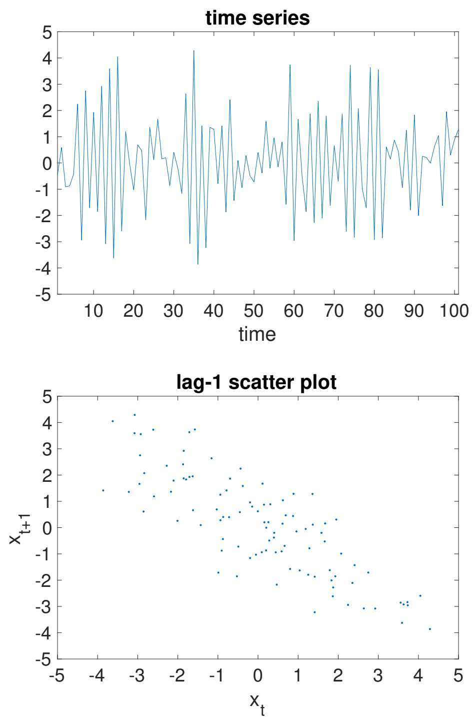
Quiz questions

Chapter 1.2

1. Consider the time series and lag-1 scatter plot shown here. Which of the following is a correct inference from visual inspection?



A Successive values are uncorrelated.

B Successive values are positively correlated.

C Negative values tend to be followed by negative values.

D Successive values are negatively correlated.

Answer: The lag one scatter plot shows that positive values for x_t tend to be followed by negative values for x_{t+1}, and negative values for x_t tend to be followed by positive values for x_{t+1}. Therefore, successive values are negatively correlated, so the answer is D.

Chapter 1.3

2. Which of the following is a valid covariance matrix?

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 4 & -1 \\ 1 & 4 \end{pmatrix} \quad C = \begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix} \quad D = \begin{pmatrix} 4 & 3 \\ 3 & 1 \end{pmatrix}$$

Answer: A is not a covariance matrix because it has a negative diagonal element (which equates to negative variance). Equivalently, it is not non-negative definite. B is not a covariance matrix as it is not symmetric. C is a covariance matrix because it is symmetric non-negative definite. This is true because

$$(x, y) \begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x^2 - 6xy + 9y^2 = (x - 3y)^2 \geq 0.$$

D is not a covariance matrix because it is not non-negative definite. For example

$$(1, -1) \begin{pmatrix} 4 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -1 < 0.$$

So the answer is C.

3. X and Y are random variables such that $Y = 2X + \epsilon$ where X is a random variable with mean 0 and variance 2, and $\epsilon \sim N(0,1)$ that is independent of X. What is $\mathrm{Corr}(X,Y)$?

   A $4/\sqrt{18}$

   B 0

   C $-2/\sqrt{5}$

   D 1

   Answer: By definition

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$$

   Considering the numerator, $\mathrm{Cov}(X,Y) = \mathrm{Cov}(X, 2X+\epsilon) = \mathrm{Cov}(X,2X)+\mathrm{Cov}(X,\epsilon) = 2\mathrm{Cov}(X,X) = 2\mathrm{Var}(X) = 4$, *because* $\mathrm{Cov}(X,\epsilon) = 0$ *due to independence. We further have* $\mathrm{Var}(Y) = 4\mathrm{Var}(X) + \mathrm{Var}(\epsilon) = 9$*. Therefore*

$$\mathrm{Corr}(X,Y) = \frac{4}{\sqrt{9 \cdot 2}} = \frac{4}{\sqrt{18}}.$$

   So the answer is A.

Chapter 2.1

4. Which of the following statements is TRUE?

   A A random process is completely stationary if it has a constant mean.

   B A random process is completely stationary if it is second order stationary.

   C A random process is completely stationary if and only if it is second order stationary.

   D A random process is second order stationary if it is completely stationary.

   E A random process is second order stationary if it has a constant variance.

Answer: A is FALSE. Take as an example the random process $\{X_t\}$*, where* $X_t = t \cdot \epsilon_t$ *and* $\{\epsilon_t\}$ *is a sequence of idependent and identically distributed* $N(0, 1)$ *random variables. The mean is 0 for all time, but the distribution of* X_t *is not equal to the distribution of* X_s*,* $t \neq s$*. B is FALSE. Second-order stationary does not imply complete stationary. Take as an example a random process* $\{X_t\}$ *of independent random variables where* $X_t \sim N(0, 1)$ *when* t *is even and* $X_t \sim unif(-\sqrt{3}, \sqrt{3})$ *when* t *is odd, this process is second-order stationary but is not completely stationary. C is FALSE because of B. D is TRUE, completely stationary implies second-order stationary. E is FALSE, take for example the random process* $\{X_t\}$*, where* $X_t = t + \epsilon_t$ *and* $\{\epsilon_t\}$ *is a sequence of idependent and identically distributed* $N(0, 1)$ *random variables. It is true that* $\mathrm{Var}(X_t) = E\{(X_t - E\{X_t\})^2\} = E\{\epsilon_t^2\} = 1$ *and hence constant for all* t*, however* $E\{X_t\} = t$ *which is a non-constant mean. Therefore* $\{X_t\}$ *is non-stationary. Therefore, the answer is D.*

5. Let $\{X_t\}$ be a stationary process with mean 1, variance σ^2 and autocovariance sequence $\{s_\tau\}$. Which of the following statements is FALSE?

A $E\{X_t + X_s\} = 2,\ s \neq t.$

B $E\{X_{10}^2\} = \sigma^2.$

C $E\{X_0\} = E\{X_{1283}\}.$

D $\mathrm{Cov}\{X_2, X_{231}\} = \mathrm{Cov}\{X_{-233}, X_{-4}\}.$

E $E\{X_{546}X_{536}\} = E\{X_{-230}X_{-220}\}.$

Answer: $E\{X_t + X_s\} = E\{X_t\} + E\{X_s\} = 2$, *so A is TRUE.* $E\{X_{10}^2\} = E\{X_{10}^2\} - E^2\{X_{10}\} + E^2\{X_{10}\} = \mathrm{Var}\{X_{10}\} + E^2\{X_{10}\} = \sigma^2 + 1$, *so B is FALSE. The mean of a stationary process is constant so* $E\{X_0\} = E\{X_{1283}\}$, *hence C is TRUE.* $\mathrm{Cov}\{X_2, X_{231}\} = s_{229} = \mathrm{Cov}\{X_{-233}, X_{-4}\}$, *hence D is TRUE.* $E\{X_{546}X_{536}\} = s_{-10} + 1 = s_{10} + 1 = E\{X_{-230}X_{-220}\}$, *hence E is TRUE. Therefore, the answer is B.*

6. Let $\{X_t\}$ be a (second-order) stationary process with mean 0 and autocovariance sequence $\{s_\tau\}$, and let $a \neq 0$ be a fixed constant. Which of the following is FALSE?

