SOLUTIONS TO EXERCISES 4

Solution 4.1. This follows from the standard Bayes rule:

$$\begin{aligned} p(x|y,z) &= \frac{p(x,y,z)}{p(y,z)}, \\ &= \frac{p(y|x,z)p(x|z)\cancel{p(z)}}{p(y|z)\cancel{p(z)}}, \\ &= \frac{p(y|x,z)p(x|z)}{p(y|z)}. \end{aligned}$$

Solution 4.2. This is just writing down the Bayes theorem and then using the definition of conditional independence: $p(y,x|z) = p(y|z)p(x|z)$

$$p(x|y,z) = \frac{p(y,x|z)}{p(y|z)} = \frac{p(y|z)p(x|z)}{p(y|z)} = p(x|z).$$

Same derivation is true if we swap x and y.

Solution 4.3. We have

$$\sup_{x' \in \mathbb{R}} \frac{p_0(x')}{q_0(x')} = K < \infty$$

which means that the following holds

$$\begin{aligned} M = \sup_{x \in \mathbb{R}^d} \frac{p(x)}{q(x)} &= \sup_{x \in \mathbb{R}^d} \frac{\prod_{i=1}^d p_0(x_i)}{\prod_{i=1}^d q_0(x_i)} \\ &= K^d. \end{aligned}$$

This means the acceptance rate

$$\frac{1}{M} = \frac{1}{K^d},$$

converges to zero as $d \to \infty$.

Let us now consider the Gaussian case. Note that, we have

$$p(x) = \mathcal{N}(x; 0, \sigma_p^2 I) = \frac{1}{(2\pi)^{d/2}\sigma_p^d} \exp\left(-\frac{1}{2\sigma_p^2}\sum_{i=1}^d x_i^2\right),$$

and similarly

$$q(x) = \mathcal{N}(x; 0, \sigma_q^2 I) = \frac{1}{(2\pi)^{d/2}\sigma_q^d} \exp\left(-\frac{1}{2\sigma_q^2}\sum_{i=1}^d x_i^2\right).$$

We compute

$$\frac{p(x)}{q(x)} = \frac{\sigma_q^d}{\sigma_p^d} \exp\left(-\frac{1}{2\sigma_p^2}\sum_{i=1}^d x_i^2 + \frac{1}{2\sigma_q^2}\sum_{i=1}^d x_i^2\right).$$

therefore we have

$$M = \sup_{x \in \mathbb{R}^d} \frac{p(x)}{q(x)} = \frac{\sigma_q^d}{\sigma_p^d} = \left(\frac{\sigma_p}{\sigma_q}\right)^d.$$

Note that for rejection sampling, we had to assume $\sigma_q > \sigma_p$ this means $\sigma_q/\sigma_p > 1$, which means that $M \to \infty$ as $d \to \infty$.

Solution 4.4. We want to estimate

$$\lambda = \mathbb{P}(0 < X < 2) = \int_0^2 \frac{1}{\pi(1+x^2)} \mathrm{d}x.$$

We can choose a uniform on $[0, 2]$, i.e.,

$$p(x) = \frac{1}{2} \qquad \text{for } x \in (0, 2),$$

and

$$\varphi(x) = \frac{2}{\pi(1+x^2)}.$$

The variance of the Monte Carlo estimate is given by

$$\begin{aligned}\frac{\mathsf{var}_p(\varphi)}{N} &= \frac{1}{N}\left(\mathbb{E}[\varphi^2(X)] - \mathbb{E}[\varphi(X)]^2\right) \\ &= \frac{1}{N}\left(\frac{4}{\pi^2}\frac{1}{2}\int_0^2 \frac{1}{(1+x^2)^2}\mathrm{d}x - (\frac{1}{2} - 0.1476)^2\right),\end{aligned}$$

where the last line uses $\lambda = \frac{1}{2} - I$ and note $\mathbb{E}[\varphi(X)] = \lambda$. Computing this, we arrive at

$$\frac{\mathsf{var}_p(\varphi)}{N} = \frac{0.0285}{N},$$

almost half of the best exampled discussed during the course.

The variance is this estimate determines the variance of I, of course, as they are connected up to a constant and addition/substraction of constants will not change variance.

Solution 4.5. The code is given as follows.

```
import numpy as np
import matplotlib.pyplot as plt

def phi(x):
    return np.sqrt((1 - x**2))

I = np.pi / 4

N_max = 100000

U = np.random.uniform(0, 1, N_max)
I_est = np.zeros(N_max - 1)
I_var = np.zeros(N_max - 1)
I_var_correct = np.zeros(N_max - 1)

fig = plt.figure(figsize=(10, 5))

K = np.array([])
k = 0

for N in range(1, N_max, 1):
    # print(N)

    I_est[k] = (1/N) * np.sum(phi(U[0:N]))
    I_var[k] = (1/(N**2)) * np.sum((phi(U[0:N]) - I_est[k])**2)
```

```

    k = k + 1

    K = np.append(K, N)

    if (N-1) % 200 == 0:
        plt.clf()
        plt.semilogx(K, I_est[0:k], 'k-', label='MC estimate')
        plt.plot(K, I_est[0:k] + np.sqrt(I_var[0:k]), 'r', label='${\\sigma}_{\\varphi,N}$', alpha=1)
        plt.plot(K, I_est[0:k] - np.sqrt(I_var[0:k]), 'r', alpha=1)
        plt.plot([0, N_max], [I, I], 'b--', label='True Value', alpha=1, linewidth=2)
        plt.legend()
        plt.xlabel('Number of samples')
        plt.ylabel('Estimate')
        plt.xlim([0, N_max])
        plt.show(block=False)
        plt.pause(0.01)
```