ExamModuleCode	Question Number	Comments for Students
M45P65	1	Some people spent too long on part (a)(ii) and in (a)(iii) it was not necessary to write out the full truth tables. Several people used valuations in b(ii) and this was not allowed - it's essentially using the (proof of the) completeness theorem, of which this result is an ingredient.
M45P65	2	Parts (a)(I,ii) were done well, but many answers to (iii) lacked clarity. (b)(iii) was hard and not many people solved this. You cannot say that one group is cyclic and the other is not in a first-order way.
M45P65	3	Many people did part (a) and (b)(i) (though some missed out some of the conditions on R in \eta_n, including that it was an equivalence relation). (b)(ii) is hard, but a few people solved it.
M45P65	4	In (a), the 4th set caused some puzzlement.In (b)(i), many people checked that A_1 \times A_2 had a least element, rather than checking that every non-empty subset of it has a least element. Most people who attempted part (b)(iii) picked up some easy marks, but missed the harder parts (eg \beta is similar to \omega\times\beta).
M45P65	5	It was good to see that most people had made some attempt at the mastery material and were able to answer some parts of this. Part (c)(ii) caused the most problems, with very few attempts.

Imperial College
London

M3P65

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2019

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Mathematical Logic

Date: Wednesday 08 May 2019
Time: 14.00 - 16.00
Time Allowed: 2 Hours
This paper has 4 Questions.
Candidates should use ONE main answer book.
Supplementary books may only be used after the relevant main book(s) are full.
All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Calculators may not be used.

Imperial College London

M4/5P65

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2019

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Mathematical Logic

Date: Wednesday 08 May 2019
Time: 14.00 - 16.30
Time Allowed: 2 Hours 30 Minutes
This paper has 5 Questions.
Candidates should use ONE main answer book.
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WORK IN ZFC THROUGHOUT.
BOTH $\mathbb{N}$ AND ω DENOTE THE SET OF NATURAL NUMBERS.

1. (a) (i) What is meant by a *truth function* of n variables (where $n \geq 1$ is a natural number)? Say how many such truth functions there are, explaining your answer.

   (ii) What is meant by saying that a set of connectives is *adequate*? Give an example of an adequate set of connectives. Show that there are precisely two 2-ary connectives $*$ with the property that the set $\{*\}$ is adequate.

   (iii) Is there a 2-ary connective $*$ such that $\{*\}$ is adequate and for all propositional formulas ϕ, ψ, χ, the formulas $((\phi * \psi) * \chi)$ and $(\phi * (\psi * \chi))$ are logically equivalent? Explain your answer.

   (b) Let L denote the formal system for propositional logic used in the lectures. Let Σ denote a set of L-formulas and let ϕ, ψ, χ denote L-formulas.

   (i) Prove that if Σ is consistent and $\Sigma \vdash_L \chi$, then $\Sigma \cup \{\chi\}$ is consistent.

   (ii) Prove that if Σ is consistent and $\Sigma \cup \{(\phi \to \psi)\}$ is inconsistent, then $\Sigma \vdash_L \phi$ and $\Sigma \vdash_L (\neg\psi)$. (In your proof, you may use results from the lectures, but you should not make use of the Completeness Theorem for L.)

2. (a) Suppose $\mathcal{L}$ is a first-order language and $\mathcal{A}$ is an $\mathcal{L}$-structure. Suppose v is a valuation in $\mathcal{A}$ and ϕ is an $\mathcal{L}$-formula.

   (i) Define what it means for v to *satisfy* ϕ (in $\mathcal{A}$).

   (ii) Define what is meant by the $\mathcal{L}$-formula $(\exists x_1)\phi$ and prove that v satisfies $(\exists x_1)\phi$ (in $\mathcal{A}$) if and only if there is a valuation v' in $\mathcal{A}$ which satisfies ϕ (in $\mathcal{A}$) and which is x_1-equivalent to v.

   (iii) Prove that the formula $((\exists x_1)(\forall x_2)\phi \to (\forall x_2)(\exists x_1)\phi)$ is logically valid.

   (b) Suppose $\mathcal{L}^=$ is a first-order language with equality having a binary function symbol m, a constant symbol e, and no other relation, function and constant symbols (apart from the equality symbol). In each of the following, groups $\mathcal{A}_i$ and $\mathcal{B}_i$ are given and are considered as normal $\mathcal{L}^=$-structures where m is interpreted as the group operation and e as the identity element. In each case, decide whether there is a closed $\mathcal{L}^=$-formula ϕ_i which is true in $\mathcal{A}_i$ but not in $\mathcal{B}_i$. Explain your answers.

