(1) Week 4 Monday:

problem class.

(1.3.1) Theorem (Soundness of L)

Suppose ϕ is a theorem of L. Then ϕ is a tautology.

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(1.3.2) Notation

A (propositional) valuation v is an assignment of truth values to the propositional variables $p_1, p_2, \ldots$

So $v(p_i) \in \{T, F\}$ (for $i \in \mathbb{N}$).

Using the truth table rules, this assigns a truth value $v(\phi) \in \{T, F\}$ to every L-fmla ϕ (satisfying

$v((\neg\phi)) \neq v(\phi)$

and $v((\phi \to \psi)) = F \iff v(\phi) = T$ & $v(\psi) = F$).

[Prob. sheet 2].

Pf of 1.3.1: By induction on the length of a pf. of ϕ it is enough to show

(a) Every axiom is a tautotology;

(b) MP preserves tautologies.

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(a) Use truth tables, or argue as follows: Do A2:

Suppose for a contradiction, v is a valuation with

$$v\Big(\big(\phi \to (\psi \to \chi)\big) \to \big((\phi \to \psi) \to (\phi \to \chi)\big)\Big) = F.$$

then:

$$v\big(\phi \to (\psi \to \chi)\big) = T \quad \ldots 1$$

& $v\big((\phi \to \psi) \to (\phi \to \chi)\big) = F \quad \ldots 2$

By 2 $\quad v\big((\phi \to \psi)\big) = T \quad \ldots 3$

& $v\big((\phi \to \chi)\big) = F \quad \ldots 4$

By 4 $\quad v(\phi) = T \quad v(\chi) = F$

By 3 $\quad v(\psi) = T$

This contradicts 1. //

A1, A3 Ex.

(b) If ϕ $(\phi \to \psi)$ are tautologies and v is a valuation then

$v(\phi) = T \qquad v(\phi \to \psi) = T$

so $v(\psi) = T$. #.

(1.3.3) <u>Thm</u> (Generalisation of Soundness)

Suppose Γ is a set of L-fmlas, ϕ is a fmla. and $\Gamma \vdash_L \phi$.

If v is a valuation with $v(\psi) = T$ for all $\psi \in \Gamma$

[write $v(\Gamma) = T$]

<u>then</u> $v(\phi) = T$.

Pf: As for 1.3.2.

(1.3.4) <u>Thm</u> (Completeness / Adequacy Thm. for L).

Suppose ϕ is a tautology. Then

$$\vdash_L \phi .$$

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<u>Steps in pf</u>:

1. Want to show: if $v(\phi) = T$ for all valuations v then $\vdash_L \phi$.

2. Generalisation:

   Suppose that for every v with $v(\Gamma) = T$ we have $v(\phi) = T$.

   Then $\Gamma \vdash_L \phi$.

   [1. is the case $\Gamma = \emptyset$].

3. Equivalently:

   If $\Gamma \nvdash_L \phi$ there is a v with $v(\Gamma) = T$ and <u>$v(\phi) = F$</u>.

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(1.3.6) <u>Def</u>: A set of L-fmlas Γ is <u>consistent</u> if there is no L-fmla ϕ with $\Gamma \vdash_L \phi$ and $\Gamma \vdash_L (\neg\phi)$.

<u>Rk</u>: By 1.3.1 (Soundness) there is <u>no</u> L-fmla ϕ with $\vdash_L \phi$ & $\vdash_L (\neg\phi)$

(Say that <u>L is consistent</u>.)

(1.3.7) <u>Proposition</u>

Suppose Γ is a consistent set of L-fmlas and $\Gamma \nvdash_L \phi$.

Then $\Gamma \cup \{(\neg\phi)\}$ is consistent.

<u>Proof</u>: Suppose not. So there is a fmla. ψ with

$$\Gamma \cup \{(\neg\phi)\} \vdash_L \psi \quad \ldots 1$$

and $\Gamma \cup \{(\neg\phi)\} \vdash_L (\neg\psi)$. ... 2

Apply DT to 2:

$$\Gamma \vdash_L ((\neg\phi) \to (\neg\psi)).$$

By A3 & this $(((\neg\phi) \to (\neg\psi)) \to (\psi \to \phi))$

$$\Gamma \vdash_L (\psi \to \phi) \quad \ldots 3$$

By 1, 3 + MP obtain:

$$\Gamma \cup \{(\neg\phi)\} \vdash_L \phi \quad \ldots 4$$

Apply DT $\Gamma \vdash_L ((\neg\phi) \to \phi)$ ... 5

By 1.2.7 (c)

$$\vdash_L (((\neg\phi) \to \phi) \to \phi)$$

By this and 5 + MP get $\Gamma \vdash_L \phi$.

Contradiction. #.

(1.3.8) Prop. (Lindenbaum Lemma)

Suppose Γ is a consistent set of L-fmlas. Then there is a consistent set of formulas $\Gamma^* \supseteq \Gamma$ such that for every L-fmla ϕ

either $\Gamma^* \vdash_L \phi$

or $\Gamma^* \vdash_L (\neg\phi)$.

[Sometimes say Γ^* is complete.]

Pf: The set of L-formulas is countable, so we can the L-formulas as $\phi_0, \phi_1, \phi_2, \ldots$

[Why countable? Alphabet $\neg \to) (\; p_1 \, p_2 \ldots$ is countable.

Formulas are certain finite sequences of symbols from the alphabet. This is a countable set.]

Define inductively sets of L-fmlas

$$\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \ldots$$

where $\Gamma_0 = \Gamma$

and $\Gamma^* = \bigcup_{i \in \mathbb{N}} \Gamma_i$.

