Exam Module Code	Question Number	Comments for Students
M45F22	1	Here quite a of students lost points in part (a) by stating the inequality $(x-K_1)^+ \le (x-K_2)^+ + K_2 - K_1$ without proving it, and/or in part (b) by proving an example or a market model which had arbitrage
M45F22	2	This questions was the one that stood out as the one in which students did best, with a high number of students getting full marks
M45F22	3	Here the most common mistakes were in part (d), where students either failed to correctly compute $\tilde{p}_n$, or to say that it being determinstic means that the $(Y_i)_i$ are independent under Q. It was also quite common to lose points in part (b), either by answering incorrectly or yb providing a correct answer without a proper explanation
M45F22	4	Here many students lost points in part (d), either not attempting it, or not being able to cite and use the independence lemma, or not writing explicitly the equations defining $v_n(y)$
M45F22	5	Here many students got full marks; the ones that didn't often didn't attempt part (b), and thus lost many points

Imperial College
London

M3F22

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2019

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Mathematical Finance: An Introduction to Option Pricing

Date: Tuesday 07 May 2019

Time: 14.00 - 16.00

Time Allowed: 2 Hours

This paper has 4 Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Calculators may not be used.

Imperial College London

M4/5F22

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2019

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Mathematical Finance: An Introduction to Option Pricing

Date: Tuesday 07 May 2019

Time: 14.00 - 16.30

Time Allowed: 2 Hours 30 Minutes

This paper has 5 Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

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1. Consider an arbitrage-free market model where trading takes place at times $t \in \{0, 1, \dots, T\} =: \mathbb{T}$, and the market is composed of a stock which has value $S_t > 0$ at time $t \in \mathbb{T}$, and of a bank account with constant interest rate $r > 0$. Denote by $(C_t(K, T))_{t \in \mathbb{T}}$ an arbitrage-free price of the European call option (on the stock) with strike price $K \geq 0$.

   (a) Prove that $0 \leq K_1 \leq K_2$ implies

$$C_0(T, K_1) \leq C_0(T, K_2) + \frac{K_2 - K_1}{(1+r)^T}. \tag{1}$$

   (b) Give an example of $r, S, \mathbb{P}$ for which the inequality (1) holds with equality. *Hint: take* $T = 1$.

2. Consider the trinomial model with time index $\{0, 1\}$, and a market made of a bank account with interest rate $r = 1$ and of one stock whose price is given by $S_0 = 2$ and

$$S_1(\omega) = \begin{cases} 2 & \text{if } \omega = x_1 \\ 4 & \text{if } \omega = x_2 \\ 10 & \text{if } \omega = x_3, \end{cases}$$

where $\Omega = \{x_1, x_2, x_3\}$ is the underlying probability space (on which is defined a probability $\mathbb{P}$ such that $\mathbb{P}(\{\omega\}) > 0$ for every $\omega \in \Omega$). Answer the following questions and justify carefully with either proofs or counterexamples.

   (a) Is this model free of arbitrage?

   (b) Consider the derivative with payoff

$$Y_1(\omega) = \begin{cases} 4 & \text{if } \omega = x_1 \\ 2 & \text{if } \omega = x_2 \\ -4 & \text{if } \omega = x_3 \end{cases}$$

   and time 1. Is Y_1 replicable?

   (c) Is this model complete?

   (d) Compute the set of all arbitrage free prices for Y_1; is this a singleton (i.e. is the arbitrage free price unique) ?

3. For $\omega := (\omega_1, \omega_2, \omega_3) \in \Omega = \{H, T\}^3$ and $i = 1, 2, 3$, let $Y_i(\omega)$ be given by

$$Y_i(\omega) := \begin{cases} 2 & \text{if } \omega_i = H \\ -1 & \text{if } \omega_i = T, \end{cases}$$

