Imperial College London

MATH97009 MATH97101

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Mathematical Finance: An Introduction to Option Pricing

Date: Thursday, 6 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS SEPARATE PDFs TO THE RELEVANT DROPBOXES ON BLACKBOARD (ONE FOR EACH QUESTION) WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

Question 1

(Total: 20 marks)

Consider the probability space $\Omega = \{\omega_1, \omega_2, \omega_3\}$, with the probability $\mathbb{P}$ such that $\mathbb{P}(\{\omega\}) = 1/3$ for every $\omega \in \Omega$. Define the random variables

ω	ω_1	ω_2	ω_3
$S_1(\omega)$	6	8	10
$X_1(\omega)$	-6	4	14
$Y_1(\omega)$	16	6	2

Consider the one-period trinomial model of the market (B, S) made of a bond B with initial price 1 (all prices in a fixed currency, say £), and interest rate $r = 1$, a stock whose initial price is $S_0 = 4$, and whose final price is S_1. Answer the following questions about the two illiquid derivatives with payoffs X_1, Y_1 in the market (B, S).

(a) Is this model free of arbitrage? (3 marks)

(b) Is X_1 replicable? (2 marks)

(c) Is Y_1 replicable ? (2 marks)

(d) What is the set of arbitrage-free prices of X_1? (4 marks)

(e) What is the set of arbitrage-free prices of Y_1? (4 marks)

(f) Consider now a derivative with payoff Z_1 which is *not* replicable in the (B, S) market. Now enlarge the (B, S) market, by assuming that Y is traded at the arbitrage-free price Y_0 at time 0. Is the set of arbitrage-free prices of Z_1 in the market (B, S, Y) a singleton? Explain your reasoning. (5 marks)

Question 2

(Total: 20 marks)

Suppose a British investor can:

1. deposit £ in a bank at the (domestic) interest rate $r = 1/2$
2. buy or sell \$ (by paying/getting paid in £) at any time n with exchange rate S_n
3. deposit \$ in a bank at the (foreign) interest rate f

The exchange rate S_n is defined as the number of units of £ needed to buy one unit of \$ at time n, and is assumed to follow the one period binomial model with $S_0 = 2$, $u = 2, d = 1/2$. Answer the following questions and justify carefully your answers.

(a) Explain why the formula (3 marks)

$$V_1^{x,h} := (V_0 - hS_0)(1 + r) + h(1 + f)S_1$$

described the total wealth $V_1 := V_1^{x,h}$ (in £) at time 1 of the investor whose initial capital (in £) is $V_0 = x$, and at time 0 buys \$ $h \in \mathbb{R}$, and then deposits his £ and his \$ in the banks.

(b) Assume from now on that $r = 1/2$. For what values of f is the above model arbitrage-free? (4 marks)

(c) From now on let $f = 1$. For what value of $\tilde{p} = \mathbb{Q}(H) \in (0, 1)$ does the RNPF (Risk Neutral Pricing Formula) $V_0 = \mathbb{E}^{\mathbb{Q}}[V_1/(1 + r)]$ provide the (only) arbitrage-free price of the payoff V_1, for any value of V_1? (4 marks)

(d) Consider a forward contract on the exchange rate. In other words, consider the agreement which has no initial cost, and which states that its buyer will buy one \$ at the price P_0 (called the forward exchange rate) at time 1, where the constant P_0 is determined by asking that the arbitrage-free price F_0 of the forward contract is zero. What is the value of P_0? (5 marks)

(e) What is the replicating strategy h of a call option (on the exchange rate) with strike price 2? (4 marks)

Question 3

(Total: 20 marks)

In the framework of the N-period binomial model with constant parameters $S_0 = 6, u = \frac{3}{2}, d = \frac{1}{2}, r = 0$, let $S = (S_n)_{n=0}^N$ be the stock price process,

$$Q_n := S_0^2 + \sum_{i=0}^{n-1} (S_{i+1} - S_i)^2$$

its quadratic variation up to time n, and $Y_n := \frac{Q_n}{S_n^2}$, where $n = 0, \dots, N$. Consider the option which at time N pays the amount $V_N = (Q_N - KS_N^2)^+$ for some $K \geq 0$, and denote by V_n its arbitrage-free price at time $n = 0, \dots, N$. As usual $\mathbb{Q}$ denotes the risk-neutral measure, $(X_n)_n$ denotes the process of coin tosses X which generates $(S_n)_n$, and we take as filtration $\mathcal{F}$ the natural filtration of X. Prove all your assertions carefully or provide counter-examples.

(a) Use the risk-neutral pricing formula to express V_n in terms of V_{n+1}. (3 marks)

(b) Express Y_{n+1} as a function of Y_n and $C_{n+1} := \frac{S_{n+1}}{S_n}$. (5 marks)

(c) Is $\frac{S_{n+1}}{S_n}$ independent of $\mathcal{F}_n$ under $\mathbb{Q}$? (2 marks)

(d) Is Y a $\mathbb{Q}$-Markov process? (4 marks)

(e) Work by backward induction to show that, for every $n = 0, \dots, N$, V_n admits the representation $V_n = S_n^2 v_n(Y_n)$, where $v_n : \mathbb{R} \to \mathbb{R}$, $n = 0, \dots, N$, are (deterministic) functions. Write explicitly v_N and an explicit formula to express v_n in terms of v_{n+1} for $n = 0, \dots, N-1$. (6 marks)

Question 4

(Total: 20 marks)

I bought at time 0 a derivative which gives me the right to purchase, at price $c_1 > 0$ and time $t_1 > 0$, a call option (on an underlying stock S) with expiration $t_2 > t_1$ and strike price K_2. Assume that the stock price $S = (S_t)_{0 \le t \le t_2}$ follows the Black-Scholes model, and denote with $c(x, \tau, K)$ the price at time t of a call option on S with expiry $T := t + \tau > t$ and strike $K > 0$, if $S_t = x$ (it can be proved that c does not depend on t, it only depends on τ). Answer the following questions and justify carefully with either proofs or counterexamples.

