Imperial College London

MATH97084 MATH70046

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Time Series Analysis

Date: 09 May 2022

Time: 09:00 – 11:30 (BST)

Time Allowed: 2:30 hours

Upload Time Allowed: 30 minutes

This paper has 5 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS SEPARATE PDFs TO THE RELEVANT DROPBOXES ON BLACKBOARD (ONE FOR EACH QUESTION) WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

Note: Throughout this paper $\{\epsilon_t\}$ is a sequence of uncorrelated random variables (white noise) having zero mean and variance σ_ϵ^2, unless stated otherwise. The term "stationary" will always be taken to mean second-order stationary. All processes are real-valued unless stated otherwise. The sample interval is unity unless stated otherwise. $\Delta = 1 - B$ denotes the difference operator, where B denotes the backward shift operator.

Throughout this paper you may state without proof any results you take from the lecture notes.

1. (a) Let $\{X_t\}$ be a stationary process with mean $\mu_X \neq 0$ and autocovariance sequence $\{s_{X,\tau}\}$. Furthermore, let white noise process $\{\epsilon_t\}$ be *independent* of $\{X_t\}$. Determine whether each of the following models for a random process $\{Y_t\}$ is stationary, justifying your answer. For those that are stationary, give the mean and autocovariance sequence in terms of μ_X, $s_{X,\tau}$ and σ_ϵ^2.

   (i) $Y_t = \alpha X_t + \beta \epsilon_t$ where α and β are non-zero constants. (2 marks)

   (ii) $Y_t = (-1)^t X_t - \mu_X$. (2 marks)

   (iii) $Y_t = \epsilon_t X_t$. (2 marks)

   (b) A complex valued process $\{Z_t\}$, defined via $Z_t = X_{R,t} + \mathrm{i}X_{I,t}$ is stationary if its real and imaginary parts $\{X_{R,t}\}$ and $\{X_{I,t}\}$, respectively, are jointly stationary real-valued processes. Show the complex valued process $\{Z_t\}$, defined as $Z_t = \epsilon_t \mathrm{e}^{\mathrm{i}t}$, is non-stationary. (3 marks)

   (c) (i) Show that if $\{s_\tau\}$ is the autocovariance sequence for a stationary process, then for any $c \geq 0$, the sequence $\{s_\tau^2 + c s_\tau\}$ is the autocovariance sequence for some stationary process.
   HINT: Consider forming the product $W_t = X_t Y_t$ of two stationary processes $\{X_t\}$ and $\{Y_t\}$ with the same autocovariance sequence but different means. (4 marks)

   (ii) Show that, when $c < 0$, the sequence $\{s_\tau^2 + c s_\tau\}$ need not be a valid autocovariance sequence. (3 marks)

   (iii) By considering the MA(1) process $X_t = \epsilon_t + \frac{1}{2}\epsilon_{t-1}$ with $\sigma_\epsilon^2 = 1$, show the autocovariance sequence which takes a value of $45/16$ at $\tau = 0$, $3/4$ at $|\tau| = 1$ and zero for all other $\tau \in \mathbb{Z}$ is a valid autocovariance sequence. (4 marks)

(Total: 20 marks)

2. (a) Show whether the following processes $\{X_t\}$ are stationary and/or invertible.

(i) $X_t = \epsilon_t + \frac{16}{9}\epsilon_{t-2}$. (2 marks)

(ii) $X_t = \frac{5}{2}X_{t-1} - X_{t-2} + \epsilon_t$. (2 marks)

(iii) $X_t = \frac{1}{4}X_{t-1} + \epsilon_t - \frac{5}{6}\epsilon_{t-1} + \frac{1}{6}\epsilon_{t-2}$. (2 marks)

(b) Let $\{X_t\}$ be the zero mean stationary and invertible ARMA(1,1) defined as

$$X_t = \frac{1}{4}X_{t-1} + \epsilon_t - \frac{1}{3}\epsilon_{t-1}.$$

(i) Express $\{X_t\}$ in General Linear Process form

$$X_t = \sum_{k=0}^{\infty} \psi_k \epsilon_{t-k}.$$

(4 marks)

(ii) Show that the l-step ahead forecast $X_t(l)$ is given as

$$X_t(l) = -\frac{1}{12} \cdot \frac{1}{4^{l-1}} \sum_{k=0}^{\infty} \frac{1}{3^k} X_{t-k}.$$

(4 marks)

(c) In general, the l-step ahead forecast for a zero mean stationary and invertible ARMA(p,q) model can be represented as $X_t(l) = \sum_{k=0}^{\infty} \pi_k X_{t-k}$. Suppose we only have N values with which to forecast and so instead use the truncated version $\tilde{X}_t(l) = \sum_{k=0}^{N-1} \pi_k X_{t-k}$, show the prediction variance $\tilde{\sigma}^2(l) = E\{(X_{t+l} - \tilde{X}_t(l))^2\}$ is

$$\tilde{\sigma}^2(l) = \sigma^2(l) + \sum_{j=N}^{\infty} \sum_{k=N}^{\infty} \pi_j \pi_k s_{|j-k|},$$

where $\sigma^2(l) = E\{(X_{t+l} - X_t(l))^2\}$ is the prediction variance associated with $X_t(l)$ and $\{s_\tau\}$ is the autocovariance sequence for $\{X_t\}$. (6 marks)

(Total: 20 marks)

3. (a) The spectral representation of a zero mean stationary process $\{X_t\}$ is given as

$$X_t = \int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}2\pi f t} \mathrm{d}Z(f).$$

where $\{Z(f)\}$ is an orthogonal increment process, defined on the interval $[-1/2, 1/2]$, with the following properties:

