Imperial College London

Department of Mathematics

M45P65 Mathematical Logic

Question	Examiner's Comments
Q 1	There was a missing) at the end of the formula in 1(a): the error was pointed out around 15-20 minutes into the exam and it did not appear to cause any problems. The question was certainly too long but reasonably well done; marks were lost in 1(b) because of insufficient reasoning or not considering all cases; part (c)(ii) was too long but all done in the lectures (but it wasn't enough to just say that). Part (c)(iii) was tricky but quite a few people got the idea.
Q 2	(a) was well done by those who had learned the definitions and recognised a standard result. It was not good to say that (a) (ii) follows from the completeness theorem. (b)(ii) was difficult but some people got this. In (b) (i) and (ii) a common mistake was to refer to specific elements of the structure in the formula: this changes the language and was not allowed. Very few people got (c), which was designed to be hard. Common attempts were to use groups or fields but largely these fell down because: no single formula can express that a group is cyclic; there are rings of prime order which are not fields; groups of order the square of a prime have to be abelian.
Q 3	(b)(ii) was corrected to '... infinite normal model of \Gamma.' in the first 15-20 minutes of the exam. This did not cause any issues. (a): something like this was on a coursework sheet, so more of you should have been able to do this. (b)(I, ii) is the easiest compactness argument there is, and again it should have been done by more people. (b)(iii) was intended to be hard and it proved to be so: very few people got this.
Q 4	(a) The answers here were often a bit rambling. Too many people wrote that S_3 is R\times R. Comparing S_1-S_4 was best done using results on cardinal arithmetic from the end of the module, but could have been done in a more elementary way. Comparing with S_5 was hard but a few of you remembered about using a basis of R over Q. Parts (b)(i-iii) were generally ok; (ii) was a trick question. Few people managed (iv), but it was nice to see a couple of people use Loewenheim - Skolem here.
Q 5	This was not well done at all. I did not get the impression that many people had engaged with the mastery material, but there may have been a bit too much of it. (c) was probably too hard for a question at the end of a long paper, but parts of (a), (b) should have been quite do-able on the basis of the mastery problem sheet.

Imperial College
London

M4/5P65

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2018

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Mathematical Logic

Date: Friday, 11 May 2018

Time: 10:00 AM - 12:30 PM

Time Allowed: 2.5 hours

This paper has 5 questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Each question carries equal weight.
- Calculators may not be used.

WORK IN ZFC THROUGHOUT.
BOTH $\mathbb{N}$ AND ω DENOTE THE NATURAL NUMBERS.

1. In this question, L denotes the formal system for propositional logic used in the lectures.

(a) Find a formula in disjunctive normal form which is logically equivalent to $(\neg\theta)$, where θ is the formula:

$$((p_1 \to (p_2 \to p_3)) \to ((p_2 \to (\neg p_1)) \to (p_3 \to (\neg p_2))),$$

where p_1, p_2, p_3 are propositional variables.

(b) What is meant by saying that a set of connectives is *adequate*? Which subsets of $\{\neg, \wedge, \to\}$ are adequate? Give reasons for your answer.

(c) (i) Suppose $\Sigma \cup \{\psi\}$ is a set of L-formulas. Define what is meant by a *deduction* of ψ from Σ. State the Deduction Theorem.

(ii) Suppose ψ, χ, ϕ are L-formulas. Prove that

$$\{\psi, (\neg\psi)\} \vdash_L \chi \text{ and } \{((\neg\phi) \to \phi)\} \vdash_L \phi.$$

Your arguments should involve deductions: you may not assume the Completeness Theorem for L.

(iii) Suppose θ is as in part (a). Give examples of formulas ψ, χ with

$$\{\psi, \chi\} \vdash_L (\neg\theta),\ \{\chi\} \not\vdash_L (\neg\theta) \text{ and } \{\psi\} \not\vdash_L (\neg\theta).$$

Justify your answers, stating clearly any results which you wish to use.

2. (a) Suppose $\mathcal{L}$ is a first-order language and $\mathcal{A}$ is an $\mathcal{L}$-structure.

   (i) Suppose v is a valuation in $\mathcal{A}$ and ϕ is an $\mathcal{L}$-formula. Define what it means for v to *satisfy* ϕ (in $\mathcal{A}$).