(a) Prove that, for any value of (τ, K), the function $x \mapsto c(x, \tau, K) \in \mathbb{R}$ is strictly increasing for $x \in (0, \infty)$. Compute its limits as $x \downarrow 0$ and as $x \uparrow \infty$. (3 marks)

(b) Write an equation whose solution is the value of $b_1 > 0$ such that at time t_1 if $S_{t_1} > b_1$ I should exercise the derivative, if $S_{t_1} < b_1$ I should not exercise it, and if $S_{t_1} = b_1$ it is irrelevant what I do. Prove that such equation admits one and only one solution. *Warning: do not try to solve the equation: it is transcendental, and so it does not have a closed-form analytic solution.* (4 marks)

(c) Write a formula, which involves the price c_1 and the pricing function c, for the value of the derivative at time t_1. (3 marks)

(d) Write down a formula for the function $f = f(x, y)$ such that $f(S_{t_1}, S_{t_2})$ is the payoff of the option at time t_2. (5 marks)

(e) Consider a derivative G on S with payoff $G_{t_2} = g(S_{t_1}, S_{t_2})$ at time t_2, for some function $g = g(x, y)$. Obtain an explicit formula for the price G_0 at time 0 of the derivative G; this formula must be of the form (5 marks)

$$\int_{\mathbb{R}^2} (g \circ h)(x, y)\ a(x, y)\ dx dy,$$

where h, a are functions which you have to determine; explain your reasoning.

Question 5

(Total: 20 marks)

On a finite sample space $\Omega = \{\omega_i\}_{i=1,\dots,n}$ endowed with some probability $\mathbb{P}$ s.t. $\mathbb{P}(\omega_i) > 0$ for all i, consider a one-period arbitrage-free market model where the bank account has interest rate $r = 0$, and so we model it with the process $B_0 = B_1 = 1$, and there are two stocks S^1, S^2. If we express the portfolio using the number $h^1, h^2 \in \mathbb{R}$ of shares of stocks S^1 and S^2, and the amount of cash $c \in \mathbb{R}$ in the bank account (not of the initial capital x), then at time $t = 0, 1$ the wealth $V_t^{c,h}$ relative to (c, h) is given by

$$V_0^{c,h} = c + h \cdot S_0, \quad V_1^{c,h} = c(1+r) + h \cdot S_1, \tag{1}$$

where $h \cdot S_t$ denoted the usual dot product between h and S_t. For the *non-replicable* derivative with payoff X_1, we consider the problem of finding the smallest initial capital p of a portfolio (c, h) super-replicating X_1 $\mathbb{P}$ a.s., i.e.

$$p := \min\{V_0^{c,h} : (c,h) \in \mathbb{R} \times \mathbb{R}^2 \text{ satisfies } V_1^{c,h}(\omega_i) \geq X_1(\omega_i) \text{ for all } i\}, \tag{2}$$

and its dual linear program, i.e.

$$d := \max\{\mathbb{E}^{\mathbb{Q}}(X_1) : \mathbb{Q} \in \mathcal{M}\}, \text{ where } \mathbb{M} := \{\mathbb{Q} \text{ probability on } \Omega : \mathbb{E}^{\mathbb{Q}}(S_1^j - S_0^j) = 0,\ j = 1, 2\} \tag{3}$$

is the set of martingale measures.

We denote by c^*, h^* any trading strategy such that $V_0^{c^*,h^*} = p$ and $V_1^{c^*,h^*} \geq X_1 \quad \mathbb{P}$ a.s., and with $\mathbb{Q}^*$ any element of $\mathbb{M}$ such that $d = \mathbb{E}^{\mathbb{Q}^*}(X_1)$, i.e. any optimisers of (2) and (3) (such optimisers always exist). Answer the following questions and justify carefully with either proofs or counterexamples.

(a) Is p an arbitrage-free price for X_1? If it is not, find an arbitrage. (3 marks)

(b) Is it true that $V_1^{c^*,h^*} = X_1$ $\mathbb{Q}^*$ a.s.? (3 marks)

(c) Is $\mathbb{Q}^*$ equivalent to $\mathbb{P}$? (2 marks)

(d) *Assume from now on that it is not possible to borrow any money from the bank.* Define what an arbitrage is in such a market, *using formulas*. (2 marks)

(e) Consider again the problem of finding the smallest initial capital $\bar{p}$ of the portfolio (c, h) super-replicating X_1 (but now without being able to borrow). Formulate this problem in a way analogous to (2), and formulate its dual problem in a way analogous to (3), and call the respective optimal values $\bar{p}$ and $\bar{d}$. (4 marks)

(f) Are $\bar{p}$ and $\bar{d}$ always equal? (3 marks)

(g) Is it true that $\bar{p} \geq p$, for any choice of X_1 and any choice of model (B, S^1, S^2)? What about $\bar{p} > p$? (3 marks)

Question 1 (Total: 20 marks)

SIMILARLY SEEN IN LECTURES AND PROBLEMS

Consider the probability space $\Omega = \{\omega_1, \omega_2, \omega_3\}$, with the probability $\mathbb{P}$ such that $\mathbb{P}(\{\omega\}) = 1/3$ for every $\omega \in \Omega$. Define the random variables

ω	ω_1	ω_2	ω_3
$S_1(\omega)$	6	8	10
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Consider the one-period trinomial model of the market (B, S) made of a bond B with initial price 1 (all prices in a fixed currency, say £), and interest rate $r = 1$, a stock whose initial price is $S_0 = 4$, and whose final price is S_1. Answer the following questions about the two illiquid derivatives with payoffs X_1, Y_1 in the market (B, S).

