

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)  
May-June 2020

This paper is also taken for the relevant examination for the  
Associateship of the Royal College of Science

**Probability Theory**

Date: 2nd June 2020

Time: 09.00am - 11.30am (BST)

Time Allowed: 2 Hours 30 Minutes

Upload Time Allowed: 30 Minutes

**This paper has 5 Questions.**

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

**SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD  
INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION  
NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.**

1. (1.a)

5 pts

seen

Give the definition of a random variable on a probability space.

(1.b) Explain giving reasons which of the following is a random variable and which is not.

2 pts

unseen

- (i) Let  $(\Omega, \Sigma, \mu)$  be a probability space defined with  $\Omega \equiv \mathbb{N}$ ,  $\Sigma \equiv$  *smallest  $\sigma$ -algebra containing subsets consisting of even numbers*, and  $\mu(\{2n\}) \equiv \frac{1}{2^n}, n \in \mathbb{N}$ . Let

$$X \equiv -\frac{1}{2}\chi_{\{4,6\}} + \frac{1}{2}\chi_{\{1,2,3\}}$$

where  $\chi_A$  denotes the characteristic function of a set  $A$ .

3 pts

unseen

- (ii) Let  $(\mathbb{R}, \Sigma_L, \gamma)$  be a probability space, where  $\Sigma_L \equiv$  *Lebesgue measurable sets in  $\mathbb{R}$* , and  $\gamma$  is the Gaussian measure with variance one and mean zero. Let

$$X_n(x) \equiv x^n.$$

- (iii) Let  $([0, 1], \Sigma_L \cap [0, 1], \lambda)$ , where  $\Sigma_L \equiv$  is as in (ii), and  $\lambda$  is the *Lebesgue measure*. Let  $A \subset [0, 1]$  and  $A \notin \Sigma_L \cap [0, 1]$ . Let

$$X(x) \equiv \chi_{A \cap \mathbb{Q}}.$$

4 pts

unseen

- (1.c) Prove or disprove that there are uncountably many distribution functions on the real line which have jumps at the rational points.

6 pts

unseen

2. (2.a)

4 pts

seen

Give the definition of mutually independent  $\sigma$ -algebras, explaining carefully all notions involved.

6 pts

unseen

- (2.b) Let  $[0, 2\pi]^{\times \mathbb{N}} \ni \omega \equiv (\omega_j)_{j \in \mathbb{N}} \mapsto \pi_j(\omega) := \omega_j$ . Are the random variables

$$\{Y_j \equiv \cos(j\pi_j) \cos((j+1)\pi_{j+1})\}_{j \in \mathbb{N}}$$

on a probability space  $([0, 2\pi], \Sigma_L \cap [0, 2\pi], \frac{1}{2\pi}\lambda)^{\otimes \mathbb{N}}$  mutually independent.

10 pts

unseen

- (2.c) Prove the Weak Law of Large Numbers for the family  $Z_k \equiv X_k X_{k+1}$ ,  $k \in \mathbb{N}$ , where  $(X_j)$  are mutually independent Gaussian random variables with mean zero and variance  $\text{Var}(X_j) = \log(1+j)$ ,  $j \in \mathbb{N}$ , on the probability space  $([0, 1], \Sigma_L \cap [0, 1], \lambda)$ .

3. (3.a)

5 pts  
seen

Let  $A_n$ ,  $n \in \mathbb{N}$ , be a sequence of events in a probability space. Define the corresponding  $\limsup_n A_n$  event. Prove that  $\omega \in \limsup_n A_n$  if and only if  $\omega$  belongs to infinitely many sets  $A_n$ .

8 pts  
unseen

(3.b) Let  $\Omega \equiv \{0, 1\}^{\mathbb{N}}$  and, for a finite sequence  $\alpha_k \in \{0, 1\}$ ,  $k = 1, \dots, 6$ , let

$$A_n \equiv \{\omega : \omega_{12n+k} = \alpha_k, k = 1, \dots, 6\}.$$

Suppose on  $\Omega$  we are given a product probability measure

$$\mu \equiv \bigotimes_{j \in \mathbb{N}} \nu_j$$

where  $\nu_j(\{0\}) = j^{-\frac{1}{3}}$ . Explain giving reasons whether or not the pattern  $\alpha_k = 0$ , for all  $k = 1, \dots, 6$ , will appear infinitely many times with probability one? State the necessary theorems without giving their proofs.

