Mathematical Logic (MATH6/70132;P65)
Problem Sheet 3: Notes on solutions

[1] The first-order language $\mathcal{L}$ has one unary function symbol f and one unary relation symbol P. Let ϕ be the formula $(\forall x_1)(P(x_1) \to P(f(x_1)))$. Give an interpretation of $\mathcal{L}$ in which ϕ is true, and one in which it is false.

Solution: For an interpretation where the formula is false take the domain of the structure to be the natural numbers, interpret $P(x)$ as the predicate $x = 0$ and f as the successor function $x \mapsto x + 1$. For an interpretation where it is true, modify this so that f is interpreted as the function $x \mapsto x$.

[2] The language $\mathcal{L}$ has a binary relation symbol E, a binary function symbol m, a unary function symbol i and a constant symbol e. Let G be a group and consider G as an $\mathcal{L}$-structure by interpreting E as equality, m as multiplication, i as inversion, and e as the identity element of G. Let v be a valuation (of $\mathcal{L}$) in G and let

$$H = \{v(t) : t \text{ is a term of } \mathcal{L}\}.$$

(a) Show that H is a subgroup of G.

(b) Show that H is generated by $\{v(x_i) : x_i \text{ is a variable of } \mathcal{L}\}$.

(c) What is H if we omit the function symbol i from the language?

Solution: (a) It is enough to show that H is non-empty, closed under multiplication and closed under inversion. It is non-empty because e is a term and $v(e) = \bar{e} \in H$, i.e. the identity of G is in H. Suppose $a, b \in H$. Then there exist terms s, t with $v(s) = a$, $v(t) = b$. Then $m(s,t)$ is a term and $v(m(s,t))$ is the product $a \cdot b$. So $a \cdot b \in H$, as required. Similarly $i(s)$ is a term and $v(i(s)) = a^{-1}$, so H is also closed under inversion.
(b) Let $K = \langle v(x_i) : x_i$ a variable of $\mathcal{L} \rangle$, the subgroup generated by the values ascribed to the variables by v. So $K \leq H$. We show by induction on the length of a term t that $v(t) \in K$. In the base case t is a variable or e, and we have what we want. For the inductive step suppose s, t are terms with $v(s), v(t) \in K$. We have to show that $v(m(s,t)), v(i(s)) \in K$ but these are just the product $v(s) \cdot v(t)$ and the inverse $v(s)^{-1}$, and these are in K as K is a subgroup of G.
(c) H is the smallest subset of G which is closed under multiplication and which contains the identity, and $v(x_i)$ for all the variables x_i. This need not be a subgroup (it need not be closed under inverses, if G is infinite).

[3] Let ϕ be a formula in a first-order language $\mathcal{L}$ and let v be a valuation (in some $\mathcal{L}$-structure $\mathcal{A}$). Suppose there is a valuation v' which is x_i-equivalent to v and satisfies ϕ. Show that v satisfies $(\exists x_i)\phi$.

Solution: We need to show that v satisfies $\neg(\forall x_i)(\neg\phi)$. Well, v' satisfies ϕ so v' does not satisfy $\neg\phi$. As v' is x_i-equivalent to v it follws that v does not satisfy $(\forall x_i)(\neg\phi)$. Thus v satisfies $\neg(\forall x_i)(\neg\phi)$, as required.

[4] Suppose F is a field. The language $\mathcal{L}_F$ appropriate for considering F-vector spaces V has a 2-ary relation symbol R (for equality); a 2-ary function symbol a (for addition in the vector space); a constant symbol 0 (for the zero vector) and, for every $\alpha \in F$, a 1-ary function symbol f_α (for scalar multiplication by α).

Convince yourself that it is possible to express the axioms for being an F-vector space as a set of formulas in this language.

