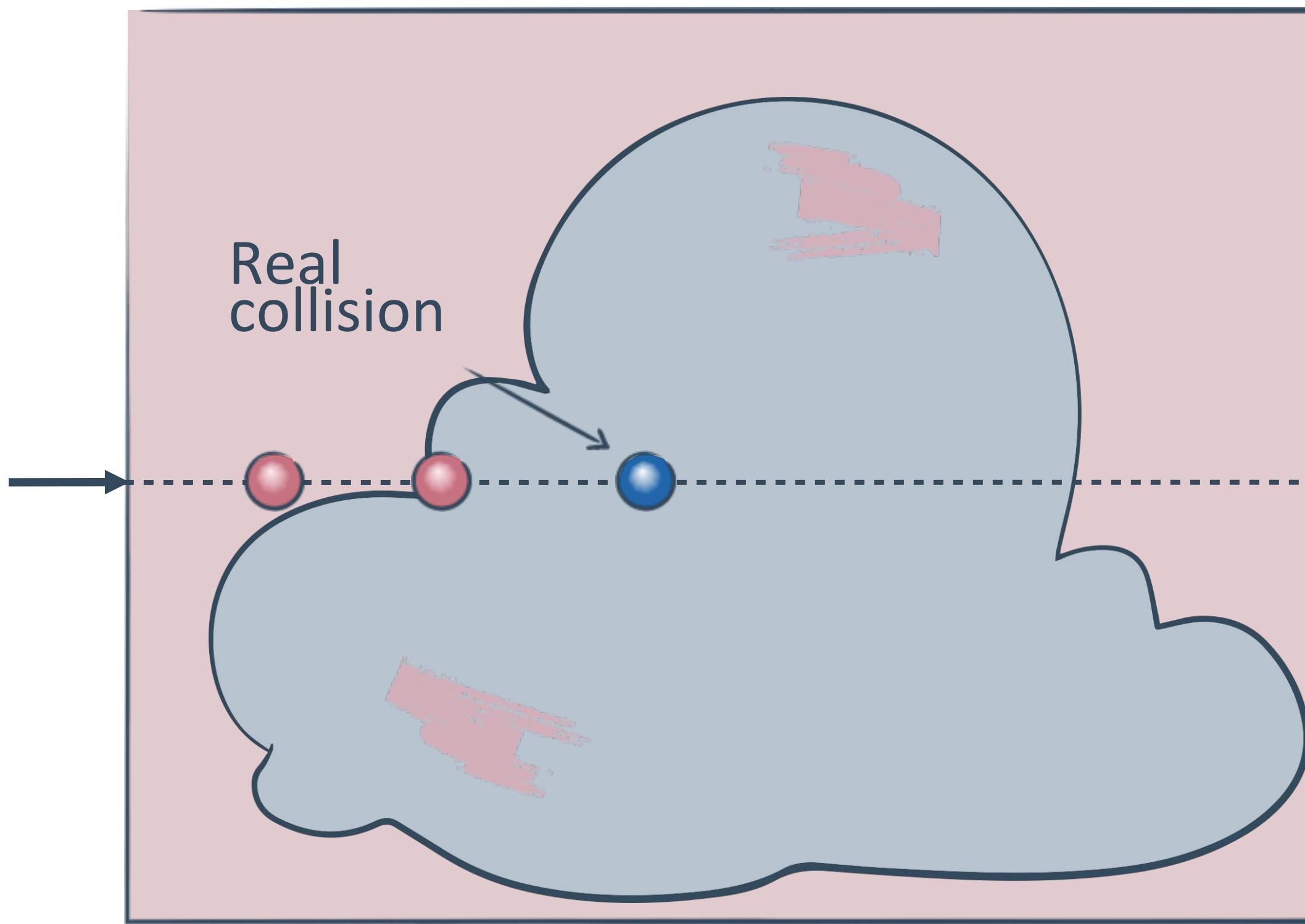
Stochastic Sampling



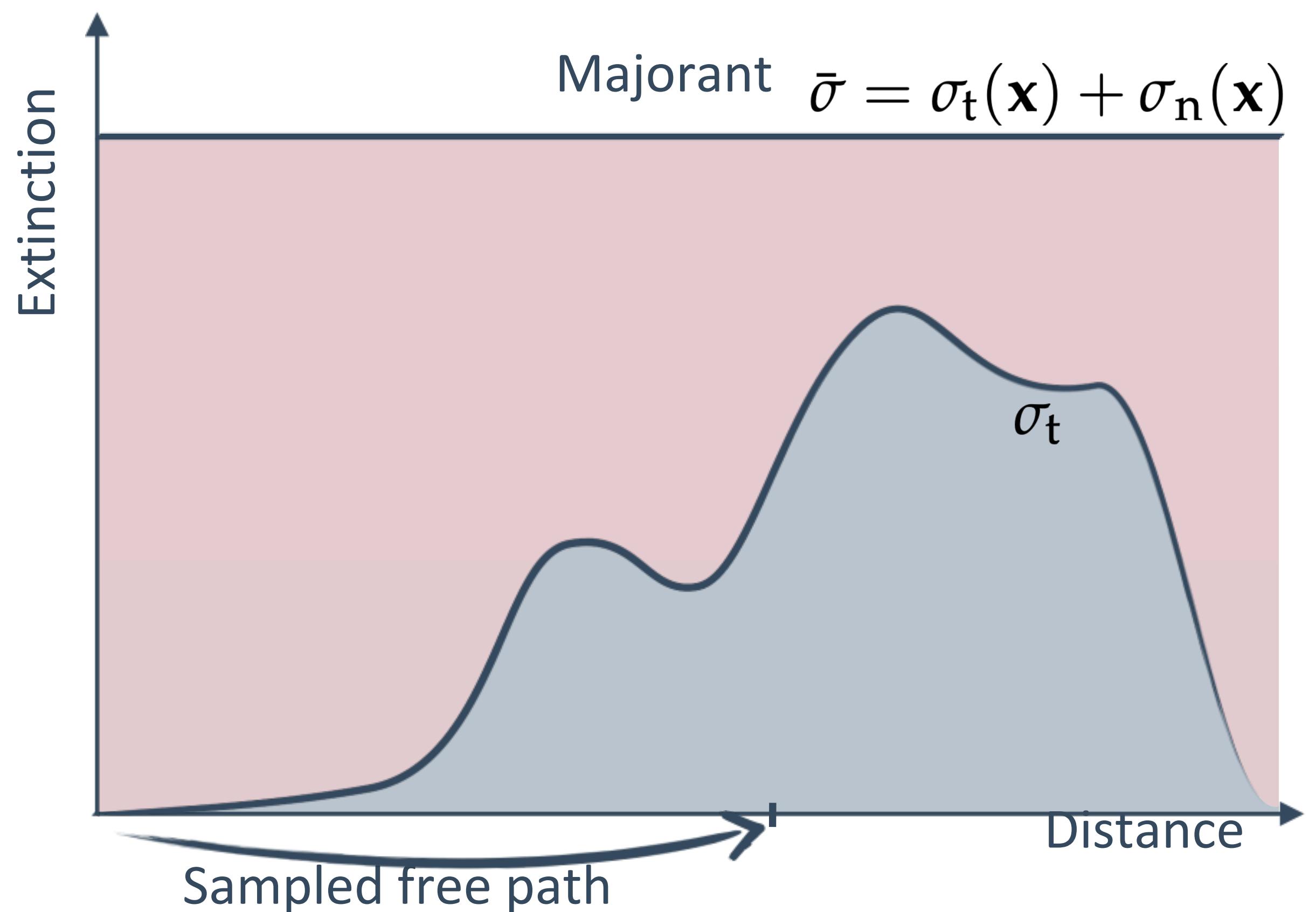
Stochastic Sampling



Stochastic Sampling

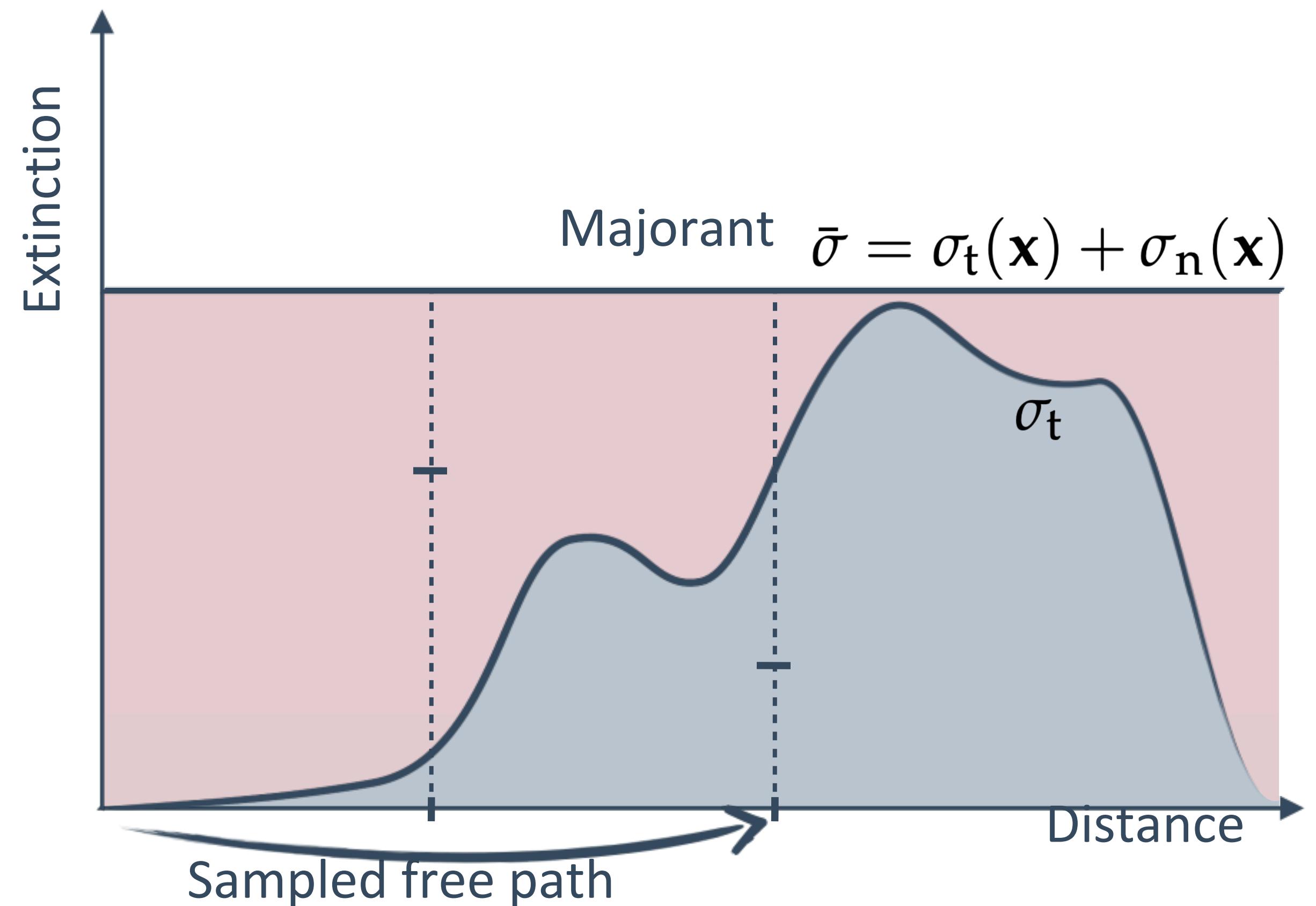


Impact of Majorant



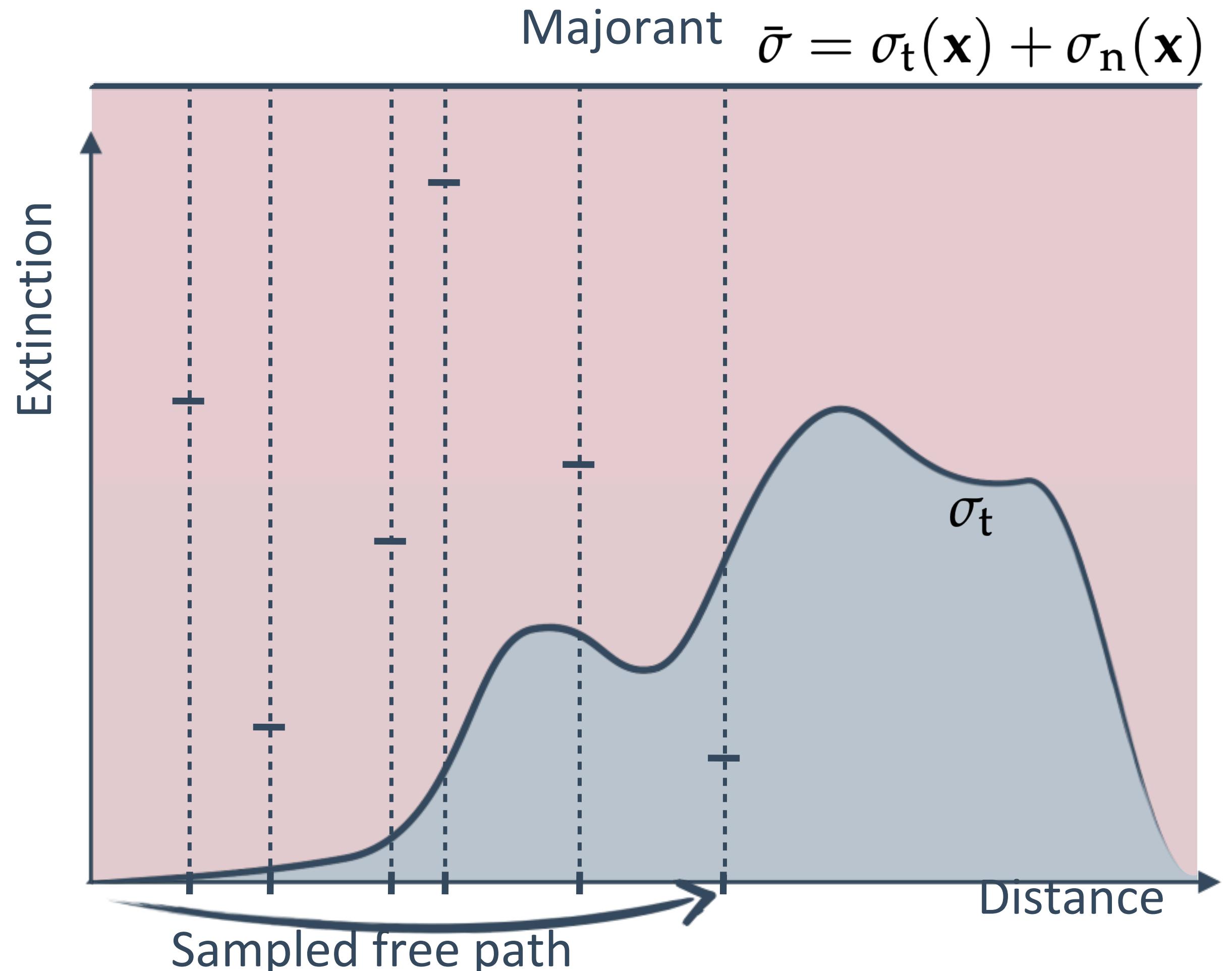
Impact of Majorant

Tight majorant = GOOD
(few rejected collisions)



Impact of Majorant

Loose majorant = BAD
(many expensive rejected collisions)



Delta Tracking

```
void preprocess()  