* $E\{\mathrm{d}Z(f)\} = 0, \ \forall \ |f| \leq 1/2.$
* $E\{|\mathrm{d}Z(f)|^2\} \equiv \mathrm{d}S^{(I)}(f)$, say, $\forall \ |f| \leq 1/2$, where $S^{(I)}(f)$ is called the integrated spectrum of $\{X_t\}$.
* For any two distinct frequencies f and $f' \in (-1/2, 1/2]$

$$\mathrm{Cov}\{\mathrm{d}Z(f'), \mathrm{d}Z(f)\} = E\{\mathrm{d}Z^*(f')\mathrm{d}Z(f)\} = 0.$$

Assume $S^{(I)}(\cdot)$ is differentiable; in this case we say it has derivative $S(\cdot)$ and we have the relationship $S^{(I)}(f) = \int_{-1/2}^{f} S(f')\mathrm{d}f'$. Furthermore, it can be shown that $S(f)$ and $\{s_\tau\}$, the symmetric autocovariance sequence for $\{X_t\}$, form a Fourier transform pair, i.e.

$$S(f) = \sum_{\tau=-\infty}^{\infty} s_\tau \mathrm{e}^{-\mathrm{i}2\pi f\tau}, \qquad s_\tau = \int_{-1/2}^{1/2} S(f)\mathrm{e}^{\mathrm{i}2\pi f\tau}\mathrm{d}f.$$

Using only the above properties and results, prove the following six statements.

(i) $S^{(I)}(-1/2) = 0.$ (2 marks)

(ii) $S^{(I)}(1/2) = s_0.$ (2 marks)

(iii) $S(f) \geq 0.$ (2 marks)

(iv) $f < f'$ implies $S^{(I)}(f) \leq S^{(I)}(f').$ (2 marks)

(v) $S(f) = S(-f).$ (2 marks)

(vi) For any $0 \leq f_0 \leq 1/2$, we must have $S^{(I)}(-f_0) + S^{(I)}(f_0) = s_0.$ (2 marks)

(b) Let $\{X_t\}$ be the stationary AR(1) process $X_t = \frac{3}{4}X_{t-1} + \epsilon_t$, where $\sigma_\epsilon^2 = 1$.

(i) Let $\{Y_t\}$ be the process $Y_t = X_t + \eta_t$ where $\{\eta_t\}$ is a white noise process with variance σ_η^2 and X_t is uncorrelated with $\eta_{t'}$ for all $t, t' \in \mathbb{Z}$. Determine the spectral density function for $\{Y_t\}$.

(3 marks)

(ii) Let $\{W_t\}$ be the ARMA(1,1) process $W_t = \frac{3}{4}W_{t-1} + \xi_t - \theta\xi_{t-1}$, where $\{\xi_t\}$ is a white noise process with variance σ_ξ^2. Letting $\sigma_\eta^2 = 1$, show $\{Y_t\}$ and $\{W_t\}$ have the same spectral density function provided $\theta = 3/(4\sigma_\xi^2)$, where σ_ξ^2 is the solution to the equation

$$\sigma_\xi^2\left(1 + \frac{9}{16\sigma_\xi^4}\right) = \frac{41}{16}.$$

(5 marks)

(Total: 20 marks)

4. (a) Consider the direct spectral estimator formed from the portion $X_1, ..., X_N$ of a stationary process $\{X_t\}$, given as

$$\hat{S}^{(d)}(f) = \left|\sum_{t=1}^{N} h_t X_t \mathrm{e}^{-\mathrm{i}2\pi f t}\right|^2,$$

where $h_1, ..., h_N$ is a data taper such that $\sum_{t=1}^{N} h_t^2 = 1$.

(i) Considering the spectral representation of $\{X_t\}$, show

$$E\left\{\hat{S}^{(d)}(f)\right\} = \int_{-1/2}^{1/2} \mathcal{H}(f - f')S(f')\mathrm{d}f',$$

where $\mathcal{H}(f) = |H(f)|^2$, with $H(f) = \sum_{t=1}^{N} h_t \mathrm{e}^{-\mathrm{i}2\pi f t}$.

(3 marks)

(ii) Show $\mathcal{H}(f)$ is periodic with period 1. (2 marks)

(iii) Let $X_1, ..., X_N$ be a portion of length N of a *white noise process* $\{X_t\}$ with variance σ^2. Show in this case that $\hat{S}^{(d)}(f)$ is an unbiased estimator of the spectral density function of $\{X_t\}$. HINT: you may use Parseval's formula

$$\int_{-1/2}^{1/2} |H(f)|^2 \mathrm{d}f = \sum_{t=1}^{N} h_t^2.$$

(2 marks)

(iv) Explain the benefits of tapering the time series when forming a direct spectral estimate. Your answer should refer to the *periodogram*, *bias*, *sidelobe-leakage* and *Fejér's kernel*.

(5 marks)

(b) (i) Given a portion $X_1, ..., X_N$ of a stationary process $\{X_t\}$, the "unbiased" estimator for the autocovariance sequence is given as

$$\hat{s}_\tau^{(u)} = \frac{1}{N - |\tau|} \sum_{t=1}^{N-|\tau|} (X_t - \bar{X})(X_{t+\tau} - \bar{X}),$$

where $\bar{X}$ is the sample mean of $X_1, ..., X_N$.
By considering the realisation $x_1 = -1$, $x_2 = 0$ and $x_3 = 1$, show that $\{\hat{s}_\tau^{(u)}\}$ may not always be non-negative definite. (4 marks)

(ii) The periodogram is formed by taking the Fourier transform of the "biased" estimator $\{\hat{s}_\tau^{(p)}\}$. If we instead take the Fourier transform of $\{\hat{s}_\tau^{(u)}\}$, show this may give negative values for the estimated spectral density function. (4 marks)

(Total: 20 marks)

5. **You may use without proof any results given in your notes and the Mastery Material.**

In addition, you may use the following version of Isserlis's Theorem for part (a). If Z_1, Z_2, Z_3 and Z_4 are four complex valued random variables with zero means, then

$$\text{Cov}\{Z_1 Z_2, Z_3 Z_4\} = \text{Cov}\{Z_1, Z_3\}\text{Cov}\{Z_2, Z_4\} + \text{Cov}\{Z_1, Z_4\}\text{Cov}\{Z_2, Z_3\}.$$

Recall: for a pair of zero mean complex random variables S and T, $\text{Cov}\{S, T\} = E\{S^* T\}$, where * denotes complex conjugation.

