(2.2.9) (ii)(c): v satisfies $(\forall x_\ell)\phi$ (in $\mathcal{A}$) iff whenever w is a valuation (in $\mathcal{A}$) which is x_ℓ-equivalent to v, then w satisfies ϕ.

// Uses:

(2.2.8) Def: Suppose v, w are valuations in an $\mathcal{L}$-str. $\mathcal{A}$ and x_ℓ is a variable. Say that v, w are x_ℓ-equivalent if $v(x_m) = w(x_m)$ when $m \neq \ell$.

[Remark: Def. 2.2.9 does not work if we allow the domain of $\mathcal{A}$ to be empty: there are no valuations.]

// If every valuation in $\mathcal{A}$ satisfies ϕ, say that ϕ is true in $\mathcal{A}$ or $\mathcal{A}$ is a model of ϕ & write $\mathcal{A} \models \phi$.

If $\mathcal{A} \models \phi$ for every $\mathcal{L}$-str. say that ϕ is logically valid and write $\models \phi$.

These are the analogues of the tautologies in prop. logic.

(2.2.10) Examples. (of using the terminology)

1) Suppose $\mathcal{L}$ has a 2-ary rel. symbol R. The $\mathcal{L}$-formula:

$$\Big(R(x_1, x_2) \to \big((R(x_2, x_3) \to R(x_1, x_3))\big)\Big)$$

is true in

$$\mathcal{A} = \langle \underbrace{\mathbb{N}}_{\text{domain}} ; \underbrace{<}_{R \text{ interpreted as } ``<"} \rangle.$$

2) The same formula is not true in the str.

$$\mathcal{B} = \langle \mathbb{N} ; \neq \rangle$$

~~(so $R(x$ is interp~~

(so $R(x_1, x_2)$ is interpreted as "$x_1 \neq x_2$")

Eg. let v be a valuation with $v(x_1) = 1 = v(x_3)$ and $v(x_2) = 2$.

then (using 2.2.9)

$$v[R(x_1, x_2)] = T$$

$$v[R(x_2, x_3)] = T$$

$$v[R(x_1, x_3)] = F.$$

So $v[\text{formula}] = F$.

3) Recall $(\exists x_1)\phi$ is shorthand for
$$(\neg(\forall x_1)(\neg\phi))$$

<u>Lemma</u>. Suppose $\mathcal{A}$ is an $\mathcal{L}$-str. and v is a valuation in $\mathcal{A}$. Then v satisfies $(\exists x_1)\phi$ (in $\mathcal{A}$) iff there is a valuation w which is x_1-equivalent to v and where $w[\phi] = T$.

<u>Pf</u>: <u>$\Rightarrow$</u>: Suppose v satisfies $(\exists x_1)\phi$ (in $\mathcal{A}$)

ie. v satisfies $(\neg(\forall x_1)(\neg\phi))$.

$\therefore$ By 2.2.9 we have
$$v[(\forall x_1)(\neg\phi)] = F.$$

Again (by 2.2.9(ii)(c)) there is a valuation w x_1-equiv. to v with
$$w[(\neg\phi)] = F.$$

But then, for this w,
$$w[\phi] = T,$$
as req^d. //.

$\Leftarrow$: Ex. #.

(2.2.11) <u>Example</u>:
$$(\forall x_1)(\exists x_2)R(x_1, x_2)$$
is true in $\langle \mathbb{Z}; < \rangle$ but not in $\langle \mathbb{N}; > \rangle$

(2.2.12) Suppose ϕ is any $\mathcal{L}$-fmla. Then

(1) $\left((\exists x_1)(\forall x_2)\phi \rightarrow (\forall x_2)(\exists x_1)\phi \right)$

is logically valid.

(2) $\left((\forall x_2)(\exists x_1)\phi \rightarrow (\exists x_1)(\forall x_2)\phi \right)$

is not necessarily logically valid.

//

(1) Ex. (using valuations).

(2) Give an example.

Some logically valid formulas. 4

Consider the ~~pr~~ propositional formula

$\chi \qquad (p_1 \rightarrow (p_2 \rightarrow p_1))$

Suppose $\mathcal{L}$ is 1^{st} order language and

ϕ_1, ϕ_2 are $\mathcal{L}$-fmlas.

Substitute $\left.\begin{array}{l}\phi_1 \text{ in place of } p_1 \\ \phi_2 \text{ in place of } p_2\end{array}\right)$ in χ

The ~~res~~ resulting:

$\theta \qquad (\phi_1 \rightarrow (\phi_2 \rightarrow \phi_1))$

is an $\mathcal{L}$-formula and

θ is logically valid.

Suppose v is a valuation in an $\mathcal{L}$-str. $\mathcal{A}$.

Suppose for a ↯ (contradiction)

that $v[\theta] = F$.

then $v[\phi_1] = T$ & $v[(\phi_2 \to \phi_1)] = F$

so $v[\phi_2] = T$

& $v[\phi_1] = F$.

Contradiction.