Imperial College
London

M4/5P6

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2017

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Probability Theory

Date: Tuesday 23 May 2017

Time: 10:00 - 12:30

Time Allowed: 2.5 Hours

This paper has 5 Questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers to each question as per the table below.

Raw Mark	Up to 12	13	14	15	16	17	18	19	20
Extra Credit	0	½	1	1 ½	2	2 ½	3	3 ½	4

- Each question carries equal weight.
- Calculators may not be used.

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1. (i) Give the definition of a random variable, its distribution and distribution function.

   (ii) Let $F(x)$ be the distribution function of a random variable. Let $a \in \mathbb{R}$. Is it possible that the limit $\lim_{x\to a} F(x)$ exists but is not equal to $F(a)$? Give a reason for your answer.

   (iii) Formulate the central limit theorem for independent identically distributed random variables.

2. (i) Give any two definitions of what it means that a sequence of random variables ζ_n converges to a random variable ζ in distribution.

   (ii) Are $f_1(t) = \cos(t/2)$, $f_2(t) = \cos^6(t)$, $f_3(t) = \exp\{-t^6\}$ characteristic functions of some random variables? In each case, if yes: (a) describe this random variable, for example, by providing its distribution; (b) find its expectation and variance.

3. (i) Let $\mathcal{P}$ be a family of probability measures on $\mathbb{R}$ which is relatively compact with respect to weak convergence. Prove that $\mathcal{P}$ is tight.

   (ii) Let ζ_n, $n = 1, 2, \ldots$, be independent Bernoulli random variables with parameters $1/n^2$, namely $P(\zeta_n = 1) = 1/n^2$, $P(\zeta_n = 0) = 1 - 1/n^2$. Prove or disprove that $\zeta_n \to 0$ a.s.

4. Let X_k, $k = 1, 2, \ldots$, be positive independent identically distributed random variables, $Y_k = \frac{X_{2k-1}}{X_{2k-1}+X_{2k}}$, and $S_n = Y_1 + \cdots + Y_n$. Is it true that $\frac{S_n}{n} \to a$ a.s. for some constant a? If yes, what is the value of a? Justify your answers.

5. (i) Let ζ_n, $n = 1, 2, \ldots$, be independent identically distributed random variables such that $E\zeta_1 = 0$, $0 < E\zeta_1^2 < \infty$. Let $S_n = \zeta_1 + \cdots + \zeta_n$. Show that $\lim_{n\to\infty} E\frac{|S_n|}{n} = 0$.

(ii) Let ζ, η be independent identically distributed random variables such that $\zeta + \eta$ has the same distribution as $\zeta + 1$. What can be said about the distribution of ζ? Justify your answer.

M345 P6 SOLUTIONS

	EXAMINATION SOLUTIONS 2016-17	Course Prob.
Question 1		Marks & seen/unseen
Parts i	Let $(\Omega, \mathcal{F}, P)$ be a probability space. A function $\xi : \Omega \to R$ is called random variable if $\xi^{-1}(B) \in \mathcal{F}$ for any $B \in \mathcal{B}(R)$ – Borel sets.	2 seen
	The distribution P_ξ of ξ is a probability measure on $(R, \mathcal{B}(R))$ given by $P_\xi(B) = P(\xi^{-1}(B))$ $\forall B \in \mathcal{B}(R)$	2 seen
	The distribution function $F_\xi(x) = P_\xi(-\infty, x]$ $\forall x \in R$	2 seen
ii	No, since distribution function is nondecreasing	6 unseen
	Setter's initials l.K.	Checker's initials	Page number 1

	EXAMINATION SOLUTIONS 2016-17	Course Prob.
Question 1		Marks & seen/unseen
Parts		
iii	Let $\zeta_1, \zeta_2, \ldots$ be i.i.d. r.v. with $0 < E\zeta_1^2 < \infty$; $V\zeta_1 \neq 0$; $S_n = \zeta_1 + \zeta_2 + \cdots + \zeta_n$.	2 seen
	Then $P\left(\frac{S_n - ES_n}{\sqrt{VS_n}} \leq x \right) \xrightarrow[n\to\infty]{} \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du$, $\forall x \in R$ i.e. $\frac{S_n - ES_n}{\sqrt{VS_n}} \xrightarrow{d} N(0,1)$.	6 seen
	Setter's initials L.K. — Checker's initials [illegible]	Page number 2

