Imperial College London

MATH97009 MATH70012

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Mathematical Finance: An Introduction to Option Pricing

Date: 19 May 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS SEPARATE PDFs TO THE RELEVANT DROPBOXES ON BLACKBOARD (ONE FOR EACH QUESTION) WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

Question 1

(Total: 20 marks)

Consider the probability space $\Omega = \{\omega_1, \omega_2, \omega_3\}$, with a probability $\mathbb{P}$ such that $\mathbb{P}(\{\omega\}) > 0$ for every $\omega \in \Omega$. For $t \in \mathbb{R}$, define the random variables

ω	ω_1	ω_2	ω_3
$S_1(\omega)$	2	4	6
$X_1(\omega)$	1	t	8

Consider the one-period trinomial model of the market (B, S) made of a bond B with initial price 1 (all prices in a fixed currency, say £), and interest rate $r = \frac{1}{4}$, a stock whose initial price is $S_0 = 4$, and whose final price is S_1. We also consider a derivative X with payoff X_1. Answer the following questions and justify carefully with either proofs or counterexamples.

(a) Is the market (B, S) free of arbitrage? (2 marks)

(b) Find all the values of t (if any) for which X_1 is replicable (in the market (B, S)). (3 marks)

(c) Is the market (B, S, X) complete when $t = 6$? (3 marks)

Suppose from now on that we extend the market (B, S), so that one can trade also X at (initial) price $X_0 := 4$.

(d) Find all the values of t for which the market (B, S, X) is free of arbitrage. (7 marks)

(e) Assume $t = 12$. Explicitly find an arbitrage. (5 marks)

Question 2

(Total: 20 marks)

In the framework of the N-period model where the bank account has interest rate r, and the stock price process $S = (S_n)_{n=0}^N$ is given by the binomial model with constant parameters $d, u, S_0 > 0$ which satisfy $0 < d < 1 + r < u$, let

$$Q_n := \sum_{i=0}^{n} S_i^2, \quad M_n := \sqrt{\frac{Q_n}{n+1}}, \quad n = 0, \ldots, N,$$

so that M_n equals the 2-mean of $(S_i)_{i=0}^n$. For $a, b \in \mathbb{R}, a < b$, define T_n as the time spent by $M = (M_n)_{n=0}^N$ in the corridor $[a, b]$ up to time n, i.e.

$$T_n := \sum_{i=0}^{n} 1_{\{a \le M_i \le b\}}.$$

Consider the option which at maturity N pays the average time $\frac{T_N}{N+1}$ which M spends inside $[a, b]$, and denote with V_n its arbitrage-free price at time $n = 0, \ldots, N$. As usual $\mathbb{Q}$ denotes the risk-neutral measure, $(X_i)_i$ the process of coin tosses X associated to S, and $\mathcal{F}$ the natural filtration of X. Prove all your assertions carefully or provide counter-examples.

(a) Is $C_{n+1} := \frac{S_{n+1}}{S_n}$ independent of $\mathcal{F}_n$ under $\mathbb{Q}$? (1 marks)

(b) Write an explicit formula for a function h_n which satisfies $M_{n+1} = h_n(S_{n+1}, M_n)$. (3 marks)

(c) Is (S, M) a $\mathbb{Q}$-Markov process? (3 marks)

(d) Is (S, T) a $\mathbb{Q}$-Markov process? (3 marks)

(e) Is (S, T, M) a $\mathbb{Q}$-Markov process? (3 marks)

(f) Which of the processes (S, T), (S, T, M) are such that, for every $n = 0, \ldots, N$, V_n can be written as a function f_n of the value of the process at time n? (3 marks)

(g) For each process in item (f), for which V_n can be written in the way described there, write explicitly f_N, and an explicit formula to express f_n in terms of f_{n+1} for $n = 0, \ldots, N-1$. (4 marks)

Question 3

(Total: 20 marks)

Consider a market composed of a domestic bank account with interest rate $r_{£} = 1/3$, and foreign bank account with interest rate $r_{\$} = 1/4$, and an exchange rate E between £ and \$ (defined as the cost of one \$ in £) modelled as

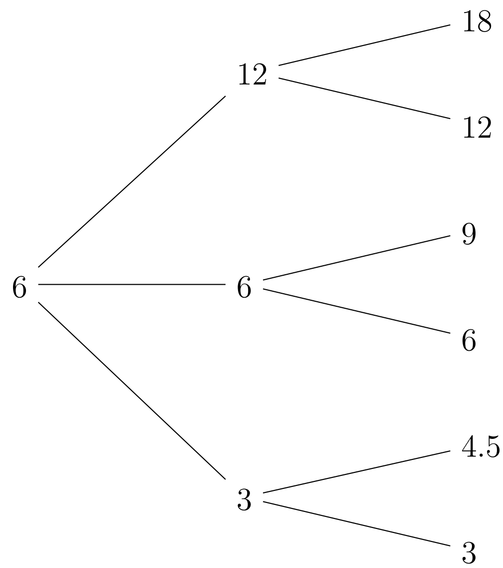

Figure 1: Tree of E.

Denote with D, F the values of the domestic and foreign bank accounts, i.e. the processes defined by $D_n = (1 + r_{£})^n$, $F_n = (1 + r_{\$})^n$ for all n.

A British investor wants to buy at time 0 a US treasury bond which states that at time 2 (s)he will receive $B_2 = \$1000$. Denote with P_n the price in £ at time n, and B_n the price in \$ at time n, of such a bond.

Answer the following questions and justify carefully with either proofs or counterexamples.

(a) State explicitly what is the underlying market in which the British investor is aiming to price the US bond, and draw the corresponding tree of prices of the traded assets. Is such a market arbitrage-free? (4 marks)

(b) Is the market complete? Is the US bond replicable? Is the exchange rate E replicable? (5 marks)

(c) Compute P_1 by converting first all prices to £, and then doing all the calculations in £. (3 marks)

(d) Write B_1 as given by the RNPF (Risk-Neutral Pricing Formula) using F as numeraire, and calculate explicitly the transition probability from time 1 to time 2 of the corresponding EMM (Equivalent Martingale Measure) $\mathbb{P}^F$. In other words, compute the EMM $\mathbb{P}^F$ for the American investor, and use it to determine B_1 by discounting using the value of the American bank account and applying the RNPF. (3 marks)

(e) Answer the same question as in part (c), but doing first the calculations in \$, and then converting the price to £. (3 marks)

(f) Compute P_0. (2 marks)

Question 4

(Total: 20 marks)

Consider a 2-period binomial model with a bank account with interest rate $r = 0$, and a stock whose price S is described by the following binomial tree.

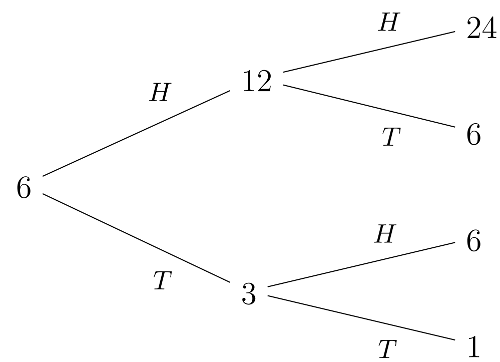

Figure 2: Tree of S.

As usual $(X_n)_n$ denotes the process of coin tosses X which generates $(S_n)_n$, and $\mathcal{F} = (\mathcal{F}_n)_n$ the natural filtration of X. Consider a Bermudan call option with strike price $K = 4$ and with possible exercise dates 1 and 2. In other words, consider the derivative which gives its owner the right to decide at time 1 whether (s)he wants to immediately get paid $(S_1 - K)^+$, or whether to wait and get paid the amount $(S_2 - K)^+$ at time 2. Answer the following questions and justify carefully with either proofs or counterexamples.

