Imperial College London

MATH97056 MATH97167

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Probability Theory

Date: Thursday, 27 May 2021

Time: 09:00 to 11:30

Time Allowed: 2.5 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. (a) Consider a probability space $([0,1], \Sigma_{Leb}, \lambda)$. Compute a probability that a number has a representation $x = \sum_{k\in\mathbb{N}} \frac{\alpha_k}{3^k}$, with $\alpha_k \in \{0,2\}$ (5 marks)

(b) Consider a probability space $([0,1], \Sigma_{Leb}, dF_C)$, where dF_C is the distribution given by the Cantor function. For $\beta > 0$, let A_β be the set of numbers described in (a) satisfying $x \geq \beta$. For what value of parameter $\beta > 0$, a probability of A_β is greater or equal $1/2$? (7 marks)

(c) Let $X : \Omega \to \mathbb{R}$ be a random variable on a probability space (Ω, Σ, μ). Suppose the following bound is satisfied

$$\mu(\{|X| > t\}) \leq e^{-t}$$

for every $t \in \mathbb{R}^+$. Prove or disprove that it has every moment finite. (8 marks)

(Total: 20 marks)

2. (a) Suppose $X_k = \cos(2\pi n_k x)$, $n_k \in \mathbb{N}$, are random variables on the probability space $([0,1], \Sigma_{Leb}, \lambda)$. Prove or disprove the Weak Law of Large Numbers for the given sequence of random variables.

(7 marks)

(b) Let $(\Omega, \Sigma, \mu) = (\{0,1\}, 2^{\{0,1\}}, \nu)^{\mathbb{N}}$, where $\nu(\{0\}) = q \in (0,1)$. Let

$$A_n \equiv \{\omega \in \Omega : \forall j \in [n, 2n]\ \omega_j = 1\}$$

Prove or disprove that

$$\mu(\limsup A_n) = 1.$$

(6 marks)

(c) Let (Ω, Σ, μ) be the same as in (2.b). Let $\pi_n : \Omega \to \{0,1\}$ denote the n-th coordinate projection. Prove that the following event, defined with $\alpha \in \mathbb{R}$,

$$A \equiv \{\omega \in \Omega : \sum_{n\in\mathbb{N}} \frac{\sin(n\alpha)}{n} \pi_n(\omega) \quad converges\}.$$

satisfies

$$\mu(A) \in \{0,1\}.$$

(7 marks)

(Total: 20 marks)

3. (a) Let $H_k(x)$, $k \in \mathbb{N}$, be Hermite polynomials associated to the measure

$$d\mu \equiv \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} d\lambda$$

where λ is the Lebesgue measure on the real line. Prove that for $\alpha > \frac{3}{4}$

$$s_n \equiv \frac{1}{n^\alpha} \sum_{j=1}^{n} e^{-j} H_j$$

converges to zero a.e. as $n \to \infty$.

Hint: You can use the following property without proving it

$$\left(\mathbb{E} \left(\sum_k e^{-k} \beta_k H_k \right)^4 \right)^{\frac{1}{4}} \leq \left(\mathbb{E} \left(\sum_k \beta_k H_k \right)^2 \right)^{\frac{1}{2}}$$

provided $\sum_k |\beta_k|^2 < \infty$.

(10 marks)

(b) On a given probability space (Ω, Σ, μ), consider a sequence $(X_j)_{j\in\mathbb{N}}$ of i.i.d. random variables with finite first moment. Define $\xi_j \equiv X_j \chi(|X_j| < j^2)$ Prove or disprove that $\{X_j \neq \xi_j\}$, $j \in \mathbb{N}$, does not hold infinitely often.