   A $\{aX_t\}$ is a stationary process.

   B $\{-X_t\}$ is a stationary process.

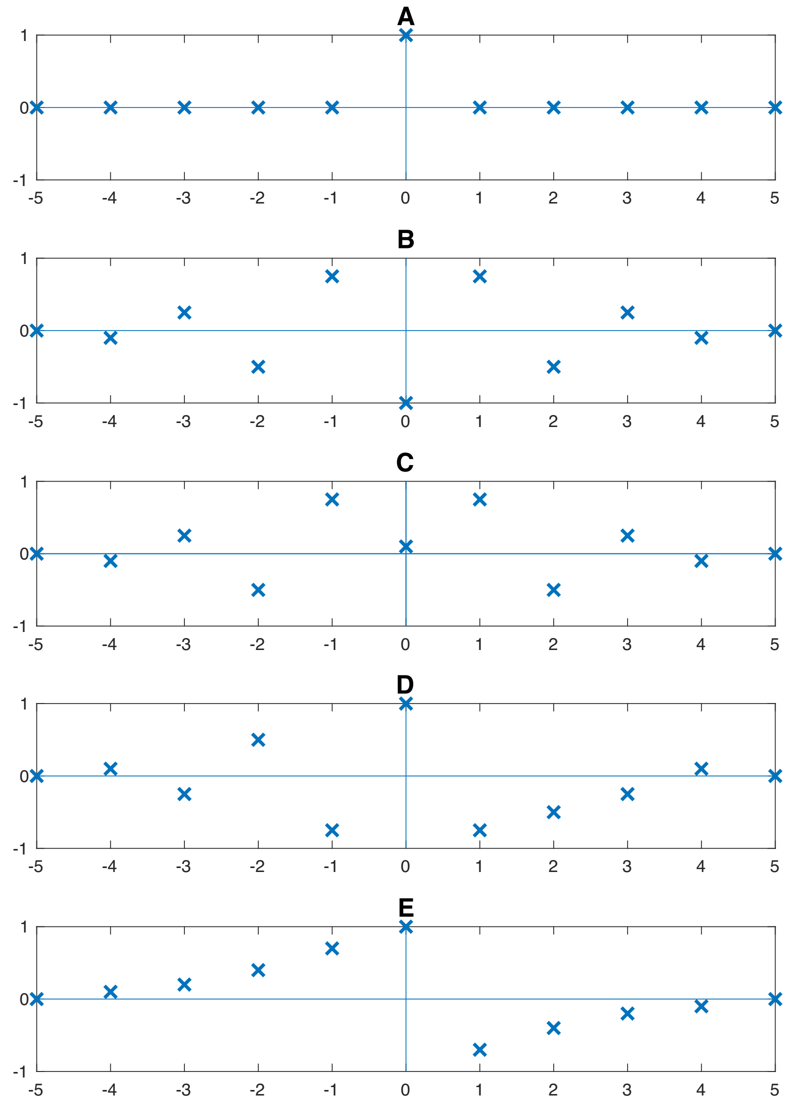
   C The autocovariance sequence of $\{aX_t\}$ is $\{a^2 s_\tau\}$

   D $\{(-1)^t X_t\}$ is a non-stationary process.

   E $\{X_t + at\}$ is a non-stationary process.

Answer: $E\{aX_t\} = aE\{X_t\} = 0$ *for all* t *and* $\mathrm{Cov}\{aX_t, aX_{t+\tau}\} = a^2\mathrm{Cov}\{X_t, X_{t+\tau}\}$, *which from stationarity of* $\{X_t\}$ *depends only on* τ, *so A is TRUE.* $E\{-X_t\} = -E\{X_t\} = 0$ *for all* t *and* $\mathrm{Cov}\{-X_t, -X_{t+\tau}\} = \mathrm{Cov}\{X_t, X_{t+\tau}\}$, *which from stationarity of* $\{X_t\}$ *depends only on* τ, *so B is TRUE. The autocovariance sequence of* $\{aX_t\}$ *is* $a^2 s_\tau$ *(see A), so C is TRUE.* $E\{(-1)^t X_t\} = (-1)^t E\{X_t\} = 0$, *and* $\mathrm{Cov}\{(-1)^t X_t, (-1)^{t+\tau} X_{t+\tau}\} = (-1)^{2t+\tau}\mathrm{Cov}\{X_t, X_{t+\tau}\} = (-1)^\tau s_\tau$, *which only depends on* τ, *hence* $\{(-1)^t X_t\}$ *is stationary and D is FALSE.* $E\{X_t + at\} = at$, *which depends on time, hence* $\{X + at\}$ *is non-stationary and E is TRUE. So the answer is D.*

7. Which of the following is a valid autocovariance sequence?



Answer: A fulfils all the requirements of an autocovariance sequence. B has $s_0 < 0$, however s_0 is the variance of the process which must be positive so this is not a valid autocovariance sequence. C is not a valid autocovariance sequence as we know from Cauchy-Swartz that $|s_\tau| \leq s_0$ which is not the case here. D and E are not valid autocovariance sequences as neither are symetric. Therefore, the answer is A.

8. Let $\{\epsilon_t\}$ be a zero mean white noise process of variance σ_ϵ^2. Which of the following statements is FALSE?

   A. $\{\epsilon_t\}$ is a stationary process.

   B. $E\{\epsilon_t \epsilon_{t+\tau}\} = 0$ for all $\tau \neq 0$.

   C. The random variable $(\epsilon_t + \epsilon_{t+1})$ is correlated with the random variable $(\epsilon_{t+1} + \epsilon_{t+2})$.