(a) Prove that the above market is arbitrage-free. (2 marks)

(b) Draw the binomial tree of (I, C), where I is the intrinsic value I the call option with strike $K = 4$ (i.e. the process $I_n := (S_n - K)^+, n = 0, 1, 2$), and $C = (C_n)_{n=0,1,2}$ is the arbitrage-free price of the call exercised at time 2, i.e. of receiving the payoff $(S_2 - K)^+$ at time 2. (5 marks)

(c) Compute the value V_1 of the Bermudan call at time 1. (*Hint: clearly V_1 equals the value at time 1 of the American call with strike K which has not been exercised at time 0*). (3 marks)

(d) If the buyer of the Bermudan chooses to exercise the derivative at the following (stopping) time (1 marks)

ω	HH	HT	TH	TT
$\tau_1(\omega)$	1	1	2	2

(s)he receives the payment $(S_{\tau_1} - K)^+$ at time τ_1. Determine the capital $V_1(\tau_1)$ required at time 1 to replicate such payment.

(e) How many stopping times $\tau_1, \tau_2, \ldots$ with values in $\{1, 2\}$ are there? List all such stopping times $(\tau_i)_{i \in I}$, along with their values, on a table (as done above for τ_1). Explain why there exist no other stopping times with values in $\{1, 2\}$. (4 marks)

(f) For each $\tau_i, i \in I$, compute the capital $V_1(\tau_i)$ required at time 1 to replicate the payment of $(S_{\tau_i} - K)^+$ at time τ_i. Then compute $\bar{V}_1 := \max_{i \in I} V_1(\tau_i)$. (3 marks)

(g) Prove that $(S_\sigma - K)^+ 1_{\{\sigma \le n\}}$ is $\mathcal{F}_n$-measurable for all $n = 0, 1, 2$ and stopping time σ with values in $\{1, 2\}$. (2 marks)

Question 5

(Total: 20 marks)

On the sample space $\Omega = \{\omega_i\}_{i=1,\dots,3}$ endowed with some probability $\mathbb{P}$ s.t. $\mathbb{P}(\omega_i) > 0$ for all i, consider a one-period market model where the bank account has interest rate $r = 0$ and there are some stocks with price $S = (S^1, \dots, S^m)$. Recall that if one short-sells some shares at time t it means that (s)he borrows them from a lender and then sells them at time t, and then at time $t+1$ (s)he has to buy them and return them to the lender. Unlike normally done in class, in this exercise we consider the fact that the lender will in reality charge a short-selling fee $f \geq 0$, proportional to the value of the shares lent, to be paid immediately (i.e. at time t); we assume that this fee is the same for each stock $S^1, \dots, S^m$. Denote with V_t^{x,h^+,h^-} the value of a portfolio at times $t = 0, 1$, as a function of the following variables: the initial capital $x \in \mathbb{R}$, and $h^+, h^- \in \mathbb{R}_+^m$, where h_j^+ (resp. h_j^-) represents the number of shares of type $j = 1, \dots, m$ bought (resp. sold) at time 0. For notational convenience write $k := (h^+, h^-)$ and $V_t^{x,k} = V_t^{x,h^+,h^-}$. You can use that the formula for $V_t^{x,k}$ is

$$V_1^{x,k} = (h^+ - h^-) \cdot S_1 + (x - (h^+ - (1-f)h^-) \cdot S_0)(1+r), \tag{1}$$

and that the following version of the FTAP (Fundamental Theorem of Asset Pricing) holds: this model is arbitrage-free if and only if the set

$$\mathcal{M}^f := \mathcal{M}^f(S) := \{\mathbb{Q} \text{ is a probability} \sim \mathbb{P} \text{ s.t. } (1+r)(1-f)S_0^j \leq \mathbb{E}^{\mathbb{Q}}[S_1^j] \leq (1+r)S_0^j \quad \forall j = 1, \dots, m\} \tag{2}$$

is not empty. As usual we identify $\mathbb{Q}$ with the vector $(q_i)_i \in \mathbb{R}^3$ s.t. $q_i = \mathbb{Q}(\{\omega_i\})$, $i = 1, 2, 3$, and so consider $\mathcal{M}^f(S)$ as a subset of $\mathbb{R}^3$. Assume now that $m = 1$ and the stock S has prices $S_0 = 4$,

ω	ω_1	ω_2	ω_3
$S_1(\omega)$	4	3	2

(a) Define what an arbitrage is in such a market, *using formulas.* (3 marks)

(b) If $f = 0$, is there an arbitrage in this market? If so, find one explicitly. (3 marks)

Assume from now on that $f = \frac{1}{8}$.

(c) Identify the set $\mathcal{M}^f(S) \subseteq \mathbb{R}^3$ using algebra, and then describe it geometrically and draw it. (6 marks)

(d) Compute the set of arbitrage-free prices for the derivative X with payoff (8 marks)

ω	ω_1	ω_2	ω_3
$X_1(\omega)$	6	4	2

assuming that the fee for short-selling X is also f. *Hint: use the FTAP and the definition of arbitrage.*

Module: MATH60012/MATH70012/MATH97009
Setter: Siorpaes
Checker: Zheng
Editor: Wu
External: Giesl
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Version: Final version **WITH SOLUTIONS INCLUDED**

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2022

MATH60012/MATH70012/MATH97009 Mathematical Finance: An Introduction to Option Pricing

The following information must be completed:

Is the paper suitable for resitting students from previous years:

No, as I include one exercise based on material which I did not teach last year

Category A marks: available for basic, routine material (excluding any mastery question) (40 percent = 32/80 for 4 questions):
1(a,b,c) 2+3+3=8 marks; 2(a,b,c) 1+3+3=7 marks; 3(a,b,c,f) 4+5+3+2=14 marks; 4(a,d) 2+1=3 marks.

Category B marks: Further 25 percent of marks (20/ 80 for 4 questions) for demonstration of a sound knowledge of a good part of the material and the solution of straightforward problems and examples with reasonable accuracy (excluding mastery question):
3(e) 3 marks; 4(b,c,e,f,g) 5+3+4+3+2=17 marks.

Category C marks: the next 15 percent of the marks (= 12/80 for 4 questions) for parts of questions at the high 2:1 or 1st class level (excluding mastery question):
1(e) 5 marks; 2(e,g) 3+4=7 marks.

Category D marks: Most challenging 20 percent (16/80 marks for 4 questions) of the paper (excluding mastery question):
1(d) 7 marks, 2(d,f) 3+3=6 marks; 3(d) 3 marks.

Signatures are required for the final version:

Setter's signature	Checker's signature	Editor's signature
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BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Mathematical Finance: An Introduction to Option Pricing

Date: Thursday, 19th May 2022

Time: 09 – 11

Time Allowed: 2 Hours for MATH96 paper; 2.5 Hours for MATH97 papers

This paper has *4 Questions (MATH96 version); 5 Questions (MATH97 versions)*.

Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.

Question 1

(Total: 20 marks)

SIMILARLY SEEN IN PROBLEMS

Consider the probability space $\Omega = \{\omega_1, \omega_2, \omega_3\}$, with a probability $\mathbb{P}$ such that $\mathbb{P}(\{\omega\}) > 0$ for every $\omega \in \Omega$. For $t \in \mathbb{R}$, define the random variables

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(c) Is the market (B, S, X) complete when $t = 6$? (3 marks)

Suppose from now on that we extend the market (B, S), so that one can trade also X at (initial) price $X_0 := 4$.

(d) Find all the values of t for which the market (B, S, X) is free of arbitrage. (7 marks)

(e) Assume $t = 12$. Explicitly find an arbitrage. (5 marks)

Solution:

(a) Recall that $\mathbb{Q}$ is an EMM (equivalent martingale measure) if $S_0(1+r) = \mathbb{E}^{\mathbb{Q}}[S_1]$, $\mathbb{Q}$ is a probability and $\mathbb{Q} \sim \mathbb{P}$, i.e. iff $q_i := \mathbb{Q}(\{\omega_i\})$ satisfy

$$\begin{cases} 4\left(1 + \dfrac{1}{4}\right) = 2q_1 + 4q_2 + 6q_3 \\ 1 = q_1 + q_2 + q_3 \\ q_i > 0 \text{ for } i = 1, 2, 3 \end{cases}$$

The system of equalities has solution $q_3 = s, q_2 = \frac{3}{2} - 2s, q_1 = -\frac{1}{2} + s$, and imposing $q_i > 0$ we obtain that $s \in (\frac{1}{2}, \frac{3}{4})$, i.e. the set of ($q$'s corresponding to the set of) EMM is

$$\mathcal{M}(B, S) = \left\{ q_s := \begin{pmatrix} -\frac{1}{2} + s \\ \frac{3}{2} - 2s \\ s \end{pmatrix} : s \in \left(\frac{1}{2}, \frac{3}{4}\right) \right\}. \qquad (1)$$

Since $\mathcal{M}$ is not empty, the model is arbitrage-free.

