Imperial College London

**MATH97006 MATH97171**

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2020

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

**Mathematical Logic**

Date: 4th May 2020

Time: 13.00pm - 15:30pm (BST)

Time Allowed: 2 Hours 30 Minutes

Upload Time Allowed: 30 Minutes

**This paper has 5 Questions.**

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

**SUBMIT YOUR ANSWERS AS SEPARATE PDFs TO THE RELEVANT DROPBOXES ON BLACKBOARD (ONE FOR EACH QUESTION) WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.**

.

1. Suppose $P$ is a non-empty set of atoms and $\mathcal{W}(P)$ is the set of propositional formulas generated by $P$.

(a) Let $\Gamma \subseteq \mathcal{W}(P)$ and $\phi \in \mathcal{W}(P)$. Define what is meant by $\Gamma \models \phi$. Suppose that $\phi, \psi, \theta \in \mathcal{W}(P)$. Prove that if $\{\phi\} \models \psi$ and $\{\psi\} \models \theta$, then $\{\phi\} \models \theta$. (4 marks)

(b) Define what it means for a set of propositional formulas $\Gamma$ to be *inconsistent*. Without using the Completeness Theorem, show that $\Gamma$ is maximally consistent if and only if $\Gamma$ is consistent and for every $\phi$ and $\psi$ in $\mathcal{W}(P)$

$$\neg\phi \wedge \neg\psi \notin \Gamma \text{ if and only if } \phi \in \Gamma \text{ or } \psi \in \Gamma.$$

(5 marks)

(c) Using natural deduction rules and providing precise rules in each step, prove that if $\Gamma \cup \{\psi\} \vdash \phi$ then $\Gamma \vdash \neg\phi \to \neg\psi$.

(3 marks)

(d) Let $\theta \in \mathcal{W}(P)$ and $p \in P$. We will use the notation $\phi[\theta/p]$ for the substitution of $\theta$ for the atom $p$ in $\phi \in \mathcal{W}(P)$. Suppose $v : \mathcal{W}(P) \to \{0, 1\}$ is a valuation function. Prove that there exists a valuation function $v^*$ with $v(\phi[\theta/p]) = v^*(\phi)$ for all $\phi \in \mathcal{W}(P)$.
[Hint: First clarify what $v^*$ is and then use induction on formulas. You may use the fact that $\{\perp, \wedge, \vee\}$ is adequate].

(4 marks)

(e) Let $\phi, \psi \in \mathcal{W}(P)$ and $p \in P$. Assume $p$ does not occur in $\psi$. Show that if $\models \phi \to \psi$ then $\models \phi[\theta/p] \to \psi$ for all $\theta \in \mathcal{W}(P)$.

(4 marks)

(Total: 20 marks)

2. Suppose $\mathcal{L}$ is a first order language and $\mathfrak{A}$ is an $\mathcal{L}$-structure.

(a) Show that if $\beta$ and $\gamma$ are two $\mathfrak{A}$-assignments that agree on variables $v_1, \ldots, v_n$, then $t^{\mathfrak{A}}[\beta] = t^{\mathfrak{A}}[\gamma]$ for every $\mathcal{L}$-term $t$ where the variables in $t$ are among $v_1, \ldots, v_n$. Explain briefly how this is used to define the notion of an $\mathcal{L}$-formula $\phi$ being *satisfied* by $\mathfrak{A}$.

(4 marks)

(b) In the language that only consists of $\doteq$ and has no function, relation and constant symbols, show that $\models (\exists v\phi \wedge \exists v\psi) \rightarrow \exists v(\phi \wedge \psi)$ does not always hold for some $\mathcal{L}$-formulas $\phi$ and $\psi$.

(2 marks)

(c) Suppose $v$ is a free variable in the $\mathcal{L}$-formula $\phi$. Show that

$$\mathfrak{A} \models \phi \text{ if and only if } \mathfrak{A} \models \forall v\phi.$$

(4 marks)

(d) In a language with exactly one binary function symbol $f$ consider two $\mathcal{L}$-structures $\mathfrak{Z}$ and $\mathfrak{Q}$ with domains $\mathbb{Z}$ and $\mathbb{Q}$, respectively, where $f$ is interpreted as the usual addition "$+$". Provide an $\mathcal{L}$-formula $\phi(v)$ such that $\mathfrak{Q} \models \phi[a]$ but $\mathfrak{Z} \not\models \phi[a]$ for some element $a \in \mathbb{Z}$ (note that $\mathbb{Z} \subset \mathbb{Q}$).

(2 marks)

(e) Define precisely what is meant by an "$\mathcal{L}$-sentence". In a language with exactly one binary relation symbol $E$ provide a sentence that has an infinite model but does not have any finite model.

