Imperial College London

MATH97056 MATH70028

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Probability Theory

Date: 31 May 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. Let $\zeta, \zeta_1, \zeta_2, \dots$ be random variables such that the sequence ζ_n converges to ζ in probability.

   (a) Is it true that $\mathbb{P}(\zeta_n \to \zeta) = 1$? (No justification needed.) (5 marks)

   (b) Is it true that $\mathbb{E}|\zeta_n - \zeta| \to 0$? (No justification needed.) (5 marks)

   (c) Is it true that $\zeta_n \to \zeta$ a.s. if we assume in addition, that the sequence ζ_n is bounded and increasing? Prove if yes and give a counterexample if no. (10 marks)

2. (a) Let F be the function defined by:

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{x^2}{3} & \text{if } 0 \le x < 1, \\ \frac{1}{3} & \text{if } 1 \le x < 2, \\ \frac{1}{6}x + \frac{1}{3} & \text{if } 2 < x < 4, \\ 1 & \text{if } x \ge 4. \end{cases}$$

   (i) Define the value $F(2)$ so that F becomes a distribution function of some random variable X. Justify your answer. (5 marks)

   (ii) Compute $\mathbb{P}\{X = 2\}$ and $\mathbb{P}\{\frac{1}{2} \le X < \frac{3}{2}\}$. (5 marks)

   (b) Let the indicator functions $\chi_{A_m} = \zeta_m$ of sets A_m, $m = 1, 2, \dots$ ($\chi_A(\omega) = 1$ if $\omega \in A$ and zero otherwise) be independent random variables. Let $S_n = \zeta_1 + \cdots + \zeta_n$ and let $\mathbb{E}S_n \to \infty$ as $n \to \infty$. Using Chebyshev inequality, show that $S_n/(\mathbb{E}S_n) \to 1$ in probability. (10 marks)

3. (a) Let $\zeta_1, \zeta_2 \dots$ be independent random variables with characteristic function $e^{-|t|^\alpha}$, $0 < \alpha < 2$. Show that $X_n = (\zeta_1 + \cdots + \zeta_n)/n^{1/\alpha}$ has the same distribution as ζ_1. (10 marks)

   (b) Let ζ be an integer-valued random variable and $\phi_\zeta(t)$ be its characteristic function. Show that:

$$\mathbb{P}(\zeta = k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-itk} \phi_\zeta(t) dt, \quad k = 0, \pm 1, \pm 2, \cdots$$

   (10 marks)

4. (a) Let $\zeta_1, \zeta_2 \dots$ be i.i.d. with distribution $F(x)$ and let a sequence of real numbers $\lambda_1, \lambda_2, \dots$ be nondecreasing and such that $\lambda_n \to \infty$. Denote $A_n = \{\max_{1 \le k \le n} \zeta_k > \lambda_n\}$. Show that $\mathbb{P}(A_n \text{ i.o.}) = 0$ if and only if $\sum_{n=1}^{\infty}(1 - F(\lambda_n)) < \infty$. (12 marks)

   (b) Let $\zeta_1, \zeta_2 \dots$ be i.i.d. standard normals $N(0, 1)$. Show that

$$\sum_{n=1}^{\infty} \frac{\zeta_n}{n}$$

   converge a.s. (8 marks)

5. Let $S_n = X_1 + \cdots + X_n$ be a simple random walk (i.e., X_k are i.i.d. random variables with $\mathbb{P}(X_k = 1) = \mathbb{P}(X_k = -1) = 1/2$), and let T_k be the hitting times of 0, that is $T_0 := 0$ and $T_{k+1} := \inf\{n > T_k : S_n = 0\}$. Prove that $T_1, T_2 - T_1, T_3 - T_2, \ldots$ are i.i.d. random variables, thus T_n is a new random walk. To complete the proof follow this way:

   (a) First show that $\mathbb{P}(T_1 = n_1, T_2 - T_1 = n_2) = \mathbb{P}(T_1 = n_1)\mathbb{P}(T_1 = n_2)$ (generalisation to m terms is trivial, conclude without proof). Conclude that T_1 and $T_2 - T_1$ are identically distributed if $\mathbb{P}(T_1 < \infty) = 1$. (10 marks)

