Imperial College London

MATH97084 MATH97185

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Time Series

Date: Friday, 21 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS SEPARATE PDFs TO THE RELEVANT DROPBOXES ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

Note: Throughout this paper $\{\epsilon_t\}$ is a sequence of uncorrelated random variables (white noise) having zero mean and variance σ_ϵ^2, unless stated otherwise. The term "stationary" will always be taken to mean second-order stationary. All processes are real-valued unless stated otherwise. The sample interval is unity unless stated otherwise. $\Delta = 1 - B$ denotes the difference operator, where B denotes the backward shift operator.

Throughout this paper you may state without proof any results you take from the lecture notes.

1. (a) Let $\{X_t\}$ be a stationary process of mean $\mu_X \neq 0$, variance $\sigma_X^2 > 0$ and autocovariance sequence $\{s_{X,\tau}\}$. Determine whether each of the following models for a random process $\{Y_t\}$ is stationary, justifying your answer. For those that are stationary, give the mean, variance and autocovariance sequence in terms of μ_X, σ_X^2 and $s_{X,\tau}$.

   (i) $Y_t = \alpha X_t + c$, where α and c are non-zero constants. (2 marks)

   (ii) $Y_t = (X_t - \mu_X)\cos(t)$. (2 marks)

   (iii) $Y_t = W + X_t$, where W is a Bernoulli(0.5) random variable, independent of $\{X_t\}$. (2 marks)

   (b) Consider the following process

$$X_t = \alpha X_{t-2} + \epsilon_t + \alpha\epsilon_{t-1},$$

   where $\alpha \neq 0$ is a fixed real-valued constant with $|\alpha| < 1$.

   (i) Is this an AR(2), ARMA(1,2), ARMA(2,1) or MA(1) process? (1 mark)

   (ii) Show that $\{X_t\}$ is both a stationary and invertible process. (2 marks)

   (iii) Show that $\mathrm{Var}\{X_t\} = \sigma_\epsilon^2\left(\frac{1+\alpha^2}{1-\alpha^2}\right)$. (4 marks)

   (iv) Determine ρ_1, the first autocorrelation of $\{X_t\}$. (2 marks)

   (v) The spectral density function of $\{X_t\}$ when $\sigma_\epsilon^2 = 1$ is

$$S(f) = \frac{13 + 12\cos(2\pi f)}{13 - 12\cos(4\pi f)}.$$

   Determine the value of α. (5 marks)

(Total: 20 marks)

2. (a) (i) Let α and β be non-zero constants. Show $L\{X_t\} = \alpha + \beta X_t$ is *not* a linear time invariant filter.

(3 marks)

(ii) Show $L\{X_t\} = X_t - X_{t-1}$ is a linear time invariant filter. (3 marks)

(iii) Let $L_1\{\cdot\}$ and $L_2\{\cdot\}$ be two linear time invariant filters with frequency response functions $G_1(f)$ and $G_2(f)$, respectively. Show $K\{\cdot\} \equiv L_1\{L_2\{\cdot\}\}$ is also a linear time invariant filter and show its frequency response function is given as $G_1(f)G_2(f)$. (6 marks)

(b) Let $X_t = \mu_t + Y_t$ where μ_t is a polynomial trend of order $d-1$ and $\{Y_t\}$ is a zero mean second order stationary process with spectral density function $S_Y(f)$.

(i) Show that the d-th differenced process $\{X_t^{(d)}\} \equiv \{\Delta^d X_t\}$ has spectral density function

$$S_{X^{(d)}}(f) = 4^d \sin^{2d}(\pi f) S_Y(f).$$

(5 marks)

(ii) Let $\{Y_t\}$ be a white noise process with variance σ_Y^2. Show

$$|s_{X^{(d)},\tau}| \leq 4^d \sigma_Y^2 \int_{-1/2}^{1/2} |\sin^{2d}(\pi f)| \mathrm{d}f,$$

where $\{s_{X^{(d)},\tau}\}$ denotes the autocovariance sequence of $\{X_t^{(d)}\}$. (3 marks)

(Total: 20 marks)

3. (a) (i) Let $\{X_t\}$ be the MA(1) process $X_t = \mu + \epsilon_t - \frac{1}{4}\epsilon_{t-1}$. Show its autocovariance sequence is given as

$$s_\tau = \begin{cases} 17\sigma_\epsilon^2/16 & \tau = 0, \\ -\sigma_\epsilon^2/4 & |\tau| = 1, \\ 0 & |\tau| > 1. \end{cases}$$

(3 marks)

(ii) Let $X_1, ..., X_N$ be a portion of this process and let $\bar{X} = N^{-1}\sum_{t=1}^{N} X_t$ be the sample mean and estimator for μ. Show

$$\mathrm{Var}\{\bar{X}\} = \sigma_\epsilon^2 \cdot \frac{8+9N}{16N^2}.$$

(4 marks)

(iii) Hence show that for some $C > 0$, to have $\mathrm{mse}\{\bar{X}\} < C$ we require

$$\frac{N^2}{8+9N} > \frac{\sigma_\epsilon^2}{16C}.$$

(3 marks)

(b) Let $\{-1, 3, 3, -1\}$ be a realisation of length $N = 4$ of a portion X_1, X_2, X_3, X_4 of a stationary process with unknown mean. Choosing the correct formula from your notes, compute the "biased" autocovariance estimates $\hat{s}_0^{(p)}, \hat{s}_1^{(p)}, \hat{s}_2^{(p)}$ and $\hat{s}_3^{(p)}$. (4 marks)

(c) Let $X_1, ..., X_N$ be a portion of a stationary process with *unknown* mean. We define the centred periodogram to be

$$\hat{S}^{(p)}(f) = \frac{1}{N}\left|\sum_{t=1}^{N}(X_t - \bar{X})\mathrm{e}^{-\mathrm{i}2\pi ft}\right|^2. \qquad (\dagger)$$

(i) Show that the centred periodogram (†) satisfies

$$\hat{S}^{(p)}(f) = \sum_{\tau=-(N-1)}^{N-1} \hat{s}_\tau^{(p)}\mathrm{e}^{-\mathrm{i}2\pi f\tau},$$

where $\hat{s}_\tau^{(p)}$ is the same estimator as used in part (b). (3 marks)

(ii) Show that $\hat{S}^{(p)}(f)$ as defined in (†) necessarily takes a value of 0 at $f = 0$. What does this imply about $\sum_{\tau=-(N-1)}^{N-1} \hat{s}_\tau^{(p)}$? (3 marks)

(Total: 20 marks)

4. (a) Let $\{X_t\}$ be a zero mean stationary process with autocovariance sequence $\{s_{X,\tau}\}$. Let $\{Y_t\}$ be the process

$$Y_t = X_t + aX_{t-d} + \eta_t,$$

where a is a non-zero real-valued constant and d is a fixed positive integer. Here, $\{\eta_t\}$ is a zero mean *white* noise process that is uncorrelated with $\{X_t\}$ and has variance σ_η^2.

