SOLUTIONS for Examples III for Time Series

1. The corresponding characteristic polynomial is $\Phi(z) = (1 + \frac{1}{12}z - \frac{1}{24}z^2)$ which can be factorized as $(1 - \frac{1}{6}z)(1 + \frac{1}{4}z)$ so that the roots are 6 and -4, which are both outside the unit circle, and therefore this AR(2) process is stationary.

2. Note first that $E\{I\} = 1/2 \times 1 + 1/2 \times (-1) = 0$. We thus have $E\{X_t\} = E\{y_t I\} = y_t E\{I\} = 0$.

   For the variance, note that $\text{var}\{I\} = E\{I^2\} = 1/2 \times 1^2 + 1/2 \times (-1)^2 = 1$. We thus have $\text{var}\{X_t\} = \text{var}\{y_t I\} = y_t^2 E\{I^2\} = 1$ since $y_t^2 = 1$ for all t.

   For the autocovariance, we have $\text{cov}\{X_t, X_{t+\tau}\} = E\{X_t X_{t+\tau}\} = y_t y_{t+\tau} E\{I^2\} = y_t y_{t+\tau}$. Now, if either (a) $t + \tau \leq 0$ and $t \leq 0$ or (b) $t + \tau > 0$ and $t > 0$, then $y_t y_{t+\tau} = 1$; otherwise, $y_t y_{t+\tau} = -1$.

   A requirement of stationarity is that $\text{cov}\{X_t, X_{t+\tau}\}$ be a finite number independent of t. This is not true for this stochastic process. For example, if $\tau = -5$ and $t = 0$, then $\text{cov}\{X_t, X_{t+\tau}\} = 1$; on the other hand, if $\tau = -5$ and $t = 1$, then $\text{cov}\{X_t, X_{t+\tau}\} = -1$. We conclude that X_t is not a stationary process.

3. (a) i. The roots of the characteristic polynomial must be outside the unit circle.

   ii. For the MA(2) process the characteristic polynomial is $\Theta(z) = (1 - \frac{9}{4}z + \frac{1}{2}z^2)$ which can be factorized as $(1 - 2z)(1 - \frac{1}{4}z)$ so that the roots are $1/2$ and 4, so one is inside the unit circle, and therefore this MA(2) process is not invertible.

   (b) i. The definition of $\{X_t\}$ implies that $\epsilon_{t-1} = \theta\epsilon_{t-2} + X_{t-1}$, $\epsilon_{t-2} =$

$\theta\epsilon_{t-3} + X_{t-2}$ and so forth. Hence we have

$$
\begin{aligned}
X_t &= \epsilon_t - \theta\epsilon_{t-1} \\
&= \epsilon_t - \theta\left(\theta\epsilon_{t-2} + X_{t-1}\right) \\
&= \epsilon_t - \theta X_{t-1} - \theta^2\epsilon_{t-2} \\
&= \epsilon_t - \theta X_{t-1} - \theta^2\left(\theta\epsilon_{t-3} + X_{t-2}\right) \\
&= \epsilon_t - \theta X_{t-1} - \theta^2 X_{t-2} - \theta^3\epsilon_{t-3} \\
&\vdots \\
&= \epsilon_t - \sum_{j=1}^{p} \theta^j X_{t-j} - \theta^{p+1}\epsilon_{t-p-1}
\end{aligned}
$$

(after p such substitutions).

ii. As $p \to \infty$, the final line above converges to the infinite autoregression

$$X_t + \sum_{j=1}^{\infty} \theta^j X_{t-j} = \epsilon_t$$

if $\lim_{p\to\infty} \theta^{p+1}\epsilon_{t-p-1} = 0$. The condition that $|\theta| < 1$ will do the trick and this is entirely consistent with 5(a)(i) since for invertibility the root of the polynomial $1 - \theta z$ must be outside the unit circle so that $|\theta| < 1$.