(4 marks)

(f) A set of $\mathcal{L}$-sentences $\Phi$ is called *independent* if $\Phi\backslash\{\varphi\} \nvdash \varphi$ for every $\varphi \in \Phi$. Assume the language $\mathcal{L}$ consist of one $2$-ary function symbol $f$ and one constant symbol $c$. Show that the following set $\Phi_{SGr}$ is independent:

$$\begin{aligned}\Phi_{SGr} := \{&\forall v_0 \forall v_1 \forall v_2\ f(f(v_0, v_1), v_2) \doteq f(v_0, f(v_1, v_2)),\\ &\forall v_0\ f(v_0, c) \doteq v_0,\\ &\forall v_0 \exists v_1\ f(v_0, v_1) \doteq c\}.\end{aligned}$$

(4 marks)

(Total: 20 marks)

3. Let $\mathcal{L}$ be a first-order language.

(a) (i) In the language that consists of $\doteq$ and has no function, relation and constant symbols, provide, for $n \in \mathbb{N}$, a sentence $\lambda_n$ such that $\lambda_n$ holds in a model if and only if there are at least $n$ elements in the model.

(1 mark)

(ii) Recall that a theory is a set of $\mathcal{L}$-sentences that is closed under deduction. Define what is a meant by a *Henkin theory*. Suppose that the language consists of only two constant symbols $c_1$ and $c_2$. Prove that $\{\theta : \{\lambda_2, \neg(c_1 \doteq c_2)\} \vdash \theta\}$ is Henkin (where $\lambda_2$ is as in part (i)).

(4 marks)

(b) Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are two $\mathcal{L}$-structures which are elementarily equivalent (that is, they satisfy exactly the same $\mathcal{L}$-sentences). Prove that $\mathfrak{A}$ is finite if and only if $\mathfrak{B}$ is finite. (2 marks)

(c) Let $\Sigma_1$ and $\Sigma_2$ be two consistent $\mathcal{L}$-theories. Suppose that $\Sigma_1 \cup \Sigma_2$ has no model. Prove that there is an $\mathcal{L}$-sentence $\theta$ such that $\Sigma_1 \models \theta$ and $\Sigma_2 \models \neg\theta$.

(4 marks)

(d) Suppose there is a set $\Sigma$ of $\mathcal{L}$-sentences that has arbitrarily large finite models. Prove that there are two consistent theories $\Sigma_1$ and $\Sigma_2$ containing $\Sigma$ that satisfy the conditions in part (c) and provide an $\mathcal{L}$-sentence $\theta$ with the property in part (c).

(5 marks)

(e) Suppose that $\{\Sigma_i : i \in \mathbb{N}\}$ is a set of $\mathcal{L}$-theories such that $\Sigma_i \neq \Sigma_{i+1}$ and $\Sigma_i \subseteq \Sigma_{i+1}$ (for $i \in \mathbb{N}$). Show that $\Sigma = \bigcup\{\Sigma_i : i \in \mathbb{N}\}$ is a theory which extends each $\Sigma_i$. Prove that if $\Sigma$ is consistent, then there is an infinite set of models of $\Sigma_0$, no two of which are elementarily equivalent.

(4 marks)

(Total: 20 marks)

4. (a) (i) Define what it means for a set to be an *ordinal*. Prove that if $\alpha$ and $\beta$ are ordinals, then $\alpha \cap \beta$ is an ordinal. (4 marks)

(ii) Prove that if $\alpha$ and $\beta$ are ordinals and $\alpha \neq \beta$, then either $\alpha \in \beta$ or $\beta \in \alpha$.
[You may use without proof the fact that if a proper subset $\delta$ of an ordinal $\gamma$ is an ordinal, then $\delta \in \gamma$.] (4 marks)

(b) (i) State Zorn's Lemma. (2 marks)
Suppose $X$ is a set and let

$$A = \{f : \alpha \text{ is an ordinal and } f : \alpha \to X \text{ is an injective function}\}.$$

(ii) Explain briefly why $A$ is a set. (2 marks)

(iii) By applying Zorn's Lemma to $A$, prove that $X$ is equinumerous to some ordinal.
[You should not assume any other consequences of the Axiom of Choice without proving them.] (4 marks)

(c) Recall that a subset $U$ of $\mathbb{R}$ is *open* if for every $a \in A$ there is an open interval $I$ with $a \in I \subseteq U$. Prove that the set $\mathcal{O}$ of open subsets of $\mathbb{R}$ is equinumerous with $\mathbb{R}$.
[Hint: Every open interval in $\mathbb{R}$ is a union of open intervals with rational endpoints.]
(4 marks)

(Total: 20 marks)

5. Let $\mathcal{L}$ be a countable first-order language.

(a) Suppose $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are two $\mathcal{L}$-structures. Define what is meant by saying that $\mathfrak{A}_1$ *is an elementary substructure of* $\mathfrak{A}_2$. Prove that every infinite structure is a proper elementary substructure of some structure of the same cardinality.

(4 marks)

(b) Assume $\mathcal{L}$ consists of only a binary relation symbol. Find $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$ which are not elementarily equivalent and such that $\mathfrak{A}$ is isomorphic to a substructure of $\mathfrak{B}$ and $\mathfrak{B}$ is isomorphic to a substructure of $\mathfrak{A}$.

