Imperial College London

MATH97009 MATH97101

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2020

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Mathematical Finance - An Introduction to Option Pricing

Date: 4th May 2020

Time: 09.00am - 11:30am (BST)

Time Allowed: 2 Hours 30 Minutes

Upload Time Allowed: 30 Minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS SEPARATE PDFs TO THE RELEVANT DROPBOXES ON BLACKBOARD (ONE FOR EACH QUESTION) WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

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1. Consider the following one period trinomial model: $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathbb{P}(\omega_i) = 1/3$ for $i = 1, 2, 3$, a bank account B with interest rate $r = 0$, and one stock S with $S_0 = 6$ and

$$S_1(\omega) = \begin{cases} 2, & \text{if } \omega = \omega_1, \\ 6, & \text{if } \omega = \omega_2, \\ 12, & \text{if } \omega = \omega_3. \end{cases}$$

We denote with $C(K)$ the European call option (on the stock) with strike price $K \geq 0$; this has payoff $C_1(K) := (S_1 - K)^+$ at time 1. Answer the following questions and justify carefully with either proofs or counterexamples.

(a) Is the market (B, S) arbitrage free? (2 marks)

(b) Is the call option $C(K_1)$ with strike $K_1 = 4$ replicable? (2 marks)

(c) Find arbitrage free prices $C_0(K_1)$ (at time 0, in the given market (B, S)) of a call with strike $K_1 = 4$. (5 marks)

(d) Consider the enlarged market $(B, S, C(K_1))$ made of: bank account, stock, call option with strike $K_1 = 4$ sold at time 0 at an arbitrage free price $C_0(K_1)$. Is this market complete? Does the answer depend on the value of $C_0(K_1)$? (5 marks)

(e) Consider the market $(B, S, C(K_1))$ made of: bank account, stock, call option with strike $K_1 = 4$ sold at time 0 at price $C_0(K_1) := \frac{13}{5}$. Now enlarge the market $(B, S, C(K_1))$ using call options with strike $K_2 = 5$, sold at time 0 at price $C_0(K_2)$. We do not assume that $C_0(K_2)$ is necessarily an arbitrage free price; instead we assume that $C_0(K_2)$ satisfies the inequalities

$$C_0(K_2) \leq C_0(K_1) \leq C_0(K_2) + K_2 - K_1 \tag{A}$$

$$(S_0 - K_2)^+ \leq C_0(K_2) \leq S_0. \tag{B}$$

It can be shown that in any market model where at least one of these inequalities fails there is an arbitrage. Does the converse hold, i.e. do our assumptions *imply* that the enlarged market $(B, S, C(K_1), C(K_2))$ is arbitrage free? If yes, prove it; if not, explicitly find values of $C_0(K_2)$ which satisfy (A), (B) and for which the market admits an arbitrage.

(6 marks)

(Total: 20 marks)

2. In the framework of the N-period binomial model with constant parameters $S_0 = 8, u = 2, d = 1/2, r = 0$, let $S = (S_n)_{n=0}^N$ be the stock price process, M_n its historical *minimum* up to time n (i.e. $M_n := \min_{i=0,\ldots,n} S_i$). Consider the down-and-in rebate option with the lower barrier $L = 6$ which expires at time N and pays 1 if S_n is less than L for any $n = 0, \ldots, N$; in other words, this derivative has a payoff $1 - Y_N$ at maturity N, where $Y_n := 1_{\{M_n \geq 6\}}$ (i.e. $Y_n = 1$ if $M_n \geq 6$, and $Y_n = 0$ otherwise). We denote with V_n the arbitrage-free price at time $n = 0, \ldots, N$ of this option.

Below, whenever we say that a process is *Markov*, we mean with respect to the risk-neutral measure $\mathbb{Q}$ and with the usual filtration $\mathcal{F}_n = \sigma(X_1, \ldots, X_n), n = 0, \ldots, N$ generated by the coin tosses $X_n(\omega) = \omega_n$ on the probability space $\Omega = \{H, T\}^N$. Answer the following questions and justify carefully with either proofs or counterexamples.

(a) Is M a Markov process? (2 marks)

(b) Is Y a Markov process? (3 marks)

(c) Is (S, M) a Markov process? (4 marks)

(d) Is (S, Y) a Markov process? (5 marks)

(e) Use the risk-neutral pricing formula to express V_n in terms of V_{n+1}. (2 marks)

(f) Work by backward induction to show that, for every $n = 0, \ldots, N$, V_n admits the representation $V_n = v_n(X_n)$, where $v_n : \mathbb{R}^k \to \mathbb{R}$, $n = 0, \ldots, N$, are some (deterministic) functions and $X = (X_n)_n$ is some Markov process with values in $\mathbb{R}^k$, for some $k \geq 1$. Determine one such X, and write explicitly v_N and an explicit formula to express v_n in terms of v_{n+1} for $n = 0, \ldots, N - 1$. (4 marks)

(Total: 20 marks)

3. Consider a market $(B_n, S_n)_{n=0,1,\dots,T}$ described by a multi-period binomial model with constant parameters $0 < d < 1 + r < u$, and as usual let $\mathcal{F}_k = \sigma(X_1, \dots, X_k), 0 \le k \le T$ be the filtration generated by the coin tosses $(X_i)_i$. Consider a *forward-start call option*, which entitles its holder to receive at time $T_0 \in \mathbb{N}, T_0 < T$ a call option (on the stock S) with maturity T and strike KS_{T_0} (where $K > 0$). Answer the following questions, and (other than in item (a)) justify carefully with proofs.

(a) Write down a formula, involving the expectation with respect to the risk-neutral measure $\mathbb{Q}$, for

$$V_0 := \text{the price at time } 0 \text{ of the forward-start call option.}$$

(5 marks)

(b) (i) Prove that the random variables

$$R_{k+1} := \frac{S_{k+1}}{S_k}, \quad k = 0, 1, \dots, T-1,$$

are IID under the EMM $\mathbb{Q}$.

(ii) Prove that, for each $k = 0, 1, \dots, T-1$, R_{k+1} is independent of $\mathcal{F}_k$.

(iii) Prove that, for each $i = 0, 1, \dots, T-1$, the random vector $V_{k+1} := (R_k)_{k=i+1,\dots,T-1}$ is independent of $\mathcal{F}_i$. *Hint: Prove that* $\mathbb{E}_i^{\mathbb{Q}}(f(R_{i+1}, \dots, R_T))$ *is constant, for any* f.

(iv) Prove that $\frac{S_T}{S_{T_0}}$ is independent of S_{T_0} under $\mathbb{Q}$.

