SOLUTIONS TO EXTRA EXERCISES FOR WEEK 1-2

Solution 2.4. In order to achieve this, we need to rely on transformation of random variables formula. However, since Y is a function of both X_1 and X_2, we need a transformation in two dimensions to be able to use transformation of random variables. In general, one chooses another auxiliary variable that makes computations easier (again, please practice transformation of random variables, that must have been the part of previous courses).

In our case, let $Y = \frac{X_1}{X_1+X_2}$ and define an auxiliary $Z = X_1 + X_2$. We aim at finding the density of $p_{y,z}(y,z)$ and this is given by

$$p_{y,z}(y,z) = p_{x_1,x_2}(g^{-1}(y,z)) \det J_{g^{-1}}. \tag{1}$$

Note that in the lecture notes, we denoted the Jacobian with $|J_{g^{-1}}|$, this is the same as $\det J_{g^{-1}}$ (for you to get used to both notations). The inverse g^{-1} can be constructed in both arguments from the fact that

$$X_1 = YZ, \quad \text{This is why we chose } Z = X_1 + X_2,$$

and

$$X_2 = Z - X_1 = Z - YZ = Z(1-Y).$$

Therefore, we obtain $g^{-1}(y,z) = (yz, z(1-y))$, therefore $g_1^{-1} = yz$ and $g_2^{-1} = z(1-y)$. Now we compute the Jacobian:

$$\begin{aligned} J_{g^{-1}} &= \begin{bmatrix} \frac{\partial g_1^{-1}}{\partial y} & \frac{\partial g_1^{-1}}{\partial z} \\ \frac{\partial g_2^{-1}}{\partial y} & \frac{\partial g_2^{-1}}{\partial z} \end{bmatrix} \\ &= \begin{bmatrix} z & y \\ -z & (1-y). \end{bmatrix} \end{aligned}$$

Therefore $\det J_{g^{-1}} = z - zy - (-zy) = z$. Therefore, the formula (1) becomes

$$\begin{aligned} p_{y,z}(y,z) &= \text{Gamma}(yz;\alpha,1)\text{Gamma}(z(1-y);\beta,1)z, \\ &= \frac{1}{\Gamma(\alpha)}(yz)^{\alpha-1}e^{-yz}\frac{1}{\Gamma(\beta)}(z(1-y))^{\beta-1}e^{-z(1-y)}z \end{aligned}$$

We are interested ultimately in $p_y(y)$, therefore, we integrate this

$$\begin{aligned} p_y(y) &= \int p_{y,z}(y,z)\mathrm{d}z \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}y^{\alpha-1}(1-y)^{\beta-1}\int e^{-z}z^{\alpha+\beta-1}\mathrm{d}z, \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}y^{\alpha-1}(1-y)^{\beta-1}. \end{aligned}$$

which is the Beta distribution as intended. The last line follows from the definition of the Gamma function:

$$\Gamma(a) = \int z^{a-1}e^{-z}\mathrm{d}z.$$

Solution 2.5. (a) We first write the ratio

$$R(x) = \frac{p(x)}{q_\alpha(x)} = \frac{1/\sqrt{2\pi}\exp(-x^2/2)}{(\alpha/2)\exp(-\alpha|x|)}.$$

This is not differentiable at 0, but we can do a piecewise computation to verify the maximum. First note that

$$R(0) = \alpha^{-1}\sqrt{\frac{2}{\pi}}$$

Since below computation excludes the case $x = 0$, we keep this in mind to determine maximum later.

For $x > 0$, we have the ratio

$$R(x) = \frac{p(x)}{q_\alpha(x)} = \frac{1/\sqrt{2\pi}\exp(-x^2/2)}{(\alpha/2)\exp(-\alpha x)},$$

Taking derivative of $\log$ and setting it to 0, we obtain

$$\frac{\mathrm{d}\log R(x)}{\mathrm{d}x} = -x + \alpha = 0,$$

hence $x^\star = \alpha$ (since the second derivative is negative). Similarly for $x < 0$, the computation shows

$$x^\star = -\alpha.$$

Note however that $R(\alpha) = R(-\alpha)$ (due to the use of square and absolute value), we obtain that

$$R(\alpha) = \alpha^{-1}\sqrt{\frac{2}{\pi}}\exp(\alpha^2/2).$$

To verify that this is the value at maximum, we also verify $R(\alpha) > R(0)$ as $\exp(\alpha^2/2) > 1$. Hence, we can conclude that

$$R(\alpha) = M_\alpha = \sup_x \frac{p(x)}{q_\alpha(x)}.$$

Next, we would like to optimise

$$M_\alpha = \alpha^{-1}\sqrt{\frac{2}{\pi}}\exp(\alpha^2/2).$$

Computing the derivative of M_α and setting it to zero,

$$\frac{\mathrm{d}\log M_\alpha}{\mathrm{d}\alpha} = -\frac{1}{\alpha} + \alpha = 0,$$

which implies $\alpha^2 = 1$. Since we assumed $\alpha > 0$, we conclude $\alpha^\star = 1$.

(b) In the lectures (Lecture 4), we have seen that

$$\hat{a} = \frac{1}{M}.$$

Our optimal M here is

$$M := M_{\alpha^\star} = \sqrt{\frac{2}{\pi}} e^{1/2}.$$

Therefore,

$$\hat{a} = \frac{1}{M} = \sqrt{\pi/2e}.$$