(a) Is this model free of arbitrage? (3 marks)

(b) Is X_1 replicable? (2 marks)

(c) Is Y_1 replicable ? (2 marks)

(d) What is the set of arbitrage-free prices of X_1? (4 marks)

(e) What is the set of arbitrage-free prices of Y_1? (4 marks)

(f) Consider now a derivative with payoff Z_1 which is *not* replicable in the (B, S) market. Now enlarge the (B, S) market, by assuming that Y is traded at the arbitrage-free price Y_0 at time 0. Is the set of arbitrage-free prices of Z_1 in the market (B, S, Y) a singleton? Explain your reasoning. (5 marks)

Solution: Recall that we identify any random variable Y with a vector y via $y_i = Y(\omega_i)$. Since $1 + r = 2$, the *discounted* stock and payoff values are

$$\bar{S}_0 = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}, \quad \bar{S}_1 = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}, \quad \bar{X}_1 = \begin{pmatrix} -3 \\ 2 \\ 7 \end{pmatrix}, \quad \bar{Y}_1 = \begin{pmatrix} 8 \\ 3 \\ 1 \end{pmatrix},$$

where as usual $\overline{W}_1 := W_1/(1 + r)$, $\overline{W}_0 := W_0$ for any process $W = (W_0, W_1)$.

(a) Recall that $\mathbb{Q}$ is an EMM (equivalent martingale measure) if $\bar{S}_0 = \mathbb{E}^{\mathbb{Q}}[\bar{S}_1]$, Q is a probability and $\mathbb{Q} \sim \mathbb{P}$, i.e. iff $q_i := \mathbb{Q}(\{\omega_i\})$ satisfy

$$\begin{cases} 4 = 3q_1 + 4q_2 + 5q_3 \\ 1 = q_1 + q_2 + q_3 \\ q_i > 0 \text{ for } i = 1, 2, 3 \end{cases}$$

The system of equalities has solution $q_2 = 1 - 2q_1, q_3 = q_1$, and imposing $q_i > 0$ we obtain that the set of (q's corresponding to the set of) EMM is

$$\mathcal{M} = \left\{ q_t := \begin{pmatrix} t \\ 1-2t \\ t \end{pmatrix} : t \in \left(0, \frac{1}{2}\right) \right\}.$$

Since $\mathcal{M}$ is not empty, the model is arbitrage-free.

(b) 1^{st} **solution:** The portfolio with initial wealth x and trading strategy h has discounted payoff $\bar{V}_1 = x + h(\bar{S}_1 - \bar{S}_0)$ equal to

$$x \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + h \cdot \begin{pmatrix} 3-4 \\ 4-4 \\ 5-4 \end{pmatrix} = \begin{pmatrix} x-h \\ x \\ x+h \end{pmatrix}.$$

Solving for $\bar{V}_1 = \bar{X}_1$ gives

$$\begin{pmatrix} -3 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} x-h \\ x \\ x+h \end{pmatrix}$$

which has solution $x = 2, h = 5$, so X_1 is replicable starting with initial wealth 2 and buying 5 stocks and putting the remaining $2 - 5 \cdot 4 = -18$ in the bank (i.e. borrowing 18 from the bank, i.e. short-selling 18 bonds).

2^{nd} **solution:** As we will prove in the next items, X_1 has a unique arbitrage-free price, and so it is replicable.

(c) 1^{st} **solution:** Solving for $\bar{V}_1 = \bar{Y}_1$ gives

$$\begin{pmatrix} 8 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} x-h \\ x \\ x+h \end{pmatrix}$$

which has no solution, so Y_1 is not replicable.

2^{nd} **solution:** As we will prove in the next items, Y_1 has a non-unique arbitrage-free price, and so it is not replicable.

(d) If q_t is as above then $\mathbb{E}^{\mathbb{Q}_t}[\bar{X}_1] = -3t + 2(1-2t) + 7t = 2$, so $\mathcal{AFP}(X_1) = \{2\}$ and X_1 is replicable because it has a unique AFP

(e) If q_t is as above then $\mathbb{E}^{\mathbb{Q}_t}[\bar{Y}_1] = 8t + 3(1-2t) + t = 3(t+1)$. In particular depends on t, so $\mathbb{E}^{\mathbb{Q}_t}[\bar{Y}_1]$ is not constant across of EMM's $\mathbb{Q}$ and Y_1 is not replicable. Since $t \in (0, 1/2)$ we get that

$$\mathcal{AFP}(Y_1) = \{3(t+1) : t \in (0, 1/2)\} = (3, 9/2).$$

(f) Since Y is not replicable, the replication equation $xB_1 + kS_1 + hY_1 = Z_1$ has 3 *independent* equations. Since it has 3 unknowns x, k, h, it has a unique solution, so Z, though not replicable in the (B, S) market, is replicable in the (B, S, Y) market. Thus, in such market it has a unique arbitrage-free price.

Question 2

(Total: 20 marks)

SIMILARLY SEEN IN PROBLEMS

Suppose a British investor can:

1. deposit £ in a bank at the (domestic) interest rate $r = 1/2$
2. buy or sell \$ (by paying/getting paid in £) at any time n with exchange rate S_n
3. deposit \$ in a bank at the (foreign) interest rate f

The exchange rate S_n is defined as the number of units of £ needed to buy one unit of \$ at time n, and is assumed to follow the one period binomial model with $S_0 = 2$, $u = 2, d = 1/2$. Answer the following questions and justify carefully your answers.

