Mathematical Logic (M345P65)
Problem Sheet 8

Work in ZFC unless otherwise stated.

[1] (i) Suppose A is a set of cardinality λ and $\kappa \leq \lambda$ is a cardinal. Show that A has a subset B with $|B| = \kappa$.

(ii) Prove that ω is equinumerous with a proper subset of itself.

(iii) Suppose X is any set. Prove that X is infinite if and only if X is equinumerous with a proper subset of itself.
(Hint: use question 2, sheet 7 for one direction.)

[2] Using Zorn's Lemma (or otherwise), prove the following.

(i) Suppose $(A; \leq_1)$ is any partially ordered set. Prove that there is a linearly ordered set $(A; \leq_2)$ with the property that for all $a, a' \in A$ we have $a \leq_1 a'$ implies $a \leq_2 a'$.

(ii) Let R be any (commutative) ring with identity element and $I \subset R$ be a proper ideal of R. Then there is a maximal proper ideal J of R with $I \subseteq J \subset R$.

(iii) Suppose G is a non-trivial group with an element g whose conjugates generate G. Prove that G has a maximal proper normal subgroup. Is this necessarily true without assuming the existence of such an element g?

[3] Suppose κ is a cardinal with $\kappa > |\mathbb{R}|$. Prove that there is a vector space V over $\mathbb{R}$ with $|V| = \kappa$. (You could use the Löwenheim - Skolem Theorem here, but it's probably also instructive to try to do this directly.) Prove that a basis of V has cardinality κ.

Prove that if V_1, V_2 are $\mathbb{R}$-vector spaces with $|V_1| = |V_2| > |\mathbb{R}|$ then there is a bijective linear map $T : V_1 \to V_2$ (i.e. V_1, V_2 are isomorphic).

[4] Let A be a non-empty set. A set F of subsets of A is called a *filter* on A if it satisfies the first three of the following properties. If it satisfies all four, it is called an *ultrafilter* on A.

UF1 $\emptyset \notin F$;

UF2 if $x \in F$ and $x \subseteq y \subseteq A$, then $y \in F$;

UF3 if $x, y \in F$ then $x \cap y \in F$;

UF4 if x is any subset of A then either x or its complement $A \setminus x$ is in F.

(i) (Nothing to do with Zorn's Lemma) Suppose A is a finite set and F an ultrafilter on A. Show that there exists $a \in A$ such that $F = \{x \subseteq A : a \in x\}$.

(ii) Show that if A is an infinite set the the set of subsets whose complements are finite forms a filter on A.

(iii) Show that if F_0 is a filter on A then the set of filters which contain it is a poset (under inclusion) which satisfies the hypotheses of Zorn's Lemma.

(iv) Show that a maximal filter satisfies (UF4).

(v) Let F be a maximal filter containing the filter in (ii). Show that F does not contain any finite set.