(4 marks)

(c) Suppose $\mathcal{L}$ consists of $\{E, f, c_0, c_1\}$ where $E$ is a binary relation symbol, $f$ is a 2-ary function symbol and $c_0$ and $c_1$ are constant symbols. Consider $\mathfrak{N}$ as an $\mathcal{L}$-structure where its domain is $\mathbb{N}$ and $f$ is interpreted as the usual multiplication and $E$ as the usual order $\leq$ of natural numbers, $c_0^{\mathfrak{N}} = 0$ and $c_1^{\mathfrak{N}} = 1$.

(i) Find an $\mathcal{L}$-formula $\psi(v)$ such that

$$\mathfrak{N} \models \psi[n] \text{ if and only if } n \text{ is a prime number.}$$

Then, using the formula $\psi$, provide $\mathcal{L}$-sentences which express the following:

- For every prime number there is a bigger prime number.
- There are infinitely many prime numbers.

(4 marks)

(ii) Prove that there is an countable $\mathcal{L}$-structure $\mathfrak{M}$ such that $\mathfrak{N}$ is an elementary substructure of $\mathfrak{M}$ and the set $\{\psi[m] : m \in M \setminus \mathbb{N}\}$ is infinite (where $M$ is the domain of $\mathfrak{M}$).

(4 marks)

Prove that $\mathfrak{M}$ and $\mathfrak{N}$ are not isomorphic.

(4 marks)

(Total: 20 marks)

# MATH96057/MATH97006/MATH97171
# Mathematical Logic, Solutions 2019-20.

*Question 1:*

(a) **(Standard, second part new but straightforward similar seen in course and coursework 1)** Propositional formula $\phi$ is a semantic consequence of $\Gamma$ if for every valuation function $v : \mathcal{W}(P) \to \{0,1\}$ where $v(\theta) = 1$ for all $\theta \in \Gamma$, then $v(\phi) = 1$. **2 marks (A)**
Suppose $v : \mathcal{W}(P) \to \{0,1\}$ is a valuation function such that $v(\phi) = 1$. Since $\{\phi\} \models \psi$ then $v(\psi) = 1$ and similarly from $\{\psi\} \models \theta$ follows $v(\theta) = 1$. Hence $\{\phi\} \models \theta$. **2 marks (A)**

(b) **(First part is standard, second part unseen but did similar things in notes and problem sheets.)** A set of propositional formulas $\Gamma$ is inconsistent if $\Gamma \vdash \perp$ or in other words $\Gamma$ proves falsity i.e. there is a proof for $\perp$ using rules of deduction. **1 mark (A)**
One solution is the following: first to show that if $\Gamma$ is maximally consistent then for each formula $\theta \in \mathcal{W}(P)$ either $\theta \in \Gamma$ or $\neg\theta \in \Gamma$ (this has been done in the course and coursework): Suppose $\theta \notin \Gamma$ then $\Gamma \cup \{\theta\} \vdash \perp$ and hence $\Gamma \vdash \neg\theta$ by ($\to$-I) rule. If $\neg\theta \notin \Gamma$ using the same argument and RA-rule we conclude $\Gamma \vdash \theta$ which contradicts with the fact that $\Gamma$ is consistent.
$\Rightarrow$) By the assumption since $\Gamma$ is maximally consistent then it is obviously consistent. Assume $\neg\phi \wedge \neg\psi \notin \Gamma$. Using the statement above if $\phi \notin \Gamma$ and $\psi \notin \Gamma$ then $\neg\phi \in \Gamma$ and $\neg\psi \in \Gamma$. Then $\neg\psi \wedge \neg\phi \in \Gamma$ using ($\wedge$-I)-rule which contradicts with the assumption $\neg\phi \wedge \neg\psi \notin \Gamma$. For the other direction, assume $\neg\phi \wedge \neg\psi \in \Gamma$ if $\phi \in \Gamma$ or $\psi \in \Gamma$ then from ($\wedge$-E) rule and MP-rule we conclude $\Gamma \vdash \perp$, which is a contradiction.
$\Leftarrow$) We want to show $\Gamma$ is maximally consistent. Let $\phi$ be a formula. Since $\neg\neg\phi \wedge \neg\phi \notin \Gamma$ then $\neg\phi \in \Gamma$ or $\phi \in \Gamma$, by the condition given. Hence $\Gamma$ is complete and therefore maximally consistent.
**4 marks (A)**

(c) **(Similar seen)** If $\Gamma \cup \{\phi\} \vdash \psi$ then by ($\to$-I) rule $\Gamma \vdash \phi \to \psi$. Now it only needs to be shown that $\vdash (\phi \to \psi) \to (\neg\psi \to \neg\phi)$. A proof of this is as follows: $\{\phi, \phi \to \psi\} \vdash \psi$ by ($\to$-E)-rule (or MP rule). Then $\{\phi, \phi \to \psi, \neg\psi\} \vdash \perp$ by MP-rule. Then from RA-rule follows $\{\phi \to \psi, \neg\psi\} \vdash \neg\phi$ and using ($\to$-I) rule we get $\{\phi \to \psi\} \vdash \neg\phi \to \neg\psi$. Once more applying ($\to$-I) rule finishes the proof. **3 marks (A)**

