Mathematical Logic (MATH6/70132; P65)
Notes on Solutions, Problem Sheet 1

**1.** Let $p$ denote 'I will pass this course,' $q$ denote 'I do my homework regularly' and $r$ denote 'I am lucky.'

(a) I will pass this course only if I do my homework regularly: $(p \to q)$.

(b) Doing homework regularly is a necessary condition for me to pass this course: $(p \to q)$

(c) If I do my homework regularly and I do not pass this course then I am unlucky:

$((q \wedge (\neg p)) \to (\neg r))$.

(d) If I do not do my homework regularly and I pass this course then I am lucky:
$(((\neg q) \wedge p) \to r)$.

**2.** Let $v$ be a valuation. Recall that $v(\phi \leftrightarrow \psi)$ is T precisely when $v(\phi) = v(\psi)$. Thus if $\phi$ and $\psi$ are logically equivalent $v(\phi \leftrightarrow \psi) = T$. Conversely, if $v(\phi \leftrightarrow \psi) = T$ then $v(\phi) = v(\psi)$ so if this holds for all $v$, $\phi$ and $\psi$ are logically equivalent. (You could also express this argument using 'truth tables' instead of the slightly more formal notion of a valuation).

**3.** This is like 1.1.5 in the notes. We can give a formal argument by induction, or we can argue informally as follows. Any assignment of truth values to the propositional variables in $\eta_1, \ldots, \eta_n$ assigns a truth value $v(\eta_i)$ to $\eta_i$. But the truth value this assigns to $\theta$ (respectively $\chi$) is the same as the value assigned to $\phi$ (respectively $\psi$) by giving $p_i$ the truth value $v(\eta_i)$. As $\phi$ and $\psi$ are logically equivalent, it follows that this is the same truth value for $\psi$ and $\phi$, hence also for $\theta$ and $\chi$.

Another way to do this is to use question 2 and 1.1.5.

**4.** (a) The formula $\phi : ((p \to q) \to ((\neg p) \wedge q))$ has truth table

| $p$ | $q$ | $\phi$ |
|---|---|---|
| T | T | F |
| T | F | T |
| F | T | T |
| F | F | F |

So the disjunctive normal form is $((p \wedge (\neg q)) \vee ((\neg p) \wedge q))$.

(b) $(\neg((p \to q) \to r))$. This has truth value T when $((p \to q) \to r)$ has value F. This happens when $r$ has value $F$ and $(p \to q)$ has value T. So the d.n.f. is (omitting brackets)

$$(p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge \neg r).$$

**5.** (a) Observe that $(\neg\phi)$ is logically equivalent to $(\phi \mid \phi)$ and $(\phi \wedge \psi)$ is logically equivalent to $((\phi \mid \psi) \mid (\phi \mid \psi))$.

(b) Suppose we have a binary connective $*$. There are 16 possibilities for the truth table for $*$. Half of these give $p * q$ truth value T when $p$, $q$ have truth value T, and such a connective cannot express $(\neg p)$ (any formula involving only such a $*$ would always take truth value T when the propositional variables in it all took value T). Of the remaining 8, half give $p * q$ truth value F when $p$ and $q$ have truth value F, and these also cannot express $(\neg p)$. Two of the remaining cases are $\mid$ and $\downarrow$, and we know that these are adequate. It remains to eliminate the other two possibilities. But in these cases $(p * q)$ is logically equivalent either to $(\neg p)$ or to $(\neg q)$, and clearly this cannot be adequate. (To see this more formally, suppose in one of these cases that $\phi$ is a formula obtained using this connective and propositional variables from $p_1, \ldots, p_n$. Then $\phi$ is logically equivalent to $p_i$ or $(\neg p_i)$, for some $i \leq n$. Thus there are at most $2n$ possibilities for the truth function of $\phi$.)

**6.** There are $2^{2^n}$ truth functions of $n$ variables (1.1.7 in the notes). Half of these take value $T$ at $(T, T, \ldots, T)$, so the number of such truth functions of $n$ variables is $2^{2^n - 1}$. (Alternaltively, argue as in the proof of 1.1.7, noting that to specify the function we need to say what value it has at the remaining $2^n - 1$ inputs apart from $(T, \ldots, T)$.)

If $f(F, \ldots, F) = T$ then $f$ cannot be expressed as the truth function of a formula constructed using connectives $\wedge, \vee$ as such a formula always takes value $F$ when the variables have value $F$.

**7.** (i) Either construct a truth table or argue as follows. If a valuation $v$ gives the formula truth value $F$, then we have $v(((p_3 \to p_2) \to p_1)) = F$ and $v((p_1 \to ((\neg p_2) \to p_3)) = T$. From the first of these, $v(p_1) = F$, $v(p_3 \to p_2) = T$, and any such valuation also satisfies $v((p_1 \to ((\neg p_2) \to p_3)) = T$. Thus the possible values for $(p_1, p_2, p_3)$ which make the original formula $F$ are

$$(F, T, T), (F, T, F), (F, F, F).$$

$\neg\theta$ has truth value $T$ iff $\theta$ has truth value $F$, so a formula in dnf which is logically equivalent to $\neg\theta$ can be obtained from a disjunction of formulas which are true precisely at the above values, ie

$$((\neg p_1) \wedge p_2 \wedge p_3) \vee ((\neg p_1) \wedge p_2 \wedge (\neg p_3)) \vee ((\neg p_1) \wedge (\neg p_2) \wedge (\neg p_3)).$$

(ii) We take $\chi$ to be the conjunction $p_1 \wedge (\neg p_2) \wedge p_3$. This has truth value $T$ iff each of the conjuncts has value $T$: ie iff $p_1, p_2, p_3$ have the indicated values.

**8.**

1. $((\neg\psi) \to ((\neg\phi) \to (\neg\psi)))$ (Axiom A1)
2. $(((\neg\phi) \to (\neg\psi)) \to (\psi \to \phi))$ (Axiom A3)

Denote this formula by $\chi$

3. $(\chi \to ((\neg\psi) \to \chi))$ (Axiom A1)
4. $((\neg\psi) \to \chi)$ (2, 3 and Modus Ponens)

Denote this formula by $\theta$

5. $(\theta \to (((\neg\psi) \to ((\neg\phi) \to (\neg\psi))) \to ((\neg\psi) \to (\psi \to \phi))))$ (Axiom A2)
6. $((\neg\psi) \to ((\neg\phi) \to (\neg\psi))) \to ((\neg\psi) \to (\psi \to \phi))$ (4, 5 and Modus Ponens)
7. $((\neg\psi) \to (\psi \to \phi))$ (1, 6 and Modus Ponens).