(7 marks)

(c) Compute the expectation of S_{T_0} under $\mathbb{Q}$. (2 marks)

(d) Show that $V_0 = c(T - T_0, Kx)$, where $c(t, x)$ is the price at time 0 of a call option with expiry t and $S_0 = x$. (6 marks)

(Total: 20 marks)

4. On the sample space $\Omega = \{\omega_i\}_{i=1,\dots,5}$ endowed with some probability $\mathbb{P}$ s.t. $\mathbb{P}(\omega_i) > 0$ for all i, consider a one-period arbitrage-free market model where the bank account has zero interest rate, and there are two stocks S^1, S^2 with prices $S_0^1 = 3, \quad S_0^2 = 3$

$$S_1^1 = \begin{pmatrix} 1 & 3 & 5 & 7 & 4 \end{pmatrix}^T, \quad S_1^2 := \begin{pmatrix} 1 & 4 & 6 & 5 & 4 \end{pmatrix}^T.$$

For the *non-replicable* derivative with payoff X_1, we consider the problem of finding

$$p := \min\{x : (x,h) \in \mathbb{R} \times \mathbb{R}^2 \text{ satisfies } V_1^{x,h}(\omega_i) \geq X_1(\omega_i) \text{ for } i = 1,2,3,4\}, \tag{1}$$

the smallest initial capital p of a portfolio (x,h) super-replicating X_1 $\mathbb{P}$ a.s., where as usual we denote with $V_1^{x,h} = x + \sum_{j=1}^2 h_j(S_1^j - S_0^j)$ the wealth relative to the initial capital x and the trading strategy $h = (h_1, h_2)$; and its dual linear program, i.e.

$$d := \max\{\mathbb{E}^{\mathbb{Q}}(X_1) : \mathbb{Q} \in \mathcal{M}\}, \text{ where } \mathcal{M} := \{\mathbb{Q} \text{ proba. on } \Omega : \mathbb{E}^{\mathbb{Q}}(S_1^j - S_0^j) = 0,\ j = 1,2\} \tag{2}$$

is the set of martingale measures. We denote with h^* any trading strategy s.t. $V_1^{p,h^*} \geq X_1 \quad \mathbb{P}$ a.s., and with $\mathbb{Q}^*$ any element of $\mathcal{M}$ such that $d = \mathbb{E}^{\mathbb{Q}^*}(X_1)$, i.e. any optimisers of (1) and (2). Answer the following questions and justify carefully with either proofs or counterexamples.

(a) Prove that the market (B, S^1, S^2) is arbitrage-free. (3 marks)

(b) Is p an arbitrage-free price for X_1? Prove your assertion. (2 marks)

(c) Prove that $V_1^{p,h^*} = X_1$ $\mathbb{Q}^*$ a.s.. (3 marks)

(d) Consider the problem of finding the smallest initial capital $\bar{p}$ of a portfolio (x,h) super-replicating X_1, choosing only among portfolios which are not short-selling the first stock (i.e. S^1). Formulate this problem in a way analogous to (1), and formulate its dual problem in a way analogous to (2). Are the respective optimal values $\bar{p}$ and $\bar{d}$ always equal? (4 marks)

(e) Suppose you own one share of the first stock, and you are allowed to trade the second stock and use the bank account to borrow and deposit money, but you cannot trade in the first stock. Consider the smallest initial capital $\tilde{p}$ which you will need to super-replicate X_1. Formulate the problem of finding $\tilde{p}$ in a way analogous to (1), and formulate its dual linear program in a way analogous to (2). (3 marks)

(f) Solve the primal and dual problems you formulated in item (e), i.e. find their optimisers and the optimal values, in the case where the derivative is given by

$$X_1 := \begin{pmatrix} 1 & 0 & 3 & 4 & -3 \end{pmatrix}^T.$$

(5 marks)

(Total: 20 marks)

5. Consider the trinomial model with time index $\{0,1\}$, and a market made of a bank account with interest rate $r=2$ and of one stock whose price is given by $S_0=3$ and

$$S_1(\omega)=\begin{cases}3 & \text{if } \omega=x_1\\ 9 & \text{if } \omega=x_2\\ 15 & \text{if } \omega=x_3,\end{cases}$$

where $\Omega=\{x_1,x_2,x_3\}$ is the probability space, endowed with a probability $\mathbb{P}$ s.t.

$$\mathbb{P}(\{x_1\})=\frac{2}{10},\quad \mathbb{P}(\{x_2\})=\frac{5}{10},\quad \mathbb{P}(\{x_3\})=\frac{3}{10}.$$

An investor with initial capital $\frac{1}{5}$ and utility function $U(X)=\ln(x), x>0$ wants to invest in this market up to maturity.

(a) Compute the set of equivalent martingale measures. (6 marks)

(b) Compute the set of the terminal wealths which the investor can attain. (4 marks)

(c) Find the optimal investment strategy for the investor, i.e. compute the number of stocks $\hat{\Delta}_0$ he needs to buy or sell at time 0 in order to maximize the expected utility of his terminal wealth. (10 marks)

(Total: 20 marks)

SOLUTION OF FINAL EXAM M5MF22 2019/2020

1. Exercise 1, similarly seen in Lectures and Problems

(a) This trinomial model is free of arbitrage since the condition $d < 1 + r < u$ is satisfied: indeed $d = 2/6 = 1/3$, $1 + r = 1$, $u = 12/6 = 2$.

(b) In the market (B, S) the call with strike K_1 is not replicable. This follows from the next item, since we find that there exists more than one AFP (arbitrage free price) for the call. Alternatively, it follows from the fact that, since $r = 0$, the replication equation is $x + h(S_1 - S_0) = C(K_1)$, which in vector notation becomes

$$x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + h \begin{pmatrix} 2-6 \\ 6-6 \\ 12-6 \end{pmatrix} = \begin{pmatrix} (2-4)^+ \\ (6-4)^+ \\ (12-4)^+ \end{pmatrix},$$

i.e.

$$x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + h \begin{pmatrix} -4 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 8 \end{pmatrix}, \tag{1}$$

which has no solution. Indeed its first equation gives $x = 4h$, its second equation gives $x = 2$, combining these gives $h = 1/2$, and these values do not solve the third equation since $2 + 6/2 = 5$ does not equal 8.