Suppose Γ_n has been defined.

If $\Gamma_n \vdash_L \phi_n$ then

let $\Gamma_{n+1} = \Gamma_n$.

If $\Gamma_n \nvdash_L \phi_n$ then

let $\Gamma_{n+1} = \Gamma_n \cup \{(\neg\phi_n)\}$.

An easy induction using Prop 1.3.7 shows that each Γ_n is consistent. Let $\Gamma^* = \bigcup_{n \in \mathbb{N}} \Gamma_n$.

Claim 1 Γ^* is consistent.

If $\Gamma^* \vdash \phi$

and $\Gamma^* \vdash (\neg \phi)$

then as deductions are finite

$\Gamma_n \vdash \phi$

and $\Gamma_n \vdash (\neg \phi)$ for some $n \in \mathbb{N}$. Contradiction

Claim 2 Γ^* is complete.

Let ϕ be any formula.

We have $\phi = \phi_n$ for some $n \in \mathbb{N}$.

Then by construction

either $\Gamma_n \vdash \phi$

or $\Gamma_{n+1} \vdash (\neg \phi)$

In the first case

$\Gamma^* \vdash \phi$

in the second case

$\Gamma^* \vdash (\neg \phi)$.

$\#$

(1.3.9) <u>Lemma</u>. Suppose Γ^* is a set of L-fmlas which is consistent and complete.

<u>Then</u> there is a valuation v such that for every L-fmla ϕ

$$\boxed{v(\phi) = T \iff \Gamma^* \vdash_L \phi .}$$

<u>Pf</u>: ~~Eat~~ Each variable p_i is an L-formula. <u>So</u> by the properties of Γ^*

either $\Gamma^* \vdash_L p_i$

or $\Gamma^* \vdash_L (\neg p_i)$

(and only one of these is the case).

~~In the first case~~

Let v be the valuation with

$$v(p_i) = T \iff \Gamma^* \vdash_L p_i$$

(for each $i \in \mathbb{N}$).

<u>Show</u> this v has the required property.

Do this by induction on the length of ϕ: show

$$v(\phi) = T \iff \Gamma^* \vdash_L \phi$$

<u>Base case</u>: ϕ is a variable. this is the def. of ~~...~~ v.

<u>Case 1</u>: ϕ is $(\neg \psi)$

<u>$\Rightarrow$</u>: Suppose $v(\phi) = T$

So $v(\psi) = F$

(v is a valuation)

By induction $\Gamma^* \nvdash \psi$.

By completeness of Γ^*

obtain $\Gamma^* \vdash (\neg \psi)$

i.e. $\Gamma^* \vdash \phi$. //

<u>$\Leftarrow$:</u> Conversely suppose

$\Gamma^* \vdash \phi$

i.e. $\Gamma^* \vdash (\neg \psi)$

By consitency $\Gamma^* \nvdash \psi$

So by inductive assumption

$v(\psi) = F$, so

$v(\neg\psi) = T$ i.e. $v(\phi) = T$.

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<u>Case 2</u> ϕ is $(\psi \to \chi)$

<u>$\Leftarrow$:</u> Suppose $v(\phi) = F$

(Show $\Gamma^* \nvdash \phi$)

then $v(\psi) = T$ & $v(\chi) = F$

By ind. hypothesis

$\Gamma^* \vdash \psi$ . . . (a)

& $\Gamma^* \nvdash \chi$.

If $\Gamma^* \vdash \phi$ then

$\Gamma^* \vdash (\psi \to \chi)$ . . . (b)

then (a), (b) give $\Gamma^* \vdash \chi$

a contradiction. So

$\Gamma^* \nvdash \phi$.

(ϕ is $(\psi \to \chi)$)

$\Rightarrow$: Suppose

$\Gamma^* \nvdash \phi$.

(show $v(\phi) = F$)

So $\Gamma^* \nvdash (\psi \to \chi)$

then: $\Gamma^* \nvdash \chi$ ... (1)

(as $\vdash (\chi \to (\psi \to \chi))$)

Also $\Gamma^* \nvdash (\neg\psi)$ ... (2)

(as $\vdash_L ((\neg\psi) \to (\psi \to \chi))$ 1.2.7(a)).

By (1), (2) & ind. hyp

$v(\chi) = F$

and $v(\neg\psi) = F$

so $v(\psi) = T$.

Thus $v(\phi) = F$. //#

(1.3.10) Cor. Suppose

Δ is a consistent set of L-formulas and $\Delta \nvdash_L \phi$.

Then there is a valuation v with $v(\Delta) = T$ and $v(\phi) = F$.

Pf: Let $\Gamma = \Delta \cup \{(\neg\phi)\}$.

By 1.3.7 Γ is also consistent.

Let 1.3.8 there is $\Gamma^* \supseteq \Gamma$ which is consistent and complete.

By 1.3.9 there is a valuation v with $v(\Gamma^*) = T$.

In particular $v(\Delta) = T$ and $v(\neg\phi) = T$

& so $v(\phi) = F$. #
(as v is a valuation)

(1.3.11) <u>Theorem</u>.
(Completeness / Adequacy Thm. for L).

<u>If</u> ϕ is an L-fmla and $v(\phi) = T$ for every valuation v

<u>then</u> $\vdash_L \phi$.

<u>Pf</u>: Suppose $\nvdash_L \phi$. ok by soundness thm.

By 1.3.10 (with $\Delta = \emptyset$) there is a valuation v with $v(\phi) = F$. $\#$

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