(d) **(Unseen but similar has been seen, the starting point is challenging)**
First $v^*$ is the valuation function where $v^*(p_i) = v(p_i)$ for all $p_i \in P$ where $p_i \neq p$ and if $p$ occurs in $\phi$ then $v^*(p) = v(\theta)$; otherwise $v^*(p) = v(p)$. **2 marks (C)**
The rest goes by induction on formulas of $\mathcal{W}(P)$. If $\phi$ is $\perp$ then $\phi[\theta/p] = \perp$ and it is obvious that $v = v^*$. Assume $\phi$ is $p_i$ for some $p_i \in P$. Then if $p_i \neq p$ then $\phi[\theta/p] = \phi$ and $v^* = v$. If $\phi = p$ then $\phi[\theta/p] = \theta$ and $v^*(p) = v(\theta) = v^*(\phi)$. Now we use the fast that $\{\perp, \wedge, \vee\}$ is adequate and assume by induction we know the result for $\phi_1$ and $\phi_2$ then if $\phi$ is of the form $\phi_1 \wedge \phi_2$ then $\phi[\theta/p] = \phi_1[\theta/p] \wedge \phi_2[\theta/p]$ by definition. Then $v(\phi[\theta/p]) = \min\{v(\phi_1[\theta/p]), v(\phi_1[\theta/p])\}$ by induction hypothesis $\min\{v(\phi_1[\theta/p]), v(\phi_1[\theta/p])\} = \min\{v^*(\phi_1), v^*(\phi_1)\} = v^*(\phi_1 \wedge \phi_2) = v^*(\phi)$. Similarly for $\vee$ and by replacing $\min$ to $\max$. **2 marks (B)**

(e) **(Unseen but the previous part is meant to be used and hence simpler)** If $p$ does not occur in $\phi$ then it is obvious. From part (c) it follows that $v(\phi[\theta/p]) = v^*(\phi)$ where $v^*(p) = v(\theta)$. Since

$p$ does not occur in $\psi$ then $v^*(\psi) = v(\psi)$ (using a fact from the course). Hence if $v(\phi[\theta/p]) = 1$ and therefore $v^*(\phi) = 1$, from $\models \phi \to \psi$ follow that $v^*(\psi) = 1$ which implies $v(\psi) = 1$. If $v(\phi[\theta/p]) = 0$ then the outcome of $v(\phi[\theta/p] \to \psi) = 1$ and nothing else needs to be checked. **4 marks (2B, 2C)**

*Question 2:*

(a) **(Standard seen in the notes)** This is proved by induction on $\mathcal{L}$-terms. If $t$ is just a variable $v_i$ where $1 \leq i \leq n$ then clearly $t^{\mathfrak{A}}[\beta] = \beta(v_i) = \gamma(v_i) = t^{\mathfrak{A}}[\gamma]$. If $t$ is a constant then $t^{\mathfrak{A}}[\beta] = c^{\mathfrak{A}} = t^{\mathfrak{A}}[\gamma]$. Suppose that $t$ is of the form $f(t_1, \ldots, t_m)$ where $f$ is an $m$-ary function symbol and assumpe $t_i^{\mathfrak{A}}[\beta] = t_i^{\mathfrak{A}}[\gamma]$ for $0 \leq i \leq n$. Then $f(t_1, \ldots, t_m)^{\mathfrak{A}}[\beta] = f^{\mathfrak{A}}(t_1^{\mathfrak{A}}[\beta], \ldots, t_m^{\mathfrak{A}}[\beta]) = f^{\mathfrak{A}}(t_1^{\mathfrak{A}}[\gamma], \ldots, t_m^{\mathfrak{A}}[\gamma]) = f(t_1, \ldots, t_m)^{\mathfrak{A}}[\gamma]$. **2 marks (A)**

Formula $\phi$ is satisfied by $\mathfrak{A}$ or $\mathfrak{A} \models \phi$ if for every $\mathfrak{A}$-assignment function $\beta : V \to A$ where $V$ is the set of variables and $A$ is the domain of $\mathfrak{A}$, the valuation function $v^{\beta}(\phi) = 1$. The fact that the valuation function $v^{\beta}$ is well-defined is followed by induction on formulas and in order to have the valuation function well-defined on atomic formulas we need the fact mentioned above about $\mathcal{L}$-terms. **2 marks (A)**

(b) **(Similar is seen in problem sheets and coursework)** Let $\phi$ be the formula $x \doteq y$ and $\psi = \neg\phi$. Consider any $\mathcal{L}$- structures with at least two elements. Then clearly $\mathfrak{A} \models \exists x\phi \wedge \exists x\psi$ but $\mathfrak{A} \not\models \exists x(\phi \wedge \psi)$. **2 marks (A)**