7 pts  
unseen

(3.c)

Let  $r_j$ ,  $j \in \mathbb{N}$ , be the Rademacher random variables on unit interval with the Lebesgue measure. Prove the Strong Law of Large Numbers for the random variables  $R_j \equiv r_j r_{j+1} r_{j+2}$ ,  $j \in \mathbb{N}$ .

4. (4.a)

5 pts  
seen

- (i) Give the definition of the characteristic function of a real valued random variable.
- (ii) Prove that the product of  $n$  characteristic functions satisfies all properties of a characteristic function.

6 pts  
unseen

(4.b) Let random variables  $r_j$ ,  $j \in \mathbb{N}$ , be as in Problem (3.c). Find a characteristic function of the following random variables

$$Y_j \equiv r_j r_{j+1}, \quad j \in \mathbb{N}$$

on the probability space  $([0, 1], \Sigma_L \cap [0, 1], \lambda)$ .

9 pts  
unseen

(4.c) Prove the Central Limit Theorem for the random variables  $\{Y_{2j}\}_{j \in \mathbb{N}}$  on a probability space  $([0, 1], \Sigma_L \cap [0, 1], \lambda)$ .

5. (5.a) State the Birkhoff Ergodic Theorem explaining carefully all notions involved.

5 pts

seen

7 pts

unseen

(5.b) Let  $r_k, k \in \mathbb{N}$ , be the Rademacher functions on  $([0, 1], \Sigma_L \cap [0, 1], \lambda)$ . Using the Birkhoff Ergodic Theorem prove the Strong Law of Large Numbers for  $r_k, k \in \mathbb{N}$ .

8 pts

unseen

(5.c) Let  $(\Omega, \Sigma, \mu) \equiv ([0, 2\pi], \Sigma_L \cap [0, 2\pi], \frac{1}{2\pi}\lambda)^\mathbb{N}$ , where  $\lambda$  denotes the Lebesgue measure on  $\Sigma_L \cap [0, 2\pi]$ .

Let

$$f_k(\omega) = \sum_{n \in \mathbb{N}} \frac{1}{n^2} \prod_{j=1}^n \omega_{j+k}^{\frac{1}{n}} \cos\left(n \sum_{j=1}^n \omega_{j+k}\right).$$

Using the Birkhoff Ergodic Theorem prove that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f_k = 0$$

in  $\mathbb{L}_1$  and almost everywhere.

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BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2020

MATH Probability Theory

*The following information must be completed:*

**Is the paper suitable for resitting students from previous years: Yes**

**Category A marks: available for basic, routine material (excluding any mastery question) (40 percent = 32/80 for 4 questions):**

eg 1(a) 6 marks; 1(b.i) 7 marks; 2(a) 5 marks; 3(a) 9 marks; 4(a) 6 marks.

For example : 1a 5 pts, 1b.i& ii 5 pts , 2a 4 pts, 3a 5 pts, 3c 7 pts, 4a 5 pts

**Category B marks: Further 25 percent of marks (20/ 80 for 4 questions) for demonstration of a sound knowledge of a good part of the material and the solution of straightforward problems and examples with reasonable accuracy (excluding mastery question):**

eg 1(b.ii) 8 marks; 3(b.i) 5 marks; 3(b.ii) 6 marks; 2(c) 1 mark computation of convolution of kernels.

For example : 1b.iii 4 pts, 2b 6 pts, 3a pts, 4b 6 pts, 4c (statement of Thm) 4 pts

**Category C marks: the next 15 percent of the marks (= 12/80 for 4 questions) for parts of questions at the high 2:1 or 1st class level (excluding mastery question):**

eg 2(b) 6 marks; 4(c.i) 6 marks.

For example : 1c 6 pts, 4c (proof of Thm) 5pts

**Category D marks: Most challenging 20 percent (16/80 marks for 4 questions) of the paper (excluding mastery question):**

eg 2(c) 8 marks; 4(c.ii) 8 marks.

For example : 2c 10 pts, 3b 8pts

*Signatures are required for the final version:*

Setter's signature	Checker's signature	Editor's signature
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BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2020

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Probability Theory

Date: ??

Time: ??

Time Allowed: 2 Hours for MATH96 paper; 2.5 Hours for MATH97 papers

This paper has *4 Questions (MATH96 version); 5 Questions (MATH97 versions)*.

