Mathematical Logic (M345P65)
Problem Class, week 5

[1] Suppose $\mathcal{L}$ is a first-order language with relation, function and constant symbols:

$$(R_i : i \in I);\ (f_j : j \in J);\ (c_k : k \in K),$$

where R_i is of arity m_i and f_j is of arity n_j.

(a) Based on what you know to be the definition of 'isomorphism' in the case of groups, or rings (or graphs or orderings), write down a definition of what it should mean for two $\mathcal{L}$-structure $\mathcal{A}$ and $\mathcal{B}$ to be isomorphic.

(b) Suppose α is an isomorphism from the $\mathcal{L}$-structure $\mathcal{A}$ to the $\mathcal{L}$-structure $\mathcal{B}$. Suppose v is a valuation in $\mathcal{A}$. Let w be the vaulation in $\mathcal{B}$ with $w(x_i) = \alpha(v(x_i))$ for all variables x_i. Prove that for every $\mathcal{L}$-formula ϕ,

$$v \text{ satisfies } \phi \text{ in } \mathcal{A} \Leftrightarrow w \text{ satisfies } \phi \text{ in } \mathcal{B}.$$

(Use induction on the length of ϕ...)

(c) With $\mathcal{A}, \mathcal{B}$ as in (b), show that for every $\mathcal{L}$-formula ϕ:

$$\mathcal{A} \models \phi \Leftrightarrow \mathcal{B} \models \phi.$$

[2] It's difficult to do any reasoning in mathematics without using the equality symbol. In 1st-order logic, we use the following terminology to handle equality. We will say more about this in week 6 or 7.

A first-order *language with equality* $\mathcal{L}^=$ is a 1st-order language with a distinguished 2-ary relation symbol $=$. An $\mathcal{L}^=$-structure $\mathcal{A}$ is *normal* if the symbol $=$ is interpreted as equality in $\mathcal{A}$.

We write the more usual '$x_1 = x_2$' instead of '$=(x_1, x_2)$' in $\mathcal{L}^=$-formulas.

Suppose $\mathcal{L}^=$ is a language with equality which also has a 2-ary relation symbol R.

(a) Suppose $n \in \mathbb{N}$. Write down a closed $\mathcal{L}^=$-formula σ_n with the property that, for every normal $\mathcal{L}^=$-structure $\mathcal{A}$ we have $\mathcal{A} \models \sigma_n$ iff the domain of $\mathcal{A}$ has at least n elements.

(b) Suppose $n \in \mathbb{N}$. Write down a closed $\mathcal{L}^=$-formula τ_n with the property that, for every normal $\mathcal{L}^=$-structure $\mathcal{A}$ we have $\mathcal{A} \models \tau_n$ iff the domain of $\mathcal{A}$ has exactly n elements.

(c) Find a closed $\mathcal{L}^=$ formula θ with the property that for every positive $n \in \mathbb{N}$:

there is a normal $\mathcal{L}^=$ structure $\mathcal{A}$ with n elements in its domain and $\mathcal{A} \models \theta$
if and only if n is even.

(d) Suppose $=$ and R are the only relation symbols in the language and $J = K = \emptyset$. Can you construct your formula θ in (c) so that any two countably infinite normal models of θ are isomorphic? Can you find two non-isomorphic, infinite normal models of such a θ?

(1)

1(a) $\mathcal{L}$-structures $\mathcal{A} = \left\langle A; (\bar{R}_i^A : i \in I), (\bar{f}_j^A : j \in J), (\bar{c}_k^A : k \in K) \right\rangle$

and $\mathcal{B} = \left\langle B; (\bar{R}_i^B : i \in I), (\bar{f}_j^B : j \in J), (\bar{c}_k^B : k \in K) \right\rangle$

are <u>isomorphic</u> if there is a bijection

$\alpha : A \to B$ with:

- For each $i \in I$ and $a_1, \ldots, a_{m_i} \in A$
$$\bar{R}_i^A(a_1, \ldots, a_{m_i}) \iff \bar{R}_i^B(\alpha(a_1), \ldots, \alpha(a_{m_i}))$$