(c) In this simple setting, the less computationally intensive way to find its AFP is probably to compute the smallest super-replication price s and largest sub-replication price i. An alternative but more commonly used way is to use the EMM (equivalent martingale measures), which we now do. Recall that $\mathbb{Q}$ is an EMM if $S_0 = \mathbb{E}^{\mathbb{Q}}[S_1/(1+r)]$, $\mathbb{Q}$ is a probability and $\mathbb{Q} \sim \mathbb{P}$, i.e. iff $q_i := \mathbb{Q}(\{\omega_i\})$ satisfy

$$\begin{cases} 6 = 2q_1 + 6q_2 + 12q_3 \\ 1 = q_1 + q_2 + q_3 \\ q_i > 0 \text{ for } i = 1, 2, 3 \end{cases}$$

The system has 2 equalities and 3 unknowns, so it has one free parameter. So we choose $q_2 = t$, and compute q_1, q_3 as $q_1 = 3(1-t)/5, q_3 = 2(1-t)/5$, and imposing $q_i > 0$ we obtain that the set of (q's corresponding to the set of) EMM is

$$\mathcal{M} := \left\{ \begin{pmatrix} 3(1-t)/5 \\ t \\ 2(1-t)/5 \end{pmatrix} : t \in (0,1) \right\}. \tag{2}$$

As $\mathcal{M}$ is not empty, this confirms that the model is arbitrage-free. The set $\mathcal{P}$ of AFP for the call with strike 4 is

$$\mathcal{P} := \{\mathbb{E}^{\mathbb{Q}}[(S_1 - 4)^+/(1 + r)] : \mathbb{Q} \in \mathcal{M}\},$$

which using (2) can be written as

$$\mathcal{P} = \left\{ \frac{3}{5}(1-t)\cdot 0 + t\cdot 2 + \frac{2}{5}(1-t)\cdot 8 : t \in (0,1) \right\}. \tag{3}$$

By evaluating the above expression at $t = 0$ and at $t = 1$ to get the values $\frac{16}{5}$ and 2, and since the function of t which appears is strictly monotone (it is affine), we get that $\mathcal{P} = (2, \frac{16}{5})$.

(d) For any choice of AFP $p := C_0(4)$, the market $(B, S, C(K_1))$ is complete. Indeed, the replication equation

$$x + h(S_1 - S_0) + k(C_1(K_1) - C_0(K_1)) = D_1$$

for a derivative with payoff D_1 corresponds to the system of equations

$$x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + h \begin{pmatrix} -4 \\ 0 \\ 6 \end{pmatrix} + k \begin{pmatrix} 0-p \\ 2-p \\ 8-p \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}, \tag{4}$$

where $d_i := D_1(\omega_i)$. which always has a solution. Indeed, the system has 3 unknowns x, h, k, and is made of independent equations, because the vectors v_1, v_2, v_3 which represent the payoff of bank account, stock and call option, are linearly independent: indeed v_1, v_2 are linearly independent (one is not a multiple of the other), and, for any choice of AFP $p := C_0(4)$, v_3 is not a linear combination of v_1, v_2 (otherwise $C_1(4)$ would have been replicable).

For an alternative solution, observe that, for any choice of AFP $C_0(4)$, the market $(B, S, C(4))$ has only one EMM (thus it is complete), which is the unique EMM $\mathbb{Q}$ for the (B, S) market s.t.

$$C_0(4) = \mathbb{E}^{\mathbb{Q}}[C_1(4)/(1+r)]. \tag{5}$$

(e) The converse does NOT hold, i.e. our assumptions do not imply that the enlarged market $(B, S, C(K_1), C(K_2))$ is arbitrage free. The reason is that, $C_0(4) := \frac{13}{5}$ is an AFP for $C(4)$, the market $(B, S, C(K_1))$ is arbitrage-free and complete, so any derivative has a *unique* AFP in $(B, S, C(K_1))$, and the market $(B, S, C(K_1))$ has only one EMM, which is the unique EMM $\mathbb{Q}$ for the (B, S) market s.t. (5) holds. We can then use such $\mathbb{Q}$ to find the unique AFP p for $C_1(5)$ in the $(B, S, C(K_1))$ market, and show that there is a value of $C_0(5)$ which satisfies the inequalities

$$C_0(K_2) \le C_0(K_1) \le C_0(K_2) + K_2 - K_1 \tag{6}$$

$$(S_0 - K_2)^+ \le C_0(K_2) \le S_0, \tag{7}$$

and yet is different from p. This happens because the above inequalities do not fix uniquely the exact value of $C_0(K_2)$, but only require it to be in some interval I, so all but at most one value of $C_0(K_2) \in I$ will result in an arbitrage. Of course, we actually have to check that the market $(B, S, C(K_1))$ is complete, and that I is not degenerate (i.e. a singleton), which would invalidate our argument above; let us do that.

The EMMs for the market $(B, S, C_1(K_1))$ are the $\mathbb{Q}$'s given by (2) and such that $C_0(K_1) = \mathbb{E}^{\mathbb{Q}}(C_1(K_1)/(1+r))$. Since since $t = \frac{1}{2}$ is the only solution of $C_0(K_1) = \mathbb{E}^{\mathbb{Q}}(C_1(K_1)/(1+r))$, i.e. of

$$\frac{13}{5} = \frac{3(1-t)}{5}\cdot 0 + t\cdot 2 + \frac{2(1-t)}{5}\cdot 8,$$

the market $(B, S, C_1(K_1))$ admits only one EMM, so it is complete.

Since the payoff $C_1(5)$ is given by

$$\begin{pmatrix} (2-5)^+ \\ (6-5)^+ \\ (12-5)^+ \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 7 \end{pmatrix},$$

the only AFP for $C_1(5)$ in the $(B, S, C(K_1))$ market is given by

$$p = \frac{3}{5}(1-\frac{1}{2}) \cdot 0 + \frac{1}{2} \cdot 1 + \frac{2}{5}(1-\frac{1}{2}) \cdot 7 = \frac{19}{10}.$$

Since (6) , (7) state that $C_0(5)$ must satisfy

$$\frac{13}{5} - 1 \leq C_0(K_2) \leq \frac{13}{5}, \qquad (6-5)^+ \leq C_0(K_2) \leq 6,$$

or equivalently that $\frac{8}{5} \leq C_0(K_2) \leq \frac{13}{5}$, we can choose for $C_0(5)$ any value in $[\frac{8}{5}, \frac{13}{5}] \setminus \{\frac{19}{10}\}$ (say $C_0(5) = 2$), with $C_0(K_1) = \frac{13}{5}$ as chosen above, and the corresponding market has an arbitrage, yet it satisfies (6) , (7).

2. Exercise 2, similarly seen in Lectures and Problems

(a) M is not Markov. Indeed after drawing its tree we immediately see the problem: $M_2(HH) = 8 = M_2(HT)$ yet $M_3(HTT) = 4$ is different from $M_3(HHH) = M_3(HHT) = M_3(HTH) = 8$, so

$$\mathbb{E}_2^{\mathbb{Q}} f(M_3)(HH) = \tilde{p} f(8) + (1-\tilde{p}) f(8), \qquad \text{does not equal}$$
$$\mathbb{E}_2^{\mathbb{Q}} f(M_3)(HT) = \tilde{p} f(8) + (1-\tilde{p}) f(4)$$

whenever $f(8) \neq f(4)$, where $\tilde{p} = \mathbb{Q}(H) = 1/3$.

