Imperial College London

M3F22

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2018

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Mathematical Finance: Introduction to Option Pricing

Date: Thursday, 24 May 2018

Time: 2:00 PM - 4:00 PM

Time Allowed: 2 hours

This paper has 4 questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Each question carries equal weight.
- Calculators may not be used.

Imperial College
London

M4/5F22

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Mathematical Finance: Introduction to Option Pricing

Date: Thursday, 24 May 2018

Time: 2:00 PM - 4:30 PM

Time Allowed: 2.5 hours

This paper has 5 questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

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1. (a) For a probability distribution F on $(0, \infty)$, define the *lack of memory property*. Find the functional equation satisfied by the tail $\overline{F}(x)$ of $F(x)$ with this property.

   (b) Solve this functional equation, quoting any results you need.

   (c) Explain the relevance of $E(\lambda)$ (the exponential distribution with parameter $\lambda > 0$) to the Poisson process with parameter (or rate) $\lambda > 0$, and the modelling of insurance claims.

   (d) Comment on the limitations of this model.

2. (a) Prove the converse part of the No-Arbitrage Theorem: that if an equivalent martingale measure P^* exists, then there is no arbitrage.

   (b) Why is the direct part more difficult?

   (c) Describe the use of the No-Arbitrage Theorem in pricing assets such as options.

   (d) To what extent do arbitrage opportunities exist in real markets?

   (e) How would market be affected by arbitrage opportunities on any sizeable scale?

3. The *Theta*, Θ, of an option is defined as the time-derivative of its value.

(a) Given the Black-Scholes formula for the price c_t of European calls,

$$c_t = S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2),$$

with S_t the stock price at time $t \in [0, T]$, K the strike price, r the riskless interest rate, σ the volatility and

$$d_1 := [\log(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)]/\sigma\sqrt{T-t}, \qquad d_2 := d_1 - \sigma\sqrt{T-t}:$$

(i) find Θ and show that $\Theta < 0$;

(ii) interpret this.

(b) Given the corresponding Black-Scholes formula for the price p_t of European puts,

$$p_t = Ke^{-r(T-t)}\Phi(-d_2) - S_t\Phi(-d_1),$$

(i) find Θ, and show that this time Θ can change sign.

(ii) Describe the conditions under which Θ will be positive, and interpret this.
You may quote that $Ke^{-r(T-t)}\phi(d_2) = S\phi(d_1)$.

4. (a) In the Cramér-Lundberg risk model, define the *safety loading* $\rho > 0$ in terms of the premium rate $c > 0$, the claim rate $\lambda > 0$ and the mean claim size $\mu \in (0, \infty)$.

(b) State without proof the *key renewal theorem*.

(c) Define the *Lundberg coefficient* $r > 0$ (assumed to exist).

(d) Define the *Esscher transform* $F \mapsto G$ of the claim-size distribution F.

(e) Given the integral equation for the ruin probability $\psi(u)$ with initial capital u:

$$\psi(u) = \frac{1}{(1+\rho)}\int_u^\infty \frac{(1-F(x))}{\mu}dx + \frac{1}{(1+\rho)} \cdot \int_0^u \psi(u-x)\frac{(1-F(x))}{\mu}dx, \qquad (*)$$

obtain the *Cramér estimate of ruin* $\psi(u) \sim Ce^{-ru}$ as $u \to \infty$, with C a constant.

(f) Comment briefly on where the assumptions here may prove to be unrealistic.

5. (a) Define *geometric Brownian motion* (GBM) with parameters μ, σ, and find the stochastic differential equation it satisfies.

   (b) Give the financial interpretation of this.

   (c) Find the quadratic variation of GBM.

   (d) Find the expected quadratic variation of GBM.

