Imperial College
London

MATH97083

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Applied Probability

Date: Wednesday, 5 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS SEPARATE PDFs TO THE RELEVANT DROPBOXES ON BLACKBOARD (ONE FOR EACH QUESTION) WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

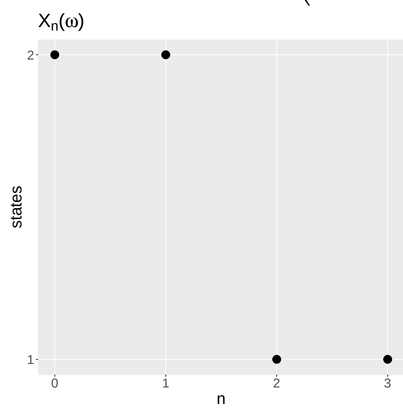
Throughout the exam, we denote by $(\Omega, \mathcal{F}, \mathrm{P})$ a probability space.

1. (a) Consider a discrete-time, time-homogeneous Markov chain $X = (X_n)_{n\in\mathbb{N}_0}$ on a countable state space E. We denote the (one-step) transition matrix by $\mathbf{P} = (p_{ij})_{i,j\in E}$ and the corresponding marginal distribution of X at time n by $\nu^{(n)}$ for $n \in \mathbb{N}_0$.

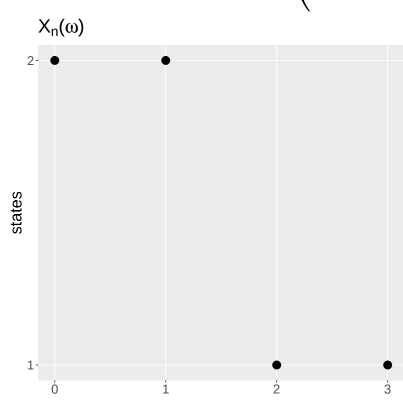
For each of the sample paths/realisations below, explain whether or not it could be a realisation of the corresponding Markov chain. How likely is each realisation?

[This question continues on the next pages.]

(i) Let $E = \{1, 2\}$, $\mathbf{P} = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$, $\boldsymbol{\nu}^{(0)} = (0.5, 0.5)$. (2 marks)



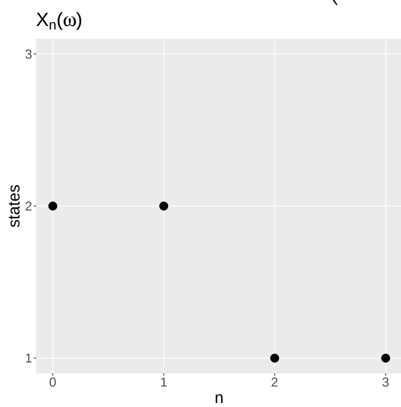
(ii) Let $E = \{1, 2\}$, $\mathbf{P} = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$, $\boldsymbol{\nu}^{(0)} = (1, 0)$. (2 marks)



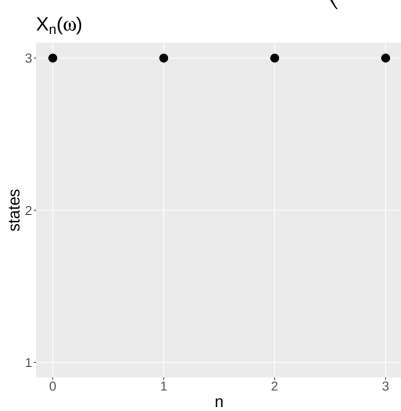
[This question continues on the next two pages.]

[Continuation of Question 1a):]

(iii) Let $E = \{1, 2, 3\}$, $\mathbf{P} = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\boldsymbol{\nu}^{(0)} = (0.25, 0.5, 0.25)$. (2 marks)



(iv) Let $E = \{1, 2, 3\}$, $\mathbf{P} = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\boldsymbol{\nu}^{(0)} = (0.25, 0.5, 0.25)$. (2 marks)



(b) Consider a discrete-time, time-homogeneous Markov chain $X = (X_n)_{n \in \mathbb{N}_0}$ on the state space $E = \{1, 2, 3, 4\}$ with (one-step) transition matrix given by

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

(i) Find the communicating classes of the Markov chain and determine their periods. (3 marks)

(ii) Find $\mathbf{P}_n$ (the matrix of the n-step transition probabilities) for all $n \in \mathbb{N}$. (3 marks)

(iii) Find all possible stationary distributions of the Markov chain. (1 mark)

[This question continues on the next page.]

[Final part of Question 1:]

(c) Does a discrete-time, time-homogeneous Markov chain on a countable state space exist which has some null-recurrent states and a stationary distribution? Justify your answer carefully.

(5 marks)

(Total: 20 marks)

2. (a) Consider a discrete-time, time-homogeneous Markov chain $X = (X_n)_{n\in\mathbb{N}_0}$ on the state space $E = \{1, 2, 3\}$ with (one-step) transition matrix given by

$$\mathbf{P} = \begin{pmatrix} \frac{1}{4} & \frac{3}{8} & \frac{3}{8} \\ \frac{1}{4} & \frac{3}{8} & \frac{3}{8} \\ \frac{1}{4} & \frac{3}{8} & \frac{3}{8} \end{pmatrix}.$$

Also, suppose that $\mathrm{P}(X_0 = 2) = 1$.

Define the stochastic process $Y = (Y_n)_{n\in\mathbb{N}_0}$, where $Y_n = (-1)^{X_n}$ for $n \in \mathbb{N}_0$.