(10 marks)

(Total: 20 marks)

4. (a) Let r_n denote the Rademacher random variable on the probability space $([0,1], \Sigma_L \cap [0,1], \lambda)$. Find the characteristic function of the following random variable $X_n \equiv r_n r_{n+1} r_{n+2}$

(8 marks)

(b) Prove Central Limit Theorem for random variables $(X_{3n})_{n\in\mathbb{N}}$

(7 marks)

(c) Show that for some $\sigma \in (0, \infty)$ and any bounded continuous function f

$$\lim_{n\to\infty} \int f\left(\frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_{4n}\right) d\mu = \int f(x) \gamma_{0,\sigma}(dx)$$

where $\gamma_{0,\sigma}$ is the standard Gaussian measure on real line. (5 marks)

(Total: 20 marks)

5. (a) Define an exponential martingale for the one-dimensional Brownian motion $(B_t)_{t\geq 0}$ and demonstrate the martingale property.

(3 marks)

(b) In the context of one-dimensional Brownian motion explain relation between Hermite polynomials and Martingales.

(6 marks)

(c) For $i = 0, .., 2^n$, let $0 \equiv t_0 < t_{i,n} < t_{i+1,n} \leq 1$, be such that $\max_i |t_{i+1,n} - t_{i,n}| \leq 2^{-n+1}$ and the subdivision with index $n+1$ is obtained by dividing the intervals into two equal parts. For $\varphi(x) = \cos(x)$ define a sequence of random variables

$$\mathcal{S}_n \equiv \sum_{i=1,..,2^{n-1}} \varphi\left(B_{t_{i,n}}\right)(B_{t_{i+1,n}} - B_{t_{i,n}}).$$

Prove that this sequence converges in $\mathbb{L}_1(\mathbb{P})$ and a.e.

(11 marks)

(Total: 20 marks)

1. (a) The set $x = \sum_{k\in\mathbb{N}} \frac{\alpha_k}{3^k}$, with $\alpha_k \in \{0,2\}$ is the tertiary Cantor set. It is constructed inductively by dividing the interval $[0,1]$ into three intervals of equal length and removing the middle one, and applying the same principle to each of remaining intervals. That is in the n-th step we have 2^{n-1} intervals each of length 3^{-n}. The total length of removed intervals is equal to

sim. seen ⇓

$$\sum_{n\in\mathbb{N}} \frac{2^{n-1}}{3^n} = \frac{2}{3}\frac{1}{1-\frac{2}{3}} = 1.$$

This means that probability of the Cantor set is zero.

5, B

(b) The Cantor distribution is constructed as a limit of monotone nondecreasing piecewise linear continuous functions which are constant on intervals which are removed from the unit interval when constructing the tertiary Cantor set. In particular it is equal to $1/2$ on the interval $[1/3, 2/3]$. It follows from this construction that for $\beta = 1/3$ the set A_β has probability $1/2$ (with the interval $[1/3, 2/3]$ having probability zero).

unseen ⇓

7, B

(c) Let $\varphi(t) := \nu([0,t))$ where ν is a Borel measure on $\mathbb{R}^+$. Then, for a random variable $f : \Omega \to \mathbb{R}$, we have

unseen ⇓

$$\int \varphi(f)d\mu = \int \int_0^{f(\omega)} \nu(dt)d\mu$$
$$= \int\int \chi_{\{t<f(\omega)\}}\nu(dt)\mu(d\omega) = \int_0^\infty \mu(\{\omega\in\Omega : f(\omega) > t\})\nu(dt)$$

where in the last step we have used Fubini theorem. In particular for $f = |X|$ and the Borel measure

3, A

$$\nu(dt) = n\, t^{n-1}\lambda(dt),$$

we have

$$\phi(t) = t^n.$$

Hence, given the bound

$$\mu(\{|X| > t\}) \le e^{-t},$$

and using the formula for the expectation derived above, for any $n \in \mathbb{N}$, we have

$$\int |X|^n d\mu = \int_0^\infty \mu(\{\omega\in\Omega : |X|(\omega) > t\})\nu(dt) \le \int_0^\infty e^{-t}\frac{1}{n}t^{n-1}\lambda(dt) < \infty$$