so that Y_i informs us of the result of the i^{th} coin toss, but it is not symmetric. On Ω consider the probability $\mathbb{P}$ for which the Y_i are independent and $\mathbb{P}(Y_i = 2) = 1/2 = \mathbb{P}(Y_i = -1)$. We take as usual the filtration $\mathcal{F}_i = \sigma(X_1, \ldots, X_i) = \sigma(Y_1, \ldots, Y_i)$ generated by the coin tosses $X_i(\omega) = \omega_i$. Let $W = (W_k)_{k \le 3}$ be the (asymmetric) random walk given by $W_0 := 0$ and $W_k := \sum_{i=1}^{k} Y_i$ for $k = 1, 2, 3$. Consider the multiperiod binomial model with expiration 3, zero interest rate and stock price $S_k := 4 + W_k$ for $k = 0, 1, 2, 3$. Let M be the running max of S, i.e. $M_k := \max_{i=0,\ldots,k} S_i$, and define Z by setting $Z_i := S_i$ for $i = 0, 1, 2$, and $Z_3 := S_1 + S_2 - 4$. Answer the following questions and justify carefully with either proofs or counterexamples.

(a) Is $S = (S_i)_{i \le 3}$ a submartingale under $\mathbb{P}$?

(b) Is $Z = (Z_i)_{i \le 3}$ Markov under $\mathbb{P}$?

(c) Is $Z = (Z_i)_{i \le 3}$ a submartingale under $\mathbb{P}$?

(d) Are the Y_1, Y_2, Y_3 independent under the risk neutral measure $\mathbb{Q}$?

4. In the framework of N-period binomial model with constant parameters $S_0, u, d, r > 0$ which satisfy $0 < d < 1 + r < u$, let $S = (S_n)_{n=0}^{N}$ be the stock price process, M_n its historical maximum up to time n (i.e. $M_n := \max_{i=0,\ldots,n} S_i$), and $Y_n := \frac{M_n}{S_n}$, where $n = 0, \ldots, N$. Consider the *lookback option*, which at time N pays the amount $V_N = M_N$, and denote with V_n its arbitrage-free price at time $n = 0, \ldots, N$.

(a) Use the risk-neutral pricing formula to express V_n in terms of V_{n+1}.

(b) Express Y_{n+1} as a function of Y_n and $\frac{S_{n+1}}{S_n}$.

(c) Prove that $\frac{S_{n+1}}{S_n}$ is independent of $S_0, \ldots, S_n$ under the risk-neutral measure $\mathbb{Q}$.

(d) Work by backward induction to show that, for every $n = 0, \ldots, N$, V_n admits the representation $V_n = S_n v_n(Y_n)$, where $v_n : \mathbb{R} \to \mathbb{R}$, $n = 0, \ldots, N$, are (deterministic) functions. Write explicitly v_N and an explicit formula to express v_n in terms of v_{n+1} for $n = 0, \ldots, N-1$.

5. Consider the trinomial model with time index $\{0, 1\}$, and a market made of a bank account with interest rate $r = 0$ and of one stock whose price is given by $S_0 = 2$ and

$$S_1(\omega) = \begin{cases} 1 & \text{if } \omega = x_1 \\ 2 & \text{if } \omega = x_2 \\ 3 & \text{if } \omega = x_3, \end{cases}$$

where $\Omega = \{x_1, x_2, x_3\}$ is the probability space, endowed with a probability $\mathbb{P}$ s.t.

$$\mathbb{P}(\{x_1\}) = \frac{1}{10}, \quad \mathbb{P}(\{x_2\}) = \frac{4}{10}, \quad \mathbb{P}(\{x_3\}) = \frac{5}{10}.$$

Suppose you want to invest in this market up to maturity, your initial capital is 1 and your attitude towards risk is represented by the utility function $U(X) = \ln(x), x > 0$.

(a) Compute the set of equivalent martingale measures, and the set of the terminal wealths which you can attain.

(b) Find your optimal investment strategy, i.e. compute the number of stocks Δ_0 you need to buy or sell at time 0 in order to maximize the expected utility of your terminal wealth.