(i) Explain why this can be considered as a signal + echo + noise model. (1 mark)

(ii) Show that $\{X_t\}$ and $\{Y_t\}$ are jointly stationary processes, deriving the cross-covariance sequence $\{s_{XY,\tau}\}$ in terms of $\{s_{X,\tau}\}$. (3 marks)

(b) (i) Consider the model for $\{X_t\}$ and $\{Y_t\}$ in Question 4(a). Now let $\{X_t\}$ be the MA(1) process

$$X_t = \epsilon_t + \frac{1}{2}\epsilon_{t-1},$$

where $\sigma_\epsilon^2 = 2$. Compute and sketch a plot of the cross covariance sequence $\{s_{XY,\tau}\}$ for $a = 1$ and $d = 2$. (6 marks)

(ii) Find the group delay between $\{X_t\}$ and $\{Y_t\}$. Comment on your result. (4 marks)

(c) Let $X_1, ..., X_N$ and $Y_1, ..., Y_N$ be portions of zero mean stationary processes $\{X_t\}$ and $\{Y_t\}$, respectively. The cross-periodogram is defined as

$$\hat{S}_{XY}^{(p)}(f) = \frac{1}{N}\left(\sum_{u=1}^{N} X_u \mathrm{e}^{\mathrm{i}2\pi f u}\right)\left(\sum_{v=1}^{N} Y_v \mathrm{e}^{-\mathrm{i}2\pi f v}\right).$$

Use the spectral representation theorem to show that

$$E\{\hat{S}_{XY}^{(p)}(f)\} = \int_{-1/2}^{1/2} \mathcal{F}(f-f')S_{XY}(f')\mathrm{d}f',$$

where $\mathcal{F}(f)$ is Fejér's kernel, defined as

$$\mathcal{F}(f) = \left|\frac{1}{\sqrt{N}}\sum_{t=1}^{N}\mathrm{e}^{-\mathrm{i}2\pi f t}\right|^2 = \frac{\sin^2(N\pi f)}{N\sin^2(\pi f)}.$$

HINT: Consider the equivalent derivation for the expected value of the standard periodogram. (6 marks)

(Total: 20 marks)

5. **You may use without proof any results given in your notes and the Mastery Material.**

In addition, you may use the following version of Isserlis' Theorem for part (b). If Z_1, Z_2, Z_3 and Z_4 are four complex valued random variables with zero means, then

$$\mathrm{Cov}\{Z_1Z_2, Z_3Z_4\} = \mathrm{Cov}\{Z_1, Z_3\}\mathrm{Cov}\{Z_2, Z_4\} + \mathrm{Cov}\{Z_1, Z_4\}\mathrm{Cov}\{Z_2, Z_3\}.$$

Recall: for a pair of zero mean complex random variables S and T, $\mathrm{Cov}\{S, T\} = E\{S^*T\}$, where * denotes complex conjugation.

(a) In this part you may assume the distributional result that

$$S^{(p)}(f) \stackrel{\mathrm{d}}{=} \begin{cases} S(f)\chi_2^2/2 & 0 < |f| < 1/2 \\ S(f)\chi_1^2 & |f| = 0 \text{ or } 1/2. \end{cases}$$

Let $\hat{s}_0^{(p)} = 2$, $\hat{s}_1^{(p)} = \hat{s}_{-1}^{(p)} = -1$, and $\hat{s}_2^{(p)} = \hat{s}_{-2}^{(p)} = 1/2$ be the "biased" estimates of the autocovariance sequence of a zero mean stationary process $\{X_t\}$ computed from a realisation of length $N = 3$. Justifying your answer, which of the following is the 90% confidence interval for $S(1/4)$?

$$A.\quad \left[-\frac{3}{\log(0.05)}, -\frac{3}{\log(0.95)}\right] \qquad B.\quad \left[-\frac{1}{\log(0.95)}, -\frac{1}{\log(0.05)}\right]$$

$$C.\quad \left[-\frac{1}{\log(0.05)}, -\frac{1}{\log(0.95)}\right] \qquad D.\quad \left[-\frac{2}{\log(0.1)}, -\frac{2}{\log(0.9)}\right]$$

(4 marks)

QUESTION CONTINUES OF NEXT PAGE

(b) Let $X_0, X_2, ..., X_{N-1}$ be a portion of a Gaussian zero mean stationary process $\{X_t\}$ with spectral density function $S(f)$. Consider the weighted multitaper estimator

$$\hat{S}^{(WMT)}(f) = \sum_{k=0}^{K-1} d_k \hat{S}_k^{(MT)}(f) \qquad \text{with} \qquad \hat{S}_k^{(MT)}(f) = \left| \sum_{t=0}^{N-1} h_{k,t} X_t \mathrm{e}^{-\mathrm{i}2\pi f t} \right|^2,$$

where $\{h_{k,t}\}$ is the data taper for the kth direct spectral estimator $\hat{S}_k^{(MT)}(f)$ and $d_0, .., d_{K-1}$ are weights with $\sum_{k=0}^{K-1} d_k = 1$. We assume $\sum_{t=0}^{N-1} h_{k,t}^2 = 1$ for all $k = 0, ..., K-1$.

