SOLUTIONS for Examples I for Time Series (S8)

1. (a) Rearranging this gives

$$X_t = \epsilon_t + 2\epsilon_{t-3}$$

which is a MA(3) process with $\theta_{0,3} = -1$ (as is convention), $\theta_{1,3} = 0$, $\theta_{2,3} = 0$ and $\theta_{3,3} = -2$. MA(q) processes (finite q) are always stationary. Immediately, $E\{X_t\} = 0$, and using the formula given in the notes, the acvs for $\{X_t\}$ is $s_0 = ((-1)^2 + (-2)^2)\sigma_\epsilon^2 = 5\sigma_\epsilon^2$, $s_1 = s_{-1} = 0$, $s_2 = s_{-2} = 0$, $s_3 = s_{-3} = (-1 \times -2) = 2\sigma_\epsilon^2$ and $s_\tau = 0$ for all $|\tau| > 3$.

(b) First check the mean:

$$E\{X_t\} = E\{\epsilon_{t-2}\epsilon_{t-1}\epsilon_t\} = E\{\epsilon_{t-2}\}E\{\epsilon_{t-1}\}E\{\epsilon_t\} = 0$$

because $\{\epsilon_t\}$ is a sequence of uncorrelated Gaussian, and hence independent, random variables with zero mean. Therefore mean is constant.

Next check autocovariance:

$$\text{cov}\{X_t, X_{t+\tau}\} = E\{X_t X_{t+\tau}\} = E\{\epsilon_{t-2}\epsilon_{t-1}\epsilon_t\epsilon_{t-2+\tau}\epsilon_{t-1+\tau}\epsilon_{t+\tau}\}.$$

First check for $\tau = 0$:

$$s_0 = \text{var}\{X_t\} = E\{\epsilon_{t-2}^2\epsilon_{t-1}^2\epsilon_t^2\} = E\{\epsilon_{t-2}^2\}E\{\epsilon_{t-1}^2\}E\{\epsilon_t^2\}$$

as $\{\epsilon_t\}$ are independent[1]. It follows from $E\{\epsilon_t^2\} = \text{var}\{\epsilon_t\} = \sigma_\epsilon^2$ that $s_0 = \sigma_\epsilon^6$.

Next consider $\tau = 1$:

$$\begin{aligned}\text{cov}\{X_t, X_{t+1}\} = E\{X_t X_{t+1}\} &= E\{\epsilon_{t-2}\epsilon_{t-1}\epsilon_t\epsilon_{t-1}\epsilon_t\epsilon_{t+1}\}\\ &= E\{\epsilon_t^2\}E\{\epsilon_{t-1}^2\}E\{\epsilon_{t-2}\}E\{\epsilon_{t+1}\}\end{aligned}$$

as $\{\epsilon_t\}$ are uncorrelated. It follows that $s_1 = s_{-1} = 1 \times 1 \times 0 \times 0 = 0$. A Similar argument follows for every $|\tau| > 0$, therefore

$$s_\tau = \begin{cases} \sigma_\epsilon^6 & \tau = 0 \\ 0 & |\tau| > 0. \end{cases}$$

[1] It is not necessarily the case that if X and Y are uncorrelated then X^2 and Y^2 are uncorrelated. Often it is true, however, there are some rather niche counterexamples of when it does not hold. Certainly, a sufficient condition is that X and Y are independent, as is Gaussianity (uncorrelated Gaussian implies independent Gaussian).

(c) Check its mean: $E\{X_t\} = E\{(-1)^t(\epsilon_t + 1)\} = (-1)^t(E\{\epsilon_t\} + 1) = (-1)^t$. This is not constant for all time, and therefore $\{X_t\}$ is non-stationary.

