Applied Probability Midterm

11 November 2022

Question 1: Let $X = (X_n)_{n\in\{0,1,2,\dots\}}$ denote a discrete-time, time-homogeneous Markov chain on a countable state space E with transition matrix $\mathbf{P} = (p_{ij})_{i,j\in E}$.

(a) (2 points) State the Markov condition.

(b) (2 points) Describe in about 3-6 sentences, using results (without proofs!) from lectures, why it makes sense to say that "recurrence is a class property".

Solution:

(a) X satisfies the *Markov condition* if

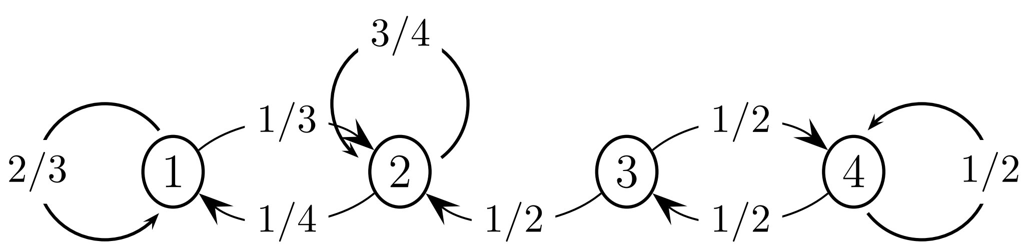
$$\mathrm{P}(X_n = j|X_{n-1} = i, X_{n-2} = x_{n-2}, \dots, X_0 = x_0) = \mathrm{P}(X_n = j|X_{n-1} = i),$$

for all $n \in \mathbb{N}$ and for all $x_0, \dots, x_{n-2}, i, j \in E$. [**2 marks**]

(b) From the lectures, we know that the state space of a Markov chain can be partitioned into communicating classes, which are irreducible. Moreover, we showed that if two states i and j, say, communicate with each other, than i is recurrent iff j is recurrent. Hence, we can conclude that all states within the same communicating class are either recurrent or transient. Hence it makes sense to say that recurrence (and also transience) is a class property.

[**1 mark for mentioning the communicating classes, 1 mark for describing that recurrence of one state is inherited by all states within the same communicating class.**]

Question 2: Let $X = (X_n)_{n\in\{0,1,2,\dots\}}$ denote a discrete-time, time-homogeneous Markov chain with state space $E = \{1, 2, 3, 4\}$ and transition diagram given by



(a) (2 points) Find the transition matrix $\mathbf{P} = (p_{ij})_{i,j\in E}$.

(b) (2 points) Determine the communicating classes.

(c) (2 points) For each class, specify whether the class is transient, positive recurrent or null recurrent and justify your answer.

(d) (1 point) Does this Markov chain have a unique stationary distribution? Justify your answer.

(e) Suppose that $\mathrm{P}(X_0 = 3) = 1$. Find the following probabilities:

i. (1 point) $\mathrm{P}(X_2 = 1)$.

ii. (1 point) $\mathrm{P}(X_2 = 2)$.

iii. (1 point) $\mathrm{P}(X_2 = 3)$.

iv. (1 point) $\mathrm{P}(X_2 = 4)$.

v. (1 point) $\mathrm{P}(X_0 = 3, X_1 = 2, X_2 = 3, X_3 = 4, X_4 = 3)$.

Solution:

(a) The transition matrix is given by

$$\mathbf{P} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

[**2 marks**]

(b) There are two communicating classes: $C := \{1, 2\}, T := \{3, 4\}$. [**2 marks**]

(c) C is finite and closed and hence positive recurrent. T is not closed and hence transient (by results from lectures). [**2 marks**]

(d) Yes, we know from lectures that, on a finite state space, there always exists a stationary distribution. Moreover, the stationary distribution is unique if and only if there is one closed communicating class, which is the case here.

(e) Using the notation from the lectures, we note that the initial marginal distribution of the Markov chain is given by $\boldsymbol{\nu}^{(0)} = (0, 0, 1, 0)$. Applying a result from the lectures, leads to

$$\boldsymbol{\nu}^{(1)} = \boldsymbol{\nu}^{(0)}\mathbf{P} = \left(0, \frac{1}{2}, 0, \frac{1}{2}\right),$$
$$\boldsymbol{\nu}^{(2)} = \boldsymbol{\nu}^{(1)}\mathbf{P} = \left(\frac{1}{8}, \frac{3}{8}, \frac{1}{4}, \frac{1}{4}\right).$$

Hence, we get that

i. $\mathrm{P}(X_2 = 1) = \frac{1}{8}$, [**1 mark**]

ii. $\mathrm{P}(X_2 = 2) = \frac{3}{8}$, [**1 mark**]

iii. $\mathrm{P}(X_2 = 3) = \frac{1}{4}$, [**1 mark**]

iv. $\mathrm{P}(X_2 = 4) = \frac{1}{4}$.[**1 mark**]

v. $\mathrm{P}(X_0 = 3, X_1 = 2, X_2 = 3, X_3 = 4, X_4 = 3) = \boldsymbol{\nu}_3^{(0)} p_{32}p_{23}p_{34}p_{43} = 0$, since $p_{23} = 0$. [**1 mark**]

Question 3: (4 points) Let $X = (X_n)_{n\in\{0,1,2,\dots\}}$ denote a discrete-time, time-homogeneous Markov chain with state space $E = \{1, 2\}$ and transition matrix

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Find

$$\lim_{n\to\infty} \mathrm{E}\left(X_n^2\right).$$

Solution: We can read off from the transition matrix that this Markov chain is irreducible and aperiodic with a finite state space. Hence we conclude from the lectures that there exists a unique stationary distribution $\boldsymbol{\pi} = (\pi_1, \pi_2)$ and that the limiting distribution is given by the stationary distribution.

We derive the stationary distribution: $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}, \pi_1 + \pi_2 = 1, \Leftrightarrow \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2 = \pi_1, \pi_1 + \pi_2 = 1, \Leftrightarrow \frac{1}{2}\pi_2 = \frac{1}{2}\pi_1, \pi_1 + \pi_2 = 1 \Leftrightarrow \pi_2 = \pi_1, \pi_2 = 1 - \pi_1 \Leftrightarrow \pi_1 = \frac{1}{2}, \pi_2 = \frac{1}{2}$. Hence, $\lim_{n\to\infty} \mathrm{P}(X_n = 1) = \frac{1}{2}$ and $\lim_{n\to\infty} \mathrm{P}(X_n = 2) = \frac{1}{2}$, which can also be written as

$$\lim_{n\to\infty} \mathrm{P}(X_n = k) = \pi_k = \frac{1}{2}$$

for $k = 1, 2$.

[2 marks up to here]

Using the definition of an expectation for a discrete random variable, we have

$$\mathrm{E}\left(X_n^2\right) = \sum_{k=1}^{2} k^2 \mathrm{P}(X_n = k) = \sum_{k=1}^{2} k^2 \mathrm{P}(X_n = k).$$

Hence we have

$$\lim_{n\to\infty} \mathrm{E}\left(X_n^2\right) = \sum_{k=1}^{2} k^2 \lim_{n\to\infty} \mathrm{P}(X_n = k) = \sum_{k=1}^{2} k^2 \pi_k = \frac{1}{2}(1+4) = \frac{5}{2}.$$

[2 marks up to here]