   (i) $\mathcal{A}_1$ is the multiplicative group of positive rational numbers $\langle \mathbb{Q}^{>0}; \cdot, 1\rangle$ and $\mathcal{B}_1$ is the additive group of rational numbers $\langle \mathbb{Q}; +, 0\rangle$.

   (ii) $\mathcal{A}_2$ is the multiplicative group of positive real numbers $\langle \mathbb{R}^{>0}; \cdot, 1\rangle$ and $\mathcal{B}_2$ is the additive group of real numbers $\langle \mathbb{R}; +, 0\rangle$.

   (iii) $\mathcal{A}_3$ is the additive group of integers $\langle \mathbb{Z}; +, 0\rangle$ and $\mathcal{B}_3$ is $\langle \mathbb{Z}^2; +, (0,0)\rangle$.

3. (a) Suppose $\mathcal{L}$ is a first-order language and $K_{\mathcal{L}}$ is the formal system used in the lectures. Suppose that Σ is a set of $\mathcal{L}$-formulas and ϕ is an $\mathcal{L}$-formula.

   (i) Give the definition of $\Sigma \vdash_{K_{\mathcal{L}}} \phi$. Is it necessarily the case that if $\Sigma \vdash_{K_{\mathcal{L}}} \phi$, then $\Sigma \vdash_{K_{\mathcal{L}}} (\forall x_i)\phi$? Explain your answer carefully.

   (ii) State and prove the Deduction Theorem for $K_{\mathcal{L}}$.

   (b) Suppose $\mathcal{L}^=$ is the first-order language with equality in which there are no function and constant symbols and the only relation symbols are equality and a binary relation symbol R.

   (i) For $n \in \mathbb{N}$, write down a closed $\mathcal{L}^=$-formula η_n with the property that a normal $\mathcal{L}^=$-structure $\mathcal{B}$ is a model of η_n if and only if the interpretation of R in $\mathcal{B}$ is an equivalence relation with all equivalence classes of size 3 and at least n equivalence classes.

   (ii) Let $\mathcal{A}$ be an infinite normal $\mathcal{L}^=$-structure in which the interpretation of R is an equivalence relation where all equivalence classes have size 3. Suppose σ is a closed $\mathcal{L}^=$-formula and $\mathcal{A} \models \sigma$. Prove that σ has a finite normal model.

   (iii) Does there exist a normal $\mathcal{L}^=$-structure $\mathcal{B}$ and a closed $\mathcal{L}^=$-formula ϕ with $\mathcal{B} \models \phi$ and such that ϕ has no finite normal model? Explain your answer.

4. (a) Compare the cardinalities of the following sets (any general results about cardinal arithmetic may be assumed, if quoted accurately):

   - $\mathbb{R}$, the set of real numbers;
   - The set of lines in $\mathbb{R}^2$;
   - The set of subsets of $\mathbb{R}$;
   - $\{(X, Y) : X, Y \subseteq \mathbb{R} \text{ and } X \cap Y \text{ is dense in } \mathbb{R}\}$;
   - The set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

   (b) (i) Suppose, for $i = 1, 2$, that $\mathcal{A}_i = (A_i, \leq_i)$ is a linearly ordered set. Define the *reverse lexicographic product* $\mathcal{A}_1 \times \mathcal{A}_2$ (you need not prove that this is a linear ordering). Show that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are well ordered sets, then $\mathcal{A}_1 \times \mathcal{A}_2$ is well ordered. Under what circumstances is the converse true?

   (ii) What does it mean to say that a set α is an *ordinal*? Let β denote the set of countable ordinals. Show that β is the least uncountable ordinal (any general results about ordinals which you wish to use should be stated clearly).

   (iii) With β as in part (b)(ii), determine all $\alpha, \gamma \in \{\beta, \beta \times \omega, \omega \times \beta, \omega\}$ with the property that α is similar to an initial segment of γ. Explain your answer.

5. In this question, $\mathcal{L}^=$ is a first-order language with equality and all structures considered ($\mathcal{M}$, $\mathcal{N}$, $\mathcal{N}_i$, $\mathcal{M}_i$, $\mathcal{A}$, $\mathcal{B}$) are normal $\mathcal{L}^=$-structures.