Candidates should start their solutions to each question in a new main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted.
- Each question carries equal weight.
- Calculators may not be used.

## Solutions

### 1. Solution

5 pts

seen

(1.a)

Let  $(\Omega, \Sigma, \mu)$  be a probability space. A function  $X : \Omega \rightarrow \mathbb{R}$  is called a random variable iff for any Borel set  $A \subset \mathbb{R}$  we have

$$X^{-1}(A) \equiv \{\omega \in \Omega : X(\omega) \in A\} \in \Sigma.$$

2 pts

unseen

(1.b) (i) Let  $\Omega \equiv \mathbb{N}$ ,  $\Sigma \equiv$  *smallest  $\sigma$ -algebra containing subsets consisting of even numbers*, and  $\mu(\{2n\}) \equiv \frac{1}{2^n}$ ,  $n \in \mathbb{N}$ . If

$$X \equiv -\frac{1}{2}\chi_{\{4,6\}} + \frac{1}{2}\chi_{\{1,2,3\}}$$

where  $\chi_A$  denotes the characteristic function of a set  $A$ , then we have

$$X^{-1}\left(\left\{\frac{1}{2}\right\}\right) = \{1, 2, 3, 4, 6\}$$

which does not belong to  $\Sigma$ . Hence  $X$  is not a random variable.

3 pts

unseen

(ii) Let  $(\mathbb{R}, \Sigma_L, \gamma)$ , where  $\Sigma_L \equiv$  *Lebesgue measurable sets in  $\mathbb{R}$* , and  $\gamma$  is the Gaussian measure with variance one and mean zero. Let

$$X_n(x) \equiv x^n.$$

Since preimage by  $X_n$  of an interval is an interval and intervals generate the Borel  $\sigma$ -algebra which is contained in  $\Sigma_L$ ,  $X_n$  is a random variable.

4 pts

unseen

(iii) Let  $([0, 1], \Sigma_L \cap [0, 1], \lambda)$ , where  $\Sigma_L \equiv$  *Lebesgue measurable sets in  $\mathbb{R}$* , and  $\lambda$  is the Lebesgue measure. Let  $A \subset [0, 1]$  and  $A \notin \Sigma_L \cap [0, 1]$ . Let

$$X(x) \equiv \chi_{A \cap \mathbb{Q}}.$$

We note that rational numbers have Lebesgue measure zero and hence  $A \cap \mathbb{Q}$  is of measure zero too. Since sets of measure zero belong to  $\Sigma_L$  by definition of the  $\sigma$ -algebra of Lebesgue measurable sets, this implies that  $X$  is a random variable.

6 pts

unseen

(1.c) Since rationals form a countable set  $\mathbb{Q} \equiv (q_n)_{n \in \mathbb{N}}$  we can assign a probability measure  $\nu(\cdot)$  on the  $\sigma$ -algebra of Borel or Lebesgue measurable sets by setting

$$\nu(\{q_n\}) = \alpha_n, \quad \text{with} \quad \alpha_n \geq 0, \quad \sum_n \alpha_n = 1$$

and consequently zero for any set not containing rational points. It is clear that there is uncountable many choices of such measures. Then the distribution function of the function *id* on  $\mathbb{R}$  has jumps on the set of rational points.

## 2. Solution

4 pts

seen

(2.a)

Let  $(\Omega, \Sigma, \mu)$  be a probability space and let  $\Sigma_i$ ,  $i = 1, \dots, n$ ,  $n \in \mathbb{N}$ , be a collection of sub  $\sigma$ -algebras in  $\Sigma$ . We say that the  $\sigma$ -algebras in this collection are mutually independent iff

$$\forall A_i \in \Sigma_i, \quad i = 1, \dots, n, \quad \mu(\cap_{i=1, \dots, n} A_i) = \prod_{i=1, \dots, n} \mu(A_i).$$