(b) Y is not Markov. Indeed after drawing its tree we immediately see the problem: $Y_2(HH) = 1 = Y_2(HT)$ yet $Y_3(HTT) = 0$ is different from $Y_3(HHH) = Y_3(HHT) = Y_3(HTH) = 1$, so

$$\mathbb{E}_2^{\mathbb{Q}} f(Y_3)(TH) = \tilde{p} f(1) + (1-\tilde{p}) f(1), \qquad \text{does not equal}$$
$$\mathbb{E}_2^{\mathbb{Q}} f(Y_3)(TT) = \tilde{p} f(1) + (1-\tilde{p}) f(0)$$

whenever $f(1) \neq f(0)$, where $\tilde{p} = \mathbb{Q}(H) = 1/3$.

(c) (S, M) is Markov, as it is easily guessed after drawing its tree. To prove it, define

$$C_{n+1} := \frac{S_{n+1}}{S_n} = \begin{cases} u, & \text{if } X_{n+1} = H \\ d, & \text{if } X_{n+1} = T \end{cases},$$

so that $M_{n+1} = M_n \wedge (S_n C_{n+1})$. Then by the independence lemma

$$\mathbb{E}_n^{\mathbb{Q}}[f(S_{n+1}, M_{n+1})] = \mathbb{E}_n^{\mathbb{Q}}[f(S_n C_{n+1}, M_n \wedge (S_n C_{n+1}))] = g(S_n, M_n),$$

where

$$g(s, m) := \mathbb{E}^{\mathbb{Q}}[f(s C_{n+1}, m \wedge (s C_{n+1}))].$$

(d) (S, Y) is Markov, as it is easily guessed after drawing its tree. To prove it, define Define $h(x) := 1_{\{x \geq 16\}}, x \in \mathbb{R}$, notice that $h(x \wedge y) = h(x)h(y)$, so that $Y_n = h(M_n)$ and so

$$Y_{n+1} = h(M_{n+1}) = h(M_n \wedge (S_n C_{n+1})) = Y_n h(S_n C_{n+1}).$$

Then by the independence lemma

$$\mathbb{E}_n^{\mathbb{Q}}[f(S_{n+1}, Y_{n+1})] = \mathbb{E}_n^{\mathbb{Q}}[f(S_n C_{n+1}, Y_n h(S_n C_{n+1}))] = z(S_n, Y_n),$$

where
$$z(s,y) := \mathbb{E}^{\mathbb{Q}}[f(sC_{n+1}, yh(sC_{n+1}))].$$

(e) The risk neutral pricing formula gives
$$V_n = \mathbb{E}_n^{\mathbb{Q}}[\frac{V_{n+1}}{1+r}]$$

(f) One could choose X to be either (S,M) or (S,Y).

1st solution If we take $X = (S,M)$ then, using the results of item (3) we get that
$$v_n(S_n, M_n) = V_n = \mathbb{E}_n^{\mathbb{Q}}[\frac{V_{n+1}}{1+r}] = \frac{1}{1+r}\mathbb{E}_n^{\mathbb{Q}}[v_{n+1}(S_{n+1}, M_{n+1})] = \frac{1}{1+r}g(S_n, M_n),$$
where
$$g(s,m) := \mathbb{E}^{\mathbb{Q}}[v_{n+1}(sC_{n+1}, m \wedge (sC_{n+1}))],$$
which leads to the formula
$$v_n(s,m) = \frac{1}{3(1+r)}v_{n+1}(su, m\wedge(su)) + \frac{2}{3(1+r)}v_{n+1}(sd, m\wedge(sd)), \quad 0 \le n \le N-1,$$
where of course $v_N(s,m) = 1_{\{m \ge 16\}}$.

2nd solution If we take $X = (S,Y)$ then, using the results of item (4) we get that
$$v_n(S_n, Y_n) = V_n = \mathbb{E}_n^{\mathbb{Q}}[\frac{V_{n+1}}{1+r}] = \frac{1}{1+r}\mathbb{E}_n^{\mathbb{Q}}[v_{n+1}(S_{n+1}, Y_{n+1})] = \frac{1}{1+r}z(S_n, Y_n),$$
where
$$z(s,y) := \mathbb{E}^{\mathbb{Q}}[v_{n+1}(sC_{n+1}, yh(sC_{n+1}))],$$
which leads to the formula
$$v_n(s,y) = \frac{1}{3(1+r)}v_{n+1}(su, yh(su)) + \frac{2}{3(1+r)}v_{n+1}(sd, yh(sd)), \quad 0 \le n \le N-1,$$
where of course $v_N(s,y) = 1 - y$.

3. Exercise 3, similarly seen in Lectures and Problems

(a) The risk neutral pricing formula gives that the price of the forward-start is
$$V_0 = \frac{1}{(1+r)^T}\mathbb{E}^{\mathbb{Q}}\left((S_T - KS_{T_0})^+\right).$$

(b) The following implies at once (i) and (ii):

(i and ii) It is enough to show that, for each $i = 0, 1 \dots, T-1$, under the EMM $\mathbb{Q}$ the random variables
$$R_{k+1} := \frac{S_{k+1}}{S_k}, \quad k = i, i+1, \dots, T-1,$$
are IID, and each R_{k+1} is independent of $\mathcal{F}_i$. This holds since, by definition of $\mathbb{Q}$
$$\mathbb{Q}(R_{k+1} = u | \omega_1, \dots, \omega_k) = \tilde{p} := \frac{(1+r) - d}{u - d}$$
does not depend on $\omega_1, \dots, \omega_k$ (i.e. R_{k+1} is independent of $\mathcal{F}_k$) and $(R_1, \dots, R_k)$ is a function of $\omega_1, \dots, \omega_k$ (i.e. it is $\mathcal{F}_k$-measurable).

(iii) From (i) *and (ii)* it follows that the random vector $(R_k)_{k=i+1,\dots,T-1}$ is independent of $\mathcal{F}_i$: for example because the independence lemma gives that for any function $f_T : \mathbb{R}^{T-k} \to \mathbb{R}$

$$\mathbb{E}^{\mathbb{Q}}_{T-1}(f_T(R_{k+1},\dots,R_T)) = f_{T-1}(R_{k+1},\dots,R_{T-1}), \tag{8}$$

where

$$f_{T-1}(r_{k+1},\dots,r_{T-1}) := \mathbb{E}^{\mathbb{Q}}[f_T(r_{k+1},\dots,r_{T-1},R_T)].$$

Thus using the tower property of the conditional expectation we get

$$\mathbb{E}^{\mathbb{Q}}_k(f_T(R_{k+1},\dots,R_T)) = \mathbb{E}^{\mathbb{Q}}_k(f_{T-1}(R_{k+1},\dots,R_{T-1})),$$

and working by induction (/iterating the above reasoning) we get that

$$\mathbb{E}^{\mathbb{Q}}_k(f_T(R_{k+1},\dots,R_T)) = \mathbb{E}^{\mathbb{Q}}_k(f_{k+1}(R_{k+1})),$$

and the latter is constant and equals $c := \mathbb{E}^{\mathbb{Q}}(f_{k+1}(R_{k+1}))$, since R_{k+1} is independent of $\mathcal{F}_k$.