(a) Say what it means for normal $\mathcal{L}^=$-structures $\mathcal{M}, \mathcal{N}$ to be *elementarily equivalent* (denoted by $\mathcal{M} \equiv \mathcal{N}$) and for $\mathcal{N}$ to be an *elementary substructure* of $\mathcal{M}$ (denoted by $\mathcal{N} \preceq \mathcal{M}$).

(b) Decide whether each of the following is true or false. Justify your answers fully.

(i) If $\mathcal{N}$ is a substructure of $\mathcal{M}$ and $\mathcal{N} \equiv \mathcal{M}$, then $\mathcal{N} \preceq \mathcal{M}$.

(ii) If $\mathcal{N}_1 \preceq \mathcal{N}_2 \preceq \mathcal{N}_3$, then $\mathcal{N}_1 \preceq \mathcal{N}_3$.

(iii) If, for $i \in \mathbb{N}$, we have $\mathcal{M}_i \preceq \mathcal{M}$ and $\mathcal{M}_i \subseteq \mathcal{M}_{i+1}$, then $\bigcup_{i\in\mathbb{N}} \mathcal{M}_i \preceq \mathcal{M}$.

(c) Suppose that the language $\mathcal{L}^=$ has, apart from the symbol for equality, a single 2-ary relation symbol and no other relation, function or constant symbols. Suppose that $\mathcal{A} = \langle A; \leq \rangle$ is an infinite well-ordered set, considered as an $\mathcal{L}^=$-structure.

(i) Show that there is a countable well-ordering $\mathcal{C}$ with $\mathcal{C} \preceq \mathcal{A}$.

(ii) Show that there is a linear order $\mathcal{B}$ which is elementarily equivalent to $\mathcal{A}$ and which is not a well-ordering.

(d) Suppose that $\mathcal{F}$ is a non-principal ultrafilter on $\mathbb{N}$. Let $\mathcal{M}$ be an $\mathcal{L}^=$-structure and let $\mathcal{N}$ be the ultrapower $(\prod_{i\in\mathbb{N}} \mathcal{M})/\mathcal{F}$. Suppose that $\phi_i(x)$ is an $\mathcal{L}^=$-formula with a single free variable x (for $i \in \mathbb{N}$). Suppose further that for all $n \in \mathbb{N}$ we have:

$$\mathcal{M} \models (\exists x) \bigwedge_{i \leq n} \phi_i(x).$$

(i) Give an example where there is no $a \in \mathcal{M}$ with $\mathcal{M} \models \phi_i[a]$ for all $i \in \mathbb{N}$.

(ii) Show that there is $b \in \mathcal{N}$ such that for all $i \in \mathbb{N}$ we have $\mathcal{N} \models \phi_i[b]$.

M345P65 Mathematical Logic Solutions 2018-19.

Question 1:

(a) **((i) bookwork; (ii) on problem sheet; (iii) unseen)**

(i) A truth function of n variables is a function $F : \{T, F\}^n \to \{T, F\}$. As $|\{T, F\}| = 2^n$ and there are two possible values for $F(x)$ for each $x \in \{T, F\}^n$, the number of these is 2^{2^n}.

3 marks (3A)

(ii) A set S of connectives is adequate if, for every $n > 0$, every truth function of n variables can be expressed as the truth function of a propositional formula constructed using these connectives (and n variables). The set of connectives $\{\neg, \wedge\}$ is adequate.

Suppose $\{*\}$ is adequate and let $G(p_1, p_2)$ be the truth function of $*$. We must have $G(T, T) = F$, as otherwise $F(T, \ldots, T) = T$ whenever F is the truth function of a formula built using $*$. Similarly $G(F, F) = T$. If $G(T, F) = F$ and $G(F, T) = T$ then G is the truth function of $(\neg p_1)$ and so if F is the truth function of a formula built using $*$, then F is either the truth function of the first variable in the formula, or its negation - in particular it depends on only one variable. So in this case $\{*\}$ would not be adequate. Similarly, if $G(T, F) = T$ and $G(F, T) = F$, then $\{*\}$ is not adequate.

Thus we have two cases remaining: Case 1, where $G(T, F) = T$ and $G(F, T) = T$ and Case 2 where $G(T, F) = F = G(F, T)$. In each case we show that the connective is adequate by expressing $\neg$ and $\wedge$ in terms of $*$.

