ExamModuleCode	Question Number	Comments for Students
M45S8	1	Spotting which of the processes were stationary andwhich were non-stationary caused the most problems in this question. Care, particularly with the second and third processes, was needed when considering the autocovariance, and this was oftern lacking.
M45S8	2	On the whole, this question was answered very well. There were two methods for computing the spectral density function at the end. The easiest was to consider the SDF of the MA(1) process, and then the frequency response function of the difference operator. Those of who who decided to turn the MA(1) into an MA(3) and then compute it's spectrum often became stuck on the messy algebraic manipulation
M45S8	3	It is clear that the final part caused people the most difficulty. Very few people got full marks for this bit. In particular, either the spectral representation did not include the mean term, or cross tems in the multiplication were not rigorously dealt with
M45S8	4	This question was answered very well. Many of you arrived at the correct answer for the coherence calcuation at the end. The part that caused the most problems was showing coherence is the abs squared of the correlation between dZ_X(f) and dZ_Y(f). This is very straight forward by considering the definitions of the spectral functions with regards to the orthogonal increments.
M45S8	5	This question was answered pretty well on the whole. There were quite a few marks available for the short essay type question at the beginning, and many of you didn't give the detail that was requred.

Imperial College London

M3S8

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2019

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Time Series

Date: Thursday 09 May 2019
Time: 10.00 - 12.00
Time Allowed: 2 Hours
This paper has 4 Questions.
Candidates should use ONE main answer book.
Supplementary books may only be used after the relevant main book(s) are full.
All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Calculators may not be used.

Imperial College London

M4/5S8

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2019

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Time Series

Date: Thursday 09 May 2019

Time: 10.00 - 12.30

Time Allowed: 2 Hours 30 Minutes

This paper has 5 Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
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- Calculators may not be used.

Note: Throughout this paper $\{\epsilon_t\}$ is a sequence of uncorrelated random variables (white noise) having zero mean and variance σ_ϵ^2, unless stated otherwise. The unqualified term "stationary" will always be taken to mean second-order stationary. All processes are real-valued unless stated otherwise. The sample interval is unity unless stated otherwise. B denotes the backward shift operator.

1. (a) What is meant by saying that a stochastic process is stationary?

   (b) Determine whether each of the following models for a random process $\{X_t\}$ is stationary, justifying your answer. For those that are stationary, specify their mean and autocovariance sequence.

   (i) $X_t = \epsilon_t \cos(ct)$, where $c \neq 0$ is a fixed constant.

   (ii) $X_t = \epsilon_t \epsilon_{t-1}$.

   (iii) $X_t = W y_t$, where W is a random variable with distribution $N(0,1)$ and

$$y_t = \begin{cases} +1 & t \text{ even} \\ -1 & t \text{ odd.} \end{cases}$$

   (c) An ARMA(p,q) process $\{X_t\}$ can be represented by the equation

$$\Phi(B)X_t = \Theta(B)\epsilon_t$$

   where $\Phi(z)$ and $\Theta(z)$ are p and q order z-polynomials, respectively. State conditions on $\Phi(z)$ and $\Theta(z)$ for $\{X_t\}$ to be

   (i) stationary,

   (ii) invertible.

   (d) Consider the process $\{X_t\}$ defined through the model

$$X_t - \epsilon_t = \frac{1}{2}X_{t-1} + \frac{1}{4}\epsilon_{t-1}.$$

   (i) Show that $\{X_t\}$ is both stationary and invertible.

   (ii) Express $\{X_t\}$ in its *general linear process* form and hence show $\text{var}\{X_t\} = \frac{7}{4}\sigma_\epsilon^2$.

2. (a) Let $\{X_t\}$ be the MA(1) process

$$X_t = \epsilon_t - \theta\epsilon_{t-1}.$$

(i) Derive the form of the autocorrelation sequence $\{\rho_\tau\}$ for $\{X_t\}$.

(ii) Show that $|\rho_1| \leq 1/2$ for any value of θ. For which values of θ does ρ_1 attain its maximum and minimum?

(b) The difference operator is defined as $\Delta = 1 - B$. You may use without proof:

$$X_t^{(d)} \equiv \Delta^d X_t = \sum_{k=0}^{d} \binom{d}{k} (-1)^k X_{t-k}.$$

Consider the process $\{X_t\}$ defined as

$$X_t = \alpha + \beta t + Y_t,$$

where $\{Y_t\}$ is a zero-mean stationary process.

(i) Show $\{X_t^{(2)}\}$ is a zero-mean stationary process.

(ii) State the three conditions that must be satisfied by a linear time invariant (LTI) digital filter and hence show that Δ^d is an LTI filter.

(iii) Find the frequency response function $G(f)$ associated with the LTI filter Δ^d.

(iv) If $\{Y_t\}$ is the MA(1) process $Y_t = \epsilon_t - \theta\epsilon_{t-1}$, show the spectral density function of $X_t^{(2)}$ is

$$S_{X^{(2)}}(f) = \sigma_\epsilon^2(6 - 8\cos(2\pi f) + 2\cos(4\pi f))(1 + \theta^2 - 2\theta\cos(2\pi f)).$$

3. (a) Let $X_1, ..., X_N$ be a realisation from a stationary process $\{X_t\}$ with mean μ and autocovariance sequence $\{s_\tau\}$.

Show that the sample mean

$$\bar{X} = \frac{1}{N}\sum_{t=1}^{N} X_t$$

is an unbiased estimator for μ and

$$\mathrm{var}\{\bar{X}\} = \frac{1}{N}\sum_{\tau=-(N-1)}^{(N-1)} \left(1 - \frac{|\tau|}{N}\right) s_\tau.$$

HINT: when confronted with a double sum, instead of summing across "rows", sum across "diagonals".