4. (a) First note

$$S(0) = \sum_{\tau=-\infty}^{\infty} s_\tau \mathrm{e}^{-\mathrm{i}2\pi 0\tau} = \sum_{\tau=-\infty}^{\infty} s_\tau.$$

Now, $S(f)$ is always non-negative, therefore

$$S(f) = |S(f)| = \left|\sum_{\tau=-\infty}^{\infty} s_\tau \mathrm{e}^{-\mathrm{i}2\pi f\tau}\right| \le \sum_{\tau=-\infty}^{\infty} |s_\tau \mathrm{e}^{-\mathrm{i}2\pi f\tau}| = \sum_{\tau=-\infty}^{\infty} |s_\tau|.$$

If s_τ is positive for all τ, this gives

$$S(f) \le \sum_{\tau=-\infty}^{\infty} s_\tau = S(0).$$

(b) The AR(1) process $X_t = \phi X_{t-1} + \epsilon_t$ has the autocovariance sequence (see notes) $s_\tau = s_0\phi^{|\tau|}$ where

$$s_0 = \frac{\sigma_\epsilon^2}{1-\phi^2}.$$

We therefore notice that when $0 < \phi < 1$ we have $s_\tau > 0$ for all τ. Using part (a), this gives $S(f) < S(0)$ for all $f \in [-1/2, 1/2)$.

Furthermore, using the fact $s_\tau = s_{-\tau}$,

$$
\begin{aligned}
S(0) = \sum_{\tau=-\infty}^{\infty} s_\tau &= \left(2\sum_{\tau=0}^{\infty} s_\tau\right) - s_0 \\
&= \left(2\sum_{\tau=0}^{\infty} s_0\phi^\tau\right) - s_0 \\
&= s_0\left(2\left(\sum_{\tau=0}^{\infty}\phi^\tau\right) - 1\right) \\
&= s_0\left(\frac{2}{1-\phi} - 1\right) \\
&= s_0\left(\frac{2}{1-\phi} - \frac{1-\phi}{1-\phi}\right) \\
&= s_0\left(\frac{1+\phi}{1-\phi}\right) \\
&= \sigma_\epsilon^2 \frac{1}{1-\phi^2}\frac{1+\phi}{1-\phi} \\
&= \sigma_\epsilon^2 \frac{1+\phi}{(1-\phi)(1+\phi)(1-\phi)} \\
&= \frac{\sigma_\epsilon^2}{(1-\phi)^2}.
\end{aligned}
$$

The result follows.

5. (a) We will attempt to show

$$s_\tau = \int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}2\pi f\tau}\mathrm{d}S^{(I)}(f). \tag{1}$$

To do so, we first need to derive s_τ. Let us first consider $E\{X_t\}$. Given A_k and C_k are independent, it follows that

$$E\{X_t\} = E\{\epsilon_t\} + \sum_{k=1}^{K} E\{A_k\cos(2\pi f_k t + C_k)\} = E\{\epsilon_t\} + \sum_{k=1}^{K} E\{A_k\}E\{\cos(2\pi f_k t + C_k)\} = 0.$$

Therefore

$$\begin{aligned}
\mathrm{cov}\{X_t, X_{t+\tau}\} &= E\{X_t X_{t+\tau}\} \\
&= E\left\{\left(\epsilon_t + \sum_{k=1}^{K} A_k \cos(2\pi f_k t + C_k)\right)\left(\epsilon_{t+\tau} + \sum_{k=1}^{K} A_k \cos(2\pi f_k (t+\tau) + C_k)\right)\right\} \\
&= E\{\epsilon_t \epsilon_{t+\tau}\} + \sum_{k=1}^{K}\sum_{k'=1}^{K} E\{A_k A_{k'} \cos(2\pi f_k t + C_k)\cos(2\pi f_{k'}(t+\tau) + C_{k'})\} \\
&= E\{\epsilon_t \epsilon_{t+\tau}\} + \sum_{k=1}^{K}\sum_{k'=1}^{K} E\{A_k A_{k'}\} E\{\cos(2\pi f_k t + C_k)\cos(2\pi f_{k'}(t+\tau) + C_{k'})\} \\
&= \sigma_\epsilon^2 \delta_{0,\tau} + \sum_{k=1}^{K} \sigma_A^2 E\{\cos(2\pi f_k t + C_k)\cos(2\pi f_k(t+\tau) + C_k)\},
\end{aligned}$$

where $\delta_{i,j}$ is the Kronecker delta.
The term $E\{\cos(2\pi f_k t + C_k)\cos(2\pi f_k(t+\tau) + C_k)\}$

$$\begin{aligned}
&= \frac{1}{2\pi}\int_0^{2\pi} \cos(2\pi f_k t + c)\cos(2\pi f_k(t+\tau) + c)\mathrm{d}c \\
&= \frac{1}{2\pi}\cos(2\pi f_k \tau)\int_0^{2\pi} \cos^2(2\pi f_k t + c)\mathrm{d}c \qquad \text{c.f. Exercises I, Q1} \\
&= \frac{1}{2}\cos(2\pi f_k \tau).
\end{aligned}$$