In both cases $(\neg p_1)$ is l.e. to $(p_1 * p_1)$. In case 1, $p_1 \wedge p_2$ is l.e. to $(p_1 * p_2) * (p_1 * p_2)$. In case 2, $p_1 \wedge p_2$ is l.e. to $(((p_1 * p_2) * (p_1 * p_2)) * ((p_1 * p_2) * (p_1 * p_2)))$. **5 marks (3A, 2B)**

(iii) No: neither of the connectives in (ii) has this property as $(p_1 * (p_2 * p_3))$ and $(p_1 * p_2) * p_3)$ have different truth values when $(p_1, p_2, p_3) = (F, F, T)$. **3 marks (3B)**

(b) **(Used in notes; (ii) is harder)**

(i) As $\Sigma \vdash_L \chi$ we can convert a deduction from $\Sigma \cup \{\chi\}$ into a deduction of the same formula from Σ by including a deduction of χ from Σ in the deduction. So any consequence of $\Sigma \cup \{\chi\}$ is also a consequence of Σ and thus, if $\Sigma \cup \{\chi\}$ is inconsistent, so is Σ. **3 marks (3A)**

(ii) Suppose that $\Sigma \nvdash_L \phi$. Then, by a result in lectures, $\Sigma \cup \{(\neg\phi)\}$ is consistent (as Σ is consistent). But $((\neg\phi) \to (\phi \to \psi))$ is a theorem of L (in lectures), so $\Sigma \cup \{(\neg\phi)\} \vdash_L (\phi \to \psi)$. By (i), it then follows that $\Sigma \cup \{(\phi \to \psi)\}$ is consistent: a contradiction.

Now suppose that $\Sigma \nvdash_L (\neg\psi)$. Then $\Sigma \cup \{(\neg(\neg\psi))\}$ is consistent. As $(\neg\neg\psi \to \psi)$ is a theorem of L, it follows (by (i)) that $\Sigma \cup \{\psi\}$ is consistent. But then as $(\psi \to (\phi \to \psi))$ is a theorem of L it follows that $\Sigma \cup \{(\phi \to \psi)\}$ is consistent - a contradiction.

6 marks (3C,3D)

Question 2:

(a) **((i) Standard definition; (ii), (iii) on problem sheets)**

(i) This is defined by induction on the number of connectives and quantifiers in ϕ (we write $v[\phi] = T$ to indicate that v satisfies ϕ in $\mathcal{A}$, as in lectures).
If ϕ is an atomic formula $R(t_1, \ldots, t_n)$ (where t_i are terms of $\mathcal{L}$), then $v[\phi] = T$ iff $\bar{R}(v(t_1), \ldots, v(t_n))$ holds in $\mathcal{A}$ (where $\bar{R}$ is the interpretation of R in $\mathcal{A}$).
If ϕ is $(\neg\psi)$ then $v[\phi] = T$ iff $v[\psi] = F$.
If ϕ is $(\psi \to \chi)$, then $v[\phi] = F$ iff $v[\psi] = T$ and $v[\chi] = F$.
If ϕ is $(\forall x_i)\psi$ then $v[\phi] = T$ iff $w[\psi] = T$ whenever w is a valuation in $\mathcal{A}$ which is x_i-equivalent to v (meaning $v(x_j) = w(x_j)$ whenever $j \neq i$).

3 marks (3A)

(ii) This is shorthand for $(\neg(\forall x_1)(\neg\phi))$. Using the definitions in part (i):

$$v[(\neg(\forall x_1)(\neg\phi))] = T \Leftrightarrow v[(\forall x_1)(\neg\phi)] = F$$

and this happens iff there is a valuation v' x_1-equivalent to v with $v'[(\neg\phi)] = F$. Equivalently, there is a valuation v' x_1-equivalent to v with $v'[\phi] = T$, as required. **3 marks (3A)**

(iii) Suppose v satisfies $(\exists x_1)(\forall x_2)\phi$. We must show v satisfies $(\forall x_2)(\exists x_1)\phi$. So suppose v' is x_2-equivalent to v. We need to show v' satisfies $(\exists x_1)\phi$. Now, there is a valuation v'' which is x_1-equivalent to v and which satisfies $(\forall x_2)\phi$ (by (ii)). Let w be the valuation given by

$$w(x_i) = \begin{cases} v''(x_1) & \text{if } i = 1 \\ v'(x_2) & \text{if } i = 2 \\ v(x_i) & \text{if } i > 2 \end{cases}.$$