M3F22/M4F22/M5F22 EXAMINATION SOLUTIONS 2017-18

Q1 (*Lack of memory and the exponential laws*).
(i) Consider a probability distribution (law) F on $(0,\infty)$, interpreted as the lifetime law of components, say. Then F has the *lack-of-memory property* iff the components show no aging – that is, if a component still in use behaves as if new. The condition for this is

$$P(X > s+t|X > s) = P(X > t) \qquad (s,t > 0):$$

$$P(X > s+t) = P(X > s)P(X > t).$$

Writing $\overline{F}(x) := 1 - F(x)$ $(x \geq 0)$ for the *tail* of F, this says that

$$\overline{F}(s+t) = \overline{F}(s)\overline{F}(t) \qquad (s,t \geq 0). \qquad [5]$$

(ii) Obvious solutions are

$$\overline{F}(t) = e^{-\lambda t}, \qquad F(t) = 1 - e^{-\lambda t}$$

for some $\lambda > 0$ – the exponential law $E(\lambda)$. Now

$$f(s+t) = f(s)f(t) \qquad (s,t \geq 0) \qquad (CFE)$$

is a 'functional equation' – the *Cauchy functional equation* (CFE) – and (we quote) these are the *only* bounded solutions, (indeed, the only ones subject to any – minimal – regularity condition).

So the exponential laws $E(\lambda)$ are *characterized* by the lack-of-memory property. [5]
(iii) The Poisson point process $Ppp(\lambda)$ with rate $\lambda > 0$ is defined to have the inter-arrival times independent $E(\lambda)$. It is the lack-of-memory property of the $E(\lambda)$ that makes the Poisson process the basic model for events occurring 'out of the blue'. Typical examples are accidents, insurance claims, hospital admissions, earthquakes, volcanic eruptions etc. [5]
(iv) *Limitations.* The weakness in this model for insurance claims is that a major catastrophe produces a *cluster* of claims. The independence assumption will fail badly *within* clusters, though it may still work well *between* clusters. [5]
[(i)-(iii): seen – lectures; (iv): mainly unseen]

Q2 (*No-Arbitrage Theorem (NA Theorem)*).
(i) *Proof.* $\Leftarrow$. In discrete time: we take the state space Ω to be discrete also; we can then retain only sample points ω with positive probability, $P(\omega) > 0$.

Assume such an equivalent martingale measure (EMM) P^* exists. For any self-financing strategy H, we have

$$\tilde{V}_n(H) = V_0(H) + \Sigma_1^n H_j.\Delta\tilde{S}_j$$

(at the jth trade, the gain in value $\Delta V_j(H)$ is the amount H_j of the jth asset that we buy, times the gain ΔS_j in its price; similarly for $\tilde{V}_j$, $\tilde{S}_j$ with discounting). This gives $\tilde{V}_n(H)$ as the *martingale transform* of the P^*-martingale $\tilde{S}_j$ by $H = (H_n)$, so $\tilde{V}_n(H)$ is a P^*-martingale. So the initial and final P^*-expectations are the same: using E^* for P^*-expectation,

$$E^*[\tilde{V}_N(H)] = E^*[\tilde{V}_0(H)].$$

If the strategy is admissible and its initial value – the RHS above – is zero, the LHS $E^*[\tilde{V}_N(H)]$ is zero, but $\tilde{V}_N(H) \geq 0$ (by admissibility). Since each $P(\{\omega\}) > 0$ (by assumption), each $P^*(\{\omega\}) > 0$ (by equivalence). This and $\tilde{V}_N(H) \geq 0$ force $\tilde{V}_N(H) = 0$ (sum of non-negatives can only be 0 if each term is 0). So no arbitrage is possible. // [6]

(ii) The direct half (no arbitrage implies existence of an EMM) needs the Separating Hyperplane Theorem. The general form of this is related to the Hahn-Banach Theorem of Functional Analysis, which needs the Axiom of Choice (AC). In a finite-dimensional setting (as in (i)), one can use Euclidean geometry – much simpler. [3]

(iii) The NA Theorem (NA iff EMMs *exist*) shows that the assumption of NA is needed to be able to *price* assets, including *options*. (Completeness is needed to make EMMs, and so prices, *unique*; real markets are incomplete; real prices are non-unique; "You'd better shop around".) In particular, one can price options without needing to know the market participant's *utility* function – i.e., his attitude to risk. This is the *Arbitrage Pricing Technique (APT)*, due to the late Steve (S. A.) Ross (1976/78): it takes the qualitative insight of the NA Theorem above, and uses it systematically to produce *quantitative* results – asset pricing, etc. (EMMs correspond to *pricing kernels*). [3]