(c) **(Seen in the course)** Suppose $\mathfrak{A} \models \phi$ then it means $v^{\beta}(\phi) = 1$ for every $\mathfrak{A}$-assignment $\beta : V \to A$. Therefore for a given $\mathfrak{A}$-assignment $\beta : V \to A$ and $a \in A$ we have $v^{\beta[a/v]}(\phi) = 1$ where $\beta[a/v] : V \to A$ is the $\mathfrak{A}$-assignment such that $\beta[a/v](v) = a$ and $\beta[a/v](w) = \beta(w)$ for $w \neq v$ and $w \in V$. Therefore $\min\{v^{\beta[v/a]}(\phi) : a \in A\} = 1$ and hence $\mathfrak{A} \models \forall v\phi$. For the other direction if $\mathfrak{A} \not\models \phi$ then it means there is an $\mathfrak{A}$-assignment $\beta$ such that $v^{\beta}(\phi) = 0$ let $a = \beta(v)$ and hence $v^{\beta[a/v]}(\phi) = 0$ and hence $\mathfrak{A} \not\models \forall v\phi$. **4 marks (A)**

(d) **(Unseen)** There are different solutions for the question one is $\phi(v) = \exists x f(x, x) \doteq y$. Then clearly $\mathfrak{Q} \models \phi[1]$ but $\mathfrak{Z} \not\models \phi[1]$. **2 marks (B)**

(e) **(First part standard, second part the example is discussed and well-known but the question is not precisely mentioned so it can be challenging)** An $\mathcal{L}$-formula $\phi$ is an $\mathcal{L}$-sentence if each occurrence of $v$ in $\phi$ is in the scope of $\forall v$ or $\exists v$; in other words variable $v$ has no free occurrence in $\phi$. **1 mark (A)**

This can have different solutions. One example that has been discussed in the course is a sentence that is built from the the finite axioms that indicates the binary relation is a dense total order: Suppose $E$ is the binary relation symbol. Consider $\phi_1$ to be

$$(\forall v \forall w E(v, w) \vee E(w, v)) \wedge (\forall v \neg E(v, v)) \wedge (\forall v \forall w E(v, w) \to \neg E(w, v)).$$

Let $\phi_2$ be the following $\forall v \forall w E(v, w) \to (\exists z E(v, z) \wedge E(z, w))$. Then the sentence $\phi_1 \wedge \phi_2$ has a model such as $\langle \mathbb{Q}, < \rangle$ and no finite model. **3 marks (B)**

(f) **(Similar seen in a coursework but in a different setting but coming up with the examples can be challenging)** There are different solutions for the question and two of the structures mentioned below have been discussed in the course and only one can be slightly challenging. First it needs to be discussed that Gödel's completeness theorem is used and in order to prove any of the axioms is independent form the rest we provide a model that witnesses it.

$\langle \mathbb{N}, ., 1\rangle \models (\forall v_0 \forall v_1 \forall v_2\ f(f(v_0, v_1), v_2)) \wedge (\forall v_0\ f(v_0, c) \doteq v_0)$ but $\langle \mathbb{N}, ., 1\rangle \not\models \forall v_0 \exists v_1\ f(v_0, v_1) \doteq c$.

$\langle \mathbb{Z}, +, 1\rangle \models (\forall v_0 \forall v_1 \forall v_2\ f(f(v_0, v_1), v_2)) \wedge (\forall v_0 \exists v_1\ f(v_0, v_1) \doteq c)$ but $m + 1 \neq m$ for all $m \in \mathbb{Z}$ and hence $\langle \mathbb{Z}, +, 1\rangle \not\models \forall v_0\ f(v_0, c) \doteq v_0$.

$\langle \mathbb{Q}, f, 0\rangle$ where $f(p, q) = p + tq$ for a fixed $t \in \mathbb{Q}$. Then $f(p, 0) = p$ and $f(p, -\frac{1}{t}p) = 0$ for every $p$. However $f(1, 1) = 1 + t$ and $f(1 + t, 1) = 1 + 2t$ and $f(1, 1 + t) = 1 + t + t^2$. Hence $\langle \mathbb{Q}, f, 0\rangle \models (\forall v_0\ f(v_0, c) \doteq v_0) \wedge (\forall v_0 \exists v_1\ f(v_0, v_1) \doteq c)$ but $\langle \mathbb{Q}, f, 0\rangle \not\models \forall v_0 \forall v_1 \forall v_2\ f(f(v_0, v_1), v_2)$. **4 marks (2C, 2D)**

*Question 3:*

(a) **((i) Seen in problem sheet; (ii) First part standard, second part similar seen.)**

(i) The formula $\lambda_n$ is as follows

$$\exists v_1 \dots v_n \bigwedge_{i \neq j, 1 \leq i, j \leq n} \neg(v_i \doteq v_j).$$

**1 mark (B)**

(ii) A theory $T$ is Henkin if for every $\mathcal{L}$-sentence of the form $\exists v \phi(v)$ there is a constant symbol $c$ in the language such that $\exists v \phi(v) \rightarrow \phi(c) \in T$. **1 mark (A)**
Clearly $\{\theta : \{\lambda_2, \neg(c_1 \doteq c_2)\} \vdash \theta\}$ is a theory. Let $\mathfrak{A}$ be a set of size 2 then $\mathfrak{A} \models \{\lambda_2, \neg(c_1 \doteq c_2)\}$ if we interpret $c_1$ and $c_2$ as two different elements of $A$. So it is consistent. Now any formula of the form $\exists v \phi(v)$ is satisfied by $\mathfrak{A}$ and hence $\phi(c_1)$ or $\phi(c_2)$. Therefore Henkin. **3 marks (D)**