Therefore

$$s_\tau = \sigma_\epsilon^2 \delta_{0,\tau} + \frac{\sigma_A^2}{2}\sum_{k=1}^{K}\cos(2\pi f_k \tau). \tag{2}$$

Now check against Fourier relationship between s_τ and $S^{(I)}(f)$ in (1). When $\tau = 0$ we have

$$s_0 = \int_{-1/2}^{1/2} \mathrm{d}S^{(I)}(f) = \sigma_\epsilon^2 \int_{-1/2}^{1/2} \mathrm{d}f + \frac{\sigma_A^2}{4}\sum_{k=1}^{K}(1+1) = \sigma_\epsilon^2 + \frac{K\sigma_A^2}{2}$$

which matches (2) for $\tau = 0$.

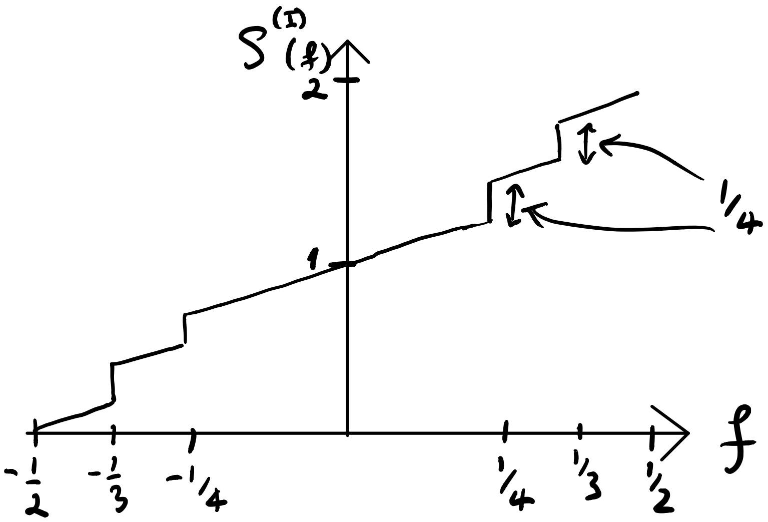
For $\tau \neq 0$, we have

$$
\begin{aligned}
s_0 &= \int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}2\pi f\tau} \mathrm{d}S^{(I)}(f) \\
&= \sigma_\epsilon^2 \int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}2\pi f\tau} \mathrm{d}f + \frac{\sigma_A^2}{4} \sum_{k=1}^{K} (\mathrm{e}^{\mathrm{i}2\pi f_k\tau} + \mathrm{e}^{-\mathrm{i}2\pi f_k\tau}) \\
&= 0 + \frac{\sigma_A^2}{4} \sum_{k=1}^{K} (\mathrm{e}^{\mathrm{i}2\pi f_k\tau} + \mathrm{e}^{-\mathrm{i}2\pi f_k\tau}) \\
&= \frac{\sigma_A^2}{2} \sum_{k=1}^{K} \cos(2\pi f_k\tau)
\end{aligned}
$$

matching the expression in (2) for $\tau \neq 0$.

(b)

$$
S^{(I)}(f) = \left(f + \tfrac{1}{2}\right) + \frac{1}{4}\left(\mathbb{1}_{[-\frac{1}{4},\frac{1}{2})}(f) + \mathbb{1}_{[\frac{1}{4},\frac{1}{2})} + \mathbb{1}_{[-\frac{1}{3},\frac{1}{2})} + \mathbb{1}_{[\frac{1}{3},\frac{1}{2})}\right).
$$



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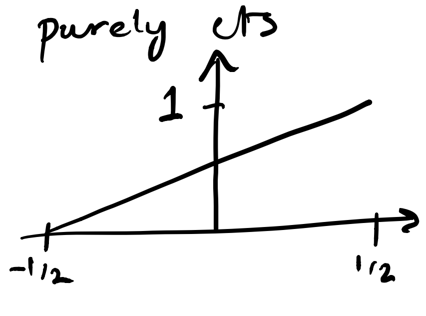

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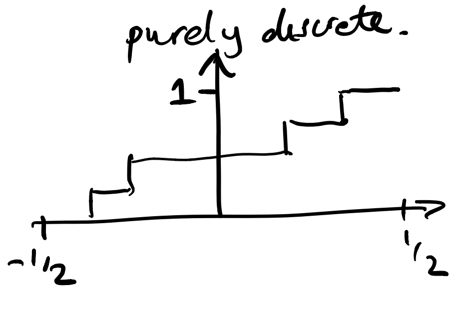

