Mathematical Logic (MATH6/70132; P65)
Problem sheet $2\frac{1}{2}$ - for problem class

[1] (Warm-up) Decide whether the following are true or false - give reasons.

1. Every L-formula is a theorem.
2. If ϕ is an L-formula, then one of ϕ, $(\neg\phi)$ is a theorem of L.
3. In every L-formula, the number of opening brackets $($ equals the number of closing brackets $)$.
4. In every L-formula, the number of opening brackets $($ is equal to the number of connectives in the formula.

[2] Show that the set of connectives $\{\neg, \leftrightarrow\}$ is not adequate.

[3] (An alternative formal system for propositional logic: natural deduction) The formal system $\widehat{L}$ has the same language and formulas as L, but it has only deduction rules and no axioms. The notion $\Gamma \vdash_{\widehat{L}} \phi$, where Γ is a set of $\widehat{L}$-formulas and ϕ is an $\widehat{L}$-formula, is defined by saying that it satisfies the following deduction rules:

- If $\phi \in \Gamma$ then $\Gamma \vdash_{\widehat{L}} \phi$;
- (Modus Ponens) If $\Gamma \vdash_{\widehat{L}} \phi$ and $\Gamma \vdash_{\widehat{L}} (\phi \to \psi)$, then $\Gamma \vdash_{\widehat{L}} \psi$;
- (Deduction Theorem) If $\Gamma \cup \{\phi\} \vdash_{\widehat{L}} \psi$, then $\Gamma \vdash_{\widehat{L}} (\phi \to \psi)$;
- (PBC) If $\Gamma \vdash_{\widehat{L}} ((\neg\phi) \to \psi)$ and $\Gamma \vdash_{\widehat{L}} ((\neg\phi) \to (\neg\psi))$, then $\Gamma \vdash_{\widehat{L}} \phi$;

and any instance of $\Gamma \vdash_{\widehat{L}} \phi$ arises after a finite number of applications of these rules.

We say that ϕ is a theorem of $\widehat{L}$ if $\emptyset \vdash_{\widehat{L}} \phi$.

Prove that:

(a) $(\phi \to \phi)$ is a theorem of $\widehat{L}$, for every $\widehat{L}$-formula ϕ.

(b) Every axiom of L is a theorem of $\widehat{L}$.

(c) The deduction rule (PBC) is valid with L in place of $\widehat{L}$.

(d) A formula ϕ is a theorem of $\widehat{L}$ if and only if it is a theorem of L.

[4] (For fun) The following is known as Hofstadter's MU puzzle. You can look at the Wikipedia entry, but first try the problem yourself.

The formal system H has: alphabet M, I, U; formulas all (finite) strings of these symbols; one axiom MI; and the following deduction rules (where x, y are any formulas):

1. from xI deduce xIU;
2. from Mx deduce Mxx;
3. from $xIIIy$ deduce xUy;
4. from $xUUy$ deduce xy.

The problem is to decide whether MU is a theorem of H. But you could first write down some theorems of H, just to test your understanding of what a formal system is.