(b) The *biased* estimator $\hat{s}_\tau^{(p)}$ of $\{s_\tau\}$ is defined as

$$\hat{s}_\tau^{(p)} = \frac{1}{N}\sum_{t=1}^{N-|\tau|} (X_t - \bar{X})(X_{t+|\tau|} - \bar{X}) \qquad \tau = 0, \pm 1, \pm 2, ..., \pm(N-1).$$

When the mean is known, $\bar{X}$ is replaced by μ.

The periodogram $\hat{S}^{(p)}(\cdot)$ can be defined as the Fourier transform of $\{\hat{s}_\tau^{(p)}\}$. Show

$$\frac{1}{N}\sum_{k=0}^{N-1} \hat{S}^{(p)}(f_k) = \hat{s}_0^{(p)},$$

where $f_k = k/N$. *HINT: you may use without proof that*

$$\sum_{k=0}^{N-1} \mathrm{e}^{\mathrm{i}kx} = \frac{1 - \mathrm{e}^{\mathrm{i}Nx}}{1 - \mathrm{e}^{\mathrm{i}x}}$$

for $x \neq 0, \pm 2\pi, \pm 4\pi, ...$

[Question 3 continues on the next page]

(c) A random process $\{X_t\}$ with non-zero mean μ has the spectral representation

$$X_t - \mu = \int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}2\pi ft} \mathrm{d}Z(f).$$

If a time series has a non-zero mean, centring it (removing the mean) before performing spectral estimation is crucial. Failure to do so can have consequences.

Let $X_1, ..., X_N$ be a realisation of a white noise process with mean and variance both equal to 1. Consider the direct spectral estimator

$$\widehat{S}^{(d)}(f) \equiv \left| \sum_{t=1}^{N} h_t X_t \mathrm{e}^{-i2\pi ft} \right|^2, \qquad |f| \leq \frac{1}{2}$$

where $\{h_t\}$ is a data taper of length N normalised such that $\sum_{t=1}^{N} h_t^2 = 1$.

Show

$$E\{\widehat{S}^{(d)}(f)\} = 1 + \mathcal{H}(f),$$

where

$$\mathcal{H}(f) = \left| \sum_{t=1}^{N} h_t \mathrm{e}^{-\mathrm{i}2\pi ft} \right|^2.$$

You may use without proof that $\int_{-1/2}^{1/2} \mathcal{H}(f - f')\mathrm{d}f' = 1$ for all $f \in [-\frac{1}{2}, \frac{1}{2}]$.

4. (a) What is meant by saying a pair of real-valued discrete-time stochastic processes are jointly stationary?

(b) Let $\{X_t\}$ and $\{Y_t\}$ be a pair of zero-mean real-valued jointly stationary processes with spectral density functions $S_X(\cdot)$ and $S_Y(\cdot)$, respectively, and cross-spectrum $S_{XY}(\cdot)$. Considering their individual spectral representations

$$X_t = \int_{-1/2}^{1/2} e^{i2\pi ft} dZ_X(f) \qquad Y_t = \int_{-1/2}^{1/2} e^{i2\pi ft} dZ_Y(f),$$

show the coherence

$$\gamma^2_{XY}(f) = \frac{|S_{XY}(f)|^2}{S_X(f)S_Y(f)}$$

at frequency $f \in [-1/2, 1/2]$ is the magnitude square of the correlation between $dZ_X(f)$ and $dZ_Y(f)$. *HINT: for a pair of zero mean complex random variables S and T, $\text{cov}(S,T) = E\{S^*T\}$, where * denotes complex conjugation.*

(c) Let $\{X_t\}$ and $\{Y_t\}$ be a pair of zero-mean real-valued stationary processes that are independent of each other. They have autocovariance sequences $\{s_{X,\tau}\}$ and $\{s_{Y,\tau}\}$ and spectral density functions $S_X(\cdot)$ and $S_Y(\cdot)$, respectively. Consider the processes $\{V_t\}$ and $\{W_t\}$ defined as

$$V_t = AX_t + BY_t$$
$$W_t = CX_t + DY_t,$$

where A, B, C and D are each unit variance, zero mean real-valued random variables that are all independent of $\{X_t\}$ and $\{Y_t\}$. The vector $(A, B, C, D)^T$ has covariance matrix

$$\Sigma = \begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & 0 & \beta \\ \alpha & 0 & 1 & 0 \\ 0 & \beta & 0 & 1 \end{pmatrix}.$$

(i) Show that $\{V_t\}$ and $\{W_t\}$ are jointly stationary and determine their autocovariance sequences $\{s_{V,\tau}\}$ and $\{s_{W,\tau}\}$, and the cross-covariance sequence $\{s_{VW,\tau}\}$ in terms of $s_{X,\tau}$, $s_{Y,\tau}$, α and β.

(ii) Give an expression for $\gamma^2_{VW}(f)$, the coherence between $\{V_t\}$ and $\{W_t\}$ at frequency f, in terms of $S_X(f)$, $S_Y(f)$, α and β.

(iii) Suppose the autocovariance sequences for $\{X_t\}$ and $\{Y_t\}$ are given as

$$s_{X,\tau} = \begin{cases} 1 & \tau = 0 \\ \frac{1}{2} & |\tau| = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad s_{Y,\tau} = \begin{cases} 2 & \tau = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Show that

$$\gamma^2_{VW}(0) = \frac{(\alpha+\beta)^2}{4}.$$

5. Let $X_1, ..., X_N$ be a realisation of a zero mean stationary process $\{X_t\}$ with autocovariance sequence $\{s_\tau\}$ and spectral density function $S(\cdot)$. The direct spectral estimator $\widehat{S}^{(d)}(\cdot)$ is as defined in 3(c).

(a) What are the reasons for smoothing the periodogram and direct spectral estimators? Justify smoothing in the frequency domain by considering the average of the periodogram over a finite set of adjacent Fourier frequencies.