(b) 1^{st} **solution:** X_1 is replicable iff $\mathbb{E}^{\mathbb{Q}(s)}[X_1]$ does not depend on $\mathbb{Q}(s) \in \mathcal{M}(B,S)$. Since

$$\mathbb{E}^{\mathbb{Q}(s)}[X_1] = \left(-\frac{1}{2}+s\right)\cdot 1 + \left(\frac{3}{2}-2s\right)\cdot t + s\cdot 8 = \frac{3t-1}{2} + s\,(9-2t)\,,$$

which does not depend on s iff $9-2t=0$, we have that X_1 is replicable iff $t=\frac{9}{2}$.

2^{nd} **solution:** X_1 is replicable iff the system $x(1+r)+h(S_1-(1+r)S_0) = X_1$ has solution (x,h). Writing the system as

$$\begin{cases} x(1+\frac{1}{4}) + h(2-\frac{5}{4}\cdot 4) = 1 \\ x(1+\frac{1}{4}) + h(4-\frac{5}{4}\cdot 4) = t \\ x(1+\frac{1}{4}) + h(6-\frac{5}{4}\cdot 4) = 8 \end{cases},$$

we find

$$\begin{cases} \frac{5}{4}x - 3h = 1 \\ \frac{5}{4}x - \ \ h = t \\ \frac{5}{4}x + \ \ h = 8 \end{cases}.$$

The only values of x,h which solve the 1st and 3rd equations are $x=5, h=\frac{7}{4}$, and plugging these into the 2nd equation gives $\frac{25}{4}-\frac{7}{4}=t$, i.e. $t=\frac{9}{2}$.

(c) 1^{st} **solution:** (B,S,X) is complete since the set $\mathcal{M}(B,S,X)$ of EMM is a singleton: indeed if $t=6$ then $\mathbb{E}^{\mathbb{Q}(s)}[X_1]$ depends on s, so the linear (in s) equation $\mathbb{E}^{\mathbb{Q}(s)}[X_1] = X_0(1+r)$ has only one solution $s=s^*$, and then $\mathcal{M}(B,S,X)=\{\mathbb{Q}(s^*)\}$.

2^{nd} **solution:** Since $t=6$, by the previous item X_1 is not replicable. Thus the 3 vectors B_1, S_1, X_1 are linearly independent elements of $\mathbb{R}^3$, and so they span the whole space $\mathbb{R}^3$, i.e. (B,S,X) is complete.

(d) If $\mathcal{P}(t)$ denotes the interval of arbitrage-free prices of X_1 in the (B,S) market, we need to determine the values of t for which $4\in\mathcal{P}(t)$.

1^{st} **solution:** Since

$$\mathcal{P}(t) := \{\frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[X_1(t)] : \mathbb{Q}\in\mathcal{M}(B,S)\},$$

using the above formulas for $\mathcal{M}(B,S)$ and $\mathbb{E}^{\mathbb{Q}}[X_1]$ we have that

$$\mathcal{P}(t) = \{\frac{4}{5}\left(\frac{3t-1}{2}+s\,(9-2t)\right) : s\in\left(\frac{1}{2},\frac{3}{4}\right)\}. \tag{2}$$

Evaluating the above expression at $s=\frac{1}{2}$ and at $s=\frac{3}{4}$ gives

$$\frac{2}{5}(8+t), \quad 5.$$

Since the function

$$s\mapsto \frac{4}{5}\left(\frac{3t-1}{2}+s\,(9-2t)\right)$$

is affine, it is either constant or strictly monotone. Thus, $\mathcal{P}(t)$ is a singleton iff

$$\frac{2}{5}(8+t) = 5,$$

i.e. iff $t = \frac{9}{2}$, in which case $\mathcal{P}(7) = \{5\}$. If $t > \frac{9}{2}$ then $\frac{2}{5}(8+t) > 5$, so $\mathcal{P}(t) = (5, \frac{2}{5}(8+t))$, and if $t < \frac{9}{2}$ then $\frac{2}{5}(8+t) < 5$, so $\mathcal{P}(t) = (\frac{2}{5}(8+t), 5)$.

It follows that $4 \notin \mathcal{P}(t)$ if $t \geq \frac{9}{2}$. If instead $t < \frac{9}{2}$ then $4 \in \mathcal{P}(t)$ holds iff $4 \in (\frac{2}{5}(8+t), 5)$, i.e. iff $4 > \frac{2}{5}(8+t)$, i.e. $2 > t$. Thus, when $X_0 = 4$, (B, S, X) is arbitrage-free iff $t < 2$.

2^{nd} **solution:** Calling $s = s(t)$ the smallest value of x for which there exists an h such that $x + h(S_1 - (1+r)S_0) \geq X_1$, and $i = i(t)$ the largest value of x for which there exists an h such that $x + h(S_1 - (1+r)S_0) \leq X_1$, we have that $i \leq s$, $\mathcal{P}(t) = (i, s)$ if $i < s$, and $\mathcal{P}(t) = \{s\}$ if $i = s$. This way one can compute $\mathcal{P}(t)$ and determine whether $4 \in \mathcal{P}(t)$.

Let us now compute s. The system $x(1+r) + h(S_1 - (1+r)S_0) \geq X_1$ is

$$\begin{cases} x(1+r) - 3h \geq 1 \\ x(1+r) - h \geq t \\ x(1+r) + h \geq 8 \end{cases},$$

and to eliminate the variable h we write it as

$$\begin{cases} h \leq \frac{5}{12}x - \frac{1}{3} \\ h \leq \frac{5}{4}x - t \\ h \geq -\frac{5}{4}x + 8 \end{cases},$$

which yields

$$\begin{cases} -\frac{5}{4}x + 8 \leq \frac{5}{4}x - t \\ -\frac{5}{4}x + 8 \leq \frac{5}{12}x - \frac{1}{3} \end{cases}, \quad \text{i.e.} \quad \begin{cases} x \geq \frac{16}{5} + \frac{2}{5}t \\ x \geq 5 \end{cases}.$$

So the smallest value for which the system has solution is $s(t) := \max(5, \frac{2}{5}(8+t))$.

Analogously i is the largest value of x for which the system $x(1+r) + h(S_1 - (1+r)S_0) \leq X_1$ has solution, i.e. the largest x for which

$$\begin{cases} x \leq 4 + \frac{1}{2}t \\ x \leq \frac{25}{4} \end{cases},$$

so $s(t) := \min(5, \frac{2}{5}(8+t))$.

Thus, $\mathcal{P}(t)$ is a singleton iff

$$\frac{2}{5}(8+t) = 5,$$

i.e. iff $t = \frac{9}{2}$, in which case $\mathcal{P}(7) = \{5\}$. If $t > \frac{9}{2}$ then $\frac{2}{5}(8+t) > 5$, so $\mathcal{P}(t) = (5, \frac{2}{5}(8+t))$, and if $t < \frac{9}{2}$ then $\frac{2}{5}(8+t) < 5$, so $\mathcal{P}(t) = (\frac{2}{5}(8+t), 5)$.

It follows that $4 \notin \mathcal{P}(t)$ if $t \geq \frac{9}{2}$. If instead $t < \frac{9}{2}$ then $4 \in \mathcal{P}(t)$ holds iff $4 \in (\frac{2}{5}(8+t), 5)$, i.e. iff $4 > \frac{2}{5}(8+t)$, i.e. $2 > t$. Thus, when $X_0 = 4$, (B, S, X) is arbitrage-free iff $t < 2$.

