SOLUTIONS for Examples II for Time Series (S8)

NB: Stationarity by itself always means 'second-order' stationarity

[1]

(a) We cannot assume the means are zero here. But, since $\{X_t\}$ and $\{Y_t\}$ are stationary,

$$E\{Z_t\} = E\{X_t + Y_t\} = \mu_X + \mu_Y = \mu_Z$$
$$E\{Z_{t+\tau}\} = E\{X_{t+\tau} + Y_{t+\tau}\} = \mu_X + \mu_Y = \mu_Z.$$

Then

$$\begin{aligned} s_{Z,\tau} &= E\{(Z_t - E\{Z_t\})(Z_{t+\tau} - E\{Z_{t+\tau}\})\} \\ &= E\{([X_t - \mu_X] + [Y_t - \mu_Y])([X_{t+\tau} - \mu_X] + [Y_{t+\tau} - \mu_Y])\} \\ &= E\{[X_t - \mu_X][X_{t+\tau} - \mu_X]\} + E\{[Y_t - \mu_Y][Y_{t+\tau} - \mu_Y]\} \\ &+ E\{[X_t - \mu_X][Y_{t+\tau} - \mu_Y]\} + E\{[Y_t - \mu_Y][X_{t+\tau} - \mu_X]\} \\ &= s_{X,\tau} + s_{Y,\tau}, \end{aligned}$$

as the last two expectations are zero because $\{X_t\}$ and $\{Y_t\}$ are uncorrelated.

(b) A white noise process with variance unity has an autocovariance which is unity at $\tau = 0$ and zero elsewhere. An MA(1) process with parameter $\theta_{1,1} = \psi, |\psi| < 1$, and innovations variance $\sigma_\epsilon^2 = 1$ has an autocovariance sequence of the form

$$s_\tau = \begin{cases} 1 + \psi^2, & \text{if } \tau = 0, \\ -\psi, & \text{if } |\tau| = 1, \\ 0, & \text{if } |\tau| > 1. \end{cases}$$

Hence

$$s_\tau = \begin{cases} 2 + \psi^2, & \text{if } \tau = 0, \\ -\psi, & |\tau| = 1, \\ 0, & \text{otherwise}, \end{cases}$$

is the autocovariance sequence corresponding to the sum of uncorrelated white noise and MA(1) processes, with the parameters as above. By part (a) it follows that the stated autocovariance is a valid autocovariance sequence.

[2]

(a)

$$\begin{aligned} s_{Y,\tau} &= E\{Y_t Y_{t+\tau}\} - E\{Y_t\}E\{Y_{t+\tau}\} \\ &= E\{X_t X_{t-1} X_{t+\tau} X_{t+\tau-1}\} - E\{X_t X_{t-1}\}E\{X_{t+\tau} X_{t+\tau-1}\} \\ &= E\{X_t X_{t-1}\}E\{X_{t+\tau} X_{t+\tau-1}\} + E\{X_t X_{t+\tau}\}E\{X_{t-1} X_{t+\tau-1}\} \\ &+ E\{X_t X_{t+\tau-1}\}E\{X_{t-1} X_{t+\tau}\} - E\{X_t X_{t-1}\}E\{X_{t+\tau} X_{t+\tau-1}\} \\ &= \operatorname{cov}\{X_t, X_{t+\tau}\} \operatorname{cov}\{X_{t-1}, X_{t+\tau-1}\} + \operatorname{cov}\{X_t, X_{t+\tau-1}\} \operatorname{cov}\{X_{t-1}, X_{t+\tau}\} \\ &= s_{X,\tau}^2 + s_{X,\tau-1} s_{X,\tau+1}. \end{aligned}$$

For an MA(1) we have

$$s_{X,\tau} = \begin{cases} \sigma_\epsilon^2(1+\theta_{1,1}^2) & \tau = 0, \\ -\sigma_\epsilon^2\theta_{1,1}, & |\tau| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

So

$$s_{Y,\tau} = \begin{cases} \sigma_\epsilon^4(1+3\theta_{1,1}^2+\theta_{1,1}^4) & \tau = 0, \\ \sigma_\epsilon^4\theta_{1,1}^2, & |\tau| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

(b) $s_{X,\tau} = E\{X_t X_{t+\tau}\}$ since the mean is zero. Then,

$$\begin{aligned} s_{Y,\tau} &= E\{X_t^2 X_{t+\tau}^2\} - E\{X_t^2\}E\{X_{t+\tau}^2\} \\ &= E\{X_t^2\}E\{X_{t+\tau}^2\} + 2E^2\{X_t X_{t+\tau}\} - E\{X_t^2\}E\{X_{t+\tau}^2\} \\ &= 2E^2\{X_t X_{t+\tau}\} = 2s_{X,\tau}^2. \end{aligned}$$

