[5 marks]

[Total 17 marks + 3 for clarity]

4. We wish to model the behaviour of an internet user browsing the following web-sites Facebook (A), Reddit (B), Instagram (C), Tumblr (D), Twitter (E) and TikTok (F). We seek to model the user browsing behaviour as a time-homogeneous Markov chain on the state space $E = \{A, B, C, D, E, F\}$. From surveys we have the following table of data on user behaviour. The rows correspond to the web-pages users are currently on, the columns correspond to the web-pages users visit next, and the entries indicate the percentage of users who make this transition.

	A	B	C	D	E	F
A	75%	0%	25%	0%	0%	0%
B	0%	0%	0%	100%	0%	0%
C	50%	0%	50%	0%	0%	0%
D	0%	10%	0%	90%	0%	0%
E	0%	0%	0%	20%	0%	80%
F	0%	0%	0%	0%	70%	30%

Let $(X_n)_{n\in\{0,1,2,\dots\}}$ denote a time-homogeneous Markov chain on the state space $E = \{A, B, C, D, E, F\}$, which represents which web-page is visited by the user at time n.

(a) Write down the transition matrix $\mathbf{P}$ for the time-homogeneous Markov chain X_n.

[3 marks]

(b) Determine the communicating classes.

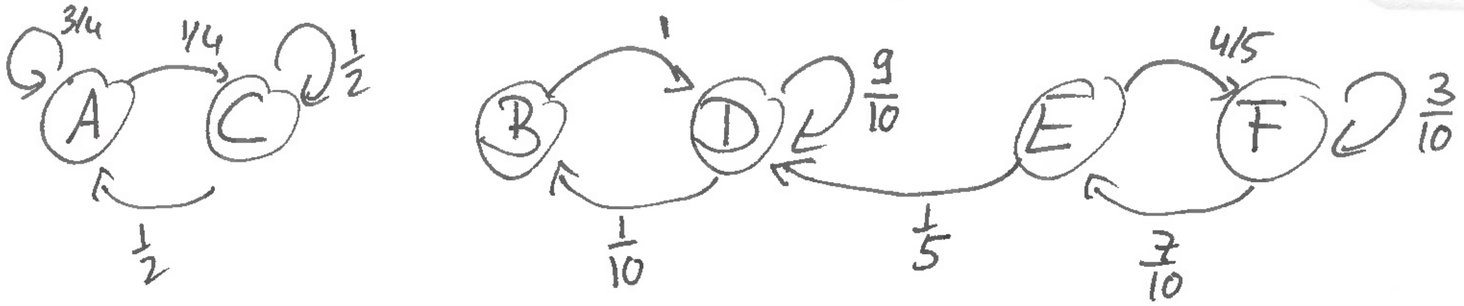
[2 marks]

(c) For each class, specify whether the class is transient, positive recurrent or null recurrent and justify your answer.

[2 marks]

(d) Derive all possible stationary distributions.

[5 marks]



(a) Write down the transition matrix $\mathbf{P}$ for the time-homogeneous Markov chain X_n.

Solution:

$$\mathbf{P} = \begin{pmatrix} \frac{3}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{10} & 0 & \frac{9}{10} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{5} & 0 & \frac{4}{5} \\ 0 & 0 & 0 & 0 & \frac{7}{10} & \frac{3}{10} \end{pmatrix}.$$

[3 marks] [Seen similar]

(b) Determine the communicating classes.

Solution: From the transition matrix we deduce that the Markov chain has three communicating classes $C_1 = \{A, C\}$, $C_2 = \{B, D\}$ and $T = \{E, F\}$.

[2 marks] [Seen similar]

(c) For each class, specify whether the class is transient, positive recurrent or null recurrent and justify your answer.

Solution: C_1 and C_2 are both finite and closed and hence positive recurrent and T is not closed and hence transient.

[2 marks] [Seen similar]

(d) Derive all possible stationary distributions.

Solution: From lectures we know that the elements of the stationary distribution corresponding to the inessential states are equal to 0. Hence the stationary distribution has to be of the form $\pi = (\pi_1, \pi_2, \pi_3, \pi_4, 0, 0)$. We find the remaining elements by solving two systems of equations:

$$(\pi_1, \pi_3) \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (\pi_1, \pi_3),$$

which implies that $\pi_3 = \frac{1}{2}\pi_1$. Similarly

$\pi_1 + \pi_3 = 1 \Leftrightarrow \underbrace{\pi_1 + \frac{1}{2}\pi_1 = \frac{3}{2}\pi_1}_{=1}$

$\Leftrightarrow \pi_1 = \frac{2}{3} \Rightarrow \pi_3 = \frac{1}{3}$

$$(\pi_2, \pi_4) \begin{pmatrix} 0 & 1 \\ \frac{1}{10} & \frac{9}{10} \end{pmatrix} = (\pi_2, \pi_4),$$

$\pi_4 \cdot \frac{1}{10} = \pi_2$

$\pi_2 + \pi_4 = 1 \Leftrightarrow \left(\frac{1}{10} + \frac{10}{10}\right)\pi_4 = 1$

$\Leftrightarrow \pi_4 = \frac{10}{11} \Rightarrow \pi_2 = \frac{1}{11}$

so that [illegible] Hence all stationary distributions are given by

[illegible]

for all $\pi_1, \pi_2 \geq 0$ such that $\frac{3}{2}\pi_1 + \frac{19}{9}\pi_2 = 1$.

[5 marks] [Seen similar]

Hence all stat. distr. are given by

$$w \cdot \left(\frac{2}{3}, 0, \frac{1}{3}, 0, 0, 0\right) + (1-w)\left(0, \frac{1}{11}, 0, \frac{10}{11}, 0, 0\right)$$

for $w \in [0, 1]$.

(e) You wish to pay for an online advertising campaign of your product. What's the least number of web-sites you can advertise on to ensure that a user will eventually encounter the advert, and which websites are they? Justify your answer.

Solution: C_1 and C_2 are only two closed communicating classes. By the definition of closed, once X_n enters C_1 or C_2 there is zero probability of exiting the respective class. Moreover, the states within C_1 and C_2 are irreducible and recurrent, so that the probability of reaching any state within a given class is 1. To ensure the eventual visibility of the advert, we must therefore put one ad in class C_1 and one ad in class C_2. For example, advertising on Instagram and Reddit would ensure this.