6 pts

unseen

(2.b) Let  $[0, 2\pi]^{\times \mathbb{N}} \ni \omega \equiv (\omega_j)_{j \in \mathbb{N}} \mapsto \pi_j(\omega) := \omega_j$ . By definition of the product measure the  $\sigma$ -algebra  $\pi_j^{-1}(\mathcal{B}_{\mathbb{R}})$  are mutually independent. Hence all random variables  $Y_j$  and  $Y_k$  for  $|j - k| \geq 2$  are mutually independent. Next we note that

$$E(Y_j^2) = E(\cos^2(j\pi_j) \cos^2((j+1)\pi_{j+1})) = E\left(\frac{1}{2}(\cos(2j\pi_j) + 1) \frac{1}{2}(\cos(2(j+1)\pi_{j+1}) + 1)\right) = \frac{1}{4}$$

and so

$$E(Y_{j-1}^2) E(Y_j^2) = \frac{1}{16}$$

On the other hand

$$\begin{aligned} E(Y_{j-1}^2 Y_j^2) &= E(\cos^2((j-1)\pi_{j-1}) \cos^4(j\pi_j) \cos^2((j+1)\pi_{j+1})) \\ &= E\left(\frac{1}{2}(\cos(2(j-1)\pi_{j-1}) + 1) \left(\frac{1}{2}(\cos(2j\pi_j) + 1)\right)^2 \frac{1}{2}(\cos(2(j+1)\pi_{j+1}) + 1)\right) \\ &= E\left(\frac{1}{2}(\cos(2(j-1)\pi_{j-1}) + 1)\right) E\left(\frac{1}{2}(\cos(2j\pi_j) + 1)\right)^2 E\left(\frac{1}{2}(\cos(2(j+1)\pi_{j+1}) + 1)\right) = \frac{3}{16} \end{aligned}$$

Thus

$$E(Y_{j-1}^2 Y_j^2) \neq E(Y_{j-1}^2) E(Y_j^2)$$

and hence the random variables  $Y_{j-1}$  and  $Y_j$  as well as  $Y_{j+1}$  and  $Y_j$  are not mutually independent;

10 pts

unseen

(2.c) Using mutual independence of the Gaussian random variables  $X_j$  and  $EX_j = 0$ , we have

$$E\left(\frac{1}{n} \sum_{j=1}^n Y_j\right)^2 = \frac{1}{n^2} \sum_{j=1}^n E(Y_j^2).$$

We also note that

$$E(Y_j^2) = EX_j^2 X_{j+1}^2 = \log(1+j) \log(2+j).$$

Hence

$$E\left(\frac{1}{n} \sum_{j=1}^n Y_j\right)^2 = \frac{1}{n^2} \sum_{j=1}^n \log(1+j) \log(2+j) \leq \frac{\log(1+n) \log(2+n)}{n}$$

which goes to zero as  $n \rightarrow \infty$ . This implies that

$$\frac{1}{n} \sum_{j=1}^n Y_j \rightarrow_{n \rightarrow \infty} 0$$

in probability.



3. (3.a)

5 pts

seen

For be a sequence of events  $A_n$ ,  $n \in \mathbb{N}$ , in some probability space  $(\Omega, \Sigma, \mu)$ , the corresponding  $\limsup_n A_n$  event is defined as follows

$$\limsup_n A_n := \bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} A_k$$

By definition of the intersection of sets in the set theory, we have

$$\bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} A_k = \{\omega \in \Omega : \forall n \in \mathbb{N} \quad \omega \in \bigcup_{k \geq n} A_k\}$$

so, using the definition of the union of the sets, we get the following equivalence

$$\omega \in \limsup_n A_n \iff \forall n \in \mathbb{N} \quad \exists k_n \geq n \quad \omega \in A_{k_n}$$

i.e. exists an infinite sequence  $A_{k_n}$ ,  $k_n \in \mathbb{N}$ , such that for  $\omega \in \limsup_n A_n$ , we have  $\omega \in A_{k_n}$  for infinitely many sets  $A_{k_n}$ .

8 pts

unseen

(3.b) Let  $\Omega \equiv \{0, 1\}^{\mathbb{N}}$  and, for a finite sequence  $\alpha_k \in \{0, 1\}$ ,  $k = 1, \dots, 6$ , let

$$A_n \equiv \{\omega : \omega_{12n+k} = \alpha_k, k = 1, \dots, 6\}.$$

Suppose we are given a product probability measure

$$\mu \equiv \bigotimes_{j \in \mathbb{N}} \nu_j$$

where  $\nu_j(\{0\}) = j^{-\frac{1}{3}}$ . Then we have

$$\mu(A_n) = \mu\left(\bigcap_{k=1, \dots, 6} \{\omega_{12n+k} = 0\}\right) = \prod_{k=1, \dots, 6} \nu_{12n+k}(\{0\}) = \prod_{k=1, \dots, 6} (12n+k)^{-\frac{1}{3}} \leq (12n)^{-2}$$

Since  $\sum_n (12n)^{-2} < \infty$ , we get

$$\sum_n \mu(A_n) < \infty.$$

Next we use the following theorem.