Thus, if $W := f_T(V_{k+1})$, for any $\mathcal{F}_k$-measurable random variable Y we get that

$$\mathbb{E}^{\mathbb{Q}}[YW] = \mathbb{E}^{\mathbb{Q}}[\mathbb{E}^{\mathbb{Q}}_k[YW]] = \mathbb{E}^{\mathbb{Q}}[Y\mathbb{E}^{\mathbb{Q}}_k[W]] = \mathbb{E}^{\mathbb{Q}}[Yc] = c\mathbb{E}^{\mathbb{Q}}[Y] = \mathbb{E}^{\mathbb{Q}}[W]\mathbb{E}^{\mathbb{Q}}[Y],$$

so V_{k+1} is independent of $\mathcal{F}_k$.

(iv) Since $\frac{S_T}{S_{T_0}} = R_{T_0+1}\cdots R_T$ is a function $(R_{T_0+1},\dots,R_T)$, it is independent of $\mathcal{F}_{T_0}$.

(c) Since the discounted stock price is a martingale under $\tilde{\mathbb{P}}$,

$$\frac{1}{(1+r)^{T_0}}\tilde{\mathbb{E}}\left(S_{T_0}\right) = \frac{1}{(1+r)^0}\tilde{\mathbb{E}}\left(S_0\right) = S_0, \text{ and so } \tilde{\mathbb{E}}\left(S_{T_0}\right) = S_0(1+r)^{T_0}.$$

(d) Since $S_{T_0} > 0$ we can write $(S_T - KS_{T_0})^+$ as the product of the two independent random variables S_{T_0} and $\left(\frac{S_T}{S_{T_0}} - K\right)^+$, and so

$$V_0 = \frac{1}{(1+r)^T}\tilde{\mathbb{E}}\left(S_{T_0}\right)\tilde{\mathbb{E}}\left[\left(\frac{S_T}{S_{T_0}} - K\right)^+\right],$$

which one could of course also have derived using the independence lemma. Since $\tilde{\mathbb{E}}\left(S_{T_0}\right) = x(1+r)^{T_0}$ and $x > 0$ we get that

$$V_0 = x\frac{1}{(1+r)^{T-T_0}}\tilde{\mathbb{E}}\left[\left(\frac{S_T}{S_{T_0}} - K\right)^+\right] = \frac{1}{(1+r)^{T-T_0}}\tilde{\mathbb{E}}\left[\left(x\frac{S_T}{S_{T_0}} - Kx\right)^+\right].$$

Since the $(R_k)_k$ are IIDs, the random variables

$$x\frac{S_T}{S_{T_0}} = xR_{T_0+1}\cdots R_T, \quad S_{T-T_0} = S_0R_1\cdots R_{T-T_0}$$

have the same law (under $\mathbb{Q}$), and so

$$V_0 = \frac{1}{(1+r)^{T-T_0}}\tilde{\mathbb{E}}\left[(S_{T-T_0} - Kx)^+\right] = c(T-T_0, Kx).$$

4. Exercise 4, unseen

(a) The market is arbitrage-free by the FTAP, since the set of EMM is

$$\mathcal{M} = \{(-1+5t+5s, 2-8t-6s, 2t, t, s) : t, s \in \mathbb{R}\} \cap (0,\infty)^5;$$

this set non-empty iff such is $\mathcal{N} \cap (0,\infty)^2$, where

$$\mathcal{N} := \{(-1+5t+5s, 2-8t-6s) : t, s \geq 0\}.$$

Drawing $\mathcal{N}$ allows to verify that $\mathcal{N} \cap (0,\infty)^2 \neq \emptyset$; for example one can choose $t = 1/20, s = 1/5$ and thus obtain

$$\mathbb{Q} = (-1+5t+5s, 2-8t-6s, 2t, t, s) = \left(\frac{1}{4}, \frac{2}{5}, \frac{1}{10}, \frac{1}{20}, \frac{1}{5}\right)^T \in \mathcal{M}.$$

(b) Since X_1 is not replicable, p is not an arbitrage-free price. To build an arbitrage, consider h^* s.t. $V_1^{p,h^*} \geq X_1 \quad \mathbb{P}$ a.s., and do this: short-sell one derivative at price p, buy h^* underlying S, and deposit the remaining money $p - S_0^1 - S_0^2$ in the bank. This leads to the final payoff $V_1^{p,h^*} - X_1$, which by definition is positive, and is not a.s. zero (otherwise X_1 would be replicable).

(c) Since $\mathbb{Q}^*$ is a martingale measure it satisfies $p = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}^*}(V_1^{p,h^*})$. Since $V_1^{p,h^*} \geq X_1$ $\mathbb{Q}^*$ a.s., the identity $\mathbb{E}^{\mathbb{Q}^*}(X_1) = \mathbb{E}^{\mathbb{Q}^*}(V_1^{p,h^*})$ shows that

$$V_1^{p,h^*} = X_1 \ \mathbb{Q}^* \text{ a.s..} \tag{9}$$

(d) The first problem could be written either as

$$\bar{p} := \min\{x : (x,h) \in \mathbb{R} \times \mathbb{R}^2 \text{ satisfies } V_1^{x,h} \geq X_1 \text{ a.s.}, \quad h_1 \geq 0\}, \tag{10}$$

or as

$$\bar{p} := \min\{x : (x,h_1,h_2) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \text{ satisfies } V_1^{x,h} \geq X_1 \text{ a.s.}\}, \tag{11}$$

We choose to consider the second form, since it leads to a simpler-looking dual problem (with one variable less than the other). The dual of (11) is then

$$\bar{d} := \max\{\mathbb{E}^{\mathbb{Q}}(X_1) : \mathbb{Q} \in \bar{\mathcal{M}}\}, \tag{12}$$

where

$$\bar{\mathcal{M}} := \{\mathbb{Q} \text{ proba on } \Omega : \quad \mathbb{E}^{\mathbb{Q}}(S_1^1 - S_0^1) \leq 0, \quad \mathbb{E}^{\mathbb{Q}}(S_1^2 - S_0^2) = 0\}.$$

Since $\bar{\mathcal{M}} \supseteq \mathcal{M} \neq \emptyset$, $\bar{\mathcal{M}}$ is not empty, so the dual problem is feasible. To apply the strong duality theorem and conclude that the two problems have solution and $\bar{p} = \bar{d}$, one can either observe that the primal is feasible[1], or that the dual is solvable[2].