(a) Explain why the formula (3 marks)

$$V_1^{x,h} := (V_0 - hS_0)(1+r) + h(1+f)S_1$$

described the total wealth $V_1 := V_1^{x,h}$ (in £) at time 1 of the investor whose initial capital (in £) is $V_0 = x$, and at time 0 buys \$ $h \in \mathbb{R}$, and then deposits his £ and his \$ in the banks.

(b) Assume from now on that $r = 1/2$. For what values of f is the above model arbitrage-free? (4 marks)

(c) From now on let $f = 1$. For what value of $\tilde{p} = \mathbb{Q}(H) \in (0,1)$ does the RNPF (Risk Neutral Pricing Formula) $V_0 = \mathbb{E}^{\mathbb{Q}}[V_1/(1+r)]$ provide the (only) arbitrage-free price of the payoff V_1, for any value of V_1? (4 marks)

(d) Consider a forward contract on the exchange rate. In other words, consider the agreement which has no initial cost, and which states that its buyer will buy one \$ at the price P_0 (called the forward exchange rate) at time 1, where the constant P_0 is determined by asking that the arbitrage-free price F_0 of the forward contract is zero. What is the value of P_0? (5 marks)

(e) What is the replicating strategy h of a call option (on the exchange rate) with strike price 2? (4 marks)

Solution:

(a) Buying \$ h the investor spends £ hS_0, so he deposits £ $V_0 - hS_0$ in the British bank; his \$ h deposited in the American bank will accrue interest with the foreign interest rate, so they will become \$ $h(1+f)$ at time 1, which are worth £ $h(1+f)S_1$. Thus the final wealth in £ is

$$V_1 = V_1^{x,h} = (V_0 - hS_0)(1+r) + h(1+f)S_1. \tag{1}$$

In other words, if we define $S_0^* := S_0, S_1^* := (1+f)S_1$, which corresponds to the value in £ of an investment in one \$, then we obtain the usual pricing formula, but with respect to the asset S^* instead of S.

(b) **First solution** As observed in the above item, we can simply pretend we are trading in a stock modelled by S^* and a bank account in £. Denoting as usual with $d < u$ the two values taken by S_1/S_0 in the binomial model, using the (already known) condition for No Arbitrage in the binomial model thus gives

$$d(1+f) < (1+r) < u(1+f),$$

i.e. there is No Arbitrage iff $f \in (-\frac{1}{4}, 2)$.
Second solution Setting $V_0 = 0$ we see that

$$V_1 = h((1+f)S_1 - (1+r)S_0)$$

Thus $h > 0$ is an arbitrage iff

$$Y_1 := ((1+f)S_1 - (1+r)S_0)$$

is positive (meaning ≥ 0) but not identically 0, and analogously $h < 0$ is an arbitrage iff $-Y_1$ is positive but not identically 0; of course $h = 0$ is never an arbitrage. Since

$$((1+f)S_1 - (1+r)S_0)(\omega) = \begin{cases} 4(1+f) - 3 & \text{if } \omega = H \\ (1+f) - 3 & \text{if } \omega = T, \end{cases}$$

is means the model is arbitrage-free iff

$$4(1+f) - 3 > 0 > (1+f) - 3,$$

i.e. iff $f \in (-\frac{1}{4}, 2)$.

(c) Since (1) gives that $V_1 = V_0(1+r) + hY_1$, taking $\mathbb{E}^{\mathbb{Q}}$ gives the RNPF for any final payoff iff $\mathbb{E}^{\mathbb{Q}} Y_1 = 0$, i.e iff

$$\mathbb{E}^{\mathbb{Q}}(1+f)S_1 = (1+f)(\tilde{p}S_0 u + (1-\tilde{p})S_0 d) \quad \text{equals} \quad (1+r)S_0,$$

where as usual $\tilde{p} = \mathbb{Q}(H)$. In other words

$$(u-d)\tilde{p} = \frac{1+r}{1+f} - d,$$

which substituting the numerical values becomes $\frac{3}{2}\tilde{p} = \frac{1}{4}$, and so $\tilde{p} = \frac{1}{6}$.

(d) Clearly the value in £ of the forward at time 1 is

$$F_1 = S_1 - P_0.$$

Here P_0 is the constant determined by requiring that this payoff can be replicated at cost 0. To determine P_0, suppose that we start with zero initial capital and buy \$ $\frac{1}{1+f}$, while borrowing £ $\frac{1}{1+f}S_0$ from the bank to do so. As both the investment in \$ and the debt in £ will accrue interest, at time one the investor will own \$ 1 (which is worth £ S_1) and owe the bank the constant (meaning non-random) value £ $\frac{1+r}{1+f}S_0$; thus

$$P_0 = \frac{1+r}{1+f}S_0.$$

(e) **First solution** The payoff of the call option is

$$C_1 = (S_1 - 2)^+ = \begin{cases} 2 & \text{if } \omega = H, \\ 0 & \text{if } \omega = T. \end{cases}$$

We can calculate the replicating strategy using the usual formula

$$h = \frac{C_1(H) - C_1(T)}{S_1^*(H) - S_1^*(T)},$$

where we have however replaced S by S^* (see item (a)); this gives $h = 1/3$. The arbitrage-free price can analogously be calculated by using the RNPF, where $\tilde{p} = \frac{1}{6}$ is determined in item (c), and in particular it not given by the same formula as for $f = 0$. This gives

$$x = \mathbb{E}^{\mathbb{Q}}\left[\frac{C_1}{1+r}\right] = \frac{1}{1+\frac{1}{2}}\left(\frac{1}{6} \cdot 2 + \left(1 - \frac{1}{6}\right) \cdot 0\right) = 2/9.$$

Second solution Using (1) we get $V_1 = \frac{3}{2}(x - 2h) + 2hS_1$, and to replicate C_1 we set $V_1^{x,h} = C_1$, which leads to

$$= \begin{cases} \dfrac{3}{2}(x - 2h) + 8h & \text{if } \omega = H, \\ \dfrac{3}{2}(x - 2h) + 2h & \text{if } \omega = T. \end{cases}$$

Solving for x, h gives that the replicating strategy is $h = 1/3$ and the arbitrage-free price is $x = 2/9$.