majorant = findMaximumExtinction()  
  
void sampleFreePath(x,  $\omega$ )  
  
t = 0  
  
do:  
  
    // Sample distance to next tentative collision  
    t += - $\ln(1 - \text{randf}()) / \text{majorant}$   
  
    // Compute probability of a real collision  
     $Pr$  = getExtinction(x +  $t^*\omega$ ) / majorant  
  
while  $Pr < \text{randf}()$   
  
return t
```

Delta Tracking Summary

Unbiased, see [Coleman 68] for a proof

NUCLEAR SCIENCE AND ENGINEERING: 32, 76-81 (1968)

Mathematical Verification of a Certain Monte Carlo Sampling Technique and Applications of the Technique to Radiation Transport Problems

W. A. Coleman
Oak Ridge National Laboratory, Oak Ridge, Tennessee 37830
Received September 27, 1967
Revised November 10, 1967

The first section of this paper is a mathematical construction of a certain Monte Carlo procedure for sampling from the distribution

$$F(X) = \int_0^X \Sigma(x) \exp[-\int_0^x \Sigma(v) dv] dx, \quad 0 \leq X.$$

The construction begins by defining a particular random variable λ . The distribution function of λ is developed and found to be identical to $F(X)$. The definition of λ describes the sampling procedure. Depending on the behavior of $\Sigma(x)$, it may be more efficient to sample from $F(X)$ by obtaining realizations of λ than by the more conventional procedure described in the paper.

