

Mathematical Logic (MATH6/70132;P65)
Problem Sheet 4

[1] In each of the following cases a first-order language \mathcal{L}_i and two \mathcal{L}_i -structures $\mathcal{A}_i, \mathcal{B}_i$ are given. In each case, write down a sentence of \mathcal{L}_i which is true in \mathcal{A}_i but not in \mathcal{B}_i . Explain your answers briefly (your argument need not involve valuations).

(a) \mathcal{L}_1 has a single binary relation symbol R . The domain of \mathcal{A}_1 is \mathbb{N} and $R(x_1, x_2)$ is interpreted as $x_1 \leq x_2$. The domain of \mathcal{B}_1 is \mathbb{Z} and $R(x_1, x_2)$ is interpreted as $x_1 \leq x_2$.

(b) \mathcal{L}_2 has a single binary relation symbol R . The domain of \mathcal{A}_2 is \mathbb{Z} and $R(x_1, x_2)$ is interpreted as $x_1 < x_2$. The domain of \mathcal{B}_2 is \mathbb{Q} (the set of rational numbers) and $R(x_1, x_2)$ is interpreted as $x_1 < x_2$.

(c) \mathcal{L}_3 has a single unary function symbol f and a single binary relation symbol E . The domain of \mathcal{A}_3 is \mathbb{N} and f is interpreted as the function $x_1 \mapsto x_1 + 1$. The domain of \mathcal{B}_3 is \mathbb{Z} and f is interpreted as the function $x_1 \mapsto x_1 + 2$. In both structures E is interpreted as equality.

(d) \mathcal{L}_4 has a single binary relation symbol R . The domain of \mathcal{A}_4 is \mathbb{N} and $R(x_1, x_2)$ is interpreted as ' x_1, x_2 are congruent modulo 3'. The domain of \mathcal{B}_4 is \mathbb{N} and $R(x_1, x_2)$ is interpreted as ' x_1, x_2 are congruent modulo 5'.

[2] (a) Show (by giving an argument involving valuations) that for any formula ϕ the following formula is logically valid:

$$((\exists x_1)(\forall x_2)\phi \rightarrow (\forall x_2)(\exists x_1)\phi).$$

(b) Give an example of a formula ϕ and an interpretation where the following is false:

$$((\forall x_1)(\exists x_2)\phi \rightarrow (\exists x_2)(\forall x_1)\phi).$$

[3] Let \mathcal{L} be a first-order language with a binary relation symbol R . A *strict partial order* is an \mathcal{L} -structure which is a model of the closed formula ϕ :

$$(\forall x_1)(\forall x_2)(\forall x_3)((\neg R(x_1, x_1)) \wedge ((R(x_1, x_2) \wedge R(x_2, x_3)) \rightarrow R(x_1, x_3))).$$

(So in a model of this formula, the interpretation of R behaves like $<$.)

(a) Show that in any model of ϕ the formula ψ given by:

$$(\forall x_1)(\forall x_2)(R(x_1, x_2) \rightarrow (\neg R(x_2, x_1)))$$

is true.

(b) Write down an \mathcal{L} -formula χ which has a model and is such that any \mathcal{L} -structure which is a model of χ is infinite.

[4] Suppose \mathcal{L} is a first-order language and $\phi(x_1)$ is an \mathcal{L} -formula with a free variable x_1 and possibly other free variables. Under what circumstances is the formula

$$((\forall x_1)\phi(x_1) \rightarrow (\forall x_2)\phi(x_2))$$

logically valid? Justify your answer.

David Evans, October 2022.