(1)

(2.2.13) Def. Suppose χ is an L-formula involving prop. vars. $p_1, \ldots, p_n$. Suppose $\mathcal{L}$ is a 1^{st} order language and $\phi_1, \ldots, \phi_n$ are $\mathcal{L}$-formulas.

A substitution instance of χ is obtained by replacing each p_i in χ by ϕ_i (for $i = 1, \ldots, n$).

Call the result θ.

(2.2.14) Thm (1) θ is an $\mathcal{L}$-formula.

(2) Let v be an valuation in an $\mathcal{L}$-str. $\mathcal{A}$.

Let w be a prop. valuation with $w(p_i) = v[\phi_i]$ (for $i = 1, \ldots, n$).

Then $\underbrace{v[\theta]}_{\mathcal{L}\text{-formula}} = \underbrace{w(\chi)}_{L\text{-formula}}$

(3) If χ is a tautology then θ is logically valid.

Pf: (Sketch) (1) Omit.

(3) follows from (2) (as $w(\chi) = T$ in this case).

(2) By induction on the number of connectives in χ.

Base case Ex.

Inductive step:

(a) χ is $(\sim\alpha)$

(b) χ is $(\alpha \to \beta)$

[for L-formulas α, β].

(a) Ex.

(b) θ is of the form

$$(\theta_1 \to \theta_2)$$

where θ_1 is obtained from α by substituting ϕ_i in place of p_i in α; similarly for θ_2.

$w(\chi) = F$

$\iff w(\alpha) = T$ & $w(\beta) = F$

$\underset{\text{induction}}{\iff} v[\theta_1] = T$ & $v[\theta_2] = F$

$\iff v[(\theta_1 \to \theta_2)] = F$

$\iff v[\theta] = F$.

which does the ind. step in this case. //. #.

Note: Not all logically valid formulas arise in this way

Eg. $((\exists x_2)(\forall x_1)\phi \to (\forall x_1)(\exists x_2)\phi)$

is logically valid, but not a subst. instance of a prop. tautology.

(2.3) Bound and free variables.

1) Eg: ψ_1

free free bound free bound

$$(R_1(x_1, x_2) \to (\forall x_3)\underbrace{R_2(x_1, x_3)}_{\text{scope of}})$$

(2.3.1) Def. Suppose ϕ, ψ are $\mathcal{L}$-formulas and $(\forall x_i)\phi$ occurs as a subformula of ψ

i.e. ψ is $\ldots\ldots(\forall x_i)\phi\ldots\ldots$

We say that ϕ is the scope of the quantifier $(\forall x_i)$ here.

A occurrence of a variable x_j in ψ is bound if it is in the scope of a quantifier $(\forall x_j)$ in ψ (or it is the x_j here).

Otherwise, it is a free occurrence of in ψ. Variables having a free occurrence in ψ are called free variables of ψ.

A formula with no free variables is called a closed formula (or an $\mathcal{L}$-sentence).

Examples

2) ψ_2:

bound free free free

$$((\forall x_1)\underbrace{R_1(x_1, x_2)}_{\text{scope}} \to R_2(x_1, x_2)))$$

Compare with:

bound

$$(\forall x_1)\underbrace{\left(R_1(x_1, x_2) \to R_2(x_1, x_2)\right)}_{\text{scope}}$$

3) ψ_3:

free bound free

$$\left((\exists x_3)\underbrace{R_1(x_1, x_2)}_{\text{scope}} \to (\forall x_2)\underbrace{R_2(x_2, x_3)}_{\text{scope}}\right)$$

//

(2.3.2) <u>Notation</u>:

If ψ is an $\mathcal{L}$-formula with free variables amongst $x_1, \dots, x_n$ write $\psi(x_1, \dots, x_n)$ (instead of ψ).

If $t_1, \dots, t_n$ are terms then by $\psi(t_1, \dots, t_n)$ we mean the $\mathcal{L}$-formula obtained by replacing each <u>free</u> occurrence of x_i in ψ by t_i (for $i = 1, \dots, n$).

Eg

$$\psi(x_1, x_2)$$

$$\left(\left(\forall x_1\right) R_1(x_1, \overset{\text{free}}{x_2}) \to \left(\forall x_3\right) R_2(\overset{\text{free}}{x_1, x_2}, x_3)\right)$$

t_1 $\qquad$ $f_1(x_1)$

t_2 $\qquad$ $f_2(x_1, x_2)$

$$\psi(t_1, t_2)$$

$$\left(\left(\forall x_1\right) R_1(x_1, f(x_1, x_2)) \to \left(\forall x_3\right) R_2(f_1(x_1), f(x_1, x_2), x_3)\right)$$

(2.3.3) **Theorem**

Suppose ϕ is a closed $\mathcal{L}$-formula and $\mathcal{A}$ is an $\mathcal{L}$-str. Then either

$$\mathcal{A} \models \phi$$

or

$$\mathcal{A} \models (\neg \phi)$$

More generally, if ϕ has free variables amongst $x_1, \ldots, x_n$ and v, w are valuations in $\mathcal{A}$ with

$$v(x_i) = w(x_i) \text{ for } i = 1, \ldots, n$$

then $v[\phi] = T \iff w[\phi] = T$

(allow $n = 0$ here : no free vars.).

Pf: Note that the first statement follows from the general statement. If ϕ has no free variables then for any valuations v, w (in $\mathcal{A}$) they agree on the free variables, so

$$v[\phi] = w[\phi].$$

//

Prove generalization by ind. on number of connectives + quantifiers in ϕ.