Borel-Cantelli Lemma: For a sequence of events  $A_n$ ,  $n \in \mathbb{N}$ , in a probability space  $(\Omega, \Sigma, \mu)$ , if

$$\sum_n \mu(A_n) < \infty,$$

then

$$\mu(\limsup_n A_n) = 0.$$

This implies that in our case the pattern of six zeros appears infinitely many times with probability zero.

(3.c)

Let  $r_j$ ,  $j \in \mathbb{N}$ , be the Rademacher random variables on unit interval with the Lebesgue measure and let  $R_j \equiv r_j r_{j+1} r_{j+2}$ ,  $j \in \mathbb{N}$ . Define

$$s_n \equiv \frac{1}{n} \sum_{j=1, \dots, n} R_j.$$

We have

$$E(s_n^4) = \frac{1}{n^4} \sum_{i,j,k,l=1, \dots, n} E(R_i R_j R_k R_l)$$

Since the Rademacher random variables are mutually independent, have mean zero and their square equals to one, we have

$$\begin{aligned} n^4 \cdot E(s_n^4) &= \sum_{i=j=k=l=1, \dots, n} E(R_i R_j R_k R_l) + \sum_{i=j, k=l \in \{1, \dots, n\}} E(R_i R_j R_k R_l) \\ &\quad + \sum_{i=k, j=l \in \{1, \dots, n\}} E(R_i R_j R_k R_l) + \sum_{i=l, j=k \in \{1, \dots, n\}} E(R_i R_j R_k R_l) \\ &= n + 3n^2. \end{aligned}$$

This implies that

$$E(s_n^4) \leq \frac{1}{n^4} (n + 3n^2) \leq 4 \frac{1}{n^2}$$

and so we have

$$\sum_n E(s_n^4) < \infty.$$

From this, using the monotone convergence theorem and necessary condition for convergence of series, we conclude that

$$s_n \xrightarrow{n \rightarrow \infty} 0$$

with probability one.

4. (4.a)

5 pts
seen

- (i) Let  $X$  be a real valued random variable on a probability space  $(\Omega, \Sigma, \mu)$ . A characteristic function of  $X$  is defined by

$$\varphi_X(t) := \int e^{itX} d\mu.$$

- (ii) Let  $\varphi_i(t)$ ,  $i = 1, \dots, n$ , be characteristic functions of some random variables  $X_i$  on some probability spaces  $(\Omega_i, \Sigma_i, \mu_i)$ . Let

$$\varphi(t) \equiv \prod_{j=1, \dots, n} \varphi_j(t).$$

Since all  $\varphi_i(t)$  are uniformly continuous and have value one at  $t = 0$ , so does the function  $\varphi(t)$ . The uniform continuity follows from the following bound

$$\begin{aligned} |\varphi(t+h) - \varphi(t)| &\leq |\varphi_1(t+h) - \varphi_1(t)| \prod_{k>1} |\varphi_k(t)| \\ &+ \sum_{j=1, \dots, n-1} \prod_{k<j} |\varphi_k(t+h)| \prod_{k>j} |\varphi_k(t)| \cdot |\varphi_j(t+h) - \varphi_j(t)| + \\ &\quad \prod_{k<n} |\varphi_k(t+h)| \cdot |\varphi_n(t+h) - \varphi_n(t)| \\ &\leq \sum_{j=1, \dots, n} |\varphi_j(t+h) - \varphi_j(t)| \end{aligned}$$

where in the last step one uses the fact that any characteristic function satisfies  $|\varphi_k(t)| \leq 1$ . To show positive definiteness, it is sufficient to show that for  $n = 2$  and use the mathematical induction for the general case. We note that any positive definite matrix  $C$  can be represented as follows

$$C = A^* A$$

with some matrix  $A$  and with  $A^*$  denoting the adjoint of  $A$ . Hence if

$$\varphi_1(t_i - t_j) = (A^* A)_{ij} \quad \text{and} \quad \varphi_2(t_i - t_j) = (B^* B)_{ij},$$

we have

$$\begin{aligned} \sum_{i,j} \varphi_1(t_i - t_j) \varphi_2(t_i - t_j) \bar{z}_i z_j &= \sum_{i,j} (A^* A)_{ij} (B^* B)_{ij} \bar{z}_i z_j \\ &= \sum_{k,l} \sum_{i,j} A_{ik}^* A_{kj} B_{il}^* B_{lj} \bar{z}_i z_j = \sum_{k,l} \left| \sum_j A_{kj} B_{lj} z_j \right|^2 \geq 0 \end{aligned}$$

That is the product of characteristic functions satisfy the positive definiteness condition.