(e) By plugging $t = 12$ into the above formula for $\mathcal{P}(t)$, or in eq. (2), gives that $\mathcal{P} = \mathcal{P}(12) = (5, 8)$. Thus, we see that the price 4 for the derivative X is too small to be fair. Thus, let $h \in \mathbb{R}$ the number of shares of S required to super-hedge X_1 while starting from the smallest possible initial capital (which is $\sup \mathcal{P} = 8$). Then the following is an arbitrage: hold $-h$ shares, and buy one derivative X at price 4, and put the remaining $8 - 4 = 4$ units of currency in the bank. This results in a final payoff which is $\geq 4(1 + r) = 5$, so it is an arbitrage. It only remains to determine h explicitly. To do so, one has to solve the system $8 \cdot \frac{5}{4} + h(S_1 - (1 + r)S_0) \geq X_1$, i.e.

$$\begin{cases} 10 + h(2 - \frac{5}{4} \cdot 4) \ \geq \ 1 \\ 10 + h(4 - \frac{5}{4} \cdot 4) \geq 12 \\ 10 + h(6 - \frac{5}{4} \cdot 4) \ \geq \ 8 \end{cases},$$

i.e. the system

$$\begin{cases} -3h \geq -9 \\ \ \ -h \ \geq \ 2 \\ \ \ \ \ h \geq -2 \end{cases},$$

which has solution $h = -2$.

Question 2

(Total: 20 marks)

SIMILARLY SEEN IN LECTURES AND PROBLEMS

In the framework of the N-period model where the bank account has interest rate r, and the stock price process $S = (S_n)_{n=0}^N$ is given by the binomial model with constant parameters $d, u, S_0 > 0$ which satisfy $0 < d < 1 + r < u$, let

$$Q_n := \sum_{i=0}^{n} S_i^2, \quad M_n := \sqrt{\frac{Q_n}{n+1}}, \quad n = 0, \dots, N,$$

so that M_n equals the 2-mean of $(S_i)_{i=0}^n$. For $a, b \in \mathbb{R}, a < b$, define T_n as the time spent by $M = (M_n)_{n=0}^N$ in the corridor $[a, b]$ up to time n, i.e.

$$T_n := \sum_{i=0}^{n} 1_{\{a \le M_i \le b\}}.$$

Consider the option which at maturity N pays the average time $\frac{T_N}{N+1}$ which M spends inside $[a, b]$, and denote with V_n its arbitrage-free price at time $n = 0, \dots, N$. As usual $\mathbb{Q}$ denotes the risk-neutral measure, $(X_i)_i$ the process of coin tosses X associated to S, and $\mathcal{F}$ the natural filtration of X. Prove all your assertions carefully or provide counter-examples.

(a) Is $C_{n+1} := \frac{S_{n+1}}{S_n}$ independent of $\mathcal{F}_n$ under $\mathbb{Q}$? (1 marks)

(b) Write an explicit formula for a function h_n which satisfies $M_{n+1} = h_n(S_{n+1}, M_n)$. (3 marks)

(c) Is (S, M) a $\mathbb{Q}$-Markov process? (3 marks)

(d) Is (S, T) a $\mathbb{Q}$-Markov process? (3 marks)

(e) Is (S, T, M) a $\mathbb{Q}$-Markov process? (3 marks)

(f) Which of the processes (S, T), (S, T, M) are such that, for every $n = 0, \dots, N$, V_n can be written as a function f_n of the value of the process at time n? (3 marks)

(g) For each process in item (f), for which V_n can be written in the way described there, write explicitly f_N, and an explicit formula to express f_n in terms of f_{n+1} for $n = 0, \dots, N-1$. (4 marks)

Solution:

(a) Notice that $\frac{S_{n+1}}{S_n}$ is independent (under the risk neutral measure $\mathbb{Q}$) of the filtration $\mathcal{F}_n$ generated by the first n coin tosses, since it only depends on the last coin toss ω_{n+1} and the coin tosses are independent under $\mathbb{Q}$ since

$$\mathbb{Q}(\omega_{n+1} = H | \omega_1, \dots, \omega_n) = \tilde{p} = \frac{(1+r) - d}{u - d} \tag{3}$$

does not depend on $\omega_1, \dots, \omega_n$. Since $\sigma(S_0, \dots, S_n) \subseteq \mathcal{F}_n$, this shows that $\frac{S_{n+1}}{S_n}$ is independent of $S_0, \dots, S_n$ under $\mathbb{Q}$.

(b) Clearly

$$Q_{n+1} = \sum_{i=0}^{n+1} S_i^2 = S_{n+1}^2 + \sum_{i=0}^{n} S_i^2 = S_{n+1}^2 + Q_n,$$

i.e.

$$Q_{n+1} = g_n(S_{n+1}, Q_n) \quad \text{for } g_n(x,q) := x^2 + q. \tag{4}$$

Using the formula for g_n, from $(n+1)M_n^2 = Q_n$ we get that

$$M_{n+1} = \sqrt{\frac{Q_{n+1}}{n+2}} = \sqrt{\frac{S_{n+1}^2 + Q_n}{n+2}} = \sqrt{\frac{S_{n+1}^2 + (n+1)M_n^2}{n+2}}$$

i.e.

$$M_{n+1} = h_n(S_{n+1}, M_n) \quad \text{for } h_n(x,m) := \sqrt{\frac{x^2 + (n+1)m^2}{n+2}}. \tag{5}$$

(c) (S, M) is easily seen to be Markov. Indeed, since C_{n+1} is independent of $\mathcal{F}_n$ under $\mathbb{Q}$, and S_n, M_n are $\mathcal{F}_n$-measurable, it follows from eq. (5) and $S_{n+1} = S_n C_{n+1}$, by applying the independence lemma, that $\mathbb{E}_n[f(S_{n+1}, M_{n+1})] = g(S_n, M_n)$ with

$$g(s,m) := \mathbb{E}[f(sC_{n+1}, h_n(sC_{n+1}, m))].$$

(d) (S, T) is not in general Markov. It is easy to guess this, looking at the expression $T_{n+1} = T_n + 1_{\{a \leq M_{n+1} \leq b\}}$, which suggests that to predict T_{n+1} one needs to know also M_n, not just S_n. To build a counter-example is easy, but a tad time consuming. For example, taking

$$S_0 = 4, \quad u = 1/d = 2,$$

we get the following tree (for which we only draw the branches that matters to us)

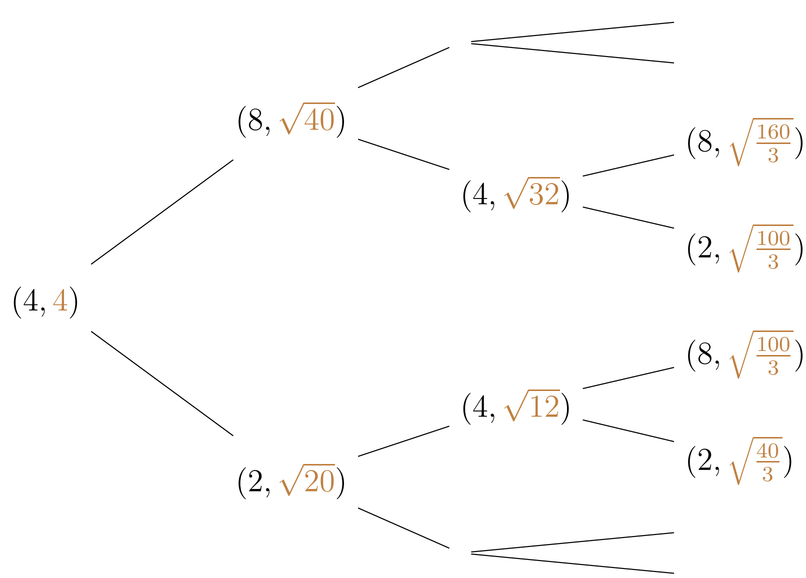

Figure 1: Partial view of the binary tree of the process (S, M)

Thus, choosing

$$a = 2, \quad b = 7$$

which clearly satisfy

$$a \leq \min(4, \sqrt{20}, \sqrt{12}, \sqrt{\frac{40}{3}}), \quad \max(4, \sqrt{40}, \sqrt{32}, \sqrt{\frac{100}{3}}) \leq b < \sqrt{\frac{160}{3}}$$

we find that while $(S_2, T_2)(HT) = (S_2, T_2)(TH) = (4, 3)$, we have that $T_3(HTH) = 3$ does not equal $T_3(HTT) = T_3(THH) = T_3(THT) = 4$, and thus for $p := \mathbb{Q}(X_n = H) = \frac{1/2}{3/2} = \frac{1}{3}$ we get that

$$\mathbb{E}_2^{\mathbb{Q}}(f(S_3, T_3))(HT) = pf(8,3) + (1-p)f(2,4)$$

differs from

$$\mathbb{E}_2^{\mathbb{Q}}(f(S_3, T_3))(TH) = pf(8,4) + (1-p)f(2,4)$$

for any f such that $f(8,3) \neq f(8,4)$. This shows that $\mathbb{E}_2(f(S_3, T_3))$ is not $\sigma(S_2, T_2)$-measurable, and so (S, T) is not Markov.