QUESTION 5 CONTINUES ON NEXT PAGE.

(a) Let $G_0, ..., G_{N-1}$ be a portion of the Gaussian white noise process $\{G_t\}$ with variance σ^2. Let

$$I(f_k) = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} G_t \mathrm{e}^{-\mathrm{i}2\pi f_k t} \qquad \text{and} \qquad J(f_k) = \sum_{t=0}^{N-1} h_t G_t \mathrm{e}^{-\mathrm{i}2\pi f_k t},$$

where $f_k = k/N$ are the Fourier frequencies satisfying $0 \le f_k \le 1/2$, where k is an integer. The data taper here is $h_t = C[1 - \cos(2\pi t)/N]$ with C chosen such that $\sum_{t=0}^{N-1} h_t^2 = 1$.

(i) Show $\mathrm{Cov}\{I(f_k), I(f_j)\} = \mathrm{Cov}\{I(f_k), I^*(f_j)\} = 0$ for all $f_j \neq f_k$.
HINT: Consider writing $I(f_k) = A(f_k) + \mathrm{i}B(f_k)$ and using results provided in the Mastery Material. (3 marks)

(ii) Use Isserlis's Theorem to show $\mathrm{Corr}\{\hat{S}_G^{(p)}(f_j), \hat{S}_G^{(p)}(f_k)\} = 0$ for all $f_j \neq f_k$. (3 marks)

(iii) You may take without proof that for this taper

$$J(f_k) = CN^{1/2}\left[-\frac{1}{2}I(f_{k-1}) + I(f_k) - \frac{1}{2}I(f_{k+1})\right].$$

Using this, show

$$\mathrm{Cov}\{J(f_k), J(f_{k+\tau})\} = \begin{cases} 3C^2N\sigma^2/2 & \tau = 0 \\ -C^2N\sigma^2 & \tau = 1; \\ C^2N\sigma^2/4 & \tau = 2; \\ 0 & \tau = 3, \end{cases}$$

for $1 < k < \lfloor N/2 \rfloor - 3$. (5 marks)

(iv) Given the result in 5(a)(iii) holds for $\mathrm{Cov}\{J^*(f_k), J^*(f_{k+\tau})\}$, $\mathrm{Cov}\{J(f_k), J^*(f_{k+\tau})\}$ and $\mathrm{Cov}\{J^*(f_k), J(f_{k+\tau})\}$, show

$$\mathrm{Corr}\left\{\hat{S}_G^{(d)}(f_k), \hat{S}_G^{(d)}(f_{k+\tau})\right\} = \begin{cases} 4/9 & \tau = 1; \\ 1/36 & \tau = 2; \\ 0 & \tau = 3, \end{cases}$$

for $1 < k < \lfloor N/2 \rfloor - 3$, where $\hat{S}_G^{(d)}(\cdot)$ is the direct spectral estimator that uses the taper $\{h_t\}$ as defined above. (4 marks)

(b) Explain the core principles and benefits of multi-tapering. Your answer should refer to the variance and bias of a multitaper estimator, its asymptotic distribution and how multi-tapering affects the confidence intervals.

(5 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2022

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MATH60046/70046/97084

Time Series Analysis (Solutions)

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1. (a) (i) First check the mean: $E\{Y_t\} = \alpha E\{X_t\} + \beta E\{\epsilon_t\} = \alpha \mu_X$, which is constant in time. sim. seen ⇓

Now check the autocovariance sequence: $\text{cov}\{Y_t, Y_{t+\tau}\} = \text{cov}\{\alpha X_t + \beta \epsilon_t, \alpha X_{t+\tau} + \beta \epsilon_{t+\tau}\} = \alpha^2 \text{cov}\{X_t, X_{t+\tau}\} + \alpha\beta \text{cov}\{X_t, \epsilon_{t+\tau}\} + \alpha\beta \text{cov}\{\epsilon_t, X_{t+\tau}\} + \beta^2 \text{cov}\{\epsilon_t, \epsilon_{t+\tau}\} = \alpha^2 \text{cov}\{X_t, X_{t+\tau}\} + \beta^2 \text{cov}\{\epsilon_t, \epsilon_{t+\tau}\}$, due to the independence of $\{X_t\}$ and $\{\epsilon_t\}$. This will depend only on τ because $\{X_t\}$ and $\{\epsilon_t\}$ are themselves stationary and we have

$$s_\tau = \begin{cases} \alpha^2 s_{X,\tau} + \beta^2 \sigma_\epsilon^2 & \tau = 0 \\ \alpha^2 s_{X,\tau} & \tau \neq 0. \end{cases}$$

2, A

(ii) First check the mean: $E\{Y_t\} = (-1)^t \mu_X - \mu_X$. This equals $-2\mu_X$ when t is odd, and equals 0 when t is even. Therefore $E\{Y_t\}$ depends on time and $\{Y_t\}$ is non-stationary. 2, A