(b) The lag-window estimator of $S(f)$ is defined as

$$\widehat{S}^{(lw)}(f) = \int_{-1/2}^{1/2} W_m(f-\phi)\widehat{S}^{(d)}(\phi)\mathrm{d}\phi$$

where $W_m(\cdot)$ is a symmetric real-valued periodic (period 1) function which is square integrable over $[-\frac{1}{2}, \frac{1}{2}]$ and whose smoothing properties can be controlled by a parameter m. Furthermore, $W_m(\cdot)$ is normalised such that $\int_{-1/2}^{1/2} W_m(f)\mathrm{d}f = 1$.
Using a data taper $\{h_t\}$, it is true that

$$E\{\widehat{S}^{(lw)}(f)\} = \int_{-1/2}^{1/2} \mathcal{U}_m(f-\phi)S(\phi)\mathrm{d}\phi,$$

where $\mathcal{U}_m(f) = \int_{-1/2}^{1/2} W_m(f-f')\mathcal{H}(f')\mathrm{d}f'$ and $\mathcal{H}(\cdot)$ is as defined in Question 3(c).

(i) Show that

$$\mathcal{U}_m(f) = \sum_{\tau=-(N-1)}^{N-1} w_{\tau,m}\left(\sum_{t=1}^{N-|\tau|} h_t h_{t+|\tau|}\right) \mathrm{e}^{-\mathrm{i}2\pi f\tau},$$

where $W_m(\cdot)$ and $\{w_{\tau,m}\}$ form the Fourier transform pair

$$w_{\tau,m} = \int_{-1/2}^{1/2} W_m(f)\mathrm{e}^{\mathrm{i}2\pi f\tau}\mathrm{d}\phi \qquad W_m(f) = \sum_{\tau=-(N-1)}^{N-1} w_{\tau,m}\mathrm{e}^{-\mathrm{i}2\pi f\tau}.$$

HINT: it is true that

$$\left|\sum_{t=1}^{N} h_t \mathrm{e}^{-\mathrm{i}2\pi ft}\right|^2 = \sum_{\tau=-(N-1)}^{N-1}\sum_{t=1}^{N-|\tau|} h_t h_{t+|\tau|}\mathrm{e}^{-\mathrm{i}2\pi f\tau}.$$

(ii) Show that

$$E\{\widehat{S}^{(lw)}(f)\} = \sum_{\tau=-(N-1)}^{N-1}\left(w_{\tau,m}s_\tau \sum_{t=1}^{N-|\tau|} h_t h_{t+|\tau|}\right)\mathrm{e}^{-\mathrm{i}2\pi f\tau}.$$

(iii) Let $X_1, ..., X_N$ be a realisation of the zero mean process $\{X_t\}$ given in Question 4(c)(iii). Using the rectangular data-taper $h_t = N^{-1/2}$, $t = 1, ..., N$, show that

$$E\{\widehat{S}^{(lw)}(f)\} - S(f) = \left(w_{1,m}\frac{N-1}{N} - 1\right)\cos(2\pi f).$$

HINT: $\{w_{m,\tau}\}$ is symmetric about $\tau = 0$.

M345S8 SOLUTIONS

1. (a) $\{X_t\}$ is second-order stationary if $E\{X_t\}$ is a finite constant for all t, var$\{X_t\}$ is a finite constant for all t, and cov$\{X_t, X_{t+\tau}\}$, is a finite quantity depending only on τ and not on t. seen ⇓ 3(A)

(b) (i) This is non-stationary. The mean is constant as $E\{X_t\} = 0$ for all t, however $E\{X_t^2\} = \sigma_\epsilon^2 \cos^2(ct)$ which depends on t. 2(A)

(ii) This is stationary and is shown by considering the mean and autocovariance. sim. seen ⇓

Mean: $E\{X_t\} = E\{\epsilon_t\epsilon_{t-1}\} = E\{\epsilon_t\}E\{\epsilon_{t-1}\} = 0$.

Autocovariance: $E\{X_tX_{t+\tau}\} = E\{\epsilon_t\epsilon_{t-1}\epsilon_{t+\tau}\epsilon_{t-1+\tau}\}$. First consider $\tau = 0$, then $E\{\epsilon_t\epsilon_{t-1}\epsilon_t\epsilon_{t-1}\} = E\{\epsilon_t^2\}E\{\epsilon_{t-1}^2\} = \sigma_\epsilon^4$. When $\tau = 1$ (and similarly for $\tau = -1$), $E\{\epsilon_t\epsilon_{t-1}\epsilon_{t+1}\epsilon_t\} = E\{\epsilon_t^2\}E\{\epsilon_{t+1}\}E\{\epsilon_{t-1}\} = 0$. Likewise, for all $|\tau| > 1$, $E\{X_tX_{t+\tau}\} = 0$. Therefore

$$E\{X_tX_{t+\tau}\} = \begin{cases} \sigma_\epsilon^4 & \tau = 0 \\ 0 & \text{otherwise} \end{cases}$$

and the process is stationary. 3(B)

(iii) This is stationary and is shown by considering the mean and autocovariance. unseen ⇓

Mean: $E\{X_t\} = y_tE\{W\} = 0$.

