MATH60132 Coursework 1

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1. Consider the valuation function as $v(T)=1$, $v(F)=0$. then the valuation would be on the field $\mathbb{Z}/2\mathbb{Z}$

We need to construct ϕ which satisfies

$$v(\phi)=1 \quad \text{iff exactly 3 of } v(p_1), v(p_2), v(p_3), v(p_4) \text{ are } 1.$$

Examine the propositional connectives, then we have

$$v((F\wedge G)) = v(F)\cdot v(G)$$
$$v((F\leftrightarrow G)) = 1+v(F)+v(G)$$, which could be checked easily.
$$v(((F\wedge\neg G)\vee(\neg F\wedge G))) = v(F)+v(G)$$

Then we construct as below $v(\phi)$

$$v(\phi) = \left(1+v(p_1)\cdot v(p_2)\cdot v(p_3)+v(p_2)v(p_3)\cdot v(p_4)\right)+\left(1+v(p_3)\cdot v(p_4)\cdot v(p_1)+v(p_4)\cdot v(p_1)\cdot v(p_2)\right) \quad (*)$$

Now we check that $v(\phi)=1$ iff exactly 3 of $v(p_1), v(p_2), v(p_3), v(p_4)$ are 1.

As $(*)$ could also be written as

$$v(\phi) = v(p_1)v(p_2)\cdot v(p_3)+v(p_2)\cdot v(p_3)\cdot v(p_4)+v(p_3)\cdot v(p_4)\cdot v(p_1)+v(p_4)\cdot v(p_2)\cdot v(p_1)$$

we could find that $v(\phi)$ is symmetric for p_1, p_2, p_3, p_4.

For the same hierarchy of p_1, p_2, p_3, p_4, we examine the value of $v(\phi)$ without the loss of generality,

	$v(p_1)$	$v(p_2)$	$v(p_3)$	$v(p_4)$	$v(\phi)$
4 Ts	1	1	1	1	$1+1+1+1=0$
3 Ts	1	1	1	0	$1+0+0+0=1$
2 Ts	1	1	0	0	$0+0+0+0=0$
1 T	1	0	0	0	$0+0+0+0=0$
0 T	0	0	0	0	$0+0+0+0=0$.

Therefore, we do have $v(\phi)=1$ when exactly 3 of $v(p_1)\, v(p_2)\, v(p_3)\, v(p_4)$ are 1.

So we construct ϕ from $(*)$ according to our exam on propositional connectives.

Denote $F = ((p_1\wedge p_2\wedge p_3)\leftrightarrow(p_2\wedge p_3\wedge p_4))$

$G = ((p_3\wedge p_4\wedge p_1)\leftrightarrow(p_4\wedge p_1\wedge p_2))$

Then $\phi = (F\wedge\neg G)\vee(\neg F\wedge G)$

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2. As we are unable to use $((\neg\phi)\to\psi)$ for $(\phi\lor\psi)$, we try to substitute $\neg, \land$ for $\to$.
With easy check of the truth table, we have $\neg\phi\lor\psi$ to substitute $\phi\to\psi$.

We first prove (ii) is a theorem of $\mathcal{L}$.

According to A3 in $\mathcal{L}$, we have $(\neg\phi\to\psi)\to(\neg\psi\to\phi)$

Substitute $\neg\phi\lor\psi$ for $\phi\to\psi$, then we have

$$(\neg\neg\phi\lor\psi)\to(\neg\neg\psi\lor\phi). \qquad (*)$$

Now, we prove that $\phi\leftrightarrow\neg\neg\phi$ in our formal system.