SOLUTION OF FINAL EXAM M3F22/M4F22/M5F22 2018/2019

1. Exercise 1, similarly seen in Problems

(1) For any $x \in \mathbb{R}$ there holds

$$(x - K_1)^+ \leq (x - K_2)^+ + (K_2 - K_1), \tag{1}$$

as it can easily be verified by looking separately at the three cases $x < K_1, x \in [K_1, K_2], x > K_2$. Replacing x with S_T in (1) shows that the time T payoff of a call with strike K_1 is smaller than that of a call with strike K_2 plus the amount of money in the bank at time T resulting from having deposited $\frac{K_2 - K_1}{(1+r)^T}$ at time 0. By the domination property, it follows that the time 0 values of these portfolios satisfy the same inequality, concluding the proof.

[12 Points]

(2) Notice that the inequality (1) holds with equality if $x \geq K_2$. Thus, in any model where S_T satisfies $S_T \geq K_2$ $\mathbb{P}$ a.s. the inequality proved in item (1) holds with equality $\mathbb{P}$ a.s.. So, one can take any binomial model, and then choose K_2 to be smaller than both values of $S_1 > 0$; so for example one can assume $\mathbb{P}(H) = 1/2 = \mathbb{P}(T), S_0 = 4, S_1(H) = 8, S_1(T) = 2, K = 1$.

[8 Points]

2. Exercise 2, similarly seen in Lectures and Problems

(1) The model is free of arbitrage since the down, middle and up factors d, m, u are respectively $1, 2, 5$ and so $1 + r = 2$ satisfies $d < 1 + r < u$.

Alternatively one can compute the set $\mathcal{M}$ of equivalent martingale measures and show that it is not empty. Recall that $\mathbb{Q} \in \mathcal{M}$ if $\bar{S}_0 = \mathbb{E}^{\mathbb{Q}}[\bar{S}_1]$, Q is a probability and $\mathbb{Q} \sim \mathbb{P}$, i.e. iff $q_i := \mathbb{Q}(\{x_i\})$ satisfy

$$\begin{cases} 2 = q_1 + 2q_2 + 5q_3 \\ 1 = q_1 + q_2 + q_3 \\ q_i > 0 \text{ for } i = 1, 2, 3 \end{cases}$$

Subtracting twice the second line from the first line we get $0 = -q_1 + 3q_3$ and so $q_1 = 3q_3$ and the second line now gives $q_2 = 1 - q_1 - q_3 = 1 - 4q_3$. Imposing $q_i > 0$ we obtain that the set of q_i's corresponding to $\mathcal{M}$ is

$$\left\{ q_t := \begin{pmatrix} 3t \\ 1 - 4t \\ t \end{pmatrix} : t \in \left(0, \frac{1}{4}\right) \right\}, \tag{EMM}$$

which is non-empty. **[4 Points]**

(2) One possible approach is to compute explicitly the solution to the replication equation, which we will presently do. If X is a process, we denote with $\bar{X}$ the discounted process $(X_0, \frac{X_1}{1+r})$. The portfolio with initial wealth

x and trading strategy Δ has discounted payoff $\bar{V}_1 = x + \Delta(\bar{S}_1 - \bar{S}_0)$ equal to

$$x \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \Delta \cdot \begin{pmatrix} 1-2 \\ 2-2 \\ 5-2 \end{pmatrix} = \begin{pmatrix} x-\Delta \\ x \\ x+3\Delta \end{pmatrix}.$$

Solving for $\bar{Y}_1 = \bar{V}_1$ gives

$$\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} x-\Delta \\ x \\ x+3\Delta \end{pmatrix}$$

which has solution $x = 1, \Delta = -1$, so Y_1 is replicable (starting with initial wealth 1 and short-selling 1 stock, and depositing the remaining $x - \Delta \cdot S_0 = 1 - (-1) \cdot 2 = 3$ in the bank).

Another possible solution is to show that $\mathbb{E}^{\mathbb{Q}}[\bar{Y}_1]$ is constant across all $\mathbb{Q} \in \mathcal{M}$. Using (EMM) this means that $2(3t) + 1(1-4t) + (-2)t = 1$ is constant over $t \in \left(0, \frac{1}{4}\right)$, which is clearly true. **[8 Points]**

(3) Clearly the market is not complete, since the vector space generated by

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$$

has dimension 2 while the vector space of all possible values of derivatives is in this example $\mathbb{R}^3$, which is strictly bigger; in other words, the equation $\bar{X}_1 = \bar{V}_1$ does not have solution for arbitrary X_1, since it corresponds to a system of 3 linearly independent equations in 2 unknowns (which does not always have a solution).