Then $s_{Y,0} = 2s_{X,0}^2$, so that

$$\rho_{Y,\tau} = \frac{s_{Y,\tau}}{s_{Y,0}} = \frac{2s_{X,\tau}^2}{2s_{X,0}^2} = \rho_{X,\tau}^2.$$

We take the most obvious possible model for our one case: an MA(2) process with parameters $\theta_{0,2} = -1$, (standard) and $\theta_{1,2} = 0$, and $\theta_{2,2}$. For convenience define $\theta \equiv \theta_{2,2}$.

$$s_{X,\tau} = \begin{cases} [1+\theta^2]\sigma_\epsilon^2, & \text{if } \tau = 0, \\ 0, & \text{if } |\tau| = 1, \\ -\theta\sigma_\epsilon^2, & \text{if } |\tau| = 2, \\ 0, & \text{if } |\tau| > 2, \end{cases} \quad \text{so} \quad \rho_{X,\tau} = \begin{cases} 1, & \text{if } \tau = 0, \\ 0, & \text{if } |\tau| = 1, \\ -\frac{\theta}{[1+\theta^2]}, & \text{if } |\tau| = 2, \\ 0, & \text{if } |\tau| > 2. \end{cases}$$

But

$$s_{Y,\tau} = \begin{cases} 200, & \text{if } \tau = 0, \\ 0, & \text{if } |\tau| = 1, \\ 18, & \text{if } |\tau| = 2, \\ 0, & \text{if } |\tau| > 2, \end{cases} \quad \text{so} \quad \rho_{X,\tau} = \begin{cases} 1, & \text{if } \tau = 0, \\ 0, & \text{if } |\tau| = 1, \\ \pm\frac{3}{10}, & \text{if } |\tau| = 2, \\ 0, & \text{if } |\tau| > 2. \end{cases}$$

Let $\rho = \rho_{X,2}$, then we see that $\rho\theta^2 + \theta + \rho = 0$ which has solutions

$$\theta = \frac{-1 \pm \sqrt{[1-4\rho^2]}}{2\rho} = \frac{-1 \pm \frac{8}{10}}{\pm\frac{6}{10}} = \pm\frac{1}{3} \text{ or } \pm 3.$$

Now $s_{Y,0} = 200 = 2s_{X,0}^2 = 2[1+\theta^2]^2\sigma_\epsilon^4$. Four possible parameter combinations arise:

$$\theta = \pm 1/3 \Rightarrow \sigma_\epsilon^2 = 9; \quad \theta = \pm 3 \Rightarrow \sigma_\epsilon^2 = 1.$$

3 (a)

$$
\begin{aligned}
X_t^{(d)} &= \Delta^d X_t \\
&= \sum_{k=0}^{d} \left(\begin{array}{c} d \\ k \end{array} \right) (-1)^k Y_{t-k} \\
&= \sum_{k=0}^{d} \left(\begin{array}{c} d \\ k \end{array} \right) (-1)^k \left(\sum_{j=1}^{p} \phi_{j,p} Y_{t-k-j} + \epsilon_{t-k} \right) \\
&= \sum_{j=1}^{p} \phi_{j,p} \left(\sum_{k=0}^{d} \left(\begin{array}{c} d \\ k \end{array} \right) (-1)^k Y_{t-k-j} \right) + \sum_{k=0}^{d} \left(\begin{array}{c} d \\ k \end{array} \right) (-1)^k \epsilon_{t-k} \\
&= \sum_{j=1}^{p} \phi_{j,p} X_{t-j}^{(d)} + \sum_{k=0}^{d} \left(\begin{array}{c} d \\ k \end{array} \right) (-1)^k \epsilon_{t-k}
\end{aligned}
$$

which is in ARMA(p,d) form, where $\theta_{k,d} = \left(\begin{array}{c} d \\ k \end{array} \right) (-1)^k$.

We showed in lectures that for an ARMA(p,q) of general form

$$\Phi(B)X_t = \Theta(B)\epsilon_t$$

to be invertible, we require the roots of $\Theta(z)$ to lie outside the unit circle. For $\{X_t^{(d)}\}$ we have

$$
\begin{aligned}
\Phi(B) &= 1 - \phi_{1,p}B - \phi_{2,p}B^2 - ... - \phi_{p,p}B^p \\
\Theta(B) &= (1-B)^d.
\end{aligned}
$$

The roots of $\Theta(B) = (1-z)^d$ lie *on* the unit circle, therefore it is not invertible.