Autocovariance: $E\{X_tX_{t+\tau}\} = E\{W^2\}y_ty_{t+\tau} = y_ty_{t+\tau}$. For any fixed t, when τ is even $y_ty_{t+\tau} = 1$ and when τ is odd $y_ty_{t+\tau} = -1$, therefore

$$E\{X_tX_{t+\tau}\} = \begin{cases} 1 & \tau \text{ even} \\ -1 & \tau \text{ odd} \end{cases}$$

and it is not dependent t. Therefore it is stationary. 3(B)

(c) (i) Roots of $\Phi(z)$ lie outside the unit circle. seen ⇓

(ii) Roots of $\Theta(z)$ lie outside the unit circle. 2(A)

(d) (i) The process can be formulated as $\boxed{\text{sim. seen} \Downarrow}$

$$X_t - \frac{1}{2}X_{t-1} = \epsilon_t + \frac{1}{4}\epsilon_{t-1}$$

which is in ARMA(1,1) form where $\Phi(z) = 1 - \frac{1}{2}z$ and $\Theta(z) = 1 + \frac{1}{4}z$. The roots of $\Phi(z)$ and $\Theta(z)$ are 2 and -4, respectively, which both lie outside the unit circle. Process $\{X_t\}$ is therefore stationary and invertible. $\boxed{2(\text{A})}$

(ii) We are required to put it in general linear form $X_t = G(B)\epsilon_t$. Here

$$\begin{aligned}
G(z) &= \Phi^{-1}(z)\Theta(z) \\
&= (1 + \frac{1}{2}z + \frac{1}{4}z^2 + \ldots)(1 + \frac{1}{4}z) \\
&= (1 + \frac{1}{2}z + \frac{1}{4}z^2 + \ldots) + (\frac{1}{4}z + \frac{1}{8}z^2 + \ldots) \\
&= 1 + \sum_{k=1}^{\infty} \frac{3}{2^{k+1}} z^k.
\end{aligned}$$

Therefore, general linear process form is

$$X_t = \epsilon_t + \sum_{k=1}^{\infty} \frac{3}{2^{k+1}} \epsilon_{t-k}.$$

$\boxed{2(\text{B})}$

For a general linear process $X_t = \sum_{k=0}^{\infty} g_k \epsilon_{t-k}$, we have $\text{var}\{X_t\} = \sigma_\epsilon^2 \sum_{k=0}^{\infty} g_k^2$. Therefore, $\text{var}\{X_t\} = \sigma_\epsilon^2(1 + \sum_{k=1}^{\infty} \frac{9}{4^{k+1}}) = \sigma_\epsilon^2(1 + 9(\frac{1}{1-\frac{1}{4}} - 1 - \frac{1}{4})) = \frac{7}{4}\sigma_\epsilon^2$. $\boxed{3(\text{B})}$

2. (a) We immediately have $E\{X_t\} = 0$, therefore $\boxed{\text{seen} \Downarrow}$

$$\begin{aligned} s_\tau &= E\{X_t X_{t+\tau}\} = E\{(\epsilon_t - \theta\epsilon_{t-1})(\epsilon_{t+\tau} - \theta\epsilon_{t-1+\tau})\} \\ &= E\{\epsilon_t\epsilon_{t+\tau}\} - \theta(E\{\epsilon_t\epsilon_{t-1+\tau}\} + E\{\epsilon_{t-1}\epsilon_{t+\tau}\}) + \theta^2 E\{\epsilon_{t-1}\epsilon_{t-1+\tau}\} \end{aligned}$$

This gives

$$\begin{aligned} s_0 &= \sigma_\epsilon^2(1+\theta^2), \\ s_1 &= -\sigma_\epsilon^2\theta = s_{-1} \end{aligned}$$

and $s_\tau = 0$ for all $|\tau| > 1$. Therefore the autocorrelation sequence, defined as $\rho_\tau = s_\tau / s_0$ is

$$\rho_\tau = \begin{cases} 1 & \tau = 0 \\ -\frac{\theta}{1+\theta^2} & |\tau| = 1 \\ 0 & |\tau| > 1 \end{cases}$$

$\boxed{3(\text{A})}$

It follows that $\frac{d\rho_1}{d\theta} = \frac{d}{d\theta}\frac{-\theta}{1+\theta^2} = \frac{-(1+\theta)(1-\theta)}{(1+\theta^2)^2}$. This equals zero when $\theta = 1$ (minimum, $\rho_1 = -1/2$) and -1 (maximum, $\rho_1 = 1/2$.) $\boxed{\text{unseen} \Downarrow}$ $\boxed{1(\text{A})}$

(b) (i) $\boxed{\text{seen} \Downarrow}$

$$\begin{aligned} X_t^{(2)} &= \Delta^2 X_t \\ &= \Delta(\alpha - \alpha + \beta - \beta(t-1) + \Delta Y_t) \\ &= \Delta^2 Y_t \\ &= Y_t - 2Y_{t-1} + Y_{t-2}. \end{aligned}$$

This is stationary because, by the stationarity of $\{Y_t\}$,

$$E\{X_t^{(2)}\} = E\{Y_t - 2Y_{t-1} + Y_{t-2}\} = E\{Y_t\} - 2E\{Y_{t-1}\} + E\{Y_{t-2}\} = 0$$

and

$$\begin{aligned} E\{X_t^{(2)} X_{t+\tau}^{(2)}\} &= E\{(Y_t - 2Y_{t-1} + Y_{t-2})(Y_{t+\tau} - 2Y_{t-1+\tau} + Y_{t-2+\tau})\} \\ &= s_{Y,\tau} - 2s_{Y,\tau-1} + s_{Y,\tau-2} - 2s_{Y,\tau+1} + 4s_{Y,\tau} - 2s_{Y,\tau-1} + s_{Y,\tau+2} - 2s_{Y,\tau+1} + s_{Y,\tau} \\ &= s_{Y,\tau-2} - 4s_{Y,\tau-1} + 6s_{Y,\tau} - 4s_{Y,\tau+1} + s_{Y,\tau+2} \end{aligned}$$

which depends only on τ. $\boxed{3(\text{B})}$

(ii) The conditions for an LTI digital filter are: $\boxed{\text{sim. seen} \Downarrow}$

1. Scale preservation:
$$L\{\alpha x_t\} = \alpha L\{x_t\}.$$
This is satisfied by Δ^d because
$$\Delta^d \alpha x_t = \sum_{k=0}^{d} \binom{d}{k} (-1)^k \alpha x_{t-k} = \alpha \sum_{k=0}^{d} \binom{d}{k} (-1)^k x_{t-k} = \alpha \Delta^d x_t.$$

