SOLUTIONS for Examples IV for Time Series

NB: Stationarity by itself always means 'second-order' stationarity

1. (a) X_t, and hence $\bar{X}$, is a sum of Gaussian distributed random variables and is therefore Gaussian. To parameterise the Gaussian distribution we need to know $E\{\bar{X}\}$ and $\mathrm{var}\{\bar{X}\}$. The expected value is

$$E\{\bar{X}\} = \frac{1}{100}\sum_{t=1}^{100} E\{X_t\} = \mu.$$

To compute the variance, we use the formula

$$\mathrm{var}\{\bar{X}\} = \frac{1}{N}\sum_{\tau=-(N-1)}^{N-1}\left(1 - \frac{|\tau|}{N}\right) s_\tau. \tag{1}$$

The acvs terms for the MA(2) model is $s_0 = 1^2 + (1/2)^2 + (-1/2)^2 = 3/2$, $s_{-1} = s_1 = 1\cdot 1/2 + (1/2\cdot -1/2) = 1/4$, and $s_{-2} = s_2 = 1\cdot(-1/2) = -1/2$. All other terms are zero, i.e. $s_\tau = 0$ for all $|\tau| > 2$. Therefore

$$\begin{aligned}
\mathrm{var}\{\bar{X}\} &= \frac{1}{100}\left(-\left(1-\frac{2}{100}\right)\frac{1}{2} + \left(1-\frac{1}{100}\right)\frac{1}{4} + \frac{3}{2} + \left(1-\frac{1}{100}\right)\frac{1}{4} - \left(1-\frac{2}{100}\right)\frac{1}{2}\right) \\
&= \frac{1}{100}\left(-\frac{98}{100}\cdot\frac{1}{2} + \frac{99}{100}\cdot\frac{1}{4} + \frac{3}{2} + \frac{99}{100}\cdot\frac{1}{4} - \frac{98}{100}\cdot\frac{1}{2}\right) \\
&= \frac{1}{100}\left(-\frac{98}{100} + \frac{99}{200} + \frac{3}{2}\right) \\
&= \frac{1}{100}\left(\frac{-196+99+300}{200}\right) = \frac{203}{20000}.
\end{aligned}$$

Therefore $X \sim N(\mu, \frac{203}{20000})$.

(b) Again, X_t, and hence $\bar{X}$, is a sum of Gaussian distributed random variables and is therefore Gaussian. $\{X_t\}$ is a zero mean AR(1) process, so $E\{\bar{X}\} = 0$. To compute $\mathrm{var}(\bar{X})$ we use (1) again. For this AR(1) process (c.f. derivation of AR(p) acvs in notes),

$$s_0 = \frac{1}{1-(\frac{1}{2})^2} = 4/3$$

and $s_\tau = \frac{4}{3}(\frac{1}{2})^{|\tau|}$. Therefore

$$\begin{aligned}
\text{var}\{\bar{X}\} &= \frac{1}{4}\left(2\left(1-\frac{3}{4}\right)\frac{4}{3}\cdot\frac{1}{8} + 2\left(1-\frac{2}{4}\right)\frac{4}{3}\cdot\frac{1}{4} + 2\left(1-\frac{1}{4}\right)\frac{4}{3}\cdot\frac{1}{2} + \frac{4}{3}\right) \\
&= \frac{1}{4}\left(\frac{1}{2}\cdot\frac{4}{3}\cdot\frac{1}{8} + \frac{4}{3}\cdot\frac{1}{4} + \frac{3}{2}\cdot\frac{4}{3}\cdot\frac{1}{2} + \frac{4}{3}\right) \\
&= \frac{1}{48} + \frac{1}{12} + \frac{3}{12} + \frac{1}{3} = \frac{1+4+12+16}{48} = 33/48
\end{aligned}$$

2. (a) Now

$$\hat{S}^{(p)}(f) = \sum_{\tau=-(N-1)}^{N-1} \hat{s}_\tau^{(p)} \mathrm{e}^{-\mathrm{i}2\pi f\tau}.$$

So

$$\int_{-1/2}^{1/2} \hat{S}^{(p)}(f)\,\mathrm{d}f = \sum_{\tau=-(N-1)}^{N-1} s_\tau^{(p)} \int_{-1/2}^{1/2} \mathrm{e}^{-\mathrm{i}2\pi f\tau}\,\mathrm{d}f.$$