Alternatively (EMM) shows that $\mathcal{M}$ is not a singleton, which implies that the market is not complete. **[4 Points]**

(4) Since Y_1 is replicable, it has a unique arbitrage free price Y_0, which corresponds to the initial wealth x of any replicating portfolio. So, by item 2 $Y_0 = 1$.

Alternatively, Y_0 equals the constant value of $\mathbb{E}^{\mathbb{Q}}[\bar{Y}_1]$ for any $\mathbb{Q} \in \mathcal{M}$ This value was computed to be 1 in item 2. **[4 Points]**

3. Exercise 3, similarly seen in Lectures and Problems

(1) S is a $\mathbb{P}$-submartingale since $S_{i+1} = S_i + Y_{i+1}$, where S_i is $\mathcal{F}_i$-measurable and Y_{i+1} is independent of $\mathcal{F}_i$, and so

$$\mathbb{E}[S_{i+1}|\mathcal{F}_i] = \mathbb{E}[S_i|\mathcal{F}_i] + \mathbb{E}[Y_{i+1}|\mathcal{F}_i] = S_i + \mathbb{E}[Y_{i+1}] = S_i + 1/2 \geq S_i \tag{2}$$

[4 Points]

(2) Z is not $\mathbb{P}$-Markov since $Z_2(HT) = 5 = Z_2(TH)$ and yet the branches of Z emanating from HT and TH differ; more precisely, since $Z_3 = S_1 + S_2 - 4$ is $\mathcal{F}_2$-measurable we get that $\mathbb{E}[f(Z_3)|\mathcal{F}_2] = f(Z_3) = f(S_1 + S_2 - 4)$ and so

$$\mathbb{E}[f(Z_3)|\mathcal{F}_2](HT) = f(6+5-4) = f(7),$$
$$\mathbb{E}[f(Z_3)|\mathcal{F}_2](TH) = f(3+5-4) = f(4).$$

So, choosing any function f such that $f(7) \neq f(4)$ we see that $\mathbb{E}[f(Z_3)|\mathcal{F}_2]$ cannot equal a function of Z_2 since $Z_2(HT) = 5 = Z_2(TH)$ and yet

$$\mathbb{E}[f(Z_3)|\mathcal{F}_2](HT) \neq \mathbb{E}[f(Z_3)|\mathcal{F}_2](TH).$$

[6 Points]

(3) Z is not a $\mathbb{P}$-submartingale since $\mathbb{E}[Z_3|\mathcal{F}_2] = S_2 + (S_1 - 4)$ is not always bigger than (or equal to) $Z_2 = S_2$: for example if $\omega = TTT$ then $S_2(\omega) = 1, S_1(\omega) - 4 = -1$ and so $\mathbb{E}[Z_3|\mathcal{F}_2](TTT) = 0 < Z_2(TTT) = 1$

[4 Points]

(4) Let us compute $\mathbb{Q}(\omega_i = H|\omega_1, \dots, \omega_{i-1})$ and show that it is deterministic, i.e., it does not depend on $(\omega_1, \dots, \omega_{i-1})$; this shows that the coin tosses and thus the Y_i's are independent under $\mathbb{Q}$.

$$\mathbb{Q}(\omega_i = H|\omega_1, \dots, \omega_{i-1}) = \frac{1 + r_{i-1} - d_{i-1}}{u_{i-1} - d_{i-1}}(\omega_1, \dots, \omega_{i-1})$$

and in our case $r_{i-1} = 0$ and if $s := S_{i-1}(\omega_1, \dots, \omega_{i-1})$ then we can compute the up and down factors u_{i-1}, d_{i-1} as

$$u_{i-1}(\omega_1, \dots, \omega_{i-1}) = \frac{S_i}{S_{i-1}}(\omega_1, \dots, \omega_{i-1}, H) = \frac{s+2}{s} = 1 + \frac{2}{s}$$

