Mathematical Logic (MATH60132 and MATH70132)
Coursework 1

This coursework is worth 5 percent of the module. The deadline for submitting the work is 1300 on Monday 7 November 2022. The coursework is marked out of 20 and the marks per question are indicated below.
Any quotation of a result from the notes or problem sheets must be clear. If you use any source (including internet or books) other than the lecture notes and problem sheets, you must provide a full reference for your source. Failure to do so could constitute plagiarism.

[1] Give a propositional formula ϕ in propositional variables p_1, p_2, p_3, p_4 (using any of the connectives $\neg, \to, \wedge, \vee$) which has the property that, for every propositional valuation v:

$$v(\phi) = T \text{ if and only if exactly } 3 \text{ of } v(p_1), v(p_2), v(p_3), v(p_4) \text{ have value } T.$$

Justify your answer concisely (so, without writing down the full truth table of ϕ).

Solution: There are various ways of doing this. Here is one which follows the idea of disjunctive normal form (which was the point of the question).

The formula σ_1 given by $(\neg p_1) \wedge p_2 \wedge p_3 \wedge p_4$ has the property that $v(\sigma_1) = T$ iff $v(p_1, p_2, p_3, p_4) = (F, T, T, T)$ (where v is a propositional valuation and we are abusing notation slightly). We define σ_i (for $i = 2, 3, 4$) similarly so that $v(\sigma_i) = T$ iff $v(p_j) = T$ precisely when $j \neq i$. Then ϕ given by $\sigma_1 \vee \sigma_2 \vee \sigma_3 \vee \sigma_4$ has the required property.

[2] Suppose ϕ, ψ and χ are L-formulas. In the lectures we defined $(\phi \vee \psi)$ to be shorthand for $((\neg\phi) \to \psi)$. For each of the following, write the formula without using this shorthand and prove that it is a theorem of L. You may use results and theorems of L from your notes, but do not use the Completeness Theorem for L.

(i) $(\phi \vee (\neg\phi))$;

(ii) $((\phi \vee \psi) \to (\psi \vee \phi))$;

Solution: (i) The formula is $((\neg\phi) \to (\neg\phi))$. This is a theorem of L (by 1.2.3 in the notes).

(ii) The formula is (omitting some brackets) $((\neg\phi \to \psi) \to (\neg\psi \to \phi))$. One way of showing that this is a theorem of L is as follows.

1. By Qu1(c) on Sheet 2, $\vdash_L \psi \to \neg\neg\psi$.
2. By (1) and Hypothetical Syllogism (HS) $\neg\phi \to \psi \vdash_L \neg\phi \to \neg\neg\psi$.
3. By (2) and axiom A3 we have $\neg\phi \to \psi \vdash_L \neg\psi \to \phi$.
4. By (3) and the Deduction Theorem $\vdash_L ((\neg\phi \to \psi) \to (\neg\psi \to \phi))$, as required.

[3] Here are two definitions which are not in the notes:

- Suppose $\Sigma \cup \{\psi\}$ is a set of L-formulas. We say that ψ is *independent from* Σ if and only if $\Sigma \nvdash_L \psi$ and $\Sigma \nvdash_L (\neg\psi)$.
- A set Δ of L-formulas is *independent* if and only if Δ is consistent and for every $\psi \in \Delta$ we have that ψ is independent from $\Delta \setminus \{\psi\}$.

In the following, you may use any results from the notes and problem sheets.

(i) Suppose Σ is a consistent set of L-formulas and ψ is an L-formula. Show that ψ is independent from Σ if and only if there exist valuations v, v' such that $v(\Sigma) = v'(\Sigma) = T$ and $v(\psi) = T$, $v'(\psi) = F$.

(ii) Using (i), give a formulation in terms of valuations of what it means for a set Δ of L-formulas to be independent.

(iii) Is the following set of L-formulas independent? Justify your answer.

$$\Delta = \{(p_1 \to ((\neg p_2) \to p_3)),\ ((\neg p_3) \to (p_2 \to (\neg p_1))),\ (p_3 \to (p_1 \to (\neg p_2)))\}.$$

(iv) In the following, we write 0 for F and 1 for T; addition and multiplication on $\{0,1\}$ are modulo 2, giving the field $\mathbb{F}_2$ with two elements (as in the problem class in week 4). The set $V(n)$ of truth functions $F : \{0,1\}^n \to \{0,1\}$ is then a vector space over the field $\mathbb{F}_2$.

Suppose that $\sigma_1, \ldots, \sigma_k$ are L-formulas in the propositional variables $p_1, \ldots, p_n$ and let $F_1, \ldots, F_k \in V(n)$ denote the truth functions of these. Prove that if $\{\sigma_1, \ldots, \sigma_k\}$ is an independent set of formulas, then $F_1, \ldots, F_k$ are linearly independent.

Solution: (i) Suppose ψ is independent from Σ, so $\Sigma \not\vdash_L \psi$ and $\Sigma \not\vdash_L (\neg\psi)$. Then by Corollary 1.3.9 in the notes, applied twice, there are valuations v', v with $v'(\Sigma) = T$, $v'(\psi) = F$, and $v(\Sigma) = T$, $v(\neg\psi) = F$. These have the required properties.

Conversely suppose that we have valuations v, v' such that $v(\Sigma) = v'(\Sigma) = T$ and $v(\psi) = T$, $v'(\psi) = F$. It follows from Question 2 on problem sheet 2 that $\Sigma \not\vdash_L \neg\psi$ (using v) and $\Sigma \not\vdash_L \psi$ (using v').

(ii) The set Δ is independent if and only if there is a valuation v such that $v(\Delta) = T$ (so Δ is consistent) and for every $\delta \in \Delta$, there is a valuation v' with $v'(\delta) = F$ and $v'(\Delta \setminus \{\delta\}) = T$.

(iii) The set Δ is independent. We can do this by finding valuations showing that the criterion at the end of part (ii) is satisfied. We could do this by inspecting the truth tables of the three formulas. Another way is to argue as follows. Call the three formulas $\delta_1, \delta_2, \delta_3$.

It is easy to show that, for a valuation v,

$v(\delta_1) = F$ iff $v(p_1, p_2, p_3) = (T, F, F)$;

$v(\delta_2) = F$ iff $v(p_1, p_2, p_3) = (T, T, F)$;

$v(\delta_3) = F$ iff $v(p_1, p_2, p_3) = (T, T, T)$.

Thus the valuation $v(p_1, p_2, p_3) = (F, F, F)$ satisfies $v(\Delta) = T$, showing that Δ is consistent.

The valuation $v_1(p_1, p_2, p_3) = (T, F, F)$, shows that δ_1 is independent from $\{\delta_2, \delta_3\}$; the other independences are obtained similarly from the above.

(iv) We can prove this by induction on k. For the base case $k = 1$, we need to show that $F_1 \neq 0$. As $\{\sigma_1\}$ is an independent set, it is consistent, so there is a valuation with $v(\sigma_1) = T$. Thus $F_1 \neq 0$, as required.

For the inductive step suppose that the result holds for smaller values of k and that $F_1, \ldots, F_k$ are linearly dependent. The only scalars are 0 and 1, so using the inductive assumption, the only possible linear dependence is that $F_1 + \ldots + F_k = 0$. So if v is a valuation with $v(\sigma_i) = T$ for $i < k$, then $F_k(v) = k - 1 \bmod 2$. In particular, $v(\sigma_k)$ is determined and therefore σ_k cannot be independent from $\{\sigma_i : i < k\}$, a contradiction.