5, B

2. (a) Let

$$s_n \equiv \frac{1}{n}\sum_{k=1}^{n} X_k$$

where $X_k = \cos(2\pi n_k x)$, $n_k \in \mathbb{N}$, are bounded random variables with mean zero. We need to check if the sequence s_n converges to 0 in probability. Given $\varepsilon > 0$, by Chebyshev inequality, we have

sim. seen ⇓

$$\lambda(\{|s_n| > \varepsilon\}) \leq \frac{1}{\varepsilon^2}\int_{[0,1]} s_n^2 d\lambda = \frac{1}{n^2\varepsilon^2}\sum_{j,k=1}^{n}\int_{[0,1]} \cos(2\pi n_j x)\cos(2\pi n_k x) d\lambda$$

If $n_k \neq n_j$ for $k \neq j$, the integral on the right hand side vanishes and we only have n nonzero terms in the sum. This case is the standard one and we get convergence $s_n \to 0$ in probability. In general, the right hand side may not converge to zero, e.g. if the sequence is periodic or a fraction of the n_k takes on the same value. 4, A

unseen ⇓

(b) We note that

$$\mu(A_n) = \prod_{j=n,..,2n} \nu(\{\omega_j = 1\}) = q^{n+1}$$

2, B

unseen ⇓

From this we have

4, A

$$\sum_n \mu(A_n) = \sum_n q^{n+1} = \frac{q}{1-q} < \infty.$$

Hence from Borel-Cantelli lemma the set

$$\limsup A_n \equiv \bigcap_{n\in\mathbb{N}}\bigcup_{k\geq n} A_k$$

has probability zero. 1, B

(c) We note that for any $N \in \mathbb{N}$ the set

$$\{\omega \in \Omega : \sum_{n\in\mathbb{N}} \frac{\sin(n\alpha)}{n}\pi_n(\omega) \quad converges\} = \{\omega \in \Omega : \sum_{\substack{n\in\mathbb{N}\\ n\geq N}} \frac{\sin(n\alpha)}{n}\pi_n(\omega) \quad converges\}$$

is included into the following set 2, A

$$\left\{\omega \in \Omega : \forall\varepsilon > 0\, \exists M \in \mathbb{N}\; M \geq N\; \forall n, m > M \quad \left|\sum_{k=m}^{n} \frac{\sin(k\alpha)}{k}\pi_k(\omega)\right| < \varepsilon\right\}$$

which is equal to

$$\bigcap_{\varepsilon>0}\bigcup_{M\in\mathbb{N}\; M\geq N}\bigcap_{n>m>M}\left\{\omega \in \Omega : \left|\sum_{k=m}^{n} \frac{\sin(k\alpha)}{k}\pi_k(\omega)\right| < \varepsilon\right\}.$$

(where in fact we can take ε to be rational). Next we see that for any $K \in \mathbb{N}$, $K < N$, we have 3, C

$$\bigcap_{n>m>N}\left\{\omega \in \Omega : \left|\sum_{k=m}^{n} \frac{\sin(k\alpha)}{k}\pi_k(\omega)\right| < \varepsilon\right\} \subset \bigcap_{n>m>K+1}\left\{\omega \in \Omega : \left|\sum_{k=m}^{n} \frac{\sin(k\alpha)}{k}\pi_k(\omega)\right| < \varepsilon\right\}$$

and so it is mutually independent of $\sigma(\{\pi_j : j = 1,..,K\})$. (the σ-algebra generated by the random variables $\{\pi_j : j = 1,..,K\}$). Since this can be satisfied for any $K \in \mathbb{N}$, by taking $N \in \mathbb{N}$, $N > K$, this means the set in question belongs to the tail σ-algebra and, by Kolmogorov zero-one law, it can either have probability one or zero. $\boxed{4, \text{D}}$

