Applied Probability Progress Test 2

28 November 2019

Question 1: (5 points) Let $X = (X_n)_{n\in\{0,1,2,\dots\}}$ denote a discrete-time, homogeneous Markov chain on the state space $E = \{1,2,3,4,5\}$ with transition matrix given by

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Find all possible stationary distributions.

Question 2: (5 points) Let $(X_n)_{n\in\{0,1,2,\dots\}}$ denote a discrete-time, homogeneous Markov chain on the state space $E = \{1,2,3,4\}$ with transition matrix given by

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}.$$

Is the Markov chain time-reversible? Please justify your answer carefully.

Question 3: (a) (3 points) For $n \in \mathbb{N}$, let $X_1, \dots, X_n$ be independent random variables where $X_i \sim \text{Exp}(\lambda_i)$, for $\lambda_i > 0$ and $i = 1, \dots, n$. Let $Y = \min\{X_1, \dots, X_n\}$. Determine the distribution of Y.

(b) (3 points) Three students are working independently on their Applied Probability homework. All three start at 8am on a certain day and each takes an exponential time with mean 6 hours to complete the homework. What is the earliest time, on average, at which all three students will have completed their homework?

Hint: If $X \sim \text{Exp}(\lambda)$ for $\lambda > 0$, then the density of X is given by $f_X(x) = \lambda e^{-\lambda x}$ for $x > 0$ and 0 otherwise. Also, $\mathbb{E}(X) = 1/\lambda$.

Question 4: (4 points) Define a compound Poisson process.