(b) **(Similar is seen and part (a) makes it easier)**

Suppose $\mathfrak{A}$ is finite and assume it has $n$-many elements. Then $\mathfrak{A} \models \neg\lambda_{n+1}$ where $\lambda_{n+1}$ is defined in (a). Since $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent therefore $\mathfrak{B}$ have to satisfy the same sentence which implies $\mathfrak{B}$ is finite. The other direction is similar. **2 marks (B)**

(c) **(Similar seen)** Suppose that is not the case and for every $\theta \in \Sigma_1$ we have $\Sigma_2 \not\models \neg\theta$ then there is a model $\mathfrak{A}_2$ of $\Sigma_2$ such that $\mathfrak{A}_2 \not\models \neg\theta$ and hence $\mathfrak{A}_2 \models \theta$. Then this means $\Sigma_1 \cup \Sigma_2$ is finitely consistent (since $\Sigma_1$ and $\Sigma_2$ are both theories and consistency of every finite subset in $\Sigma_1$ is equivalent to the consistency of a single formula in $\Sigma_1$). Therefore by Compactness theorem $\Sigma_1 \cup \Sigma_2$ has a model; contradiction with the assumption. **4 marks (1C, 3B)**

(d) **(Unseen, however the steps for it is given above.)** By the assumption $\Sigma$ has a finite model and let $\mathfrak{A}$ be one of such models of $\Sigma$. Then $\Sigma_1 := Th(\mathfrak{A}) = \{\theta : \theta \text{ is } \mathcal{L} - \text{sentence}, \mathfrak{A} \models \theta\}$ contains $\Sigma$ and it is closed under deduction hence it is a theory; simply because every $\mathcal{L}$-sentence or its negation have to be satisfied in any $\mathcal{L}$-structure. Consider now $\Gamma := \Sigma \cup \{\lambda_n : n \in \mathbb{N}\}$ where $\lambda_n$ is defined as part (a). Note that $\models \lambda_m \rightarrow \lambda_n$ when $m \geq n$. Now we use Compactness theorem to show $\Gamma$ is consistent. Assume $\Gamma_0 \subseteq \Gamma$ is a subset that has only finite many elements

$\lambda_{n_i}$ where $1 \leq i \leq m$ let $n^* := \max\{n_i : 1 \leq i \leq m\}$. Since by assumption $\Sigma$ has arbitrary large finite models, there is a model $\mathfrak{B}$ of $\Sigma$ with at least $n^*$-many elements and then $\mathfrak{B} \models \Gamma_0$. Hence $\Gamma$ is consistent and note that every model of $\Gamma$ is infinite. Let $\mathfrak{C} \models \Gamma$ and consider $\Sigma_2 := Th(\mathfrak{C})$. Now $\Sigma_1$ and $\Sigma_2$ are each consistent $\mathcal{L}$-theories where $\Sigma_1 \cup \Sigma_2$ has no model and the sentence $\theta$ that is asked is exactly the sentence that in part (b) and indicates that $\mathfrak{A}$ has exactly $|A|$-many elements. **5 marks (2C, 3D)**

(e) **(Unseen, therefore harder.)** In order to show $\Sigma$ is a theory we need to show if $\Sigma \vdash \phi$ then $\phi \in \Sigma$. If $\Sigma \vdash \phi$ then there is a finite subset $\Delta$ of $\Sigma$ such that $\Delta \vdash \phi$ and therefore there is $i \in \mathbb{N}$ such that $\Delta \subseteq \Sigma_i$. Since by our assumption $\Sigma_i$ is a theory then $\phi \in \Sigma_i$ and hence $\phi \in \Sigma$.
When $\Sigma$ is consistent then every $\Sigma_i$ is consistent. Let $i \in \mathbb{N}$. Since $\Sigma_i \neq \Sigma_{i+1}$ there exists $\phi_i \in \Sigma_{i+1} \backslash \Sigma_i$ such that $\Sigma_i \nvdash \phi_i$, therefore by Gödel's Completeness theorem there is a model $\mathfrak{A}_i$ of $\Sigma_i$ such that $\mathfrak{A}_i \not\models \Sigma_{n+1}$. Now given $\mathfrak{A}_i$ and $\mathfrak{A}_j$ if $i \neq j$ then $\mathfrak{A}_i \not\models \phi_t$ but $\mathfrak{A}_j \models \phi_t$ where $t = \max\{i, j\} - 1$. So we are done. **4 marks (D)**

*Question 4:*

(a) **((i) Exercise from notes plus a standard definition; (ii) from the notes.)**