- For each $j \in J$ and $a_1, \ldots, a_{n_j} \in A$, $c \in A$
$$\bar{f}_j^A(a_1, \ldots, a_{n_j}) = c \iff \bar{f}_j^B(\alpha(a_1), \ldots, \alpha(a_{n_j})) = \alpha(c).$$

- For each $k \in K$
$$\alpha(\bar{c}_k^A) = \bar{c}_k^B.$$

<u>Note</u>: (1) α is called an isomorphism from $\mathcal{A}$ to $\mathcal{B}$.

(2) α^{-1} is an isomorphism from $\mathcal{B}$ to $\mathcal{A}$

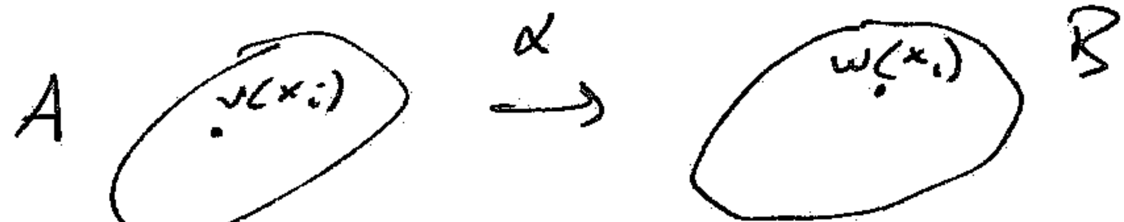
(2)

1(b) v a valuation in $\mathcal{A}$

(Sketch) Suppose α is an iso. $\mathcal{A} \to \mathcal{B}$

Let w be the val. in $\mathcal{B}$

with $w(x_i) = \alpha(v(x_i))$



By conditions on fns. + constants

$$w(t) = \alpha(v(t)) \quad \text{for each term } t.$$

Eg. $t = f(x_1, x_2)$

$$\begin{aligned} w(t) &= f^{\mathcal{B}}(w(x_1), w(x_2)) \\ &= f^{\mathcal{B}}(\alpha(v(x_1)), \alpha(v(x_2))) \\ &= f^{A}(v(x_1), v(x_2)). \end{aligned}$$

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By conditions on relations, for ϕ an atomic fmla $R(t_1, \dots, t_m)$:

$\bar{R}^{A}(v(t_1), \dots, v(t_m))$ holds in $\mathcal{A}$

iff $\bar{R}^{\mathcal{B}}(\alpha(v(t_1)), \dots, \alpha(v(t_m)))$ in $\mathcal{B}$

iff $\bar{R}^{B}(w(t_1), \dots, w(t_m))$ in $\mathcal{B}$.

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Induction ....

(3) [2] $\mathcal{L}^=$

(a) σ_n

$$(\exists x_1)\dots(\exists x_n)\bigwedge_{1\le i<j\le n}(\neg(x_i = x_j)))$$

denotes the conjunction of all these formulas.

(b) τ_n $\qquad \sigma_n \wedge (\neg \sigma_{n+1})$

(c) "R is an eq. rel. with classes of size 2."

Or "R gives a graph

where every vertex is in exactly one edge." For the first of these:

$$\big((\forall x_1)(\forall x_2)(\forall x_3)\; R(x_1,x_1) \wedge (R(x_1,x_2) \to R(x_2,x_1))$$
$$\wedge (R(x_1,x_2) \wedge R(x_2,x_3) \to R(x_1,x_3))\big)$$
$$\wedge\ (\forall x_1)(\exists x_2)\; R(x_1,x_2) \wedge (\neg(x_1 = x_2)))$$
$$\wedge\ \neg(\exists x_1)(\exists x_2)(\exists x_3)\ (\dots\text{different}\dots$$
$$\text{and} \quad R(x_1,x_2) \wedge R(x_2,x_3)))$$