I. $\phi\to\neg\neg\phi$

1. $(\neg\phi\to\neg\phi)\to((\neg\phi\to\phi)\to\neg\neg\phi)$ A3
2. $\neg\phi\to\neg\phi$ Example 1.2.3 in note
3. $(\neg\phi\to\phi)\to\neg\neg\phi$ 1.2.MP
4. $((\neg\phi\to\phi)\to\neg\neg\phi)\to(\phi\to((\neg\phi\to\phi)\to\neg\neg\phi))$ A1
5. $\phi\to((\neg\phi\to\phi)\to\neg\neg\phi)$ 3.4.MP
6. $(\phi\to((\neg\phi\to\phi)\to\neg\neg\phi))\to((\phi\to(\neg\phi\to\phi))\to(\phi\to\neg\neg\phi))$ A2
7. $(\phi\to(\neg\phi\to\phi))\to(\phi\to\neg\neg\phi)$ 5.6.MP
8. $\phi\to(\neg\phi\to\phi)$ A1
9. $\phi\to\neg\neg\phi$ 7.8.MP

II. $\neg\neg\phi\to\phi$

1. $(\neg\phi\to\neg\phi)\to((\neg\phi\to\neg\neg\phi)\to\phi)$ A3
2. $\neg\phi\to\neg\phi$ Example 1.2.3 in note
3. $(\neg\phi\to\neg\neg\phi)\to\phi$ 1.2.MP
4. $((\neg\phi\to\neg\neg\phi)\to\phi)\to\neg\neg\phi\to((\neg\phi\to\neg\neg\phi)\to\phi)$ A1
5. $\neg\neg\phi\to((\neg\phi\to\neg\neg\phi)\to\phi)$ 3.4.MP
6. $(\neg\neg\phi\to((\neg\phi\to\neg\neg\phi)\to\phi))\to((\neg\neg\phi\to(\neg\phi\to\neg\neg\phi))\to(\neg\neg\phi\to\phi))$ A2
7. $(\neg\neg\phi\to(\neg\phi\to\neg\neg\phi))\to(\neg\neg\phi\to\phi)$ 5.6.MP
8. $\neg\neg\phi\to(\neg\phi\to\neg\neg\neg\phi)$ A1
9. $\neg\neg\phi\to\phi$ 7.8.MP.

As ϕ and $\neg\neg\phi$ are logically equivalent, according to Remark 1.1.5-2 in the note,

(*) would be $(\phi\lor\psi)\to(\psi\lor\phi)$ as a theorem in $\mathcal{L}$ for logically equivalent substitute. ∎

Then for (i), as in Example 1.2.3, we have $\phi\to\phi$ as a theorem.
substitute $\to$, we have $\neg\phi\lor\phi$.
As we have proved (ii) $(\phi\lor\psi)\to(\psi\lor\phi)$, we change the order of $\neg\phi$ and ϕ.
so we have $\phi\lor(\neg\phi)$ using Modus Ponens. $\phi\lor(\neg\phi)$ is a theorem in $\mathcal{L}$. ∎

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3. (i) $\Sigma \nvDash_{\mathcal{L}} \psi$ means that there exists valuation v s.t. $v(\Sigma)=T$ while $v(\psi)=F$.

$\Sigma \nvDash_{\mathcal{L}} (\neg\psi)$ means that there exists valuation v' s.t. $v'(\Sigma)=T$ while $v'(\neg\psi)=F$, which means that $v'(\psi)=T$

Therefore, according to the definition of 'ψ is independent',

ψ is independent from Σ iff there exists valuations v, v' st.

$$v(\Sigma)=v'(\Sigma)=T \text{ and } v(\psi)=T.\ v'(\psi)=F.$$

(ii) $(\exists v \text{ s.t. } \forall \psi\in\Delta.\ v(\psi)=T) \wedge (\forall\psi\in\Delta,\ \exists v,v' \text{ s.t. } v(\Delta\setminus\{\psi\})=v'(\Delta\setminus\{\psi\})=T.\ v(\psi)=T.\ v'(\psi)=F\}$.

(iii) Yes. Δ is independent.

As $\Delta = \left\{\begin{array}{l}(p_1\to(\neg p_2\to p_3)), \\ (\neg p_3\to(p_2\to\neg p_1)), \\ (p_3\to(p_1\to\neg p_2))\end{array}\right\}$ $\begin{array}{l}① \\ ② \\ ③\end{array}$, denote 3 formulas in Δ as ① ② ③.

I. consistent

Take $v(p_1)=v(p_2)=v(p_3)=0$. then $v(①)=v(②)=v(③)=T$.