(i) Show that if $\hat{S}_k^{(MT)}(f)$ is an unbiased estimator of $S(f)$ for all $k = 0, ..., K-1$, then $\hat{S}^{(WMT)}(f)$ is also an unbiased estimator of $S(f)$. (2 marks)

(ii) Show

$$\mathrm{Var}\{\hat{S}^{(WMT)}(f)\} = \sum_{k=0}^{K-1} d_k^2 \ \mathrm{Var}\{\hat{S}^{(MT)}(f)\} + 2 \sum_{j<k} d_j d_k \ \mathrm{Cov}\{\hat{S}_j^{(MT)}(f), \hat{S}_k^{(MT)}(f)\}.$$

(3 marks)

(iii) Let $J_k(f) = \sum_{t=0}^{N-1} h_{k,t} X_t \mathrm{e}^{-\mathrm{i}2\pi f t}$. Show

$$\mathrm{Cov}\{\hat{S}_j^{(MT)}(f), \hat{S}_k^{(MT)}(f')\} = |E\{J_j(f) J_k^*(f')\}|^2 + |E\{J_j(f) J_k(f')\}|^2,$$

where f and f' are two fixed frequencies between 0 and $1/2$. (4 marks)

(iv) Let $\{h_{j,t}\}$ and $\{h_{k,t}\}$ be orthogonal for all $j \neq k$. It can in fact be shown that $\mathrm{Cov}\{\hat{S}_j^{(MT)}(f), \hat{S}_k^{(MT)}(f)\} \approx S^2(f) \left| \sum_{t=0}^{N-1} h_{j,t} h_{k,t} \right|^2$, for $0 < f < 1/2$. Show that

$$\mathrm{Var}\{\hat{S}^{(WMT)}(f)\} \approx S^2(f) \sum_{k=0}^{K-1} d_k^2.$$

(4 marks)

(v) Argue that if $K \geq 2$ and the weights d_k are all positive then $\mathrm{Var}\{\hat{S}^{(WMT)}(f)\}$ must be less than $S^2(f)$. (3 marks)

(Total: 20 marks)

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2021

MATH96053/MATH97084/MATH97185

Time Series Analysis [SOLUTIONS]

1. (a) (i) First check expected value: $E\{Y_t\} = \alpha E\{X_t\} + c = \alpha\mu_X + c$. This is constant in time. Now check variance: $\text{var}\{Y_t\} = \alpha^2\sigma_\epsilon^2$, which is also constant in time. Now check autocovariance: $\text{cov}\{Y_t, Y_{t+\tau}\} = \alpha^2 s_{X,\tau}$ which clearly depends only on τ. Therefore $\{Y_t\}$ is stationary. sim. seen ⇓ 2 (A)

(ii) The variance is given as $\text{var}\{Y_t\} = E\{Y_t^2\} = E\{(X_t - \mu_X)^2\}\cos^2(t) = \sigma_X^2\cos^2(t)$, which is not constant in time. Therefore $\{Y_t\}$ is non-stationary. 2 (A)

(iii) First check expected value: $E\{Y_t\} = E\{W\} + E\{X_t\} = 0.5 + \mu_X$. This is constant in time. Now check variance: $\text{var}\{Y_t\} = \text{var}\{W\} + \text{var}\{X_t\}$ through independence of W and $\{X_t\}$. Therefore $\text{var}\{Y_t\} = 0.25 + \sigma_X^2$, which is constant in time. Now check autocovariance: $\text{cov}\{Y_t, Y_{t+\tau}\} = \text{cov}\{W, W\} + \text{cov}\{W, X_{t+\tau}\} + \text{cov}\{X_t, W\} + \text{cov}\{X_t, X_{t+\tau}\} = 0.25 + 0 + 0 + s_{X,\tau} = 0.25 + s_{X,\tau}$, which is constant in time. Therefore $\{Y_t\}$ is stationary. 2 (A)

(b) (i) This is an ARMA(2,1) process. 1 (A)

(ii) Rearranging gives $X_t - \alpha X_{t-2} = \epsilon_t + \alpha\epsilon_{t-1}$, which we can express in the form $\Phi(B)X_t = \Theta(B)\epsilon_t$ form where $\Phi(z) = 1 - \alpha z^2$ and $\Theta(z) = 1 + \alpha z$. We can write $\Phi(z) = (1 - \alpha^{1/2}z)(1 + \alpha^{1/2}z)$. The roots of $\Phi(z)$ are $\pm\alpha^{-1/2}$. These are real for positive α and imaginary for negative α. Either way, with $|\alpha| < 1$, these lie outside of the unit circle and hence $\{X_t\}$ is stationary. The root of $\Theta(z)$ is $-1/\alpha$ which also lies outside of the unit circle for $|\alpha| < 1$, therefore $\{X_t\}$ is invertible. 2 (A)

(iii) Putting $\{X_t\}$ into GLP form we have

$$\begin{aligned}
X_t &= (1 + \alpha B)(1 - \alpha B^2)^{-1}\epsilon_t \\
&= (1 + \alpha B)(1 + \alpha B^2 + \alpha^2 B^4 + \alpha^3 B^6 + \ldots)\epsilon_t \\
&= (1 + \alpha B^2 + \alpha^2 B^4 + \alpha^3 B^6 + \ldots + \alpha B + \alpha^2 B^3 + \alpha^3 B^5 + \alpha^4 B^7 + \ldots)\epsilon_t \\
&= (1 + \alpha B + \alpha B^2 + \alpha^2 B^3 + \alpha^2 B^4 + \alpha^3 B^5 + \alpha^3 B^6 + \ldots)\epsilon_t.
\end{aligned}$$

So we have $X_t = \sum_{k=0}^{\infty} g_k \epsilon_{t-k}$, where $g_0 = 1$, $g_1 = \alpha$, $g_2 = \alpha$, $g_3 = \alpha^2$, $g_4 = \alpha^2$, etc. We know that $\mathrm{var}\{X_t\} = \sigma_\epsilon^2 \sum_{k=0}^{\infty} g_k^2$, therefore

$$\begin{aligned}\mathrm{var}\{X_t\} = \sigma_\epsilon^2(1+\alpha^2+\alpha^2+\alpha^4+\alpha^4+\ldots) &= \sigma_\epsilon^2\left(2\sum_{k=0}^{\infty}\alpha^{2k}-1\right)\\ &= \sigma_\epsilon^2\left(\frac{2}{1-\alpha^2}-1\right)\\ &= \sigma_\epsilon^2\left(\frac{1+\alpha^2}{1-\alpha^2}\right).\end{aligned}$$

$\boxed{4\ (\mathrm{B})}$

(iv) Using the formula $s_1 = \sigma_\epsilon^2 \sum_{k=0}^{\infty} g_k g_{k+1}$, gives

$$s_1 = \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \ldots = \sigma_\epsilon^2 \sum_{k=1}^{\infty} \alpha^k = \sigma_\epsilon^2 \left(\frac{1}{1-\alpha} - 1\right) = \sigma_\epsilon^2 \frac{\alpha}{1-\alpha}.$$

$\boxed{2\ (\mathrm{C})}$

From definition, $\rho_1 = s_1/s_0 = \frac{\alpha(1-\alpha^2)}{(1-\alpha)(1+\alpha^2)} = \frac{\alpha(1+\alpha)}{1+\alpha^2}$.

