Examples II for Time Series (S8)

NB: Stationarity by itself always means 'second-order' stationarity

A general hint for some parts of the two questions: the autocovariance sequence of an MA(q) process is given by

$$s_\tau = \begin{cases} \sigma_\epsilon^2 \sum_{j=0}^{q-|\tau|} \theta_{j,q}\theta_{j+|\tau|,q}, & \text{if } |\tau| \le q, \\ 0, & \text{if } |\tau| > q. \end{cases}$$

1. (a) If $\{X_t\}$ and $\{Y_t\}$ are uncorrelated stationary sequences — i.e., X_s and Y_t are uncorrelated for every s and t — show that $\{Z_t\}$ defined by $Z_t = X_t + Y_t$, is stationary with autocovariance sequence given by $s_{Z,\tau} = s_{X,\tau} + s_{Y,\tau}$, where $\{s_{X,\tau}\}$ is the autocovariance sequence for $\{X_t\}$ and $\{s_{Y,\tau}\}$ is the autocovariance sequence for $\{Y_t\}$.

   (b) Determine, with full justification, whether

$$s_\tau = \begin{cases} 2+\psi^2, & \tau = 0 \\ -\psi, & |\tau| = 1, \\ 0, & \text{otherwise,} \end{cases}$$

   (where ψ is a constant with $|\psi| < 1$), is a valid autocovariance sequence.

2. Let $\{X_t\}$ be a Gaussian (normal) stationary process with a mean of zero and autocorrelation sequence $\rho_{X,\tau}$.

   You will need to use the following version of the Isserlis Theorem: If X_j, X_k, X_l, X_m are any four real-valued Gaussian random variables with zero mean then

$$E\{X_jX_kX_lX_m\} = E\{X_jX_k\}E\{X_lX_m\} + E\{X_jX_l\}E\{X_kX_m\} + E\{X_jX_m\}E\{X_kX_l\}.$$

   (a) (a) Define $Y_t = X_tX_{t-1}$. Find the autocovariance sequence $s_{Y,\tau}$ of $\{Y_t\}$ in terms of the autocovariance sequence $s_{X,\tau}$ of $\{X_t\}$.

   If $\{X_t\}$ is an MA(1) process, give the form of $s_{Y,\tau}$ in terms of $\theta_{1,1}$ and σ_ϵ^2.

   (b) (b) Show that the autocorrelation sequence for $Y_t = X_t^2$ is given by $\rho_{Y,\tau} = \rho_{X,\tau}^2$.

Find *one* model for $\{X_t\}$, such that $Y_t = X_t^2$ has autocovariance sequence $\{s_{Y,\tau}\}$ given by

$$s_{Y,\tau} = \begin{cases} 200, & \tau = 0; \\ 0, & |\tau| = 1; \\ 18, & |\tau| = 2; \\ 0, & |\tau| > 2, \end{cases}$$

giving all parameter value combinations that satisfy the stated form of $\{s_{Y,\tau}\}$.

3. This question will be concerned with the time series model

$$X_t = \mu_t + Y_t \tag{1}$$

where μ_t is a polynomial trend of order $d-1$ ($d \geq 1$), and Y_t is a zero mean AR(p) process of the form

$$Y_t = \sum_{j=1}^{p} \phi_{j,p} Y_{t-j} + \epsilon_t.$$

(a) Show the de-trended process $\{X_t^{(d)}\}$, where $X_t^{(d)} \equiv \Delta^d X_t$, is ARMA($p,d$). Determine whether $\{X_t^{(d)}\}$ is invertible.

We will now focus on the case where $\mu_t = \alpha + \beta t$ (i.e. it is a linear trend), and Y_t is an AR(1) process of the form

$$Y_t = \phi_{1,1} Y_{t-1} + \epsilon_t$$

where $|\phi_{1,1}| < 1$.

(b) You have shown in part (a) that $\{X_t^{(2)}\}$ will be ARMA(1,2). Explain why $\{X_t^{(2)}\}$ is stationary, and write it in the form of a General Linear Process, providing expressions for the coefficients $\{g_k, k = 0, 1, ...\}$ in terms of $\phi_{1,1}$.

(c) Find expressions for the first three terms of the autocovariance sequence (s_0, s_1 and s_2) for $\{X_t^{(2)}\}$.
[HINT: All three can be written in terms of $\phi_{1,1}$ without the need for infinite sums.]