   (ii) Suppose ϕ, ψ are $\mathcal{L}$-formulas. Prove that if the variable x_1 is not free in ϕ then

$$((\forall x_1)(\phi \to \psi) \to (\phi \to (\forall x_1)\psi))$$

   is logically valid.

(b) Suppose $\mathcal{L}^=$ is a first-order language with equality having a single 1-ary function symbol f (and no other relation, function or constant symbols). In each of the following, two normal $\mathcal{L}^=$-structures $\mathcal{A}_i$ and $\mathcal{B}_i$ are given. Decide whether there is a closed $\mathcal{L}^=$-formula ϕ_i which is true in $\mathcal{A}_i$ but not in $\mathcal{B}_i$. Explain your answers.

   (i) The domain of $\mathcal{A}_1$ and $\mathcal{B}_1$ is $\mathbb{N}$. The function symbol f is interpreted as the function $x \mapsto x+2$ in $\mathcal{A}_1$ and as $x \mapsto x+3$ in $\mathcal{B}_1$.

   (ii) The domain of $\mathcal{A}_2$ and $\mathcal{B}_2$ is $\mathbb{N}$. The function symbol f is interpreted as the function $x \mapsto x^2$ in $\mathcal{A}_2$ and as $x \mapsto x^3$ in $\mathcal{B}_2$.

   (iii) The domain of $\mathcal{A}_3$ is $\mathbb{R}$ and the domain of $\mathcal{B}_3$ is $\mathbb{Q}$. In both cases, f is interpreted as the function $x \mapsto x^3$.

(c) Give an example of a language with equality $\mathcal{L}^=$ and a closed $\mathcal{L}^=$-formula θ with the property that for every non-zero natural number n, there is a normal model of θ with domain of size n if and only if n is not a prime number. Explain your answer.

3. (a) Suppose $\mathcal{L}$ is a first-order language, ϕ is an $\mathcal{L}$-formula and t is a term of $\mathcal{L}$. Define what is meant by t being *free for the variable* x_i in ϕ. Suppose $\mathcal{L}$ has a single 2-ary relation symbol R and no other relation, function or constant symbols. Give an example of a formula $\psi(x_1)$ with a single free variable x_1 such that for some variable x_i, the formula

$$((\forall x_1)\psi(x_1) \to (\forall x_i)\psi(x_i))$$

   is not logically valid. Explain your answer and comment on how many possibilities for the variable x_i there are.

(b) Suppose $\mathcal{L}^=$ is a first-order language with equality and Γ is a set of closed $\mathcal{L}^=$-formulas with arbitrarily large finite normal models.

   (i) Prove that Γ has an infinite normal model. Any result which you need from the lectures should be quoted carefully.

   (ii) Show that there is a set Δ of closed $\mathcal{L}^=$-formulas with $\Gamma \subseteq \Delta$ and such that the normal models of Δ are the infinite models of Γ.

   (iii) Prove that every set Δ with the property in part (ii) has to contain infinitely many formulas which are not in Γ (in other words, the set of formulas $\Delta \setminus \Gamma$ is infinite).

4. (a) Compare the cardinalities of the following five sets $S_1 - S_5$, giving reasons for your answers:

- $S_1 = \mathbb{R}$;
- S_2 is the set of sequences of 0's and 1's;
- S_3 is the set of functions $f : \mathbb{R} \to \mathbb{R}$;
- S_4 is the set of sequences of real numbers;
- S_5 is the set of functions $g : \mathbb{R} \to \mathbb{R}$ which satisfy $g(x+y) = g(x) + g(y)$ for all $x, y \in \mathbb{R}$.

Any general results on cardinal arithmetic which you use should be stated clearly.

(b) For each of the following statements, determine whether it is true or false. Give reasons for your answers.

(i) Every infinite set has a countably infinite subset.

(ii) Every set has a maximal countable subset.

(iii) If A, B are non-empty sets, then $|A| \leq |B|$ if and only if there is a surjection $f : B \to A$.

(iv) If A is an infinite set, then there is a 2-ary relation $\leq$ on A such that $\langle A; \leq \rangle$ is a dense linear ordering.

5. (a) Suppose that the first-order language with equality $\mathcal{L}^=$ has cardinality κ and $\mathcal{N}$ is a normal $\mathcal{L}^=$-structure of cardinality at least κ. Say what it means for a substructure $\mathcal{M}$ of $\mathcal{N}$ to be an *elementary* substructure and state the Tarski - Vaught Test for being an elementary substructure. Using this, prove that $\mathcal{N}$ has an elementary substructure of cardinality κ.