   D. $\sum_{\tau=-\infty}^{\infty} s_\tau = \sigma_\epsilon^2$.

   E. $\mathrm{Cov}\{\epsilon_{t+\tau}, \epsilon_{t-\tau}\} = 0$ for all τ.

A. White noise processes are always stationary because their autocovariance depends only on the lag τ ($s_\tau = 0$ for all $\tau \neq 0$ and $s_0 = \sigma_\epsilon^2$), so A is TRUE. B. A white noise process always has $\mathrm{Cov}\{\epsilon_t, \epsilon_{t+\tau}\} = E\{\epsilon_t \epsilon_{t+\tau}\} - \cancel{E\{\epsilon_t\}E\{\epsilon_{t+\tau}\}} = 0$ $\tau \neq 0$. *Therefore, because* $E\{\epsilon_t\} = E\{\epsilon_{t+\tau}\} = 0$ *as stated, we must have* $E\{\epsilon_t \epsilon_{t+\tau}\} = 0$, *so B is TRUE. C. We have* $\mathrm{Cov}\{(\epsilon_t + \epsilon_{t+1}), (\epsilon_{t+1} + \epsilon_{t+2})\} = \mathrm{Cov}\{\epsilon_t, \epsilon_{t+1}\} + \mathrm{Cov}\{\epsilon_t, \epsilon_{t+2}\} + \mathrm{Cov}\{\epsilon_{t+1}, \epsilon_{t+1}\} + \mathrm{Cov}\{\epsilon_{t+1}, \epsilon_{t+2}\} = 0 + 0 + \mathrm{Var}\{\epsilon_{t+1}\} + 0 = \sigma_\epsilon^2$. *Therefore they are correlated and C is TRUE. D. The sum presented is* $\ldots 0 + 0 + 0 + \sigma_\epsilon^2 + 0 + 0 + 0 + \ldots = \sigma_\epsilon^2$, *so D is TRUE. E.* $\mathrm{Cov}\{\epsilon_{t+\tau}, \epsilon_{t-\tau}\} = 0$ *for all* $\tau \neq 0$, *however* $\mathrm{Cov}\{\epsilon_{t+\tau}, \epsilon_{t-\tau}\} = \sigma_\epsilon^2$ *when* $\tau = 0$, *so E is FALSE and is the answer.*

9. Let $\{X_t\}$ be the MA(2) process

$$X_t = \epsilon_t - \frac{1}{2}\epsilon_{t-1} + \frac{1}{4}\epsilon_{t-2}, \quad \epsilon_t \sim N(0,1).$$

Which of the following statements is FALSE?

A. $s_2 = \frac{1}{4}$

B. $s_{-3} = 0$

C. X_t is Gaussian distributed.

D. $s_0 = \frac{7}{4}$

E. $E\{X_t\} = 0$.

Answer: A. $s_2 = \sigma_\epsilon^2 \sum_{j=0}^{2-2} \theta_{j,2}\theta_{j+2,2} = 1 \cdot (\theta_{0,2}\theta_{2,2}) = 1 \cdot (-1) \cdot (-\frac{1}{4}) = 1/4$, *so A is TRUE. B.* $s_\tau = 0$ *for all* $\tau > q$. *Here,* $q = 2$, *therefore* $s_{-3} = 0$, *so B is TRUE.* X_t *is the sum of Gaussian random variables, and therefore itself Gaussian. D.* $s_0 = \sigma_\epsilon^2 \sum_{j=0}^{2} \theta_{j,2}^2 = 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 = 21/16$, *so D is FALSE. E.* $E\{X_t\} = E\{\epsilon_t\} - \frac{1}{2}E\{\epsilon_{t-1}\} + \frac{1}{4}E\{\epsilon_{t-2}\} = 0$, *hence E is TRUE. Therefore, the answer is D.*

10. Consider the *stationary* AR(2) process

$$X_t = \frac{1}{2}X_{t-1} + \frac{1}{8}X_{t-2} + \epsilon_t + c,$$

where $\{\epsilon_t\}$ is a zero mean white noise process and c is a fixed constant. What is $\mu = E\{X_t\}$?

A. 0

B. $8c/3$

C. $3c/5$

D. $1 + c$

E. c.

Answer: Taking expectations of both sides, gives $E\{X_t\} = \frac{1}{2}E\{X_{t-1}\} + \frac{1}{8}E\{X_{t-2}\} + E\{\epsilon_t\} + c$*. It is given that this is stationary, therefore* $E\{X_t\} = \mu$ *is constant for all time, including times* $t-1$ *and* $t-2$*. Therefore,* $\mu = \frac{1}{2}\mu + \frac{1}{8}\mu + 0 + c$*. Solving, gives* $\mu = \frac{8c}{3}$*, so the answer is B.*

11. Consider an AR(1) process $X_t = \phi X_{t-1} + \epsilon_t$, where $0 < \phi < 1$ and $\{\epsilon_t\}$ is a zero mean white noise process with $\sigma_\epsilon^2 = 1$. What is $\sum_{\tau=-\infty}^{\infty} s_\tau$ equal to?

   A. 1

   B. $(\phi - 1)/(1 - \phi)$

   C. $1/(1 - \phi)^2$

   D. $\phi/(1 - \phi)$

   E. $2/(1 - \phi)$.

Answer: From the notes, we have $s_\tau = s_0 \phi^{|\tau|}$ and $s_0 = 1/(1 - \phi^2)$. Therefore,

$$\begin{aligned}
\sum_{\tau=-\infty}^{\infty} s_\tau &= \left(\sum_{\tau=-\infty}^{0} s_\tau \right) + \left(\sum_{\tau=0}^{\infty} s_\tau \right) - s_0 \\
&= \left(2 \sum_{\tau=0}^{\infty} s_\tau \right) - s_0 \\
&= s_0 \left(\left(2 \sum_{\tau=0}^{\infty} \phi^\tau \right) - 1 \right) \\
&= s_0 \left(\frac{2}{1 - \phi} - \frac{1 - \phi}{1 - \phi} \right) \\
&= s_0 \left(\frac{1 + \phi}{1 - \phi} \right) \\
&= \frac{1}{1 - \phi^2} \frac{1 + \phi}{1 - \phi} \\
&= \frac{1}{(1 - \phi)^2}
\end{aligned}$$

So the answer is C.

