Mathematical Logic (MATH6/70132;P65)
Solutions to Problem Sheet 6

1. Suppose $f : A \to B$ is a bijection. Use f to construct functions $g : A \times A \to B \times B$ and $h : \mathcal{P}(A) \to \mathcal{P}(B)$ which are bijections. In the case of h, give a careful proof that your function is a bijection.

Solution: Note that as f is a bijection it has an inverse $f^{-1} : B \to A$.

We can define the bijection g by letting $g((a_1, a_2)) = (f(a_1), f(a_2))$. (To see this is a bijection check that the function $g_1 : B \times B \to A \times A$ given by $g_1((b_1, b_2)) = (f^{-1}(b_1), f^{-1}(b_2))$ is an inverse of g.)

We can define the function h by letting $h(X) = \{f(a) : a \in X\}$, for $X \subseteq A$. We show that this is a bijection by showing that it has an inverse $h_1 : \mathcal{P}(B) \to \mathcal{P}(A)$ given by $h_1(Y) = \{f^{-1}(b) : b \in Y\}$, for $Y \subseteq B$. To see this, note that $h_1(h(X)) = h_1(\{f(a) : a \in X\}) = \{f^{-1}(f(a)) : a \in X\} = X$ and similarly $h(h_1(Y)) = Y$ for $X \subseteq A$ and $Y \subseteq B$.

2. Decide whether the following functions f_1, f_2, f_3 are injective or surjective (or both). Give reasons for your answers.
(i) X is some set; A is the set of finite sequences of elements of X; B is the set of finite subsets of X; $f_1 : A \to B$ is given by $f_1((a_1, \ldots, a_n)) = \{a_1, \ldots, a_n\}$.
(ii) $f_2 : \mathbb{R}^{\mathbb{R}} \times \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}$ is given by composition: $f_2(\alpha, \beta) = \alpha \circ \beta$ for $\alpha, \beta \in \mathbb{R}^{\mathbb{R}}$ (the set of functions from $\mathbb{R}$ to $\mathbb{R}$).
(iii) Recall that $\mathbb{N}^{\mathbb{N}}$ can be thought of as the set of sequences of natural numbers. Define the function $f_3 : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ to be the function which sends the pair of sequences $a = (a_0, a_1, a_2, \ldots)$, $b = (b_0, b_1, b_2, \ldots)$ to the sequence $c = (a_0, b_0, a_1, b_1, a_2, b_2, \ldots)$.

Solution: (i) This is surjective: the finite sequence $(a_1, \ldots, a_n)$ gets sent to the finite set $\{a_1, \ldots, a_n\}$ and the empty sequence gets sent to the empty set, so f_1 is surjective. As long as X is non-empty, f_1 is not injective: take any $a \in X$, then $f_1((a, a)) = f_1((a, a, a))$.

(ii) This is surjective but not injective. Let $\iota \in \mathbb{R}^{\mathbb{R}}$ be the identity function and $o \in \mathbb{R}^{\mathbb{R}}$ the zero function ($o(x) = 0$ for all $x \in \mathbb{R}$). Then for any $f \in \mathbb{R}^{\mathbb{R}}$ we have $f_2(\iota, f) = f$ and $f_2(o, f) = o$.

(iii) This is a bijection. One way to see this is to write down the inverse function $g : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$. If $c = (c_0, c_1, c_2, \ldots) \in \mathbb{N}^{\mathbb{N}}$, then $g(c)$ splits c into its even numbered terms and odd numbered terms:

$$g(c) = ((c_0, c_2, c_4, \ldots), (c_1, c_3, c_5, \ldots)).$$

3. (i) Show that the following sets are countable (you may use any of the results in the notes):
(a) The set of finite subsets of $\mathbb{N}$.
(b) The set of subsets of $\mathbb{N}$ with finite complement.
(c) The set of real numbers which are roots of non-zero polynomial equations with rational coefficients.
(ii) Use (c) to deduce that there is some real number which is not a root of any non-zero polynomial equation with rational coefficients.

Solution: (i) (a) Let F denote the set of finite subsets of $\mathbb{N}$ and S the set of finite sequences of natural numbers. By 3.1.3 in the notes, S is countable, and by Problem 2(i), there is a surjection from S to F. It follows that F is countable (by a result which you should be able to prove).

(b) Let I denote the set of subsets of $\mathbb{N}$ with finite complement. With F as in (a), there is a bijection $\alpha : F \to I$ given by $\alpha(X) = \mathbb{N} \setminus X$. So as F is countable, so is I.

(c) Let P denote the set of non-zero polynomial equations with rational coefficients i.e. $P = \{a_0 + a_1 x + \ldots + a_n x^n : n \in \mathbb{N},\ a_i \in \mathbb{Q}$ not all zero $\}$. There is an obvious surjection from the set of all finite sequences of rational numbers (excluding sequences of zeros, and the empty sequence) to P. So as $\mathbb{Q}$ is countable, P is countable. Now, each polynomial in P has finitely many roots in $\mathbb{R}$. Thus the set A consisting of roots of polynomials in P is a countable union of finite sets: so it is countable, by 3.1.3.