Then w is x_2-equivalent to v'' and so satisfies ϕ. But w is also x_1-equivalent to v', so (by (ii)) v' satisfies $(\exists x_1)\phi$, as required. **5 marks (5B)**

(b) **(Seen similar; (iii) harder)**

(i) We can take ϕ_1 to be $(\exists x_1)(\forall x_2)(m(x_2, x_2) \neq x_1)$. Every element in $\mathcal{B}_1$ has a 'square root', but this is not the case in $\mathcal{A}_1$. **2 marks (2A)**

(ii) The groups are isomorphic (use the exponential map $\mathbb{R} \to \mathbb{R}^{>0}$ given by $x \mapsto e^x$) and so there is no such formula ϕ_2. **3 marks (3C)**

(iii) Take ϕ_3 to express that, for an abelian group $(G; +, 0)$ there are only 2 cosets of the subgroup $2G$ in G. This is true in $\mathcal{A}_3$ but not in $\mathcal{B}_3$. For readability, write the group operation as $+$ and write $2z$ for $z + z$. Then ϕ_3 is

$$(\exists x_1)(\exists x_2)(\forall y)(\exists z)((y = x_1 + 2z) \vee (y = x_2 + 2z)).$$

4 marks (4D)

Question 3:

(a) **(Mostly bookwork)**

(i) This means that there is a deduction of ϕ from Σ. So this is a finite sequence of formulas, ending with ϕ, such that each formula is an axiom, or a formula in Σ or is obtained from earlier formulas in the sequence by applying a deduction rule: either Modus Ponens or Generlisation (from ϕ deduce $(\forall x_i)\phi$). The latter can only be applied when x_i does not occur freely in a formula of Σ. If x_i does not occur freely in a formula in Σ and if $\Sigma \vdash_{K_\mathcal{L}} \phi$, then $\Sigma \vdash_{K_\mathcal{L}} (\forall x_i)\phi$, using Generalisation. If x_i is not free in ϕ then $(\phi \to (\forall x_i)\phi)$ is an axiom and so the result also holds in this case. Otherwise, the result does not necessarily hold. **4 marks (2A,2B)**

(ii) Suppose $\Sigma \cup \{\phi, \psi\}$ is a set of $\mathcal{L}$-formulas. The Deduction Theorem states that if $\Sigma \cup \{\phi\} \vdash \psi$ then $\Sigma \vdash (\phi \to \psi)$ (where we write $\vdash$ instead of $\vdash_{K_\mathcal{L}}$). The proof is standard bookwork and is by induction in the number of steps n in a deduction of ψ from $\Sigma \cup \{\phi\}$.

6 marks (4A,2C)

(b) **((i) routine; (ii) unseen; (iii) on exercise sheet)**

(i) Let γ be the closed formula which says that R is an equivalence relation:

$$(\forall x)(\forall y)(\forall z)(R(x,x) \wedge (R(x,y) \to R(y,x)) \wedge (R(x,y) \wedge R(y,z) \to R(x,z)))).$$

Let α say that all classes have size 3:

$$\gamma \wedge (\forall y)(\exists x_1)(\exists x_2)(\exists x_3)((\bigwedge_{1\leq i<j\leq 3} (R(y,x_i) \wedge (x_i \neq x_j)) \wedge (\forall z)(R(z,y) \to (z = x_1) \vee (z = x_2) \vee (z = x_3))).$$

Let η_n be:

$$\alpha \wedge (\exists x_1)\dots(\exists x_n) \bigwedge_{1\leq i<j\leq n} (\neg R(x_i, x_j)).$$

3 marks (3A)

(ii) Let $\Sigma = \{\eta_n : n \in \mathbb{N}\}$. Note that any two countable normal models of Σ are isomorphic and so it follows that for every closed formula ϕ, either $\Sigma \vdash \phi$ or $\Sigma \vdash \neg\phi$ (this is the Los-Vaught test which was mentioned in the lectures). So as $\mathcal{A} \models \Sigma$ and $\mathcal{A} \models \sigma$ we have $\Sigma \vdash \sigma$. There is therefore a finite subset of Σ with σ as a consequence. So there is some n with $\eta_n \vdash \sigma$. Now, η_n has a finite normal model, so this is also a model of σ. **5 marks (5D)**

(iii) Yes. For example, take ϕ to say that R gives a strict partial order with no greatest element:

$$(\forall x)(\forall y)(\forall z)(\neg R(x,x) \wedge (R(x,y) \wedge R(y,z) \to R(x,z)) \wedge (\exists w)R(x,w)).$$

This has a normal model (for example $(\mathbb{N}; <)$) and any normal model is infinite (by definition, structures are non-empty). **2 marks (2B)**

Question 4:

(a) **(Mostly unseen, but seen similar)** Call the sets $S_1 - S_5$ and let $\lambda = |\mathbb{R}| = 2^\omega$.