3. (a) Using hint we note that $\boxed{\text{unseen} \Downarrow}$

$$\left(\mathbb{E}(s_n^4)\right)^{\frac{1}{4}} = \left(\mathbb{E}\left(\frac{1}{n^\alpha}\sum_{j=1}^n e^{-j}H_j\right)^4\right)^{\frac{1}{4}}$$
$$\leq \left(\frac{1}{n^{2\alpha}}\sum_{i,j=1}^n \mathbb{E}(H_iH_j)\right)^{\frac{1}{2}}$$
$$= \left(\frac{1}{n^{2\alpha-1}}\right)^{\frac{1}{2}}.$$

That is we have $\boxed{2, \text{D}}$

$$\mathbb{E}(s_n^4) \leq \frac{1}{n^{2(2\alpha-1)}}.$$

Since

$$2(2\alpha-1) > 1 \iff \alpha > 3/4,$$

for such α we get

$$\sum_n \mathbb{E}(s_n^4) < \infty.$$

Hence, using monotone convergence theorem we have $\boxed{5, \text{A}}$

$$\mathbb{E}(\sum_n s_n^4) = \sum_n \mathbb{E}(s_n^4) < \infty.$$

Thus $\sum_n(s_n^4)$ is integrable and so almost everywhere convergent. Hence using the necessary condition for convergence of the series, we conclude that

$$\lim_{n\to\infty} s_n = 0$$

almost everywhere. $\boxed{3, \text{C}}$

(b) Define $\boxed{\text{unseen} \Downarrow}$

$$A_n \equiv \{\xi_n \neq X_n\}.$$

Then we have

$$\mu(A_n) = \mu(\{|X_n| \geq n^2\}) \leq n^{-2}\int |X_n|d\mu = n^{-2}\int |X_1|d\mu$$

where we have used Chebyshev inequality and the fact that the random variables in questions are i.i.d.. Hence we have $\boxed{6, \text{A}}$

$$\sum_n \mu(A_n) \leq \int |X_1|d\mu \sum_n n^{-2} < \infty$$

Thus by first Borel-Cantelli lemma we have

$$\mu(\limsup A_n) = 0,$$

that is the event $\{\xi_n \neq X_n\}$ happens finitely many times with probability one. $\boxed{4, \text{D}}$

4. (a) Let r_n be the Rademacher random variable on the probability space $([0,1], \Sigma_L \cap [0,1], \lambda)$. It is known that they take on values ± 1 with probability $1/2$ and that they are mutually independent. For the random variable $X_n \equiv r_n r_{n+1} r_{n+2}$ we have $\boxed{\text{unseen} \Downarrow}$ $\boxed{2, \text{A}}$

$$\varphi_{X_n}(t) = \int_{[0,1]} e^{itr_n r_{n+1} r_{n+2}} d\lambda = \int_{[0,1]} \sum_{k=0}^{\infty} \frac{(it)^k}{k!} r_n^k r_{n+1}^k r_{n+2}^k d\lambda$$
$$= \sum_{k=0}^{\infty} \frac{(it)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(it)^{2k-1}}{(2k-1)!} \int_{[0,1]} r_n^k r_{n+1}^k r_{n+2}^k d\lambda$$

Using the mutual independence and the fact that all r_n has mean value zero, we arrive at $\boxed{4, \text{C}}$

$$\varphi_{X_n}(t) = \sum_{k=0}^{\infty} \frac{(it)^{2k}}{(2k)!} = \cos(t)$$

$\boxed{2, \text{D}}$

(b) First of all we observe that by definition $X_{3k} = r_{3k} r_{3k+1} r_{3k+2}$ and for $k \neq m$, the random variables $r_{3k} r_{3k+1} r_{3k+2}$ and $r_{3m} r_{3m+1} r_{3m+2}$ are mutually independent because they involve Rademacher functions with indices different from each other, and are identically distributed as all of the r_n's have the same Bernoulli distribution. Define $\boxed{\text{unseen} \Downarrow}$ $\boxed{2, \text{C}}$

$$\mathcal{S}_n \equiv \frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_{3k}.$$

