

Mathematical Logic (MATH6/70132;P65)
Problem Sheet 7

[1] Find subsets of \mathbb{Q} which (with their induced orderings from \mathbb{Q}) are similar to:

- (i) $\mathbb{N} + \mathbb{N} + \mathbb{N}$;
- (ii) $\mathbb{N} \times \mathbb{Z}$;
- (iii) $\mathbb{N} + \mathbb{N}^*$ (where \mathbb{N}^* is the reverse ordering on \mathbb{N}).

You do not need to write down the similarities involved here.

[2] We say that a set is *finite* if and only if it is equinumerous with some natural number $n \in \omega$. Otherwise it is *infinite*.

- (i) Suppose β is an infinite ordinal. Using results from 3.4 of the notes, prove that $\omega \leq \beta$. Deduce that $|\beta^\dagger| = |\beta|$.
- (ii) Prove that if $m, n \in \omega$ are equinumerous then $m = n$.
- (iii) Suppose X is a non-empty finite set of ordinals. Prove that X has a largest element.
- (iv) Suppose α is a finite ordinal. Prove that $\alpha \in \omega$.
- (v) Suppose $x \subseteq n \in \omega$. Then x is finite.

[Hint: You can use results on ordinals in Section 3.4. For (ii), it suffices to prove by induction on n that if $x \subseteq n$ and x is equinumerous with n , then $x = n$.]

[3] Suppose X is a non-empty set of ordinals. From the notes, you know that $\bigcup X$ and $\bigcap X$ are ordinals and $\bigcap X \leq \alpha \leq \bigcup X$ for all $\alpha \in X$.

- (i) Show that if β is an ordinal with $\alpha \leq \beta$ for all $\alpha \in X$, then $\bigcup X \leq \beta$.
- (ii) Formulate and prove a similar statement about $\bigcap X$.

[4] Suppose α and β are ordinals with α similar to $\omega + \omega$ and β similar to $\omega \times \omega$ (with the orderings as defined in 3.3.3). Which of $\alpha < \beta$, $\alpha = \beta$ or $\beta < \alpha$ holds?

[5] Let β be the set of all countable ordinals.

- (i) Show that β is an ordinal.
- (ii) Show that β is uncountable.
- (iii) Show that if γ is an uncountable ordinal then $\beta \leq \gamma$.

[6] (i) Suppose α is an ordinal and $X \subset \alpha$ is a proper initial segment of α . Prove that there is $\beta \in \alpha$ with $X = \beta$.

(ii) Suppose that $\gamma \neq \delta$ are ordinals. Prove that γ and δ are not similar.

[7] A *cardinal* is an ordinal α with the property that for all ordinals $\beta < \alpha$ we have that α and β are not equinumerous.

- (i) Prove that every natural number is a cardinal and ω is a cardinal.
- (ii) Prove that the ordinal β in question 5 is a cardinal.
- (iii) Show that if γ is any ordinal, there is a unique cardinal α which is equinumerous with γ .