12. Which of the following is an ARMA(1,2)?

   A. $X_{t-1} + \epsilon_t = \epsilon_{t-2} + X_t$
   B. $X_t - X_{t-1} = \epsilon_t$
   C. $\epsilon_{t-1} + X_t = X_{t-1} + \epsilon_t$
   D. $X_{t-2} + \epsilon_t = \epsilon_{t-2} + X_t$
   E. $\epsilon_{t-2} = 2X_{t-2} + X_t - \epsilon_t$.

   Answer: A. is ARMA(1,2), B. is AR(1), C. is ARMA(1,1), D. is ARMA(2,2), E. is ARMA(2,2). So the answer is A.

13. Which of the following is the correct model name for this process?

$$\frac{1}{2}X_{t-3} - \epsilon_t = 3\epsilon_{t-2} - X_t$$

A. MA(2)

B. ARMA(3,2)

C. AR(5)

D. AR(3,2)

E. ARMA(1/2,3)

Answer: We can re-write this process as

$$X_t + \frac{1}{2}X_{t-3} = \epsilon_t + 3\epsilon_{t-2}. \tag{1}$$

This is an ARMA(3,2) process, so the answer is B. It has a $p = 3$ *order AR part, and the* $q = 2$ *order MA part, where* $\phi_{1,3} = 0$, $\phi_{2,3} = 0$, $\phi_{3,3} = 1/2$, $\theta_{1,2} = 0$, $\theta_{2,2} = -3$. *Note: AR(3,2) and ARMA(1/2,3) are not processes!*

Chapter 2.2

14. Consider the model $X_t = \mu_t + Y_t$, where $\{Y_t\}$ is a zero mean stationary process. Which one of the following statements is FALSE?

    A. $\{Y_t^{(d)}\}$ is stationary for all $d \geq 1$.

    B. If $\mu_t = \alpha + \beta t$, then $\{X_t^{(1)}\}$ is a stationary process.

    C. If $\mu_t = \alpha + \beta t$, then $\Delta X_t = \Delta Y_t$.

    D. $B^d X_t = \mu_{t-d} + Y_{t-d}$.

    E. $\Delta^3 = 1 - 3B + 3B^2 - B^3$.

    A. $\Delta Y_t = Y_t - Y_{t-1}$. *Therefore,* $E\{Y_t^{(1)}\} = E\{Y_t\} - E\{Y_{t-1}\} = 0$, *so has constant mean. Furthermore,* $\mathrm{Cov}\{Y_t^{(1)}, Y_{t+\tau}^{(1)}\} = \mathrm{Cov}\{Y_t - Y_{t-1}, Y_{t+\tau} - Y_{t+\tau-1}\} = \mathrm{Cov}\{Y_t, Y_{t+\tau}\} - \mathrm{Cov}\{Y_t, Y_{t+\tau-1}\} - \mathrm{Cov}\{Y_{t-1}, Y_{t+\tau}\} + \mathrm{Cov}\{Y_{t-1}, Y_{t+\tau-1}\} = s_\tau - s_{\tau-1} - s_{\tau+1} + s_\tau$, *where* $\{s_\tau\}$ *is the autocovariance sequence of* $\{Y_t\}$. *This only depends on* τ, *therefore* $\{\Delta Y_t\}$ *is a stationary process. By induction the result follows, therefore A is TRUE.*
    B. From notes, $X_t^{(1)} = \beta + Y_t^{(1)}$. *From A we know* $Y_t^{(1)}$ *is a zero mean stationary processes, therefore* $X_t^{(1)}$ *is a stationary process with mean* β. *So B is TRUE.*
    C. From B we know C is FALSE.
    D. $B^d X_t = X_{t-d} = \mu_{t-d} + Y_{t-d}$, *so D is TRUE.*
    E. $\Delta^3 = (1 - B)^3 = 1 - 3B + 3B^2 - B^3$. *So E is TRUE.*
    The answer is C.

15. Consider the model $X_t = \nu_t + Y_t$, where $\{Y_t\}$ is a zero mean stationary process, and $\nu_t = \sin(\pi t/10)$. Which of the following operations would remove seasonality?

A. $1 - B^{10}$

B. $1 - B^5$

C. $1 - B^{12}$

D. Δ^{10}

E. $1 - B^{20}$

F. Δ^5

G. B^{10}

H. Δ^{20}

$\nu_t = \sin(\pi t/10)$ *is a sinusoid with a time period of* 20. *Therefore, from the notes,* $1 - B^{20}$ *is the correct operator to remove seasonality, so the answer is E.*

16. Consider the model

$$X_t = \mu_t + \nu_t + Y_t$$

where $\{Y_t\}$ is a zero mean stationary process, μ_t is a second order polynomial trend and ν_t is a seasonality component with period 52. Which of the following is the resulting process when the operator $(1 - B^{52})\Delta^3$ is applied to $\{X_t\}$?

A. $Y_t^{(3)} - Y_{t-52}^{(3)}$

B. $Y_t^{(3)}$

C. $Y_t - Y_{t-52}$

D. $Y_t^{(3)} + \nu_t^{(3)}$

E. $Y_t^{(3)} + \mu_t - \mu_{t-52}$

We have that

$$\begin{aligned}
((1 - B^{52})\Delta^3)X_t &= (1 - B^{52})(\Delta^3 X_t) = (1 - B^{52})X_t^{(3)} \\
&= (1 - B^{52})(Y_t^{(3)} + \nu_t - 3\nu_{t-1} + 3\nu_{t-2} - \nu_{t-3}) \\
&= Y_t^{(3)} - Y_{t-52}^{(3)} + \nu_t - \nu_{t-52} - 3\nu_{t-1} + 3\nu_{t-53} + 3\nu_{t-2} - 3\nu_{t-54} - \nu_{t-3} + \nu_{t-55} \\
&= Y_t^{(3)} - Y_{t-52}^{(3)},
\end{aligned}$$

so the answer is A.

