CODING SESSION EXERCISE SHEET – 25TH OCT 2022

For those of you who understood the solutions of the exercises posted earlier, here are a few more exercises to be worked on during the coding session.

Q1: SAMPLING FROM $\mathcal{N}(0, 1)$ USING LAPLACE

In the sheet "Extra Exercises for Week 1-2", Exercise 2.5, we derived a rejection sampler for $\mathcal{N}(0, 1)$ using the density

$$q_\alpha(x) = (\alpha/2)\exp(-\alpha|x|).$$

This density is called *Laplace density*. A sample from this density can be taken using[1]

```
x = np.random.laplace(0, 1/alpha).
```

Using this density as a proposal, implement the sampler for $\mathcal{N}(0, 1)$ as described in Exercise 2.5. Sampling from the proposal can be done using the above Python function, but you need to implement the acceptance step for it.

In short,

1. Write a function for the density of $p(x) = \mathcal{N}(0, 1)$:

$$p(x) = \frac{1}{\sqrt{2\pi}}\exp(-x^2/2)$$

   Use `np.exp` for this. Similarly, then, write a function for $q_\alpha(x)$,

$$q_\alpha(x) = (\alpha/2)\exp(-\alpha|x|).$$

2. Note that the optimal M was derived in Exercise 2.5 (Extra Exercises for Week 1-2 sheet)

$$M = \sqrt{2e/\pi}.$$

   Use this below.

3. Sample $X' \sim q_\alpha(x)$ using

```
x_proposed = np.random.laplace(0, 1/alpha).
```

   (here `x_proposed` is denoted above as X'). Then compute your acceptance probability

$$a(X') = \frac{p(X')}{Mq_\alpha(X')}.$$

4. Draw a uniform $u \sim \text{Unif}(0, 1)$
5. If $u \leq a(X')$, then accept X' (add it to the list of accepted samples)
6. Repeat

[1] You do not need any more details of this density beyond using the `numpy` function as a sampler in this exercise

For demonstrating the results

1. Plot your histogram against $\mathcal{N}(0, 1)$ density
2. Compute your empirical acceptance rate (the number of accepted samples/the number of total samples) and compare it to the acceptance rate derived in Exercise 2.5(b) (i.e. compare it to $1/M$).

Q2: SAMPLING FROM THE MIXTURE OF TRUNCATED GAUSSIANS

Consider two truncated Gaussians:

$$p_1(x) \propto \mathcal{N}(x; \mu_1, \sigma_1^2)\mathbf{1}_{\{x:\mu_1 - a_1 \le x \le \mu_1 + a_1\}}(x),$$

i.e. a Gaussian density $\mathcal{N}(x; \mu_1, \sigma_1^2)$ truncated to $[\mu_1 - a_1, \mu_1 + a_1]$ and

$$p_2(x) \propto \mathcal{N}(x; \mu_2, \sigma_2^2)\mathbf{1}_{\{x:\mu_2 - a_2 \le x \le \mu_2 + a_2\}}(x),$$

i.e. a Gaussian density $\mathcal{N}(x; \mu_2, \sigma_2^2)$ truncated to $[\mu_2 - a_2, \mu_2 + a_2]$. Consider the mixture density

$$p(x) = w_1 p_1(x) + w_2 p_2(x).$$

Our aim is to sample from this mixture, each sample taken from the components (p_1 or p_2) would be coming from a rejection sampler.

1. Implement a rejection sampler for truncated Gaussian of the form given above. The function takes four parameters `(mu, sigma, a, n)` which are the mean, the standard deviation, truncation from the mean, and number of samples respectively. (You can use the one if you have already implemented). `np.random.normal` is allowed for the proposal.
2. Implement a sampler that samples from a discrete distribution with weights $[w_1, w_2]$. In particular, in this case, you can sample 1 with w_1 probability and 2 with w_2 probability. You can then choose the component to sample using an `if` statement.
3. Sample from the above density with $w_1 = 0.8$, $w_2 = 0.2$, $\mu_1 = 2$, $\mu_2 = -2$, $\sigma_1 = 0.2$, $\sigma_2 = 0.2$, $a_1 = 0.5$, and $a_2 = 0.5$.
    - Sample a discrete variable k (1 or 2) using inversion from the discrete distribution $[w_1, w_2]$
    - Then sample from $p_k(x)$ using your rejection sampler function.
4. Plot the histogram

Q3: SAMPLING FROM THE MARGINAL OF THE UNIFORM DISTRIBUTION ON THE CIRCLE

This question is simple, but requires you to remember the discussion during the lecture. Recall that we discussed how to sample uniformly from a circle. Let the uniform distribution on the circle be

$$p_{x_1,x_2}(x_1, x_2) = \frac{1}{\pi}, \qquad x_1^2 + x_2^2 \le 1.$$

(i) Compute the marginals in x_1, x_2, i.e., write the expressions of $p_{x_1}(x_1)$ and $p_{x_2}(x_2)$. **Hint:** Recall how you compute marginals from joints and just apply the rule.

(ii) Sample from these marginals and plot the density vs. histograms to prove that your samples are coming from the correct marginals. **Hint:** How do you obtain samples from marginals given the samples from the joint? Do not forget to scatter plot your samples in 2d to ensure that your joint samples are uniform.