Section II is a discussion of applications of the technique to problems in radiation transport where $F(X)$ is frequently encountered as the distribution function for nuclear collisions. The first application is in charged particle transport where $\Sigma(x)$ is essentially a continuous function of x . An application in complex geometries where $\Sigma(x)$ is a step function, and changes values numerous times over a mean path, is also cited. Finally, it is pointed out that the technique has been used to improve the efficiency of estimating certain quantities, such as the number of absorptions in a material.

INTRODUCTION

In certain Monte Carlo problems it is necessary to obtain realizations (sample values) of a random variable having a distribution function^a given by

$$F(X) = \int_0^X \Sigma(x) \exp[-\int_0^x \Sigma(v) dv] dx, \quad 0 \leq X, \quad (1)$$

where $\Sigma(x)$ is any real valued function having the properties:

- (a) $0 \leq \Sigma(x)$ for $0 \leq x$.
- (b) $\lim_{y \rightarrow \infty} \int_0^y \Sigma(x) dx = \infty$.
- (c) $\Sigma(x)$ is bounded; there is an $M > 0$ with $0 \leq \Sigma(x) \leq M$ for all x .

^aIf $F(X)$ is a distribution function it is nondecreasing, $F(-\infty) = 0$, and $F(\infty) = 1$. Many authors refer to such functions as cumulative distribution functions.

$$\theta_1 = \phi^{-1}(\eta_1). \quad (2)$$

MONTE CARLO SAMPLING TECHNIQUE

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Sampling from $F_\eta(Y)$ is common practice in Monte Carlo calculations. However, the solution of Eq. (2) for θ_1 , given η_1 , may be rather laborious.

In practice it is often easier to obtain realizations from Eq. (1) by another procedure. This procedure is described in Sec. I in terms of the definition of a certain random variable λ , whose distribution is identical to that given in Eq. (1). In most applications it is fairly easy to argue that λ must be distributed according to Eq. (1) for physical reasons. The development in Sec. I is intended to provide a mathematical perspective for understanding existing applications and to encourage recognition of new applications. Section II is a summary of three current applications.

I. DEVELOPING THE DISTRIBUTION FUNCTION FOR λ

The purpose of this section is to construct the distribution function of a random variable λ whose values are the termination points of a certain random walk to be described presently. The construction is based on the following hypotheses:

A. Let $\Sigma(x)$ be as described in conjunction with the distribution in Eq. (1).

B. Let $(\xi_1, \xi_2, \dots, \xi_n, \dots)$ denote an infinite sequence of totally independent random variables having a common distribution function,

$$P(\xi_i \leq X) = F_\xi(X) = \int_0^X M e^{-Mx} dx,$$

$$0 \leq X; \quad i = 1, 2, \dots$$

where M is a fixed upper bound of $\Sigma(x)$.

C. Define $\sigma(x) = \Sigma(x)/M$ and $\alpha(x) = 1 - \sigma(x)$, where $0 < x$, to simplify notation.

D. Let $(\rho_1, \rho_2, \dots, \rho_n, \dots)$ denote an infinite sequence of totally independent random variables having a common uniform distribution function,

$$P(\rho_i \leq R) = F_\rho(R) = R, \quad 0 \leq R \leq 1;$$

$$i = 1, 2, \dots$$

E. Let $(\xi_1, \xi_2, \dots, \xi_n, \dots)$ denote the infinite sequence of random variables which are the cumulative sums of the ξ_i :

$$\xi_i = \sum_{j=1}^i \xi_j = \xi_{i-1} + \xi_i, \quad i = 1, 2, \dots,$$

$$\xi_0 \equiv \xi_0 = 0.$$

F. Denote the minimum value of n for which

$$\rho_n \leq \sigma(\xi_n), \quad n = 1, 2, \dots,$$

by N .

The random variable θ has the distribution $F(X)$ given in Eq. (1). To obtain a realization of θ one might first sample from $F_\eta(Y)$, realizing η_1 . Then

$$\theta_1 = \phi^{-1}(\eta_1). \quad (2)$$

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G. Let λ denote the random variable ξ_N . The values of λ are defined as those values of the ξ_n for which n takes on the value N .