EXAMINATION SOLUTIONS 2016-17

Course: Prob

Question 2

Marks & seen/unseen

Parts

i

$\zeta_n \xrightarrow{d} \zeta$ if

$E f(\zeta_n) \to E f(\zeta)$

for any bounded continuous function $f(x)$. — 2 seen

Equivalently, if the distribution functions

$F_{\zeta_n}(x) \to F_\zeta(x)$ at any point of continuity of $F_\zeta(x)$. — 2 seen

ii

$f_1(t) = \cos t/2$ is the characteristic function of Bernoulli r.v. ζ such that — 6 unseen

$P(\zeta = 1/2) = P(\zeta = -1/2) = 1/2$

Setter's initials: I.K.

Checker's initials: [illegible]

	EXAMINATION SOLUTIONS 2016-17	Course Prob
Question 2		Marks & seen/unseen
Parts	$E\xi = 0$, $V\xi = E\xi^2 = 1/4$. $f_2(t) = \cos^6 t$ is the characteristic function of the sum of 6 i.i.d. Bernoulli r.v. such that $P(\xi=1) = P(\xi=-1) = 1/2$; $E(\sum_{j=1}^{6} \xi_j) = 0$, $V(\sum_{j=1}^{6} \xi_j) = \sum_{j=1}^{6} V\xi_j =$ $= 6 \cdot 1 = 6$ (by independence) $f_3(t) = e^{-t^6}$ is not a characteristic function by Marcinkiewicz theorem.	6 unseen 4 unseen
	Setter's initials I.K. Checker's initials	Page number 4

	EXAMINATION SOLUTIONS 2016-17	Course Prob
Question 3		Marks & seen/unseen
Parts i	Suppose $\mathcal{P}$ is not tight, i.e. $\exists\, \varepsilon > 0$ s.t. for any compact $K \subset \mathbb{R}$, $\sup_{\alpha} P_\alpha(\mathbb{R} \setminus K) > \varepsilon$, where $\mathcal{P} = \{P_\alpha\}$. Take $K = [-n, n]$. Therefore, for any n $\exists\, P_{\alpha_n} \in \mathcal{P}$ s.t. $P_{\alpha_n}(\mathbb{R} \setminus (-n, n)) > \varepsilon$. (1) By relative compactness, there exists a subsequence $P_{\alpha_{n_k}}$ converging weakly to a probability measure Q.	10 seen
	Setter's initials: L.K. Checker's initials: [illegible]	Page number 5

By one of the definitions of weak convergence,

$$\limsup P_{d_{n_k}}(R \setminus (-n,n)) \le$$
$$\le Q(R \setminus (-n,n))$$

But $Q(R \setminus (-n,n)) \to 0$ as $n \to \infty$, which contradicts inequality (1).

Therefore, $\mathcal{P}$ is tight.

	EXAMINATION SOLUTIONS 2016-17	Course Prob
Question 3		Marks & seen/unseen
Parts		
ii	We have that $$\sum_{n=1}^{\infty} P(\zeta_n = 1) = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty .$$ Therefore, by Borel-Cantelli lemma, $P(\zeta_n = 1 \ i.o.) = 0$ The probability of the complement event $P(\zeta_n = 0 \ ev.) = 1$ i.e. $\zeta_n \to 0$ a.s.	10 unseen
	Setter's initials I.K.	Checker's initials
		Page number 7

	EXAMINATION SOLUTIONS 2016-17	Course Prob.
Question 4		Marks & seen/unseen

Parts

1) y_k's are identically distributed, indeed

$$P(y_k \in B) = \int \chi_{\{\frac{t}{t+s} \in B\}} dP_{(X_{2k-1}, X_{2k})}(t,s)$$

$$= \int \chi_{\{\frac{t}{t+s} \in B\}} dP(t) dP(s)$$

since X_k's are independent, identically distributed.

2) y_k's are independent, since, similarly,

$$P(y_k \in B_k, y_j \in B_j) =$$

$$= \int \chi_{\{\frac{t}{t+s} \in B_k\}} \chi_{\{\frac{u}{u+v} \in B_j\}} dP_{(X_{2k-1}, X_{2k}, X_{2j-1}, X_{2j})}$$

$$= P(y_k \in B_k) P(y_j \in B_j),$$

$k \neq j$.

