Mathematical Logic (M345P65)
Problem Sheet 3

[1] The first-order language $\mathcal{L}$ has one unary function symbol f and one unary relation symbol P. Let ϕ be the formula $(\forall x_1)(P(x_1) \to P(f(x_1)))$. Give an interpretation of $\mathcal{L}$ in which ϕ is true, and one in which it is false.

[2] In each of the following cases a first-order language $\mathcal{L}_i$ and two $\mathcal{L}_i$-structures $\mathcal{A}_i$, $\mathcal{B}_i$ are given. In each case, write down a sentence of $\mathcal{L}_i$ which is true in $\mathcal{A}_i$ but not in $\mathcal{B}_i$. Explain your answers briefly (your argument need not involve valuations).
(a) $\mathcal{L}_1$ has a single binary relation symbol R. The domain of $\mathcal{A}_1$ is $\mathbb{N}$ and $R(x_1, x_2)$ is interpreted as $x_1 \leq x_2$. The domain of $\mathcal{B}_1$ is $\mathbb{Z}$ and $R(x_1, x_2)$ is interpreted as $x_1 \leq x_2$.
(b) $\mathcal{L}_2$ has a single binary relation symbol R. The domain of $\mathcal{A}_2$ is $\mathbb{Z}$ and $R(x_1, x_2)$ is interpreted as $x_1 < x_2$. The domain of $\mathcal{B}_2$ is $\mathbb{Q}$ (the set of rational numbers) and $R(x_1, x_2)$ is interpreted as $x_1 < x_2$.
(c) $\mathcal{L}_3$ has a single unary function symbol f and a single binary relation symbol E. The domain of $\mathcal{A}_3$ is $\mathbb{N}$ and f is interpreted as the function $x_1 \mapsto x_1 + 1$. The domain of $\mathcal{B}_3$ is $\mathbb{Z}$ and f is interpreted as the function $x_1 \mapsto x_1 + 2$. In both structures E is interpreted as equality.
(d) $\mathcal{L}_4$ has a single binary relation symbol R. The domain of $\mathcal{A}_4$ is $\mathbb{N}$ and $R(x_1, x_2)$ is interpreted as 'x_1, x_2 are congruent modulo 3'. The domain of $\mathcal{B}_4$ is $\mathbb{N}$ and $R(x_1, x_2)$ is interpreted as 'x_1, x_2 are congruent modulo 5'.

[3] The language $\mathcal{L}$ has a binary relation symbol E, a binary function symbol m, a unary function symbol i and a constant symbol e. Let G be a group and consider G as an $\mathcal{L}$-structure by interpreting E as equality, m as multiplication, i as inversion, and e as the identity element of G. Let v be a valuation (of $\mathcal{L}$) in G and let

$$H = \{v(t) : t \text{ is a term of } \mathcal{L}\}.$$

(a) Show that H is a subgroup of G.

(b) Show that H is generated by $\{v(x_i) : x_i \text{ is a variable of } \mathcal{L}\}$.

(c) What is H if we omit the function symbol i from the language?

[4] Let ϕ be a formula in a first-order language $\mathcal{L}$ and let v be a valuation (in some $\mathcal{L}$-structure $\mathcal{A}$). Suppose there is a valuation v' which is x_i-equivalent to v and satisfies ϕ. Show that v satisfies $(\exists x_i)\phi$.

[5] Suppose F is a field. The language $\mathcal{L}_F$ appropriate for considering F-vector spaces V has a 2-ary relation symbol R (for equality); a 2-ary function symbol a (for addition in the vector space); a constant symbol 0 (for the zero vector) and, for every $\alpha \in F$, a 1-ary function symbol f_α (for scalar multiplication by α).

Convince yourself that it is possible to express the axioms for being an F-vector space as a set of formulas in this language.

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