(v) The spectral density function for an ARMA(p,q) process is given in the notes as

$$S(f) = \sigma_\epsilon^2 \frac{|1-\theta_{1,q}e^{-i2\pi f} - \ldots - \theta_{q,q}e^{-i2\pi fq}|^2}{|1-\phi_{1,p}e^{-i2\pi f} - \ldots - \phi_{p,p}e^{-i2\pi fp}|^2}.$$

For the ARMA(2,1) model given, we identify $\phi_{1,2} = 0$, $\phi_{2,2} = \alpha$, and $\theta_{1,1} = -\alpha$. Therefore, with $\sigma_\epsilon^2 = 1$, we have

$$S(f) = \frac{|1+\alpha \mathrm{e}^{-\mathrm{i}2\pi f}|^2}{|1-\alpha \mathrm{e}^{-\mathrm{i}4\pi f}|^2} = \frac{1+2\alpha\cos(2\pi f)+\alpha^2}{1-2\alpha\cos(4\pi f)+\alpha^2}$$

$\boxed{\text{unseen} \Downarrow}$

Therefore, we will have

$$\frac{1+2\alpha\cos(2\pi f)+\alpha^2}{1-2\alpha\cos(4\pi f)+\alpha^2} = \frac{13+12\cos(2\pi f)}{13-12\cos(4\pi f)},$$

so we have $(1+\alpha^2)c = 13$ and $2\alpha c = 12$. With $c = 6/\alpha$, we have $(1+\alpha^2)6/\alpha = 13$ which gives $6\alpha^2 - 13\alpha + 6 = 0$. Solving gives

$$\alpha = \frac{13 \pm \sqrt{169-144}}{12} = \frac{13 \pm 5}{12} = \frac{3}{2} \text{ or } \frac{2}{3}.$$

$\boxed{5\ (\mathrm{D})}$

However, we know that $|\alpha| < 1$, resulting in $\alpha = 2/3$.

2. (a) (i) Consider the three requirements of a LTI filter. We are only required to show $L\{X_t\} = \alpha + \beta X_t$ violates at least one of them. First consider the scale-preservation condition. We have $L\{\gamma X_t\} = \alpha + \gamma\beta X_t$ whereas $\gamma L\{X_t\} = \gamma\alpha + \gamma\beta X_t$, therefore $L\{\gamma X_t\} = \gamma L\{X(t)\} + \alpha(1-\gamma)$ and the condition is not satisfied. It also violates the superposition condition because sim. seen ⇓

$$L\{X_{1,t} + X_{2,t}\} = \alpha + \beta(X_{1,t} + X_{2,t})$$

whereas

$$L\{X_{1,t}\} + L\{X_{2,t}\} = 2\alpha + \beta(X_{1,t}) + \beta(X_{2,t}).$$

Therefore $L\{X_{1,t} + X_{2,t}\} = L\{X_{1,t}\} + L\{X_{2,t}\} - \alpha$, and the condition does not hold. Note that time invariance does hold. 3 (A)

(ii) Consider each necessary condition in turn.

1. Scale-preservation:

$$L\{\alpha X_t\} = \alpha X_t - \alpha X_{t-1} = \alpha(X_t - X_{t-1}) = \alpha L\{X_t\}.$$

2. Super-position:

$$\begin{aligned} L\{X_{1,t} + X_{2,t}\} &= X_{1,t} + X_{2,t} - (X_{1,t-1} + X_{2,t-1}) \\ &= X_{1,t} - X_{1,t-1} + X_{2,t} - X_{2,t-1} \\ &= L\{X_{1,t}\} + L\{X_{2,t}\}. \end{aligned}$$

3. Time-invariance: letting $Y_t = L\{X_t\}$, we have $L\{X_{t+\tau}\} = X_{t+\tau} - X_{t+\tau-1} = Y_{t+\tau}$. 3 (A)

(iii) Consider each necessary condition in turn. unseen ⇓

1. Scale-preservation:

$$K\{X_t\} = L_1\{L_2\{\alpha X_t\}\} = L_1\{\alpha L_2\{X_t\}\} = \alpha L_1\{L_2\{X_t\}\} = \alpha K\{\cdot\}.$$

by the scale-preservation properties of $L_1\{\}$ and $L_2\{\}$, respectively.

2. Super-position:

$$\begin{aligned} L_1\{L_2\{X_{1,t} + X_{2,t}\}\} &= L_1\{L_2\{X_{1,t}\} + L_2\{X_{2,t}\}\} \\ &= L_1\{L_2\{X_{1,t}\}\} + L_1\{L_2\{X_{2,t}\}\} \\ &= K\{X_{1,t}\} + K\{X_{2,t}\}. \end{aligned}$$

3. Time-invariance: letting $Y_t = K\{X_t\}$, we have $K\{X_{t+\tau}\} = L_1\{L_2\{X_{t+\tau}\}\} = Y_{t+\tau}$ by the time-invariance of both L_1 and L_2. 3 (C)

Considering the frequency response function, we have

$$\begin{aligned}K\{\mathrm{e}^{\mathrm{i}2\pi ft}\} &= L_1\{L_2\{\mathrm{e}^{\mathrm{i}2\pi ft}\}\} \\ &= L_1\{\mathrm{e}^{\mathrm{i}2\pi ft}G_2(f)\} \\ &= G_2(f)L_1\{\mathrm{e}^{\mathrm{i}2\pi ft}\} \\ &= G_2(f)(\mathrm{e}^{\mathrm{i}2\pi ft}G_1(f)) \\ &= \mathrm{e}^{\mathrm{i}2\pi ft}G_1(f)G_2(f),\end{aligned}$$

