L2 + L3

$$\Big(\underbrace{\big(((p\to q)\wedge(q\to(\neg p))\big)}_{\psi}\to\underbrace{(\neg p)}_{\chi}\Big) \quad : \phi$$

p	q	ψ	χ	ϕ
T	T	F	F	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

(1.1.4) Def ① A formula ϕ is a tautology if its truth function F_ϕ always has value T.

② Say that formulas ψ, χ with variables amongst $p_1, \dots, p_n$ are logically equivalent (l.e.) if they have the same truth function, i.e. $F_\psi = F_\chi$ (as functions of n variables)

Eg. $((p \to q) \to (q \to (\neg p)))$ is l.e. to $(\neg p)$.

(1.1.5) Remark

1) ψ, χ are l.e. if and only if $(\psi \leftrightarrow \chi)$ is a tautology.

2) Suppose ϕ is a formula with variables $p_1, \dots, p_n$ and $\phi_1, \dots, \phi_n$ are fmlas with variables $q_1, \dots, q_m$. For each $i \leq n$ substitute ϕ_i for p_i in ϕ. Then

(i) the result is a formula θ +

(ii) if ϕ is a tautology, then so is θ.

(1.1.6) Example Check

$(((\neg p_2) \to (\neg p_1)) \to (p_1 \to p_2))$

is a tautology. So if ϕ_1, ϕ_2 are any formulas, then

$(((\neg \phi_2) \to (\neg \phi_1)) \to (\phi_1 \to \phi_2))$

is a formula + in fact is a tautology.

Pf of 1.1.5(2).

(i) By induction on the number of connectives in ϕ.

(ii) Prove

$$F_\theta(q_1, \dots, q_m) = F_\phi(F_{\phi_1}(q_1, \dots, q_m), \dots, F_{\phi_n}(q_1, \dots, q_m))$$

by induction on the number of connectives in ϕ. #.

Ex: $(p_1 \to (\neg p_1))$ not a tautology, but you can find ϕ_1 with $(\phi_1 \to (\neg \phi_1))$ being a tautology.

Examples of l.e. formulas.

1) $(p_1 \wedge (p_2 \wedge p_3))$ is l.e. $((p_1 \wedge p_2) \wedge p_3)$

- usually omit brackets.

2) Similar with $\vee$

3) $(p_1 \vee (p_2 \wedge p_3))$ is l.e. $((p_1 \vee p_2) \wedge (p_1 \vee p_3))$

3') Similar with $\vee, \wedge$ interchanged

4) $(\neg(\neg p_1))$ is l.e. p_1

5) $(\neg(p_1 \wedge p_2))$ l.e. to $((\neg p_1) \vee (\neg p_2))$

5') ...

By 1.1.5 we obtain e.g. for formulas ϕ, ψ, χ

$(\phi \wedge (\psi \wedge \chi))$ is l.e. to $((\phi \wedge \psi) \wedge \chi)$ etc.

[See P. sheet 1.]

(1.1.7) Lemma. There are 2^{2^n} truth functions of n variables.

Pf: A truth fn. is a fn.

$$G : \{T, F\}^n \to \{T, F\}.$$

$$|\{T, F\}^n|^* = 2^n$$

and each $G(\bar{v})$ for $\bar{v} \in \{T, F\}^n$ has two possible values. #

(1.1.8) Def. Say that a set of connectives is adequate if for every $n \geq 1$, every truth fn. of n variables is the truth fn. of some formula which involves only connectives from the set and variables $p_1, \dots, p_n$.

(1.1.9) Thm The set $\{\neg, \wedge, \vee\}$ is adequate.

Disjunctive normal form

Proof: Let $G : \{T, F\}^n \to \{T, F\}$.

Case 1 $G(\bar{v}) = F$ for all $\bar{v} \in \{T, F\}^n$.

Take ϕ to be $(p_1 \wedge (\neg p_1))$.

Case 2 List the $\bar{v}$ with $G(\bar{v}) = T$ as $\bar{v}_1, \ldots, \bar{v}_r$.

Write $\bar{v}_i = (v_{i1}, \ldots, v_{in})$ where each $v_{ij} \in \{T, F\}$.

Define

$$q_{ij} = \begin{cases} p_j & \text{if } v_{ij} = T \\ (\neg p_j) & \text{if } v_{ij} = F \end{cases}$$

Let ψ_i be $(q_{i1} \wedge q_{i2} \wedge \ldots \wedge q_{in})$

then $F_{\psi_i}(\bar{v}) = T$

$\iff \bar{v} = \bar{v}_i$.

[$F_{\psi_i}(\bar{v}) = T \iff$ each q_{ij} is T

$\iff \bar{v} = \bar{v}_i$.]

Now let ϕ be $\psi_1 \vee \psi_2 \vee \ldots \vee \psi_r$.

