M4P70 Markov Processes, Problems 2

Lecture 5, 14 Oct; Solutions Lecture 8, 21 Oct

We quote *Edgeworth's Theorem*: if a random vector $X = (X_1, \cdots, X_d)$ has a non-singular multivariate normal (multinormal) distribution $N(\mu, \Sigma)$ in d dimensions with mean the d-vector μ and covariance the $d \times d$ matrix $\Sigma = (\sigma_{ij})$, the density is given by Edgeworth's formula (1893)

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{1}{2}n}|\Sigma|^{\frac{1}{2}}} \exp\{-\frac{1}{2}(\mathbf{x}-\mu)^T \mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)\}$$

(if $|\Sigma| = 0$, the distribution is supported in some m-dimensional subspace with $m < d$; it will have a density there, but not in d dimensions).

The matrix $K = (k_{ij}) := \Sigma^{-1}$ is called the *concentration* matrix (or *precision* matrix) – 'K for Konzentration'.

Q1. Show that for $i \neq j$, X_i, X_j are *uncorrelated* (have 0 correlation) iff $\sigma_{ij} = 0$.

Q2 (Dempster's Theorem, 1969). Show that for $i \neq j$, X_i, X_j are *conditionally independent given all the other X_k*) iff $k_{ij} = 0$.

You may quote the *Gaussian Regression Formula (GRF)*: with Σ, K, μ (conformably) partitioned:

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}, \quad K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix},$$

(GRF) gives the conditional distribution of x_1 given x_2 as

$$x_1|x_2 \sim N(\mu_1 - K_{11}^{-1}K_{12}(x_2 - \mu_2), K_{11}^{-1}) :$$

$$x_1|x_2 \sim N(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}). \qquad (GRF)$$

(So the conditional mean is *linear* in x_2 – the basis of the *linear model* in statistics – and the conditional variance is independent of x_2. See e.g. T. W. Anderson, *An introduction to multivariate statistical analysis*, 3rd ed., Wiley, 2003 [Th. 2.5.1].)

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