(1) office hour (in Huxley 661)
1330 – 1400 , 1415 – 1445 .

(1.2.4) Def. Suppose Γ is a set of L-formulas. A deduction from Γ is a finite sequence of L-fmlas

$$\phi_1, \phi_2, \dots, \phi_n$$

such that each ϕ_i is either

an axiom,
a formula in Γ,

or is obtained from previous formulas $\phi_1, \dots, \phi_{i-1}$ using the deduction rule MP.

(n here is the length of the deduction).

Write $\Gamma \vdash_L \phi$

if there is a deduction from Γ ending with ϕ. Say that ϕ is a consequence of Γ.

Note: $\emptyset \vdash_L \phi$ is the same

empty set

as 'ϕ is a theorem of L'.

$$\vdash_L \phi .$$

(1.2.5) Thm (The Deduction Theorem)
Suppose Γ is a set of L-formulas and ϕ, ψ are L-fmlas. Suppose

$$\Gamma \cup \{\phi\} \vdash_L \psi$$

Then $\Gamma \vdash_L (\phi \to \psi)$.

(1.2.6) Cor. (Hypothetical Syllogism HS)

Suppose ϕ, ψ, χ are L-fmlas.

and $\vdash_L (\phi \to \psi)$

and $\vdash_L (\psi \to \chi)$.

Then $\vdash_L (\phi \to \chi)$.

Proof: Use DT with $\Gamma = \emptyset$.

Show $\{\phi\} \vdash_L \chi$

[Then by DT $\emptyset \vdash_L (\phi \to \chi)$

so $\vdash_L (\phi \to \chi)$.].

" Show there is a deduction of χ from $\{\phi\}$

1. $(\phi \to \psi)$ (theorem of L)
2. $(\psi \to \chi)$ (thm. of L)
3. ϕ (Deduction from ϕ)
4. ψ (1,3 + MP).
5. χ (2,4 + MP).

Thus: $\{\phi\} \vdash_L \chi$. #

(1.2.7) Proposition Suppose ϕ, ψ are L-formulas. Then:

(a) $\vdash_L ((\neg\psi) \to (\psi \to \phi))$

(b) $\{(\neg\psi), \psi\} \vdash_L \phi$

(c) $\vdash_L (((\neg\phi) \to \phi) \to \phi)$

Pf: (a) P. sheet 1.

(b) Deduction of ϕ from $\{(\neg\psi), \psi\}$:

~~Us~~ use (a) & MP twice to obtain ϕ.

(c) Suppose χ is any formula. Then by (b) & MP

$$\{(\neg\phi), ((\neg\phi) \to \phi)\} \vdash_L \chi \quad (3)$$

Let α be an axiom and let χ be $(\neg\alpha)$.

So $\{(\neg\phi), ((\neg\phi) \to \phi)\} \vdash_L (\neg\alpha)$ ... (1)

By DT:

$$\{((\neg\phi) \to \phi)\} \vdash_L ((\neg\phi) \to (\neg\alpha))$$

Axiom A3:

$$\vdash_L (((\neg\phi) \to (\neg\alpha)) \to (\alpha \to \phi)) \quad \ldots (2)$$

(1), (2) + MP gives

$$\{((\neg\phi) \to \phi)\} \vdash_L (\alpha \to \phi)$$

As α is an axiom we get (by MP)

$$\{((\neg\phi) \to \phi)\} \vdash_L \phi.$$

Use DT. #

Pf. of Deduction Thm: Show:

if $\Gamma \cup \{\phi\} \vdash_L \psi$

then $\Gamma \vdash_L (\phi \to \psi)$.

Suppose $\Gamma \cup \{\phi\} \vdash_L \psi$ using a deduction of length n.

Prove by induction on n that

$\Gamma \vdash_L (\phi \to \psi)$.

Base step $n = 1$. In this case ψ is either an axiom

or in Γ

or it is ϕ.

In the first two cases

$$\Gamma \vdash_L \psi$$

Then use the A1 axiom

$$\vdash_L (\psi \to (\phi \to \psi))$$

& MP to get

$$\Gamma \vdash_L (\phi \to \psi).$$

If ψ is ϕ then

$$\Gamma \vdash_L (\phi \to \phi)$$

by 1.2.3. This does the base case.

Inductive step: Suppose the result holds for shorter deductions (i.e. length $< n$).

In our deduction $\Gamma \cup \{\phi\} \vdash_L \psi$

either:

(a) ψ is an axiom, or in Γ

or is equal to ϕ

or (b) ψ is obtained from earlier formulas χ, $(\chi \to \psi)$ in the deduction using MP.

In case (a), we argue as in the base case to get

$$\Gamma \vdash_L (\phi \to \psi).$$

In case (b) we have

$$\Gamma \cup \{\phi\} \vdash_L \chi$$

and $\Gamma \cup \{\phi\} \vdash_L (\chi \to \psi)$

using shorter deductions

So by inductive hypothesis:

$$\Gamma \vdash (\phi \to \chi) \quad \ldots 1$$

& $\Gamma \vdash (\phi \to (\chi \to \psi))$ ... 2

Now use A2

$$\vdash_L ((\phi \to (\chi \to \psi)) \to ((\phi \to \chi) \to (\phi \to \psi)))$$ 5

to get using 2 & MP:

$$\Gamma \vdash ((\phi \to \chi) \to (\phi \to \psi)).$$

Using 1 & MP gives:

$$\Gamma \vdash (\phi \to \psi),$$

as required. ~~//~~