Question 3

(Total: 20 marks)

SIMILARLY SEEN IN LECTURES AND PROBLEMS

In the framework of the N-period binomial model with constant parameters $S_0 = 6, u = \frac{3}{2}, d = \frac{1}{2}, r = 0$, let $S = (S_n)_{n=0}^N$ be the stock price process,

$$Q_n := S_0^2 + \sum_{i=0}^{n-1}(S_{i+1} - S_i)^2$$

its quadratic variation up to time n, and $Y_n := \frac{Q_n}{S_n^2}$, where $n = 0, \dots, N$. Consider the option which at time N pays the amount $V_N = (Q_N - KS_N^2)^+$ for some $K \geq 0$, and denote by V_n its arbitrage-free price at time $n = 0, \dots, N$. As usual $\mathbb{Q}$ denotes the risk-neutral measure, $(X_n)_n$ denotes the process of coin tosses X which generates $(S_n)_n$, and we take as filtration $\mathcal{F}$ the natural filtration of X. Prove all your assertions carefully or provide counter-examples.

(a) Use the risk-neutral pricing formula to express V_n in terms of V_{n+1}. (3 marks)

(b) Express Y_{n+1} as a function of Y_n and $C_{n+1} := \frac{S_{n+1}}{S_n}$. (5 marks)

(c) Is $\frac{S_{n+1}}{S_n}$ independent of $\mathcal{F}_n$ under $\mathbb{Q}$? (2 marks)

(d) Is Y a $\mathbb{Q}$-Markov process? (4 marks)

(e) Work by backward induction to show that, for every $n = 0, \dots, N$, V_n admits the representation $V_n = S_n^2 v_n(Y_n)$, where $v_n : \mathbb{R} \to \mathbb{R}$, $n = 0, \dots, N$, are (deterministic) functions. Write explicitly v_N and an explicit formula to express v_n in terms of v_{n+1} for $n = 0, \dots, N-1$. (6 marks)

Solution:

(a) The risk neutral pricing formula gives

$$V_n = \mathbb{E}^{\mathbb{Q}}\left(\frac{V_{n+1}}{1+r}|\mathcal{F}_n\right) \tag{2}$$

(b) Since S_{n+1}/S_n takes values $u = 3/2$ and $d = 1/2$, $(\frac{S_{n+1}}{S_n} - 1)^2 = 1/4$, and so

$$Q_{n+1} = Q_n + (S_{n+1} - S_n)^2 = Q_n + S_n^2(\frac{S_{n+1}}{S_n} - 1)^2 = Q_n + \frac{S_n^2}{4}$$

we get

$$Y_{n+1} = \frac{Q_n + \frac{S_n^2}{4}}{S_n^2(\frac{S_{n+1}}{S_n})^2} = \frac{Y_n + \frac{1}{4}}{(\frac{S_{n+1}}{S_n})^2}$$

i.e.

$$Y_{n+1} = h_n\left(Y_n, \frac{S_{n+1}}{S_n}\right) \text{ for } h_n(y, c) := \frac{y + \frac{1}{4}}{c^2}. \tag{3}$$

(c) Notice that $\frac{S_{n+1}}{S_n}$ is independent (under the risk neutral measure $\mathbb{Q}$) of the filtration $\mathcal{F}_n$ generated by the first n coin tosses, since it only depends on the last coin toss X_{n+1} and the coin tosses are independent under $\mathbb{Q}$ since

$$\mathbb{Q}(X_{n+1} = H|\omega_1, \dots, \omega_n) = \tilde{p} = \frac{(1+r) - d}{u - d} \tag{4}$$

does not depend on $\omega_1, \ldots, \omega_n \in \{H, T\}$. Since $\sigma(S_0, \ldots, S_n) \subseteq \mathcal{F}_n$, this shows that $\frac{S_{n+1}}{S_n}$ is independent of $S_0, \ldots, S_n$ under $\mathbb{Q}$.

(d) Since $\frac{S_{n+1}}{S_n}$ is independent of $\mathcal{F}_n$ under $\mathbb{Q}$, and Y_n is $\mathcal{F}_n$-measurable it follows from eq. (3) that Y is Markov (by applying the independence lemma)

(e) Let us prove by backward induction that $\frac{V_n}{S_n^2} = v_n(Y_n)$ for all n and for some function v_n (which we will determine). By definition on V_N, $\frac{V_n}{S_n^2} = v_n(Y_n)$ holds for $n = N$ with

$$v_N(y) = (y - K)^+. \tag{5}$$

Now, assume by inductive hypothesis that $\frac{V_k}{S_k} = v_k(Y_k)$ holds for $k = n+1$ and let us show that it holds for $k = n$ (by induction, this will show that $\frac{V_n}{S_n} = v_n(Y_n)$ for all n). Using (2) we get

$$(1 + r)V_n = \mathbb{E}^{\mathbb{Q}}[V_{n+1}|\mathcal{F}_n] = \mathbb{E}^{\mathbb{Q}}[S_{n+1}^2 v_{n+1}(Y_{n+1})|\mathcal{F}_n]$$

and so using (3) we get

$$\frac{V_n}{S_n^2} = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}\left[\frac{S_{n+1}^2}{S_n^2} v_{n+1}\left(h_n\left(Y_n, \frac{S_{n+1}}{S_n}\right)\right)|\mathcal{F}_n\right].$$