(i) Find the probability mass function of X_n for $n \in \mathbb{N}_0$. (3 marks)

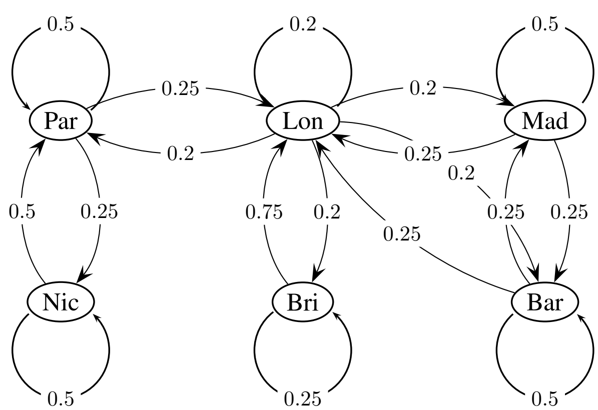
(ii) Sketch two distinct realisations of $(Y_n)_{n\in\{0,1,\dots,7\}}$. (2 marks)

(iii) Find the probability mass function of Y_n for $n \in \mathbb{N}_0$. (4 marks)

(iv) Are the random variables $Y_n, n \in \mathbb{N}_0$ independent? Justify your answer carefully. (4 marks)

(v) Is Y a Markov chain? Justify your answer carefully. If Y is a Markov chain, find the one-step transition probabilities. (3 marks)

(b) After finishing her final exams, a graduate would like to travel in Europe. Her itinerary includes six destinations: three city destinations (Paris (Par), London (Lon) and Madrid (Mad)) and three beach destinations (Nice (Nic), Brighton (Bri) and Barcelona (Bar)). Suppose her location each day can be modelled by a discrete-time, time-homogenous Markov chain and the one-step transition diagram is given by



(i) Suppose the graduate starts her journey at time $n = 0$ in London. Find the probability that she will be at a beach (i.e. either in Nice, Brighton or Barcelona) at time $n = 2$. (2 marks)

(ii) Discuss briefly any shortcomings of our modelling assumptions in this particular application. (2 marks)

(Total: 20 marks)

3. (a) Suppose that a speed camera installed on a highway records that there are on average five cars per hour which drive faster than the speed limit. Suppose you want to compute the probability, p, that there will be exactly one car driving faster than the speed limit in the next 30 minutes.

   (i) Write down a model which could be used to answer this question. (2 marks)

   (ii) Find p in your model. (2 marks)

   (iii) Discuss any limitations of your modelling assumptions. (2 marks)

(b) Consider a non-homogeneous Poisson process $N = (N_t)_{t\geq 0}$ with rate $\lambda(t) = a + bt$ for $a > 0, b > 0$. Suppose that $\mathrm{P}(N_1 = 0) = \exp(-2)$ and $\mathrm{P}(N_2 = 0) = \exp(-6)$. Find the constants a and b. (3 marks)

(c) Consider three independent Poisson processes $N = (N_t)_{t\geq 0}, N^{(1)} = (N_t^{(1)})_{t\geq 0}, N^{(2)} = (N_t^{(2)})_{t\geq 0}$ with rates $\lambda > 0, \lambda^{(1)} > 0$, and $\lambda^{(2)} > 0$, respectively.

Define two new stochastic processes as $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$, where

$$X_t = N_t + N_t^{(1)}, \qquad Y_t = N_t + N_t^{(2)}, \qquad t \geq 0.$$

   (i) Are X and Y Poisson processes? If so, what are their rates? (2 marks)

   (ii) Find $\mathrm{E}(u^{X_t})$ and $\mathrm{E}(v^{Y_t})$ for $|u| \leq 1, |v| \leq 1$. (2 marks)

   (iii) Find $\mathrm{E}(u^{X_t} v^{Y_t})$, for $|u| \leq 1, |v| \leq 1$. (3 marks)

   (iv) Explain whether the process $Z = (Z_t)_{t\geq 0}$ with $Z_t = X_t + Y_t$ is a Poisson process. (4 marks)

(Total: 20 marks)

4. (a) Consider a continuous-time, time-homogeneous, minimal Markov chain $X = (X_t)_{t \geq 0}$ on the state space $E = \{1, 2, 3, 4\}$ with generator given by

$$\mathbf{G} = \begin{pmatrix} -\frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & \frac{1}{10} & -\frac{1}{5} & \frac{1}{10} \\ 1 & 1 & 1 & -3 \end{pmatrix}.$$

(i) Find the (one-step) transition matrix of the embedded jump chain. (3 marks)

(ii) For each state in the state space, justify whether it is recurrent or transient for X. (3 marks)

(iii) Suppose you want to simulate a sample path of the continuous-time Markov chain X. Describe briefly the key steps for such a simulation algorithm and mention if you need to make any additional assumptions.
[You may assume that your computer can generate realisations of random variables with any distribution.] (3 marks)

(b) It is time for the annual flu jab. People arrive at a GP's surgery according to a Poisson process of rate of one person every three minutes and join a queue. If there are already five people in the queue, any newly arriving people will go away and try again at a later point in time. The GP's service time for one person, consisting of filling in the necessary paperwork and administering the flu jab, follows an exponential distribution, at a rate of one person every six minutes. Suppose that the arrival times of the people and the GP's service times are independent. For $t \geq 0$, let X_t denote the number of people in the queue at the GP's surgery, which is a continuous-time Markov chain. Let $X = (X_t)_{t \geq 0}$.

(i) Find the state space of X. (1 mark)

(ii) Draw the transition diagram of X and justify the transition rates. (4 marks)

(iii) Find the holding time distribution for each state of X. (3 marks)

(iv) Find the transition matrix of the embedded jump chain. (3 marks)

(Total: 20 marks)

5. (a) Consider the first definition of a Poisson process in the lecture notes: A Poisson process $\{N_t\}_{t\geq 0}$ of rate $\lambda > 0$ is a stochastic process with values in $\mathbb{N}_0$ satisfying: 1) $N_0 = 0$ almost surely. 2) The increments are independent. 3) The increments are stationary. 4) There is a 'single arrival', i.e. for any $t \geq 0$ and $\delta > 0$: $\mathrm{P}(N_{t+\delta} - N_t = 0) = 1 - \lambda\delta + o(\delta), \mathrm{P}(N_{t+\delta} - N_t = 1) = \lambda\delta + o(\delta), \mathrm{P}(N_{t+\delta} - N_t \geq 2) = o(\delta)$. Set $p_n(t) = \mathrm{P}(N_t = n)$ for $n \in \mathbb{N}_0$ and $t \geq 0$.

(i) Show that p_0 is continuous. (3 marks)

(ii) Show that p_0 is differentiable. (3 marks)

(b) Suppose $N = (N_t)_{t\geq 0}$ is a Poisson process with rate $\lambda > 0$. Suppose that $T \geq 0$ is a stopping time of N with $\mathrm{P}(T < \infty) = 1$.