We know that $|\mathbb{R}^3| = \lambda$ (by the Fundamental Theorem of Cardinal Arithmetic) and there is a surjection $\mathbb{R}^3 \setminus \{(0,0,0)\} \to S_2$ obtained by sending (a,b,c) to $\{(x,y) \in \mathbb{R}^2 : ax + by = c\}$. It follows that $|S_2| = \lambda$.

We have $|S_3| = 2^\lambda > \lambda$ (by Cantor's Theorem).

As $S_4 \subseteq S_3^2$ we have $|S_4| \leq 2^\lambda$ (by FTCA). Note that there is an injective function from the set of subsets of $\mathbb{R} \setminus \mathbb{Q}$ to S_4 given by $Z \mapsto (Z \cup \mathbb{Q}, \mathbb{Q})$. As $|\mathbb{R} \setminus \mathbb{Q}| = \lambda$ we therefore have $2^\lambda \leq |S_4|$ and so $|S_4| = 2^\lambda$.

By considering constant functions, we have $\lambda \leq |S_5|$. Any $f \in S_5$ is determined by its restriction to $\mathbb{Q}$, so we have an injective function $S_5 \to \mathbb{R}^\mathbb{Q}$ (given by restriction). Now, $|\mathbb{R}^\mathbb{Q}| = (2^\omega)^\omega = 2^{\omega.\omega} = 2^\omega = \lambda$. It follows that $|S_5| = \lambda$. **8 marks (2A,2B,4C)**

(b) **(i) bookwork; (ii) on exercise sheet; (iii) unseen)**

(i) This is the following relation on $A_1 \times A_2$. Say that $(a,b) \leq (c,d)$ if either $b <_2 d$ or $b = d$ and $a \leq_1 c$. If $\mathcal{A}_i$ are wosets and S is a non-empty subset of $A_1 \times A_2$, consider $S_2 = \{b : (\exists a)((a,b) \in S)\}$. This is a non-empty subset of A_2 so has a least element d (w.r.t. $\leq_2$). Let $S_1 = \{a : (\exists a)((a,d) \in S)\}$. This is a non-empty subset of A_1 so has a least element c. Then (c,d) is the least element of S.

If A_1, A_2 are non-empty then each of them is similar to a subset of $\mathcal{A}_1 \times \mathcal{A}_2$. So if the latter is well-ordered, so are $\mathcal{A}_1, \mathcal{A}_2$. If either is empty, then $\mathcal{A}_1 \times \mathcal{A}_2$ is empty and so well ordered, but the other $\mathcal{A}_i$ need not be well ordered. **4 marks (4A)**

(ii) A set α is an ordinal if it is a transitive set (that is, if $x \in y \in \alpha$ then $x \in \alpha$) and the membership relation $\in$ on α is a strict well ordering on α.

Suppose $x \in y \in \beta$. Then y is an ordinal. As $x \in y$ we have $x \subset y$ and so x is countable. Thus $x \in \beta$ and so β is transitive. As β is a set of ordinals, membership is a well ordering on β. So β is an ordinal.

We have $\beta \notin \beta$ (this is true of all ordinals, by definition), so β is uncountable. Moreover, if α is an uncountable ordinal, then any countable ordinal γ satisfies $\gamma < \alpha$, so $\gamma \in \alpha$. Thus $\beta \subseteq \alpha$ and therefore $\beta \leq \alpha$, as required. **4 marks (4B)**