Since Y_n is $\mathcal{F}_n$-measurable and $\frac{S_{n+1}}{S_n}$ is independent of $\mathcal{F}_n$ under $\mathbb{Q}$, we can apply the independence lemma and obtain that $\frac{V_n}{S_n^2} = v_n(Y_n)$ for

$$v_n(y) := \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}\left[\frac{S_{n+1}^2}{S_n^2} v_{n+1}\left(h_n\left(y, \frac{S_{n+1}}{S_n}\right)\right)\right] \tag{6}$$

and so

$$v_n(y) = \frac{1}{1+r}\left(\tilde{p}u^2 v_{n+1}\left(\frac{y + \frac{1}{4}}{u^2}\right) + (1-\tilde{p})d^2 v_{n+1}\left(\frac{y + \frac{1}{4}}{d^2}\right)\right), \tag{7}$$

where $\tilde{p}$ is given by (4).

Question 4 (Total: 20 marks)

UNSEEN

I bought at time 0 a derivative which gives me the right to purchase, at price $c_1 > 0$ and time $t_1 > 0$, a call option (on an underlying stock S) with expiration $t_2 > t_1$ and strike price K_2. Assume that the stock price $S = (S_t)_{0 \le t \le t_2}$ follows the Black-Scholes model, and denote with $c(x, \tau, K)$ the price at time t of a call option on S with expiry $T := t + \tau > t$ and strike $K > 0$, if $S_t = x$ (it can be proved that c does not depend on t, it only depends on τ). Answer the following questions and justify carefully with either proofs or counterexamples.

(a) Prove that, for any value of (τ, K), the function $x \mapsto c(x, \tau, K) \in \mathbb{R}$ is strictly increasing for $x \in (0, \infty)$. Compute its limits as $x \downarrow 0$ and as $x \uparrow \infty$. (3 marks)

(b) Write an equation whose solution is the value of $b_1 > 0$ such that at time t_1 if $S_{t_1} > b_1$ I should exercise the derivative, if $S_{t_1} < b_1$ I should not exercise it, and if $S_{t_1} = b_1$ it is irrelevant what I do. Prove that such equation admits one and only one solution. *Warning: do not try to solve the equation: it is transcendental, and so it does not have a closed-form analytic solution.* (4 marks)

(c) Write a formula, which involves the price c_1 and the pricing function c, for the value of the derivative at time t_1. (3 marks)

(d) Write down a formula for the function $f = f(x, y)$ such that $f(S_{t_1}, S_{t_2})$ is the payoff of the option at time t_2. (5 marks)

(e) Consider a derivative G on S with payoff $G_{t_2} = g(S_{t_1}, S_{t_2})$ at time t_2, for some function $g = g(x, y)$. Obtain an explicit formula for the price G_0 at time 0 of the derivative G; this formula must be of the form (5 marks)

$$\int_{\mathbb{R}^2} (g \circ h)(x, y)\ a(x, y)\ dxdy,$$

where h, a are functions which you have to determine; explain your reasoning.

Solution:

(a) Let us prove that $c(0+, \tau, K) = 0, c(\infty, \tau, K) = \infty$. Taking limits in the explicit call option price formula

$$c(x, \tau, K) = xN(d_+(\tau, x)) - Ke^{-rt}N(d_-(\tau, x)), \tag{8}$$

using the fact that $\mathcal{N}(-\infty) = 0$, $\mathcal{N}(\infty) = 1$ and

$$d_\pm(\tau, x) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K} + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right]$$

has limits $d_\pm(\tau, 0+) = -\infty, d_\pm(\tau, \infty) = \infty$.

Let us now prove that c is strictly increasing in x.

1st method Write $S_T = xZ_T$ with $Z_t := v_{\sigma,t}(W_t)$ for

$$v_{\sigma,t}(x) := \exp(\sigma x - \frac{1}{2}\sigma^2 t) \tag{9}$$

and W being a Brownian Motion. Since $Z_t > 0$, if $x > y > 0$ then $xZ_T > yZ_T$, so $(xZ_\tau - K)^+ \geq (yZ_\tau - K)^+$, with strict inequality iff $xZ_\tau - K > 0$. Since $xZ_\tau - K > 0$ has strictly positive probability (under $\mathbb{P}$ and thus under $\mathbb{Q} \sim \mathbb{P}$) we conclude that

$$c(x, \tau, K) = \mathbb{E}^{\mathbb{Q}}[(xZ_\tau - K)^+] > \mathbb{E}^{\mathbb{Q}}[(yZ_\tau - K)^+] = c(y, \tau, K).$$

2nd method The student might remember the formula

$$\frac{\partial}{\partial x} c(x, \tau, K) = N\left(d_+(\tau, x)\right)$$

for the delta of the call option (here N is the CDF of a standard gaussian), which shows that $\frac{\partial}{\partial x} c(x, \tau, K) > 0$ and thus c is strictly increasing in x.

(b) The equation is

$$c(b_1, t_2 - t_1, K_2) = c_1.$$

Notice that c is continuous, as it follows from the explicit formula (8) for c which we derived in class. Since moreover c is strictly increasing in x, $c(0+, \tau, K) = 0, c(\infty, \tau, K) = \infty$, the equation $c(b_1, t_2 - t_1, K_2) = c_1$ has one and only one solution for any $c_1 > 0$.