(i) A set $\gamma$ is an ordinal if: (i) it is a transitive set (that is, every element of $\gamma$ is a subset of $\gamma$), and (ii) the relation of membership between elements of $\gamma$ defines a strict well ordering on $\gamma$. If $\delta \in \alpha \cap \beta$ then $\delta \subseteq \alpha$ and $\delta \subseteq \beta$ (as $\alpha, \beta$ satisfy (i)), thus $\delta \subseteq \alpha \cap \beta$. So $\alpha \cap \beta$ satisfies (i). As the membership relation on $\alpha \cap \beta$ is just the restriction of the membership relation in $\alpha$, it gives a well ordering on $\alpha \cap \beta$ (as a subset of a well ordered set is well ordered). So $\alpha \cap \beta$ also satisfies (ii). **4 marks (A)**

(ii) Suppose not. Then by the given fact, we have $\alpha \not\subseteq \beta$ and $\beta \not\subseteq \alpha$. Thus $\delta = \alpha \cap \beta$ is a proper subset of both $\alpha$ and $\beta$. By (i) $\delta$ is an ordinal and so by the given fact, $\delta \in \alpha \cap \beta$. Thus $\delta \in \delta$, which is impossible (as $\delta$ is an ordinal and so membership is a strict well ordering on its elements). **4 marks (A)**

(b) **((i) Bookwork; (ii) seen similar; (iii) unseen, but did similar things in notes and problem sheets.)**

(i) Zorn's Lemma: Suppose $(A, \leq)$ is a non-empty partially ordered set in which each chain in $A$ has an upper bound in $A$. Then $A$ has a maximal element. **2 marks (B)**

(ii) By Hartogs' Lemma, there is an ordinal $\beta$ such that there is no injective function $\beta \to X$. Thus in the definition of $A$ we can take $\alpha \in \beta$ and then we see that $A$ is a set produced using the Axiom of Specification (and the other ZF axioms). **2 marks (B)**

(iii) We consider $A$ with the partial ordering of inclusion (regarding functions as sets of ordered pairs in the usual way). This is non-empty as we have the empty function $\emptyset \to X$ in $A$. If $C$ is a chain in $A$ then $g = \bigcup C$ is a union of a chain of injective functions and is therefore an injective function. Its domain is a union of a chain of ordinals and is therefore an ordinal. Thus $g \in A$. As $g \supseteq f$ for every $f \in C$, it follows that the hypotheses of ZL hold.

So let $f \in A$ be maximal in $A$. Thus $f$ is an injective function $\alpha \to X$ for some ordinal $\alpha$. We claim that $f$ is surjective (and therefore a bijection, as required). If there is $x \in X$

which is not in the image of $f$ then there is an injective function $f' : \alpha^\dagger \to X$ extending $f$ with $f'(\alpha) = x$ (where $\alpha^\dagger$ is the successor ordinal $\alpha \cup \{\alpha\}$). So $f \subset f' \in A$, contradicting maximality of $f$. **4 marks (1B, 3C)**

(c) **(Unseen, therefore harder.)**

By definition, any $U \in \mathcal{O}$ is the union of the open intervals which are contained in $U$. However, by denseness of the rationals in the reals, an open interval is the union of the open intervals with rational endpoints which it contains. Thus the same is true of every $U \in \mathcal{O}$.

So let $\mathcal{Q}$ denote the set of open intervals with rational endpoints. This is countable. By the above, we have an injective function $\mathcal{O} \to \mathcal{P}(\mathcal{Q})$ with $U \mapsto \{I \in \mathcal{Q} : I \subseteq U\}$. Thus $|\mathcal{O}| \leq 2^\omega = |\mathbb{R}|$. The other direction $|\mathbb{R}| \leq |\mathcal{O}|$ is easy (eg. consider the function $x \mapsto (x, x+1)_\mathbb{R}$).

**4 marks (D)**

*Question 5:*

(a) **(Standard in bookwork)** It means the $\mathfrak{A}_1$ is a substructure of $\mathfrak{A}_2$ and for every $\mathcal{L}$-formula $\phi(v_1, \ldots, v_n)$ and elements $a_1, \ldots, a_n \in A_1$ ($A_1$ is the domain of $\mathfrak{A}_2$) we have

$$\mathfrak{A}_1 \models \psi[a_1, \ldots, a_n] \text{ if and only if } \mathfrak{A}_2 \models \psi[a_1 \ldots a_2].$$

**1 mark**

Let $\mathfrak{A}$ be an $\mathcal{L}$-structure. Extend the language $\mathcal{L}$ to $\mathcal{L}_c := \mathcal{L} \cup \{c\}$ where $c$ is a new constant symbol. Consider now $Diag^+(\mathfrak{A}) \cup \{\neg(c \doteq a) : a \in A\}$ where $A$ the domain of $\mathfrak{A}$ and by $Diag^+(\mathfrak{A})$ we mean the elementary diagram of $\mathfrak{A}$. Using Compactness theorem one can show it is consistent and hence it has a model and using downward Löwenheim-Skolem theorem we find a model of the same carnality as $|A|$. **3 marks**

(b) **(Unseen, however one possible solution have been mentioned in the course)**