(e) The first problem could be written either as

$$\tilde{p} := \min\{x : (x,h) \in \mathbb{R} \times \mathbb{R}^2 \text{ satisfies } \quad V_1^{x,h} \geq X_1 \text{ a.s.}, \quad h_1 = 1\}, \tag{13}$$

or as

$$\tilde{p} := \min\{x : (x,h_2) \in \mathbb{R} \times \mathbb{R} \text{ satisfies : } \quad V_1^{x,(1,h_2)} \geq X_1 \text{ a.s.}\}. \tag{14}$$

---

[1] For $h = 0$ and x big enough we have $V_1^{x,h} = x \geq X_1(\omega_i)$ for all i

[2] Since the set of $q \in \mathbb{R}_+^4$ s.t. $\sum_{i=1}^4 q_i = 1$ (which corresponds to the set of probabilities on Ω) is bounded, so is its subset $\bar{\mathcal{M}}$. Thus the dual problem (16) is solvable, since it is the problem of maximising a continuous (in fact, linear) function on the closed and bounded (and thus compact) set $\bar{\mathcal{M}}$.

We choose to consider the second form, since it leads to a simpler-looking dual problem: one with 2 constraints and 4 variables, instead of one with 3 constraints and 5 variables. In fact, since

$$V_1^{x,(1,h_2)} \geq X_1 \iff x + h_2(S_1^2 - S_0^2) \geq X_1 - (S_1^1 - S_0^1),$$

we can (and choose to) rewrite (14) as the problem of super-hedging the derivative with payoff $X_1 - (S_1^1 - S_0^1)$ by trading in the (B, S^2) market, since we are more familiar with it. Thus we consider the problem

$$\tilde{p} := \min\{x : (x, h_2) \in \mathbb{R} \times \mathbb{R} \text{ satisfies : } \quad V_1^{x,h_2} \geq X_1 - (S_1^1 - S_0^1) \text{ a.s.}\}, \tag{15}$$

whose dual is

$$\tilde{d} := \max\{\mathbb{E}^{\mathbb{Q}}(X_1 - (S_1^1 - S_0^1)) : \mathbb{Q} \in \tilde{\mathcal{M}}\}, \tag{16}$$

where

$$\tilde{\mathcal{M}} := \{\mathbb{Q} \text{ proba on } \Omega : \quad \mathbb{E}^{\mathbb{Q}}(S_1^2 - S_0^2) = 0\}.$$

is the set of martingale measures for the market (B, S_2).

(f) Since the primal (15) has only two variables, we can easily solve it by graphical inspection, which we will presently do. Let us write (15) explicitly as the problem of minimising x over

$$(x, h_2) \in D := \left\{ (x, h_2) \in \mathbb{R}^2 : x \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + h_2 \begin{pmatrix} -2 \\ 1 \\ 3 \\ 2 \\ 1 \end{pmatrix} \geq \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \\ -4 \end{pmatrix} \right\}$$

Drawing the set D shows that the first and third inequality imply the others, so that D is the unbounded 'triangle'

$$D = \{(x, h_2) \in \mathbb{R}^2 : x \begin{pmatrix} 1 \\ 1 \end{pmatrix} + h_2 \begin{pmatrix} -2 \\ 3 \end{pmatrix} \geq \begin{pmatrix} 3 \\ 1 \end{pmatrix}\}$$

with vertex $(x, h_2) = \frac{1}{5}(11, -2)$. Thus the optimal value is $\tilde{p} = \frac{11}{5}$, and the minimiser is $(\tilde{x}, \tilde{h}_2) = \frac{1}{5}(11, -2)$. Notice (for later use) that it satisfies the 1st and 3rd inequality defining D with equality; and that instead it satisfies the 2nd, 4th and 5th inequalities with strict inequality.

The dual problem has 5 variables which satisfy 2 constraints[3], so there are only 3 free variables. One could also solve it applying two iterations of the Fourier-Motzkin elimination algorithm (seen in class), or use other LP methods (e.g. the simplex method, or just evaluate the objective function at all extreme points) which we didn't cover them in class since it would have taken too long. Let us instead implement a simpler solution. Notice that, having solved the primal problem, we have a lot of information about the solution of the dual problem. First, by the strong duality theorem we have that the optimal value $\tilde{d}$ of the dual problem also equals $\tilde{p} = \frac{11}{5}$, so the dual solution $\tilde{\mathbb{Q}}$ satisfies

$$\mathbb{E}^{\tilde{\mathbb{Q}}}[X_1 - (S_1^1 - S_0^1)] = \frac{11}{5}. \tag{17}$$

This brings down the number of free variables from 3 to 2, so we would already be in a position to solve this problem by graphical inspection; however, we

[3]The 5 variables $q_i \in \mathbb{R}_+, i \leq 5$ satisfy the 2 constraints $\sum_i q_i = 1, \mathbb{E}^{\mathbb{Q}}(S_1^2 - S_0^2) = 0$.

can simplify it further. Indeed, by the 'complementary slackness' condition condition

$$\tilde{x} + \tilde{h}_2(S_1^2 - S_0^2) = X_1 - (S_1^1 - S_0^1) \quad \tilde{\mathbb{Q}} \text{ a.s. },$$

which was proved to hold in item (3), we have that $\tilde{q}_2 = \tilde{q}_4 = \tilde{q}_5 = 0$, since

$$\tilde{x} + \tilde{h}_2(S_1^2 - S_0^2)(\omega_i) > (X_1 - (S_1^1 - S_0^1))(\omega_i) \quad \text{ for } i = 2, 4, 5$$

as mentioned before. Asking that $\tilde{q} \in \tilde{\mathcal{M}}$ satisfies (17) we get that $\tilde{q}_1, \tilde{q}_3$ satisfy

$$\tilde{q}_1 + \tilde{q}_3 = 1, \quad -2\tilde{q}_1 + 3\tilde{q}_3 = 0 \quad 3\tilde{q}_1 + \tilde{q}_3 = \frac{11}{5}.$$

whose unique solution is $\tilde{q}_1 = \frac{3}{5}, \tilde{q}_3 = \frac{2}{5}$. Thus the dual optimiser is

$$\tilde{q} = \left(\frac{2}{5}, 0, \frac{3}{5}, 0, 0\right)^T.$$

5. Exercise 5, mastery question, seen in problems

(a) We compute first the discounted stocks values as $\bar{S}_0 = S_0 = 3$ and $\bar{S}_1 = S_1/(1+r)$, i.e.

$$\bar{S}_1(\omega) = \begin{cases} 1 & \text{if } \omega = x_1 \\ 3 & \text{if } \omega = x_2 \\ 5 & \text{if } \omega = x_3. \end{cases}, \qquad \text{so that } (\bar{S}_1 - \bar{S}_0)(\omega) = \begin{cases} -2 & \text{if } \omega = x_1 \\ 0 & \text{if } \omega = x_2 \\ 2 & \text{if } \omega = x_3. \end{cases}$$