(e) Writing $T_{n+1} = T_n + 1_{\{a \leq M_{n+1} \leq b\}}$ and applying the independence lemma we that (S, T, M) is Markov, since $\mathbb{E}_n[c(S_{n+1}, T_{n+1}, M_{n+1})] = d(S_n, T_n, M_n)$ with

$$d(s, t, m) := \mathbb{E}[c(sC_{n+1}, t + 1_{\{a \leq h_n(sC_{n+1}, m) \leq b\}}, h_n(sC_{n+1}, m))]. \tag{6}$$

(f) We will denote f_n with v_n. Since (S, T, M) is Markov and $V_N = v_N(S_N, T_N, M_N)$ for $v_N(s, t, m) := t/(N+1)$, there exists v_n such that $V_n = v_n(S_n, T_n)$ for all n.

As (S, T) is not Markov, one would certainly expect that there does *not* exist v_n such that $V_n = v_n(S_n, T_n)$ for all n. To actually prove this fact, let us consider the numerical example in part (d), assume $r = 0$, and compute V_2 using the RNPF. Since $V_3 = T_3/4$, and as we already computed

ω	HTH	HTT	THH	THT
$T_3(\omega)$	3	1	1	1

$$\tilde{p} = \frac{1}{3}$$

we get that

$$\mathbb{E}_2^{\mathbb{Q}}(V_3)(HT) = \frac{1}{3} \cdot \frac{3}{4} + \frac{2}{3} \cdot 1 = \frac{11}{12}, \qquad \mathbb{E}_2^{\mathbb{Q}}(V_3)(HT) = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot 1 = 1$$

which shows that $V_2 = \mathbb{E}_2(f(S_3, T_3))$ is not a function of (S_2, T_2), since $(S_2, T_2)(HT) = (4, 3) = (S_2, T_2)(TH)$, whereas $V_2(HT) = \frac{11}{12} \neq 1 = V_2(TH)$.

(g) Let us compute explicitly v_n s.t. $V_n = v_n(S_n, T_n, M_n)$. By definition of V_N, $V_n = v_n(S_n, T_n, M_n)$ holds for $n = N$ with

$$v_N(s, t, m) = \frac{t}{N+1}. \tag{7}$$

Now, assume by inductive hypothesis that $V_k = v_k(S_k, T_k, M_k)$ holds for $k = n+1$ and let us show that it holds for $k = n$, and explicitly compute v_k. Using the RNPF and eq. (6) we get

$$V_n = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[V_{n+1}|\mathcal{F}_n] = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[v_{n+1}(S_{n+1}, T_{n+1}, M_{n+1})|\mathcal{F}_n] = v_n(S_n, T_n, M_n)$$

with

$$v_n(s,t,m) := \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[v_{n+1}(sC_{n+1}, t + 1_{\{a \le h_n(sC_{n+1},m) \le b\}}, h_n(sC_{n+1}, m))]$$

i.e.

$$\begin{aligned} v_n(s,t,m) :=& \frac{\tilde{p}}{1+r} v_{n+1}(su, t + 1_{\{a \le h_n(su,m) \le b\}}, h_n(su, m)) + \\ & \frac{1-\tilde{p}}{1+r} v_{n+1}(sd, t + 1_{\{a \le h_n(sd,m) \le b\}}, h_n(sd, m)) \end{aligned}$$

where $\tilde{p}$ is given by (3).

Question 3

(Total: 20 marks)

UNSEEN

Consider a market composed of a domestic bank account with interest rate $r_{£} = 1/3$, and foreign bank account with interest rate $r_{\$} = 1/4$, and an exchange rate E between £ and \$ (defined as the cost of one \$ in £) modelled as

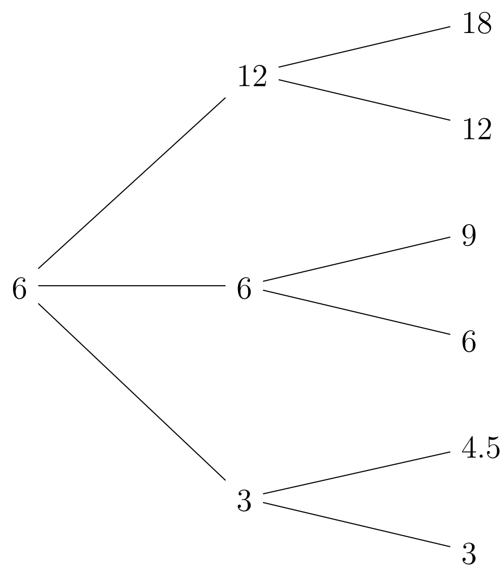

Figure 2: Tree of E.

Denote with D, F the values of the domestic and foreign bank accounts, i.e. the processes defined by $D_n = (1 + r_{£})^n$, $F_n = (1 + r_{\$})^n$ for all n.

A British investor wants to buy at time 0 a US treasury bond which states that at time 2 (s)he will receive $B_2 = \$1000$. Denote with P_n the price in £ at time n, and B_n the price in \$ at time n, of such a bond.

Answer the following questions and justify carefully with either proofs or counterexamples.

(a) State explicitly what is the underlying market in which the British investor is aiming to price the US bond, and draw the corresponding tree of prices of the traded assets. Is such a market arbitrage-free? (4 marks)

(b) Is the market complete? Is the US bond replicable? Is the exchange rate E replicable? (5 marks)

(c) Compute P_1 by converting first all prices to £, and then doing all the calculations in £. (3 marks)

(d) Write B_1 as given by the RNPF (Risk-Neutral Pricing Formula) using F as numeraire, and calculate explicitly the transition probability from time 1 to time 2 of the corresponding EMM (Equivalent Martingale Measure) $\mathbb{P}^F$. In other words, compute the EMM $\mathbb{P}^F$ for the American investor, and use it to determine B_1 by discounting using the value of the American bank account and applying the RNPF. (3 marks)

(e) Answer the same question as in part (c), but doing first the calculations in \$, and then converting the price to £. (3 marks)

(f) Compute P_0. (2 marks)

Solution:

(a) Since the value in £ of investing in \$ is $W = EF$, the market is (D, W) depicted in the following tree

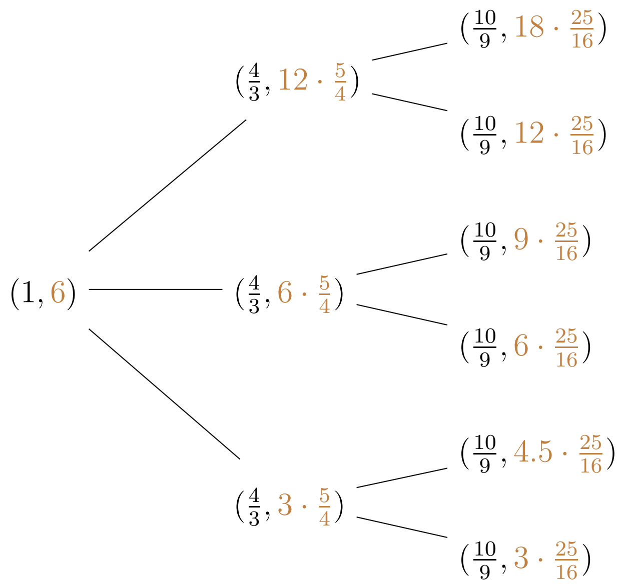

Figure 3: Tree of (D, EF).