There are different solutions: Let $\mathfrak{A}$ be an countable $\mathcal{L}$ where $E$ is an equivalence relation that portions $A$ into infinite many classes each infinite. Let $\mathfrak{B}$ be another countable structure where $E$ is an equivalence relation that partitions $B$ into infinitely many classes each infinite plus one equivalence class with only one element. Since $\mathfrak{B}$ has a class with one element and it is expressible as an $\mathcal{L}$-sentence therefore $\mathfrak{A}$ and $\mathfrak{B}$ are not elementary equivalent. **3 marks**

(c) **(Similar seen and standard)**

(i) The formula $\psi(v)$ is

$$\neg(v \doteq c_0) \wedge \neg(v \doteq c_1) \wedge \forall v_1 v_2 (f(v_1, v_2) \doteq v \to (v_1 \doteq v \vee v_2 \doteq v)).$$

**2 marks**

First $\mathcal{L}$-sentence:

$$\forall v \psi(v) \to \exists w E(w, v) \wedge \neg(v \doteq w) \wedge \psi(w).$$

**1 mark**

Second $\mathcal{L}$-sentence:

$$\exists v \psi(v) \wedge \forall v(\psi(v) \rightarrow \exists w E(v, w) \wedge \neg(v \doteq w) \wedge \psi(w)).$$

**1 mark**

(ii) **(Similar seen and standard, coming up with the precise formulas might be challenging, for the second part similar is seen)** First part: For this consider $Diag^{+}(\mathfrak{N})$ (i.e. the elementary diagram of $\mathfrak{N}$) and extend the language $\mathcal{L}$ to $\mathcal{L}_C := \mathcal{L} \cup \{c_i : i \in \mathbb{N}\}$ where $c_i$'s are new constant symbols. Consider now

$$Diag^{+}(\mathfrak{N}) \cup \{\psi(c_i) : i \in \mathbb{N}\} \cup \{\neg(c_i \doteq n) : i, n \in \mathbb{N}\} \cup \{\neg(c_i \doteq c_j) : i \neq j, i, j \in \mathbb{N}\}.$$

Clearly $\mathfrak{N}$ is a model of every finite subset of the above set of axioms. Hence using Compactness theorem the whole set of $\mathcal{L}_C$-sentences mentioned above has a model and downward Löwenheim-Skolem theorem proves the existence of a countable model. **4 marks**

Second part **(Unseen, partially challenging)**: In order to show no isomorphism is possible between $\mathfrak{N}$ and $\mathfrak{M}$ we prove it by contradiction. Suppose there is a an isomorphism $f$ between $\mathfrak{M}$ and $\mathfrak{N}$. Since an isomorphism is a bijection then $f^{-1}(m) \in \mathbb{N}$ where $m$ is a *non-standard* prime (elements from the set in the statement). But this leads to a contradiction because $f^{-1}(m)$ is a finite prime number and there are finitely many primes (or numbers between) $f^{-1}(m)$ and $c_1$ but $m$ is an infinite (non-standard) prime. **4 marks**

**If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once, for each question.**

**Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.**

| **ExamModuleCode** | **Question** | **Comments for Students** | |
|---|---|---|---|
| MATH97006 MATH97171 | 1 | Induction for propositional formulas have to start in the right place: namely from \bot and then atoms. In few cases people used deduction rules for "\vee" which was not covered or discussed in the course. Some people still used Completeness in Q1.b. although explicitly mentioned not use it. Q1.d almost every one missed the case where p does not occur in \phi. | |
| MATH97006 MATH97171 | 2 | A structure is always attached to an assignment. So using assignment arguments without specifying the structure is not correct. Many have managed to come up with the right formula for Q2.b. but have not argued why that fails. Their answer actually hold in a structure with one element. In the same question few people used constants that are not in the language. Note that in Q2. c. Completeness theorem can not be used to say a formula is deduced for a structure. In Q.2.e many failed to say in their formula the relation is not reflexive. | |
| MATH97006 MATH97171 | 3 | In our setting and as specified in the question theories are deduction closed. However that does not mean the union of two theories is also deduction closed (or is a theory). Furthermore a consistent theory does not need to be maximally consistent. There was an error in Q3.ii. the formula had to be \lambda_2\wedge \neg\lambda_3 not only \lambda_2. The solution in the exam solution is given for that case. The issue has been discussed with colleagues for effects on marking. In Q3.d having arbitrarily large finite model does not mean it has models of every size (finite). | |
| MATH97006 MATH97171 | 4 | In 4 (a)(i) and (ii) several people gave answers which used results whose proofs depend on the results you are asked to prove in the question! Doing this demonstrates a lack of understanding of the material and it was not allowed. In (b)(iii) for the ordering on functions in A, it is not enough to say that the domains extend - the functions also have to agree on elements where they are both defined. This guarantees that when you take the union of a chain, you still have a function. | |
| MATH97006 MATH97171 | 5 | Q5a it is clearly asked that the cardinality of the substructure have to be the same as the structure. Q5.b based on Cantor-Schroeder-Berstein one should immediately conclude A and B have the same cardinality. | |