6 pts

unseen

(4.b) Let

$$Y_j \equiv r_j r_{j+1}, \quad j \in \mathbb{N}$$

on the probability space  $([0, 1], \Sigma_L \cap [0, 1], \lambda)$ . Since  $r_j$  are mutually independent Bernoulli random variable taking on values  $\pm 1$  with equal probability, we have

$$\begin{aligned} \int e^{itY_j} d\lambda &= e^{it} \lambda(\{Y_j = 1\}) + e^{-it} \lambda(\{Y_j = -1\}) \\ &= e^{it} \lambda(\{r_j = 1, r_{j+1} = 1\} \cup \{r_j = -1, r_{j+1} = -1\}) \\ &\quad + e^{-it} \lambda(\{r_j = 1, r_{j+1} = -1\} \cup \{r_j = -1, r_{j+1} = 1\}) \\ &= \cos(t) \end{aligned}$$

where we have used

$$\begin{aligned} \lambda(\{r_j = 1, r_{j+1} = 1\} \cup \{r_j = -1, r_{j+1} = -1\}) \\ &= \lambda(\{r_j = 1, r_{j+1} = 1\}) + \lambda(\{r_j = -1, r_{j+1} = -1\}) \\ &= \lambda(\{r_j = 1\})\lambda(\{r_{j+1} = 1\}) + \lambda(\{r_j = -1\})\lambda(\{r_{j+1} = -1\}) \\ &= \frac{1}{2} \end{aligned}$$

and similarly for the probability of  $\{Y_j = -1\}$ .

9 pts

unseen

(4.c) Central Limit Theorem : If  $\{Y_{2j} \equiv r_{2j} r_{2j+1}\}_{j \in \mathbb{N}}$  are the random variables on a probability space  $([0, 1], \Sigma_L \cap [0, 1], \lambda)$ , then

$$\frac{1}{\sqrt{n}} \sum_{j=1, \dots, n} Y_{2j}$$

converges in distribution to the Gaussian random variable with mean zero and covariance one.

*Proof* : By definition and mutual independence of the Rademacher functions  $r_k$ ,  $k \in \mathbb{N}$ , the random variables  $Y_{2j} \equiv r_{2j} r_{2j+1}$ ,  $j \in \mathbb{N}$ , are i.i.d. with mean zero and variance one. Using (4.b) we have

$$\int \exp \left( it \frac{1}{\sqrt{n}} \sum_{j=1, \dots, n} Y_{2j} \right) d\lambda = \left( \cos \left( \frac{t}{\sqrt{n}} \right) \right)^n = \left( 1 - \frac{t^2}{2n} + \mathcal{O} \left( \frac{1}{n^{3/2}} \right) \right)^n \rightarrow_{n \rightarrow \infty} e^{-\frac{1}{2} t^2}.$$

and hence by Levy's continuity theorem  $\frac{1}{\sqrt{n}} \sum_{j=1, \dots, n} Y_{2j}$  converges in distribution to the Gaussian random variable with mean zero and covariance one.

5. (5.a)

5 pts

Let  $(\Omega, \Sigma, \mu)$  be a probability space. A map  $T : \Omega \rightarrow \Omega$  is called measure preserving if for any  $A \in \Sigma$

seen

$$\mu(T^{-1}A) = \mu(A).$$

A set  $A \in \Sigma$  is called  $T$ -invariant iff the symmetric difference  $A \setminus T^{-1}A \cup T^{-1}A \setminus A$  has probability zero. Let  $\mathcal{I}$  denote  $\sigma$ -algebra of  $T$ -invariant sets and let  $E_\mu(\cdot|\mathcal{I})$  be the corresponding conditional expectation associated to the probability measure  $\mu$ .

**Birkhoff's Ergodic Theorem** : Suppose  $X \in \mathbb{L}_1(\Omega, \Sigma, \mu)$ . Then

$$\frac{1}{n} \sum_{j=0}^n X(T^j \omega) \rightarrow_{n \rightarrow \infty} E_\mu(X|\mathcal{I})$$

in  $\mathbb{L}_1(\Omega, \Sigma, \mu)$  and almost surely.