Can you find a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that the process $X = (X_t)_{t\geq 0}$ with

$$X_t = f(N_{T+t}, N_T)$$

is a Poisson process?

(4 marks)

(c) Let $X = (X_t)_{t\geq 0}$ be a minimal continuous-time, time-homogeneous Markov chain on the state space $E = \{1, 2\}$ with generator

$$\mathbf{G} = \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}.$$

(i) Find the stationary distribution of X and justify your answer. (4 marks)

(ii) Show that the transition matrix $\mathbf{P}_t = (p_{ij}(t))_{i,j\in E}$ for all $t \geq 0$ of X is given by

$$\mathbf{P}_t = \frac{1}{2}\begin{pmatrix} 1 + e^{-\frac{2}{3}t} & 1 - e^{-\frac{2}{3}t} \\ 1 - e^{-\frac{2}{3}t} & 1 + e^{-\frac{2}{3}t} \end{pmatrix}.$$

(4 marks)

(iii) Compute $\lim_{t\to\infty} \mathbf{P}_t$. How does this limit relate to the stationary distribution of X?
(2 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2021

This paper is also taken for the relevant examination for the Associateship.

MATH96052/MATH97083

Applied Probability (Solutions)

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1. (a) (i) This is a possible realisation of the Markov chain since all one-step transition probabilities are positive, the chain can either stay or move to the other state in each step. Also, we note that, by a result from lectures, meth seen ⇓

$$\mathrm{P}(X_0 = 2, X_1 = 2, X_2 = 1, X_3 = 1) = \nu_2^{(0)} p_{22} p_{21} p_{11} = \left(\frac{1}{2}\right)^4 = \frac{1}{16}.$$

2, A

(ii) The choice of $\nu^{(0)}$ indicates that the chain starts in state 1 with probability 1. Hence,

$$\mathrm{P}(X_0 = 2, X_1 = 2, X_2 = 1, X_3 = 1) = \nu_2^{(0)} p_{22} p_{21} p_{11} = 0 \left(\frac{1}{2}\right)^3 = 0,$$

so the probability that the realisation depicted in the picture occurs is 0. 2, A

(iii) This is a possible realisation of the Markov chain since it can start in any state and if it starts in state 1 or 2, it will either stay in or switch to the states 1 or 2. If it starts in state 3, then it will stay there. Also, we note that, by a result from lectures,

$$\mathrm{P}(X_0 = 2, X_1 = 2, X_2 = 1, X_3 = 1) = \nu_2^{(0)} p_{22} p_{21} p_{11} = \left(\frac{1}{2}\right)^4 = \frac{1}{16}.$$

2, A

(iv) As in (iii), this is a possible realisation of the Markov chain since it can start in any state and if it starts in state 1 or 2, it will either stay in or switch to the states 1 or 2. If it starts in state 3, then it will stay there. Also, we note that, by a result from lectures,

$$\mathrm{P}(X_0 = 3, X_1 = 3, X_2 = 3, X_3 = 3) = \nu_3^{(0)} p_{33} p_{33} p_{33} = \frac{1}{4} 1^3 = \frac{1}{4}.$$

2, A

(b) (i) We can read off from the transition matrix (or the transition diagram) that all four states communicate with each other. Hence the Markov chain is irreducible and there is just one communicating class given by E. It is sufficient to determine the period of one state, since the Markov chain is irreducible, all states share the same period (by lectures). For instance, let us focus on state 1: Using the notation from the lecture notes, meth seen ⇓ 1, A

$$d(1) = gcd\{n : p_{11}(n) > 0\} = gcd\{2, 4, 6, 8, \ldots\} = 2.$$

So, all states have period 2. 2, A

(ii) By the Chapman-Kolmogorov equations, we know that $\mathbf{P}_n = \mathbf{P}^n$ for all $n \in \mathbb{N}$. Now we can either use the results from (i) or direct calculations to observe that $\mathbf{P}_n = \mathbf{P}^n = \mathbf{P}$ for odd n and

$$\mathbf{P}_n = \mathbf{P}^n = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

for even n. 3, B

(iii) Since the Markov chain is irreducible and has a finite state space, we know that there is a unique stationary distribution. We observe that the transition matrix is doubly-stochastic. Hence, by lectures, the discrete uniform distribution on E is a stationary distribution. So we conclude that $\boldsymbol{\pi} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ is the unique stationary distribution of the Markov chain. $\boxed{1, \mathrm{B}}$

(c) We extend the example of a simple random walk on the integers as follows: Consider the countable state space $E = \mathbb{Z} \cup \{0.5\}$. We set the one-step transition probabilities as $\boxed{\text{unseen} \Downarrow}$

$$p_{ij} = \begin{cases} 0.5 & if\ i, j \in \mathbb{Z}, |j - i| = 1 \\ 0 & if i, j \in \mathbb{Z}, |j - i| \neq 1. \end{cases}$$

and $p_{0.5,0.5} = 1$, $p_{0.5,i} = 0 = p_{i,0.5}$ for all $i \in \mathbb{Z}$.

We observe that the Markov chain has two closed communicating classes given by $\mathbb{Z}$ and $\{0.5\}$. On $\mathbb{Z}$, we are effectively considering a simple symmetric random walk, hence by lectures we conclude that all states $i \in \mathbb{Z}$ are null-recurrent.