Setter's initials: l.K. Checker's initials: [illegible] Page number 8

	EXAMINATION SOLUTIONS 2016-17	Course Prob
Question 4		Marks & seen/unseen
Parts	3) $E\lvert Y_k\rvert = E\left\lvert \frac{X_{2k-1}}{X_{2k-1}+X_{2k}} \right\rvert \leq$ $\leq E(1) = 1 < \infty$	unseen
	Therefore, by Kolmogorov's strong LLN, $S_n/n \longrightarrow EY_1$ a.s.	12
	$2EY_1 = 2\int \frac{t}{t+s}\, dP(t)\, dP(s)$ $= \int \frac{t}{t+s}\, dP(t)\, dP(s) + \int \frac{s}{t+s}\, dP(t)\, dP(s) = 1$ $\Rightarrow EY_1 = 1/2$.	8
	Setter's initials I.K. Checker's initials	Page number 9

EXAMINATION SOLUTIONS 2016-17

Course: Prob, Masters q.

Question 5

Marks & seen/unseen

Parts

i

$\zeta_1, \zeta_2, \dots$ i.i.d.

$E\zeta_1 = 0, \quad 0 < E\zeta_1^2 < \infty$.

$S_n = \zeta_1 + \dots + \zeta_n$

By Lyapunov inequality,

$$\left(E\frac{|S_n|}{n}\right)^2 \le E\frac{S_n^2}{n^2} =$$

$$= \frac{1}{n^2} E\left(\sum_{j=1}^n \zeta_j^2 + \sum_{j \ne k} \zeta_j \zeta_k\right)$$

$$= \frac{1}{n^2}\left(n E\zeta_1^2 + \sum_{j\ne k} E\zeta_j \cdot E\zeta_k\right)$$

(by independence)

$$= \frac{1}{n} E\zeta_1^2, \quad (\text{since } E\zeta_j = 0)$$

$$\to 0, \text{ as } n \to \infty$$

Thus, $E\frac{|S_n|}{n} \to 0, \ n \to \infty$. 12 unseen

Setter's initials: I.K.

Checker's initials: [illegible]

Parts ii

We have for the characteristic functions:

$$\varphi_\xi^2(t) = \varphi_{\xi+\eta}(t) = \varphi_{\xi+1}(t)$$
$$= e^{it}\varphi_\xi(t).$$

Therefore for any t, either $\varphi_\xi(t) = e^{it}$ or $\varphi_\xi(t)=0$. But since $\varphi_\xi(t)$ is a characteristic function, $\varphi_\xi(0)=1$, and since $|e^{it}|=1$, continuity of characteristic function implies that $\varphi_\xi(t)=e^{it}$ $\forall t$. Hence, by uniqueness of characteristic functions, $\xi = 1$ a.s.

8

unseen

P6.

Imperial College London
Department of Mathematics

Examiner's Comments

Exam: _________ Probability_____________________ Session: 2016-2107

Question 1

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

Generally well done

Marker: ____________________________

Signature:__________________________ **Date:** ____________________

Please return with exam marks (one report per marker)

Imperial College London
Department of Mathematics

Examiner's Comments

Exam: ________ Probability______________________ Session: 2016-2107

Question 2

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

There were sometimes difficulties
with part 2(ii)

Marker: ____________________________

Signature:__________________________ **Date:** ____________________

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Department of Mathematics

Examiner's Comments

Exam: ______ **probability**________________________
Session: 2016-2107

Question 3

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

Part 3(i) is the class material.
Some difficulties with part 3(ii)

Marker: ______________________________

Signature:____________________________ **Date:** ______________________

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Imperial College London
Department of Mathematics

Examiner's Comments

Exam: _____ probability_________________________
Session: 2016-2107

Question 4

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

Often the value of a was guessed,
a complete proof was rarely done.

Marker: ______________________________

Signature:______________________________ **Date:** ______________________

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Imperial College London
Department of Mathematics

Examiner's Comments

Exam: _____ probability__________________________
Session: 2016-2107

Question 5

Please use the space below to comment on the candidates' overall performance in the exam. A brief paragraph highlighting common mistakes and parts of questions done badly (or well) is sufficient. Do not refer to individual candidates. The purpose of this exercise is to provide guidance to the external examiners, and to the candidates themselves, on how you feel the cohort faired. Your comments will be available to students online.

This was less well done than the other questions

Marker: ______________________________

Signature:____________________________ **Date:** ______________________

Please return with exam marks (one report per marker)