Mathematical Logic (M345P65)
Problem Sheet 7

[1] Find subsets of $\mathbb{Q}$ which (with their induced orderings from $\mathbb{Q}$) are similar to:
(i) $\mathbb{N}+\mathbb{N}+\mathbb{N}$;
(ii) $\mathbb{N}\times\mathbb{Z}$;
(iii) $\mathbb{N}+\mathbb{N}^*$ (where $\mathbb{N}^*$ is the reverse ordering on $\mathbb{N}$).
You do not need to write down the similarities involved here.

[2] A set is finite if and only if it is equinumerous with some natural number $n \in \omega$. Otherwise it is infinite.
(i) Prove that if $m, n \in \omega$ are equinumerous then $m = n$.
(ii) Suppose X is a non-empty finite set of ordinals. Prove that X has a largest element.
(iii) Suppose α is a finite ordinal. Prove that $\alpha \in \omega$.
(iv) Suppose β is an infinite ordinal. Prove that $\omega \leq \beta$ and $|\beta^\dagger| = |\beta|$

[Hint: You can use results on ordinals in Section 3.4. For (i), it suffices to prove by induction on n that if $x \subseteq n$ and x is equinumerous with n, then $x = n$.]

[3] Suppose X is a non-empty set of ordinals. From the notes, you know that $\bigcup X$ and $\bigcap X$ are ordinals and $\bigcap X \leq \alpha \leq \bigcup X$ for all $\alpha \in X$.
(i) Show that if β is an ordinal with $\alpha \leq \beta$ for all $\alpha \in X$, then $\bigcup X \leq \beta$.
(ii) Formulate and prove a similar statement about $\bigcap X$.

[4] Suppose α and β are ordinals with α similar to $\omega+\omega$ and β similar to $\omega\times\omega$ (with the orderings as defined in 3.3.3). Which of $\alpha < \beta$, $\alpha = \beta$ or $\beta < \alpha$ holds?

[5] Let β be the set of all countable ordinals.
(i) Show that β is an ordinal.
(ii) Show that β is uncountable.
(iii) Show that if γ is an uncountable ordinal then $\beta \leq \gamma$.

[6] A *cardinal* is an ordinal α with the property that for all ordinals $\beta < \alpha$ we have that α and β are not equinumerous.
(i) Prove that every natural number is a cardinal and ω is a cardinal.
(ii) Prove that the ordinal β in question 5 is a cardinal.
(iii) Show that if γ is any ordinal, there is a unique cardinal α which is equinumerous with γ.