When $\tau = 0$, the integral is just 1; when $\tau \neq 0$, the integral is always zero because we have (letting $\nu = 2\pi f\tau$)

$$\begin{aligned}
\int_{-1/2}^{1/2} \mathrm{e}^{-\mathrm{i}2\pi f\tau}\,\mathrm{d}f &= \left\{\frac{1}{2\pi\tau}\int_{-\pi\tau}^{\pi\tau}\cos(\nu)\,\mathrm{d}\nu - \mathrm{i}\int_{-\pi\tau}^{\pi\tau}\sin(\nu)\,\mathrm{d}\nu\right\} \\
&= \frac{1}{2\pi\tau}\left\{[\sin(\pi\tau) - \sin(-\pi\tau)] + \mathrm{i}\,[\cos(\pi\tau) - \cos(-\pi\tau)]\right\} = 0.
\end{aligned}$$

Hence only one term of the summation is nonzero, and we obtain

$$\int_{-1/2}^{1/2} \hat{S}^{(p)}(f)\,\mathrm{d}f = \frac{1}{N}\sum_{t=1}^{N}(X_t - \overline{X})^2,$$

as required.

(b) Here $\{X_t\}$ is a stationary process with mean value $\mu = E\{X_t\}$, and variance $s_0 = \sigma^2$. Let $\hat{s}_0 = \hat{s}_0^{(p)} = \hat{s}_0^{(u)}$. Then

$$\begin{aligned}
\hat{s}_0 &= \frac{1}{N}\sum_{t=1}^{N}(X_t - \overline{X})^2 \\
&= \frac{1}{N}\sum_{t=1}^{N}([X_t - \mu] - [\overline{X} - \mu])^2 \\
&= \frac{1}{N}\sum_{t=1}^{N}\left([X_t - \mu]^2 - 2[X_t - \mu][\overline{X} - \mu] + [\overline{X} - \mu]^2\right) \\
&= \frac{1}{N}\sum_{t=1}^{N}[X_t - \mu]^2 - 2[\overline{X} - \mu][\overline{X} - \mu] + [\overline{X} - \mu]^2 \\
&= \frac{1}{N}\sum_{t=1}^{[} X_t - \mu]^2 - [\overline{X} - \mu]^2.
\end{aligned}$$

Taking the expectation of both sides and noting that $E\{\overline{X}\} = \mu$ yields

$$\begin{aligned}
E\{\hat{s}_0\} &= \frac{1}{N}\sum_{t=1}^{N} E\{[X_t - \mu]^2\} - E\{[\overline{X} - \mu]^2\} \\
&= \operatorname{var}\{X_t\} - \operatorname{var}\{\overline{X}\} = s_0 - \operatorname{var}\{\overline{X}\},
\end{aligned}$$

the desired result.

The ‘unbiased’ and ‘biased’ estimator coincide for lag $\tau = 0$; if $\overline{X}$ was replaced in the estimator by μ then both estimators would be unbiased.

3. (a) Since $\{X_t\}$ is a white noise process, we have, for $|\tau| \neq 0$ and $t = 1, \ldots, N - |\tau|$,

$$\begin{aligned}
E\{(X_t - \bar{X})(X_{t+|\tau|} - \bar{X})\} &= E\{([X_t - \mu] - [\bar{X} - \mu])([X_{t+|\tau|} - \mu] - [\bar{X} - \mu])\} \\
&= E\{(X_t - \mu)(X_{t+|\tau|} - \mu)\} \\
&\quad - \frac{1}{N}\sum_{u=1}^{N} E\{(X_t - \mu)(X_u - \mu)\} \\
&\quad - \frac{1}{N}\sum_{u=1}^{N} E\{(X_{t+|\tau|} - \mu)(X_u - \mu)\} \\
&\quad + \frac{1}{N^2}\sum_{u=1}^{N}\sum_{v=1}^{N} E\{(X_u - \mu)(X_v - \mu)\} \\
&= 0 - \frac{1}{N}\sigma^2 - \frac{1}{N}\sigma^2 + \frac{1}{N^2}N\sigma^2 = -\frac{\sigma^2}{N}.
\end{aligned}$$