In practice it is often easier to obtain realizations from Eq. (1) by another procedure. This procedure is described in Sec. I in terms of the definition of a certain random variable λ , whose distribution is identical to that given in Eq. (1). In most applications it is fairly easy to argue that λ must be distributed according to Eq. (1) for physical reasons. The development in Sec. I is intended to provide a mathematical perspective for understanding existing applications and to encourage recognition of new applications. Section II is a summary of three current applications.

1) Assign i the value 1, z_0 the value 0.

2) Generate x_i and r_i .

3) Calculate $z_i = z_{i-1} + x_i$.

4) If $r_i \leq \sigma(z_i)$, stop and assign L the value z_i ; otherwise increment i by 1 and proceed to step 2.

For brevity in all of the discussion that follows, the procedure outlined above will be referred to as the λ procedure. The distribution function for λ will now be constructed using the hypotheses A through G.

Denote the event for which $N = 1$ and $\lambda \leq Z$ by

$$E_1 = \{\rho_1 < \sigma(\xi_1), \xi_1 \leq Z\},$$

where Z is an arbitrary fixed value in the range of λ . Similarly denote the event for which $N = 2$ and $\lambda \leq Z$ by

$$E_2 = \{\rho_1 > \sigma(\xi_1), \rho_2 < \sigma(\xi_2), \xi_2 \leq Z\}.$$

This notation is extended to describe the events for general $N > 1$ and $\lambda \leq Z$:

$$E_n = \{\rho_1 > \sigma(\xi_1), \rho_2 > \sigma(\xi_2), \dots, \rho_{n-1} > \sigma(\xi_{n-1}),$$

$$\rho_n \leq \sigma(\xi_n), \xi_n \leq Z\}.$$

The event $\{\lambda \leq Z\}$ can occur in any of the mutually exclusive ways $E_1, E_2, \dots, E_n, \dots$. Hence, the distribution function for λ may be written as

$$P[\lambda \leq Z] = F_\lambda(Z) = \sum_{n=1}^{\infty} P(E_n). \quad (3)$$

Each of the joint probabilities $P(E_n)$, $n = 1, 2, \dots$, may be expressed in terms of the random walk increments ξ_i , $i = 1, 2, \dots$:

$$P(E_1) = P[\rho_1 < \sigma(\xi_1), \xi_1 \leq Z]$$

$$P(E_n) = P[\rho_1 > \sigma(\xi_1), \dots, \rho_{n-1} > \sigma(\sum_{i=1}^{n-1} \xi_i),$$

$$\rho_n \leq \sigma(\sum_{i=1}^n \xi_i),$$

$$\xi_n \leq Z - \sum_{i=1}^{n-1} \xi_i]$$

for each value of η define

$$\theta = \phi^{-1}(\eta),$$

where

$$\eta = \phi(\theta) = \int_0^\theta \Sigma(u) du.$$

The random variable θ has the distribution $F(X)$ given in Eq. (1). To obtain a realization of θ one might first sample from $F_\eta(Y)$, realizing η_1 . Then

$$\theta_1 = \phi^{-1}(\eta_1). \quad (2)$$

^aSee for example, WILLIAM FELLER, *An Introduction to Probability Theory and its Applications*, Vol. II, p. 154 ff (1966).

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The probability that $\rho_1 \leq \sigma(\xi_1)$ and $\xi_1 \leq Z$ may be expressed as the integral of the conditional probability that $\rho_1 \leq \sigma(\xi_1)$ given $\xi_1 = x_1$ with respect to the marginal distribution¹ $F_\xi(x_1)$:

$$P(E_1) = \int_0^Z P[\rho_1 \leq \sigma(\xi_1) | \xi_1 = x_1] dF_\xi(x_1) = \int_0^Z \sigma(x_1) M e^{-Mx_1} dx_1. \quad (4)$$

Similarly,

$$P(E_n) = \int_0^Z \int_0^{Z-x_1} \dots \int_0^{Z-\sum_{i=1}^{n-1} x_i} P[\rho_1 > \sigma(\xi_1), \dots, \rho_{n-1} > \sigma(\sum_{i=1}^{n-1} \xi_i),$$

$$\rho_n \leq \sigma(\sum_{i=1}^n \xi_i) | \xi_1 = x_1, \dots, \xi_n = x_n] dF_{\xi_1, \dots, \xi_n}(x_1, \dots, x_n). \quad (5)$$

$F_{\xi_1, \dots, \xi_n}(x_1, \dots, x_n)$ denotes the joint distribution function of the variables ξ_1, \dots, ξ_n . The integral limits in Eq. (5) are determined by first noting that $0 < \xi_i$, and hence $\xi_{i-1} < \xi_i$, $i = 1, 2, \dots$. For the event E_n to occur, it is necessary that $\xi_n < Z$, which implies $\xi_1 < \dots < \xi_n < Z$. In terms of x_i , it is necessary that $x_i < Z - \sum_{j=1}^{i-1} \xi_j$ for $i = 2, 3, \dots, n$.