Base case : ϕ is atomic

$R(t_1, \ldots, t_m)$ t_j terms.

The t_j only involve ~~f~~ variables from $x_1, \ldots, x_n$. So

$$v(t_j) = w(t_j) \text{ for } j = 1, \ldots, m$$

(compare 2.2.6).

then (2.2.9 ?)

$v[R(t_1,\dots,t_m)] = T$

$\Leftrightarrow \bar{R}(v(t_1),\dots,v(t_m))$ holds in $\mathcal{A}$

$\Leftrightarrow \bar{R}(w(t_1),\dots,w(t_m))$ holds in $\mathcal{A}$

$\Leftrightarrow w[R(t_1,\dots,t_m)] = T$.

//

Ind. step. ϕ is $(\neg\psi)$, $(\psi \to \chi)$ or $(\forall x_i)\psi$.

First two cases: Ex.

Suppose ϕ is $(\forall x_i)\psi$

Suppose $v[\phi] = F$.

Want to show $w[\phi] = F$.

(By symmetry, this is enough.)

By Def 2.2.9(ii)(c) there is a valuation v' x_i-equiv. to v with $v'[\psi] = F$.

The free variables of ψ are amongst $x_1,\dots,x_n, x_i$.

Let w' be the valuation x_i-equiv. to w with $w'(x_i) = v'(x_i)$.

Then v', w' agree on the free variables of ψ.

By ind. hyp. (on ψ)

$$v'[\psi] = w'[\psi]$$

so $w'[\psi] = F$. As w' is x_i-equiv. to w, we have $w[(\forall x_i)\psi] = F$, ie. $w[\phi] = F$.

//

#

Notation: If $\mathcal{A}$ is an $\mathcal{L}$-structure and $\psi(x_1, \ldots, x_n)$ is an $\mathcal{L}$-fmla (whose free vars. are amongst $x_1, \ldots, x_n$) and $a_1, \ldots, a_n \in A$ (domain of $\mathcal{A}$) then write

$$\mathcal{A} \models \psi(a_1, \ldots, a_n)$$

to mean $v[\psi] = T$ whenever v is a valuation with $v(x_i) = a_i$ for $i = 1, \ldots, n$.

(Note: By pf. of 2.3.3 this holds if $v[\psi] = T$ for some such v.)

(2.3.4) Warning Example

An example where

$$\mathcal{A} \models (\forall x_1)\, \phi(x_1)$$

but where we a term t_1 and a valuation v in $\mathcal{A}$ with $v[\phi(t_1)] = F$.

(?? Expect $v[\phi(t_1)] = T$)
But no.

$$\phi(x_1): ((\forall x_2) R(\underset{\text{free}}{x_1}, x_2) \to S(\underset{\text{free}}{x_1})))$$

t_1 is the term x_2.

$$\phi(t_1): ((\forall x_2) R(x_2, x_2) \to S(x_2)))$$

$\mathcal{A}$ Domain $\mathbb{N} = \{0, 1, 2, \dots\}$

$R(x_1, x_2)$ interpreted as '$x_1 \leq x_2$'

$S(x_1)$ interpreted as '$x_1 = 0$'

<u>So</u> $\mathcal{A} \models (\forall x_1)\, \phi(x_1)$

$(\mathcal{A} \models (\forall x_1)((\forall x_2) R(x_1, x_2) \to S(x_1))))$

<u>BUT</u> if $v(x_2) = 1$ then

$v[\phi(t_1)] = F$. //

(2.3.5) <u>Def</u>. Let ϕ be an $\mathcal{L}$-formula, x_i a variable and t a term of $\mathcal{L}$. We say that <u>t is free for x_i in ϕ</u> if there is no variable x_j in t such that x_i occurs free within the scope of a quantifier $(\forall x_j)$ in ϕ.

NOT free for x_i is ϕ:

$$\phi: \quad \dots\dots (\forall x_j)\underbrace{\dots \overset{\text{free}}{x_i}}_{\text{scope}}$$

$$t \qquad \dots x_j \dots$$

In example: t_1 is <u>not</u> free for x_1 in ϕ.

(2.3.6) **Thm**

Suppose $\phi(x_1)$ is an $\mathcal{L}$-fmla (possibly with other free variables).
Let t be a term free for x_1 in ϕ.
Then $\models ((\forall x_1)\phi(x_1) \to \phi(t))$.

In particular if $\mathcal{A}$ is a $\mathcal{L}$-str. with $\mathcal{A} \models (\forall x_1)\phi(x_1)$ then $\mathcal{A} \models \phi(t)$.

Eg: take $t = x_1$ here. So if $\mathcal{A} \models (\forall x_1)\phi(x_1)$ then $\mathcal{A} \models \phi(x_1)$.

Follows from:

(2.3.7) **Lemma** Suppose v is a valuation in $\mathcal{A}$. Let v' be the val. in $\mathcal{A}$ which is x_1-equiv. to v with $v'(x_1) = v(t)$.
Then $v'[\phi(x_1)] = T$
$\Leftrightarrow v[\phi(t)] = T$.

Lemma $\Rightarrow$ Thm:

Suppose v is a valuation with $v[\phi(t)] = F$. Show $v[(\forall x_1)\phi(x_1)] = F$.

Take v' as in the Lemma.
Then by Lemma $v'[\phi(x_1)] = F$
So $v[(\forall x_1)\phi(x_1)] = F$ as v' is x_1-equiv. to v. //