Mathematical Logic (MATH60132 and MATH70132)
Coursework 2

This coursework is worth 5 percent of the module. The deadline for submitting the work is 1300 on Monday 5 December 2022. The coursework is marked out of 20 and the marks per question are indicated below.
The work which you submit should be your own, unaided work. Unless stated otherwise, you may quote results from the notes or problem sheets. If you use any source (including internet or books) other than the lecture notes and problem sheets, you must provide a full reference for your source. Failure to do so could constitute plagiarism.

[1] (10 marks) Let $\mathcal{L}$ be a first-order language and let $\mathcal{F}$ denote the set of closed $\mathcal{L}$-formulas. If $\Sigma \subseteq \mathcal{F}$, let

$$c(\Sigma) = \{\phi \in \mathcal{F} : \Sigma \vdash_{K_{\mathcal{L}}} \phi\},$$

(the set of closed consequences of Σ). Prove that for all $\Sigma \subseteq \mathcal{F}$ the following hold.

(i) $c(\Sigma) \supseteq \Sigma$. **(1 mark)**

(ii) $c(\Sigma) = \mathcal{F}$ if and only if Σ is inconsistent. **(1 mark)**

(iii) $c(\Sigma) = \bigcup\{c(\Sigma_0) : \Sigma_0 \subseteq \Sigma \text{ is finite}\}$. **(1 mark)**

(iv) $(c(c(\Sigma)) = c(\Sigma)$. **(1 mark)**

(v) **(2 marks)** If Σ is consistent and $\mathcal{L}$ is countable, then

$$c(\Sigma) = \bigcap\{\mathrm{Th}(\mathcal{A}) : \mathcal{A} \text{ is a countable model of } \Sigma\}.$$

Here, $\mathrm{Th}(\mathcal{A}) = \{\psi \in \mathcal{F} : \mathcal{A} \models \psi\}$ is the theory of $\mathcal{A}$. In (i) - (iv) you should use syntactic arguments (based on the definition of $\vdash_{K_{\mathcal{L}}}$): answers for (i) - (iv) which involve structures will not receive full marks.

(vi) Suppose that $\mathcal{L}$ is countable. Show that Σ is consistent and complete if and only if there is an $\mathcal{L}$-structure $\mathcal{A}$ such that $c(\Sigma) = \mathrm{Th}(\mathcal{A})$. **(2 marks)**

(vii) **(2 marks)** Decide whether or not the following statement is always true, giving a proof or a counterexample:

If $\phi, \psi \in \mathcal{F}$ and $\psi \in c(\Sigma \cup \{\phi\}) \setminus c(\Sigma)$, then $\phi \in c(\Sigma \cup \{\psi\})$.

[2] (10 marks) Let $\mathcal{L}^=$ be a language (with equality) for rings with an ordering, having binary function symbols $+$, $\cdot$, constant symbols $0, 1$ and a 2-ary relation symbol $\leq$. Consider the real numbers as an $\mathcal{L}^=$-structure

$$\mathcal{R} = \langle \mathbb{R}; +, \cdot, 0, 1, \leq \rangle$$

in the usual way. Let $\Sigma = \mathrm{Th}(\mathcal{R})$. Let $\mathcal{L}^=_c$ consist of $\mathcal{L}^=$ with an extra constant symbol c. For $n \in \mathbb{N}$, let δ_n be the closed $\mathcal{L}^=_c$-formula

$$\underbrace{1 + \ldots + 1}_{n\, times} \leq c$$

and $\Delta = \{\delta_n : n \in \mathbb{N}\}$.

(i) **(3 marks)** Prove that $\Sigma \cup \Delta$ has a normal model.

(ii) Using (i), prove that there is an $\mathcal{L}^=$-structure $\mathcal{K} = \langle K; +, \cdot, 0_K, 1_K, \leq_K \rangle$ with all of the following properties:

(a) **(1 mark)** $\mathcal{K} \models \Sigma$ (so in particular, $\mathcal{K}$ is an ordered field).

(b) **(2 marks)** If $p(X)$ a polynomial of odd degree in the variable X with coefficients in K, then there is $a \in K$ with $p(a) = 0_K$.

(c) **(2 marks)** There is an element $d \in K$ with $n_K \leq d$ for all $n \in \mathbb{N}$ (where $n_K \in K$ is $\underbrace{1_K + \ldots + 1_K}_{n\, times}$).

(d) **(2 marks)** There is an element $e \in K$ with $0_K <_K e \leq_K n_K^{-1}$ for all $n \in \mathbb{N}$.