(ii) We know that $\mathbb{R}$ is not countable and $A \subseteq \mathbb{R}$. As A is countable, we therefore have $A \neq \mathbb{R}$: there is some real number not in A.

4. Let S be the set of sequences of zeros and ones (that is, functions $s : \mathbb{N} \to \{0, 1\}$), and F the set of functions from $\mathbb{R}$ to $\mathbb{R}$.
(a) Construct an injective function $i : S \times S \to S$, and hence show that S and $S \times S$ are equinumerous. Deduce that $\mathbb{R}$ and $\mathbb{R} \times \mathbb{R}$ are equinumerous.
(b) Construct an injective function from F to $\mathcal{P}(\mathbb{R} \times \mathbb{R})$ and an injective function from $\mathcal{P}(\mathbb{R})$ to F. Deduce that F and $\mathcal{P}(\mathbb{R})$ are equinumerous.

Solution: (a) This is similar to Problem 2 (iii). Define $F : S \times S \to S$ to be the function which sends the pair of sequences $(a_i)_{i\in\mathbb{N}}, (b_i)_{i\in\mathbb{N}}$ to the single sequence $(a_0, b_0, a_1, b_1, \ldots)$. In fact, this is a bijection, so S and $S \times S$ are equinumerous. By 3.1.7 in the notes, S and $\mathbb{R}$ are equinumerous. So by Problem 1, $\mathbb{R} \times \mathbb{R} \approx S \times S$. Thus $\mathbb{R} \approx S \approx S \times S \approx \mathbb{R} \times \mathbb{R}$ and therefore $\mathbb{R} \approx \mathbb{R} \times \mathbb{R}$.

(b) Any function $\mathbb{R} \to \mathbb{R}$ is actually a subset of $\mathbb{R} \times \mathbb{R}$. Thus $F \subseteq \mathbb{R} \times \mathbb{R}$, so $|F| \leq |\mathcal{P}(\mathbb{R} \times \mathbb{R})|$. On the other hand, the function which sends a subset of $\mathbb{R}$ to its characteristic function is an injective function from $\mathcal{P}(\mathbb{R})$ to F. Thus $|\mathcal{P}(\mathbb{R})| \leq |F|$.

Now, by (a) and problem 1, we know that $\mathcal{P}(\mathbb{R})$ and $\mathcal{P}(\mathbb{R} \times \mathbb{R})$ are equinumerous. So we also have $|F| \leq |\mathcal{P}(\mathbb{R})|$. It follows from the Cantor-Schröder-Bernstein Theorem that $|F| = |\mathcal{P}(\mathbb{R})|$.

5. Suppose A_1, A_2, B_1, B_2 are sets with $A_1 \approx A_2$ and $B_1 \approx B_2$. Write down bijections which show:
(i) $A_1^{B_1} \approx A_1^{B_2}$;
(ii) $A_1^{B_1} \approx A_2^{B_1}$;
and deduce:
(iii) $A_1^{B_1} \approx A_2^{B_2}$.

Solution: Let $\alpha : A_1 \to A_2$ and $\beta : B_1 \to B_2$ be bijections.

(i) Define the function $\gamma : A_1^{B_1} \to A_1^{B_2}$ as follows. If $f \in A_1^{B_1}$ then $\gamma(f)$ is the function $B_2 \to A_1$ given by $f \circ \beta^{-1}$. Note that γ has an inverse function: the function δ which sends $g \in A_1^{B_2}$ to $g \circ \beta$ (check: $\gamma(\delta(g)) = g \circ \beta \circ \beta^{-1} = g$, etc.)

(ii) Similar: define $\eta : A_1^{B_1} \to A_2^{B_1}$ to be the function which sends $f \in A_1^{B_1}$ to $\alpha \circ f$.

(iii) By (i) $A_1^{B_1} \approx A_1^{B_2}$. By (ii) $A_1^{B_2} \approx A_2^{B_2}$.

6. Again, let S denote the set of sequences of zeros and ones.
(a) Construct a bijection from $S^{\mathbb{N}}$ to S. (Note and Hint: $S^{\mathbb{N}}$ consists of functions $f : \mathbb{N} \to S$. Thus f is a sequence of sequences of zeros and ones. Turn such a thing into a single sequence s_f of zeros and ones in such a way that the original f is recoverable from s_f.)
(b) Deduce that if A is a countably infinite set then $\mathbb{R}^A$ is equinumerous with $\mathbb{R}$.
(c) Let C be the set of *continuous* functions from $\mathbb{R}$ to $\mathbb{R}$. Show that C is equinumerous with $\mathbb{R}$.
(d) What can you say about the relationship between the cardinalities of C here and F in Question 4?

Solution: (a) Let $\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be any bijection. Define $\alpha : S^{\mathbb{N}} \to S$ as follows. If $f \in S^{\mathbb{N}}$ then $f = (f_i)_{i\in\mathbb{N}}$ is a sequence of sequences of zeros and ones: write each f_i as $(f_{ij})_{j\in\mathbb{N}}$. Now let $\alpha(f)$ be the sequence $(a_n)_{n\in\mathbb{N}}$ where a_n is equal to $f_{\pi^{-1}(n)}$. Note that from this sequence we can easily recover the original sequences as $f_{ij} = a_{\pi(i,j)}$. So α is a bijection.