(b) Suppose $\mathcal{L}_c^=$ is a first-order language with equality having countably many constant symbols $(c_i : i < \omega)$ and no other relation, function or constant symbols. Let Σ be the set of $\mathcal{L}_c^=$-formulas:

$$\Sigma = \{(c_i \neq c_j) : i < j < \omega\}.$$

(i) Define what it means for two normal models $\mathcal{M}, \mathcal{N}$ of Σ to be *isomorphic*.

(ii) For each infinite cardinal λ, determine how many isomorphism types of normal models of Σ of cardinality λ there are. Explain your answer.

(iii) Suppose $\mathcal{M} = \langle M; (a_i : i < \omega)\rangle$ is a normal model of Σ and $M \setminus \{a_i : i < \omega\}$ is infinite. Suppose $\mathcal{N}$ is a normal model of Σ and $\mathcal{M}$ is a substructure of $\mathcal{N}$. Prove that $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$.

(c) Let $\mathcal{R}$ denote the real numbers considered as a structure

$$\mathcal{R} = \langle \mathbb{R}; \leq, +, -, \cdot, 0, 1\rangle$$

in the usual language of rings with an ordering. Let $\mathcal{R}^*$ be a elementary extension of $\mathcal{R}$ with domain M. Suppose that $M \neq \mathbb{R}$. Prove that there is $a \in M$ with the property that $0 < a < r$ for all $r \in \mathbb{R}$.

1. (a) **(Similar example on problem sheet)** Using the truth table for implication, one checks that for a propositional valuation v, we have $v(\theta) = F$ iff $v(p_1) = F$, $v(p_2) = T$ and $v(p_3) = T$. Thus $(\neg\theta)$ is logically equivalent to

$$(\neg p_1) \wedge p_2 \wedge p_3$$

which is in disjunctive normal form. **4 marks**

(b) **(Bookwork and standard examples)** A set of connectives is *adequate* if, for every natural number n, every truth function of n variables can be expressed as the truth function of a formula involving only the variables and the connectives from the set.

By the Disjunctive Normal Form Theorem, the set $\{\neg, \wedge, \to\}$ is adequate, as $\vee$ can be expressed in terms of $\neg$ and $\wedge$: $\phi \vee \psi$ is logically equivalent to $(\neg((\neg\phi) \wedge (\neg\psi)))$. Similarly, $\{\neg, \wedge\}$ is adequate. As $(\phi \wedge \psi)$ is logically equivalent to $\neg(\phi \to (\neg\psi))$, the subset $\{\neg, \to\}$ is also adequate.

Any formula constructed using only $\{\wedge, \to\}$ takes the value T when the variable have value T, so this subset is not adequate. Thus any subset of this is not adequate.

The single connective $\neg$ is not adequate as it can only express $2n$ truth functions of n variables (a variable or its negation). **5 marks**

(c) (i) **(Standard Bookwork)** A *deduction* of ψ from Σ is a finite sequence of formulas $\psi_0, \psi_1, \ldots, \psi_n$ with the property that ψ is ψ_n and for all $i \leq n$, either ψ_i is an axiom of L, or a formula in Σ, or follows from formulas ψ_j with $j < i$ using the deduction rule Modus Ponens.

We write $\Sigma \vdash_L \psi$ to indicate that there is a deduction of ψ from Σ. The Deduction Theorem states that if $\Sigma \cup \{\phi, \psi\}$ is a set of L-formulas and $\Sigma \cup \{\phi\} \vdash_L \psi$, then $\Sigma \vdash_L (\phi \to \psi)$. **3 marks**

(ii) **(Examples done in lectures and on problem sheet)** First, we give a (condensed) deduction of χ from $\Sigma_1 = \{\psi, (\neg\psi)\}$.