(iii) First check the mean:
by independence $E\{Y_t\} = E\{X_t \epsilon_t\} = E\{X_t\}E\{\epsilon_t\} = 0$. Now check the autocovariance sequence: $\text{cov}\{Y_t, Y_{t+\tau}\} = E\{Y_t Y_{t+\tau}\} = E\{X_t \epsilon_t X_{t+\tau} \epsilon_{t+\tau}\} = E\{X_t X_{t+\tau}\} E\{\epsilon_t \epsilon_{t+\tau}\}$. We have $E\{X_t X_{t+\tau}\} = \text{cov}\{X_t, X_{t+\tau}\} + \mu_X^2$ and $E\{\epsilon_t \epsilon_{t+\tau}\} = \sigma_\epsilon^2 \delta_{\tau,0}$. Therefore

$$s_\tau = \begin{cases} \sigma_\epsilon^2 (s_{X,\tau} + \mu_X^2) & \tau = 0 \\ 0 & \tau \neq 0, \end{cases}$$

which depends only on τ, hence $\{Y_\tau\}$ is stationary. 2, B

(b) We can write $Z_t = X_{R,t} + \mathrm{i} X_{I,t}$ where $X_{R,t} = \epsilon_t \cos(t)$ and $X_{I,t} = \epsilon_t \sin(t)$. To show $\{Z_t\}$ is non-stationary we need to show $\{X_{R,t}\}$ and $\{X_{I,t}\}$ are not jointly stationary. This is straightforward because $\text{var}\{X_{R,t}\} = \sigma_\epsilon^2 \cos^2(t)$, which clearly changes in time. Therefore $\{X_{R,t}\}$ is not individually stationary so $\{X_{R,t}\}$ and $\{X_{I,t}\}$ are not jointly stationary. Alternatively, one could make a similar argument based on $\text{var}\{X_{I,t}\}$. 3, A

(c) (i) Using the hint: let $\{X_t\}$ be a stationary process with mean 0 and acvs $\{s_\tau\}$ and let $\{Y_t\}$ be a stationary sequence independent of $\{X_t\}$ with mean $\pm\sqrt{c}$ and acvs $\{s_\tau\}$. Letting $W_t = X_t Y_t$, we have $E\{X_t Y_t\} = E\{X_t\}E\{Y_t\} = 0$, and unseen ⇓

$$\begin{aligned} s_{W,\tau} &= E\{W_t W_{t+\tau}\} - E^2\{W_t\} \\ &= E\{X_t Y_t X_{t+\tau} Y_{t+\tau}\} \\ &= E\{X_t X_{t+\tau}\} E\{Y_t Y_{t+\tau}\} \\ &= \text{cov}\{X_t, X_{t+\tau}\}(\text{cov}\{Y_t, Y_{t+\tau}\} + c) \\ &= s_\tau^2 + c s_\tau. \end{aligned}$$

Therefore, $\{s_\tau^2 + c s_\tau\}$ is a valid autocovariance sequence. 4, D

(ii) Suppose $c = -1$ and $s_0 < 1$, then the $\tau = 0$ term would be $s_0^2 + cs_0 < 0$. An acvs always has a positive value at $\tau = 0$ so therefore $\{s_\tau^2 + cs_\tau\}$ is not always a valid acvs. $\boxed{3, \text{B}}$

(iii) Let $\{X_t\}$ be the MA(1) process with zero mean of the form given then it has autocovariance sequence

$$s_\tau = \begin{cases} 5/4 & \tau = 0 \\ 1/2 & |\tau| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

We know from (c)(i) that $\{s_\tau^2 + cs_\tau\}$ will be a valid autocovariance sequence for $c \geq 0$. Setting $c = 1$ gives the desired result. Note, this is the autocovariance sequence for the product of two independent MA(1) processes with the parameters as given; one with mean zero and the other with mean ± 1. $\boxed{4, \text{C}}$

2. (a) (i) This is an MA(2) process. All MA(q) processes are stationary, so we have invertibility left to check. To do this, we need to consider the roots of the characteristic polynomial $\Theta(z) = 1 + \frac{16}{9}z^2$. $1 + \frac{16}{9}z^2 = 0 \implies z = \pm \mathrm{i}\frac{3}{4}$. Both roots lie inside the unit circle and therefore it is <u>not</u> invertible. $\boxed{\text{sim. seen} \Downarrow}$ $\boxed{2, \mathrm{A}}$

(ii) This is an AR(2) process. All AR(p) processes are invertible, so we have stationarity left to check. To do this, we need to consider the roots of the characterisic polynomial $\Phi(z) = 1 - \frac{5}{2}z + z^2$. $1 - \frac{5}{2}z + z^2 = 0 \implies (z-2)(z-\frac{1}{2}) = 0 \implies z = 2, \frac{1}{2}$. One of these roots lies inside the unit circle, therefore the process is <u>not</u> stationary. $\boxed{2, \mathrm{A}}$

(iii) To check invertibility, we check the roots of the characteristic polynomial $\Theta(z) = 1 - \frac{5}{6}z + \frac{1}{6}z^2$. $1 - \frac{5}{6}z + \frac{1}{6}z^2 = 0 \implies z^2 - 5z + 6 = 0 \implies (z-3)(z-2) = 0 \implies z = 2, 3$. Both these lie outside of the unit circle, therefore the process is invertible. To check stationarity, we check the roots of the characteristic polynomial $\Phi(z) = 1 - \frac{1}{4}z$. The root of this is $z = 4$, which lies outside of the unit circle, so the process is stationary. $\boxed{2, \mathrm{A}}$

(b) (i) The process can be written as $\Phi(B)X_t = \Theta(B)\epsilon_t$, where $\Phi(z) = 1 - \frac{1}{4}z$ and $\Theta(z) = 1 - \frac{1}{3}z$. The GLP form is

$$\begin{aligned}
X_t &= \frac{\Theta(B)}{\Phi(B)}\epsilon_t \\
&= \frac{1 - \frac{1}{3}B}{1 - \frac{1}{4}B}\epsilon_t \\
&= (1 - \frac{1}{3}B)(1 + \frac{1}{4}B + \frac{1}{4^2}B^2 + \ldots)\epsilon_t \\
&= \left[1 + \left\{\frac{1}{4} - \frac{1}{3}\right\}B + \left\{\frac{1}{4^2} - \frac{1}{3}\cdot\frac{1}{4}\right\}B^2 + \left\{\frac{1}{4^3} - \frac{1}{3}\cdot\frac{1}{4^2}\right\}B^3 + \ldots\right]\epsilon_t \\
&= \left\{1 + \left(\frac{1}{4} - \frac{1}{3}\right)\sum_{k=1}^{\infty}\frac{1}{4^{k-1}}B^k\right\}\epsilon_t = \left\{1 - \frac{1}{12}\sum_{k=1}^{\infty}\frac{1}{4^{k-1}}B^k\right\}\epsilon_t.
\end{aligned}$$