Also, since $\{0.5\}$ is a finite, closed class, the state 0.5 is positive recurrent and we find that the (infinite-dimensional) row vector $\boldsymbol{\pi} = (\pi_i)_{i \in E}$ with $\pi_i = 0$ for all $i \in \mathbb{Z}$ and $\pi_{0.5} = 1$ is nonnegative, $\sum_{j \in E} \pi_j = 1$, and satisfies

$$\sum_{i \in E} \pi_i p_{ij} = \pi_j$$

for all $j \in E$, since

$$\sum_{i \in E} \pi_i p_{i0.5} = \pi_{0.5} p_{0.5,0.5} = 1 = \pi_{0.5}$$

and

$$\sum_{i \in E} \pi_i p_{ij} = \pi_{0.5} p_{0.5,j} = p_{0.5,j} = 0 = \pi_j$$

for all $j \in \mathbb{Z}$. $\boxed{5, \mathrm{D}}$

2. (a) In this question, we use the following notation from the lecture notes: We write $\nu^{(n)} = (\mathrm{P}(X_n = 1), \mathrm{P}(X_n = 2), \mathrm{P}(X_n = 3))$ for $n \in \mathbb{N}_0$. meth seen $\Downarrow$

(i) We recall from lectures that $\nu^{(n)} = \nu^{(0)}\mathbf{P}^n$. We observe that (similar to the problem in the second progress test), the transition matrix $\mathbf{P}$ is idempotent since $\mathbf{P}^2 = \mathbf{P}$. Hence $\mathbf{P}^n = \mathbf{P}$ for all $n \in \mathbb{N}$. Hence the probability mass function of X_n can be expressed in terms of the marginal distribution of X_n:

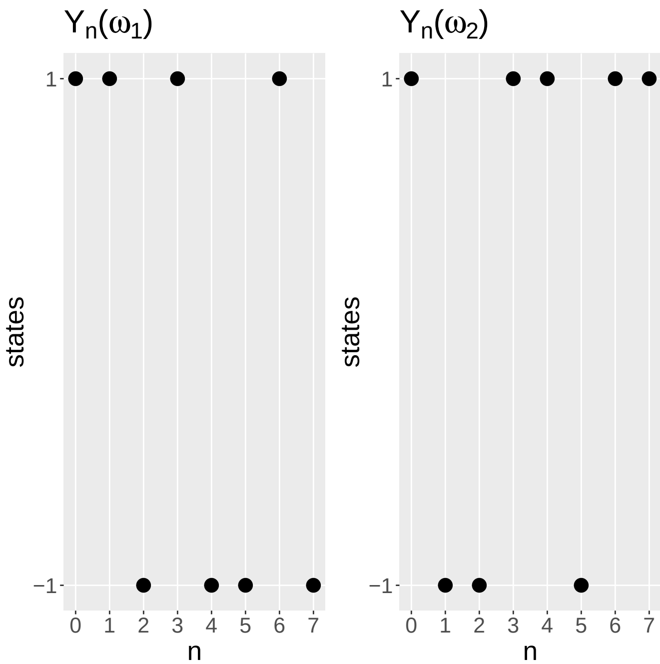
$$\nu^{(0)} = (\mathrm{P}(X_0 = 1), \mathrm{P}(X_0 = 2), \mathrm{P}(X_0 = 3)) = (0, 1, 0).$$

For $n \in \mathbb{N}$, we have

$$\nu^{(n)} = (\mathrm{P}(X_n = 1), \mathrm{P}(X_n = 2), \mathrm{P}(X_n = 3)) = \nu^{(0)}\mathbf{P} = \left(\frac{1}{4}, \frac{3}{8}, \frac{3}{8}\right).$$

Also, $\mathrm{P}(X_n = x) = 0$ for all $x \notin \{1, 2, 3\}$ and $n \in \mathbb{N}_0$. 3, B

(ii) The process Y can take the values -1 and 1. Also, $\mathrm{P}(Y_0 = 1) = 1$. Two distinct realisations of Y are given below: meth seen $\Downarrow$



2, A

(iii) Recall that the state space of Y is given by $E_Y = \{-1, 1\}$. Then, using the notation and results from lectures, meth seen $\Downarrow$

$$\mathrm{P}(Y_n = 1) = \mathrm{P}((-1)^{X_n} = 1) = \mathrm{P}(X_n \text{ even}) = \mathrm{P}(X_n = 2) = \nu_2^{(n)} = (\nu^{(0)}\mathbf{P}^n)_2.$$

Since the transition matrix $\mathbf{P}$ is idempotent, $\mathbf{P}^n = \mathbf{P}$ for all $n \in \mathbb{N}$. Hence, for $n \in \mathbb{N}$,

$$\mathrm{P}(Y_n = 1) = (\nu^{(0)}\mathbf{P}^n)_2 = \frac{3}{8}.$$

Then

$$\mathrm{P}(Y_n = -1) = 1 - \frac{3}{8} = \frac{5}{8},$$

and $\mathrm{P}(Y_n = y) = 0$ for all $y \notin \{-1, 1\}$.
For $n = 0$, we have $\mathrm{P}(Y_0 = 1) = 1$ and $\mathrm{P}(Y_0 = y) = 0$ for all $y \neq 1$. 4, D

(iv) First we show that the random variables $X_n, n \in \mathbb{N}_0$ are independent (which is a consequence of the structure of $\mathbf{P}$): For any $J \in \mathbb{N}$ and any $i_1, \dots i_J \in \mathbb{N}_0$ with $i_1 < i_2 < \cdots < i_J$, and $x_1, \dots, x_J \in E$, we have, by lectures, since X is a Markov chain, $\boxed{\text{unseen} \Downarrow}$

$$
\begin{aligned}
&\mathrm{P}(X_{i_1} = x_1, \cdots, X_{i_J} = x_J) \\
&= \mathrm{P}(X_{i_J} = x_J | X_{i_{J-1}} = x_{J-1}) \mathrm{P}(X_{i_{J-1}} = x_{J-1} | X_{i_{J-2}} = x_{J-2}) \\
&\qquad \cdots \mathrm{P}(X_{i_2} = x_2 | X_{i_1} = x_1) \mathrm{P}(X_{i_1} = x_1) \\
&= \mathrm{P}(X_{i_J} = x_J) \mathrm{P}(X_{i_{J-1}} = x_{J-1}) \cdots \mathrm{P}(X_{i_2} = x_2) \mathrm{P}(X_{i_1} = x_1),
\end{aligned}
$$

where, in the last step, we used the fact that $\mathbf{P}$ is idempotent, and the transition probability of going to a particular state does not depend on the current state. Hence the random variables $X_n, n \in \mathbb{N}_0$ are independent. Hence the transformed random variables $Y_n, n \in \mathbb{N}_0$ are independent, too (by lectures). $\boxed{4, \mathrm{C}}$

