L8 Class rep? ✓.

①

2. First-order logic

~~Propositional~~ (Predicate logic).

Plans

Semantics

1) Introduce 1^{st}-order structures

2) 1^{st}-order languages

Syntax

3) Describe a formal system

4) Show that the theorems of the formal system are logically valid formulas i.e. ones true in all structures.

4: Gödel's completeness thm.

2.1 Structures.

(2.1.1) Def: Suppose A is a set and $n \geq 1$. An n-ary relation (on A) is a subset

$$\bar{R} \subseteq A^n$$

(where $A^n = \{(a_1, \ldots, a_n) : a_i \in A\}$

n-tuple

An n-ary function on A is a function $\bar{f} : A^n \to A$

Examples.

a) ordering $\leq$ on $\mathbb{R}$:

2-ary relation on $\mathbb{R}$

b) $+$ on $\mathbb{C}$: 2-ary function on $\mathbb{C}$

c) $\bar{P} \subseteq \mathbb{Z}$ $\quad \bar{P} = \{x \in \mathbb{Z} : x \text{ is even}\}$

1-ary relation on $\mathbb{Z}$

(sometimes see 'predicates' rather than 'relations').

Notation: $\bar{R} \subseteq A^n$ is an n-ary rel. on A and $(a_1, \ldots, a_n) \in A^n$ write $\bar{R}(a_1, \ldots, a_n)$ to mean $(a_1, \ldots, a_n) \in \bar{R}$.

(2.1.2) Def. A first-order structure $\mathcal{A}$ consists of:

1) A non-empty set A (the domain of $\mathcal{A}$)

2) A set $\{\bar{R}_i : i \in I\}$ of relations on A, $\bar{R}_i \subseteq A^{n_i}$.

3) A set $\{\bar{f}_j : j \in J\}$ of functions on A, $\bar{f}_j : A^{m_j} \to A$

4) A set $\{\bar{c}_k : k \in K\}$ of constants: just elements of A.

The sets I, J, K indexing sets (can be empty). Usually, subsets of $\mathbb{N}$.

the information:

$$\left.\begin{array}{l} (n_i : i \in I) \\ (m_j : j \in J) \\ \text{the set } K \end{array}\right\} \text{ is called the \underline{signature} of } \mathcal{A}.$$

Might denote the structure by:

$$\mathcal{A} = \langle \underset{\text{domain}}{A} ; \underset{\text{relations}}{(\bar{R}_i : i \in I)}, \underset{\text{fns.}}{(\bar{f}_j : j \in J)}, \underset{\text{constants}}{(\bar{c}_k : k \in K)} \rangle$$

(2.1.3) **Examples**.

① **Orderings**

$A = \mathbb{N}, \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$

and $I = \{1\}$, $n_1 = 2$, $J, K = \emptyset$

$\bar{R}_1(a_1, a_2)$ to mean $a_1 < a_2$.

② **Groups** Could use the signature:

$\bar{R}$ 2-ary relation of equality,

$\bar{m}$ 2-ary function (for multiplication)

$\bar{i}$ 1-ary function (for inversion)

$\bar{e}$ constant (for the identity element).

(3) **Rings** Signature:

$\bar{R}$ 2-ary relation for equality

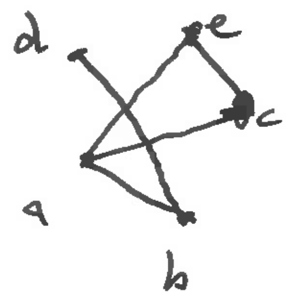
$\bar{m}$ 2-ary function for multiplication

$\bar{a}$ 2-ary function for addition

$\bar{n}$ 1-ary function $x \longmapsto -x$

$\bar{0}, \bar{1}$ constants for zero and one.

(4) Graphs.



$\bar{R}$ 2-ary relation for equality

$\bar{E}$ 2-ary relation for 'adjacency'

eg. $\bar{E}(a, b)$

(4)

(2.2) First-order languages.

(2.2.1) Def. A first order language $\mathcal{L}$ has an alphabet of symbols:

variables: $x_0, x_1, x_2, \dots$

punctuation: ( ) ,

connectives: $\neg$ $\rightarrow$

quantifier: $\forall$

relation symbols: R_i $(i \in I)$

function symbols: f_j $(j \in J)$

constant symbols: c_k $(k \in K)$

Here I, J, K are indexing sets (could have J or K being $\emptyset$).

Each R_i comes with an <u>arity</u> n_i

Each f_j comes with an arity m_j.

The information $(n_i : i \in I), (m_j : j \in J), K$ is called the <u>signature</u> of $\mathcal{L}$.

A first-order str. $\mathcal{A}$ with the same signature as $\mathcal{L}$ is called an <u>$\mathcal{L}$-structure</u>.

(2.2.2) <u>Def</u>. A <u>term</u> of $\mathcal{L}$ is defined as follows

i) any variable is a term;

ii) any constant symbol is a term;

iii) If f is an m-ary fn. symbol and $t_1, \dots, t_m$ are terms then $f(t_1, \dots, t_m)$ is a term.

iv) any term arises in this way.

<u>Example</u>: $\mathcal{L}$ has a 2-ary fn. symbol f and constant symbols c_1, c_2.

<u>Some terms</u>: c_1, c_2, x_1

$f(c_2, x_1)$; $f(f(c_2, x_1), x_2)$

<u>Not terms</u>: ffx_1x_2 $\quad g(x_1)$ not in $\mathcal{L}$

Typo in c/w que. 2 : will repair.

(2.2.3) Def. (Formulas).