1. $((\neg\psi) \to ((\neg\chi) \to (\neg\psi)))$ (Axiom A1)
2. $((\neg\chi) \to (\neg\psi))$ (1, MP and formula $\neg\psi \in \Sigma_1$)
3. $(\psi \to \chi)$ (Axiom A3, 2, and MP)
4. χ (3, MP and formula $\psi \in \Sigma_1$)

For the second part, let α be an axiom. By the above and MP we have $\{(\neg\phi), ((\neg\phi) \to \phi)\} \vdash_L (\neg\alpha)$ so by DT, $\{((\neg\phi) \to \phi)\} \vdash_L ((\neg\phi) \to (\neg\alpha))$. By this, Axiom A3 and MP, we obtain $\{((\neg\phi) \to \phi)\} \vdash_L (\alpha \to \phi)$. Thus, as α is an axiom, we can use MP to obtain $\{((\neg\phi) \to \phi)\} \vdash_L \phi$. **5 marks**

(iii) **(Unseen)** We use the fact that $\Sigma \vdash_L \phi$ if and only if every propositional valuation which satisfies all the formulas in Σ also satisfies ϕ (this is the generalised Soundness and Completeness Theorem for L). Using this and the solution to part (a), it is clear that we may take ψ as $((\neg p_1) \wedge p_2)$ and χ as p_3. Any valuation satisfying both of these satisfies $\neg\theta$, but a valuation v with $v(p_1) = F$, $v(p_2) = T$, $v(p_3) = F$ satisfies ψ and does not satisfy $\neg\theta$. Similarly a valuation w with $w(p_1) = T$ and $w(p_3) = T$ satisfies χ and does not satisfy $\neg\theta$. **3 marks**

2. (a) (i) **(Standard definition)** This is defined by induction on the number of connectives and quantifiers in ϕ (we write $v[\phi] = T$ to indicate that v satisfies ϕ in $\mathcal{A}$, as in lectures).
If ϕ is an atomic formula $R(t_1, \ldots, t_n)$ (where t_i are terms of $\mathcal{L}$), then $v[\phi] = T$ iff $\bar{R}(v(t_1), \ldots, v(t_n))$ holds in $\mathcal{A}$ (where $\bar{R}$ is the interpretation of R in $\mathcal{A}$).
If ϕ is $(\neg\psi)$ then $v[\phi] = T$ iff $v[\psi] = F$.
If ϕ is $(\psi \to \chi)$, then $v[\phi] = F$ iff $v[\psi] = T$ and $v[\chi] = F$.
If ϕ is $(\forall x_i)\psi$ then $v[\phi] = T$ iff $w[\psi] = T$ whenever w is a valuation in $\mathcal{A}$ which is x_i-equivalent to v (meaning $v(x_j) = w(x_j)$ whenever $j \neq i$). **3 marks**

(ii) **(Done in lectures)** Suppose for a contradiction that there is a valuation v in some $\mathcal{L}$-structure $\mathcal{A}$ which does not satisfy the given formula. Then $v[(\forall x_1)(\phi \to \psi)] = T$, $v[\phi] = T$ and $v[(\forall x_i)\psi] = F$. By the last of these there is w which is x_1-equivalent to v with $w[\psi] = F$. As x_1 is not free in ϕ and v, w are x_1-equivalent, v, w agree on all free variables of ϕ, so by a result proved in lectures $w[\phi] = T$. Thus $w[(\phi \to \psi)] = F$. As w, v are x_1-equivalent, this contradicts $v[(\forall x_1)(\phi \to \psi)] = T$. **5 marks**

(b) **(Seen similar questions to (i), (iii); part (ii) is unseen)**

(i) We can take ϕ_1 to be a formula which expresses that in $\mathcal{A}_1$ there are at most two elements not in the image of f:

$$(\exists x_1)(\exists x_2)(\forall x)(\neg((\exists y)(f(y) = x) \to ((x = x_1) \vee (x = x_2)))).$$

2 marks

(ii) The two structures are isomorphic, so there is no such formula ϕ_2. For the isomorphism, note that in each case the function is injective, there is a unique fixed-point and there are infinitely many points not in the image. In each case, if $x \notin \mathrm{im} f$ then the points $x, f(x), f(f(x)), \ldots$ are all distinct and sets like this partition $\mathbb{N}$ (apart from the fixed point). This is enough to give an isomorphsm (any bijection between the points not in the image extends in a unique way to an isomorphism). **3 marks**

(iii) In $\mathcal{A}_3$ the function is surjective, but this is not the case in $\mathcal{B}_3$. So we can take ϕ_3 to be:

$$(\forall x_1)(\exists x_2)(f(x_2) = x_1).$$

2 marks

(c) **(Unseen)** Various possible solutions here.

