EXTRA EXERCISES FOR WEEK 1-2

Exercise 2.4 (Transformation of random variables)**.** Define

$$\text{Gamma}(x; \alpha, \beta) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \beta^{\alpha}, \quad \text{for } x > 0, \quad \alpha, \beta > 0.$$

Recall the Beta density

$$\text{Beta}(x; \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}.$$

Prove that if $X_1 \sim \text{Gamma}(\alpha, 1)$ and $X_2 \sim \text{Gamma}(\beta, 1)$, then

$$Y = \frac{X_1}{X_1 + X_2} \sim \text{Beta}(\alpha, \beta).$$

(In general, return to your previous probability material and study *transformation of random variables* formula)

Exercise 2.5 (Exercise 2.8 from *Introducing Monte Carlo Methods with R*, Christian Robert, George Casella)**.** Consider the rejection sampling method for $p(x) = \mathcal{N}(x; 0, 1)$ and choose

$$q_\alpha(x) = (\alpha/2)\exp(-\alpha|x|)$$

as the proposal with $\alpha > 0$.

(a) Show that

$$M_\alpha = \sup_x \frac{p(x)}{q_\alpha(x)} = \sqrt{\frac{2}{\pi}} \alpha^{-1} e^{\alpha^2/2}.$$

Prove that the minimum of this supremum (in α) is attained for $\alpha = 1$, i.e., find

$$\alpha^\star = \arg\min_\alpha M_\alpha.$$

(b) Show that the acceptance rate $\hat{a}$ as defined during lectures is then $\sqrt{\pi/2e} = .76$.