Such market is arbitrage-free, because it is composed by a bank account D and *one* risky asset W which at every time step $n = 0, 1$ satisfy the conditions $d_n < 1 + r_£ < u_n$, where d_n/u_n is the smallest/largest values of W_{n+1}/W_n: indeed $D_1 = 1 + r_£ = \frac{4}{3}$ and

$$d_0 = \frac{1}{2} \cdot \frac{5}{4}, \quad u_0 = 2 \cdot \frac{5}{4}, \quad d_1 = \frac{5}{4}, \quad u_1 = \frac{3}{2} \cdot \frac{5}{4}.$$

This implies that each one-period sub-model is arbitrage-free, and so the 2-period model is arbitrage-free.

(b) The market is not complete, since there are only 2 traded assets, but 3 possible outcomes at time 1, so the replication equation results in 3 equations in two unknowns and is thus not always solvable. The US bond is replicable: to replicate it, one can simply invest $\$\frac{1000}{(1+\frac{1}{4})^2}$ in the US bank account. The exchange rate E is not replicable (which also shows that the market is not complete), since the replication equation $kD_1 + hW_1 = E_1$ reads

$$\begin{cases} \frac{4}{3}k + 12 \cdot \frac{5}{4}h = 12 \\ \frac{4}{3}k + \ \ 6 \cdot \frac{5}{4}h = \ \ 6 \\ \frac{4}{3}k + \ \ 3 \cdot \frac{5}{4}h = \ \ 3 \end{cases},$$

which using the variables $k' := \frac{4}{3}k, h' := \frac{5}{4}h'$ becomes

$$\begin{cases} k' + 12 \cdot h' = 12 \\ k' + \ \ 6 \cdot h' = \ \ 6 \\ k' + \ \ 3 \cdot h' = \ \ 3 \end{cases},$$

which is easily seen to have no solution.

(c) The value in £ at time 2 of the US bond is $P_2 = 1000E_2$. The RNPF in £ is $P_n = \frac{\mathbb{E}^{\mathbb{Q}}[P_{n+1}|\mathcal{F}_n]}{1+r_£}$; let us use it to compute the possible values of P_1, P_0. Since between time 1 and 2 the model evolves like a binomial one, it is easy to compute P_1 by computing the transition probability

$$\tilde{p}_1 := \mathbb{Q}(\{\frac{W_2}{W_1} = u_1\}|\mathcal{F}_1) = \mathbb{Q}(\{\frac{E_2}{E_1} = \frac{3}{2}\}|\mathcal{F}_1) = \frac{1 + r_£ - d_1}{u_1 - d_1} = \frac{\frac{4}{3} - \frac{5}{4}}{\frac{5}{4}\cdot\frac{3}{2} - \frac{5}{4}} = \frac{2}{15},$$

and using it to evaluate

$$X_1 = 1000 \cdot \frac{3}{4}\left(\frac{2}{15}\cdot\begin{pmatrix}18\\9\\4.5\end{pmatrix} + (1 - \frac{2}{15})\cdot\begin{pmatrix}12\\6\\3\end{pmatrix}\right) = 2400 \cdot \begin{pmatrix}4\\2\\1\end{pmatrix}$$

(d) Since the value of the bond in £ is P_n, its value in \$ is $B_n = P_n/E_n$. The RNPF in \$ is $B_1 = \mathbb{E}^{\mathbb{P}^F}[\frac{B_2}{1+r_\$}|\mathcal{F}_1]$. The probability $\mathbb{P}^F$ is the EMM for the American investor. Since the values in \$ of a £ investment in a British bank is $Y := \frac{D}{E}$, the market from the point of view of an American investor is $(F, \frac{D}{E})$, and so $\mathbb{P}^F$ is identified by the value of the transition probability

$$\tilde{p}^F := \mathbb{P}^F(\{\frac{Y_2}{Y_1} = u_1^F\}|\mathcal{F}_1) = \mathbb{P}^F(\{\frac{E_1}{E_2} = 1\}|\mathcal{F}_1) = \frac{1 + r_\$ - d_1^F}{u_1^F - d_1^F}.$$

where d_n^F/u_n^F are the smallest/largest values of Y_{n+1}/Y_n. Thus

$$u_1^F = \frac{4}{3}, \quad d_1^F = \frac{8}{9}, \quad \tilde{p}^F = \frac{\frac{5}{4} - \frac{8}{9}}{\frac{4}{3} - \frac{8}{9}} = \frac{13}{16}.$$

(e) Since B_2 equals the constant \$1000, the RNPF trivially gives $B_1 = B_2/(1+r_\$) = \frac{4}{5}\cdot 1000 = 800$ (without even the need to explicitly compute $\mathbb{P}^F$)), and so

$$P_1 = B_1E_1 = 800 \cdot \begin{pmatrix}12\\6\\3\end{pmatrix} = 2400 \cdot \begin{pmatrix}4\\2\\1\end{pmatrix}.$$

(f) Since B_1 equals the constant \$800, the RNPF trivially gives $B_0 = B_1/(1 + r_\$) = \frac{4}{5}\cdot 800 = 640$ (without even the need to explicitly compute $\mathbb{P}^F$)), and so $P_0 = B_0E_0 = 6\cdot 640 = 3840$.

Question 4 (Total: 20 marks)

SIMILARLY SEEN IN PROBLEMS

Consider a 2-period binomial model with a bank account with interest rate $r = 0$, and a stock whose price S is described by the following binomial tree.

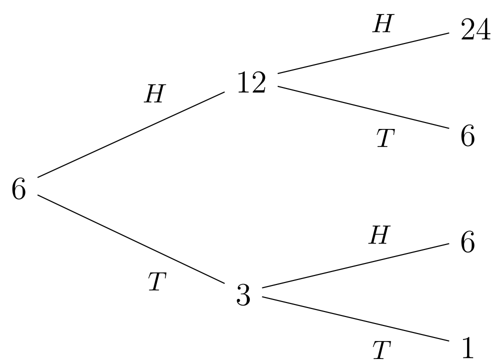

Figure 4: Tree of S.

As usual $(X_n)_n$ denotes the process of coin tosses X which generates $(S_n)_n$, and $\mathcal{F} = (\mathcal{F}_n)_n$ the natural filtration of X. Consider a Bermudan call option with strike price $K = 4$ and with possible exercise dates 1 and 2. In other words, consider the derivative which gives its owner the right to decide at time 1 whether (s)he wants to immediately get paid $(S_1 - K)^+$, or whether to wait and get paid the amount $(S_2 - K)^+$ at time 2. Answer the following questions and justify carefully with either proofs or counterexamples.

(a) Prove that the above market is arbitrage-free. (2 marks)

(b) Draw the binomial tree of (I, C), where I is the intrinsic value I the call option with strike $K = 4$ (i.e. the process $I_n := (S_n - K)^+, n = 0, 1, 2$), and $C = (C_n)_{n=0,1,2}$ is the arbitrage-free price of the call exercised at time 2, i.e. of receiving the payoff $(S_2 - K)^+$ at time 2. (5 marks)

(c) Compute the value V_1 of the Bermudan call at time 1. (*Hint: clearly V_1 equals the value at time 1 of the American call with strike K which has not been exercised at time 0*). (3 marks)

(d) If the buyer of the Bermudan chooses to exercise the derivative at the following (stopping) time (1 marks)

ω	HH	HT	TH	TT
$\tau_1(\omega)$	1	1	2	2

(s)he receives the payment $(S_{\tau_1} - K)^+$ at time τ_1. Determine the capital $V_1(\tau_1)$ required at time 1 to replicate such payment.