One possibility is to take a language for groups with an extra 1-ary relation symbol. The formula says that we have a group and the 1-ary relation gives a proper, non-trivial subgroup. Another possbility is that the language has two 2-ary relation symbols E_1, E_2 and θ expresses that: these are equivalence relations, every E_1-class intersects every E_2-class in exactly one element, and there are at least two E_i-classes for $i = 1, 2$. If $\mathcal{A}$ is a finite normal model of this with m_i E_i-classes then A has $m_1 m_2$ elements. Conversely given $m, n > 1$ we can constrct a normal model of θ of size mn by considering an $m \times n$ grid. **5 marks**

3. (a) **(Standard definitions and examples)** This means that there is no variable x_j in t such that x_i occurs free within the scope of a quantifier $(\forall x_j)$ in ϕ. **2 marks**

The formula $((\forall x_1)\psi(x_1) \to (\forall x_i)\psi(x_i))$ is logically valid if x_i is free for x_1 in $\psi(x_1)$. Thus, the number of variables x_i for which this is not the case is finite - any such variable must appear (as a bound variable) in ψ. **2 marks**

For an example in the given language, consider the formula $\psi(x_1)$ given by $(\exists x_2)R(x_1, x_2)$ and take x_i to be x_2. Let $\mathcal{A}$ be the $\mathcal{L}$-structure with domain $\mathbb{N}$ in which $R(x_1, x_2)$ is interpreted as $x_1 < x_2$ (in the usual ordering on $\mathbb{N}$). Then $\mathcal{A} \models (\forall x_1)(\exists x_2)R(x_1, x_2)$, but $\mathcal{A} \not\models (\forall x_2)(\exists x_2)R(x_2, x_2)$. So $\mathcal{A} \not\models ((\forall x_1)\psi(x_1) \to (\forall x_2)\psi(x_2))$. **4 marks**

(b) (i) **(Bookwork, seen similar)** We use the Compactness Theorem for normal models which states that if Σ is a set of closed $\mathcal{L}^=$-formulas with the property that every finite subset of Σ has a normal model, then Σ has a normal model.

For $n \in \mathbb{N}$ with $n \geq 2$, let σ_n be the $\mathcal{L}^=$-formula

$$\bigwedge_{1 \leq i < j \leq n} (x_i \neq x_j).$$

Let $\Gamma_1 = \Gamma \cup \{\sigma_n \,:\, n \geq 2\}$. Every finite subset of this has a normal model, by the assumption - we take a sufficiently large finite normal models of Γ. So by the Compactness Theorem this has a normal model $\mathcal{A}$. This is an infinite normal model of Γ. **5 marks**

(ii) We can take Δ to be the set Γ_1 from part (i). **1 mark**

(iii) **(Unseen)** Suppose for a contradiction that Δ is as in (ii) and $\Delta \setminus \Gamma = \{\delta_1, \ldots, \delta_n\}$ for some $n \in \mathbb{N}$. Let δ be the formula $\delta_1 \wedge \ldots \wedge \delta_n$ and consider the set $\Gamma_2 = \Gamma \cup \{(\neg\delta)\}$. By the assumption on Δ, the normal models of Γ_2 are precisely the finite normal models of Γ. Indeed, a finite normal model of Γ does not satisfy some formula in $\Delta \setminus \Gamma$ and so is a model of Γ_2. Conversely a normal model of Γ_2 cannot be a model of Δ and so is not infinite. Thus Γ_2 has arbitrarily large finite normal models and so by (i), it has an infinite normal model. This is a contradiction. **6 marks**

4. (a) **($S_1 - S_4$ seen similar in lectures; S_5 unseen)** We will use the facts that if λ is an infinite cardinal and $2 \leq \kappa \leq \lambda$ then $\kappa.\lambda = \lambda$ and $\kappa^\lambda = 2^\lambda$. By considering decimal expansions, there are injective functions from $S_1 = \mathbb{R}$ to $\mathbb{Z}^\mathbb{N}$ (sequences of integers) and from $S_2 = 2^\mathbb{N}$ to $\mathbb{R}$. Thus $|\mathbb{R}| \leq \omega^\omega = 2^\omega \leq |\mathbb{R}|$. So S_1, S_2 have the same cardinality. The set S_4 has cardinality $|\mathbb{R}^\omega|$ and this is $(2^\omega)^\omega = 2^\omega$. So this is the same cardinality as S_1.