Therefore, frequency response function for $K\{\cdot\}$ is $G_1(f)G_2(f)$. $\boxed{3 \text{ (C)}}$

(b) (i) We know from the notes that when μ_t is a $d-1$ polynomial trend then $X^{(d)}(t) = Y^{(d)}(t)$ is a stationary process. Defining $L\{X_t\} = \Delta X_t = X_t - X_{t-1}$, which we know from (a)(ii) is an LTI filter, we have

$$K\{X_t\} \equiv \underbrace{L\{...L\{}_{d} X_t\}\} = \Delta^d X_t = X_t^{(d)} = Y_t^{(d)} = \Delta^d Y_t = K\{Y_t\}.$$

Using (a)(iii), the frequency response function of $K\{\}$ is $G_K(f) = \underbrace{G(f)\cdots G(f)}_{d}$, where $G(f)$ is the frequency response function of $L\{\cdot\}$.

Therefore, the spectral density function of $\{X_t^{(d)}\}$ is

$$S_{X^{(d)}}(f) = |G_K(f)|^2 S_Y(f) = |G(f)|^{2d} S_Y(f).$$

It is given in the notes as $|G(f)|^2 = 4\sin^2(\pi f)$. Therefore, $S_{X^{(d)}}(f) = 4^d \sin^{2d}(\pi f) S_Y(f)$ $\boxed{5 \text{ (D)}}$

(ii) If $\{Y_t\}$ is a white noise process, then $S_{X^{(d)}}(f) = \sigma_Y^2 4^d \sin^{2d}(\pi f)$. We know from the Fourier relationship between the autocovariance sequence and the spectral density function that $\boxed{\text{sim. seen} \Downarrow}$

$$\begin{aligned}|s_{X^{(d)},\tau}| &= \left| \int_{-1/2}^{1/2} S_{X^{(d)}}(f)\mathrm{e}^{\mathrm{i}2\pi f\tau}\mathrm{d}f \right| \\ &= 4^d\sigma_Y^2 \left| \int_{-1/2}^{1/2} \sin^{2d}(\pi f)\mathrm{e}^{\mathrm{i}2\pi f\tau}\mathrm{d}f \right| \\ &\le 4^d\sigma_Y^2 \int_{-1/2}^{1/2} \left|\sin^{2d}(\pi f)\mathrm{e}^{\mathrm{i}2\pi f\tau}\right| \mathrm{d}f \\ &= 4^d\sigma_Y^2 \int_{-1/2}^{1/2} \left|\sin^{2d}(\pi f)\right| \mathrm{d}f.\end{aligned}$$

$\boxed{3 \text{ (B)}}$

3. (a) (i) Under the standard model $X_t = \mu - \theta_{0,1}\epsilon_t - \theta_{1,1}\epsilon_{t-1}$, the MA(1) process stated has $\theta_{0,1} = -1$ and $\theta_{1,1} = \frac{1}{4}$. From the notes the acvs for an MA(q) process is given as

sim. seen ⇓

$$s_\tau = \begin{cases} \sigma_\epsilon^2 \sum_{j=0}^{q-|\tau|} \theta_{j,q}\theta_{j+|\tau|,q} & |\tau| \le q \\ 0 & |\tau| > q \end{cases}.$$

This gives $s_0 = \sigma_\epsilon^2(\theta_{0,1}^2 + \theta_{1,1}^2) = \sigma_\epsilon^2(1 + \frac{1}{16}) = 17\sigma_\epsilon^2/16$, $s_1 = \sigma_\epsilon^2\theta_{0,1}\theta_{1,1} = -\sigma_\epsilon^2/4$, and $s_\tau = 0$ for $|\tau| > 1$.

3 (A)

(ii) In the notes, it is shown that $\mathrm{var}\{\bar{X}\} = \frac{1}{N}\sum_{\tau=-(N-1)}^{N-1}\left(1 - \frac{|\tau|}{N}\right)s_\tau$. Therefore, for the process given we have

$$\begin{aligned} \mathrm{var}\{\bar{X}\} &= \frac{\sigma_\epsilon^2}{N}\left(\frac{N-1}{N}\cdot\frac{-1}{4} + \frac{17}{16} + \frac{N-1}{N}\cdot\frac{-1}{4}\right) \\ &= \frac{\sigma_\epsilon^2}{N}\left(\frac{1-N}{2N} + \frac{17}{16}\right) \\ &= \frac{\sigma_\epsilon^2}{N}\cdot\frac{16 - 16N + 34N}{32N} = \frac{\sigma_\epsilon^2}{N}\cdot\frac{16 + 18N}{32N} = \sigma_\epsilon^2\cdot\frac{8 + 9N}{16N^2}. \end{aligned}$$

4 (B)

(iii) Recall that $\mathrm{mse}\{\bar{X}\} = \mathrm{bias}^2\{\bar{X}\} + \mathrm{var}\{\bar{X}\}$. It is the case that $\bar{X}$ is an unbiased estimator for μ (proved in notes) and therefore

$$\mathrm{mse}\{\bar{X}\} = \sigma_\epsilon^2\cdot\frac{8 + 9N}{16N^2}.$$

Therefore to have $\mathrm{mse}\{\bar{X}\} < C$, we require

$$\sigma_\epsilon^2\cdot\frac{8 + 9N}{16N^2} < C \implies \frac{N^2}{8 + 9N} > \frac{\sigma_\epsilon^2}{16C}.$$

3 (B)

(b) The required formula is that

$$\widehat{s}_\tau^{(p)} = \frac{1}{N}\sum_{t=1}^{N-|\tau|}(X_t - \bar{X})(X_{t+\tau} - \bar{X}).$$