2. Superposition:
$$L\{x_{1,t} + x_{2,t}\} = L\{x_{1,t}\} + L\{x_{2,t}\}.$$
This is satisfied by Δ^d because
$$\Delta^d (x_{1,t} + x_{2,t}) = \Delta^d x_{1,t} + \Delta^d x_{2,t}.$$

3. Time invariance: if $L\{x_t\} = y_t$ then $L\{x_{t+\tau}\} = y_{t+\tau}$.
This is satisfied by Δ^d because if $\Delta^d x_t = y_t$
$$\Delta^d x_{t+\tau} = \sum_{k=0}^{d} \binom{d}{k} (-1)^k x_{t+\tau-k} = \sum_{k=0}^{d} \binom{d}{k} (-1)^k x_{t'-k} = y_{t'} = y_{t+\tau}.$$
$\boxed{5(\mathrm{A})}$

(iii) Frequency response function is computed by considering
$$\Delta^d e^{i2\pi ft} = \sum_{k=0}^{d} \binom{d}{k} (-1)^k e^{i2\pi f(t-k)} = e^{i2\pi ft} \sum_{k=0}^{d} \binom{d}{k} (-1)^k e^{-i2\pi fk}.$$

Therefore, $G(f) = \sum_{k=0}^{d} \binom{d}{k} (-1)^k e^{-i2\pi fk} = (1 - e^{-i2\pi f})^d.$ $\boxed{2(\mathrm{A})}$

(iv) We have that $X_t^{(2)} = Y_t^{(2)} = \Delta^2 Y_t$. Therefore, $S_{X^{(2)}}(f) = |G(f)|^2 S_Y(f)$, where from 2b(iii) the frequency response function of LTI filter Δ^2 is given as $G(f) = 1 - 2\mathrm{e}^{-\mathrm{i}2\pi f} + \mathrm{e}^{-\mathrm{i}4\pi f}$. $\boxed{\text{unseen} \Downarrow}$

We first have to compute the spectral density function of $\{Y_t\}$. This can either be done via a Fourier transform of the autocovariance sequence (see 2(a)), or by considering an MA(1) process to be a linear filter on a white noise process. Here is the Fourier transform method:

$$\begin{aligned} S_Y(f) &= \sum_{\tau=-\infty}^{\infty} s_{Y,\tau} \mathrm{e}^{-\mathrm{i}2\pi f\tau} \\ &= \sigma_\epsilon^2(-\theta \mathrm{e}^{\mathrm{i}2\pi f} + 1 + \theta^2 - \theta \mathrm{e}^{-\mathrm{i}2\pi f}) \\ &= \sigma_\epsilon^2(1 + \theta^2 - 2\theta\cos(2\pi f t)). \end{aligned}$$

With

$$\begin{aligned} |G(f)|^2 &= (1 - 2\mathrm{e}^{-\mathrm{i}2\pi f} + \mathrm{e}^{-\mathrm{i}4\pi f})(1 - 2\mathrm{e}^{\mathrm{i}2\pi f} + \mathrm{e}^{\mathrm{i}4\pi f}) \\ &= 6 - 8\cos(2\pi f) + 2\cos(4\pi f), \end{aligned}$$

it follows that

$$S_{X^{(2)}}(f) = \sigma_\epsilon^2(6 - 8\cos(2\pi f) + 2\cos(4\pi f))(1 + \theta^2 - 2\theta\cos(2\pi f t)).$$

$\boxed{6(\mathrm{D})}$

3. (a) To show unbiasedness: $\boxed{\text{seen} \Downarrow}$

$$\mathsf{E}\{\bar{X}\} = \frac{1}{N}\sum_{t=1}^{n}\mathsf{E}\{X_t\} = \frac{1}{N}N\mu = \mu,$$

therefore $\bar{X}$ is an unbiased estimator of μ. $\boxed{2(\mathrm{A})}$

Now,

$$\begin{aligned}
\mathsf{var}\{\bar{X}\} &= \mathsf{E}\{(\bar{X}-\mu)^2\} \\
&= \mathsf{E}\left\{\left(\frac{1}{N}\sum_{i=1}^{N}(X_i-\mu)\right)^2\right\} \\
&= \frac{1}{N^2}\sum_{t=1}^{N}\sum_{u=1}^{N}\mathsf{E}\{(X_t-\mu)(X_u-\mu)\} \\
&= \frac{1}{N^2}\sum_{t=1}^{N}\sum_{u=1}^{N}s_{u-t} \\
&= \frac{1}{N^2}\sum_{\tau=-(N-1)}^{N-1}\sum_{k=1}^{N-|\tau|}s_\tau \\
&= \frac{1}{N^2}\sum_{\tau=-(N-1)}^{N-1}(N-|\tau|)s_\tau \\
&= \frac{1}{N}\sum_{\tau=-(N-1)}^{N-1}\left(1-\frac{|\tau|}{N}\right)s_\tau.
\end{aligned}$$

The summation interchange merely swaps row sums for diagonal sums. $\boxed{5(\mathrm{A})}$

(b) Using the Fourier relationship $\boxed{\text{unseen} \Downarrow}$

$$\widehat{S}^{(p)}(f) = \sum_{\tau=-(N-1)}^{(N-1)}\widehat{s}_\tau^{(p)}\mathrm{e}^{-\mathrm{i}2\pi f\tau},$$

it follows that

$$\begin{aligned}
\frac{1}{N}\sum_{k=0}^{N-1}\widehat{S}^{(p)}(f_k) &= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{\tau=-(N-1)}^{(N-1)}\widehat{s}_\tau^{(p)}\mathrm{e}^{-\mathrm{i}2\pi f_k\tau} \\
&= \frac{1}{N}\sum_{\tau=-(N-1)}^{(N-1)}\widehat{s}_\tau^{(p)}\sum_{k=0}^{N-1}\mathrm{e}^{-\mathrm{i}2\pi f_k\tau} \\
&= \frac{1}{N}\sum_{\tau=-(N-1)}^{(N-1)}\widehat{s}_\tau^{(p)}\sum_{k=0}^{N-1}\mathrm{e}^{-\mathrm{i}k\frac{2\pi\tau}{N}}
\end{aligned}$$