(d) First consider its mean:

$$E\{X_t\} = E\{\cos(t + Y)\} = \frac{1}{2\pi}\int_0^{2\pi} \cos(t + y)\,\mathrm{d}\,y$$

as Y has pdf

$$f_Y(y) = \begin{cases} \frac{1}{2\pi} & 0 \le y < 2\pi \\ 0 & \text{otherwise.} \end{cases}$$

it follows that

$$E\{X_t\} = \frac{1}{2\pi}\int_0^{2\pi} \cos(t + y)\,\mathrm{d}\,y = \frac{1}{2\pi}(-\sin(t + 2\pi) + \sin(t)) = 0$$

for all t. Next, check the autocovariance:

$$\begin{aligned}
E\{X_t X_{t+\tau}\} &= E\{\cos(t + Y)\cos(t + \tau + Y)\} \\
&= \frac{1}{2\pi}\int_0^{2\pi} \cos(t + y)\cos(t + \tau + y)\,\mathrm{d}\,y \\
&= \frac{1}{2\pi}\int_0^{2\pi} \cos(t + y)\left(\cos(t + y)\cos(\tau) - \sin(t + y)\sin(\tau)\right)\mathrm{d}\,y \\
&= \frac{\cos(\tau)}{2\pi}\int_0^{2\pi} \cos^2(t + y)\,\mathrm{d}\,y - \frac{\sin(\tau)}{2\pi}\int_0^{2\pi} \sin(t + y)\cos(t + y)\,\mathrm{d}\,y \\
&= \frac{\cos(\tau)}{4\pi}\int_0^{2\pi} 1 + \cos(2(t + y))\,\mathrm{d}\,y - \frac{\sin(\tau)}{4\pi}\int_0^{2\pi} \sin(2(t + y))\,\mathrm{d}\,y \\
&= \frac{\cos(\tau)}{4\pi}\int_0^{2\pi} 1\,\mathrm{d}\,y = \frac{1}{2}\cos(\tau),
\end{aligned}$$

using the fact that $\int_0^{2\pi}\cos(2(t+y))\,\mathrm{d}\,y = 0$ and $\int_0^{2\pi}\sin(2(t+y))\,\mathrm{d}\,y = 0$. Therefore $s_\tau = \frac{1}{2}\cos(\tau)$ is dependent only on τ and hence $\{X_t\}$ stationary.

2. (a) i.

$$E\{X_t\} = E\{Y_1\}\cos(ct) + E\{Y_2\}\sin(ct) = 0.$$

Also for the covariance (which for $\tau = 0$ gives the variance),

$$\begin{aligned}
E\{X_t X_{t+\tau}\} &= E\{[Y_1\cos(ct) + Y_2\sin(ct)][Y_1\cos(c[t + \tau]) + Y_2\sin(c[t + \tau])]\} \\
&= E\{Y_1^2\}\cos(ct)\cos(c[t + \tau]) + E\{Y_1Y_2\}\cos(ct)\sin(c[t + \tau]) \\
&\quad + E\{Y_2Y_1\}\sin(ct)\cos(c[t + \tau]) + E\{Y_2^2\}\sin(ct)\sin(c[t + \tau]) \\
&= \sigma^2\cos(ct)\cos(c[t + \tau]) + \sigma^2\sin(ct)\sin(c[t + \tau]).
\end{aligned}$$

But, since $\cos(a-b) = \cos a \cos b + \sin a \sin b$,

$$E\{X_t X_{t+\tau}\} = \sigma^2 \cos(c\tau) = s_\tau.$$

Therefore the process is always stationary.

ii. Firstly suppose that $\{X_t\}$ is strictly stationary. Then the marginal distribution of X_t is independent of $t \in \mathbb{Z}$. With $c = \pi/4$ the cases $t = 0$ and 1 give $X_0 = Y_1$ and $X_1 = (Y_1 + Y_2)/\sqrt{2}$ so that Y_1 and $(Y_1 + Y_2)/\sqrt{2}$ have the same distribution. We know that Y_1 and Y_2 are IID. From Bernstein's theorem we can conclude that Y_1 and Y_2 are Gaussian.
Now suppose that Y_1 and Y_2 are Gaussian, then $\{X_t\}$ is a Gaussian process, (all finite-dimensional marginal distributions are multivariate Gaussian). The process is (second-order) stationary by part (i), and we know that a stationary Gaussian process is strictly stationary.