With $\bar{X} = 1$, we have

$$\begin{aligned} \widehat{s}_0^{(p)} &= \frac{1}{4}((-1-1)^2 + (3-1)^2 + (3-1)^2 + (-1-1)^2) = \frac{1}{4}(4+4+4+4) = 4 \\ \widehat{s}_1^{(p)} &= \frac{1}{4}((-1-1)(3-1) + (3-1)^2 + (3-1)(-1-1)) = \frac{1}{4}(-4+4-4) = -1 \\ \widehat{s}_2^{(p)} &= \frac{1}{4}((-1-1)(3-1) + (3-1)(-1-1)) = \frac{1}{4}(-4-4) = -2 \\ \widehat{s}_3^{(p)} &= \frac{1}{4}((-1-1)(-1-1)) = \frac{1}{4}(4) = 1. \end{aligned}$$

4 (A)

(c) (i) We show equality in an analogous way as in the notes $\boxed{\text{unseen} \Downarrow}$

$$\widehat{S}^{(p)}(f) = \sum_{\tau=-(N-1)}^{(N-1)} \widehat{s}_\tau^{(p)} \mathrm{e}^{-\mathrm{i}2\pi f\tau} = \frac{1}{N} \sum_{\tau=-(N-1)}^{(N-1)} \sum_{t=1}^{N-|\tau|} (X_t - \bar{X})(X_{t+|\tau|} - \bar{X}) \mathrm{e}^{-\mathrm{i}2\pi f\tau}$$

$$= \frac{1}{N} \sum_{j=1}^{N} \sum_{k=1}^{N} (X_j - \bar{X})(X_k - \bar{X}) e^{-i2\pi f(k-j)}$$

$$= \frac{1}{N} \left| \sum_{t=1}^{N} (X_t - \bar{X}) \mathrm{e}^{-\mathrm{i}2\pi f t} \right|^2,$$

where the summation interchange has merely swapped diagonal sums for row sums. $\boxed{3 \text{ (C)}}$

(ii) Using this formula, we have

$$\widehat{S}^{(p)}(0) = \frac{1}{N} \left| \sum_{t=1}^{N} (X_t - \bar{X}) \right|^2 = \frac{1}{N} \left| \sum_{t=1}^{N} X_t - N\bar{X} \right|^2 = \frac{1}{N} \left| \sum_{t=1}^{N} X_t - N\frac{1}{N} \sum_{t=1}^{N} X_t \right|^2 = 0.$$

Given we also have

$$\widehat{S}^{(p)}(f) = \sum_{\tau=-(N-1)}^{(N-1)} \widehat{s}_\tau^{(p)} \mathrm{e}^{-\mathrm{i}2\pi f\tau},$$

it follows that

$$\widehat{S}^{(p)}(0) = \sum_{\tau=-(N-1)}^{(N-1)} \widehat{s}_\tau^{(p)} = 0.$$

$\boxed{3 \text{ (D)}}$

4. (a) (i) This is a superposition, where the X_t component is the signal, the aX_{t-d} component is a scaled version of the signal d steps ago, and hence can be considered as an echo term, and η_t is noise. seen ⇓ 1 (A)

(ii) $\{X_t\}$ is stationary by definition. $\{Y_t\}$ is the the sum of stationary processes and therefore stationary. We are therefore left to check that $\text{cov}\{X_t, Y_{t+\tau}\}$ depends only on τ. With both $\{X_t\}$ and $\{Y_t\}$ being zero mean, we have

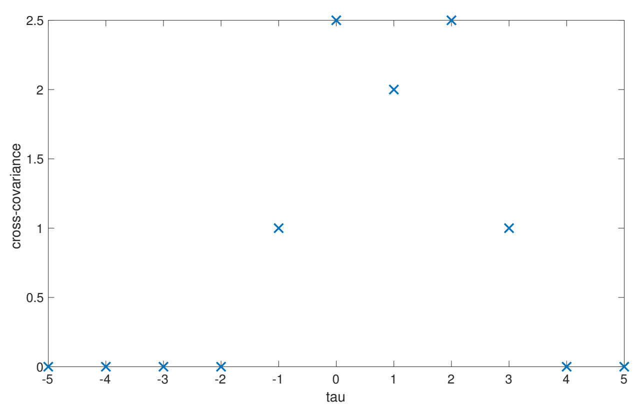
$$\begin{aligned}
\text{cov}\{X_t, Y_{t+\tau}\} &= E\{X_t Y_{t+\tau}\} \\
&= E\{X_t(X_{t+\tau} + aX_{t-d+\tau} + \eta_{t+\tau})\} \\
&= E\{X_t X_{t+\tau}\} + aE\{X_t X_{t-d+\tau}\} + E\{X_t \eta_{t+\tau}\} \\
&= s_{X,\tau} + a s_{X,\tau-d} + 0
\end{aligned}$$

because $\{X_t\}$ and $\{\eta_{t+\tau}\}$ are uncorrelated. This only depends on τ and therefore $\{X_t\}$ and $\{Y_t\}$ are jointly stationary with cross-covariance sequence $s_{XY,\tau} = s_{X,\tau} + a s_{X,\tau-d}$. 3 (A)

(b) (i) First consider the acvs of the MA(1) process $\{X_t\}$. We have, $s_0 = 2(1+\frac{1}{4}) = \frac{5}{2}$, $s_1 = 2.\frac{1}{2} = 1$ and $s_\tau = 0$ for $|\tau| > 1$.

$$s_\tau = \begin{cases} 5/2 & \tau = 0 \\ 1 & |\tau| = 1 \\ 0 & |\tau| > 1. \end{cases} .$$

Therefore, for $a = 1$ and $d = 2$, we have $s_{XY,\tau} = s_{X,\tau} + s_{X,\tau-2}$. This gives $s_{XY,-1} = s_{X,-1} + s_{X,-3} = 1 + 0 = 1$, $s_{XY,0} = s_{X,0} + s_{X,-2} = 5/2 + 0 = 5/2$, $s_{XY,1} = s_{X,1} + s_{X,-1} = 1 + 1 = 2$, $s_{XY,2} = s_{X,2} + s_{X,0} = 0 + 5/2 = 5/2$, $s_{XY,3} = s_{X,3} + s_{X,1} = 0 + 1 = 1$. Also, $s_{XY,\tau} = 0$ for all $\tau > 3$ and $\tau < -1$. 6 (B)