The set S_3 has cardinality λ^λ where $\lambda = 2^\omega$ is the cardinality of $\mathbb{R}$. This is equal to 2^λ (by the quoted result) and so is equal to the cardinality of $\mathcal{P}(\mathbb{R})$, the set of subsets of $\mathbb{R}$. So by Cantor's theorem $\lambda < 2^\lambda$.

As $S_5 \subseteq S_3$ its cardinality is at most 2^λ. Consider $\mathbb{R}$ as a vector space over $\mathbb{Q}$. Any $\mathbb{Q}$-linear map $\mathbb{R} \to \mathbb{R}$ is in S_3. If B is a basis of $\mathbb{R}$ over $\mathbb{Q}$ then any function $B \to \mathbb{R}$ extends to a $\mathbb{Q}$-linear map $\mathbb{R} \to \mathbb{R}$. Thus $|S_3| \geq |\mathbb{R}^B|$. But $|B| = \lambda$ as any element of $\mathbb{R}$ can be represented by a finite sequence of rationals and a finite sequence of elements of B. It follows that $|S_5| = 2^\lambda$. So $|S_1| = |S_2| = |S_4| < |S_3| = |S_5|$. **10 marks**

(b) (i) **(Seen on a problem sheet)** True. Call the set A and let $\kappa = |A|$. So κ is an infinite cardinal and there is a bijection $f : \kappa \to A$. We have $\omega \leq \kappa$ (as κ is infinite), and $\{f(n) : n \in \omega\}$ is a countably infinite subset of A. **2 marks**

(ii) **(Unseen)** False. Let B be any uncountable set. If $A \subseteq B$ is countable then $A \neq B$, and so there is some $b \in B \setminus A$. Then $A \subset A \cup \{b\} \subseteq B$ and $A \cup \{b\}$ is countable. So A is not maximal amongst countable subsets of B. **2 marks**

(iii) **(Done in a problem class)** True. If $|A| \leq |B|$ there is an injective function $g : A \to B$. Let $a \in A$ and define $f : B \to A$ by $f(b) = g^{-1}(b)$ if b is in the image of g and $f(b) = a$ otherwise. This is a surjective function. Conversely, suppose there is a surjection $g : B \to A$. Let h be a choice function on the non-empty subsets of B. For $a \in A$ define $f(a) = h(g^{-1}(a))$. Then $f : A \to B$ is an injective function, so $|A| \leq |B|$. **2 marks**

(iv) **(Unseen)** True. It suffices to find a dense linear ordering $\langle B; \leq_B \rangle$ for some set B of the same cardinality as A. Then there is a bijection $f : A \to B$ and we define $a \leq a'$ iff $f(a) \leq_B f(a')$ for $a, a' \in A$. Take $\langle A; \leq_A \rangle$ to be any linear ordering of A (eg a well ordering) and consider $B = \mathbb{Q} \times A$ with the reverse lexicographic ordering (essentially this replaces each element of A by a copy of $\mathbb{Q}$). This is a dense linear ordering of B and B has cardinality $\omega.|A| = |A|$. **4 marks**

5. (a) **(Bookwork: Standard definition and results)** A substructure $\mathcal{M}$ of $\mathcal{N}$ is an *elementary substructure* if, for every $\mathcal{L}^=$-formula $\phi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in M^n$ we have

$$\mathcal{M} \models \phi(a_1, \ldots, a_n) \Leftrightarrow \mathcal{N} \models \phi(a_1, \ldots, a_n).$$

(Here, M is the domain of $\mathcal{M}$.)

Tarski-Vaught Test: with the above notation, suppose that for every $\mathcal{L}^=$-formula $\psi(y, x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in M$ with $\mathcal{N} \models (\exists y)\psi(y, a_1, \ldots, a_n)$, there is $b \in M$ with $\mathcal{N} \models \psi(b, a_1, \ldots, a_n)$. Then $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$.