This is in the form $X_t = \sum_{k=0}^{\infty}\psi_k\epsilon_{t-k}$ where $\psi_0 = 1$ and $\psi_k = -\frac{1}{12}\cdot\frac{1}{4^{k-1}}$ for $k > 0$. $\boxed{4, \mathrm{A}}$

(ii) The l-step forecast is given as $X_t(l) = \Psi^{(l)}(B)\Psi^{-1}(B)X_t$, where $\Psi^{(l)}(z) = \sum_{k=0}^{\infty}\psi_{k+l}z^k$. Therefore, $\Psi^{(l)}(z) = -\frac{1}{12}\sum_{k=0}^{\infty}\frac{1}{4^{k+l-1}}z^k = -\frac{1}{12}\cdot\frac{1}{4^{l-1}}\sum_{k=0}^{\infty}\frac{1}{4^k}z^k = -\frac{1}{12}\cdot\frac{1}{4^{l-1}}\frac{1}{\Phi(z)}$. This gives

$$X_t(l) = -\frac{1}{12}\cdot\frac{1}{4^{l-1}}\frac{1}{\Phi(B)}\cdot\frac{\Phi(B)}{\Theta(B)}X_t = -\frac{1}{12}\cdot\frac{1}{4^{l-1}}\frac{1}{1 - \frac{1}{3}B}X_t = -\frac{1}{12}\cdot\frac{1}{4^{l-1}}\sum_{k=0}^{\infty}\frac{1}{3^k}X_{t-k}.$$

$\boxed{4, \mathrm{B}}$

unseen $\Downarrow$

(c) We have

$$\begin{aligned}
\widetilde{\sigma}^2(l) &= E\{(X_{t+l} - \widetilde{X}_t(l))^2\} \\
&= E\{(X_{t+l} - X_t(l) + X_t(l) - \widetilde{X}_t(l))^2\} \\
&= E\{(X_{t+l} - X_t(l))^2\} + 2E\{(X_{t+l} - X_t(l))(X_t(l) - \widetilde{X}_t(l))\} \\
&\qquad + E\{(X_t(l) - \widetilde{X}_t(l))^2\}.
\end{aligned}$$

The first term is just $E\{(X_{t+l} - X_t(l))^2\} = \sigma^2(l)$. $(X_{t+l} - X_t(l))$ is the forecast error $e_t(l) = \sum_{k=0}^{l-1} \psi_k \epsilon_{t+l-k}$ which is zero mean and consists of just future terms of $\{\epsilon_t\}$. Furthermore $(X_t(l) - \widetilde{X}_t(l)) = \sum_{k=0}^{\infty} \pi_k X_{t-k} - \sum_{k=0}^{N-1} \pi_k X_{t-k} = \sum_{k=N}^{\infty} \pi_k X_{t-k}$ which is zero mean and consists of just past terms, and hence is uncorrelated with $(X_{t+l} - X_t(l))$. Therefore we have

$$\begin{aligned}
\widetilde{\sigma}^2(l) &= \sigma^2(l) + E\left\{\left(\sum_{k=N}^{\infty} \pi_k X_{t-k}\right)^2\right\} \\
&= \sigma^2(l) + \sum_{j=N}^{\infty} \sum_{k=N}^{\infty} \pi_j \pi_k E\{X_{t-j} X_{t-k}\} \\
&= \sigma^2(l) + \sum_{j=N}^{\infty} \pi_j \pi_k s_{|j-k|}.
\end{aligned}$$

6, D

3. (a) (i) $S^{(I)}(-1/2) = 0$ because $S^{(I)}(-1/2) = \int_{-1/2}^{-1/2} S(f)\mathrm{d}f = 0$. seen ⇓

(ii) $S^{(I)}(1/2) = \int_{-1/2}^{1/2} S(f)\mathrm{d}f = s_0$ by setting $\tau = 0$ in the Fourier relationship. 2, A

(iii) $S(f)\mathrm{d}f = E\{|\mathrm{d}Z(f)|^2\}$, and we must have $\mathrm{d}f > 0$ and $E\{|\mathrm{d}Z(f)|^2\} \geq 0$, therefore $S(f) \geq 0$. 2, A

2, B

unseen ⇓

(iv) For $f < f'$, $S^{(I)}(f') - S^{(I)}(f) = \int_f^{f'} S(f)\mathrm{d}f$ which must be non-negative because $S(f)$ is non-negative. 2, B

seen ⇓

(v) By the Fourier relationship $S(f) = \sum_{\tau=-\infty}^{\infty} s_\tau \mathrm{e}^{-\mathrm{i}2\pi f\tau} = \sum_{\tau=-\infty}^{\infty} s_{-\tau}\mathrm{e}^{\mathrm{i}2\pi f\tau} = \sum_{\tau=-\infty}^{\infty} s_\tau \mathrm{e}^{\mathrm{i}2\pi f\tau} = S(-f)$, by symmetry of $\{s_\tau\}$. 2, A

unseen ⇓

(vi) $S^{(I)}(-f_0) + S^{(I)}(f_0) = \int_{-1/2}^{-f_0} S(f)\mathrm{d}f + \int_{-1/2}^{f_0} S(f)\mathrm{d}f = \int_{f_0}^{1/2} S(f)\mathrm{d}f + \int_{-1/2}^{f_0} S(f)\mathrm{d}f$ by symmetry of $S(f)$. Therefore $S^{(I)}(-f_0) + S^{(I)}(f_0) = \int_{-1/2}^{1/2} S(f)\mathrm{d}f = s_0$. 2, B

(b) (i) Using the formula for the spectral density of an AR process, the spectral density function for $\{X_t\}$ is sim. seen ⇓

$$S_X(f) = \frac{1}{|1 - \frac{3}{4}\mathrm{e}^{-\mathrm{i}2\pi f}|^2} = \frac{1}{1 - \frac{3}{2}\cos(2\pi f) + \frac{9}{16}}.$$

