SOLUTIONS TO PRELIMINARY EXERCISES

Solution 0.1.

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \le y) = P(X^2 \le y) \\ &= \mathbb{P}(-\sqrt{y} \le X \le \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) \\ \Rightarrow f_Y(y) &= \frac{\mathrm{d}}{\mathrm{d}_y} F_Y(y) = \frac{\mathrm{d}}{\mathrm{d}_y}[F_X(\sqrt{y}) - F_X(-\sqrt{y})]. \end{aligned}$$

Since f_X is the derivative of F_X

$$\begin{aligned} f_Y(y) &= \tfrac{1}{2\sqrt{y}}[f_X(\sqrt{y}) + f_X(-\sqrt{y})] \\ &= \tfrac{1}{2\sqrt{y}}\left[\tfrac{1}{\sqrt{2\pi}}e^{-y/2} + \tfrac{1}{\sqrt{2\pi}}e^{-y/2}\right] \\ &= \tfrac{1}{\pi}y^{-1/2}e^{-y/2}. \end{aligned}$$

which is the density of χ^2 distribution. This is useful for random variate generation, since if we can generate from a normal distribution, we can square this variate and obtain a χ^2_1 variate.

Solution 0.2. (i)

$$\begin{aligned} \phi_X(t) &= \exp(t\mu_1 + \tfrac{1}{2}t^2\sigma_1^2) \\ \phi_{X+Y}(t) &= \phi_X(t)\phi_Y(t) \\ \Rightarrow X+Y &= \mathrm{N}(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2). \end{aligned}$$

(ii)

$$\begin{aligned} \phi_X(t) &= e^{\lambda(e^t-1)} \\ \Rightarrow \phi_{X+Y}(t) &= e^{\lambda(e^t-1)}e^{\mu(e^t-1)} \\ &= e^{(\lambda+\mu)(e^t-1)} \\ \Rightarrow X+Y &\sim \mathrm{Poisson}(\lambda+\mu). \end{aligned}$$

(iii)

$$X \sim \mathrm{Exp}(\lambda) \equiv \mathrm{Ga}(1,\lambda) \Rightarrow \phi_X(t) = \frac{\lambda}{\lambda - t}$$

$$\Rightarrow \phi_{X+Y}(t) = \frac{\lambda}{\lambda-t}\frac{\mu}{\mu-t}.$$

Partial fractions:

$$\left(\frac{\lambda\mu}{\mu-\lambda}\right)\frac{1}{\lambda-t} + \left(\frac{\lambda\mu}{\lambda-\mu}\right)\frac{1}{\mu-t}$$

but

$$\frac{1}{\lambda-t} = \int_0^\infty e^{tz}e^{-\lambda z}\mathrm{d}z \qquad \frac{1}{\mu-t} = \int_0^\infty e^{tz}e^{-\mu z}\mathrm{d}z$$

so

$$\phi_{X+Y}(t) = \int_0^\infty e^{tz}\left[\frac{\lambda\mu}{\mu-\lambda}e^{-\lambda z} + \frac{\lambda\mu}{\lambda-\mu}e^{-\mu z}\right]\mathrm{d}z$$

$$\Rightarrow f_{X+Y}(z) = \frac{\lambda\mu}{\mu-\lambda}e^{-\lambda z} + \frac{\lambda\mu}{\lambda-\mu}e^{-\mu z}$$

i.e. a MIXTURE of exponential distributions (weights sum to one, but one is negative).

$\lambda = \mu$ in (iii),

$$\phi_{X+Y}(t) = \left(\frac{\lambda}{\lambda-t}\right)^2 \Rightarrow X+Y \sim \mathrm{Ga}(2,\lambda).$$

Generalization of (i):

$$X_i \sim \mathrm{N}(\mu_i, \sigma_i^2) \Rightarrow \sum_{i=1}^{n} X_i \sim \mathrm{N}\left(\sum \mu_i, \sum \sigma_i^2\right).$$

Generalization of (ii):

$$X_i \sim \text{Poisson}(\lambda_i) \Rightarrow \sum_{i=1}^{n} X_i \sim \text{Poisson}\left(\sum \lambda_i\right).$$

Straightforward using mgf techniques.

Solution 0.3.

$$\begin{aligned} F_X(x) &= \mathrm{P}(X \leq x) = \mathrm{P}(-\lambda^{-1}\log(U) \leq x) \\ &= \mathrm{P}(U \geq e^{-\lambda x}) = 1 - \mathrm{P}(U \leq e^{-\lambda x}) \\ &= 1 - F_U(e^{-\lambda x}) = 1 - e^{-\lambda x} \end{aligned}$$

$$\Rightarrow f_X(x) = \lambda e^{-\lambda x} \Rightarrow X \sim \text{Exp}(\lambda).$$

Solution 0.4. Recall that we are interested $Y = \max(X_1, \ldots, X_n)$. The CDF is given by

$$F_Y(y) = \mathbb{P}(Y \leq y).$$

Intuitively, the probability of the maximum of a set of random variables being less than y is equal to the probability of all of them being equal to zero (due to independence). Based on this,

$$\begin{aligned} F_Y(y) &= \mathrm{P}(Y \leq y) = \mathrm{P}\left(\bigcap_{i=1}^{n}(X_i \leq y)\right) \\ \prod_{i=1}^{n} F_{X_i}(y) &= (1-e^{-y})^n \\ \Rightarrow f_Y(y) &= n(1-e^{-y})^{n-1}e^{-y}. \end{aligned}$$

Solution 0.5. Approximate distribution is N(0,1) since

$$\begin{aligned} \mathrm{E}(U_i) &= \tfrac{1}{2}, \quad \text{var}(U_i) = \tfrac{1}{12} \Rightarrow \\ \mathrm{E}(X) &= 0, \quad \text{var}(X) = 1. \end{aligned}$$

and the central limit theorem $\Rightarrow X \sim$ N(0,1).

Solution 0.6. $Y = \lfloor X \rfloor$.

$$\begin{aligned} \mathrm{P}(Y = r) &= \mathrm{P}(r \leq X < r+1) \\ &= \textstyle\int_r^{r+1} \lambda e^{-\lambda x}\,\mathrm{d}x \\ &= e^{-\lambda r} - e^{-\lambda(r+1)} \quad r = 0, 1, \ldots \\ &= \theta(1-\theta)^r \end{aligned}$$

Where $\theta = 1 - e^{-\lambda}$, i.e. a geometric distribution. Therefore, a geometric random variable θ can be simulated by drawing an exponential random variable with $\lambda = -\log(1-\theta)$ and taking the integer part.