Since (X_{3k}), $k \in \mathbb{N}$, are mutually independent and identically distributed random variables, the corresponding characteristic function is given by

$$\phi_{\mathcal{S}_n}(t) = \left(\varphi_{X_1}(\frac{t}{\sqrt{n}}) \right)^n = \left(\cos(\frac{t}{\sqrt{n}}) \right)^n$$

Since for small values of $\frac{t}{\sqrt{n}}$, we have $\boxed{2, \text{A}}$

$$\cos(\frac{t}{\sqrt{n}}) = 1 - \frac{1}{2}(\frac{t}{\sqrt{n}})^2 + o((\frac{t}{\sqrt{n}})^4)$$

we get

$$\lim_{n\to\infty} \phi_{\mathcal{S}_n}(t) = \lim_{n\to\infty} \left(1 - \frac{1}{2}(\frac{t}{\sqrt{n}})^2 + o((\frac{t}{\sqrt{n}})^4) \right)^n = e^{-\frac{t^2}{2}}$$

$\boxed{2, \text{D}}$

(c) Similar arguments as in (b) apply to the family of random variables (X_{4n}) with the same characteristic functions. Since the sequence of characteristic functions in (b) converges to the characteristic function of the standard Gaussian random variable $N(0,1)$, therefore by the Lévy continuity theorem the sequence $\mathcal{S}_n$ converges in distribution. Therefore the corresponding sequence of integrals satisfies $\boxed{\text{unseen} \Downarrow}$ $\boxed{4, \text{A}}$

$$\lim_{n\to\infty} \int f(\frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_{4n}) d\mu = \int f(x) \gamma_{0,1}(dx)$$

for any bounded continuous function f. $\boxed{2, \text{D}}$

5. (a) Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{B})$ be a probability space with a filtration $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$, $0 \le s \le t$, and let $B_t : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$ be the Brownian motion adapted to the given filtration. The exponential martingale of Brownian motion B_t is defined by

$$M_\theta(t) \equiv e^{\theta B_t - \frac{1}{2}t\theta^2}.$$

seen ⇓

To check the martingale property

$$\mathbb{E}(M_\theta(t)|\mathcal{F}_s) = M_\theta(s)$$

for $0 \le s \le t$, we note that

$$M_\theta(t) = e^{\theta(B_t - B_s) + \frac{1}{2}s\theta^2 - \frac{1}{2}t\theta^2} e^{\theta B_s - \frac{1}{2}s\theta^2} = e^{\theta(B_t - B_s) + \frac{1}{2}s\theta^2 - \frac{1}{2}t\theta^2} M_\theta(s).$$

Next, since $B_t - B_s$ is independent of $\mathcal{F}_s$ while $M_\theta(s)$ is $\mathcal{F}_s$ measurable, we get

$$\mathbb{E}(M_\theta(t)|\mathcal{F}_s) = M_\theta(s)\mathbb{E}(e^{\theta(B_t - B_s) + s\theta^2 - t\theta^2}).$$

Finally using the fact that $B_t - B_s$ is Gaussian random variable with mean zero and covariance $t - s$, we have

$$\mathbb{E}(e^{\theta(B_t - B_s) + \frac{1}{2}s\theta^2 - \frac{1}{2}t\theta^2}) = e^{\frac{1}{2}(t-s)\theta^2 + \frac{1}{2}s\theta^2 - \frac{1}{2}t\theta^2} = 1.$$