Using the given hint,

$$\sum_{k=0}^{N-1} \mathrm{e}^{-\mathrm{i}k\frac{2\pi\tau}{N}} = \frac{1-\mathrm{e}^{\mathrm{i}2\pi\tau}}{1-\mathrm{e}^{-\mathrm{i}\frac{2\pi\tau}{N}}} = 0 \qquad \text{for } \tau = \pm 1, \pm 2, \ldots \pm (N-1)$$

and clearly for $\tau = 0$

$$\sum_{k=0}^{N-1} \mathrm{e}^{-\mathrm{i}k\frac{2\pi\tau}{N}} = \sum_{k=0}^{N-1} 1 = N.$$

Therefore

$$\frac{1}{N}\sum_{\tau=-(N-1)}^{(N-1)} \widehat{s}_\tau^{(p)} \sum_{k=0}^{N-1} \mathrm{e}^{-\mathrm{i}k\frac{2\pi\tau}{N}} = \frac{1}{N} N \widehat{s}_0^{(p)} = \widehat{s}_0^{(p)}.$$

$\boxed{5(\mathrm{C})}$

$\boxed{\text{sim. seen} \Downarrow}$

$\boxed{\text{unseen} \Downarrow}$

(c) Considering:

$$\mathrm{E}\{|J(f)|^2\} \quad \text{where} \quad J(f) = \sum_{t=1}^{N} h_t X_t \mathrm{e}^{-\mathrm{i}2\pi f t}, \quad |f| \leq \frac{1}{2}.$$

$[\widehat{S}^{(d)}(f) = |J(f)|^2.]$
We know from the spectral representation theorem that,

$$X_t = 1 + \int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}2\pi f' t}\, \mathrm{d}Z(f'),$$

so that,

$$\begin{aligned} J(f) &= \sum_{t=1}^{N}\left(\int_{-1/2}^{1/2} h_t \mathrm{e}^{\mathrm{i}2\pi f' t}\, \mathrm{d}Z(f')\right)\mathrm{e}^{-\mathrm{i}2\pi f t} + \sum_{t=1}^{N} h_t \mathrm{e}^{-\mathrm{i}2\pi f t} \\ &= \int_{-1/2}^{1/2} \sum_{t=1}^{N} h_t \mathrm{e}^{-\mathrm{i}2\pi (f-f')t}\, \mathrm{d}Z(f') + \sum_{t=1}^{N} h_t \mathrm{e}^{-\mathrm{i}2\pi f t} \end{aligned}$$

Then

$$
\begin{aligned}
\mathsf{E}\{\widehat{S}^{(p)}(f)\} &= \mathsf{E}\{|J(f)|^2\} = \mathsf{E}\{J^*(f)J(f)\} \\
&= \mathsf{E}\left\{\left(\int_{-1/2}^{1/2}\sum_{t=1}^{N} h_t \mathrm{e}^{\mathrm{i}2\pi(f-f')t}\,\mathrm{d}Z(f') + \sum_{t=1}^{N} h_t \mathrm{e}^{\mathrm{i}2\pi ft}\right)\right. \\
&\qquad \left.\times\left(\int_{-1/2}^{1/2}\sum_{t=1}^{N} h_t \mathrm{e}^{-\mathrm{i}2\pi(f-f')t}\,\mathrm{d}Z(f') + \sum_{t=1}^{N} h_t \mathrm{e}^{-\mathrm{i}2\pi ft}\right)\right\} \\
&= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\sum_{t=1}^{N} h_t \mathrm{e}^{\mathrm{i}2\pi(f-f')t}\sum_{s=1}^{N} h_s \mathrm{e}^{-\mathrm{i}2\pi(f-f'')s}\mathsf{E}\{\mathrm{d}Z^*(f')\,\mathrm{d}Z(f'')\} \\
&\qquad + \sum_{t=1}^{N} h_t \mathrm{e}^{-\mathrm{i}2\pi ft}\int_{-1/2}^{1/2}\sum_{s=1}^{N} h_s \mathrm{e}^{\mathrm{i}2\pi(f-f')s}\,E\{\mathrm{d}Z(f')\} \\
&\qquad + \sum_{t=1}^{N} h_t \mathrm{e}^{\mathrm{i}2\pi ft}\int_{-1/2}^{1/2}\sum_{s=1}^{N} h_s \mathrm{e}^{-\mathrm{i}2\pi(f-f')s}\,E\{\mathrm{d}Z(f')\} \\
&\qquad + \left|\sum_{t=1}^{N} h_t \mathrm{e}^{-\mathrm{i}2\pi ft}\right|^2 \\
&= \int_{-1/2}^{1/2}\mathcal{H}(f-f')\cdot 1\,\mathrm{d}f' + \mathcal{H}(f) \\
&= 1 + \mathcal{H}(f)
\end{aligned}
$$

using properties of orthogonal increment process that $E\{\mathrm{d}Z(f')\} = 0$ and $E\{\mathrm{d}Z^*(f')\mathrm{d}Z(f'')\} = 0$ if $f' \neq f''$ and equals $S(f) = 1$ (unit variance white noise) if $f' = f''$. $\boxed{8(\mathrm{D})}$

4. (a) Two real-valued discrete-time processes $\{X_t\}$ and $\{Y_t\}$ are said to be jointly stationary stochastic processes if each are separately second-order stationary processes, and $\text{cov}\{X_t, Y_{t+\tau}\}$ is a function of τ only. seen ⇓

3(A)