Chapter 2.3

17. Consider the ARMA process

$$X_t = \sum_{k=1}^{3} \frac{1}{2^k} X_{t-k} + \sum_{l=0}^{3} \frac{1}{2^l} \epsilon_{t-l}$$

Which of the following is the correct form of the characteristic polynomial $\Theta(B)$?

A. $\Theta(B) = 1 - \frac{1}{2}B - \frac{1}{4}B^2 - \frac{1}{8}B^3$
B. $\Theta(B) = 1 + \frac{1}{2}B + \frac{1}{4}B^2 + \frac{1}{8}B^3$
C. $\Theta(B) = 1 - \frac{1}{2}B + \frac{1}{4}B^2 - \frac{1}{8}B^3$
D. $\Theta(B) = 1 - \frac{1}{2}B$
E. $\Theta(B) = 1 - \frac{1}{8}B^3$

The ARMA process can be written as

$$\begin{aligned}
X_t - \sum_{k=1}^{3} \frac{1}{2^k} X_{t-k} &= \sum_{l=0}^{3} \frac{1}{2^l} \epsilon_{t-l} \\
X_t - \frac{1}{2}X_{t-1} - \frac{1}{4}X_{t-2} - \frac{1}{8}X_{t-3} &= \epsilon_t + \frac{1}{2}\epsilon_{t-1} + \frac{1}{4}\epsilon_{t-2} + \frac{1}{8}\epsilon_{t-3} \\
(1 - \frac{1}{2}B - \frac{1}{4}B^2 - \frac{1}{8}B^3)X_t &= (1 + \frac{1}{2}B + \frac{1}{4}B^2 + \frac{1}{8}B^3)\epsilon_t \\
\Phi(B)X_t &= \Theta(B)\epsilon_t
\end{aligned}$$

So the answer is B.

18. Consider the General Linear Process

$$X_t = \sum_{k=0}^{\infty} \frac{1}{2^k} \epsilon_{t-k}$$

where $\{\epsilon_t\}$ is a white noise process with variance σ_ϵ^2. What is the variance of $\{X_t\}$?

A. $2\sigma_\epsilon^2$

B. $4\sigma_\epsilon^2/3$

C. 1

D. σ_ϵ^2

E. $4\sigma_\epsilon^2/5$

From the notes: $s_0 = \sigma_\epsilon^2 \sum_{k=0}^{\infty} g_k^2 = \sigma_\epsilon^2 \sum_{k=0}^{\infty} 1/2^{2k} = \sigma_\epsilon^2 \sum_{k=0}^{\infty} 1/4^k = \sigma_\epsilon^2/(1-1/4) = 4\sigma_\epsilon^2/3$. *So the answer is B.*

19. Consider again the General Linear Process

$$X_t = \sum_{k=0}^{\infty} \frac{1}{2^k} \epsilon_{t-k}$$

where $\{\epsilon_t\}$ is a white noise process with variance σ_ϵ^2. What is $s_1 \equiv \text{Cov}\{X_t, X_{t+1}\}$?

A. $2\sigma_\epsilon^2/3$

B. $4\sigma_\epsilon^2/3$

C. $1/2$

D. $\sigma_\epsilon^2/2$

E. $2\sigma_\epsilon^2/5$

Again, from the notes:

$$\begin{aligned}
s_1 &= \sigma_\epsilon^2 \sum_{k=0}^{\infty} g_k g_{k+1} \\
&= \sigma_\epsilon^2 \sum_{k=0}^{\infty} \frac{1}{2^k} \cdot \frac{1}{2^{k+1}} \\
&= \frac{\sigma_\epsilon^2}{2} \sum_{k=0}^{\infty} \frac{1}{2^{2k}} \\
&= \frac{\sigma_\epsilon^2}{2} \sum_{k=0}^{\infty} \frac{1}{4^k} = \frac{\sigma_\epsilon^2}{2} \cdot \frac{4}{3} = \frac{2\sigma_\epsilon^2}{3},
\end{aligned}$$

so the answer is A.

20. Consider a process of the form

$$X_t = \frac{G_1(B)}{G_2(B)}\epsilon_t.$$

Which of the following can be written in General Linear Process form?

A. $G_1(z) = z^2 + 3z - 4$, $G_2(z) = z^2 - z + \frac{1}{4}$.

B. $G_1(z) = z^2 - 1$, $G_2(z) = z^2 + \frac{3}{2}z - 1$.

C. $G_1(z) = z^2 + 3z + 2$, $G_2(z) = z^2 - 4$.

We need to check that the poles of $G(z)$ lie outside of the unit circle. That's equivalent to checking if the roots of $G_2(z)$ lie outside of the unit circle. A. $G_2(z) = (z-\frac{1}{2})(z-\frac{1}{2})$, so a double root at $z = \frac{1}{2}$, which is insider the unit circle. This CANNOT be written in GLP form. B. $G_2(z) = (z+2)(z-\frac{1}{2})$, which has roots $z = -2$ and $z = \frac{1}{2}$. One of these is inside the unit circle. This CANNOT be written in GLP form. C. $G_2(z) = (z-2)(z+2)$, which has roots $z = \pm 2$. Both are outside the unit circle. This CAN be written in GLP form. Therefore, the answer is C.

21. Consider the MA(2) process

$$X_t = \epsilon_t + \frac{1}{4}\epsilon_{t-2}.$$

Which of the following statements is TRUE?

A. It is stationary and invertible.

B. It is stationary but not invertible.

C. It is not stationary but is invertible.

D. It is not stationary and it is not invertible.

All MA(q) processes are stationary. To check invertibility, we consider the form

$$X_t = \left(1 + \frac{1}{4}B^2\right)\epsilon_t$$

Therefore,

$$\Theta(z) = 1 + \frac{1}{4}z^2.$$

To find roots of $\Theta(z)$, *we have* $1 + \frac{1}{4}z^2 = 0 \implies 4 + z^2 = 0 \implies (z-2i)(z+2i) = 0 \implies z = \pm 2i$. *These both lie outside of the unit circle, therefore the process is invertible and the answer is A.*

22. Consider the ARMA(2,1) process

$$X_t = \frac{31}{20}X_{t-1} - \frac{3}{5}X_{t-2} + \epsilon_t - \frac{4}{3}\epsilon_{t-1}.$$

Which of the following statements is TRUE?