Furthermore, $\{\eta_t\}$ is a white noise process so its spectral density function is $S_\eta(f) = \sigma_\eta^2$. As they are uncorrelated we have

$$S_Y(f) = S_X(f) + S_\eta(f) = \frac{1}{1 - \frac{3}{2}\cos(2\pi f) + \frac{9}{16}} + \sigma_\eta^2.$$

3, B

(ii) Using the formula for the spectral density of an ARMA process, the spectral density function for $\{W_t\}$ is given as unseen ⇓

$$S_W(f) = \sigma_\xi^2 \frac{|1 - \theta\mathrm{e}^{-\mathrm{i}2\pi f}|^2}{|1 - \frac{3}{4}\mathrm{e}^{-\mathrm{i}2\pi f}|^2} = \sigma_\xi^2 \frac{1 - 2\theta\cos(2\pi f) + \theta^2}{1 - \frac{3}{2}\cos(2\pi f) + \frac{9}{16}}.$$

With $\sigma_\eta^2 = 1$, we require

$$\sigma_\xi^2 \frac{1 - 2\theta\cos(2\pi f) + \theta^2}{1 - \frac{3}{2}\cos(2\pi f) + \frac{9}{16}} = \frac{1}{1 - \frac{3}{2}\cos(2\pi f) + \frac{9}{16}} + 1,$$

and therefore require

$$\sigma_\xi^2(1 - 2\theta\cos(2\pi f) + \theta^2) = 2 - \frac{3}{2}\cos(2\pi f) + \frac{9}{16}.$$

Substituting in $\theta = 3/(4\sigma_\xi^2)$, gives the requirement that

$$\sigma_\xi^2 - \frac{3}{2}\cos(2\pi f) + \frac{9}{16\sigma_\xi^2} = 2 - \frac{3}{2}\cos(2\pi f) + \frac{9}{16},$$

and hence that $\sigma_\xi^2 \left(1 + \frac{9}{16\sigma_\xi^4}\right) = \frac{41}{16}$. $\boxed{5,\ \mathrm{D}}$

4. (a) (i) $\boxed{\text{seen} \Downarrow}$

$$E\{\widehat{S}^{(d)}(f)\} = E\{|J(f)|^2\} \quad \text{where} \quad J(f) = \sum_{t=1}^{N} h_t X_t \mathrm{e}^{-\mathrm{i}2\pi f t}, \quad |f| \le \frac{1}{2}.$$

We know from the spectral representation theorem that,

$$X_t = \int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}2\pi f' t}\,\mathrm{d}Z(f'),$$

so that,

$$\begin{aligned} J(f) &= \sum_{t=1}^{N} \left(\int_{-1/2}^{1/2} h_t \mathrm{e}^{\mathrm{i}2\pi f' t}\,\mathrm{d}Z(f') \right) \mathrm{e}^{-\mathrm{i}2\pi f t} \\ &= \int_{-1/2}^{1/2} \sum_{t=1}^{N} h_t \mathrm{e}^{-\mathrm{i}2\pi (f-f')t}\,\mathrm{d}Z(f') = \int_{-1/2}^{1/2} H(f-f')\mathrm{d}Z(f'). \end{aligned}$$

Then

$$\begin{aligned} E\{\widehat{S}^{(d)}(f)\} &= E\{|J(f)|^2\} = E\{J^*(f)J(f)\} \\ &= E\left\{ \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} H^*(f-f')H(f-f'')\mathrm{d}Z^*(f')\mathrm{d}Z(f'') \right\} \\ &= \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} H^*(f-f')H(f-f'')E\left\{\mathrm{d}Z^*(f')\mathrm{d}Z(f'')\right\} \\ &= \int_{-1/2}^{1/2} |H(f-f')|^2 S(f')\mathrm{d}f' = \int_{-1/2}^{1/2} \mathcal{H}(f-f')S(f')\mathrm{d}f'. \end{aligned}$$

$\boxed{3, \mathrm{B}}$

(ii) $\mathcal{H}(f+1) = \left|\sum_{t=1}^{N} h_t \mathrm{e}^{-\mathrm{i}2\pi(f+1)t}\right|^2 = \left|\sum_{t=1}^{N} \mathrm{e}^{-i2\pi t} h_t \mathrm{e}^{-\mathrm{i}2\pi f t}\right|^2 = \left|\sum_{t=1}^{N} h_t \mathrm{e}^{-\mathrm{i}2\pi f t}\right|^2 = \mathcal{H}(f)$ as $\mathrm{e}^{-\mathrm{i}2\pi t} = 1$ for all $t \in \mathbb{Z}$. $\boxed{2, \mathrm{A}}$

(iii) For a white noise process $\{X_t\}$ we have $S(f) = \sigma^2$, and therefore $E\{\widehat{S}^{(d)}(f)\} = \sigma^2 \int_{-1/2}^{1/2} \mathcal{H}(f-f')\mathrm{d}f'$. From Parseval's formula and the periodicity of $\mathcal{H}(f)$, it is the case that $\int_{-1/2}^{1/2} \mathcal{H}(f-f')\mathrm{d}f' = 1$ and therefore $E\{\widehat{S}^{(d)}(f)\} = \sigma^2$ and hence $\widehat{S}^{(d)}(f)$ is unbiased. $\boxed{2, \mathrm{A}}$

(iv) The answer should include the following points:

- $E\{\widehat{S}^{(p)}(f)\} = \int_{-1/2}^{1/2} \mathcal{F}(f-f')S(f')\mathrm{d}f'$, i.e. the expected value of the periodogram is a convolution of the true spectrum with Fejér's kernel.
- Fejér's kernel has prominent side-lobes, especially for small N, and this blurs the spectrum by moving energy from areas of high energy to areas of low energy, causing significant bias. This is known as side-lobe leakage.
- This effect is particularly pronounced in the processes with high dynamic range.
- Tapering replaces Fejèr's kernel with $\mathcal{H}(\cdot)$.
- Tapers are typically chosen such that $\mathcal{H}(\cdot)$ has significantly reduced side-lobes when compared with $\mathcal{F}(\cdot)$, thus reducing the bias caused by side-lobe leakage.