(b) The cross-spectrum is defined as

$$S_{XY}(f)\mathrm{d}f = E\{\mathrm{d}Z_X^*(f)\mathrm{d}Z_Y(f)\},$$

and the sdfs as $S_X(f)\mathrm{d}f = E\{|\mathrm{d}Z_X(f)|^2\}$ and $S_Y(f)\mathrm{d}f = E\{|\mathrm{d}Z_Y(f)|^2\}$, so

$$\begin{aligned}\gamma^2_{XY}(f) &= \frac{E\{\mathrm{d}Z_X^*(f)\mathrm{d}Z_Y(f)\}}{E\{|\mathrm{d}Z_X(f)|^2\}E\{|\mathrm{d}Z_Y(f)|^2\}}\\ &= \frac{\text{cov}\{\mathrm{d}Z_X(f), \mathrm{d}Z_Y(f)\}}{\text{var}\{\mathrm{d}Z_X(f)\}\,\text{var}\{\mathrm{d}Z_Y(f)\}}\end{aligned}$$

which is the magnitude square of the correlation between $\mathrm{d}Z_X(f)$ and $\mathrm{d}Z_Y(f)$. 3(A)

(c) (i) We are first required to check that $\{V_t\}$ and $\{W_t\}$ are individually stationary. Considering $\{V_t\}$: sim. seen ⇓

Mean: $E\{V_t\} = E\{AX_t + BY_t\} = E\{AX_t\} + E\{BY_t\} = E\{A\}E\{X_t\} + E\{B\}E\{Y_t\} = 0.$

Autocovariance: $s_{V,\tau} = \text{cov}\{V_t, V_{t+\tau}\} = E\{V_tV_{t+\tau}\} = E\{(AX_t + BY_t)(AX_{t+\tau} + BY_{t+\tau})\} = E\{A^2\}E\{X_tX_{t+\tau}\} + E\{AB\}E\{X_tY_{t+\tau}\} + E\{AB\}E\{X_{t+\tau}Y_t\} + E\{B^2\}E\{Y_tY_{t+\tau}\} = 1\cdot s_{X,\tau} + 0 + 0 + 1\cdot s_{Y,\tau} = s_{X,\tau} + s_{Y,\tau}$. Therefore $\{V_t\}$ is stationary, as is $\{W_t\}$ by an identical argument with also $s_{W,\tau} = s_{X,\tau} + s_{Y,\tau}$.

It is next required that $\text{cov}\{V_t, W_{t+\tau}\}$ depends only on τ.

$\text{cov}\{V_t, W_{t+\tau}\} = E\{V_tW_{t+\tau}\} = E\{(AX_t + BY_t)(CX_{t+\tau} + DY_{t+\tau})\} = E\{AC\}E\{X_tX_{t+\tau}\} + E\{AD\}E\{X_tY_{t+\tau}\} + E\{BC\}E\{X_{t+\tau}Y_t\} + E\{BD\}E\{Y_tY_{t+\tau}\} = \alpha s_{X,\tau} + 0 + 0 + \beta s_{Y,\tau} = \alpha s_{X,\tau} + \beta s_{Y,\tau}$. 5(B)

(ii) The cross spectrum for $\{V_t\}$ and $\{W_t\}$ is the Fourier transform of the $s_{VW,\tau}$. It is shown in the (c)(i) that $s_{VW,\tau} = \alpha s_{X,\tau} + \beta s_{Y,\tau}$, therefore, taking the Fourier transform, we have $S_{VW}(f) = \alpha S_X(f) + \beta S_Y(f)$, which we recognise as being real valued. Using an analogous argument, we have $S_V(f) = S_W(f) = S_X(f) + S_Y(f)$. Therefore the coherence is

$$\gamma^2_{VW}(f) = \left(\frac{\alpha S_X(f) + \beta S_{Y(f)}}{S_X(f) + S_Y(f)}\right)^2.$$

5(C)

(iii) The spectral density functions are computed by taking the Fourier transform of the respective autocovariance sequence. Specifically, $\boxed{\text{unseen} \Downarrow}$

$$\begin{aligned} S_X(f) &= \sum_{\tau=-\infty}^{\infty} s_{X,\tau} \mathrm{e}^{-\mathrm{i}2\pi f\tau} \\ &= \frac{1}{2}\mathrm{e}^{\mathrm{i}2\pi f} + 1 + \frac{1}{2}\mathrm{e}^{-\mathrm{i}2\pi f} \\ &= 1 + \cos(2\pi f) \end{aligned}$$

and

$$\begin{aligned} S_Y(f) &= \sum_{\tau=-\infty}^{\infty} s_{Y,\tau} \mathrm{e}^{-\mathrm{i}2\pi f\tau} \\ &= 2. \end{aligned}$$

Therefore

$$\gamma^2_{VW}(f) = \left(\frac{\alpha(1+\cos(2\pi f)) + 2\beta}{3+\cos(2\pi f)}\right)^2.$$

When $f = 0$, this becomes

$$\gamma^2_{VW}(0) = ((2\alpha + 2\beta)/4)^2 = (\alpha+\beta)^2/4.$$

$\boxed{4(\mathrm{C})}$

5. (a) - The periodogram has poor bias and variance properties. $\boxed{\text{seen} \Downarrow}$

- For spectra with large dynamic range, the bias can be significantly reduced by tapering. However, variance problems still persist.
- Traditional approaches to this problem look to smooth $\widehat{S}^{(d)}(\cdot)$ across frequencies.
- Suppose N is large enough so that the periodogram $\widehat{S}^{(p)}(\cdot)$ can reasonably be considered an unbiased estimator of $S(\cdot)$ and is pair-wise uncorrelated at the Fourier frequencies $f_k = k/N$. If $S(\cdot)$ is slowly varying in the neighbourhood of, for example, f_k, then

$$S(f_{k-M}) \approx \cdots \approx S(f_k) \approx \cdots \approx S(f_{k+M})$$

are a set of $2M+1$ unbiased and uncorrelated estimators of $S(f_k)$. Therefore the average of these, namely

$$\bar{S}(f_k) \equiv \frac{1}{2M+1} \sum_{j=-M}^{M} \widehat{S}^{(p)}(f_{k-j})$$

will have

$$E\{\bar{S}(f_k)\} \approx S(f_k)$$

and

$$\operatorname{var} \bar{S}(f_k) \approx \frac{\operatorname{var}\{\widehat{S}^{(p)}(f_k)\}}{2M+1}.$$