(e) How many stopping times $\tau_1, \tau_2, \ldots$ with values in $\{1, 2\}$ are there? List all such stopping times $(\tau_i)_{i \in I}$, along with their values, on a table (as done above for τ_1). Explain why there exist no other stopping times with values in $\{1, 2\}$. (4 marks)

(f) For each $\tau_i, i \in I$, compute the capital $V_1(\tau_i)$ required at time 1 to replicate the payment of $(S_{\tau_i} - K)^+$ at time τ_i. Then compute $\bar{V}_1 := \max_{i \in I} V_1(\tau_i)$. (3 marks)

(g) Prove that $(S_\sigma - K)^+ 1_{\{\sigma \le n\}}$ is $\mathcal{F}_n$-measurable for all $n = 0, 1, 2$ and stopping time σ with values in $\{1, 2\}$. (2 marks)

Solution:

(a) Since there is only one risky asset, this follows from the inequalities $d_n < 1 + r_n < u_n$ satisfied by the interest rate and the down and up factors, since $r_0 = r_1 = r = 0$ and

$$d_0 = \frac{1}{2}, \quad u_0 = 2, \quad d_1(H) = \frac{1}{2}, \quad d_1(T) = \frac{1}{3}, \quad u_1(H) = 2, \quad u_1(T) = 2.$$

(b) Plugging the values of S into the formula $I_n = (S_n - K)^+$ gives its values. As for C, we have that $C_2 := I_2$, while C_1 and C_0 are given by the RNPF. Since $\tilde{p}_n := \frac{(1+r_n)-d_n}{u_n - d_n}$ we get

$$\tilde{p}_0 = \tilde{p}_1(H) = \frac{1 - \frac{1}{2}}{2 - \frac{1}{2}} = \frac{1}{3}, \quad \tilde{p}_1(T) = \frac{1 - \frac{1}{3}}{2 - \frac{1}{3}} = \frac{2}{5},$$

and using $r = 0$ the RNPF gives

$$C_1(H) = \tilde{p}_1(H) C_2(HH) + (1 - \tilde{p}_1(H)) C_2(HT) = \frac{1}{3} \cdot 20 + \frac{2}{3} \cdot 2 = \frac{24}{3} = 8$$

and analogously

$$C_1(T) = \tilde{p}_1(T) C_2(TH) + (1 - \tilde{p}_1(T)) C_2(TT) = \frac{2}{5} \cdot 2 + \frac{3}{5} \cdot 0 = \frac{4}{5},$$

and finally

$$C_0 = \tilde{p}_0 C_1(H) + (1 - \tilde{p}_0) C_1(T) = \frac{1}{3} \cdot 8 + \frac{2}{3} \cdot \frac{4}{5} = \frac{16}{5},$$

so we can draw the following tree

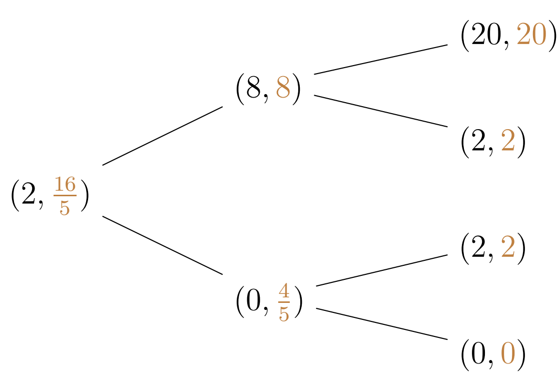

Figure 5: Tree of (I, C).

(c) Since $V_1 = \max(I_1, C_1)$ we get

$$V_1(H) = \max(8, 8) = 8, \quad V_1(T) = \max(0, \frac{4}{5}) = \frac{4}{5}.$$

(d) Trivially since $\tau_1(H) = 1$ we get $V_1(\tau_1)(H) = (S_1 - K)_1^+(H) = I_1(H) = 8$. Since $\tau_1(T) = 2$ we get $V_1(\tau_1)(T) = C_1(H) = \frac{4}{5}$.

(e) If σ is a stopping time then $\{\sigma = 1\}$ must be $\mathcal{F}_1 = \sigma(X_1)$ measurable. Such then is also $\{\sigma = 2\}$, since it is the complement of $\{\sigma = 1\}$. Thus, σ is constant on the set $\{HH, HT\}$ and on the set $\{TH, TT\}$, which are the atoms of $\sigma(X_1)$. So, there are only 4 possible stopping times with values in $\{1, 2\}$:

ω	HH	HT	TH	TT
$\tau_1(\omega)$	1	1	2	2
$\tau_2(\omega)$	2	2	1	1
$\tau_3(\omega)$	1	1	1	1
$\tau_4(\omega)$	2	2	2	2

(f) Given a stopping time σ, on the set $\{\sigma = 1\}$ we have $V_1(\sigma) = I_1$ and on its complement $\{\sigma = 2\}$ we have $V_1(\sigma) = C_1$. Thus we get

ω	H	T
$V(\tau_1)(\omega)$	8	$\frac{4}{5}$
$V(\tau_2)(\omega)$	8	0
$V(\tau_3)(\omega)$	8	0
$V(\tau_4)(\omega)$	8	$\frac{4}{5}$

and so $\bar{V}_1 = V_1$, as it was clear also from the interpretation of each stopping time being a choice of a time at which to exercise, and so $\bar{V}_1 = V_1$ means that the price of the Bermudan option equals the value of the best possible choice of time at which to exercise.

(g) This follow from

$$(S_\sigma - K)^+ 1_{\{\sigma \le n\}} = \sum_{k=0}^{n} (S_\sigma - K)^+ 1_{\{\sigma = k\}} = \sum_{k=0}^{n} (S_k - K)^+ 1_{\{\sigma = k\}}$$

and the fact that S_k and $\{\sigma = k\}$ are $\mathcal{F}_k$-measurable (since S is adapted and σ a stopping time).

Question 5

(Total: 20 marks)

SIMILARLY SEEN IN PROBLEMS

On the sample space $\Omega = \{\omega_i\}_{i=1,\dots,3}$ endowed with some probability $\mathbb{P}$ s.t. $\mathbb{P}(\omega_i) > 0$ for all i, consider a one-period market model where the bank account has interest rate $r = 0$ and there are some stocks with price $S = (S^1, \dots, S^m)$. Recall that if one short-sells some shares at time t it means that (s)he borrows them from a lender and then sells them at time t, and then at time $t+1$ (s)he has to buy them and return them to the lender. Unlike normally done in class, in this exercise we consider the fact that the lender will in reality charge a short-selling fee $f \geq 0$, proportional to the value of the shares lent, to be paid immediately (i.e. at time t); we assume that this fee is the same for each stock $S^1, \dots, S^m$. Denote with V_t^{x,h^+,h^-} the value of a portfolio at times $t = 0, 1$, as a function of the following variables: the initial capital $x \in \mathbb{R}$, and $h^+, h^- \in \mathbb{R}_+^m$, where h_j^+ (resp. h_j^-) represents the number of shares of type $j = 1, \dots, m$ bought (resp. sold) at time 0. For notational convenience write $k := (h^+, h^-)$ and $V_t^{x,k} = V_t^{x,h^+,h^-}$. You can use that the formula for $V_t^{x,k}$ is

$$V_1^{x,k} = (h^+ - h^-) \cdot S_1 + (x - (h^+ - (1-f)h^-) \cdot S_0)(1+r), \tag{8}$$

and that the following version of the FTAP (Fundamental Theorem of Asset Pricing) holds: this model is arbitrage-free if and only if the set

$$\mathcal{M}^f := \mathcal{M}^f(S) := \{\mathbb{Q} \text{ is a probability} \sim \mathbb{P} \text{ s.t. } (1+r)(1-f)S_0^j \leq \mathbb{E}^{\mathbb{Q}}[S_1^j] \leq (1+r)S_0^j \quad \forall j = 1, \dots, m\} \tag{9}$$

is not empty. As usual we identify $\mathbb{Q}$ with the vector $(q_i)_i \in \mathbb{R}^3$ s.t. $q_i = \mathbb{Q}(\{\omega_i\})$, $i = 1, 2, 3$, and so consider $\mathcal{M}^f(S)$ as a subset of $\mathbb{R}^3$. Assume now that $m = 1$ and the stock S has prices $S_0 = 4$,

ω	ω_1	ω_2	ω_3
$S_1(\omega)$	4	3	2

(a) Define what an arbitrage is in such a market, *using formulas*. (3 marks)

(b) If $f = 0$, is there an arbitrage in this market? If so, find one explicitly. (3 marks)

Assume from now on that $f = \frac{1}{8}$.