A. It is stationary and invertible.

B. It is stationary but not invertible.

C. It is not stationary but is invertible.

D. It is not stationary and it is not invertible.

We can rewrite this as

$$\begin{aligned} X_t - \frac{31}{20}X_{t-1} + \frac{3}{5}X_{t-2} &= \epsilon_t - \frac{4}{3}\epsilon_{t-1} \\ (1 - \frac{31}{20}B + \frac{3}{5}B^2)X_t &= (1 - \frac{4}{3}B)\epsilon_t \end{aligned}$$

Therefore, in characteristic polynomial form we have

$$\Phi(z) = 1 - \frac{31}{20}z + \frac{3}{5}z^2; \qquad \Theta(z) = 1 - \frac{4}{3}z.$$

The roots of $\Phi(z)$ are $\frac{31\pm1}{24} = \frac{4}{3}, \frac{5}{4}$. These are both outside the unit circle so the process is stationary.

The root of $\Theta(z)$ is 3/4 which is inside the unit circle so the process is not invertible (i.e., not representable as a well-defined autoregression). Therefore, the answer is B.

23. Consider the stationary and invertible (check this if you want) ARMA(1,1) process

$$X_t = \frac{1}{2}X_{t-1} + \epsilon_t + \frac{1}{8}\epsilon_{t-1},$$

where $\{\epsilon_t\}$ is a white noise process with variance σ_ϵ^2. What is the variance of $\{X_t\}$?

A. $\frac{72}{48}\sigma_\epsilon^2$.

B. $\frac{73}{49}\sigma_\epsilon^2$.

C. $\frac{71}{47}\sigma_\epsilon^2$.

D. $\frac{73}{48}\sigma_\epsilon^2$.

E. $\frac{74}{49}\sigma_\epsilon^2$.

The general linear process form is $X_t = G(B)\epsilon_t$ where $G(z) = \frac{\Theta(z)}{\Phi(z)} = \frac{1+\frac{1}{8}z}{1-\frac{1}{2}z}$. Expanding gives

$$\begin{aligned}
G(z) &= \left(1+\frac{1}{8}z\right)\left(1+\frac{1}{2}z+\frac{1}{4}z^2+\frac{1}{8}z^3+...\right) \\
&= 1+\left(\frac{1}{2}+\frac{1}{8}\right)z+\left(\frac{1}{4}+\frac{1}{16}\right)z^2+\left(\frac{1}{8}+\frac{1}{32}\right)z^3+... \\
&= 1+\frac{5}{4}\sum_{k=1}^{\infty}\frac{1}{2^k}z^k.
\end{aligned}$$

Therefore, the general linear process form is $X_t = \epsilon_t + \frac{5}{4}\sum_{k=1}^{\infty}\frac{1}{2^k}\epsilon_{t-k}$. For a process $\{X_t\}$ in general linear process form $X_t = \sum_{k=0}^{\infty} g_k\epsilon_{t-k}$, we have $\mathrm{Var}\{X_t\} = \sigma_\epsilon^2\sum_{k=0}^{\infty} g_k^2$. Therefore

$$\begin{aligned}
\mathrm{Var}\{X_t\} &= \sigma_\epsilon^2\left(1+\sum_{k=1}^{\infty}\left(\frac{5}{4}\right)^2\cdot\left(\frac{1}{2^k}\right)^2\right) \\
&= \sigma_\epsilon^2\left(1+\frac{25}{16}\sum_{k=1}^{\infty}\frac{1}{4^k}\right) \\
&= \sigma_\epsilon^2\left(1+\frac{25}{16}\cdot\frac{1}{4}\sum_{k=0}^{\infty}\frac{1}{4^k}\right) \\
&= \sigma_\epsilon^2\left(1+\frac{25}{64}\cdot\frac{1}{1-\frac{1}{4}}\right) \\
&= \sigma_\epsilon^2\left(1+\frac{25}{64}\cdot\frac{4}{3}\right) \\
&= \sigma_\epsilon^2\left(1+\frac{25}{48}\right) = \frac{73}{48}\sigma_\epsilon^2.
\end{aligned}$$

So the answer is D.

Chapter 3.1

24. Let $\{X_t\}$ be a stationary random process with mean zero, autocovariance sequence $\{s_\tau\}$, integrated spectrum $S^{(I)}(f)$ and spectral density function $S(f)$. Which of the following statements is FALSE?

   A. $S(0) = \sum_{\tau=-\infty}^{\infty} s_\tau$.

   B. $\frac{\mathrm{d}}{\mathrm{df}} S^{(I)}(f)$ is an even function.

   C. $S^{(I)}(f)$ is an even function.

   D. $\int_{-1/2}^{0} S(f)\mathrm{df} = \mathrm{s}_0/2$.

A. From the Fourier relationship

$$S(f) = \sum_{\tau=-\infty}^{\infty} s_\tau \mathrm{e}^{-\mathrm{i}2\pi f\tau},$$

therefore, $S(0) = \sum_{\tau=-\infty}^{\infty} s_\tau \mathrm{e}^0 = \sum_{\tau=-\infty}^{\infty} s_\tau$*, so A is TRUE.*

B. $\frac{\mathrm{d}}{\mathrm{df}} S^{(I)}(f) = S(f)$*, and we know that* $S(f)$ *is even, therefore, B is TRUE.*

C. $S^{(I)}(f)$ *is a monotonically non-decreasing function with* $S^{(I)}(-1/2) = 0$ *and* $S^{(I)}(1/2) = s_0$*. Therefore is is not even and C is FALSE.*

D. From the Fourier relationship $s_0 = \int_{-1/2}^{1/2} S(f)\mathrm{df}$*, and* $S(f)$ *is an even function, therefore* $\int_{-1/2}^{1/2} S(f)\mathrm{df} = \mathrm{s}_0/2$*, so D is TRUE.*

The answer is C.

25. Let $\{X_t\}$ is a stationary random process with mean zero and autocovariance sequence

$$s_\tau = \begin{cases} 1 & \tau = 0 \\ -1/2 & |\tau| = 1 \\ 0 & |\tau| > 1. \end{cases}$$

Which of the following is the spectral density function for $\{X_t\}$?