   (b) To see that $\mathbb{P}(T_1 < \infty) = 1$, first show that $\mathbb{P}(N \geq m) = \mathbb{P}(T_m < \infty) = \mathbb{P}(T_1 < \infty)^m$, where N is the total number of returns of the random walk to zero, thus $\mathbb{E}N = \sum_{m=1}^{\infty} \mathbb{P}(N \geq m) = \infty$ if and only if $\mathbb{P}(T_1 < \infty) = 1$. Now show that $\mathbb{E}N = \sum_{n=1}^{\infty} \mathbb{P}(S_n = 0)$ and show that this series diverges. (10 marks)

Probability exam 2022
Solutions

1a no [5 marks]

1b no [5 marks]

1c Since ξ_n is bounded, increasing, $\xi_n(\omega) \to \eta(\omega)$ pointwise, where η is a r.v.

In particular, $\xi_n \xrightarrow{P} \eta$

We show that $\xi = \eta$ a.e.

We have $|\xi - \eta| \le |\xi_n - \xi| + |\xi_n - \eta|$

So for any $\varepsilon > 0$,

$$P(|\xi - \eta| \geq \varepsilon) \leq \underbrace{P(|\xi_n - \xi| \geq \tfrac{\varepsilon}{2})}_{\to 0} + \underbrace{P(|\xi_n - \eta| \geq \tfrac{\varepsilon}{2})}_{\to 0}$$

$$\Rightarrow P(|\xi - \eta| \geq \varepsilon) = 0 \quad \forall \varepsilon > 0$$

$$P(|\xi - \eta| > 0) = P\left(\bigcup_{n=1}^{\infty} \{|\xi - \eta| > 1/n\}\right) =$$

$$= \lim_{n \to \infty} P(|\xi - \eta| > 1/n) = 0$$ [10 marks]

$$\Rightarrow P(|\xi - \eta| = 0) = 1$$ i.e. $\xi = \eta$ a.e.

so $\xi_n \to \xi$ a.s.

2a (i) $F(2) = 2/3$ [5 marks]

2a (ii) $P(X=2) = 2/3 - 1/3 = 1/3$

$P(1/2 \leq X < 3/2) = 1/3 - (\frac{1}{2})^2 \frac{1}{3} = 1/4$ [5 marks]

2b. By Chebyshev inequality

$$P(|S_n - ES_n| \geq \delta ES_n) \leq \frac{VS_n}{(\delta ES_n)^2}$$

But by independence,

$VS_n = V\xi_1 + \dots + V\xi_n$.

Moreover, we have $V\xi_j \leq E\xi_j^2 = E\xi_j$

$\Rightarrow VS_n \leq ES_n$

So $P(|S_n - ES_n| \geq \delta ES_n) \leq$

$$\leq \frac{1}{\delta^2 ES_n} \longrightarrow 0 \quad \text{since } ES_n \to \infty$$

i.e. $S_n / ES_n \xrightarrow{P} 1$. [10 marks]

3a. By independence, [10 marks]

$$\varphi_{X_n}(t) = \prod_{j=1}^{n} \varphi_{\xi_j/n^{1/\alpha}}(t) =$$

$$= \prod_{j=1}^{n} \varphi_{\xi_j}(t/n^{1/\alpha}) = \prod_{j=1}^{n} e^{-\frac{|t|^\alpha}{n}} = e^{-|t|^\alpha}.$$

Since X_n and ξ_1 has the same characteristic function, they have the same distribution.

3b Since ξ is integer-valued,

$$\int_{-\pi}^{\pi} e^{-itk} \varphi_\xi(t)\,dt = \int_{-\pi}^{\pi} e^{-itk}\Big(\sum_{k=-\infty}^{\infty} e^{itn} P(\xi=n)\Big)dt$$

$$= \sum_{k=-\infty}^{\infty} \int_{-\pi}^{\pi} e^{it(n-k)}\,dt\; P(\xi=n) =$$

$$= 2\pi\, P(\xi=k)$$

by Fubini thm. [10 marks]

4a.