II. $\forall\psi\in\Delta$, ψ is independent from $\Delta\setminus\{\psi\}$.

ψ as ①: take $v(p_1)=1.\ v(p_2)=0.\ v(p_3)=0$.

Then $v(①)=F.\ v(②)=T.\ v(③)=T$. ① is independent from $\Delta\setminus\{\psi\}$

ψ as ②: take $v(p_1)=1.\ v(p_2)=1.\ v(p_3)=0$.

Then $v(①)=T.\ v(②)=F,\ v(③)=T$. ② is independent from $\Delta\setminus\{\psi\}$

ψ as ③: take $v(p_1)=1.\ v(p_2)=0.\ v(p_3)=1$

Then $v(①)=T.\ v(②)=T.\ v(③)=F$. ③ is independent from $\Delta\setminus\{\psi\}$.

Therefore, Δ is an independent set of $\mathcal{L}$-formulas.

(iv) ~~To prove $F_1, F_2, \ldots, F_k$ are linearly independent, then we need to prove that:~~

~~$\forall 1\le i\le k,\ \sum_{j=1}^{k} \varepsilon_j\cdot F_j(\sigma_i)\neq 0$ $\varepsilon_i\neq 0$ in the field $\mathbb{F}_2$.~~

~~Therefore we also have $\forall 1\le i\le k.\ \varepsilon_i=1$~~

~~So we need to prove that~~

~~$\forall 1\le i\le k,\ \sum_{j=1}^{k} F_j(\sigma_i)=1$.~~

~~As $\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ is an independent set of formulas,~~

~~for each $1\le i\le k$, there exists F_i st. $v(\sigma_i)=$~~

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3. (iv) The ~~statement~~ statement in the problem requires us to prove that

if $\{\sigma_1, \sigma_2, \cdots, \sigma_k\}$ is an independent set of formulas, $F_1, F_2 \cdots F_k$ are the truth functions of those,

then ~~[illegible]~~ if $\varepsilon_1 F_1(\sigma_i) + \varepsilon_2 F_2(\sigma_i) + \cdots + \varepsilon_k F_k(\sigma_i) = 0$ holds for every $\sigma_i \in \{\sigma_1, \sigma_2, \cdots, \sigma_k\}$,

we would have $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_k = 0$.

As $\{\sigma_1, \sigma_2, \cdots, \sigma_k\}$ is an independent set and $F_1, F_2 \cdots F_k$ are the truth functions of those,

we have for every σ_i in $\{\sigma_1, \sigma_2, \cdots, \sigma_k\}$, $F_i(\sigma_j) = \begin{cases} 0, & j = i \\ 1, & j \neq i, \quad 1 \le i \le k \end{cases}$

Then we would have k equalities as below for $\varepsilon_1 F_1(\sigma_i) + \varepsilon_2 F_2(\sigma_i) + \cdots + \varepsilon_k F_k(\sigma_i) = 0$ for every $\sigma_i \in \{\sigma_1, \sigma_2, \cdots, \sigma_k\}$

$$0 + \varepsilon_2 + \varepsilon_3 + \cdots + \varepsilon_k = 0 \qquad (1)$$
$$\varepsilon_1 + 0 + \varepsilon_3 + \cdots + \varepsilon_k = 0 \qquad (2)$$
$$\varepsilon_1 + \varepsilon_2 + 0 + \cdots + \varepsilon_k = 0 \qquad (3)$$
$$\vdots \qquad\qquad \vdots$$
$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \cdots + 0 = 0 \qquad (k)$$

Add from (1) to (k), we would have $(k-1)(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_k) = 0$.

As $k \geq 2$, or there would be no sense for the linear independence of $F_1, F_2, \cdots F_k$,

we have $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_k = 0$. $\qquad (\Delta)$

Compare (Δ) with (1), (2), $\cdots$, (k), we could find $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_k = 0$.

Then the statement is proved, that if $\{\sigma_1, \sigma_2, \cdots, \sigma_k\}$ is an independent set of formulas, then $F_1, F_2, \cdots, F_k$ are linearly independent.

$\blacksquare$