Hence

$$E\{\hat{s}_\tau^{(u)}\} = \frac{1}{N-|\tau|}\sum_{t=1}^{N-|\tau|} E\{(X_t - \bar{X})(X_{t+|\tau|} - \bar{X})\} = -\frac{\sigma^2}{N},$$

and

$$E\{\hat{s}_\tau^{(p)}\} = \frac{1}{N}\sum_{t=1}^{N-|\tau|} E\{(X_t - \bar{X})(X_{t+|\tau|} - \bar{X})\} = -\left(1 - \frac{|\tau|}{N}\right)\frac{\sigma^2}{N}.$$

Since for a white noise process $s_\tau = 0$ when $|\tau| \neq 0$, the magnitude of the bias for $\hat{s}_\tau^{(u)}$ is σ^2/N, while the magnitude of the bias for $\hat{s}_\tau^{(p)}$ is $(1 - |\tau|/N)\,\sigma^2/N$, which is strictly less than σ^2/N (unless $\sigma^2 = 0$, an uninteresting special case).

(b) Let

$$Q_\tau \equiv \sum_{t=1}^{N-|\tau|} (X_t - \bar{X})(X_{t+|\tau|} - \bar{X})$$

so that $\hat{s}_\tau^{(u)} = Q_\tau/(N - |\tau|)$ and $\hat{s}_\tau^{(p)} = Q_\tau/N$. Then we have

$$\mathrm{var}\,\{\hat{s}_\tau^{(u)}\} = \frac{\mathrm{var}\,\{Q_\tau\}}{(N-|\tau|)^2} \geq \frac{\mathrm{var}\,\{Q_\tau\}}{N^2} = \mathrm{var}\,\{\hat{s}_\tau^{(p)}\}.$$

Hence, recalling that mse is given by variance plus bias squared,

$$\mathrm{mse}\,\{\hat{s}_\tau^{(u)}\} = \frac{\mathrm{var}\,\{Q_\tau\}}{(N-|\tau|)^2} + \frac{\sigma^4}{N^2} > \frac{\mathrm{var}\,\{Q_\tau\}}{N^2} + \left(1 - \frac{|\tau|}{N}\right)^2 \frac{\sigma^4}{N^2} = \mathrm{mse}\,\{\hat{s}_\tau^{(p)}\}.$$

(c) The sum of the elements on the main diagonal is $\sum_{t=1}^{N}(X_t - \bar{X})^2 = N\hat{s}_0^{(p)}$. The sum of the elements on the τth diagonal is $N\hat{s}_\tau^{(p)}$ for $\tau = 1, \ldots, N-1$. Since $\hat{s}_\tau^{(p)} = \hat{s}_{-\tau}^{(p)}$, the sum of all the elements in the matrix is $N\sum_{\tau=-(N-1)}^{(N-1)} \hat{s}_\tau^{(p)}$. However, the sum of the uth row is $(X_u - \bar{X})\sum_{v=1}^{N}(X_v - \bar{X})$ which is identically zero for all u. Hence the result.
Since $\hat{s}_0^{(p)} > 0$, it follows that $\hat{s}_\tau^{(p)}$ must be negative for some value(s) of τ (a property not necessarily shared by the true autocovariance sequence).

4. (a) Using

$$\hat{s}_\tau^{(p)} = \frac{1}{N}\sum_{t=1}^{N-|\tau|} X_t X_{t+\tau}, \qquad \tau = 0, \pm 1, ..., \pm(N-1),$$

and recalling that $\hat{s}_\tau^{(p)} \equiv 0$ for $|\tau| \geq N$, we obtain

$$\begin{aligned}
\hat{s}_0^{(p)} &= \frac{1}{3}(a^2 + 0^2 + a^2) = \frac{2a^2}{3} \\
\hat{s}_1^{(p)} &= \frac{1}{3}(a \cdot 0 + 0 \cdot (-a)) = 0 = \hat{s}_{-1}^{(p)} \\
\hat{s}_2^{(p)} &= \frac{1}{3}a \cdot (-a) = -\frac{a^2}{3} = \hat{s}_{-2}^{(p)} \\
\hat{s}_\tau^{(p)} &= 0 \qquad |\tau| \geq 3.
\end{aligned}$$

