MATH6/70132; P65: Notes on Lemma 2.3.7

(2.3.7) Lemma: Suppose $\mathcal{L}$ is a first-order language and $\phi(x_1)$ is an $\mathcal{L}$-formula (possibly with other free variables). Let t be a term free for x_1 in $\phi(x_1)$.

Suppose $\mathcal{A}$ is an $\mathcal{L}$-structure and v is a valuation in $\mathcal{A}$. Let v' be the valuation in $\mathcal{A}$ which is x_1-equivalent to v with $v'(x_1) = v(t)$. Then

$$v'[\phi(x_1)] = T \Leftrightarrow v[\phi(t)] = T.$$

Proof: This is by induction on the number of connectives and quantifiers in ϕ.

Base case: ϕ is an atomic formula $R(u_1, \ldots, u_m)$ where R is an m-ary relation symbol and $u_1, \ldots, u_m$ are terms.
Let u_i^* be the result of substituting t for x_1 in u_i. Then, by induction on the length of the terms, each u_i^* is a term and $v'(u_i) = v(u_i^*)$. Moreover, $\phi(t)$ is $R(u_1^*, \ldots, u_m^*)$. Then:

$$v'[\phi(x_1)] = T \Leftrightarrow \mathcal{A} \models R(v'(u_1), \ldots, v(u_m)) \Leftrightarrow \mathcal{A} \models R(v(u_1^*), \ldots, v(u_m^*)) \Leftrightarrow v[\phi(t)] = T.$$

Inductive step: There are 3 cases:
Case 1: ϕ is $(\neg\psi)$;
Case 2: ϕ is $\psi \to \chi)$;
Case 3: ϕ is $(\forall x_i)\psi$.
We leave the fist two cases as exercises and do the third.
We can assume that $i \neq 1$. Otherwise x_1 is not free in ϕ and $\phi(t)$ is just ϕ. The lemma then follows from 2.3.3.
Note also that as t is free for x_1 in $(\forall x_i)\psi$, it follows that t is free for x_1 in ψ and x_i is not a variable in t.
Suppose first that $v'[\phi(x_1)] = F$. We show that $v[\phi(t)] = F$.
By Definition 2.2.9, there is a valuation w' which is x_i-equvalent to v' with $w'[\psi(x_1)] = F$. Note that as $i \neq 1$:

$$w'(x_1) = v'(x_1) = v(t). \tag{1}$$

Define a valuation w by:

$$w(x_j) = \begin{cases} v(x_j) & \text{if } j \neq 1, i \\ w'(x_i) & \text{if } j = i \\ v(x_1) & \text{if } j = 1 \end{cases}.$$

So w is x_1-equivalent to w' and x_i-equivalent to v (noting that v, v' are x_i-equivalent and w, v' are x_i-equivalent).
As x_i does not occur in t we have, by (1),

$$w(t) = v(t) = w'(x_1).$$

We can now apply the induction hypothesis to w, w' and ψ. We obtain that $w[\psi(t)] = w'[\psi(x_1)] = F$.
As w, v are x_i-equivalent, it follows that

$$v[(\forall x_i)\psi(t)] = F.$$

So $v[\phi(t)] = F$, as required.

We now prove the converse direction (we cannot argue by symmetry here). So suppose $v[\phi(t)] = F$. There is a valuation w which is x_i-equivalent to v with $w[\psi(t)] = F$. Let w' be the valuation x_1-equivalent to w with

$$w'(x_1) = w(t) = v(t) = v'(x_1).$$

(The fact that $w(t) = v(t)$ is as before.)
By the inductive hypothesis, $w'[\psi(x_1)] = w[\psi(t)] = F$. As w' is x_i-equivalent to v' we have

$$v'[(\forall x_i)\psi(x_1)] = F.$$

So $v'[\phi(x_1)] = F$. This completes the inductive step. □