(b) If A is countably infinite, then $A \approx \mathbb{N}$. We also know that $S \approx \mathbb{R}$. So by Problem 5, $\mathbb{R}^A \approx S^{\mathbb{N}}$. By part (a), $S^{\mathbb{N}} \approx S$. So we have $\mathbb{R}^A \approx S \approx \mathbb{R}$, as required.

(c) Define the function $\rho : C \to \mathbb{R}^{\mathbb{Q}}$ by restriction: if $f \in C$ then $\rho(f)$ is f restricted to $\mathbb{Q}$. As $\mathbb{Q}$ is dense in $\mathbb{R}$ it follows that ρ is injective (consider the effect of f on sequences of rational numbers), so $C \preceq \mathbb{R}^{\mathbb{Q}}$. As $\mathbb{Q}$ is countable, part (b) then gives $|C| \leq |\mathbb{R}|$. On the other hand we can find an injective function from $\mathbb{R}$ to C: just take the real number r to the constant function f_r with $f_r(x) = r$ (for all $x \in \mathbb{R}$). So $|\mathbb{R}| \leq |C|$. Thus, by Cantor-Schröder-Bernstein, $|C| = |\mathbb{R}|$.

(d) By Problem 4, $|F| = |\mathcal{P}(\mathbb{R})|$ and by Cantor's Theorem (3.1.4), $|\mathcal{P}(\mathbb{R})| > |\mathbb{R}|$. Thus (using (c)), $|C| < |F|$.

7. Suppose $\mathbf{A_1} = (A_1, \leq_1)$ and $\mathbf{A_2} = (A_2, \leq_2)$ are linearly ordered sets.

(i) Show that the reverse-lexicographic product $\mathbf{A_1} \times \mathbf{A_2}$ (as defined in the notes) is a linearly ordered set.
(ii) Suppose $\mathbf{B_1} = (B_1, \leq_1')$ and $\mathbf{B_2} = (B_1, \leq_2')$ are linearly ordered sets which are similar to $\mathbf{A_1}$ and $\mathbf{A_2}$ respectively. Show that $\mathbf{B_1} \times \mathbf{B_2}$ is similar to $\mathbf{A_1} \times \mathbf{A_2}$.
(Hint: Take similarities $f_i : A_i \to B_i$ for $i = 1, 2$ and show carefully from the definitions that $h : A_1 \times A_2 \to B_1 \times B_2$ given by $h(a_1, a_2) = (f_1(a_1), f_2(a_2))$ (for $a_i \in A_i$) is a similarity.)

Solution: (i) It is clear that if $(a_1, a_2) \in A_1 \times A_2$ then $(a_1, a_2) \leq (a_1, a_2)$. Suppose that $(a_1, a_2) \leq (a_1', a_2')$ and $(a_1', a_2') \leq (a_1, a_2)$. Then $a_2 \leq_2 a_2'$ and $a_2' \leq_2 a_2$. So $a_2 = a_2'$. It then follows that $a_1 \leq_1 a_1'$ and $a_1' \leq_1 a_1$: so $a_1 = a_1'$.

Now suppose $(a_1, a_2) \leq (a_1', a_2') \leq (a_1'', a_2'')$. Then $a_2 \leq_2 a_2' \leq_2 a_2''$. So $a_2 \leq_2 a_2''$. If $a_2 = a_2''$, then $a_2 = a_2' = a_2''$, so $a_1 \leq_1 a_1' \leq_1 a_1''$. Thus $a_1 \leq_1 a_1''$ and therefore $(a_1, a_2) \leq (a_1'', a_2'')$. If $a_2 <_2 a_2''$, then also $(a_1, a_2) \leq (a_1'', a_2'')$.

So far, this has shown that $A_1 \times A_2$ is a partial order. To show that it is a total order, take $(a_1, a_2), (a_1', a_2') \in A_1 \times A_2$. Without loss, we may assume $a_2 \leq_2 a_2'$. If $a_2 <_2 a_2'$ then $(a_1, a_2) < (a_1', a_2')$. If $a_2 = a_2'$, then $(a_1, a_2) \leq (a_1', a_2')$ or $(a_1', a_2') \leq (a_1, a_2)$ depending on whether $a_1 \leq_1 a_1'$ or $a_1' \leq a_1$.

(ii) We skip the proof that h is a bijection as it is similar to problem 1.

Suppose $(a_1, a_2) \leq (a_1', a_2')$. If $a_2 < a_2'$ then $f_2(a_2) < f_2(a_2')$, so $h(a_1, a_2) < h(a_1', a_2')$. If $a_2 = a_2'$ and $a_1 \leq a_1'$, then $f_2(a_2) = f_2(a_2')$ and $f_1(a_1) \leq f_1(a_1')$. So again $h(a_1, a_2) \leq h(a_1', a_2')$.

A similar argument shows that if $h(a_1, a_2) \leq h(a_1', a_2')$, then $(a_1, a_2) \leq (a_1', a_2')$.