(v) We recall that a sequence of independent random variables is trivially a Markov chain (as discussed in lectures). Hence we can conclude from our findings in (iv) that Y is a Markov chain. The state space of Y is given by $E_Y = \{-1, 1\}$. Since the Y_n are independent, the transition probabilities are equal to the marginal probabilities, i.e. $\boxed{\text{unseen} \Downarrow}$

$$\mathrm{P}(Y_n = i | Y_{n-1} = j) = \mathrm{P}(Y_n = i)$$

for all $i, j \in \{-1, 1\}$. Hence the one-step transition matrix of Y is given by

$$\boldsymbol{P}^Y = \begin{pmatrix} \frac{5}{8} & \frac{3}{8} \\ \frac{5}{8} & \frac{3}{8} \end{pmatrix}.$$

$\boxed{3, \mathrm{C}}$

(b) (i) First, we read off the transition matrix using the following labels for the states: 1 - Paris, 2 - London, 3 - Madrid, 4 - Nice, 5 - Brighton, 6 - Barcelona. Then

meth seen ⇓

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & \frac{1}{5} & \frac{1}{5} \\ 0 & \frac{1}{4} & \frac{1}{2} & 0 & 0 & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{4} & 0 & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

First, we compute the marginal distribution of the Markov chain at time $n = 2$ (using the results from the lectures)

$$\nu^{(2)} = \nu^{(0)}\mathbf{P}^2 = (\nu^{(0)}\mathbf{P})\mathbf{P} = ((0,1,0,0,0,0)\mathbf{P})\mathbf{P} = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 0, \frac{1}{5}, \frac{1}{5}\right)\mathbf{P}$$
$$= \left(\frac{14}{100}, \frac{34}{100}, \frac{19}{100}, \frac{5}{100}, \frac{9}{100}, \frac{19}{100}\right).$$

Hence the probability that the graduate will be on the beach at time $n = 2$, will be

$$\sum_{j=4}^{6} \nu_j^{(2)} = \frac{5+9+19}{100} = \frac{33}{100}.$$

2, A

[Alternatively, students could read off the corresponding transition probabilities directly from the transition diagram and state that

$$p_{\text{Lon,Nic}}(2) = 0.2 \cdot 0.25 = \frac{1}{20},$$
$$p_{\text{Lon,Bri}}(2) = 0.2 \cdot 0.25 + 0.2 \cdot 0.2 = \frac{9}{100},$$
$$p_{\text{Lon,Bar}}(2) = 0.2 \cdot 0.5 + 0.2 \cdot 0.2 + 0.2 \cdot 0.25 = \frac{19}{100}.$$

Hence, the probability that the graduate will be at the beach at time $n = 2$ will be equal to $\frac{1}{20} + \frac{9}{100} + \frac{19}{100} = \frac{33}{100}$.]

unseen ⇓

(ii) Various of our modelling assumptions are not very realistic, for instance,

- Markov property: The Markov property might not hold, e.g. the future given the present might not be independent of the past. For instance, if the person has already been to three beach locations, this might change the probability of going to London in the next step, compared to if they have only been to one beach destination etc.
- The time-homogeneity assumption might need to be relaxed: For instance, we could imagine that the transition probabilities change with the seasons/weather.
- The choice of the transition probabilities is arbitrary and is not based on any empirical information. It is not clear, why it should not be possible to travel from London to Nice in one step, say. Also, the transition probabilities for staying in some locations (eg in London) is rather low (1/5) which is probably not very realistic.

2, B

3. (a) (i) We can model the arrivals of the cars driving above the speed limit by a Poisson process $N = (N_t)_{t \geq 0}$ with rate $\lambda > 0$. By lectures, we know that $\mathrm{E}(N_t) = \lambda t$. Here, time is recorded in hours, and we have $\mathrm{E}(N_1) = 5$, hence $\lambda = 5$.

meth seen ⇓

1, A

(ii) By the stationarity of the increments of the Poisson process, we find that

1, B

$$p = \mathrm{P}(N_{0.5} = 1) = \frac{(0.5 \cdot 5)^1}{1!} e^{-0.5 \cdot 5} = 2.5e^{-2.5}.$$

2, A

(iii) We could imagine that the number of cars driving above the speed limit might depend on the time of the day, e.g. drivers might drive significantly faster at night time when the highway is not very busy. A Poisson process cannot take such features into account, whereas a non-homogeneous Poisson process could.

2, A

(b) From lectures we know that $N_t \sim \mathrm{Poi}(m(t))$, where

meth seen ⇓

$$m(t) = \int_0^t \lambda(s)ds = \int_0^t (a + bs)ds = at + \frac{b}{2}t^2.$$

1, A

Hence,

$$\mathrm{P}(N_t = 0) = \exp(-(m(t)) = \exp\left(-at - \frac{b}{2}t^2\right).$$

We obtain the system of equations

$$\mathrm{P}(N_1 = 0) = \exp(-(m(1)) = \exp\left(-a - \frac{b}{2}\right) = \exp(-2),$$
$$\mathrm{P}(N_2 = 0) = \exp(-(m(2)) = \exp\left(-2a - \frac{b}{2}2^2\right) = \exp(-2(a + b)) = \exp(-6),$$

which is equivalent to

$$a + \frac{b}{2} = 2, 2(a + b) = 6 \Leftrightarrow a = 2 - \frac{b}{2}, a = 3 - b$$
$$\Leftrightarrow 2 - \frac{b}{2} = 3 - b, a = 3 - b \Leftrightarrow a = 1, b = 2.$$

2, C

(c) (i) Yes, according to lectures, the sum (superposition) of two independent Poisson processes is a Poisson process and the rates add up. I.e. X is a Poisson process with rate $\lambda + \lambda^{(1)}$ and Y is a Poisson process with rate $\lambda + \lambda^{(2)}$.

seen ⇓

2, A

meth seen ⇓

(ii) Let $|u| \leq 1, |v| \leq 1$. We have $X_t \sim \mathrm{Poi}(t(\lambda + \lambda^{(1)}))$ and $Y_t \sim \mathrm{Poi}(t(\lambda + \lambda^{(2)}))$. Hence

$$\mathrm{E}(u^{X_t}) = \sum_{x=0}^{\infty} u^x \mathrm{P}(X_t = x) = \sum_{x=0}^{\infty} u^x \frac{[t(\lambda + \lambda^{(1)})]^x}{x!} e^{-[t(\lambda + \lambda^{(1)})]}$$
$$= \exp[t(\lambda + \lambda^{(1)})(u - 1)].$$