$\boxed{5, \text{A}}$

(b) (i) Using the provided formula, we have $\widehat{s}_0^{(u)} = (1/3)((-1)^2 + 0^2 + (1)^2) = 2/3$, $\widehat{s}_1^{(u)} = \widehat{s}_{-1}^{(u)} = (1/2)((-1)(0) + (0)(1)) = 0$, $\widehat{s}_2^{(u)} = \widehat{s}_{-2}^{(u)} = 1 \cdot (-1)(1) = -1$. Therefore we have $|\widehat{s}_2^{(u)}| > s_0$, and hence $\widehat{s}_\tau^{(u)}$ is not non-negative definite. For example, take $t_1 = 1$, $t_2 = 3$, $a_1 = 1$ and $a_2 = 1$, then $\boxed{\text{unseen} \Downarrow}$

$$\sum_{j=1}^{2}\sum_{k=1}^{2} a_j a_k \widehat{s}_{|t_j - t_k|}^{(u)} < 0.$$

$\boxed{4, \text{C}}$

(ii) Consider the Fourier transform of $\{\widehat{s}_\tau^{(u)}\}$ computed in part (i). We have

$$\widehat{S}^{(u)}(f) = \sum_{\tau=-2}^{2} \widehat{s}_\tau^{(u)} \mathrm{e}^{-\mathrm{i}2\pi f\tau} = \frac{2}{3} - 2\cos(4\pi f).$$

Take $f = 0$, for example, then $\widehat{S}^{(u)}(0) = 2/3 - 2 < 0$. $\boxed{4, \text{C}}$

5. (a) (i) $\boxed{\text{unseen} \Downarrow}$

$$\begin{aligned}\operatorname{cov}\{I(f_j), I(f_k)\} &= \operatorname{cov}\{A(f_j) + \mathrm{i}B(f_j), A(f_k) + \mathrm{i}B(f_k)\} \\ &= \operatorname{cov}\{A(f_j), A(f_k)\} + \mathrm{i}\operatorname{cov}\{A(f_j), B(f_k)\} \\ &\qquad - \mathrm{i}\operatorname{cov}\{B(f_j), A(f_k)\} + \operatorname{cov}\{B(f_j), B(f_k)\}.\end{aligned}$$

Using the provided results in the Mastery Material, all these covariances are zero for $f_j \neq f_k$. An analogous argument holds for $\operatorname{cov}\{I(f_j), I(f_k)^*\}$. $\boxed{3}$

(ii) Consider $\operatorname{cov}\{\widehat{S}^{(p)}(f_j), \widehat{S}^{(p)}(f_k)\}$. Using Isserlis's Theorem, we have

$$\begin{aligned}&\operatorname{cov}\{\widehat{S}^{(p)}(f_j), \widehat{S}^{(p)}(f_k)\} = \operatorname{cov}\{I(f_j)I^*(f_j), I(f_k)I^*(f_k)\} \\ &= \operatorname{cov}\{I(f_j), I(f_k)\}\operatorname{cov}\{I^*(f_j), I^*(f_k)\} + \operatorname{cov}\{I(f_j), I^*(f_k)\}\operatorname{cov}\{I^*(f_j), I(f_k)\}.\end{aligned}$$

We know from 5(a)(ii) that these are all zero for $f_j \neq f_k$, and therefore $\operatorname{Corr}\{\widehat{S}^{(p)}(f_j), \widehat{S}^{(p)}(f_k)\} = 0$ for $f_j \neq f_k$. $\boxed{3}$

(iii) Using the relationship between $J(\cdot)$ and $I(\cdot)$ given in the question,

$$\begin{aligned}&\operatorname{cov}\{J(f_k), J(f_{k+\tau})\} \\ &= C^2 N \operatorname{cov}\left\{\left[-\frac{1}{2}I(f_{k-1}) + I(f_k) - \frac{1}{2}I(f_{k+1})\right], \left[-\frac{1}{2}I(f_{k+\tau-1}) + I(f_{k+\tau}) - \frac{1}{2}I(f_{k+1+\tau})\right]\right\} \\ &= C^2 N \left(\frac{1}{4}\operatorname{cov}\{I(f_{k-1}), I(f_{k+\tau-1})\} - \frac{1}{2}\operatorname{cov}\{I(f_{k-1}), I(f_{k+\tau})\} + \frac{1}{4}\operatorname{cov}\{I(f_{k-1}), I(f_{k+1+\tau})\}\right. \\ &-\frac{1}{2}\operatorname{cov}\{I(f_k), I(f_{k+\tau-1})\} + \operatorname{cov}\{I(f_k), I(f_{k+\tau})\} - \frac{1}{2}\operatorname{cov}\{I(f_k), I(f_{k+1+\tau})\} \\ &\left.+\frac{1}{4}\operatorname{cov}\{I(f_{k+1}), I(f_{k+\tau-1})\} - \frac{1}{2}\operatorname{cov}\{I(f_{k+1}), I(f_{k+\tau})\} + \frac{1}{4}\operatorname{cov}\{I(f_{k+1}), I(f_{k+1+\tau})\}\right). (\dagger)\end{aligned}$$

When $\tau = 0$, ($\dagger$) becomes

$$\operatorname{cov}\{J(f_k), J(f_k)\} = C^2 N\left(\frac{1}{4}\sigma^2 + \sigma^2 + \frac{1}{4}\sigma^2\right) = 3C^2N\sigma^2/2,$$

because $\operatorname{var}\{I(f_k)\} = \operatorname{var}\{I(f_{k+1})\} = E\{|I(f_k)|^2\} = \sigma^2$.
When $\tau = 1$, ($\dagger$) becomes

$$\operatorname{cov}\{J(f_k), J(f_{k+1})\} = C^2 N\left(-\frac{1}{2}\operatorname{var}\{I(f_k)\} - \frac{1}{2}\operatorname{var}\{I(f_{k+1})\}\right) = -C^2N\sigma^2.$$

