Mathematical Logic (MATH6/70132; P65), Problem Sheet 1

[1] Let p denote 'I will pass this course,' q denote 'I do my homework regularly' and r denote 'I am lucky.' Express in symbolic form each of the following propositions:
(a) I will pass this course only if I do my homework regularly.
(b) Doing homework regularly is a necessary condition for me to pass this course.
(c) If I do my homework regularly and I do not pass this course then I am unlucky.
(d) If I do not do my homework regularly and I pass this course then I am lucky.

[2] Show that ϕ and ψ are logically equivalent if and only if $(\phi \leftrightarrow \psi)$ is a tautology.

[3] Suppose ϕ and ψ are logically equivalent propositional formulas whose propositional variables are amongst $p_1, \ldots, p_n$. Let $\eta_1, \ldots, \eta_n$ be formulas and let θ and χ be the formulas obtained by replacing each occurrence of p_i by η_i in ϕ and ψ respectively. Show that θ and χ are logically equivalent.

[4] Suppose p, q, r are propositional variables. For each of the following formulas find a formula in disjunctive normal form (- see 1.1.9 in notes) which is logically equivalent to it.

(a) $((p \to q) \to ((\neg p) \wedge q))$.

(b) $(\neg((p \to q) \to r))$.

[5] The connective $\mid$ has truth table:

ϕ	ψ	$(\phi \mid \psi)$
T	T	F
T	F	T
F	T	T
F	F	T

(a) Show that this connective is adequate.

(b) Show that $\mid$ and the NOR connective $\downarrow$ are the only binary connectives which are adequate.

[6] How many truth functions f of n propositional variables are there with the property that $f(T, T, \ldots, T) = T$? Can all of these be expressed as the truth function of a formula of n variables using only the connectives $\wedge, \vee$? Explain your answer.

[7] (i) For which values of the propositional variables p_1, p_2, p_3 does the following propositional formula θ have truth value F:

$$((p_1 \to ((\neg p_2) \to p_3)) \to ((p_3 \to p_2) \to p_1))?$$

Find a formula in disjunctive normal form which is logically equivalent to $(\neg \theta)$.

(ii) Find a formula χ with three propositional variables p_1, p_2, p_3 whose truth value is T if and only if p_1, p_2, p_3 have truth values T, F, T (respectively). Justify your answer.

[8] Below is the start of a proof that if ϕ, ψ are propositional formulas, then $((\neg \psi) \to (\psi \to \phi))$ is a theorem (of L). Write this out again, but at each stage give the reasoning, and then complete the proof:

1. $((\neg \psi) \to ((\neg \phi) \to (\neg \psi)))$
2. $(((\neg \phi) \to (\neg \psi)) \to (\psi \to \phi))$

Denote this formula by χ

3. $(\chi \to ((\neg \psi) \to \chi))$

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