Since $\rho_1, \rho_2, \dots, \rho_n$ are totally independent, the integrand in Eq. (5) is equal to

$$P[\rho_1 > \sigma(\xi_1) | \xi_1 = x_1] \dots P[\rho_{n-1} > \sigma(\sum_{i=1}^{n-1} \xi_i) | \xi_1 = x_1, \dots, \xi_{n-1} = x_{n-1}]$$

$$\times P[\rho_n \leq \sigma(\sum_{i=1}^n \xi_i) | \xi_1 = x_1, \dots, \xi_n = x_n].$$

Also ξ_1, \dots, ξ_n are totally independent and have a common distribution function, so that

$$F_{\xi_1, \dots, \xi_n}(x_1, \dots, x_n) = F_\xi(x_1) \dots F_\xi(x_n) = F_\xi(x_1) \dots F_\xi(x_n).$$

Substituting these relations into Eq. (5) gives

$$P(E_n) = \int_0^Z \int_0^{Z-x_1} \dots \int_0^{Z-\sum_{i=1}^{n-1} x_i} P[\rho_1 > \sigma(\xi_1) | \xi_1 = x_1]$$

$$\dots P[\rho_{n-1} > \sigma(\sum_{i=1}^{n-1} \xi_i) | \xi_1 = x_1, \dots, \xi_{n-1} = x_{n-1}]$$

$$\times P[\rho_n \leq \sigma(\sum_{i=1}^n \xi_i) | \xi_1 = x_1, \dots, \xi_n = x_n] dF_\xi(x_1) \dots dF_\xi(x_n)$$

$$= \int_0^Z \int_0^{Z-x_1} \dots \int_0^{Z-\sum_{i=1}^{n-1} x_i} [1 - \sigma(x_1)]$$

$$\dots [1 - \sigma(\sum_{i=1}^{n-1} x_i)] \sigma(\sum_{i=1}^n \xi_i) M e^{-Mx_1} \dots M e^{-Mx_n}.$$

It is convenient to proceed with the probabilities expressed in terms of the variables ξ_1, \dots, ξ_n . The transformations from ξ_1, \dots, ξ_n are direct. Introducing $\alpha(x)$ for brevity, the expressions for $P(E_1)$ and $P(E_n)$, $n \geq 2$, become

$$P(E_1) = \int_0^Z \sigma(z_1) M e^{-Mz_1} dz_1,$$

and

$$P(E_n) = \int_0^Z dz_n M^n \sigma(z_n) e^{-Mz_n} \int_0^{z_n} dz_{n-1} \sigma(z_{n-1}) \int_0^{z_{n-1}} dz_{n-2} \dots \int_0^{z_2} dz_1 \sigma(z_1)$$

$$= \int_0^Z dz_n \sigma(z_n) M^n e^{-Mz_n} \int_0^{z_n} dz_2 \sigma(z_2) \int_0^{z_2} dz_3 \dots \int_0^{z_{n-1}} dz_n \sigma(z_n). \quad (6)$$

¹See for example, WILLIAM FELLER, *An Introduction to Probability Theory and its Applications*, Vol. II, p. 154 ff (1966).

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It will now be proved that

$$\int_0^{z_1} dz_2 \sigma(z_2) \int_0^{z_2} dz_3 \dots \int_0^{z_{n-1}} dz_n \sigma(z_n) = \frac{\left[\int_0^{z_1} \alpha(v) dv \right]^{n-1}}{(n-1)!}, \quad 2 \leq n. \quad (7)$$

Equation (7) is true for $n = 2$ by inspection. For $n = 3$,

Delta Tracking Summary

Unbiased, see [Coleman 68] for a proof

Majorant extinction

- defines the combined homogeneous volume
- must bound the real extinction
- loose majorants lead to many fictitious collisions