□ 3, M

unseen ⇓

(b) We note that

$$G(x, \lambda) \equiv e^{\lambda x - \frac{1}{2}\lambda^2}$$

is the generating function for Hermite polynomials He_n, $n \in \mathbb{Z}^+$, i.e. we have

$$G(x, \lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} He_n(x).$$

Thus we have

$$G\left(\frac{x}{\sqrt{t}}, \sqrt{t}\theta\right) = e^{\theta x - \frac{1}{2}t\theta^2} = \sum_{n=0}^{\infty} \frac{\theta^n}{n!} \left(t^{\frac{n}{2}} He_n\left(\frac{x}{\sqrt{t}}\right)\right).$$

Hence by the exponential martingale property

$$\mathbb{E}(e^{\theta B_t - \frac{1}{2}t\theta^2}|\mathcal{F}_s) = e^{\theta B_s - \frac{1}{2}s\theta^2},$$

we have

$$\left(\partial_\theta^n \mathbb{E}(e^{\theta B_t - \frac{1}{2}t\theta^2}|\mathcal{F}_s)\right)_{|\theta=0} = \left(\partial_\theta^n e^{\theta B_s - \frac{1}{2}s\theta^2}\right)_{|\theta=0} = t^{\frac{n}{2}} He_n\left(\frac{x}{\sqrt{t}}\right),$$

and hence, using the fact that here we can interchange the differentiation and conditional expectation, we conclude that

$$\mathbb{E}\left(\left(\partial_\theta^n e^{\theta B_t - \frac{1}{2}t\theta^2}\right)_{|\theta=0} |\mathcal{F}_s\right) = \mathbb{E}\left(t^{\frac{n}{2}} He_n\left(\frac{B_t}{\sqrt{t}}\right)|\mathcal{F}_s\right) = s^{\frac{n}{2}} He_n\left(\frac{B_s}{\sqrt{s}}\right),$$

i.e. for every $n \in \mathbb{N}$, the process $t^{\frac{n}{2}} He_n\left(\frac{B_t}{\sqrt{t}}\right)$ is the martingale adapted to the filtration given in the probability space. □ 6, M

(c) For $i = 0, .., 2^n$, let $0 \equiv t_0 < t_{i,n} < t_{i+1,n} \leq 1$, be such that $\max_i |t_{i+1,n} - t_{i,n}| \leq 2^{-n+1}$ and the subdivision with index $n+1$ is obtained by dividing the intervals into two equal parts. For $\varphi(x) = \cos(x)$, let

unseen ⇓

$$\mathcal{S}_n \equiv \sum_{i=1,..,2^{n-1}} \varphi\left(B_{\{t_{i,n}\}}\right)(B_{t_{\{i+1,n\}}} - B_{t_{\{i,n\}}}).$$

If

$$t'_{\{i,n\}} \equiv t_{\{i,n\}} + \frac{t_{\{i,n+1\}} - t_{\{i,n\}}}{2}$$

is the mid point of the interval $[t_{\{i,n\}}t, t_{\{i,n+1\}}]$ we have

$$\mathcal{S}_{n+1} - \mathcal{S}_n \equiv \sum_{i=1,..,2^{n-1}} \left(\varphi\left(B_{t'_{\{i,n\}}}\right) - \varphi\left(B_{t_{\{i,n\}}}\right)\right)\left(B_{t_{\{i+1,n\}}} - B_{t'_{\{i,n\}}}\right).$$

Hence

$$\mathbb{E}\left(\mathcal{S}_{n+1} - \mathcal{S}_n\right)^2 = \sum_{i,j=1,..,2^{n-1}} \mathbb{E}\left(\left(\varphi\left(B_{t'_{\{i,n\}}}\right) - \varphi\left(B_{t_{\{i,n\}}}\right)\right)\left(\varphi\left(B_{t'_{\{j,n\}}}\right) - \varphi\left(B_{t_{\{j,n\}}}\right)\right)\right.$$
$$\left.\cdot\left(B_{t_{\{i+1,n\}}} - B_{t'_{\{i,n\}}}\right)\left(B_{t_{\{j+1,n\}}} - B_{t'_{\{j,n\}}}\right)\right)$$

