Mathematical Logic (MATH6/70132;P65)
Problem Sheet 6 (Basic set theory)

1. Suppose $f : A \to B$ is a bijection. Use f to construct functions $g : A \times A \to B \times B$ and $h : \mathcal{P}(A) \to \mathcal{P}(B)$ which are bijections. In the case of h, give a careful proof that your function is a bijection.

2. Decide whether the following functions f_1, f_2, f_3 are injective or surjective (or both). Give reasons for your answers.
(i) X is some set; A is the set of finite sequences of elements of X; B is the set of finite subsets of X; $f_1 : A \to B$ is given by $f_1((a_1, \ldots, a_n)) = \{a_1, \ldots, a_n\}$.
(ii) $f_2 : \mathbb{R}^{\mathbb{R}} \times \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}$ is given by composition: $f_2(\alpha, \beta) = \alpha \circ \beta$ for $\alpha, \beta \in \mathbb{R}^{\mathbb{R}}$ (the set of functions from $\mathbb{R}$ to $\mathbb{R}$).
(iii) Recall that $\mathbb{N}^{\mathbb{N}}$ can be thought of as the set of sequences of natural numbers. Define the function $f_3 : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ to be the function which sends the pair of sequences $a = (a_0, a_1, a_2, \ldots)$, $b = (b_0, b_1, b_2, \ldots)$ to the sequence $c = (a_0, b_0, a_1, b_1, a_2, b_2, \ldots)$.

3. (i) Show that the following sets are countable (you may use any of the results in the notes):
(a) The set of finite subsets of $\mathbb{N}$.
(b) The set of subsets of $\mathbb{N}$ with finite complement.
(c) The set of real numbers which are roots of non-zero polynomial equations with rational coefficients.
(ii) Use (c) to deduce that there is some real number which is not a root of any non-zero polynomial equation with rational coefficients.

4. Let S be the set of sequences of zeros and ones (that is, functions $s : \mathbb{N} \to \{0, 1\}$), and F the set of functions from $\mathbb{R}$ to $\mathbb{R}$.
(a) Construct an injective function $i : S \times S \to S$, and hence show that S and $S \times S$ are equinumerous. Deduce that $\mathbb{R}$ and $\mathbb{R} \times \mathbb{R}$ are equinumerous.
(b) Construct an injective function from F to $\mathcal{P}(\mathbb{R} \times \mathbb{R})$ and an injective function from $\mathcal{P}(\mathbb{R})$ to F. Deduce that F and $\mathcal{P}(\mathbb{R})$ are equinumerous.

5. Suppose A_1, A_2, B_1, B_2 are sets with $A_1 \approx A_2$ and $B_1 \approx B_2$. Write down bijections which show:
(i) $A_1^{B_1} \approx A_1^{B_2}$;
(ii) $A_1^{B_1} \approx A_2^{B_1}$;
and deduce:
(iii) $A_1^{B_1} \approx A_2^{B_2}$.

6. Again, let S denote the set of sequences of zeros and ones.
(a) Construct a bijection from $S^{\mathbb{N}}$ to S. (Note and Hint: $S^{\mathbb{N}}$ consists of functions $f : \mathbb{N} \to S$. Thus f is a sequence of sequences of zeros and ones. Turn such a thing into a single sequence s_f of zeros and ones in such a way that the original f is recoverable from s_f.)
(b) Deduce that if A is a countably infinite set then $\mathbb{R}^A$ is equinumerous with $\mathbb{R}$.
(c) Let C be the set of *continuous* functions from $\mathbb{R}$ to $\mathbb{R}$. Show that C is equinumerous with $\mathbb{R}$.
(d) What can you say about the relationship between the cardinalities of C here and F in Question 4?

7. Suppose $\mathbf{A_1} = (A_1, \leq_1)$ and $\mathbf{A_2} = (A_2, \leq_2)$ are linearly ordered sets.
(i) Show that the reverse-lexicographic product $\mathbf{A_1} \times \mathbf{A_2}$ (as defined in the notes) is a linearly ordered set.
(ii) Suppose $\mathbf{B_1} = (B_1, \leq_1')$ and $\mathbf{B_2} = (B_1, \leq_2')$ are linearly ordered sets which are similar to $\mathbf{A_1}$ and $\mathbf{A_2}$ respectively. Show that $\mathbf{B_1} \times \mathbf{B_2}$ is similar to $\mathbf{A_1} \times \mathbf{A_2}$.
(Hint: Take similarities $f_i : A_i \to B_i$ for $i = 1, 2$ and show carefully from the definitions that $h : A_1 \times A_2 \to B_1 \times B_2$ given by $h(a_1, a_2) = (f_1(a_1), f_2(a_2))$ (for $a_i \in A_i$) is a similarity.)