and analogously

$$d_{i-1}(\omega_1, \dots, \omega_{i-1}) = \frac{S_i}{S_{i-1}}(\omega_1, \dots, \omega_{i-1}, T) = \frac{s-1}{s} = 1 - \frac{1}{s}$$

and so we obtain

$$\mathbb{Q}(\omega_i = H|\omega_1, \dots, \omega_{i-1}) = \frac{1 + 0 - (1 - \frac{1}{s})}{(1 + \frac{2}{s}) - (1 - \frac{1}{s})} = \frac{\frac{1}{s}}{\frac{3}{s}} = \frac{1}{3};$$

since this does not depend on s, it does not depend on $(\omega_1, \dots, \omega_{i-1})$.

[6 Points]

4. Exercise 4, similarly seen in Problems

(1) The risk neutral pricing formula gives

(3) $$V_n = \mathbb{E}^{\mathbb{Q}}\left(\frac{V_{n+1}}{1+r}|\mathcal{F}_n\right)$$

[4 Points]

(2) Write

$$Y_{n+1} = \frac{S_0 \vee \ldots \vee S_{n+1}}{S_{n+1}} = \frac{M_n \vee S_{n+1}}{S_{n+1}} = \frac{M_n}{S_{n+1}} \vee \frac{S_{n+1}}{S_{n+1}} = \frac{Y_n S_n}{S_{n+1}} \vee 1,$$

i.e.

(4) $$Y_{n+1} = h_n\left(Y_n, \frac{S_{n+1}}{S_n}\right) \text{ for } h_n(y, s) := \max\left(\frac{y}{s}, 1\right).$$

[5 Points]

(3) Notice that $\frac{S_{n+1}}{S_n}$ is independent (under the risk neutral measure $\mathbb{Q}$) of the filtration $\mathcal{F}_n$ generated by the first n coin tosses, since it only depends on the last coin toss ω_{n+1} and the coin tosses are independent under $\mathbb{Q}$ since

(5) $$\mathbb{Q}(\omega_{n+1} = H|\omega_1, \dots, \omega_n) = \tilde{p} = \frac{(1+r) - d}{u - d}$$

does not depend on $\omega_1, \dots, \omega_n$. Since $\sigma(S_0, \dots, S_n) \subseteq \mathcal{F}_n$, this shows that $\frac{S_{n+1}}{S_n}$ is independent of $S_0, \dots, S_n$ under $\mathbb{Q}$. **[3 Points]**

(4) Let us prove by backward induction that $\frac{V_n}{S_n} = v_n(Y_n)$ for all n and for some function v_n (which we will determine). By definition on V_N, $\frac{V_n}{S_n} = v_n(Y_n)$ holds for $n = N$ with

$$v_N(y) = y. \tag{6}$$

Now, assume by inductive hypothesis that $\frac{V_k}{S_k} = v_k(Y_k)$ holds for $k = n+1$ and let us show that it holds for $k = n$ (by induction, this will show that $\frac{V_n}{S_n} = v_n(Y_n)$ for all n). Using (3) we get

$$(1+r)V_n = \mathbb{E}^{\mathbb{Q}}[V_{n+1}|\mathcal{F}_n] = \mathbb{E}^{\mathbb{Q}}[S_{n+1}v_{n+1}(Y_{n+1})|\mathcal{F}_n]$$

and so using (4) we get

$$\frac{V_n}{S_n} = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}\left[\frac{S_{n+1}}{S_n}v_{n+1}\left(h_n\left(Y_n, \frac{S_{n+1}}{S_n}\right)\right)|\mathcal{F}_n\right].$$