(iv) Arbitrage opportunities do exist in reality – and professional arbitrageurs hunt for them. They are a 'second-order effect': anyone opening himself to arbitrage is in effect offering the market free money; the market will take the free money without limit until he withdraws from the market (bankrupt or otherwise), or at least withdraws the arbitrage opportunity – which is thus 'arbitraged away'. [4]

(v) With EMMs, we can price assets (albeit non-uniquely without completeness – to within an interval, the 'bid-ask spread'). But without NA and EMMs, pricing cannot be done systematically at all. If assets cannot be *priced* reliably, they will not be *traded*, in any significant quantity. So option exchanges (such as CBOE), where options can be traded in quantity and so as liquid assets, could not have been developed. So the existence of a mass market in options and other assets (an essential aspect of the City of London and other global financial centres) depends on the no-arbitrage assumption. [4]

[(i)-(iv): seen; (v): unseen]

Q3 (*Theta*). Given
$$Ke^{-r(T-t)}\phi(d_2) = S\phi(d_1): \qquad (*)$$
(i) *Calls.* Given the Black-Scholes formula for the price c_t of European calls,
$$c_t = S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2),$$
$$d_{1,2} := [\log(S/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)]/\sigma\sqrt{T-t}: \qquad d_2 = d_1 - \sigma\sqrt{T-t}:$$
(a) Differentiating and using $(*)$: as
$$\partial(d_1 - d_2)/\partial t = \partial(\sigma\sqrt{T-t})/\partial t = -\frac{1}{2}\sigma/\sqrt{T-t}:$$
$$\Theta = \partial c_t/\partial t = S\phi(d_1)\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)}\Phi(d_2) - Ke^{-r(T-t)}\phi(d_2)\frac{\partial d_2}{\partial t}:$$
$$\Theta = Ke^{-r(T-t)}[\phi(d_2)\frac{\partial(d_1 - d_2)}{\partial t} - r\Phi(d_2)]:$$
$$\Theta = -Ke^{-r(T-t)}[\phi(d_2)\cdot\frac{\frac{1}{2}\sigma}{\sqrt{T-t}} + r\Phi(d_2)] < 0. \qquad [6]$$
(b) Interpretation: an option is (partly) an insurance against future uncertainty. As time passes, there is less future (till expiry) to protect against, so such protection becomes less valuable. [4]
(ii) *Puts.* Given the corresponding BS formula for European puts,
$$p_t = Ke^{-r(T-t)}\Phi(-d_2) - S_t\Phi(-d_1),$$
(a) As above, as $\phi(-x) = \phi(x)$,
$$\Theta = \partial p_t/\partial t = rKe^{-r(T-t)}\Phi(-d_2) + Ke^{-r(T-t)}\phi(d_2)\frac{\partial(-d_2)}{\partial t} - S\phi(d_1)\frac{\partial(-d_1)}{\partial t}:$$
$$\Theta = Ke^{-r(T-t)}[r\Phi(-d_2)+\phi(d_2)\frac{\partial(d_1-d_2)}{\partial t}] = Ke^{-r(T-t)}[r\Phi(-d_2)-\phi(d_2)\cdot\frac{\frac{1}{2}\sigma}{\sqrt{T-t}}].$$
This can change sign! [6]
(b) The situation with puts is different, because of the different role of the strike K (fixed, while S varies). But for large enough K (when a put option – the right to *sell* at price K – will be deeply in the money), the option stands to make a large profit – so the nearer this is to being realised, the better. [4]
[(ia), (iia): similar to seen; (ib), (iib): unseen]

Q4 (*Renewal theory and ruin theory*).