Recall that $\mathbb{Q} \in \mathcal{M}$ if $\bar{S}_0 = \mathbb{E}^{\mathbb{Q}}[\bar{S}_1]$, Q is a probability and $\mathbb{Q} \sim \mathbb{P}$, i.e. iff $q_i := \mathbb{Q}(\{x_i\})$ satisfy

$$\begin{cases} 3 = q_1 + 3q_2 + 5q_3 \\ 1 = q_1 + q_2 + q_3 \\ q_i > 0 \text{ for } i = 1, 2, 3 \end{cases}$$

Subtracting the second line from the first line we get $2 = 2q_2 + 4q_3$, so writing $s := q_3$ we get $q_2 = 1 - 2s$ and the second line now gives

$$q_1 = 1 - (1 - 2s) - s = s.$$

Imposing $q_i > 0$ we obtain that the set of q_i's corresponding to $\mathcal{M}$ is

$$\left\{ q(s) := \begin{pmatrix} s \\ 1 - 2s \\ s \end{pmatrix} : s \in \left(0, \frac{1}{2}\right) \right\}. \tag{EMM}$$

Notice for later use that

$$q(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad q(\frac{1}{2}) = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} \tag{18}$$

We will denote with $\mathbb{Q}_s$ the probability such that $(\mathbb{Q}_s(\{\omega_i\}))_i$ is the vector $q(s)$, and with $\mathbb{P}_0, \mathbb{P}_1$ the end-points of the segment $\bar{\mathcal{M}}$, i.e. the probabilities $\mathbb{Q}_s$ with $s = 0, \frac{1}{2}$ corresponding to the vectors (18).

(b) The set of undiscounted attainable wealths is the affine subspace

$$\left\{ (1+r)(x + \Delta_0(\bar{S}_1 - \bar{S}_0)) = 3 \begin{pmatrix} 1/5 - 2\Delta_0 \\ 1/5 \\ 1/5 + 2\Delta_0 \end{pmatrix} : \Delta_0 \in \mathbb{R} \right\}. \tag{19}$$

(c) **1st solution** Now let us instead simply solve the maximization problem by taking derivatives. We need to solve the optimization problem

$$\max_{\Delta_0\in\mathbb{R}} \mathbb{E}u(B_1(x+\Delta_0(\bar{S}_1-\bar{S}_0))).$$

We then use (19) and see that we need to maximize

$$f(\Delta_0) := \frac{2}{10}\ln(\frac{3}{5}-6\Delta_0)+\frac{5}{10}\ln(\frac{3}{5})+\frac{3}{10}\ln(\frac{3}{5}+6\Delta_0).$$

To do this, we compute the first derivative

$$f'(\Delta_0) = \frac{2}{10}\cdot\frac{-6}{\frac{3}{5}-6\Delta_0}+\frac{3}{10}\cdot\frac{6}{\frac{3}{5}+6\Delta_0},$$

and set it equal to zero. This gives

$$\frac{\frac{3}{5}-6\Delta_0}{2} = \frac{\frac{3}{5}+6\Delta_0}{3},$$

from which we get $\frac{3}{5}(\frac{1}{2}-\frac{1}{3}) = 5\Delta_0$ and so the optimal strategy is $\bar{\Delta}_0 = \frac{1}{50}$. To actually make sure this is a maximimum, we should also check that $f''(\bar{\Delta}_0) < 0$; alternatively, it suffices to notice that f is a concave function (since such is the logarithm), so $f'(\bar{\Delta}_0) = 0$ implies that $\bar{\Delta}_0$ is a maximizer.

Notice how, in this example (as it often happens in very simple examples), the standard optimization approach leads to very simple computations; this generally fails to be true where there are multiple time steps, and does not handle constraints well (one needs to introduce Lagrange multipliers). Moreover, this method is not well suited for problems in continuous time, plus it does not allow to develop any theory.

2nd and 3rd solution Let us now see two additional (related) ways to solve the problem, using martingale measures. The terminal wealth of the strategy Δ_0 is given by

$$X_1(\Delta_0) := (1+r)(x+\Delta_0(\bar{S}_1-\bar{S}_0)), \tag{20}$$

where $x = 1/5$ is the initial capital. Following Pliska's textbook, to the optimal final wealth $\hat{X}_1$ satisfies

$$U'(\hat{X}_1) = \frac{1}{1+r}\left(\lambda_0\frac{d\mathbb{P}_0}{d\mathbb{P}}+\lambda_1\frac{d\mathbb{P}_1}{d\mathbb{P}}\right) \tag{21}$$

for some $\lambda_i > 0, i = 0,1$, which are determined by asking that $\hat{X}_1$ is replicable. We now have two choices, and two corresponding ways to solve the problem. Replicability can be expressed by asking that $\hat{X}_1(\lambda_0,\lambda_1)$ (found by inverting the above equation) satisfies

$$\mathbb{E}^{\mathbb{P}_i}\left(\frac{\hat{X}_1(\lambda_0,\lambda_1)}{1+r}\right) = x, \qquad i = 0,1, \tag{22}$$

which is a system of two unknowns in the two variables λ_0,λ_1. Having thus found the optimal final wealth $\hat{X}_1$, one can then solve the equation $\hat{X}_1 =$

$X_1(\Delta_0)$ to find the optimal trading strategy $\hat{\Delta}_0$. Let us do this explicitly. Using $U'(x) = 1/x$ we write (21) as

$$\frac{\hat{X}_1}{1+r} = \frac{1}{\left(\lambda_0 \frac{d\mathbb{P}_0}{d\mathbb{P}} + \lambda_1 \frac{d\mathbb{P}_1}{d\mathbb{P}}\right)} \tag{23}$$

and then using (EMM) and (18) we write this as

$$\begin{cases} x_0 = \dfrac{2}{5\lambda_1} \\ x_1 = \dfrac{1}{2\lambda_0} \\ x_2 = \dfrac{3}{5\lambda_1} \end{cases} \tag{24}$$

where $x_i := \frac{\hat{X}_1}{1+r}(\omega_i)$ and in the first equality we used that

$$1/\left(\frac{0\lambda_0 + \frac{1}{2}\lambda_1}{\frac{2}{10}}\right) = \frac{2}{5\lambda_1}$$

and analogous calculations are used for the other two equalities. Using (18) we compute

$$\mathbb{E}^{\mathbb{P}_0}\left(\frac{\hat{X}_1(\lambda_0,\lambda_1)}{1+r}\right) = x_1, \quad \mathbb{E}^{\mathbb{P}_1}\left(\frac{\hat{X}_1(\lambda_0,\lambda_1)}{1+r}\right) = \frac{1}{2}x_0 + \frac{1}{2}x_2,$$

and then, using (24), (22) becomes

$$\frac{1}{2\lambda_0} = x, \qquad \frac{1}{2\lambda_1} = x, \tag{25}$$

and since $x = 1/5$ this gives $\lambda_0 = \frac{5}{2} = \lambda_1$. Plugging this into (24) gives

$$\hat{X}_1 = \frac{3}{25}\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \tag{26}$$

and finally we solve $\hat{X}_1 = X_1(\Delta_0)$ using (19), finding that $\Delta_0 = 1/50$; thus the optimal strategy is $\hat{\Delta}_0 = 1/50$.