When $\tau = 2$, ($\dagger$) becomes

$$\operatorname{cov}\{J(f_k), J(f_{k+2})\} = C^2 N\frac{1}{4}\operatorname{cov}\{I_{k+1}, I_{k+1}\} = \frac{\sigma^2 C^2 N}{4}.$$

When $|\tau| = 3$, all terms in ($\dagger$) are zero. $\boxed{5}$

(iv) Using Isserlis's Theorem, we have

$$\begin{aligned}
&\text{cov}\{\widehat{S}^{(d)}(f_k), \widehat{S}^{(d)}(f_{k+\tau})\} = \text{cov}\{J(f_k)J^*(f_k), J(f_{k+\tau})J^*(f_{k+\tau})\} \\
&= \text{cov}\{J(f_k), J(f_{k+\tau})\}\,\text{cov}\{J^*(f_k), J^*(f_{k+\tau})\} + \text{cov}\{J(f_k), J^*(f_{k+\tau})\}\,\text{cov}\{J^*(f_k), J(f_{k+\tau})\}.
\end{aligned}$$

Using the statement in the question, $\text{cov}\{\widehat{S}^{(d)}(f_k), \widehat{S}^{(d)}(f_{k+\tau})\} = 2\left[\text{cov}\{J(f_k), J(f_{k+\tau})\}\right]^2$, and hence

$$\text{cov}\{\widehat{S}^{(d)}(f_k), \widehat{S}^{(d)}(f_{k+\tau})\} = \begin{cases} 9C^4N^2\sigma^4/2 & \tau = 0 \\ -2C^4N^2\sigma^4 & \tau = 1; \\ C^4N^2\sigma^4/8 & \tau = 2; \\ 0 & \tau = 3, \end{cases}$$

for $1 < k < \lfloor N/2 \rfloor - 3$. Given $\text{var}\{\widehat{S}^{(d)}(f_k)\} = \text{var}\{\widehat{S}^{(d)}(f_{k+\tau})\}$, we have

$$\text{Corr}\{\widehat{S}^{(d)}(f_k), \widehat{S}^{(d)}(f_{k+\tau})\} = \frac{\text{cov}\{\widehat{S}^{(d)}(f_k), \widehat{S}^{(d)}(f_{k+\tau})\}}{\text{var}\{\widehat{S}^{(d)}(f_k)\}} = \frac{\text{cov}\{\widehat{S}^{(d)}(f_k), \widehat{S}^{(d)}(f_{k+\tau})\}}{\text{cov}\{\widehat{S}^{(d)}(f_k), \widehat{S}^{(d)}(f_k)\}},$$

giving

$$\text{Corr}\left\{\widehat{S}_G^{(d)}(f_k), \widehat{S}_G^{(d)}(f_{k+\tau})\right\} = \begin{cases} 4/9 & \tau = 1; \\ 1/36 & \tau = 2; \\ 0 & \tau = 3. \end{cases}$$

$\boxed{4}$

(b) Answers should contain the following key points.

* A set of K orthogonal tapers are used to create K approximately uncorrelated direct spectral estimators $\widehat{S}_k^{(MT)}(f)$, $k = 0, ..., K-1$.
* Each of these retain the property of being approximately unbiased.
* The multi-taper spectral estimator is formed by taking the average of these K direct spectral estimators, namely,

$$\widehat{S}^{(MT)}(f) = \frac{1}{K}\sum_{k=0}^{K-1} \widehat{S}_k^{(MT)}(f).$$

* This has the effect of reducing the variance of a single direct spectral estimator by a factor of K.
* The asymptotic distribution of the multi-taper spectral estimator is now

$$\widehat{S}^{(MT)}(f) \sim \begin{cases} S(f)\chi^2/2K & 0 < f < 1/2 \\ S(f)\chi^2/K & f = 0 \text{ or } 1/2. \end{cases}$$

* The reduction in variance and tightening in distribution of the estimator gives smaller confidence intervals.

$\boxed{5}$

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a separate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
Time Series Analysis_MATH60046 MATH97084 MATH70046	1	The question had overall been responded to rather well. There were very frequent slips in computations involving covariances, though. In particular, one should be careful with the formula Cov[X,Y] = E[XY] which applies only if E[X] = 0 or E[Y]=0.
Time Series Analysis_MATH60046 MATH97084 MATH70046	2	This question was addressed reasonably well by most candidates. Part (a) was generally done well, with few issues with wrong characteristic polynomials and/or wrong roots. Part (b) was generally done well too, though some students struggled to find compact expressions for Xt and/or Xt(l). There were also some mistakes in the general expressions of the polynomial coefficients. Part (c) was proved challenging for some, with the majority of candidates dropping marks for failing to sufficiently show details and explain the steps, and a number of students leaving the question unanswered.
Time Series Analysis_MATH60046 MATH97084 MATH70046	3	This question was addressed reasonably well by most candidates. Part (a) was generally done well, with few issues with wrong justifications and/or some statements left unproven. Part (b) was proved challenging for many students. It was asked to determine the spectral density function for {Yt}. A number of candidates found the density function, but not the spectral one. In such cases no marks were computed. Another common mistake, particularly in part (b)(ii) was failing to sufficiently show what was asked. In these cases, partial credit was given, as more detail was required to get a full mark.
Time Series Analysis_MATH60046 MATH97084 MATH70046	4	The question was well answered. Parts i (a), and ii (d) could be answered in more directly and in more detail. The rest parts were very well answered.
Time Series Analysis_MATH60046 MATH97084 MATH70046	5	The question was well answered. All serious attempts were well answered, part a iii was more variable, but rest were mostly fine.