We construct an elementary substructure of $\mathcal{N}$ of cardinality κ as a union of a chain of subsets $A_0 \subseteq B_0 \subseteq A_1 \subseteq B_1 \subseteq \ldots$ of size κ. We let $M = \bigcup_{i<\omega} A_i = \bigcup_{i<\omega} B_i$. We take A_0 to be any subset of $\mathbb{N}$ of cardinality κ. For every $i < \omega$, we let B_i be the substructure generated by A_i. So if $|A_i| = \kappa$, then $|B_i| = \kappa$. Given B_i we construct A_{i+1} so that if $a_1, \ldots, a_n \in B_i$ and there is $b \in N$ with $\mathcal{N} \models \psi(b, a_1, \ldots, a_n)$, then there is such a b in A_{i+1}. Note that if B_i is of cardinality κ then we can do this so that $|A_{i+1}| = \kappa$ as there are κ possibilities for $\psi(y, a_1, \ldots, a_n)$ here. As the B_i are substructures, M is the domain of a substructure $\mathcal{M}$ of $\mathcal{N}$. Moreover, by the construction of A_i, the set M satisfies the condition in the Tarski-Vaught Test. So $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$. As each A_i is of cardinality κ, M is of cardinality κ. **8 marks**

(b) (i) **(Special case of standard definition)** Suppose $\mathcal{M} = \langle M; (a_i : i < \omega)\rangle$ and $\mathcal{N} = \langle N; (b_i : i < \omega)\rangle$ are normal models of Σ (so a_i, b_i are the interpretations of c_i in $\mathcal{M}, \mathcal{N}$ respectively). An isomorphism $\mathcal{M} \to \mathcal{N}$ is a bijection $f : M \to N$ with $f(a_i) = b_i$ for all $i < \omega$. **1 marks**

(ii) **(Unseen)** Suppose $\mathcal{M} = \langle M; (a_i : i < \omega)\rangle$ and $\mathcal{N} = \langle N; (b_i : i < \omega)\rangle$ are normal models of Σ of cardinality $\lambda \geq \omega$. Then $\mathcal{M}$ and $\mathcal{N}$ are isomorphic if and only if $M \setminus \{a_i : i < \omega\}$ and $N \setminus \{b_i : i < \omega\}$ are of the same cardinality κ. If $\lambda = \omega$, then κ can be any of $0, 1, 2, \ldots, \omega$. If $\lambda > \omega$ then $\kappa + \omega = \lambda$, so $\kappa = \lambda$. Thus there are countably many isomorphism types of countable normal models of Σ and a single isomorphism type of normal model of Σ of cardinality λ when $\lambda > \omega$. **3 marks**

(iii) **(Idea seen on problem sheet)** Suppose $\psi(y, x_1, \ldots, x_n)$ is an $\mathcal{L}_c^=$-formula and $m_1, \ldots, m_n \in M$ are such that $\mathcal{N} \models (\exists y)\psi(y, m_1, \ldots, m_n)$. Let $b \in \mathcal{N}$ be such that $\mathcal{N} \models \psi(b, m_1, \ldots, m_n)$. To apply Tarski-Vaught we may assume $b \notin M$. Using the assumption on $\mathcal{M}$, let b' be an element of M not equal to any m_i or a_j. There is an isomorphism $\mathcal{N} \to \mathcal{N}$ fixing all m_i and sending b to b'. Then, applying this isomorphism, $\mathcal{N} \models \psi(b', m_1, \ldots, m_n)$, so the condition in Tarski-Vaught holds. **3 marks**

(c) **(Unseen)** If there is $b \in M$ with $b > r$ for all $r \in \mathbb{R}$, then take $a = 1/b$. Suppose for a contradiction that there is no such b and let $b \in M \setminus \mathbb{R}$. We may assume $b > 0$. By assumption, there is $n \in \mathbb{R}$ with $b < n$ and therefore $\{c \in \mathbb{R} : c < b\}$ is bounded above in $\mathbb{R}$. Let $c_1 \in \mathbb{R}$ denote the supremum of this set in $\mathbb{R}$. Note that $c_1 \neq b$.

If $c_1 < b$ let $a = b - c_1$. Then $a > 0$ and there is no $r \in \mathbb{R}$ with $a > r > 0$. So $0 < a < r$ for all $r \in \mathbb{R}$ with $r > 0$. This is a contradiction, by considering $1/a$.

If $c_1 > b$ let $a = c_1 - b$. Then $a > 0$. If $c \in \mathbb{R}$ and $c < c_1$, then $c < b$, so $a < r$ for all $r \in \mathbb{R}$ with $r > 0$ (otherwise $c_1 - r > b$). Again, this is a contradiction. **5 marks**