Solution: The main point here is to understand what to do about the axioms which involve scalar multiplication. For example, to express the usual vector space axiom

'for all $\alpha, \beta \in F$ and $v \in V$ we have $(\alpha + \beta)v = \alpha v + \beta v$'

we need one formula of the following form for each possible choice of $\alpha, \beta \in F$:

$$(\forall x)R(f_{\alpha+\beta}(x), a(f_\alpha(x), f_\beta(x)).$$

The issue here is not the unfamiliar notation. It is that we cannot quantify over the elements of F.

Note that if the field is the real numbers, then the language is uncountable and we have uncountably many axioms here.

[5] In each of the following formulas, indicate which of the occurrences of the variables x_1 and x_2 are bound and which are free:
(a) $(\forall x_2)(R_2(x_1, x_2) \to R_2(x_2, c_1))$;
(b) $(R_1(x_2) \to (\forall x_1)(\forall x_2)R_3(x_1, x_2, c_1))$;
(c) $((\forall x_1)R_1(f(x_1, x_2)) \to (\forall x_2)R_2(f(x_1, x_2), x_1))$.

Decide whether the term $f(x_1, x_1)$ is free for x_2 in each of the above formulas (explain briefly your answer).

Solution: (a) $(\forall x_2)(R_2(x_1, x_2) \to R_2(x_2, c_1))$;
All occurrences of x_2 are bound; all occurrences of x_1 are free.

(b) $(R_1(x_2) \to (\forall x_1)(\forall x_2)R_3(x_1, x_2, c_1))$;
Only the first occurrence of x_2 is free.

(c) I'll put a hat over the free variables: $((\forall x_1)R_1(f(x_1, \hat{x_2})) \to (\forall x_2)R_2(f(\hat{x_1}, x_2), \hat{x_1}))$.

Decide whether the term $f(x_1, x_1)$ is free for x_2 in each of the above formulas (explain briefly your answer):

The only variable in the term is x_1. So this term is *not* free for x_2 in a formula if x_2 occurs free within the scope of a quantifier $(\forall x_1)$ in the formula. Thus $f(x_1, x_1)$ is free for x_2 in the first two formulas. However, it is not free for x_2 in the third, because of:

$$((\forall x_1)R_1(f(x_1, \hat{x_2})) \ldots$$

[6] For the formula $\phi(x_2)$ given by $((\exists x_1)R(x_1, f(x_1, x_2)) \to (\forall x_1)R(x_1, x_2))$ (in a particular language $\mathcal{L}$) give an example of a term t which is not free for x_2 in $\phi(x_2)$. Find an $\mathcal{L}$-structure $\mathcal{A}$ in which $(\forall x_2)\phi(x_2)$ is true and a valuation v in $\mathcal{A}$ which does not satisfy $\phi(t)$.

Solution: Consider the term x_1. Now, in the formula $((\exists x_1)R(x_1, f(x_1, x_2)) \to (\forall x_1)R(x_1, x_2))$ the variable x_2 occurs free in the scope of the quantifier $\exists x_1$ (and the quantifier $\forall x_1$). So the term x_1 is not free for x_2 in the formula $\phi(x_2)$.

The formulas $(\forall x_2)\phi(x_2)$ and $\phi(x_1)$ are then:
$(\forall x_2)((\exists x_1)R(x_1, f(x_1, x_2)) \to (\forall x_1)R(x_1, x_2))$ and
$((\exists x_1)R(x_1, f(x_1, x_1)) \to (\forall x_1)R(x_1, x_1))$. To find an interpretation where the first is true, but the second is false, try to keep things as simple as possible. Take:
Domain $\mathbb{N}$;
$\bar{R}(x_1, x_2) \Leftrightarrow x_2 = 0$;
$\bar{f}(x_1, x_2) = x_2$.

Then the first formula says 'if $x_2 = 0$, then $x_2 = 0$' (which is true!), whereas the second formula says 'if there exists some value of x_1 with $x_1 = 0$, then $x_1 = 0$ for all values of x_1, ' which is false.

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