Applied Probability Progress Test 1 **29 October 2019**

Question 1: Let $\{X_n\}_{n\in\{0,1,2,\dots\}}$ denote a discrete-time homogeneous Markov chain with transition matrix $\mathbf{P} = (p_{ij})_{i,j\in E}$ for a state space $E \subseteq \mathbb{Z}$.

(a) (1 point) Define what it means that state j is *accessible* from state i (for $i, j \in E$).

(b) (1 point) Define what it means that states i and j *communicate* (for $i, j \in E$).

(c) (2 points) Describe (in about two to three sentences) why it can be useful to find the communicating classes of a Markov chain.

Solution:

(a) We say that state j is **accessible** from state i, written $i \to j$, if the chain may ever visit state j, with positive probability, starting from i. In other words, $i \to j$ if there exist $m \geq 0$ such that $p_{ij}(m) > 0$.

(b) States i and j **communicate** if $i \to j$ and $j \to i$, written $i \leftrightarrow j$, i.e. if there exists $n, m \geq 0$ such that $p_{ij}(n) > 0$ and $p_{ji}(m) > 0$.

(c) Various (good!) reasons exist! For example, you might want to determine which states are (positive/null) recurrent and which ones are transient. If you determine the communicating classes first, you only need to check one state in each class, and then you can conclude from a theorem from lectures that all states within the same class inherit the same property. I.e. if one state in a communicating class is recurrent, then all states within the same communicating class are recurrent, too.

Question 2: A die is rolled repeatedly. We denote by Y_n the number of fours in n rolls for $n \in \mathbb{N}$.

(a) (3 points) Show that $Y = (Y_n)_{n\in\mathbb{N}}$ is a Markov chain.

(b) (3 points) State the transition probabilities of the Markov chain Y.

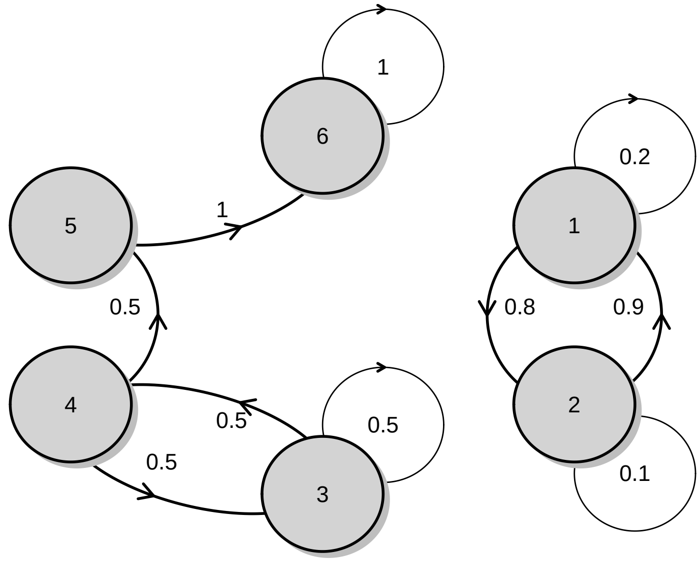
Solution:

(a) Define the random variable $X_n = 1$ if we roll a four in the nth experiment and $X_n = 0$ otherwise. Then $Y_n = \sum_{i=1}^{n} X_n$. We can write $Y_n = Y_{n-1} + X_n$ for $n \in \mathbb{N}$. By the set-up of the experiment, we have that all X_n ($n \in \mathbb{N}$) are independent, hence X_n and Y_{n-1} are independent as well. Hence, Y_n depends on $(Y_1, \dots, Y_{n-1})$ only through Y_{n-1} and X_n is independent of $X_1, \dots, X_{n-2}$, hence the Markov property holds.

(b) The state space of the Markov chain is given by $E = \mathbb{N}$ and the transition probabilities satisfy

$$p_{ij} = \begin{cases} \frac{5}{6}, & \text{if } j = i, \\ \frac{1}{6}, & \text{if } j = i+1, \\ 0, & \text{otherwise} . \end{cases}$$

Question 3: Let $X = (X_n)_{n\in\{0,1,2,\dots\}}$ denote a discrete-time, homogeneous Markov chain with state space $E = \{1, 2, 3, 4, 5, 6\}$ and transition diagram given by



(a) (2 points) Find the transition matrix $\mathbf{P} = (p_{ij})_{i,j\in E}$.

(b) (2 points) Determine the communicating classes.

(c) (2 points) For each class, specify whether the class is transient, positive recurrent or null recurrent and justify your answer.

(d) Suppose that the Markov chain starts (at time 0) in state 1 with probability 1/2 and it starts in state 2 with probability 1/2. Find the following probabilities:

i. (1 point) $\mathbb{P}(X_1 = 0.5)$.

ii. (1 point) $\mathbb{P}(X_0 = 1, X_1 = 1)$.

iii. (2 points) $\mathbb{P}(X_0 = 1, X_2 = 1)$.

Solution:

(a) The transition matrix is given by

$$\mathbf{P} = \begin{pmatrix} \frac{1}{5} & \frac{4}{5} & 0 & 0 & 0 & 0 \\ \frac{9}{10} & \frac{1}{10} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) There are four communicating classes: $C_1 := \{1, 2\}, C_2 := \{6\}$, $T_1 := \{3, 4\}, T_2 := \{5\}$.

(c) C_1, C_2 are finite and closed and hence positive recurrent. T_1, T_2 are not closed and hence transient.

(d) i. $\mathbb{P}(X_1 = 0.5) = 0$ since $0.5 \notin E$.

ii. $\mathbb{P}(X_0 = 1, X_1 = 1) = \mathbb{P}(X_1 = 1|X_0 = 1)\mathbb{P}(X_0 = 1) = \frac{1}{5} \cdot \frac{1}{2} = \frac{1}{10}$.

iii. The initial marginal distribution of the Markov chain is given by $\boldsymbol{\nu}^{(0)} = (1/2, 1/2, 0, 0, 0, 0)$. Then using the law of total probability and a result from the problem class, we have

$$\begin{aligned}
\mathbb{P}(X_0 = 1, X_2 = 1) &= \sum_{i=1}^{6} \mathbb{P}(X_0 = 1, X_1 = i, X_2 = 1) \\
&= \sum_{i=1}^{6} \mathbb{P}(X_2 = 1 | X_1 = i)\mathbb{P}(X_1 = i | X_0 = 1)\mathbb{P}(X_0 = 1) \\
&= \sum_{i=1}^{2} p_{i1} p_{1i} \nu_1^{(0)} \\
&= \frac{1}{5} \cdot \frac{1}{5} \cdot \frac{1}{2} + \frac{9}{10} \cdot \frac{4}{5} \cdot \frac{1}{2} = \frac{19}{50}.
\end{aligned}$$