Define $\mathcal{L}$-formulas inductively :

1 An atomic formula of $\mathcal{L}$ is of the form

$$R(t_1, \dots, t_m) \quad \text{with}$$

R an m-ary relation symbol and $t_1, \dots, t_m$ terms of $\mathcal{L}$.

2 i) Any atomic formula is an $\mathcal{L}$-formula

ii) if ϕ, ψ are $\mathcal{L}$-formulas then

$$(\neg \phi)$$
$$(\phi \to \psi)$$
$$(\forall x)\, \phi$$

are $\mathcal{L}$-formulas (where x is any variable)

Example Suppose $\mathcal{L}$ has

2-ary fn. symbol f
1-ary fn. symbol g
2-ary rel. symbol R
constant symbol c_1

Some terms

$$x_1, \; x_2, \; c_1, \; f(g(x_2), c_1)$$

Atomic formulas

$$R(x_1, f(g(x_2), c_1))$$
$$R(g(x_3), x_4)$$

$$(\forall x_4)\, R(g(x_3), x_4) \quad \dots$$

<u>Ex</u>: Take the signature for groups in 2.1.3 (2) write down some terms + atomic formulas.
Can you write the group axioms?

Eg. $(\forall x_1) R\,(m(i(x_1), x_1), e)$

(2.2.4) <u>Def</u>. (Short-hand)
Suppose ϕ, ψ are $\mathcal{L}$-fmlas

$(\exists x)\,\phi$ means
$$(\neg(\forall x)(\neg\phi))$$

$(\phi \vee \psi)$ means
$$((\neg\phi) \to \psi).$$

(2.2.5) <u>Def</u>. Suppose

$\mathcal{L}$ is as in 2.2.1 and

$$\mathcal{A} = \langle A; (\bar{R}_i : i \in I), (\bar{f}_j : j \in J), (\bar{c}_k : k \in K)\rangle$$

is an $\mathcal{L}$-structure.

The correspondence between the relation, function + constant <u>symbols</u> in $\mathcal{L}$ and the actual relations, functions and contants of $\mathcal{A}$ (with matching airties) is called an <u>interpretation</u> of $\mathcal{L}$.
(or say $\mathcal{A}$ is an <u>interpretation</u> of $\mathcal{L}$).

(2.2.6) <u>Def</u>. Suppose $\mathcal{A}$ is an $\mathcal{L}$-structure. A <u>valuation</u> in $\mathcal{A}$ is a function v from the set of terms of $\mathcal{L}$ to A (the domain of $\mathcal{A}$) satisfying:

a) $v(c_k) = \bar{c}_k$

b) if $t_1, \dots, t_m$ are terms and f is an m-ary fn. symbol (of $\mathcal{L}$) then

$$v(f(t_1, \dots, t_m)) = \bar{f}(v(t_1), \dots, v(t_m))$$

(2.2.7) <u>Lemma</u>: Suppose $\mathcal{A}$ is an $\mathcal{L}$-str. and $a_0, a_1, \dots \in A$. Then there is a unique valuation v (in $\mathcal{A}$) with

~~all~~ $v(x_\ell) = a_\ell$ (for all $\ell \in \mathbb{N}$).

[the variables are $x_0, x_1, \dots$].

<u>Pf</u>: (Sketch) By induction on the length of terms: show that if we let

i) $v(x_\ell) = a_\ell$ (for $\ell \in \mathbb{N}$)

ii) $v(c_k) = \bar{c}_k$ ($k \in K$)

iii) $v(f(t_1, \dots, t_m)) = \bar{f}(v(t_1), \dots, v(t_m))$

then this is a well-defined valuation. ∎

<u>Example</u> Groups

Signature $\left.\begin{array}{l} R \\ m \\ i \\ e \end{array}\right\}$ as in 2.1.3.

Let $\mathcal{G}$ be a group

$= \langle G ; \bar{R} ; \bar{m}, \bar{i}, \bar{e} \rangle$

Let $g, h \in G$.

Let v be a valuation with

$v(x_0) = g$, $v(x_1) = h$, ...

$$v\big(m(m(x_0, x_1), i(x_0))\big)$$

$$= ghg^{-1}.$$

$$v\big(m(x_0, x_1)\big)$$

$$= \bar{m}\big(v(x_0), v(x_1)\big)$$

$$= gh \quad \text{etc.}$$

(2.2.9) Def.

Suppose $\mathcal{A}$ is an $\mathcal{L}$-str. and v is a valuation in $\mathcal{A}$

define, inductively, for an $\mathcal{L}$-formula ϕ what's meant by

v satisfies ϕ (in $\mathcal{A}$)

(abbreviated as $v[\phi] = T$)

(negation: $v[\phi] = F$ for v does not satisfy ϕ (in $\mathcal{A}$)).

(i) ~~Sta~~ Atomic formulas:

Suppose R is an n-ary relation symbol and $t_1, \ldots, t_n$ are terms (of $\mathcal{L}$). Then v satisfies the atomic formula $R(t_1, \ldots, t_n)$

iff $\overline{R}(v(t_1), \ldots, v(t_n))$ holds in $\mathcal{A}$.

(ii) Suppose ϕ, ψ are $\mathcal{L}$-formulas (& we already know about valuations satisfying ϕ, ψ).

(a) Say $v[(\neg\phi)] = T$

iff $v[\phi] = F$

(i.e. v satisfies $(\neg\phi)$ iff v dns does not satisfy ϕ (in $\mathcal{A}$)).

(b) Say $v[(\phi \to \psi)] = F$

iff $v[\phi] = T$ & $v[\psi] = F$.

(c)