(c)

$$\left(c\left(S_{t_1}, t_2 - t_1, K_2\right) - c_1\right)^+ \tag{10}$$

(d)

$$\Big((S_{t_2} - K_2)^+ - c_1 e^{r(t_2 - t_1)}\Big) 1_{\{S_{t_1} > b_1\}}$$

(e) By the RNPF

$$G_0 = \mathbb{E}^{\mathbb{Q}}[e^{-rt_2} G_{t_2}] = \mathbb{E}^{\mathbb{Q}}[e^{-rt_2} g(S_{t_1}, S_{t_2})] = \mathbb{E}^{\mathbb{Q}}[e^{-rt_2} g(S_{t_1}, S_{t_1} \frac{S_{t_2}}{S_{t_1}})]. \tag{11}$$

and so, since

$$S_{t_1} = S_0 v_{\sigma, t_1}(W_{t_1}), \qquad \frac{S_{t_2}}{S_{t_1}} = v_{\sigma, t_2 - t_1}(W_{t_2} - W_{t_1}),$$

where $v_{\sigma,t}(x)$ was defined in (9), and $W_{t_1}, W_{t_2} - W_{t_1}$ are independent and have law $\mathcal{N}(0, t_1), \mathcal{N}(0, t_2 - t_1)$ respectively, we get

$$G_0 = \int_{\mathbb{R}^2} e^{-rt_2} \rho_{t_1}(x) \rho_{t_2 - t_1}(y) g\Big(S_0 v_{\sigma, t_1}(x),\ S_0 v_{\sigma, t_1}(x) v_{\sigma, t_2 - t_1}(y)\Big) dx dy,$$

where $\rho_t(x) := \frac{1}{\sqrt{2\pi t}} \exp(-\frac{1}{2}(\frac{x}{t})^2)$ is the density of $\mathcal{N}(0, t)$.

Question 5

(Total: 20 marks)

SIMILARLY SEEN IN PROBLEMS

On a finite sample space $\Omega = \{\omega_i\}_{i=1,\ldots,n}$ endowed with some probability $\mathbb{P}$ s.t. $\mathbb{P}(\omega_i) > 0$ for all i, consider a one-period arbitrage-free market model where the bank account has interest rate $r = 0$, and so we model it with the process $B_0 = B_1 = 1$, and there are two stocks S^1, S^2. If we express the portfolio using the number $h^1, h^2 \in \mathbb{R}$ of shares of stocks S^1 and S^2, and the amount of cash $c \in \mathbb{R}$ in the bank account (not of the initial capital x), then at time $t = 0, 1$ the wealth $V_t^{c,h}$ relative to (c, h) is given by

$$V_0^{c,h} = c + h \cdot S_0, \quad V_1^{c,h} = c(1 + r) + h \cdot S_1, \tag{12}$$

where $h \cdot S_t$ denoted the usual dot product between h and S_t. For the *non-replicable* derivative with payoff X_1, we consider the problem of finding the smallest initial capital p of a portfolio (c, h) super-replicating X_1 $\mathbb{P}$ a.s., i.e.

$$p := \min\{V_0^{c,h} : (c, h) \in \mathbb{R} \times \mathbb{R}^2 \text{ satisfies } V_1^{c,h}(\omega_i) \geq X_1(\omega_i) \text{ for all } i\}, \tag{13}$$

and its dual linear program, i.e.

$$d := \max\{\mathbb{E}^{\mathbb{Q}}(X_1) : \mathbb{Q} \in \mathcal{M}\}, \text{ where } \mathbb{M} := \{\mathbb{Q} \text{ probability on } \Omega : \mathbb{E}^{\mathbb{Q}}(S_1^j - S_0^j) = 0,\ j = 1, 2\} \tag{14}$$

is the set of martingale measures.

We denote by c^*, h^* any trading strategy such that $V_0^{c^*,h^*} = p$ and $V_1^{c^*,h^*} \geq X_1$ $\mathbb{P}$ a.s., and with $\mathbb{Q}^*$ any element of $\mathbb{M}$ such that $d = \mathbb{E}^{\mathbb{Q}^*}(X_1)$, i.e. any optimisers of (13) and (14) (such optimisers always exist). Answer the following questions and justify carefully with either proofs or counterexamples.

(a) Is p an arbitrage-free price for X_1? If it is not, find an arbitrage. (3 marks)

(b) Is it true that $V_1^{c^*,h^*} = X_1$ $\mathbb{Q}^*$ a.s.? (3 marks)

(c) Is $\mathbb{Q}^*$ equivalent to $\mathbb{P}$? (2 marks)

(d) *Assume from now on that it is not possible to borrow any money from the bank.* Define what an arbitrage is in such a market, *using formulas.* (2 marks)

(e) Consider again the problem of finding the smallest initial capital $\bar{p}$ of the portfolio (c, h) super-replicating X_1 (but now without being able to borrow). Formulate this problem in a way analogous to (13), and formulate its dual problem in a way analogous to (14), and call the respective optimal values $\bar{p}$ and $\bar{d}$. (4 marks)

(f) Are $\bar{p}$ and $\bar{d}$ always equal? (3 marks)

(g) Is it true that $\bar{p} \geq p$, for any choice of X_1 and any choice of model (B, S^1, S^2)? What about $\bar{p} > p$? (3 marks)

Solution:

(a) We stated in class that, since X_1 is not replicable, the set of arbitrage-free prices is an open interval whose supremum is p, so p is not an arbitrage-free price. To actually prove this, let us build an arbitrage. Consider h^* s.t. $V_0^{c^*,h^*} = p$ and $V_1^{c^*,h^*} \geq X_1 \quad \mathbb{P}$ a.s. (such h^* exists, as we stated in class). Then do this: starting with zero initial capital, short-sell one derivative at price p, buy h^* underlying S, and deposit the remaining money $c^* := p - h^* \cdot S_0$ in the bank. This leads to the final payoff $V_1^{c^*,h^*} - X_1$, which by definition is positive, and is not a.s. zero (otherwise X_1 would be replicable); since we started with zero initial capital, this is an arbitrage.