Alternatively, to express that $\hat{X}_1$ is replicable we can use that $\hat{X}_1$ belongs to the set in (19); we are thus lead to solve the system of three equations (one for each ω)

$$U'(X_1(\Delta_0)) = \frac{1}{1+r}\left(\lambda_0 \frac{d\mathbb{P}_0}{d\mathbb{P}} + \lambda_1 \frac{d\mathbb{P}_1}{d\mathbb{P}}\right) \tag{27}$$

in the three unknowns $\Delta_0, \lambda_0, \lambda_1$. Of course solving a system in 3 unknowns is slightly harder than solving one in 2 unknowns and then another in 1 unknown, so this second method is slightly harder, in general. Using $U'(x) = 1/x$ we write (27) as

$$\frac{X_1(\Delta_0)}{1+r} = \frac{1}{\left(\lambda_0 \frac{d\mathbb{P}_0}{d\mathbb{P}} + \lambda_1 \frac{d\mathbb{P}_1}{d\mathbb{P}}\right)} \tag{28}$$

and then using (EMM), (18) and (19) we write this explicitly as

$$\begin{cases} \dfrac{1}{5} - 2\Delta_0 = \dfrac{2}{5\lambda_1} \\ \dfrac{1}{5} \qquad\quad = \dfrac{1}{2\lambda_0} \\ \dfrac{1}{5} + 2\Delta_0 = \dfrac{3}{5\lambda_1} \end{cases}, \qquad \text{where } s \in \left(0, \frac{1}{2}\right), \Delta_0 \in \mathbb{R}.$$

The second equality now gives $\lambda_0 = 5/2$ and summing the first and third equality we get $\frac{2}{5} = 1/\lambda_1$ and so $\lambda_1 = \frac{5}{2}$. Plugging this into the third equality now gives $\Delta_0 = 1/50$; thus the optimal strategy is $\hat{\Delta}_0 = 1/50$.

Notice that, instead of using Lagrange multipliers λ_0, λ_1, we can reparameterize (21) in a way which you may be more familiar with. Indeed, from the theory of optimal investment you know that $U'(\hat{X}_1) = c\frac{d\mathbb{Q}_s}{d\mathbb{P}}$, and this is closely related to the above: just rewrite the right-hand side of (27) as $c\frac{d\mathbb{Q}_s}{d\mathbb{P}}$ for some[4] $c > 0$ and for some[5] $s \in \left(0, \frac{1}{2}\right)$. We can thus use c, s instead of λ_0, λ_1 as parameters. To find $\hat{\mathbb{Q}}, c$ and Δ_0 we use (19) to write the (vector) equation $\hat{X}_1(\omega) = \frac{1}{c}\frac{d\mathbb{P}}{d\hat{\mathbb{Q}}}(\omega)$, $\omega \in \Omega$ as

$$\begin{cases} 3\left(\dfrac{1}{5} - 2\Delta_0\right) = \dfrac{1}{c} \cdot \dfrac{2}{10s} \\ 3\left(\dfrac{1}{5} \qquad\quad\right) = \dfrac{1}{c} \cdot \dfrac{1}{2(1-2s)} \\ 3\left(\dfrac{1}{5} + 2\Delta_0\right) = \dfrac{1}{c} \cdot \dfrac{3}{10s} \end{cases}, \qquad \text{where } s \in \left(0, \frac{1}{2}\right), c > 0, \Delta_0 \in \mathbb{R},$$

and then find the unique solution to the above system of 3 equations in 3 variables. In fact, in the present setting of log utility things simplify a little, as we can compute the value of c without even solving the above system: since $U'(x) = 1/x$ we get $\hat{X}_1 = \frac{1}{c}\frac{d\mathbb{P}}{d\hat{\mathbb{Q}}}$, and so we can compute $\mathbb{E}^{\hat{\mathbb{Q}}}[\hat{X}_1]$ without even knowing $\hat{\mathbb{Q}}$, since

$$x(1+r) = \mathbb{E}^{\hat{\mathbb{Q}}}[\hat{X}_1] = \frac{1}{c}\mathbb{E}^{\hat{\mathbb{Q}}}[\frac{d\mathbb{P}}{d\hat{\mathbb{Q}}}] = \frac{1}{c}\mathbb{E}^{\mathbb{P}}[\frac{d\mathbb{P}}{d\hat{\mathbb{Q}}}\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}] = \frac{1}{c}.$$

This way we found that $c = 1/(x(1+r)) = \frac{5}{3}$, and plugging this into the system of equations, from the second equation we get $s = 1/4$, and so the third equation gives that the optimal strategy is $\Delta_0 = \frac{1}{50}$ (which of course also solves the first equation).

[4]Of course c is given by $c := \frac{1}{1+r}(\lambda_0 + \lambda_1)$.

[5]Since $\mathcal{M}$ is convex, any convex combination of $\mathbb{P}_0$ and $\mathbb{P}_1$ is in $\mathcal{M}$.

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionI	Comments for Students	
MATH97009 MATH97101	1	This was meant to be the simplest question to answer. It was generally quite well done, though disappointingly often easy points were lost, especially in the last item, due to confusion	
MATH97009 MATH97101	2	This turned out to be the questions with the 2nd highest average mark. Often the student did everything correctly, save for the last item, which was often left empty or only partially solved	
MATH97009 MATH97101	3	This exercise, which was meant to be of only medium difficulty, turned out to be too hard: most students solved items (a), (b)(i and ii) correctly, and nothing else. Almost nobody understood the point of item (b)(iii), which proved too much of a technical subtlety. Surprisingly few students answered (c) correctly, and fewer still managed item (d). This convinced me that next year I should spend more time doing manipulations with independent random variables and conditional expectation.	
MATH97009 MATH97101	4	This was meant to be the hardest question to answer, as the second half was mostly unseen material. In item (a), the students often waster much effort by calculating all the Martingale Measures, instead of simply exhibiting one (or proving directly that there was no arbitrage). They often got (b) wrong, even if it was enough to cite a result seen in class, and could not prove (c), which was very basic. Fortunately, by replying on their intuition they did better than I expected in items (d),(e), which were the novel parts. Item (f) was rarely attempted	
MATH97009 MATH97101	5	This surprisingly turned out to be the questions with the highest average mark. While I don't think it was the easiest, every item in it was very similar to previously solved exercises and was purely computational	

Error in the exam - in Q4 equation (1) it should say: i=1.2.3.4.5. Correction announced mid-exam