(i) *Safety loading.* With $c > 0$ the premium rate at which cash comes in, $\lambda > 0$ the rate at which claims occur, $\mu \in (0, \infty)$ the mean claim size, cash goes out at rate $\lambda\mu$, so one needs ('more in than out') $c > \lambda\mu$. The *safety loading* $\rho > 0$ is defined by

$$\frac{c}{\lambda\mu} = 1 + \rho. \qquad (SL) \quad [3]$$

(ii) *Key renewal theorem.* The *renewal equation* for F and z (both known) is the integral equation

$$Z(t) = z(t) + \int_0^t Z(t-u)dF(u) \quad (t \geq 0): \quad Z = z + F * Z. \qquad (RE)$$

Here F (the lifetime distribution) and z are given, and (RE) is to be solved for Z. Then for $U := \sum_0^\infty F^{*n}$ the renewal function of F:

Theorem (Key Renewal Theorem; W. L. Smith). If z in (RE) is directly Riemann integrable, then with U the renewal function of F,

$$\lim_{t\to\infty} Z(t) = \lim_{t\to\infty}(U * z)(t) = \frac{1}{\mu}\int_0^\infty z(x)dx. \qquad [3]$$

(iii) *The Lundberg (or adjustment) coefficient*, r. This is the point $r > 0$ (assumed to exist – a strengthening of the Small Claims Condition; it is then unique) such that the MGF of $Z = Z_1$ satisfies, writing M for M_{X_1} for short,

$$M_{Z_1}(r) := E[\exp\{r(X_1 - cW_1)\}] = M(r).\frac{\lambda}{\lambda + cr} = 1: \quad M(r) = 1 + \frac{cr}{\lambda}.$$

As s increases, $M(s) \uparrow \infty$. So (from the graph of M): the bigger r is, the bigger the strip of holomorphy of the claim-size MGF, the smaller the claim-size tails, so the smaller the chance of a damaging big claim: the bigger r is, the better. [4]

(iv) *The Esscher transform.* By above,

$$M(r) := \int_0^\infty e^{rx}dF(x) = -\int_0^\infty e^{rx}d(1-F)(x) = 1 + \frac{cr}{\lambda}.$$

Integrating by parts, the integrated term is 1, giving

$$\int_0^\infty (1 - F(x))e^{rx}dx = \frac{c}{\lambda} = (1+\rho)\mu,$$

by (SL). So

$$\frac{\lambda}{c}(1-F(x))e^{rx} = \frac{1}{(1+\rho)\mu}(1-F(x))e^{rx}$$

is a probability density on $(0,\infty)$ – of G, say. Then $F \mapsto G$ is called the *Esscher transform.* **[3]**

(v) *The Cramér estimate of ruin.* Given the integral equation for the ruin probability $\psi(u)$ $\cancel{= 1 - \phi(u)}$:

$$\psi(u) = \frac{1}{(1+\rho)}\int_u^\infty \frac{(1-F(x))}{\mu}dx + \frac{1}{(1+\rho)}\cdot\int_0^u \psi(u-x)\frac{(1-F(x))}{\mu}dx \quad (*)$$

(as $(1-F(x))/\mu$ is a probability density, so integrates to 1). This is of renewal-equation type, *except* that, as $(1-F(x))/\mu$ is a probability *density,* the factor $1/(1+\rho) < 1$ turns it into a *sub-probability* (or *defective*) density.

Theorem (Cramér's estimate of ruin, 1930).

For the Cramér-Lundberg model, with Lundberg coefficient $r > 0$ and $\psi(u)$ the probability of ruin with initial capital u,

$$e^{ru}\psi(u) \to C: \qquad \psi(u) \sim Ce^{-ru} \qquad (u \to \infty),$$

with C an (identifiable) constant.

Proof. Multiply $(*)$ by e^{ru}, and regard it as an integral equation in $\psi(u)e^{ru}$:

$$[\psi(u)e^{ru}] = e^{ru}\int_u^\infty \frac{(1-F(x))}{(1+\rho)\mu}dx + \int_0^u [\psi(u-x)e^{r(u-x)}]\frac{e^{rx}(1-F(x))}{(1+\rho)\mu}dx.$$

This is now an integral equation of renewal type (RE). So by the Key Renewal Theorem, its solution $\psi(u)e^{ru}$ has a limit, C say, as $u \to \infty$ (C can be read off from the Key Renewal Theorem). // **[4]**