By Borel-Cantelli thm, using independence, we have

$$\sum_{n=1}^{\infty} (1-F(\lambda_n)) = \sum_{n=1}^{\infty} P(\xi_n > \lambda_n) < \infty$$

iff $P(\xi_n > \lambda_n \text{ i.o.}) = 0$

Clearly $\{\xi_n > \lambda_n \text{ i.o.}\} \subset \{A_n \text{ i.o.}\}$

But if $\xi_n \leq \lambda_n \quad \forall n \geq N_0$

then $\exists N_1$ s.t. $\max_{1 \leq k \leq n} \xi_k \leq \lambda_n \quad \forall n \geq N_1$

so $\{\xi_n > \lambda_n \text{ i.o.}\} = \{A_n \text{ i.o.}\}$, and

we have

$$0 = P(\xi_n > \lambda_n \text{ i.o}) = P(A_n \text{ i.o.})$$

$$\text{iff} \quad \sum_{n=1}^{\infty} (1-F(\lambda_n)) < \infty .$$

[12 marks]

4b Since $E\xi_j = 0 \ \forall j$,

$$V\frac{\xi_j}{j} = 1/j^2,$$

[8 marks]

$$\sum_{n=1}^{\infty} V\frac{\xi_n}{n} = \sum_{n=1}^{\infty}\frac{1}{n^2} < \infty,$$

we have by 2-series theorem that $\sum_{n=1}^{\infty}\frac{\xi_n}{n}$ converge a.s.

5a. $P(T_1 = n_1, T_2 - T_1 = n_2) =$

$= P(X_1 + \cdots + X_j \neq 0 \;\forall\, 1 \leq j \leq n_1 - 1,$
$X_1 + \cdots + X_{n_1} = 0;$
$X_{n_1+1} + \cdots + X_{n_1+j} \neq 0 \;\forall\, 1 \leq j \leq n_2 - 1,$
$X_{n_1+1} + \cdots + X_{n_1+n_2} = 0) =$

by independence

$= P(X_1 + \cdots + X_j \neq 0 \;\forall\, 1 \leq j \leq n_1 - 1,$
$X_1 + \cdots + X_{n_1} = 0) \cdot$

$\cdot P(X_{n_1+1} + \cdots + X_{n_1+j} \neq 0 \;\forall\, 1 \leq j \leq n_2 - 1,$
$X_{n_1+1} + \cdots + X_{n_1+n_2} = 0)$

$= P(T_1 = n_1)\, P(T_1 = n_2)$

Summing up over n_1, we obtain
$P(T_1 < \infty, T_2 - T_1 = n_2) = P(T_1 < \infty)\, P(T_1 = n_2)$
$\Rightarrow T_1, T_2 - T_1$ are identically distributed
if $P(T_1 < \infty) = 1$.

[10 marks]

5b. Similarly,

$$P(T_1=n_1, T_2-T_1=n_2, \dots, T_m-T_{m-1}=n_m) =$$

$$= \prod_{k=1}^{m} P(T_1=n_k)$$

summing over $n_1, n_2 \dots n_m$, we have

$$P(N>m) = P(T_m<\infty) = P(T_1<\infty)^m$$

and so

$$EN = \sum_{m=1}^{\infty} P(N \geq m) = \infty \text{ iff } P(T_1<\infty)=1.$$

Since $N = \sum_{n=1}^{\infty} \chi_{\{S_n=0\}}$,

$EN = \sum_{n=1}^{\infty} P(S_n=0)$. This series diverge by De-Moivre-Laplace theorems.

Thus $P(T_1<\infty)=1$.

[10 Marks]

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a separate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
Probability Theory_MATH60028 MATH97056 MATH70028	1	a,b mostly well done c has several methods for solution
Probability Theory_MATH60028 MATH97056 MATH70028	2	a is generally well done b one uses independence and an estimate on V, here mixed results
Probability Theory_MATH60028 MATH97056 MATH70028	3	Independence for (a) and Fubini theorem for (b) are relied upon
Probability Theory_MATH60028 MATH97056 MATH70028	4	a) The key was to show that A_n i.o is the same as zeta_n> lambda_n i.o. b) 2 series thm here results were mixed
Probability Theory_MATH60028 MATH97056 MATH70028	5	b) use of indicator function and then de Moivre-Laplace thms to estimate the sum of series