

Examples I for Time Series

“Stationary” is meant to mean second order stationary unless explicitly stated otherwise.

1. Determine whether each of the following models for a random process $\{X_t\}$ is stationary, justifying your answer. For those that are stationary, specify their mean and autocovariance sequence. Here, $\{\epsilon_t\}$ is a white noise process with mean 0 and variance σ_ϵ^2 .
 - (a) $2X_t - 2\epsilon_t = 4\epsilon_{t-3}$.
 - (b) $X_t = \epsilon_{t-2}\epsilon_{t-1}\epsilon_t$, where ϵ_t is Gaussian.
 - (c) $X_t = (-1)^t(\epsilon_t + 1)$.
 - (d) $X_t = \cos(t + Y)$ where Y is a continuous random variable uniformly distributed on $[0, 2\pi)$.
2. (a) Let Y_1 and Y_2 be independent and identically distributed random variables, each with mean zero and variance σ^2 , and let c be a constant.
 - i. Find the mean and covariance of $\{X_t\}$ defined by

$$X_t = Y_1 \cos(ct) + Y_2 \sin(ct),$$

and hence show that the process is stationary.

- ii. For the case $c = \pi/4$ show that $\{X_t\}$ is *strictly* stationary if and only if Y_1 and Y_2 each have the Gaussian (normal) distribution. [You will need to use Bernstein's theorem which states that if U and V are IID random variables, and $(U + V)/\sqrt{2}$ has the same distribution as U and V , then U and V are Gaussian (normal). Also recall that $\cos(\pi/4) = 1/\sqrt{2}$.]
- (b) Let Y_1 be a random variable with mean zero and variance σ^2 , and let c be a constant.
 - i. Find the mean and covariance of $\{X_t\}$ defined by

$$X_t = Y_1 \cos(ct),$$

and hence determine when the process is stationary.

- ii. Show that when the process in (b)(i) is stationary that its auto-correlation sequence must be of the form

$$\rho_\tau = (-1)^{|\ell\tau|}, \quad \tau \in \mathbb{Z},$$

for some $\ell \in \mathbb{Z}$.

- iii. The sequence $\{\rho_\tau\}$ is positive semidefinite, if, for all $n \geq 1$, for any t_1, t_2, \dots, t_n contained in \mathbb{Z} , and for any set of nonzero real numbers a_1, a_2, \dots, a_n

$$\sum_{j=1}^n \sum_{k=1}^n \rho_{t_j - t_k} a_j a_k \geq 0.$$

Show that this inequality condition is satisfied by $\rho_\tau = (-1)^{|\ell\tau|}$ for $\ell, \tau \in \mathbb{Z}$.

3. Let $\{X_t\}$ be defined by

$$X_t = \phi X_{t-1} + \epsilon_t, \quad t = 1, 2, 3, \dots$$

where $X_0 = 0$. Find the 3×3 covariance matrix of X_1 , X_2 and X_3 . Here $\{\epsilon_t\}$ is a white noise process with mean zero and variance σ_ϵ^2 .