(c) Identify the set $\mathcal{M}^f(S) \subseteq \mathbb{R}^3$ using algebra, and then describe it geometrically and draw it. (6 marks)

(d) Compute the set of arbitrage-free prices for the derivative X with payoff (8 marks)

ω	ω_1	ω_2	ω_3
$X_1(\omega)$	6	4	2

,

assuming that the fee for short-selling X is also f. *Hint: use the FTAP and the definition of arbitrage.*

Solution:

(a) $k = (h^+, h^-) \in \mathbb{R}_+^m \times \mathbb{R}_+^m$ is an arbitrage if $V_1^{0,k} \geq 0$ and $V_1^{0,k} \neq 0$.

(b) Yes, there is arbitrage: this is the trinomial market with the down and up factors being

$$d = \frac{2}{4} = \frac{1}{2}, \quad u = \frac{4}{4} = 1,$$

and since $r = 0$, the market does not satisfy $d < 1 + r < u$ (indeed $1 + r = 1 = u$). Since this happens because the up factor u is too low (i.e. the stock is underperforming), to build an arbitrage we should short-sell the stock. Thus, we can take any $h < 0$, and to fix ideas we take $h = -1$ (i.e. $h^+ = 0, h^- = 1$), which is indeed an arbitrage since

$$V_1^{0,h} = h(S_1 - S_0(1 + r)) = (-1)(0, -1, -2) = (0, 1, 2) \geq 0, \quad (0, 1, 2) \neq 0.$$

(c) Using eq. (9) we find that $\mathbb{Q} \in \mathcal{M}^f(S)$ iff $q_i := \mathbb{Q}(\omega_i)$ satisfies

$$\begin{cases} 1 = q_1 + q_2 + q_3, \\ (1 - f)4 \leq 4q_1 + 3q_2 + 2q_3 \leq 4, \\ q_i > 0, \ i = 1, 2, 3 \end{cases}$$

Substituting $q_3 = 1 - q_1 - q_2$ we get

$$M^f = \{(q_1, q_2, 1 - q_1 - q_2) \in \mathbb{R}^3 : (q_1, q_2) \in P^f\},$$

with

$$P^f := \{(q_1, q_2) \in (0, \infty) : 2 - 4f \leq 2q_1 + q_2 \leq 2, 1 - q_1 - q_2\}.$$

Thus, P^f is described geometrically as the intersection of three sets:

(a) the closed strip C^f between the two lines $2q_1 + q_2 = 2$ and $2q_1 + q_2 = 2 - 4f$,

(b) the interior $Q := \{(q_1, q_2) : q_1 > 0, q_2 > 0\}$ of the first quadrant $\mathbb{R}_+^2$,

(c) the open half-plane $H := \{(q_1, q_2) : q_1 + q_2 < 1\}$.

If $f = \frac{1}{8}$ then

$$P^f = \{(q_1, q_2) \in (0, \infty) : 2q_1 + q_2 \geq \frac{3}{2}, q_1 + q_2 < 1\} \tag{10}$$

equals the triangle with vertices $(\frac{1}{2}, \frac{1}{2}), (\frac{3}{4}, 0), (1, 0)$, minus the two edges with $(1, 0)$ as one vertex. Thus, $M^f \subseteq \mathbb{R}^3$ is the triangle with vertices $(\frac{1}{2}, \frac{1}{2}, 0), (\frac{3}{4}, 0, \frac{1}{4}), (1, 0, 0)$, minus the two edges with $(1, 0, 0)$ as one vertex.

Out of curiosity, let us now describe more generally what happens for every possible value of $f \in [0, 1]$. If $f \in [\frac{1}{2}, 1]$, the set $C^f \cap \mathbb{R}_+^2$ is the triangle with vertices $(0, 2), (0, 0), (1, 0)$, so $P^f = C^f \cap Q \cap H$ equals the interior of the triangle with vertices $(0, 1), (0, 0), (1, 0)$. If

$f \in [0, \frac{1}{2})$, then $C^f \cap \mathbb{R}^2_+$ is the trapezoid with vertices $(0,2), (0, 2-4f), (1-2f, 0), (1, 0)$. Thus, $C^f \cap Q$ is the trapezoid $C^f \cap \mathbb{R}^2_+$ minus its two edges on the axis (the segment from $(0,2)$ to $(0, 2-4f)$, and the segment from $(1-2f, 0)$ to $(1,0)$). If $f \in (\frac{1}{4}, \frac{1}{2})$, $P^f = C^f \cap Q \cap H$ equals the trapezoid with vertices $(0,1), (0, 2-4f), (1-2f, 0), (1, 0)$, minus all the three edges other than the edge from $(0, 2-4f)$ to $(1-2f, 0)$. If $f \in (0, \frac{1}{4}]$, $P^f = C^f \cap Q \cap H$ equals the triangle with vertices $(1-4f, 4f), (1-2f, 0), (1, 0)$, minus the two edges with $(1,0)$ as one vertex. If $f = 0$, P^f is empty.

(d) If we enlarge the market as to include X, with payoff X_1 at time 1 and sold at price X_0 at time 0, then market (B, S, X) (where B is the bank account, i.e. $B_0 = 1, B_1 = (1+r)$) is arbitrage-free iff $\mathcal{M}^f(B, S, X) \neq \emptyset$, as it follows applying the version of the FTAP stated when we defined $\mathcal{M}^f$. Since $\mathcal{M}^f(B, S) \neq \emptyset$, and $\mathbb{Q} \in \mathcal{M}^f(B, S)$ also belongs to the smaller set $\mathcal{M}^f(B, S, X)$ iff

$$\mathbb{E}^{\mathbb{Q}}[\frac{X_1}{(1+r)}] \leq X_0 \leq \mathbb{E}^{\mathbb{Q}}[\frac{X_1}{(1+r)(1-f)}], \tag{11}$$

we get that X_0 is an arbitrage-free prices for X iff eq. (11) holds for some $\mathbb{Q} \in \mathcal{M}^f(B, S)$, i.e., eq. (11) is the Risk Neutral Pricing Formula in this setting.

Substituting the values of X_1 and taking $q_3 = 1 - q_1 - q_2$ we find that if $\mathbb{Q} \in \mathcal{M}^f$ then

$$\mathbb{E}^{\mathbb{Q}} X_1 = 6q_1 + 4q_2 + 2(1 - q_1 - q_2) = 4q_1 + 2q_2 + 2,$$

and so the RNPF eq. (11) gives that the set of arbitrage-free prices of X is

$$\mathcal{AFP}(X_1) = \bigcup_{(q_1, q_2) \in P^f} [4q_1 + 2q_2 + 2, \frac{8}{7}(4q_1 + 2q_2 + 2)].$$

To compute this set, notice that $4q_1 + 2q_2$ equals the dot product of (q_1, q_2) with $(4, 2)$, and thus the function $(q_1, q_2) \mapsto 4q_1 + 2q_2 + 2$ on P^f achieves its minimum 5 on the open segment from $(\frac{1}{2}, \frac{1}{2})$ to $(\frac{3}{4}, 0)$; and its supremum is not attained (because $(1,0) \notin P^f$), equals 6, and is achieved as (q_1, q_2) converges to $(1, 0)$. Thus, we can conclude that

$$\mathcal{AFP}(X_1) = \left[5, \frac{48}{7}\right).$$

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a separate pdf file with your email.

ExamModuleCode	**QuestionNumber**	**Comments for Students**
MATH60012	1	Most students got very good marks in this question; the only part which proved hard for many was item (e)
MATH60012	2	Q2, which was similar to the material covered in class, stated: Prove all your assertions carefully or provide counter-examples. A recurring mistake was that students did not provide counter-examples when their answer was negative. In addition to this, some students answered correctly but with missing details.
MATH60012	3	Part (a) of the question was answered correctly by the majority of students. Part (b) was attempted by many but only few answered it correctly. Only a small percentage managed to go beyond part (b).
MATH60012	4	This was undoubtedly the question which the students found the hardest, as many students seemed confused and managed only to work out items (a),(b), and sometimes (e)