Since Y_n is $\mathcal{F}_n$-measurable, we can apply the independence lemma and obtain that $\frac{V_n}{S_n} = v_n(Y_n)$ for

$$v_n(y) := \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}\left[\frac{S_{n+1}}{S_n}v_{n+1}\left(h_n\left(y, \frac{S_{n+1}}{S_n}\right)\right)\right] \tag{7}$$

and so

$$v_n(y) = \frac{1}{1+r}\left(\tilde{p}uv_{n+1}\left(\frac{y}{u}\vee 1\right) + (1-\tilde{p})dv_{n+1}\left(\frac{y}{d}\vee 1\right)\right), \tag{8}$$

where $\tilde{p}$ is given by (5). **[8 Points]**

5. Exercise 5, mastery question, similarly seen in problems

(1) Recall that $\mathbb{Q} \in \mathcal{M}$ if $\bar{S}_0 = \mathbb{E}^{\mathbb{Q}}[\bar{S}_1]$, Q is a probability and $\mathbb{Q} \sim \mathbb{P}$, i.e. iff $q_i := \mathbb{Q}(\{x_i\})$ satisfy

$$\begin{cases} 2 = q_1 + 2q_2 + 3q_3 \\ 1 = q_1 + q_2 + q_3 \\ q_i > 0 \text{ for } i = 1,2,3 \end{cases}$$

Subtracting twice the second line from the first line we get $0 = -q_1 + q_3$ and so $q_1 = q_3$ and the second line now gives $q_2 = 1 - q_1 - q_3 = 1 - 2q_3$. Imposing $q_i > 0$ we obtain that the set of q_i's corresponding to $\mathcal{M}$ is

$$\left\{ q(t) := \begin{pmatrix} t \\ 1-2t \\ t \end{pmatrix} : t \in \left(0, \frac{1}{2}\right) \right\}. \tag{EMM}$$

[5 Points]

The set of attainable wealths is the affine subspace

$$\left\{ x + \Delta_0(S_1 - S_0) = \begin{pmatrix} 1-\Delta_0 \\ 1 \\ 1+\Delta_0 \end{pmatrix} : \Delta_0 \in \mathbb{R} \right\}. \tag{EMM}$$

[3 Points]

(2) The terminal wealth of the optimal strategy Δ_0 is $\hat{X}_1 := x + \Delta_0(S_1 - S_0)$, where $x = 1$ is the initial capital, and as we know from the theory of optimal investment it satisfies $U'(\hat{X}_1) = c\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}$ for some $\hat{\mathbb{Q}}$ in the set $\mathcal{M}$ of equivalent martingale measures, and for the c which satisfies $\mathbb{E}^{\hat{\mathbb{Q}}}[\hat{X}_1] = x(1+r)$.

[4 Points]

Since $U'(x) = 1/x$ we get $\hat{X}_1 = \frac{1}{c}\frac{d\mathbb{P}}{d\hat{\mathbb{Q}}}$ and so

$$x(1+r) = \mathbb{E}^{\hat{\mathbb{Q}}}[\hat{X}_1] = \frac{1}{c}\mathbb{E}^{\hat{\mathbb{Q}}}[\frac{d\mathbb{P}}{d\hat{\mathbb{Q}}}] = \frac{1}{c}\mathbb{E}^{\mathbb{P}}[\frac{d\mathbb{P}}{d\hat{\mathbb{Q}}}\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}] = \frac{1}{c},$$

i.e. $c = 1/(x(1+r)) = 1$ and so $\hat{X}_1 = \frac{d\mathbb{P}}{d\hat{\mathbb{Q}}}$. **[3 Points]**

To find $\hat{\mathbb{Q}}$ and thus Δ_0 we use item (a) to write the equation $\hat{X}_1(\omega) = \frac{d\mathbb{P}}{d\hat{\mathbb{Q}}}(\omega)$, $\omega \in \Omega$ as

$$\begin{cases} 1 - \Delta_0 = \dfrac{1}{10t} \\ 1 = \dfrac{4}{10(1-2t)} \\ 1 + \Delta_0 = \dfrac{5}{10t} \end{cases}, \qquad \text{where } t \in \left(0, \frac{1}{2}\right), \Delta_0 \in \mathbb{R}.$$

The second equation gives $t = 3/10$, and so the third equation gives that the optimal strategy is $\Delta_0 = \frac{5}{3} - 1 = \frac{2}{3}$ (which of course also solves the first equation). **[5 Points]**