A. $S(f) = 1$

B. $S(f) = 1 - \cos(2\pi f)$

C. $S(f) = 1 + \frac{1}{2}\mathrm{e}^{\mathrm{i}2\pi f}$

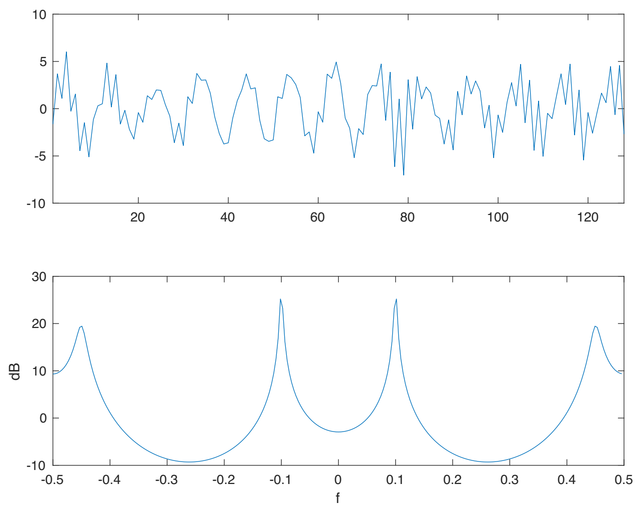
D. $S(f) = 1 + \sin(2\pi f)$

E. $S(f) = 1 - \sin(2\pi f)$

$$\begin{aligned} S(f) &= \sum_{\tau=-\infty}^{\infty} s_\tau \mathrm{e}^{-\mathrm{i}2\pi f\tau} \\ &= -\frac{1}{2}\mathrm{e}^{\mathrm{i}2\pi f} + \mathrm{e}^0 - \frac{1}{2}\mathrm{e}^{-\mathrm{i}2\pi f} \\ &= 1 - \frac{1}{2}(\mathrm{e}^{\mathrm{i}2\pi f} + \mathrm{e}^{-\mathrm{i}2\pi f}) = 1 - \cos(2\pi f) \end{aligned}$$

So the answer is B.

26. Here is shown a single realization and the spectral density function for a stationary random process.



Which of the following is a correct statement?

A. This is a white noise process.

B. This process has a purely discrete spectra.

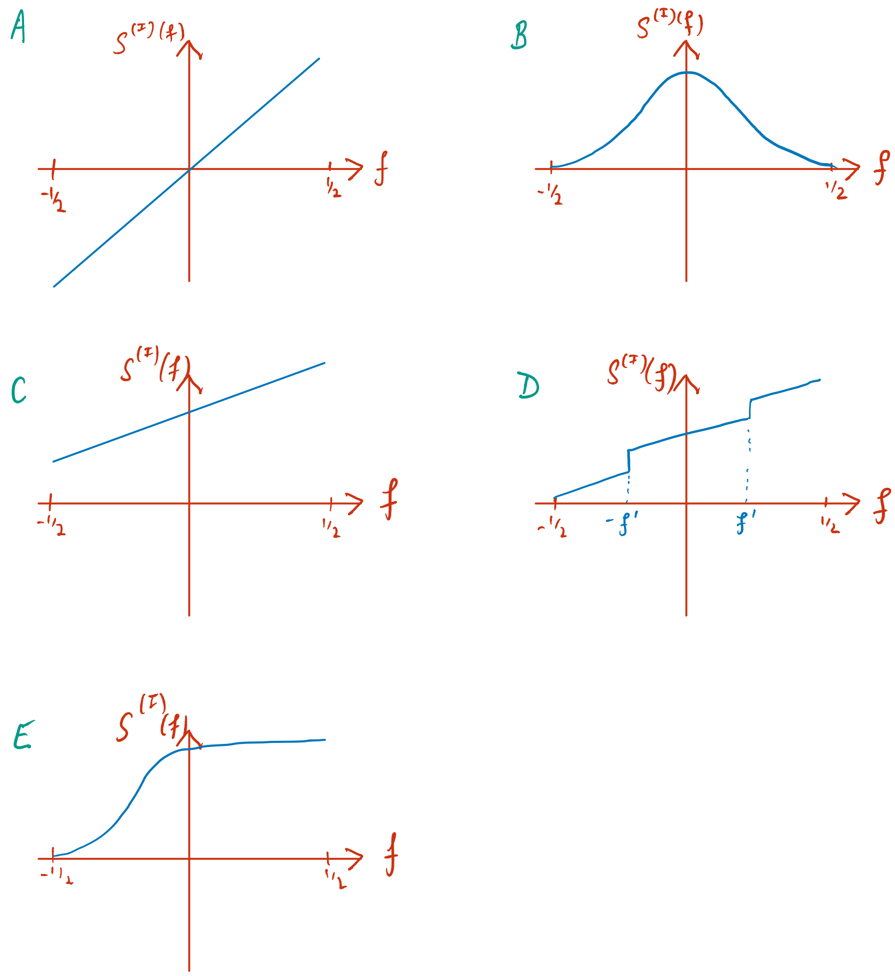
C. This process exhibits strong oscillatory behaviour at frequencies 0.1 and 0.45.

D. This process has a mixed spectra.

E. Successive values of this process are uncorrelated.

A. This is not a flat spectrum so the process is not white. B. This has a purely continuous spectrum so this is not correct. C. This process does exhibit strong oscillatory behaviour at those stated frequencies. D. This process has no discrete elements to it so it is not mixed. E. As already stated, this is not a white noise process so E is not true. The answer is C.

27. Which of the following is a valid integrated spectrum?



Integrated spectra take non-negative values only, therefore A is not valid.

Integrated spectra are monotonically non-decreasing, therefore B is not valid.

Integrated spectra have $S^{(I)}(f) = 0$*, therefore C is not valid.*

D has all the properties of an integrated spectrum.

$S(f) = \frac{\mathrm{dS^{(I)}(f)}}{\mathrm{df}}$ *must be even, therefore E is not valid.*

The answer is D

28. How would you classify the valid integrated spectrum from Question 27 (using the classification system given in the notes)?

    A. Purely continuous.

    B. Purely discrete.

    C. Mixed.

    D. Discrete.

    Integrated spectrum D is white noise with an additional discrete element oscillating at frequency f'. Therefore, this is classed as "discrete", so the answer is D.

