Lecture 4, 13.10.2022

Independence and independent copies

Recall the definition of independence:
1. Events $(A_i : i \in I)$, or $(A(i) : i \in I)$ to avoid suffices within suffices as before, are *independent* if and only if [iff], for every finite subset $(i_1, \cdots i_d)$ of I, the events $A(i_j)$, $j = 1, \cdots, d$, are independent, that is,

$$\mathbb{P}(A(i_1) \cap \cdots A(i_d)) = \prod_{j=1}^{d} \mathbb{P}(A(i_j)).$$

2. Random variables $(X_i : i \in I)$ are independent iff all events of the form $A_i := (X_i \in B_i)$, for B_i in the σ-field of the pr space on which X_i is defined, are independent.

The Daniell-Kolmogorov Theorem allows us to construct all these random variables on the same probability space (the *product* probability space), as the relevant consistency conditions are clearly satisfied.

In particular, we may have all the X_i defined on the same probability space, and so with the same distribution. The resulting set of X_i $(i \in I)$ are then *independent and identically distributed*, or *iid* for short. They are often called *independent copies* of each other – or just *copies*.
Note.

Recall von Neumann's construction of the natural numbers with zero: $\mathbb{N} := \{1, 2, 3, ...\}$, the set of *natural numbers*, and $\mathbb{N}_0 := \{0, 1, 2, ...\} = \mathbb{N} \cup \{0\}$. We can take these for granted, or proceed as follows:

$$0 \leftrightarrow \emptyset, \quad 1 \leftrightarrow \{\emptyset\}, \quad 2 \leftrightarrow \{0, 1\}, \quad 3 \leftrightarrow \{0, 1, 2\}, \cdots.$$

etc. (John von Neumann (1903-57) in 1923).

Example: Lebesgue measure and infinite coin-tossing: binary expansion

We note the simplest example of this situation. Let the x_i $(i \in \mathbb{N})$ be independent copies of a *binary* random variable (coin toss):

$$\mathbb{P}(x_i = 0) = \mathbb{P}(x_i = 1) = \frac{1}{2}, \qquad (i \in \mathbb{N}).$$

The relevant probability space for a single toss is $(\{0, 1\}, \mathcal{P}((\{0, 1\}), \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1)$, where $\mathcal{P}(A)$ is the *power set* of A, the set of all its subsets, and δ_n is the Dirac mass (unit point mass) at n, which by above we could write as $(2, \mathcal{P}(2), \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1)$. For infinitely many tosses, we use the infinite product of this, written $(2, \mathcal{P}(2), \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1)^{\mathbb{N}}$. On this, the consistency conditions are clearly satisfied (automatic with independence). So the Daniell-Kolmogorov theorem gives us infinitely many independent copies of a coin toss.

This is a familiar result in a new setting! Recall the decimal expansion of a real number $x \in \mathbb{R}$: with $[x]$ for its integer part,

$$x = [x] \,.\, x_1 x_2 \cdots x_n \cdots, \qquad [x] \in \mathbb{Z}, \quad x_n \in \{0, 1, 2, \cdots, 9\}.$$

In just the same way, we can take the *binary expansion* instead: with the point now binary, not decimal,

$$x = [x] \,.\, x_1 x_2 \cdots x_n \cdots, \qquad [x] \in \mathbb{Z}, \quad x_n \in \{0, 1\},$$

Consider now the simplest continuous probability space, the *Lebesgue space*: the unit interval $[0, 1]$ endowed with its Borel σ-field and Lebesgue measure (a probability measure here). Drawing X from $[0, 1]$ under Lebesgue measure and taking its binary expansion as above, one can check inductively that the expansion coefficients X_n are independent coin-tosses.

The converse also holds. To see this, pick any $c \in \mathbb{R}$, and translate x by c, $x \mapsto x + c$. Discarding the integer part, the fractional part (binary expansion) of $x + c$ is also a sequence of independent coin-tosses, by the previous argument. So, the measure on x is translation-invariant. So it is Lebesgue measure (defined uniquely to within a positive multiplicative constant – needed to adjust for the unit of length, cm or inches say).

Thus *the Lebesgue space is identified with the countable independent product of the coin-tossing space.* So we have here two remarkably different ways of looking at the same thing.

Note: Haar measure

Translation-invariance of Lebesgue measure is invariance under the group action of the additive group of reals. This can be generalised, up to *locally compact topological groups* [locally compact: points have compact neighbourhoods; topological group: a group with a topology under which the group operations are continuous] – Alfred Haar (1885 - 1933), in 1933. See MacTutor for Haar (and the poem on Haar and von Neumann).