Since $\hat{s}_\tau^{(u)} = \hat{s}_\tau^{(p)} N/(N - |\tau|)$ for $0 \leq |\tau| \leq N - 1$, we find that

$$\begin{aligned}
\hat{s}_0^{(u)} &= s_0^{(p)} = \frac{2a^2}{3} \\
\hat{s}_1^{(u)} &= \frac{3}{2}\hat{s}_1^{(p)} = 0 \\
\hat{s}_2^{(u)} &= 3\hat{s}_2^{(p)} = -a^2.
\end{aligned}$$

(b) Since $N - 1 = 2$

$$\hat{S}^{(p)}(f) = \sum_{\tau=-2}^{2} \hat{s}_\tau^{(p)} e^{-i2\pi f\tau} = \hat{s}_0^{(p)} + 2\sum_{\tau=1}^{2} \hat{s}_\tau^{(p)} \cos(2\pi f\tau),$$

substituting the results of part (a) yields

$$\hat{S}^{(p)}(f) = \frac{2a^2}{3} + 2 \cdot 0 \cdot \cos(2\pi f) + 2 \cdot \left(-\frac{a^2}{3}\right) \cdot \cos(4\pi f) = \frac{2a^2}{3}[1 - \cos(4\pi f)];$$

where we have made use of the result, that, since $\hat{s}_{-\tau}^{(p)} = \hat{s}_\tau^{(p)}$,

$$\hat{s}_\tau^{(p)} e^{-i2\pi f\tau} + \hat{s}_{-\tau}^{(p)} e^{i2\pi f\tau} = \hat{s}_\tau^{(p)}(e^{-i2\pi f\tau} + e^{i2\pi f\tau}) = \hat{2}s_\tau^{(p)} \cos(2\pi f\tau).$$

(c) We have

$$\sum_{t=1}^{N} X_t e^{-i2\pi ft} = ae^{-i2\pi f} - ae^{-i6\pi f} = ae^{-i2\pi f}[1 - e^{-i4\pi f}].$$

Then, with $N = 3$,

$$\begin{aligned}
\left|\sum_{t=1}^{N} X_t e^{-i2\pi ft}\right|^2 &= a^2(1 - e^{-i4\pi f})(1 - e^{4i\pi f}) \\
&= a^2[2 - (e^{i4\pi f} + e^{-i4\pi f})] \\
&= 2a^2[1 - \cos(4\pi f)].
\end{aligned}$$

Hence

$$\hat{S}^{(p)}(f) = \frac{1}{N}\left|\sum_{t=1}^{N} X_t e^{-i2\pi f t}\right|^2 = \frac{2a^2}{3}[1 - \cos(4\pi f)],$$

which agrees with the result obtained in part (a)(ii).

(d) Given that

$$E\{\hat{S}^{(p)}(f)\} = \int_{-1/2}^{1/2} \mathcal{F}(f - f')S(f')\mathrm{d}f',$$

then

$$\begin{aligned}
\int_{-1/2}^{1/2} b(f)\mathrm{d}f &= \int_{-1/2}^{1/2} E\{\hat{S}^{(p)}(f)\} - S(f)\mathrm{d}f \\
&= \int_{-1/2}^{1/2} E\{\hat{S}^{(p)}(f)\}\mathrm{d}f - \int_{-1/2}^{1/2} S(f)\mathrm{d}f \\
&= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \mathcal{F}(f - f')S(f')\mathrm{d}f'\mathrm{d}f - \int_{-1/2}^{1/2} S(f)\mathrm{d}f \\
&= \int_{-1/2}^{1/2} S(f')\left[\int_{-1/2}^{1/2} \mathcal{F}(f - f')\right]\mathrm{d}f' - \int_{-1/2}^{1/2} S(f)\mathrm{d}f \\
&= \int_{-1/2}^{1/2} S(f')\mathrm{d}f' - \int_{-1/2}^{1/2} S(f)\mathrm{d}f = 0,
\end{aligned}$$

where we have made use of the fact that, because $\mathcal{F}(\cdot)$ is a periodic function with a period of unity, $\int_{-1/2}^{1/2}\mathrm{d}f = 1$ for any f' because $\int_{-1/2}^{1/2}\mathcal{F}(f)\mathrm{d}f = 1$.