Mathematical Logic (MATH60132 and MATH70132)
Coursework 1

This coursework is worth 5 percent of the module. The deadline for submitting the work is 1300 on Monday 7 November 2022. The coursework is marked out of 20 and the marks per question are indicated below.
The work which you submit should be your own, unaided work. Any quotation of a result from the notes or problem sheets must be clear. If you use any source (including internet or books) other than the lecture notes and problem sheets, you must provide a full reference for your source. Failure to do so could constitute plagiarism.

[1] (5 marks) Give a propositional formula ϕ in propositional variables p_1, p_2, p_3, p_4 (using any of the connectives $\neg, \to, \wedge, \vee$) which has the property that, for every propositional valuation v:

$$v(\phi) = T \text{ if and only if exactly } 3 \text{ of } v(p_1), v(p_2), v(p_3), v(p_4) \text{ have value } T.$$

Justify your answer concisely (so, without writing down the full truth table of ϕ).

[2] (5 marks) Suppose ϕ, ψ and χ are L-formulas. In the lectures we defined $(\phi \vee \psi)$ to be shorthand for $((\neg\phi) \to \psi)$. For each of the following, write the formula without using this shorthand and prove that it is a theorem of L. You may use results and theorems of L from your notes, but do not use the Completeness Theorem for L.

(i) $(\phi \vee \phi)$;

(ii) $((\phi \vee \psi) \to (\psi \vee \phi))$.

[3] Here are two definitions which are not in the notes:

- Suppose $\Sigma \cup \{\psi\}$ is a set of L-formulas. We say that ψ is *independent from* Σ if and only if $\Sigma \nvdash_L \psi$ and $\Sigma \nvdash_L (\neg\psi)$.
- A set Δ of L-formulas is *independent* if and only if Δ is consistent and for every $\psi \in \Delta$ we have that ψ is independent from $\Delta \setminus \{\psi\}$.

In the following, you may use any results from the notes and problem sheets.

(i) **(2 marks)** Suppose Σ is a consistent set of L-formulas and ψ is an L-formula. Show that ψ is independent from Σ if and only if there exist valuations v, v' such that $v(\Sigma) = v'(\Sigma) = T$ and $v(\psi) = T$, $v'(\psi) = F$.

(ii) **(1 mark)** Using (i), give a formulation in terms of valuations of what it means for a set Δ of L-formulas to be independent.

(iii) **(4 marks)** Is the following set of L-formulas independent? Justify your answer.

$$\Delta = \{(p_1 \to ((\neg p_2) \to p_3)),\ ((\neg p_3) \to (p_2 \to (\neg p_1))),\ (p_3 \to (p_1 \to (\neg p_2)))\}.$$

(iv) **(3 marks)** In the following, we write 0 for F and 1 for T; addition and multiplication on $\{0, 1\}$ are modulo 2, giving the field $\mathbb{F}_2$ with two elements. The set $V(n)$ of truth functions $F : \{0, 1\}^n \to \{0, 1\}$ is then a vector space over the field $\mathbb{F}_2$.

Suppose that $\sigma_1, \ldots, \sigma_k$ are L-formulas in the propositional variables $p_1, \ldots, p_n$ and let $F_1, \ldots, F_k \in V(n)$ denote the truth functions of these. Prove that if $\{\sigma_1, \ldots, \sigma_k\}$ is an independent set of formulas, then $F_1, \ldots, F_k$ are linearly independent.