Examples IV for Time Series

"Stationary" is meant to mean second order stationary unless explicitly stated otherwise.

1. In this question, $\{\epsilon_t\}$ is a zero mean Gaussian white noise process with variance $\sigma_\epsilon^2 = 1$.

   (a) Let $X_1, X_2, ..., X_{100}$ be a portion of the process

   $$X_t = \mu + \epsilon_t + \frac{1}{2}\epsilon_{t-1} - \frac{1}{2}\epsilon_{t-2}$$

   for some fixed μ. Find the distribution of

   $$\bar{X} = \frac{1}{100}\sum_{i=1}^{100} X_i.$$

   (b) Let X_1, X_2, X_3, X_4 be a portion of the process

   $$X_t = \frac{1}{2}X_{t-1} + \epsilon_t.$$

   Find the distribution of

   $$\bar{X} = \frac{1}{4}\sum_{i=1}^{4} X_i.$$

2. Let $X_1, \ldots, X_N$ be a sample from a stationary process $\{X_t\}$ with unknown mean μ and variance σ^2. The so-called 'unbiased' and 'biased' autocovariance estimators are given, respectively, by

   $$\hat{s}_\tau^{(u)} = \frac{1}{N-|\tau|}\sum_{t=1}^{N-|\tau|}(X_t - \bar{X})(X_{t+|\tau|} - \bar{X}) \text{ and } \hat{s}_\tau^{(p)} = \frac{1}{N}\sum_{t=1}^{N-|\tau|}(X_t - \bar{X})(X_{t+|\tau|} - \bar{X}).$$

   (a) By writing the periodogram in terms of the biased autocovariance sequence estimator show that the integral of the periodogram is equal to the sample variance, i.e.,

   $$\int_{-1/2}^{1/2} S^{(p)}(f)\,\mathrm{d}f = \sum_{t=1}^{N}(X_t - \overline{X})^2/N.$$

(b) Also show that

$$E\{\hat{s}_0^{(p)}\} \equiv E\{\hat{s}_0^{(u)}\} = s_0 - \text{var}\{\overline{X}\},$$

and comment on this result.

3. Let $X_1, ..., X_N$ be a sample of size N from a *white noise process* with unknown mean μ and variance σ^2.

(a) Show that, for $0 < |\tau| < N-1$,

$$E\{\hat{s}_\tau^{(u)}\} = -\frac{\sigma^2}{N} \quad \text{and} \quad E\{\hat{s}_\tau^{(p)}\} = -\left(1 - \frac{|\tau|}{N}\right)\frac{\sigma^2}{N}.$$

and hence that, for white noise, the magnitude of the bias of the 'biased' estimator $\hat{s}_\tau^{(p)}$ is less than that of the 'unbiased' estimator $\hat{s}_\tau^{(u)}$.

(b) Show that the mean square error of $\hat{s}_\tau^{(p)}$ is less than that of $\hat{s}_\tau^{(u)}$ for $0 < |\tau| < N-1$.

(c) By considering the row and diagonal sums of the $N \times N$ matrix having (u, v)th entry $(X_u - \bar{X})(X_v - \bar{X})$ for $1 \leq u, v \leq N$, show that $\sum_{\tau=-(N-1)}^{(N-1)} \hat{s}_\tau^{(p)} = 0$.

Hence deduce that $\hat{s}_\tau^{(p)}$ must be negative for some value(s) of τ.

4. Let a be a real-valued nonzero constant, and suppose that $\{a, 0, -a\}$ is a realization of length $N = 3$ of a portion X_1, X_2, X_3 of a stationary process with a *known* mean of zero, spectral density function $S(f)$ and autocovariance sequence $\{s_\tau\}$.

(a) Show the biased estimator $\{\hat{s}_\tau^{(p)}\}$ of the autocovariance sequence for $\{X_t\}$ is given by

$$\hat{s}_\tau^{(p)} = \begin{cases} 2a^2/3, & \tau = 0; \\ 0, & |\tau| = 1; \\ -a^2/3, & |\tau| = 2; \\ 0, & |\tau| > 2. \end{cases}$$

Also determine the 'unbiased' estimator $\{\hat{s}_\tau^{(u)}\}$ of the autocovariance sequence.

(b) Use the fact that $\{\hat{s}_\tau^{(p)}\}$ and the periodogram $\hat{S}^{(p)}(f)$ are a Fourier transform pair to show that

$$S^{(p)}(f) = \frac{2a^2}{3}[1 - \cos(4\pi f)], \qquad |f| \leq 1/2.$$

(c) An equivalent way of obtaining the periodogram is via

$$\hat{S}^{(p)}(f) = \frac{1}{N}\left|\sum_{t=1}^{N} X_t e^{-i2\pi f t}\right|^2 .$$

Verify that computing the periodogram in this alternative manner gives the same results as in part (a)(ii).

(d) The quantity

$$b(f) \equiv E\{\hat{S}^{(p)}(f)\} - S(f)$$

is the bias in the periodogram at frequency f.

Use the fact that

$$E\{\hat{S}^{(p)}(f)\} = \int_{-1/2}^{1/2} \mathcal{F}(f - f')S(f')\mathrm{d}f',$$

where $\mathcal{F}(\cdot)$ is Féjer's kernel defined by

$$\mathcal{F}(f) = \left|\sum_{t=1}^{N} \frac{1}{\sqrt{N}} e^{i2\pi f t}\right|^2 = \frac{\sin^2(N\pi f)}{N\sin^2(\pi f)},$$

to show that the average value of the bias in the periodogram over the interval $[-1/2, 1/2]$ is zero. NOTE: If $g(\cdot)$ is a function defined over the interval $[a, b]$, then, by definition, $[1/(b-a)]\int_a^b g(x)\mathrm{d}x$ is the average value of $g(\cdot)$ over $[a, b]$.