- This concept can be extended to averaging over any discrete set of frequencies, or over a continuous range of frequencies through a convolution of the type

$$\widehat{S}^{(lw)}(f) = \int_{-1/2}^{1/2} V_m(f-\phi)\widehat{S}^{(d)}(\phi)\mathrm{d}\phi.$$

$\boxed{8(\mathrm{B})}$

(b) (i) $\boxed{\text{unseen} \Downarrow}$

$$\begin{aligned}
\mathcal{U}_m(f) &= \int_{-1/2}^{1/2} W_m(f-f')\mathcal{H}(f')\mathrm{d}f' \\
&= \int_{-1/2}^{1/2} \sum_{\tau=-(N-1)}^{(N-1)} w_{\tau,m} \mathrm{e}^{-\mathrm{i}2\pi(f-f')\tau} \left| \sum_{t=1}^{N} h_t \mathrm{e}^{-\mathrm{i}2\pi f' t} \right|^2 \mathrm{d}f' \\
&= \int_{-1/2}^{1/2} \sum_{\tau=-(N-1)}^{(N-1)} w_{\tau,m} \mathrm{e}^{-\mathrm{i}2\pi(f-f')\tau} \sum_{\tau'=-(N-1)}^{N-1} \sum_{t=1}^{N-|\tau'|} h_t h_{t+|\tau'|} \mathrm{e}^{-\mathrm{i}2\pi f'\tau'} \mathrm{d}f' \\
&= \sum_{\tau=-(N-1)}^{(N-1)} w_{\tau,m} \mathrm{e}^{-\mathrm{i}2\pi f\tau} \sum_{\tau'=-(N-1)}^{N-1} \sum_{t=1}^{N-|\tau'|} h_t h_{t+|\tau'|} \int_{-1/2}^{1/2} \mathrm{e}^{-\mathrm{i}2\pi f'(\tau'-\tau)} \mathrm{d}f'
\end{aligned}$$

Considering the integral, we have

$$\int_{-1/2}^{1/2} \mathrm{e}^{-\mathrm{i}2\pi f'(\tau'-\tau)}\mathrm{d}f' = \begin{cases} 1 & \tau = \tau' \\ 0 & \tau \neq \tau' \end{cases},$$

and it follows that

$$\mathcal{U}_m(f) = \sum_{\tau-(N-1)}^{(N-1)} w_{\tau,m} \left(\sum_{t=1}^{N-|\tau|} h_t h_{t+|\tau|} \right) \mathrm{e}^{-\mathrm{i}2\pi f\tau}.$$

$\boxed{5(\mathrm{D})}$

(ii) Using the result from (i), we have

$$\begin{aligned}
E\{\widehat{S}^{(lw)}(f)\} &= \int_{-1/2}^{1/2} \mathcal{U}_m(f-\phi)S(\phi)\mathrm{d}\phi \\
&= \int_{-1/2}^{1/2} \sum_{\tau-(N-1)}^{(N-1)} w_{\tau,m} \left(\sum_{t=1}^{N-|\tau|} h_t h_{t+|\tau|} \right) \mathrm{e}^{-\mathrm{i}2\pi(f-\phi)\tau} S(\phi)\mathrm{d}\phi \\
&= \sum_{\tau-(N-1)}^{(N-1)} w_{\tau,m} \left(\sum_{t=1}^{N-|\tau|} h_t h_{t+|\tau|} \right) \mathrm{e}^{-\mathrm{i}2\pi f\tau} \int_{-1/2}^{1/2} S(\phi)\mathrm{e}^{\mathrm{i}2\pi\phi\tau}\mathrm{d}\phi \\
&= \sum_{\tau=-(N-1)}^{N-1} \left(w_{\tau,m} s_\tau \sum_{t=1}^{N-|\tau|} h_t h_{t+|\tau|} \right) \mathrm{e}^{-\mathrm{i}2\pi f\tau}
\end{aligned}$$

due to $s_\tau = \int_{-1/2}^{1/2} S(\phi)\mathrm{e}^{\mathrm{i}2\pi\phi\tau}\mathrm{d}\phi$. $\boxed{4(\mathrm{D})}$

(iii) For $\{X_t\}$, $s_0 = 1$, $s_{-1} = s_1 = \frac{1}{2}$ and $s_\tau = 0$ for all $|\tau| > 1$. Therefore (with $w_{0,m} = \int_{-1/2}^{1/2} W_m(f)\mathrm{d}f = 1$),

$$\begin{aligned}
E\{\widehat{S}^{(lw)}(f)\} &= \frac{1}{2}w_{-1,m}\frac{N-1}{N}\mathrm{e}^{\mathrm{i}2\pi f} + w_{0,m} + \frac{1}{2}w_{1,m}\frac{N-1}{N}\mathrm{e}^{-\mathrm{i}2\pi f} \\
&= 1 + w_{1,m}\frac{N-1}{N}\cos(2\pi f). \\
E\{\widehat{S}^{(lw)}(f)\} - S(f) &= 1 + w_{1,m}\frac{N-1}{N}\cos(2\pi f) - (1+\cos(2\pi f)) \\
&= \left(w_{1,m}\frac{N-1}{N} - 1 \right)\cos(2\pi f).
\end{aligned}$$

$\boxed{3(\mathrm{D})}$