(b) We showed in lectures that for an ARMA(p,q) of general form

$$\Phi(B)X_t = \Theta(B)\epsilon_t$$

to be stationary, we require the roots of $\Phi(z)$ to lie outside the unit circle. The ARMA(1,2) model here is

$$(1 - \phi_{1,1}B)X_t = (1-B)^2\epsilon_t.$$

Therefore the root of $\Phi(z) = 1 - \phi_{1,1}z$ is $z = 1/\phi_{1,1}$ and lies outside the unit circle since $\phi_{1,1} < 1$.

We can express $\{X_t\}$ as $X_t = G(B)\epsilon_t$ where

$$G(z) = \frac{(1-z)^2}{1 - \phi_{11}z}.$$

We can write

$$
\begin{aligned}
G(z) &= (1-z)^2(1+\phi_{1,1}z+\phi_{1,1}^2z^2+\phi_{1,1}^3z^3+...) \\
&= (1-2z+z^2)\sum_{k=0}^{\infty}\phi_{1,1}^k z^k \\
&= \sum_{k=0}^{\infty}\phi_{1,1}^k z^k - 2\sum_{k=0}^{\infty}\phi_{1,1}^k z^{k+1} + \sum_{k=0}^{\infty}\phi_{1,1}^k z^{k+2} \\
&= 1+(\phi_{1,1}-2)z+\sum_{k=2}^{\infty}(\phi_{1,1}^k - 2\phi_{1,1}^{k-1}+\phi_{1,1}^{k-2})z^k \\
&= 1+(\phi_{1,1}-2)z+(\phi_{1,1}-1)^2\sum_{k=2}^{\infty}\phi_{1,1}^{k-2}z^k
\end{aligned}
$$

which means $X_t = \sum_{k=0}^{\infty} g_k \epsilon_{t-k}$ where $g_0 = 1$, $g_1 = \phi_{1,1} - 2$ and $g_k = (\phi_{1,1}-1)^2\phi_{1,1}^{k-2}$ for $k \geq 2$.

(c) Expression for s_0:

$$
\begin{aligned}
s_0 &= \sigma_\epsilon^2 \sum_{k=0}^{\infty} g_k^2 \\
&= \sigma_\epsilon^2\left(1+(\phi_{1,1}-2)^2+(\phi_{1,1}-1)^4\sum_{k=0}^{\infty}\phi_{1,1}^{2k}\right) \\
&= \sigma_\epsilon^2\left(1+(\phi_{1,1}-2)^2+\frac{(\phi_{1,1}-1)^4}{1-\phi_{1,1}^2}\right)
\end{aligned}
$$

Expression for s_1:

$$
\begin{aligned}
s_1 &= \sigma_\epsilon^2 \sum_{k=0}^{\infty} g_k g_{k+1} \\
&= \sigma_\epsilon^2\left((\phi_{1,1}-2)+(\phi_{1,1}-2)(\phi_{1,1}-1)^2+\phi_{1,1}(\phi_{1,1}-1)^4\sum_{k=0}^{\infty}\phi_{1,1}^{2k}\right) \\
&= \sigma_\epsilon^2\left((\phi_{1,1}-2)+(\phi_{1,1}-2)(\phi_{1,1}-1)^2+\frac{\phi_{1,1}(\phi_{1,1}-1)^4}{1-\phi_{1,1}^2}\right)
\end{aligned}
$$

Expression for s_2:

$$
\begin{aligned}
s_2 &= \sigma_\epsilon^2 \sum_{k=0}^{\infty} g_k g_{k+2} \\
&= \sigma_\epsilon^2\left((\phi_{1,1}-1)^2+\phi_{1,1}(\phi_{1,1}-2)(\phi_{1,1}-1)^2+\phi_{1,1}^2(\phi_{1,1}-1)^4\sum_{k=0}^{\infty}\phi_{1,1}^{2k}\right) \\
&= \sigma_\epsilon^2\left((\phi_{1,1}-1)^2+\phi_{1,1}(\phi_{1,1}-2)(\phi_{1,1}-1)^2+\frac{\phi_{1,1}^2(\phi_{1,1}-1)^4}{1-\phi_{1,1}^2}\right)
\end{aligned}
$$