Examples III for Time Series

"Stationary" is meant to mean second order stationary unless explicitly stated otherwise.

1. Determine whether the following process is stationary, giving your reasons.

$$X_t + \frac{1}{12}X_{t-1} = \frac{1}{24}X_{t-2} + \epsilon_t.$$

2. Define a real-valued deterministic sequence $\{y_t\}$ by

$$y_t = \begin{cases} +1, & \text{if } t = 0,\ -1,\ -2,\ \ldots, \\ -1, & \text{if } t = 1,\ 2,\ 3,\ \ldots \end{cases}$$

   Now define a stochastic process by $X_t = y_t I$, where I is a random variable taking on the values $+1$ and -1 with probability $1/2$ each.

   Find the mean, variance and autocovariance of $\{X_t\}$ and determine, with justification, whether this process is stationary.

3. (a) Consider the following MA(2) process

$$X_t = \epsilon_t - \frac{9}{4}\epsilon_{t-1} + \frac{1}{2}\epsilon_{t-2}.$$

      i. What condition must hold on the roots of the characteristic polynomial of an MA(q) process in order that the process is invertible?

      ii. Is this MA(2) process invertible?

   (b) Consider the MA(1) process defined by

$$X_t = \epsilon_t - \theta\epsilon_{t-1}.$$

      i. Show that $\{X_t\}$ can be written in terms of previous values of the process as

$$X_t = \epsilon_t - \sum_{j=1}^{p} \theta^j X_{t-j} - \theta^{p+1}\epsilon_{t-p-1}$$

      for any positive integer p.

ii. With respect to the formula in (b)(i), what condition on θ must hold in order that X_t can be expressed as an infinite-order autoregressive process? Is this consistent with 5(a)(i)?

4. (a) Let $\{X_t\}$ be a stationary process with acvs $\{s_\tau\}$ and spectral density function $S(f)$. Show $S(f) \leq S(0)$ for all $f \in [-1/2, 1/2]$ if $s_\tau \geq 0$ for all $\tau \in \mathbb{Z}$.

   (b) Let $\{X_t\}$ be the AR(1) process

$$X_t = \phi X_{t-1} + \epsilon_t$$

   where $0 < \phi < 1$ and $\{\epsilon_t\}$ is zero mean white noise process with variance σ_ϵ^2, and let $S(f)$ be its spectral density function. Show

$$\max_{f \in [-1/2,1/2]} S(f) = \frac{\sigma_\epsilon^2}{(1-\phi)^2}.$$

5. A (very) simple signal + noise model for an astronomical times series $\{X_t\}$ is

$$X_t = \epsilon_t + \sum_{k=1}^{K} A_k \cos(2\pi f_k t + C_k)$$

where $f_1, ..., f_K$ are fixed frequencies in $[0, 1/2)$. Here, $A_1, A_2, ..., A_K$ are i.i.d random variables with mean zero and variance σ_A^2, $C_1, C_2, ..., C_K$ and i.i.d. random variables uniformly distributed on $[0, 2\pi)$ and independent of A_i for all $i = 1, ..., K$. Furthermore, $\{\epsilon_t\}$ is a white noise process with mean zero and variance σ_ϵ^2 which is independent of $A_1, A_2, ..., A_K, C_1, C_2, ..., C_K$.

   (a) Show

$$S^{(I)}(f) = \sigma_\epsilon^2(f + 1/2) + \frac{\sigma_A^2}{4} \sum_{k=1}^{K} (\mathbb{1}_{[-f_k,1/2]}(f) + \mathbb{1}_{[f_k,1/2]}(f)) \qquad -1/2 \leq f \leq 1/2.$$

   is the integrated spectrum for $\{X_t\}$, where

$$\mathbb{1}_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{otherwise.} \end{cases}$$

   (b) Sketch $S^{(I)}(f)$ for $K = 2$ with $f_1 = 1/4$, $f_2 = 1/3$ and $\sigma_A^2 = \sigma_\epsilon^2 = 1$.

6. (a) A complex-valued time series Z_t is given by $Z_t = C\mathrm{e}^{\mathrm{i}(2\pi f_0 t + \theta)}$, where f_0 and C are finite real-valued constants and θ is uniformly distributed over $[-\pi, \pi]$.

   Determine, with justification, whether this process is stationary.

   [*The autocovariance for a complex-valued time series is given by* $\mathrm{cov}\{Z_t, Z_{t+\tau}\} = E\{Z_t^* Z_{t+\tau}\} - E\{Z_t^*\}E\{Z_{t+\tau}\}$, *where* * *denotes complex conjugate.*]

(b) Let $\{X_t\}$ be a real-valued zero mean stationary process with autocovariance sequence $\{s_{X,\tau}\}$ and spectral density function $S_X(f)$.

i. Define the complex-valued process $\{Z_t\}$ by

$$Z_t = X_t \mathrm{e}^{-\mathrm{i}2\pi f_0 t},$$

where f_0 is a fixed frequency such that $0 < f_0 \leq 1/2$. Show that $\{Z_t\}$ has spectral density function given by $S_Z(f) = S_X(f_0 + f)$.

ii. Now define $\{Z_t\}$ as

$$Z_t = X_t + \mathrm{i}X_{t+k},$$

for some integer k. Find the autocovariance sequence $\{s_{Z,\tau}\}$ and hence show that

$$S_Z(f) = 2[1 - \sin(2\pi f k)]S_X(f).$$