6. (a) Firstly,

$$E\{Z_t\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} C\mathrm{e}^{\mathrm{i}(2\pi f_0 t+\theta)}d\theta = C\mathrm{e}^{\mathrm{i}2\pi f_0 t}\left[\frac{\mathrm{e}^{\mathrm{i}\pi} - \mathrm{e}^{-\mathrm{i}\pi}}{\mathrm{i}2\pi}\right] = 0.$$

So,

$$\mathrm{cov}\{Z_t, Z_{t+\tau}\} = E\{C\mathrm{e}^{-\mathrm{i}(2\pi f_0 t+\theta)} \cdot C\mathrm{e}^{\mathrm{i}(2\pi f_0[t+\tau]+\theta)}\} = C^2\mathrm{e}^{\mathrm{i}2\pi f_0\tau},$$

which is finite and dependent on τ and not t. Hence, the process is stationary.

(b) i. Since

$$E\{Z_t\} = E\{X_t\mathrm{e}^{-\mathrm{i}2\pi f_0 t}\} = \mathrm{e}^{-\mathrm{i}2\pi f_0 t}E\{X_t\} = 0,$$

we have

$$\begin{aligned}\mathrm{cov}\{Z_t, Z_{t+\tau}\} &= E\{Z_t^* Z_{t+\tau}\} = E\{X_t\mathrm{e}^{\mathrm{i}2\pi f_0 t}X_{t+\tau}\mathrm{e}^{-\mathrm{i}2\pi f_0(t+\tau)}\}\\ &= \mathrm{e}^{-\mathrm{i}2\pi f_0\tau}E\{X_t X_{t+\tau}\} = \mathrm{e}^{-\mathrm{i}2\pi f_0\tau}s_{X,\tau}.\end{aligned}$$

So $\{Z_t\}$ is a complex-valued process with acvs $s_{Z,\tau} \equiv \mathrm{e}^{-\mathrm{i}2\pi f_0\tau}s_{X,\tau}$. Now

$$\begin{aligned}S_Z(f) = \sum_{\tau=-\infty}^{\infty} s_{Z,\tau}\mathrm{e}^{-\mathrm{i}2\pi f\tau} &= \sum_{\tau=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}2\pi f_0\tau}s_{X,\tau}\mathrm{e}^{-\mathrm{i}2\pi f\tau}\\ &= \sum_{\tau=-\infty}^{\infty} s_{X,\tau}\mathrm{e}^{-\mathrm{i}2\pi(f+f_0)\tau} = S_X(f+f_0),\end{aligned}$$

from which we see that the spectral density function of $\{Z_t\}$ is $S_Z(f) \equiv S_X(f+f_0)$.

ii. Since $E\{Z_t\} = E\{X_t + \mathrm{i}X_{t+k}\} = 0$, we have

$$\begin{aligned}\mathrm{cov}\{Z_t, Z_{t+\tau}\} &= E\{Z_t^* Z_{t+\tau}\} = E\{(X_t - \mathrm{i}X_{t+k})(X_{t+\tau} + \mathrm{i}X_{t+\tau+k})\}\\ &= 2s_{X,\tau} + \mathrm{i}s_{X,\tau+k} - \mathrm{i}s_{X,\tau-k}.\end{aligned}$$

So $\{Z_t\}$ is a complex-valued process with acvs $s_{Z,\tau} \equiv 2s_{X,\tau} + \mathrm{i}s_{X,\tau+k} - \mathrm{i}s_{X,\tau-k}$. Now

$$\begin{aligned}S_Z(f) = \sum_{\tau=-\infty}^{\infty} s_{Z,\tau}\mathrm{e}^{-\mathrm{i}2\pi f\tau} &= \sum_{\tau=-\infty}^{\infty} [2s_{X,\tau} + \mathrm{i}s_{X,\tau+k} - \mathrm{i}s_{X,\tau-k}]\mathrm{e}^{-\mathrm{i}2\pi f\tau}\\ &= 2S_X(f) + \mathrm{i}\mathrm{e}^{\mathrm{i}2\pi fk}S_X(f) - \mathrm{i}\mathrm{e}^{-\mathrm{i}2\pi fk}S_X(f)\\ &= 2[1 - \sin(2\pi fk)]S_X(f).\end{aligned}$$