(iii) Write $\alpha < \gamma$ if α is similar to a proper initial segment of γ. Note that this is transitive and irreflexive. If α, γ are not similar, then exactly one of $\alpha < \gamma$, $\gamma < \alpha$ holds. As ω is countable and all the others are uncountable, ω is similar to a proper initial segment of the others. Note that β is similar to $\beta \times \{0\}$ and this is a proper initial segment of $\beta \times \omega$. If X is a proper initial segment of $\omega \times \beta$ it is contained in $\omega \times \alpha$ for some $\alpha \in \beta$ and so is countable and therefore not similar to β. Any proper intial segment of β is countable (as it is an element of β), and so is not similar to $\beta \times \omega$. It follows that β and $\beta \times \omega$ are similar. **4 marks (4D)**

Question 5:

(a) **(Standard definitions)** $\mathcal{M}, \mathcal{N}$ are elementarily equivalent if for every closed $\mathcal{L}^=$-formula ψ we have

$$\mathcal{M} \models \psi \Leftrightarrow \mathcal{N} \models \psi.$$

A substructure $\mathcal{N}$ of $\mathcal{M}$ is an elementary substructure of $\mathcal{M}$ if for all $\mathcal{L}^=$-formulas $\phi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in N$ we have:

$$\mathcal{N} \models \phi[a_1, \ldots, a_n] \Leftrightarrow \mathcal{M} \models \phi[a_1, \ldots, a_n].$$

2 marks

(b) **((i, ii) standard; (iii) harder)**

(i) False. In a language for groups, let $\mathcal{M} = \langle \mathbb{Z}; + \rangle$ and let $\mathcal{N}$ be the substructure consisting of even integers. Then $\mathcal{N}$ is isomorphic to $\mathcal{M}$ and so $\mathcal{M} \equiv \mathcal{N}$. But if we let $\phi(x_1)$ be $(\exists x_2)(x_1 = x_2.x_2)$, then $\mathcal{N} \not\models \phi[2]$, but $\mathcal{M} \models \phi[2]$. **2 marks**

(ii) True: an easy argument from the definition. **2 marks**

(iii) True. By the Tarski - Vaught test it is enough to show that, for every formula $\phi(y, x_1, \ldots, x_n)$, if $a_1, \ldots, a_n \in \bigcup_{i \in \mathbb{N}} \mathcal{M}_i$ and there is c in $\mathcal{M}$ with $\mathcal{M} \models \phi[c, a_1, \ldots, a_n]$, then there is $b \in \bigcup_{i \in \mathbb{N}} \mathcal{M}_i$ with $\mathcal{M} \models \phi[b, a_1, \ldots, a_n]$. There is some i with $a_1, \ldots, a_n \in \mathcal{M}_i$ and as $\mathcal{M}_i \preceq \mathcal{M}$, we can therefore find such an element b in $\mathcal{M}_i$. **4 marks**

(c) **(In book: (i) easy; (ii) harder)**

(i) By the (downward) Löwenheim - Skolem Theorem, there is a countable $\mathcal{C} \preceq \mathcal{A}$, and a substructure of a w.o. set is well ordered. **2 marks**

(ii) Expand the language by a set of new constant symbols $\{c_i : i \in \mathbb{N}\}$ and consider the set Σ consisting of $Th(\mathcal{A})$ (the theory of $\mathcal{A}$) together with the set of closed formulas $\{(c_i < c_j : i > j\}$. Every finite subset of this has a normal model (use $\mathcal{A}$ with the constant symbols interpreted appropriately) and so there is a normal model, by the compactness theorem. This is a model of $Th(\mathcal{A})$ so, in the original language it is elementarily equivalent to $\mathcal{A}$. But the interpretation of the c_i gives a subset with no least element, so it is not a well ordered set. **3 marks**

(d) **(Unseen)**

(i) Let $\mathcal{M}$ be $\mathbb{Q}$ in the language of rings with an ordering $(+, \cdot, 0, 1, \geq)$, and $\phi_n(x)$ the formula $x \geq 1 + 1 + \cdots + 1$ (with n 1's on the right-hand side). **2 marks**

(ii) Let $a_n \in \mathcal{M}$ be such that $\mathcal{M} \models \bigwedge_{i \leq n} \phi_i[a_n]$. Let b be the image of the sequence $(a_n : n \in \mathbb{N})$ in the ultrapower $\mathcal{N}$. If $i \in \mathbb{N}$ then $\{n : \mathcal{M} \models \phi_i[a_n]\}$ contains all $n \geq i$, so is cofinite. It is therefore in $\mathcal{F}$, as $\mathcal{F}$ is non-principal. Thus, by the Los ultraproduct theorem, we have $\mathcal{N} \models \phi_i[b]$. **3 marks**