7 pts

unseen

(5.b) Let  $r_k$  be the Rademacher functions on  $([0, 1], \Sigma_L \cap [0, 1], \lambda)$ .

If

$$T(t) = \begin{cases} 2t & \text{for } 0 \leq t \leq \frac{1}{2} \\ 2(t - \frac{1}{2}) & \text{for } \frac{1}{2} < t \leq 1 \end{cases}$$

Let

$$\epsilon(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq \frac{1}{2} \\ 0 & \text{for } \frac{1}{2} < t \leq 1 \end{cases}$$

and define

$$\epsilon_k(t) \equiv \epsilon(T^{k-1}(t)).$$

Then the Rademacher functions are related to  $\epsilon_k$ 's as follows

$$r_k(t) = 1 - 2\epsilon_k(t)$$

and so

$$r_k(t) = r_1(T^{k-1}(t)).$$

We note that the corresponding  $\sigma$ -algebra at infinity is given by

$$\bigcap_n \sigma(\{r_k, k \geq n\}),$$

where  $\sigma(\{r_k, k \geq n\})$  denotes the smallest  $\sigma$ -algebra s.t. all  $\{r_k, k \geq n\}$  are measurable. By the Kolmogorov 0-1 Law this  $\sigma$ -algebra is trivial. On the other hand we note that

$$T^{-1}\sigma(\{r_k, k \geq n+1\}) = \sigma(\{r_k, k \geq n\}),$$

which implies that the  $\sigma$ -algebra at infinity contains the  $T$  invariant sets. Hence, using the triviality of  $\sigma$ -algebra at infinity, the Birkhoff's Ergodic Theorem implies the Strong Law of Large numbers for Rademacher functions.

- (5.c) Let  $(\Omega, \Sigma, \mu) \equiv ([0, 2\pi], \Sigma_L \cap [0, 2\pi], \frac{1}{2\pi}\lambda)^\mathbb{N}$ , where  $\lambda$  denotes the Lebesgue measure on  $\Sigma_L \cap [0, 2\pi]$ .

Let

$$f_k(\omega) = \sum_{n \in \mathbb{N}} \frac{1}{n^{n+2}} \prod_{j=1}^n \omega_{j+k}^{\frac{1}{n}} \cos\left(n \sum_{j=1}^n \omega_{j+k}\right)$$

We note first that

$$\int \prod_{j=1}^n \omega_j^{\frac{1}{n}} d\mu = \prod_{j=1}^n \frac{1}{2\pi} \int_0^{2\pi} \omega_j^{\frac{1}{n}} d\omega_j = n^n 2\pi$$

and so

$$\int |f_k| d\mu = \int |f_0| d\mu \leq \sum_{n \in \mathbb{N}} \frac{1}{n^{2+\varepsilon}} \int \prod_{j=1}^n \omega_j^{-1+\frac{1}{n}} d\mu \leq 2\pi \sum_{n \in \mathbb{N}} \frac{1}{n^2} < \infty$$

This means that  $f_k \in \mathbb{L}_1(\mu)$ . A map  $T : \mathbb{R}^\mathbb{N} \rightarrow \mathbb{R}^\mathbb{N}$  defined by

$$(T\omega)_j = \omega_{j+1}$$

is preserving the product measure. Since we have  $\pi_{j+1} = \pi_j \circ T$ , we get

$$f_k = f_1 \circ T^{k-1}$$

Using the Birkhoff ergodic theorem we conclude that

$$\frac{1}{n} \sum_{k=1}^n f_k = \frac{1}{n} \sum_{k=1}^n f_1 \circ T^{k-1} \xrightarrow{n \rightarrow \infty} 0$$

almost everywhere and in  $\mathbb{L}_1$ . We use here the fact that by Kolmogorov 0-1 the  $\sigma$ -algebra at infinity is trivial.

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	Question	Comments for Students	
MATH97056 MATH97167	1	Generally good, 2biii question from Lebesgue measure theory was sometimes a problem	
MATH97056 MATH97167	2	2b not that good, but 2c very well done	
MATH97056 MATH97167	3	Generally not bad, but some problems with application of Borel-Cantelli Lemma	
MATH97056 MATH97167	4	Well done	
MATH97056 MATH97167	5	Very well done	