Lecture 3. 10.10.2022

Usually, I will be infinite. If I is finite, of dimension $d < \infty$ say, the X_i are the elements of a d-dimensional random vector, which can be handled using the familiar machinery of a first course in probability, and/or statistics, particularly the vast and important area of statistics called Multivariate Analysis. The probabilistic structure here is described by the corresponding d-dimensional pr law:

$$F(\mathbf{x}), \ F_{\mathbf{X}}(\mathbf{x}) := \mathbb{P}(X_1 \leq x_1, \cdots, X_d \leq x_d), \tag{$*$}$$

labelling I here as $\{1, 2, \cdots, d\}$, as we may.

When I is infinite, we have no familiar machinery to hand, and it is far from obvious how to find an infinite-dimensional analogue of the above, or even whether there is one. There is definitely something to prove!

Finite-dimensional distributions and the Daniell-Kolmogorov theorem

To proceed: note first that I, being infinite, has (infinitely many) finite-dimensional subsets, $(i_1, \cdots, i_d)$ say, to each of which the above applies. These are called the *finite-dimensional distributions* (fi-di d/ns, fdds) of the infinite collection. Write the corresponding d-dimensional probability law as $F(i_1, \cdots, i_d)$. These satisfy two *consistency conditions*:
(C1) Each d-dimensional distribution is invariant under permutation of the d indices. For, this just permutes the conditions $X_i \leq x_i, \ i = 1, \cdots d$ in $(*)$, so their intersection is unchanged.
(C2) Recall that if $x_i \uparrow \infty$ in $(*)$, the effect is to delete the condition on x_i, so taking a d-dimensional law into a $(d-1)$-dimensional one.

It turns out that these two *Daniell-Kolmogorov consistency conditions* (C1), (C2) are not only *necessary* for the existence of a stochastic process defined in this infinite-dimensional setting, but also *sufficient*. This is the Daniell-Kolmogorov theorem (P. J. Daniell (1889 - 1946) in 1918, A. N. Kolmogorov (1903 - 1987) in 1933; also called the Daniell extension theorem and the Kolmogorov extension theorem):

Daniell-Kolmogorov Theorem, D-K. For I any infinite index set, and any collection of finite-dimensional distributions satisfying the consistency conditions (C1), (C2) above, there exists a unique measure μ defined on the

Borel subsets $\mathcal{B}(\mathbb{R}^I)$ of $\mathbb{R}^I$ such that I restricted to each $(i_1, \cdots, i_d)$ gives $F(i_1, \cdots, i_d)$.

This classic result is the fundamental *existence theorem for stochastic processes*. We shall assume it here. We cite three textbook proofs (see References, Lecture 1):
[KinT] (measure, 159 - 161, by a compactness argument; probability, 381),
[Bil] (§36; two proofs, 513 - 515 and 515 - 517),
[Kal] (projective limits, 114 - 115).

In words: whenever speaking of a stochastic process $X = (X(i) : i \in I)$ on an infinite index set I makes sense (i.e., whenever its finite-dimensional distributions satisfy the consistency conditions (C1), (C2)), it can be constructed as above, and we can use it. So: in view of D-K, existence is no problem for us: *if it could exist, it does exist.* This will suffice for us.

Path properties

The Daniell-Kolmogorov theorem clearly uses all the information in the finite-dimensional d/ns, and *only* that. But one needs to go beyond this.

The map $t \mapsto X(t, \omega)$ (or to $X_t(\omega)$, $X(t)$ or X_t) is called the *path* (or *sample path*) of the process. Path properties only arise in continuous time. They involve information going *beyond* the finite-dimensional distributions! One wants to work with as well-behaved a *version* of the process as can be *realised* (constructed), consistent with the finite-dimensional distributions. The nicest property here is (path-)*continuity* (example: *Brownian motion*, Ch. 4). The next nicest is continuity from one side and limits from the other (this usually suffices, and is all we need in this course). Of the two possibilities here, the commoner is '*continuous on the right, limits on the left*' (corlol in English) (example: the *Poisson process*, Ch. 3). But as the area was developed by Paul-André Meyer and his colleagues in the Strasbourg school, it is usual to use the French, 'continu à droite, limite à gauche' (càdlàg, or cadlag). With right and left interchanged, one has càglàd, or caglad ('collor' isn't used). One even needs both together (in stochastic integration, one needs the random integrator cadlag and the random integrand caglad).