Then $F_\phi(\bar{v}) = T \iff$

$F_{\psi_i}(\bar{v}) = T$ for some $i \leq r$

$\iff \bar{v} = \bar{v}_i$ for some $i \leq r$.

Thus $F_\phi(\bar{v}) = G(\bar{v})$ for all $\bar{v}$. #

Example: $n = 3$: $\bar{v} = (T, F, F)$

$p_1 \wedge (\neg p_2) \wedge (\neg p_3)$ has value T only at $(p_1, p_2, p_3) = (T, F, F)$.

A formula ϕ as in case 2 is said to be in <u>disjunctive normal form</u>.

(1.1.10) <u>Cor.</u> Suppose χ is a formula whose truth fn. is not always F then χ is l.e. to a formula in d.n.f. //.

[Apply Case 2 to F_χ.]

<u>Eg</u> $\chi : ((p_1 \to p_2) \to (\neg p_2))$

<u>n=2</u> $F_\chi(\bar{v}) = T$

$\Leftrightarrow \bar{v} = (T, F)$ or (F, F)

dnf:

$((p_1 \wedge (\neg p_2)) \vee ((\neg p_1) \wedge (\neg p_2)))$

(1.1.11) <u>Cor.</u> The following sets of connectives are adequate (5)

1) $\{\neg, \vee\}$

2) $\{\neg, \wedge\}$

3) $\{\neg, \to\}$

<u>Pf</u>: 1) By (1.1.9) enough to show that we can express $\wedge$ in terms of $\neg, \vee$:

$(p_1 \wedge p_2)$ is l.e. to $(\neg((\neg p_1) \vee (\neg p_2)))$

2) Similar

3) Express $\vee$ using $\neg, \to$.

$(p \vee q)$ l.e. to

$((\neg p) \to q)$ // #

(1.1.12) <u>Example</u> the following are <u>not</u> adequate:

(i) $\{\wedge, \vee\}$

(ii) $\{\neg, \leftrightarrow\}$

//

(1.1.13) <u>Example</u> NOR connective $\downarrow$ has truth table

p	q	$(p \downarrow q)$
T	T	F
T	F	F
F	T	F
F	F	T

$\{\downarrow\}$ is <u>adequate</u>:

$(\neg p)$ is l.e. to $(p \downarrow p)$

$(p \wedge q)$ is l.e. to

$((p \downarrow p) \downarrow (q \downarrow q))$.

So as $\{\neg, \wedge\}$ is adequate, $\{\downarrow\}$ is also adequate. //

(1.2) A formal system for propositional logic.

Idea: Try to generate all tautologies from certain 'basis assumptions' (axioms) using appropriate deduction rules.

(1.2.2) Def. The formal system L for propositional logic has the following:

Alphabet: variables $p_1, p_2, p_3, \ldots$
connectives $\neg$ $\to$
punctuation) (

Formulas Finite sequences ('strings') of symbols from the alphabet as follows (as in 1.1.2)

(a) Any variable p_i is a formula;
(b) If ϕ, ψ are formulas then so are $(\neg\phi)$ $(\phi \to \psi)$
(c) Any formula arises in this way.

L-formulas

Axioms Suppose ϕ, ψ, χ are L-formulas. The following are **axioms** of L:

(A1) $$(\phi \to (\psi \to \phi))$$

(A2) $$((\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi)))$$

(A3) $$(((\neg\psi) \to (\neg\phi)) \to (\phi \to \psi))$$

Deduction rule Modus Ponens (MP)

From ϕ $(\phi \to \psi)$

Deduce ψ

A **proof** in L is a finite sequence of L-formulas $\phi_1, \phi_2, \ldots, \phi_n$ such that each ϕ_i is either an **axiom** or is obtained from earlier formulas in the sequence by applying the deduction rule MP. The final formula in a proof is a **theorem** of L.

$$\ldots\ldots \phi \ldots (\phi \to \psi) \ldots \psi \ldots$$

applied MP.

Write $\vdash_L \phi$ to mean 'ϕ is a theorem of L'.

Note: ① Any axiom is a theorem of L.

② Every formula in a proof is a theorem of L.

(1.2.3) Example

Suppose ϕ is an L-formula. Then $\vdash_L (\phi \to \phi)$.

Here is a proof in L:

1. $\underbrace{(\phi \to ((\phi \to \phi) \to \phi))}_{\text{Call this } \chi}$ (A1)

2. $(\chi \to ((\phi \to (\phi \to \phi)) \to (\phi \to \phi)))$ (A2)

3. $((\phi \to (\phi \to \phi)) \to (\phi \to \phi))$ (1, 2 + MP)

4. $(\phi \to (\phi \to \phi))$ (A1)

5. $(\phi \to \phi)$ (by 3, 4 and MP).