(b) Since $V_1^{c^*,h^*} \geq X_1$ holds $\mathbb{P}$ a.s., it also holds $\mathbb{Q}^*$ a.s. (because $\mathbb{Q}^* \ll \mathbb{P}$). Since $\mathbb{Q}^*$ is a martingale measure it satisfies $V_0^{c^*,h^*} = \mathbb{E}^{\mathbb{Q}^*}(V^{c^*,h^*})$. Moreover, by assumption $p = V_0^{c^*,h^*}$ and $d = \mathbb{E}^{\mathbb{Q}^*}(X_1)$, and by the FTAP (/the strong duality theorem) $p = d$ and putting the pieces together we have that $\mathbb{E}^{\mathbb{Q}^*}(V^{c^*,h^*}) = \mathbb{E}^{\mathbb{Q}^*}(X_1)$. Since $V_1^{c^*,h^*} \geq X_1$ $\mathbb{Q}^*$ a.s., this shows that

$$V_1^{c^*,h^*} = X_1 \ \mathbb{Q}^* \text{ a.s.}. \tag{15}$$

(c) Since $V_1^{c^*,h^*} = X_1$ holds $\mathbb{Q}^*$ a.s. (by the previous item) but not $\mathbb{P}^*$ a.s. (because by assumption X is not replicable), $\mathbb{Q}$ is not equivalent to $\mathbb{P}$: as we had already discussed in class, the minimiser of the dual problem for a non-replicable derivative is a martingale measure which is not equivalent to $\mathbb{P}$.

(d) An arbitrage is a (c,h) s.t. $c \geq 0, h \in \mathbb{R}^2$, $V_0^{c,h} = 0$, $V_1^{c,h}(\omega_i) \geq 0$ for all i, and $V_1^{c,h}(\omega_i) \neq 0$ for at least one i.

(e) The first problem could be written either as

$$\bar{p} := \min\{V_0^{c,h} : (c,h) \in \mathbb{R} \times \mathbb{R}^2 \text{ satisfies } V_1^{c,h}(\omega_i) \geq X_1(\omega_i) \forall i, c \geq 0\}, \tag{16}$$

or as

$$\bar{p} := \min\{V_0^{c,h} : (c,h) \in \mathbb{R}_+ \times \mathbb{R}^2 \text{ satisfies } V_1^{c,h}(\omega_i) \geq X_1(\omega_i) \forall i\}, \tag{17}$$

We choose to consider the second form, since it leads to a simpler-looking dual problem (with one fewer variable than the other). The dual of (17) is then

$$\bar{d} := \max\{\mathbb{E}^{\mathbb{Q}}(X_1) : \mathbb{Q} \in \bar{\mathbb{M}}\}, \tag{18}$$

where

$$\bar{\mathbb{M}} := \{\mathbb{Q} \text{ sub-probability on } \Omega : \quad \mathbb{E}^{\mathbb{Q}}(S_1^j - S_0^j) = 0, j = 1,2\},$$

where we say that a positive measure $\mathbb{Q}$ is a sub-probability if $\mathbb{Q}(\Omega) \leq 1$.

(f) Since $\bar{\mathbb{M}} \supseteq \mathbb{M} \neq \emptyset$, $\bar{\mathbb{M}}$ is not empty, so the dual problem is feasible. To apply the strong duality theorem and conclude that the two problems have solution and $\bar{p} = \bar{d}$, one can either observe that the primal is feasible (for $h = 0$ and c big enough we have

$V_1^{c,h} = c \geq X_1(\omega_i)$ for all i), or that the dual is feasible (since $\bar{\mathbb{M}} \supseteq \mathbb{M}$, and $\mathbb{M} \neq \emptyset$ by item (i)). Thus the dual problem (18) is solvable, since it is the problem of maximising a continuous (in fact, linear) function on the set $\bar{\mathbb{M}}$, which is closed and bounded (indeed $\bar{\mathbb{M}}$ is a subset of the set of $q \in \mathbb{R}^4_+$ s.t. $\sum_{i=1}^4 q_i \leq 1$, which corresponds to the set of sub-probabilities on Ω, which is trivially bounded), and thus compact.

(g) $\bar{p} \geq p$ always holds: if we cannot borrow money, we have fewer available portfolios, thus the smallest initial value of all available super-replicating portfolios is bigger. $\bar{p} > p$ does not always hold: in fact $\bar{p} = p$ will hold whenever an optimiser (c^*, h^*) for the problem defining p satisfies $c^* \geq 0$, in which case it is also an optimiser for the problem defining $\bar{p}$.

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH96014	1	Almost of all the students had a good understanding of the broad techniques they needed to apply, but sometimes lacking in detail. In particular, question a) was often answered much too briefly, and when finding arbitrage free prices there was some confusion about the correct form of replicating portfolios and proper discounting. In general, solving for EMMs was done well, althoug one should be careful to show ones computations. Question f was the question that caused the most trouble and many got very little or no points from it,although a good portion solved it perfectly. For question f, there was some confusion about completeness, and in general one should be careful to justify why the payoffs are linearly independent so that all claims can be replicated.
MATH96014	2	In Q2 a surprisingly common mistake was to assume that all formulas would work just as when working with markets without forex.
MATH96014	3	The results in Q3 were somewhat higher than I anticipated. Most students correctly solved all items but for the last one
MATH96014	4	Q4 intentionally covered little seen material and was designed to be hard...it but proved too be too hard, as many students only correctly solved item (a); Q4 could have been made easier by assigning more closely-related exercises during term.
MATH96014	5	In Q5, surprisingly few students were able to correctly formulate the dual LPs in item (e), and many got tricked by the delicate question in item (c).