Similarly,

$$\mathrm{E}(v^{Y_t}) = \exp[t(\lambda + \lambda^{(2)})(v - 1)].$$

2, A

(iii) Let $|u| \leq 1, |v| \leq 1$. Then the joint probability generating function of X_t and Y_t is given by unseen $\Downarrow$

$$
\begin{aligned}
\mathrm{E}(u^{X_t} v^{Y_t}) &= \mathrm{E}(u^{N_t^{(1)}} v^{N_t^{(2)}} (uv)^{N_t}) \\
&\overset{N, N^{(1)}, N^{(2)} \text{indep.}}{=} \mathrm{E}(u^{N_t^{(1)}}) \mathrm{E}(v^{N_t^{(2)}}) \mathrm{E}((uv)^{N_t}) \\
&= \exp(\lambda^{(1)} t(u-1) + \lambda^{(2)} t(v-1) + \lambda t(uv-1)),
\end{aligned}
$$

where we used the probability generating function for a Poisson random variable derived in (ii) and noting that $N_t \sim \mathrm{Poi}(\lambda t), N_t^{(1)} \sim \mathrm{Poi}(\lambda^{(1)} t), N_t^{(2)} \sim \mathrm{Poi}(\lambda^{(2)} t)$.

3, C

(iv) The process Z is not a Poisson process since Z_t does not have the Poisson distribution. To see this, note that, from (iii), we derive for $v = u$ and $|u| \leq 1$ that unseen $\Downarrow$

$$
\begin{aligned}
\mathrm{E}(u^{Z_t}) &= \mathrm{E}(u^{X_t} u^{Y_t}) = \exp(\lambda^{(1)} t(u-1) + \lambda^{(2)} t(u-1) + \lambda t(u^2-1)) \\
&= \exp((\lambda^{(1)} + \lambda^{(2)}) t(u-1) + \lambda t(u^2-1)).
\end{aligned}
$$

Unless $\lambda = 0$ we observe that the probability generating function (pgf) of Z_t does not have the form of the pgf of a Poisson random variable. Since the pgf characterises the distribution of a discrete random variable uniquely, we conclude that Z_t does not have a Poisson distribution, hence Z cannot be a Poisson process.

4, D

4. (a) (i) The transition probabilities of the associated embedded chain are given by $p_{ij} = g_{ij}/(-g_{ii})$ for $i, j \in E, i \neq j$ (since $g_{ii} \neq 0$ for all $i \in E$) and $p_{ii} = 0$. Hence, the transition matrix of the embedded jump chain is given by meth seen ⇓

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}.$$

3, B

(ii) From the problem classes, we know that a state is recurrent (transient) for X if it is recurrent (transient) for the corresponding jump chain. We observe that the jump chain has two communicating classes: $C_1 = \{1, 2\}$ and $T = \{3, 4\}$. C_1 is finite and closed, hence (positive) recurrent and T is not closed, hence transient. So we conclude that the states 1 and 2 are recurrent for X and 3 and 4 are transient for X. 3, A

(iii) We assume that the initial distribution unseen ⇓

$$\boldsymbol{\nu}^{(0)} = (\mathrm{P}(X_0 = 1), \mathrm{P}(X_0 = 2), \mathrm{P}(X_0 = 3), \mathrm{P}(X_0 = 4))$$

is known.
First we draw the initial value i_0 of the Markov chain from the distribution $\boldsymbol{\nu}^{(0)}$. [Students may assume a fixed starting point, but should state this assumption clearly.]
We can think of the continuous-time Markov-chain in terms of jump chains and holding times. So, in each state, i say, we wait for an exponentially distributed amount of time (with rate specified by $-g_{ii}$), and then we jump to the next state, j say, according to the transition probabilities specified in the transition matrix of the corresponding jump chain $p_{ij}^Z = -g_{ij}/g_{ii}$.
We can simulate the corresponding jump-chain as follows. The initial values of the jump chain $Z = (Z_n)_{n \in \mathbb{N}_0}$ is the same as the initial value of X, i.e. we set $z_0 = x_0 = i_0$ for the corresponding realisation. For $z_n = i$, we draw a realisation j from the distribution $(p_{i1}^Z, p_{i2}^Z, p_{i3}^Z, p_{i4}^Z)$ (with the probabilities specified in (i)) and obtain the next realisation z_{n+1} of the jump chain. I.e. we set $z_{n+1} = j$ with probability p_{ij}^Z for $j \in E$ etc.
We also need to simulate the time points, when X switches between the realisations of the jump chain. For this, we simulate the sequence of realisations (h_n) of the exponential holding times corresponding to the realisations of Z. I.e. the holding times in states 1, 2, 3 and 4 follow an Exp(1/4), Exp(2), Exp(1/5) and Exp(3) distribution, respectively. Also, let $j_0 = 0$ and $j_n = \sum_{k=1}^{n} h_k$ denote the realisations of the jump times of the Markov chain. Then we set $x_0 = i_0$ and, for $z_n = i$, we set $x_t = i$ for $j_n \leq t < j_{n+1}$. We typically fix a finite time point $t^* > 0$, say, and terminate the algorithm when $j_n > t^*$. 3, D

(b) (i) The state space of X is given by $E = \{0, 1, 2, 3, 4, 5\}$ since people will not join the queue if there are already five people in the queue. unseen ⇓ 1, A

(ii) Let $N = (N_t)_{t \geq 0}$ described the Poisson process of people arriving, where $N_t \sim \mathrm{Poi}(\lambda t)$ for rate $\lambda > 0$, $\mathrm{E}(N_t) = \lambda t$. The arrival rate of the people is quoted as one person every 3 minutes, i.e. $\mathrm{E}(N_3) = 1$, hence $\lambda \cdot 3 = 1 \Leftrightarrow \lambda = 1/3$. Similarly, the parameter in the exponential distribution of the GP's service time is given by $1/6$. To see this, let H be the exponential service time with parameter $\mu > 0$, say. From lectures, we know that $\mathrm{E}(H) = 1/\mu = 6$. Hence, $\mu = 1/6$.