If $t_{\{j+1,n\}} > t_{\{i+1,n\}}$ or $t_{\{j+1,n\}} < t_{\{i+1,n\}}$, using the independence of increments of Brownian motion of the past we get

$$\mathbb{E}\left(\left(\varphi\left(B_{t'_{\{i,n\}}}\right) - \varphi\left(B_{t_{\{i,n\}}}\right)\right)\left(\varphi\left(B_{t'_{\{j,n\}}}\right) - \varphi\left(B_{t_{\{j,n\}}}\right)\right)\left(B_{t_{\{i+1,n\}}} - B_{t'_{\{i,n\}}}\right)\left(B_{t_{\{j+1,n\}}} - B_{t'_{\{j,n\}}}\right)\right) = 0.$$

Thus

$$\mathbb{E}\left(\mathcal{S}_{n+1} - \mathcal{S}_n\right)^2 = \sum_{j=1,..,2^{n-1}} \mathbb{E}\left(\left(\varphi\left(B_{t'_{\{j,n\}}}\right) - \varphi\left(B_{t_{\{j,n\}}}\right)\right)^2 \cdot \left(B_{t_{\{j+1,n\}}} - B_{t'_{\{j,n\}}}\right)^2\right)$$
$$= \sum_{j=1,..,2^{n-1}} \mathbb{E}\left(\left(\varphi\left(B_{t'_{\{j,n\}}}\right) - \varphi\left(B_{t_{\{j,n\}}}\right)\right)^2\right) \cdot \mathbb{E}\left(\left(B_{t_{\{j+1,n\}}} - B_{t'_{\{j,n\}}}\right)^2\right)$$
$$= \sum_{j=1,..,2^{n-1}} \mathbb{E}\left(\left(\varphi\left(B_{t'_{\{j,n\}}}\right) - \varphi\left(B_{t_{\{j,n\}}}\right)\right)^2\right) \cdot \left(t_{\{j+1,n\}} - t'_{\{j,n\}}\right)$$

Since fr $\varphi(x) = cos(x)$, we have

$$\left|\cos\left(B_{t'_{\{j,n\}}}\right) - \cos\left(B_{t_{\{j,n\}}}\right)\right| \leq \left|B_{t'_{\{j,n\}}} - B_{t_{\{j,n\}}}\right|$$

we get

$$\mathbb{E}\left(\left(\varphi\left(B_{t'_{\{j,n\}}}\right) - \varphi\left(B_{t_{\{j,n\}}}\right)\right)^2\right) \leq \mathbb{E}\left(\left(B_{t'_{\{j,n\}}} - B_{t_{\{j,n\}}}\right)^2\right) = |t'_{\{j,n\}} - t_{\{j,n\}}| \leq 2^{-n}.$$

Combining all the above we get

$$\mathbb{E}\left(\mathcal{S}_{n+1} - \mathcal{S}_n\right)^2 \leq 2^{-n}$$

Hence, using monotone convergence theorem we can show that a sequence

$$\Psi_n \equiv |S_1| + \sum_{k=1}^{n-1} |\mathcal{S}_{k+1} - \mathcal{S}_k|$$

converges in $\mathbb{L}_1$ (and in fact also in $\mathbb{L}_2$) and a.s.. From this by Lebesgue dominated convergence theorem it follows that also

$$S_n = S_1 + \sum_{k=1}^{n-1} (\mathcal{S}_{k+1} - \mathcal{S}_k)$$

converges in $\mathbb{L}_1$ (and in fact also in $\mathbb{L}_2$) and a.s.. $\boxed{\text{11, M}}$

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionI	Comments for Students
MATH96035 Probability Theory	1	Generally well done
MATH97056/MATH97167 Probability Theory	2	Generally well done
	3	Generally well done
	4	Generally well done
	5	Generally well done