(ii) The cross-spectrum is given as

$$\begin{aligned}
S_{XY}(f) &= \sum_{\tau=-\infty}^{\infty} s_{XY,\tau}\mathrm{e}^{-\mathrm{i}2\pi f\tau} \\
&= \mathrm{e}^{\mathrm{i}2\pi f} + \frac{5}{2} + 2\mathrm{e}^{-\mathrm{i}2\pi f} + \frac{5}{2}\mathrm{e}^{-\mathrm{i}4\pi f} + \mathrm{e}^{-\mathrm{i}6\pi f} \\
&= \mathrm{e}^{-\mathrm{i}2\pi f}\left(\mathrm{e}^{\mathrm{i}4\pi f} + \frac{5}{2}\mathrm{e}^{\mathrm{i}2\pi f} + 2 + \frac{5}{2}\mathrm{e}^{-\mathrm{i}2\pi f} + \mathrm{e}^{-\mathrm{i}4\pi f}\right) \\
&= \left(2 + 5\cos(2\pi f) + 2\cos(4\pi f)\right)\mathrm{e}^{-\mathrm{i}2\pi f}.
\end{aligned}$$

This is in $S_{XY}(f) = |S_{XY(f)}|\mathrm{e}^{\mathrm{i}\theta(f)}$ form where the phase cross-spectrum is $\theta(f) = -2\pi f$. The group delay is defined as

$$-\frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}f}\theta(f) = -\frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}f}(-2\pi f) = 1.$$

The cross-covariance sequence is centred and symmetric about $\tau = 1$. This therefore represents the average delay between the two signals. $\boxed{4\text{ (D)}}$

(c) This is a generalisation of the result shown in the notes. Define

$$J_X(f) = \frac{1}{\sqrt{N}}\sum_{t=1}^{N} X_t\mathrm{e}^{-\mathrm{i}2\pi ft} \qquad \text{and} \qquad J_Y(f) = \frac{1}{\sqrt{N}}\sum_{t=1}^{N} Y_t\mathrm{e}^{-\mathrm{i}2\pi ft} \qquad |f| \leq 1/2.$$

We know from the spectral representation theorem that,

$$X_t = \int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}2\pi f't}\, dZ_X(f') \qquad\qquad Y_t = \int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}2\pi f't}\, dZ_Y(f')$$

so that,

$$J_X(f) = \sum_{t=1}^{N}\left(\int_{-1/2}^{1/2} \frac{1}{\sqrt{N}}\mathrm{e}^{\mathrm{i}2\pi f't}\, dZ_X(f')\right)\mathrm{e}^{-\mathrm{i}2\pi ft} = \int_{-1/2}^{1/2}\sum_{t=1}^{N}\frac{1}{\sqrt{N}}\mathrm{e}^{-\mathrm{i}2\pi(f-f')t}\, dZ_X(f'),$$

and similarly for $J_Y(f)$. Then

$$\begin{aligned}
\mathsf{E}\{\widehat{S}^{(p)}(f)\} &= \mathsf{E}\{J_X^*(f)J_Y(f)\} \\
&= \mathsf{E}\left\{\int_{-1/2}^{1/2}\sum_{u=1}^{N}\frac{1}{\sqrt{N}}\mathrm{e}^{\mathrm{i}2\pi(f-f')u}\, dZ_X^*(f')\int_{-1/2}^{1/2}\sum_{v=1}^{N}\frac{1}{\sqrt{N}}\mathrm{e}^{-\mathrm{i}2\pi(f-f'')v}\, dZ_Y(f'')\right\} \\
&= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\sum_{u=1}^{N}\frac{1}{\sqrt{N}}\mathrm{e}^{\mathrm{i}2\pi(f-f')u}\sum_{v=1}^{N}\frac{1}{\sqrt{N}}\mathrm{e}^{-\mathrm{i}2\pi(f-f'')v}E\{dZ_X^*(f')\, dZ_Y(f'')\} \\
&= \int_{-1/2}^{1/2}\mathcal{F}(f-f')S_{XY}(f')\, df',
\end{aligned}$$

where $\mathcal{F}$ is Féjer's kernel defined by

$$\mathcal{F}(f) = \left|\sum_{t=1}^{N}\frac{1}{\sqrt{N}}\mathrm{e}^{-\mathrm{i}2\pi ft}\right|^2 = \frac{\sin^2(N\pi f)}{N\sin^2(\pi f)}.$$

$\boxed{6\text{ (A)}}$

5. (a) The periodogram can be computed as

$$\widehat{S}^{(p)}(f) = \sum_{\tau=-2}^{2} \widehat{s}_{\tau}^{(p)} e^{-i2\pi f\tau} = 2 - (e^{-i2\pi f} + e^{-i2\pi f}) + \frac{1}{2}(e^{-i4\pi f} + e^{-i4\pi f})$$
$$= 2 - 2\cos(2\pi f) + \cos(4\pi f).$$

Therefore $\widehat{S}^{(p)}(1/4) = 2 - 2\cos(\pi/2) + \cos(\pi) = 2 - 0 - 1 = 1$. Given the distributional result stated in the question, the Mastery material provides the following $100(1-2p)\%$ confidence interval for $S(f)$

$$\left[-\frac{\widehat{S}^{(p)}(f)}{\log(p)}, -\frac{\widehat{S}^{(p)}(f)}{\log(1-p)}\right].$$

For the 90% confidence interval we have $p = 0.05$, therefore the confidence interval is

$$\left[-\frac{1}{\log(0.05)}, -\frac{1}{\log(0.95)}\right]$$

and the answer is C. $\boxed{4}$

(b) (i)

$$E\{\widehat{S}^{(WMT)}(f)\} = \sum_{k=0}^{K-1} d_k E\{\widehat{S}_k^{(MT)}(f)\} = \sum_{k=0}^{K-1} d_k S(f) = S(f)\sum_{k=0}^{K-1} d_k = S(f).$$

Therefore $E\{\widehat{S}^{(WMT)}(f)\}$ is an unbiased estimator. $\boxed{2}$

(ii)

$$\begin{aligned}\mathrm{var}\{\widehat{S}^{(WMT)}(f)\} &= \mathrm{cov}\{\widehat{S}^{(WMT)}(f), \widehat{S}^{(WMT)}(f)\} \\ &= \mathrm{cov}\left\{\sum_{k=0}^{K-1} d_k \widehat{S}_k^{(MT)}(f), \sum_{k=0}^{K-1} d_k \widehat{S}_k^{(MT)}(f)\right\} \\ &= \sum_{j=0}^{J-1}\sum_{k=0}^{K-1} d_j d_k \,\mathrm{cov}\{\widehat{S}_j^{(MT)}(f), \widehat{S}_k^{(MT)}(f)\}.\end{aligned}$$