If there is no person in the queue, then the queue size goes up to one when a person arrives. If there are 5 people in the queue, then the queue size decreases by one if the GP has vaccinated one person and completed the paperwork.

If the queue length is either 1, 2, 3, or 4, then the queue length can either increase by one if an additional person arrives, or decrease by one if the GP completes his task.

I.e. the transition rates from state i to state j denoted by q_{ij} are given by

$$
\begin{aligned}
q_{01} = q_{12} = q_{23} = q_{34} = q_{45} = 1/3, \\
q_{10} = q_{21} = q_{32} = q_{43} = q_{54} = 1/6,
\end{aligned}
$$

and $q_{ij} = 0$ otherwise. Hence, the transition diagram is given by 3, B

$$
\begin{tikzcd}
0 \arrow[r, bend left, "1/3"] & 1 \arrow[l, bend left, "1/6"] \arrow[r, bend left, "1/3"] & 2 \arrow[l, bend left, "1/6"] \arrow[r, bend left, "1/3"] & 3 \arrow[l, bend left, "1/6"] \arrow[r, bend left, "1/3"] & 4 \arrow[l, bend left, "1/6"] \arrow[r, bend left, "1/3"] & 5 \arrow[l, bend left, "1/6"]
\end{tikzcd}
$$

1, A

(iii) According to lectures, for state $i \in E$, the holding time in state i follows an exponential distribution with rate $q_i = \sum_{j \in E} q_{ij}$. 1, A

Here we have

$$
q_0 = \frac{1}{3}, q_1 = q_2 = q_3 = q_4 = \frac{1}{2}, q_5 = \frac{1}{6}.
$$

2, B

(iv) The transition probabilities of the associated embedded chain are given by $p_{ij} = q_{ij}/q_i$. Hence, the transition matrix of the embedded jump chain is given by 1, A

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 \\
0 & 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}.
$$

2, B

5. (a) [Right-differentiability shown in lectures, extensions discussed in additional reading material, see Norris (1997), p. 77.] meth seen ⇓

(i) Let $n = 0, t \geq 0, \delta > 0$, then (as in lectures)

$$\begin{aligned}
p_0(t+\delta) &= \mathrm{P}(N_{t+\delta} = 0) = \mathrm{P}(\text{no event in } [0, t+\delta]) \\
&= \mathrm{P}(\text{no event in}[0,t] \text{ and no event in}(t, t+\delta]) \\
&= \mathrm{P}(N_t = 0, N_{t+\delta} - N_t = 0) \\
&\overset{\text{ind. incr.}}{=} \mathrm{P}(N_t = 0)\mathrm{P}(N_{t+\delta} - N_t = 0) \\
&\overset{\text{stat. incr.}}{=} \mathrm{P}(N_t = 0)\mathrm{P}(N_\delta = 0) \\
&\overset{\text{singl. arrival}}{=} p_0(t)[1 - \lambda\delta + o(\delta)].
\end{aligned}$$

Hence we have

$$p_0(t+\delta) - p_0(t) = p_0(t)[-\lambda\delta + o(\delta)],$$

taking the limit leads to

$$\lim_{\delta \to 0}(p_0(t+\delta) - p_0(t)) = 0,$$

since $p_0(t) \in [0,1]$ and $\lim_{\delta\to 0}[-\lambda\delta + o(\delta)] = 0$. Hence p_0 is right-continuous.
Setting $t = s - \delta \geq 0$ in the above computation leads to

$$p_0(s) - p_0(s-\delta) = p_0(s-\delta)[-\lambda\delta + o(\delta)],$$

taking the limit leads to

$$\lim_{\delta \to 0}(p_0(s) - p_0(s-\delta)) = 0,$$

since $p_0(s-\delta) \in [0,1]$ and $\lim_{\delta\to 0}[-\lambda\delta + o(\delta)] = 0$. Hence p_0 is left-continuous.
We have shown that p_0 is both left- and right-continuous, hence it is continuous. 3, M

(ii) Using the same notation as above, we get

$$\frac{p_0(t+\delta) - p_0(t)}{\delta} = -\lambda p_0(t) + \frac{o(\delta)}{\delta},$$

and

$$\frac{p_0(s) - p_0(s-\delta)}{\delta} = -\lambda p_0(s-\delta) + \frac{o(\delta)}{\delta}.$$

Letting $\delta \downarrow 0$, and using the left-continuity of p_0, leads to

$$\begin{aligned}
\lim_{\delta\to 0} \frac{p_0(t+\delta) - p_0(t)}{\delta} &= -\lambda p_0(t), \\
\lim_{\delta\to 0} \frac{p_0(s) - p_0(s-\delta)}{\delta} &= -\lambda p_0(s),
\end{aligned}$$

where we note that the left and the right limit coincide when $s = t$. Hence, p_0 is differentiable. 3, M

(b) According to the strong Markov property for a Poisson process [Norris (1997), p. 76], conditional on $T < \infty$ (which is satisfied by assumption), the process $X = (X_t)_{t \geq 0}$ with

$$X_t = N_{T+t} - N_T,$$

is also a Poisson process of rate λ, independent of $(N_s)_{s \leq T}$. Hence, we can choose

$$f(x, y) = x - y,$$

which implies that $X_t = N_{T+t} - N_T$.

$\boxed{4, \mathrm{M}}$

(c) (i) We denote the stationary distribution of X by $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$. We solve $\boldsymbol{\lambda}\mathbf{G} = \mathbf{0}$, for $\lambda_1, \lambda_2 \geq 0, \lambda_1 + \lambda_2 = 1$. Here we have $-\frac{\lambda_1}{3} + \frac{\lambda_2}{3} = 0 \Leftrightarrow \lambda_1 = \lambda_2$. Then $1 = \lambda_1 + \lambda_2 \Rightarrow 1 = 2\lambda_2 \Rightarrow \lambda_1 = \frac{1}{2}, \lambda_2 = \frac{1}{2}$.

$\boxed{2, \mathrm{M}}$

Since $\mathbf{G}$ is irreducible and recurrent, we get that $\boldsymbol{\lambda}\mathbf{G} = \mathbf{0} \Leftrightarrow \boldsymbol{\lambda}\mathbf{P}_t = \boldsymbol{\lambda}$ for all $t \geq 0$, where $(\mathbf{P}_t)_{t \geq 0}$ denotes the matrix of transition probabilities associated with X.