(vi) The most unrealistic assumption here is that different claims are independent. A natural disaster will produce a cluster of claims, heavily dependent. This can be handled by treating the clusters as 'points' in a Poisson process. **[3]**

[(i) - (v): Seen – lectures; (vi): unseen]

Q5 (Mastery question: *Geometric Brownian motion and its quadratic variation*).
(i) Consider the process

$$X_t = f(t, B_t) := x_0. \exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t\} : \quad \log X_t = const + (\mu - \frac{1}{2}\sigma^2)t + \sigma B_t, \qquad (*)$$

with $B = (B_t)$ Brownian motion (BM). Here, since

$$f(t, x) = x_0. \exp\{(\mu - \frac{1}{2}s^2)t + \sigma x\},$$

$$f_1 = (\mu - \frac{1}{2}\sigma^2)f, \qquad f_2 = \sigma f, \qquad f_{22} = \sigma^2 f.$$

By Itô's Lemma,

$$dX_t = f_1 dx + f_2 dB_t + \frac{1}{2} f_{22} (dB_t)^2 :$$

$$dX_t = df = [(\mu - \frac{1}{2}\sigma^2)f + \frac{1}{2}\sigma^2 f]dt + \sigma f dB_t :$$
$$dX_t = \mu f dt + \sigma f dB_t = \mu X_t dt + \sigma X_t dB_t :$$

X satisfies the SDE

$$dX_t = X_t(\mu dt + \sigma dB_t) : \qquad dX_t/X_t = \mu dt + \sigma dB_t, \qquad (GBM)$$

geometric Brownian motion (GBM). It is used to model (stock) *price processes* in the Black-Scholes model – where, by $(*)$, *log-prices* $\log X_t$ *are normally distributed*, so *prices* are *log-normally distributed.* [8]
(ii) *Interpretation.* The μdt term on the RHS corresponds to a *riskless* asset with *return* rate μ. The σdB_t term corresponds to a *risky* asset with *volatility* σ; the Brownian motion (B_t) models the uncertainty driving the economic/financial environment; the volatility σ represents how sensitive this particular stock is to this. [3]
(iii) *Quadratic variation.* Recall $(dB_t)^2 = dt$ (Itô: differential form of Lévy's theorem on quadratic variation of BM). So

$$(dX_t)^2 = X_t^2(\mu^2(dt)^2 + 2\mu\sigma dt dB_t + \sigma^2(dB_t)^2) : \qquad (dX_t)^2 = \sigma^2 X_t^2 dt,$$

as above. So, as with BM, GBM has quadratic variation (QV)

$$\sigma^2 \int_0^t X_s^2 ds. \qquad [3]$$

(iv) *Expected QV.* By Fubini's theorem, it has expected QV

$$E[\sigma^2 \int_0^t X_s^2 ds] = \sigma^2 \int_0^t E[X_s^2] ds,$$

By (*), as $B_t \sim \sqrt{t} Z$ with $Z \sim N(0,1)$ with MGF $\exp\{\frac{1}{2}t^2\}$,

$X_t^2 = x_0^2 \cdot \exp\{(2\mu - \sigma^2)t\} \cdot \exp\{2\sigma B_t\} \sim x_0^2 \cdot \exp\{(2\mu - \sigma^2)t\} \cdot \exp\{2\sigma\sqrt{t} Z\}.$

By the normal MGF, the last term has expectation $\exp\{\frac{1}{2}(2\sigma\sqrt{t})^2\} = \exp\{2\sigma^2 t\}$. Combining,

$$E[X_t^2] = x_0^2 \cdot \exp\{(2\mu - \sigma^2)t\} \cdot \exp\{2\sigma^2 t\} = x_0^2 \cdot \exp\{(2\mu + \sigma^2)t\}.$$

So the expected QV of GBM is

$$x_0^2 \sigma^2 \frac{\exp\{(2\mu + \sigma^2)t\} - 1}{2\mu + \sigma^2}. \qquad [6]$$

[(i), (ii): seen, lectures; (iii), (iv): unseen] N. H. Bingham