When $j = k$ we have $\mathrm{cov}\{\widehat{S}_j^{(MT)}(f), \widehat{S}_k^{(MT)}(f)\} = \mathrm{var}\{\widehat{S}_k^{(MT)}(f)$. We also have $\mathrm{cov}\{\widehat{S}_j^{(MT)}(f), \widehat{S}_k^{(MT)}(f)\} = \mathrm{cov}\{\widehat{S}_k^{(MT)}(f), \widehat{S}_j^{(MT)}(f)\}$. Therefore,

$$\sum_{j=0}^{J-1}\sum_{k=0}^{K-1} d_j d_k \,\mathrm{cov}\{\widehat{S}_j^{(MT)}(f), \widehat{S}_k^{(MT)}(f)\} =$$
$$\sum_{k=0}^{K-1} d_k^2 \;\mathrm{var}\{\widehat{S}^{(MT)}(f)\} + 2\sum_{j<k} d_j d_k \;\mathrm{cov}\{\widehat{S}_j^{(MT)}(f), \widehat{S}_k^{(MT)}(f)\}.$$

$\boxed{3}$

(iii) Because $\{X_t\}$ is zero mean Gaussian, $J_k(f)$ is zero mean complex-Gaussian. Therefore, making use of Isserlis' Theorem as provided, we have

$$\begin{aligned}
\operatorname{cov}\{\widehat{S}_j^{(MT)}(f), \widehat{S}_k^{(MT)}(f')\} &= \operatorname{cov}\{|J_j(f)|^2, |J_k(f')|^2\} \\
&= \operatorname{cov}\{J_j(f)J_j^*(f), J_k(f')J_k^*(f')\} \\
&= \operatorname{cov}\{J_j(f), J_k(f')\} \operatorname{cov}\{J_j^*(f), J_k^*(f')\} \\
&\qquad + \operatorname{cov}\{J_j(f), J_k^*(f')\} \operatorname{cov}\{J_j^*(f), J_k(f')\} \\
&= E\{J_j^*(f)J_k(f')\}E\{J_j(f)J_k^*(f')\} \\
&\qquad + E\{J_j^*(f)J_k^*(f')\}E\{J_j(f)J_k(f')\} \\
&= |E\{J_j(f)J_k^*(f')\}|^2 + |E\{J_j(f)J_k(f')\}|^2 .
\end{aligned}$$

$\boxed{4}$

(iv) If $\{h_{j,t}\}$ and $\{h_{k,t}\}$ are orthogonal then $\operatorname{cov}\{\widehat{S}_j^{(MT)}(f), \widehat{S}_k^{(MT)}(f)\} \approx 0$ when $j \neq k$, and $\operatorname{cov}\{\widehat{S}_j^{(MT)}(f), \widehat{S}_k^{(MT)}(f)\} \approx S^2(f)$ when $j = k$. Therefore, from part (ii) we have

$$\begin{aligned}
\operatorname{var}\{\widehat{S}^{(WMT)}(f)\} &= \sum_{k=0}^{K-1} d_k^2 \ \operatorname{var}\{\widehat{S}^{(MT)}(f)\} + 2\sum_{j<k} d_j d_k \ \operatorname{cov}\{\widehat{S}_j^{(MT)}(f), \widehat{S}_k^{(MT)}(f)\} \\
&\approx S^2(f) \sum_{k=0}^{K-1} d_k^2.
\end{aligned}$$

$\boxed{4}$

(v) We have

$$\sum_{k=0}^{K-1} d_k^2 \leq \sum_{k=0}^{K-1} d_k^2 + 2\sum_{k<j} d_k d_j = \left(\sum_{k=0}^{K-1} d_k\right)^2 = 1.$$

$\boxed{3}$

Therefore $S^2(f)\sum_{k=0}^{K-1} d_k^2 \leq S^2(f)$.

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH96053 MATH97084 MATH97185	1	Part (a) was answered well. Although, while most students correctly identified which processes were stationary and which weren't, deriving the acvs for the stationary processes caused a few problems. Often errors could have been avoided by working with covariance directly, instead of decomposing into expectations. Parts (b)(i) and (ii) were answered very well. (iii) caused a few more difficulties: there were two valid approaches to this. The first was to use the GLP representation, the second was to work with the process directly. If doing the latter of these, students needed to demonstrate how they dealt with the cross terms in the variance to get full marks. Part (v) caused the most problems. The easiest and most elegant method was to use the expression for an ARMA spectrum given in the notes and match the terms - remembering that a/b = c/d does NOT imply a=c and b=d (a common error). An alternative method was to integrate the given spectrum and match it to the variance.
MATH96053 MATH97084 MATH97185	2	Part a was generally tackled very well, with students confident in verifying the properties of a linear time invariant filter. Part b proved more difficult: a good number were able to make progress, although not always using the most efficient methods
MATH96053 MATH97084 MATH97185	3	Many students answered parts (a)(i) and (ii) from first principles, rather than assuming relevant equations for the MA derived in the notes. In part (b), most who answered the question incorrectly had been careless calculating the sample mean. Part (c) was done very well, mirroring a derivation in the lecture notes.

MATH96053 MATH97084 MATH97185	4	Part (a) was dealt with well on the whole. A common error in part (b) was to assume the ccvs was symmetric. It was explicitly stated in lectures that it is not, as was the case in this example. Part (c) was a direct adaptation of the result for the periodogram given in the notes. A common mistake here was to assign the same orgthogonal increment process to both X_t and Y_t. They have their own orthogonal increment process Z_X(f) and Z_Y(f).
MATH96053 MATH97084 MATH97185	5	The mastery material answers were a little mixed. Part (a) was answered well, although it was a little surprising how many people thought cos(pi) = 1. A lot of what followed was adapting the results given in the book for the MT estimator for the WMT case. It is worth noting that (b)(ii) is much easier to answer using the covariance operator directly than to consider decomposing it into expectations.