$\boxed{2, \mathrm{M}}$

(ii) First, we want to represent the generator as $\mathbf{G} = \mathbf{O}\mathbf{D}\mathbf{O}^{-1}$ for a diagonal matrix $\mathbf{D}$ and an invertible matrix $\mathbf{O}$. We derive the eigenvalues of $\mathbf{G}$:

$$0 = \det(\mathbf{G} - \lambda\mathbf{I}) = \det\begin{pmatrix} -\frac{1}{3} - \lambda & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} - \lambda \end{pmatrix} = \left(-\frac{1}{3} - \lambda\right)^2 - \frac{1}{9} = \lambda\left(\lambda + \frac{2}{3}\right).$$

Hence the eigenvalues are given by $\lambda_1 = 0, \lambda_2 = -\frac{2}{3}$. Hence $\mathbf{D} = \text{diag}\left(0, -\frac{2}{3}\right)$.
Next, we compute the eigenvectors $\mathbf{v}^{(\lambda_i)}$ for $i = 1, 2$. We have $\lambda_1\mathbf{v}^{(\lambda_1)} = \mathbf{G}\mathbf{v}^{(\lambda_1)} \Leftrightarrow \mathbf{G}\mathbf{v}^{(\lambda_1)} = \mathbf{0} \Leftrightarrow -\frac{1}{3}v_1^{(\lambda_1)} + \frac{1}{3}v_2^{(\lambda_1)} = 0 \Leftrightarrow v_1^{(\lambda_1)} = v_2^{(\lambda_1)}$. We choose $\mathbf{v}^{(\lambda_1)} = (1, 1)^\top$.
Also, $\lambda_2\mathbf{v}^{(\lambda_2)} = \mathbf{G}\mathbf{v}^{(\lambda_2)} \Leftrightarrow -\frac{2}{3}v_1^{(\lambda_2)} = -\frac{1}{3}v_1^{(\lambda_2)} + \frac{1}{3}v_2^{(\lambda_2)} \Leftrightarrow -v_1^{(\lambda_2)} = v_2^{(\lambda_2)}$. We choose $\mathbf{v}^{(\lambda_2)} = (1, -1)^\top$.
Set $\mathbf{O} = (\mathbf{v}^{(\lambda_1)}, \mathbf{v}^{(\lambda_2)}) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Then $\mathbf{O}^{-1} = \frac{1}{-2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$. From lectures, we know that, for $t \geq 0$, we have

$\boxed{2, \mathrm{M}}$

$$\begin{aligned}
\mathbf{P}_t = e^{t\mathbf{G}} &= \sum_{n=0}^{\infty} \frac{t^n}{n!}\mathbf{O}\mathbf{D}^n\mathbf{O}^{-1} = \mathbf{O}\sum_{n=0}^{\infty} \frac{t^n}{n!}\mathbf{D}^n\mathbf{O}^{-1} \\
&= \mathbf{O}\text{diag}\left(\sum_{n=0}^{\infty} \frac{\lambda_1^n t^n}{n!}, \sum_{n=0}^{\infty} \frac{\lambda_2^n t^n}{n!}\right)\mathbf{O}^{-1} = \mathbf{O}\text{diag}\left(1, e^{-\frac{2}{3}t}\right)\mathbf{O}^{-1} \\
&= \begin{pmatrix} 1 & e^{-\frac{2}{3}t} \\ 1 & -e^{-\frac{2}{3}t} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 + e^{-\frac{2}{3}t} & 1 - e^{-\frac{2}{3}t} \\ 1 - e^{-\frac{2}{3}t} & 1 + e^{-\frac{2}{3}t} \end{pmatrix}.
\end{aligned}$$

$\boxed{2, \mathrm{M}}$

(iii) We have

$$\lim_{t\to\infty} \mathbf{P}_t = \lim_{t\to\infty} \frac{1}{2} \begin{pmatrix} 1+e^{-\frac{2}{3}t} & 1-e^{-\frac{2}{3}t} \\ 1-e^{-\frac{2}{3}t} & 1+e^{-\frac{2}{3}t} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

$\boxed{1, \mathrm{M}}$

We know from a theorem from the lectures that for an irreducible Markov chain which has a stationary distribution (in our case given by $\boldsymbol{\lambda} = (\frac{1}{2}, \frac{1}{2})$) the transition probabilities converge: $\lim_{t\to\infty} p_{ij}(t) = \lambda_j$ for $i, j \in E$. This is exactly what we observe in our example, where each row of $\mathbf{P}_t$ converges to the stationary distribution. $\boxed{1, \mathrm{M}}$

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
Applied Probability: MATH96052 and MATH97083	1	Q1 was generally answered well. Some students did not provide sufficient justifications for their answers in a) and b). Part c) was the most difficult part of this question. Very few students got this question completely right, but many good attempts have been made.
	2	Many students managed to answer 2a) i), ii), iii), v) correctly. Many struggled with iv). Those who struggled with iv) were mostly not aware how independence between multiple random variables is defined and hence focused only on proving pairwise independence. Part b) was generally answered well, with some calculation errors in i) and some students not providing sufficient mathematical detail in ii).
	3	(a) Part (i) most students incorrectly used a Poisson model instead of a Poisson process. Most students answered part (ii) and (iii) correctly. (b) Most students did well in this part by identifying that for a non-homogeneous process the distribution of N1 and N2 can be found by integrating the rate. (c) Most students made the error of assuming that Xt and Yt are independent in parts (iii) and (iv). But since they are not independent, one needs to write Xt and Yt in terms of the Nt's to find the expectation in part (iii). Then you can use the result in part (iii) to find the pgf of Zt in part (iv) to observe that it is not a Poisson process.
	4	Many students lost marks because they either did not provide sufficient explanation or did not define symbols not already introduced.
	5	Good attempts have been made in aswering the mastery question. Part a) was often done well, but several students forgot to check the left limits in their proofs. Part b) was also done well, but some students gave incomplete justificiations. Part c) was a routine question and generally done well apart from occasional minor calculation errors.