(b) i. $E\{X_t\} = E\{Y_1\}\cos(ct) = 0$. Taking $Y_2 \equiv 0$ in (a), gives

$$E\{X_t X_{t+\tau}\} = \sigma^2 \cos(ct)\cos(c[t+\tau]).$$

Since t and τ are integers, the process is stationary for $c = \ell\pi, \ell \in \mathbb{Z}$ and non-stationary otherwise, i.e.,

$$s_\tau = \sigma^2 \cos(\ell\pi t)\cos(\ell\pi[t+\tau]).$$

ii. Now $\cos(\ell\pi t) = (-1)^{\ell t}$ and $\cos(\ell\pi[t+\tau]) = (-1)^{\ell(t+\tau)}$ so that

$$s_\tau = \sigma^2(-1)^{\ell t}(-1)^{\ell(t+\tau)} = \sigma^2(-1)^{\ell\tau},$$

for some choice $\ell \in \mathbb{Z}$. Hence $s_0 = \sigma^2$ and by symmetry $\rho_\tau = s_\tau/s_0 = (-1)^{|\ell\tau|},\ \tau \in \mathbb{Z}$.

iii.

$$\begin{aligned}
\sum_{j=1}^{n}\sum_{k=1}^{n}\rho_{t_j-t_k}a_j a_k &= \sum_{j=1}^{n}\sum_{k=1}^{n}(-1)^{\ell(t_j-t_k)}a_j a_k \\
&= \sum_{j=1}^{n}(-1)^{\ell t_j}a_j \sum_{k=1}^{n}(-1)^{\ell t_k}a_k = [\sum_{j=1}^{n}(-1)^{\ell t_j}a_j]^2 \geq 0.
\end{aligned}$$

3.

$$\begin{aligned}
X_1 &= \phi X_0 + \epsilon_1 = \epsilon_1 \\
X_2 &= \phi X_1 + \epsilon_2 = \phi\epsilon_1 + \epsilon_2 \\
X_3 &= \phi X_2 + \epsilon_3 = \phi(\phi\epsilon_1 + \epsilon_2) + \epsilon_3 = \phi^2\epsilon_1 + \phi\epsilon_2 + \epsilon_3.
\end{aligned}$$

So $E\{X_j\} = 0$ for $j = 1, 2, 3$. Then

$$
\begin{aligned}
E\{X_1^2\} &= E\{\epsilon_1^2\} = \sigma_\epsilon^2 \\
E\{X_2^2\} &= E\{[\phi\epsilon_1 + \epsilon_2]^2\} = [1 + \phi^2]\sigma_\epsilon^2 \\
E\{X_3^2\} &= E\{[\phi^2\epsilon_1 + \phi\epsilon_2 + \epsilon_3]^2\} = [1 + \phi^2 + \phi^4]\sigma_\epsilon^2 \\
E\{X_1X_2\} &= E\{\epsilon_1[\phi\epsilon_1 + \epsilon_2]\} = \phi\sigma_\epsilon^2 \\
E\{X_1X_3\} &= E\{\epsilon_1[\phi^2\epsilon_1 + \phi\epsilon_2 + \epsilon_3]\} = \phi^2\sigma_\epsilon^2 \\
E\{X_2X_3\} &= E\{[\phi\epsilon_1 + \epsilon_2][\phi^2\epsilon_1 + \phi\epsilon_2 + \epsilon_3]\} = \phi^3\sigma_\epsilon^2 + \phi\sigma_\epsilon^2.
\end{aligned}
$$

So covariance matrix is

$$
\sigma_\epsilon^2 \begin{bmatrix} 1 & \phi & \phi^2 \\ \phi & 1 + \phi^2 & \phi(1 + \phi^2) \\ \phi^2 & \phi(1 + \phi^2) & 1 + \phi^2 + \phi^4 \end{bmatrix}.
$$

This is not Toeplitz. The generated variables are not part of a stationary sequence.

This is an important example from the point of simulation. For simulation using this sort of recursive scheme (with zero boundary conditions such as $X_0 = 0$) you would have to throw away 1000's of values to be sure of removing the 'start-up transients' before keeping the generated values. Alternatively it is possible to work-out special 'stationary boundary values' so that *all* the generated sequence is stationary.

In the theory the only boundary values that can be set to zero are those at $-\infty$; strictly speaking stochastic processes run from $-\infty$ to ∞.