Chapter 3.2

29. I sample a continuous time random process at 9am GMT every Monday. What is the Nyquist frequency in units hour^{-1}?

    A. 1/14.
    B. 1/336.
    C. 1/48.
    D. 1/168.
    E. 1/7.

    The sampling rate is $\Delta t = 1$ *week, which is equivalent to* $\Delta t = 168$ *hours. The Nyquist frequency is therefore* $1/(2\Delta t) = 1/336$ *hour*$^{-1}$*. So the answer is B.*

30. A continuous-time stationary process $\{X(t)\}$, with t in seconds (s), has spectral density function

$$S_{X(t)}(f) = \begin{cases} 1 - \frac{1}{4}(|f| - 6), & 6 < |f| \leq 10, \\ 0, & \text{otherwise}, \end{cases}$$

with f in cycles/s. It is sampled with a sample interval $\Delta t = 0.1$s to produce the discrete-time process $\{X_t\}$.

What is the spectral density function $S_{X_t}(f)$ of $\{X_t\}$ for $|f| < f_{\mathcal{N}}$, where $f_{\mathcal{N}}$ is the Nyquist frequency?

A.
$$S_{X_t}(f) = \begin{cases} \frac{1}{4}|f|, & |f| \leq 4, \\ 0, & 4 < |f| \leq 5 \end{cases}$$

B.
$$S_{X_t}(f) = \begin{cases} 1 - \frac{1}{4}|f|, & |f| \leq 4, \\ 0, & 4 < |f| \leq 5 \end{cases}$$

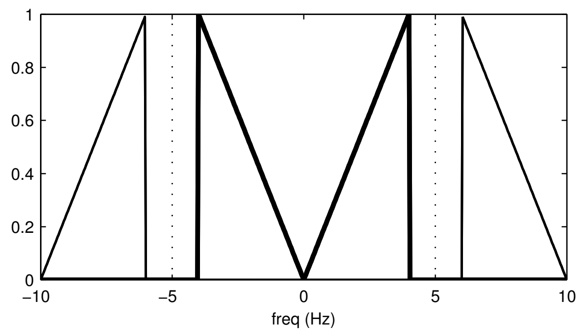
C.
$$S_{X_t}(f) = S_{X(t)}(f), \qquad |f| < 10.$$

D.
$$S_{X_t}(f) = 1 - |f|, \qquad |f| < 10.$$

E.
$$S_{X_t}(f) = \frac{1}{4}|f|, \qquad |f| < 5.$$

See Figure. The non-zero spectrum of the continuous-time process is delineated by the thin lines making up two triangles symmetric about zero. The aliasing (Nyquist) frequency is given by $f_{\mathcal{N}} = 1/(2\Delta t) = 5Hz$ *when* $\Delta t = 0.1s$*. The non-zero spectrum of the discretely-sampled process is found by reflecting about the Nyquist frequency, giving the two triangles delineated by the heavy lines. The spectrum for* $|f| < f_{\mathcal{N}}$ *is thus*

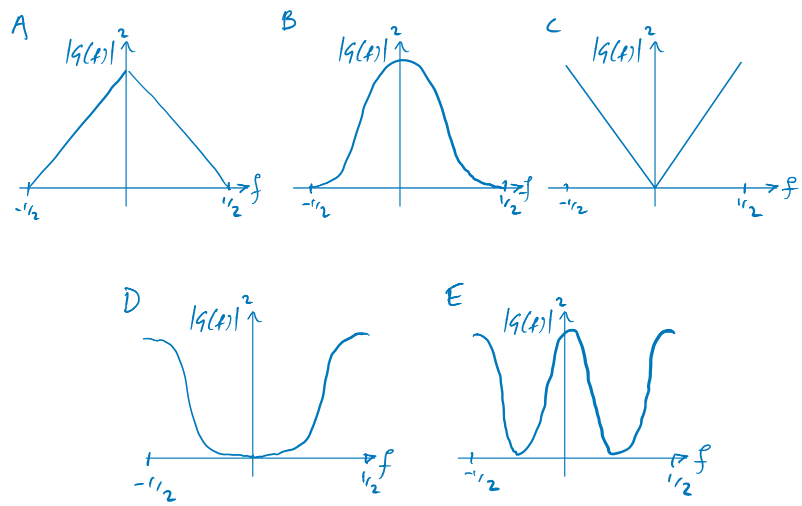
$$S(f) = \begin{cases} \frac{1}{4}|f|, & |f| \le 4, \\ 0, & 4 < |f| \le 5 \end{cases}$$



So the answer is A.

Chapter 3.3

31. Consider the linear filter $L\{X_t\} = X_t - \frac{1}{2}X_{t-1} - \frac{1}{2}X_{t+1}$.
Which of the following is $|G(f)|^2$?



There are two equivalent methods:

(i)

$$
\begin{aligned}
L\{e^{i2\pi ft}\} &= e^{i2\pi ft} - \frac{1}{2}e^{i2\pi f(t-1)} + -\frac{1}{2}e^{i2\pi f(t+1)} \\
&= e^{i2\pi ft}\left(1 - \frac{1}{2}e^{-i2\pi f} - \frac{1}{2}e^{i2\pi f}\right) \\
&= e^{i2\pi ft}\,(\underbrace{1 - \cos(2\pi f)}_{y_{\text{[illegible]}} f \,=\, G(f)})
\end{aligned}
$$

Therefore, $G(f) = 1 - \cos(2\pi f)$.

(ii) $g_{-1} = g_1 = -\frac{1}{2}$, $g_0 = 1$, $g_u = 0$ *for all* $|u| > 1$. *Taking the Fourier transform gives*

$$
G(f) = \sum_{u=-\infty}^{\infty} g_u e^{-i2\pi fu} = 1 - \frac{1}{2}e^{-i2\pi f} - \frac{1}{2}e^{i2\pi f} = 1 - \cos(2\pi f).
$$

Therefore, $|G(f)|^2 = (1 - \cos(2\pi f))^2$ *and the answer is D.*

32. The filter from Question 31 is which of the following types?

    A. Low band-pass filter.

    B. High band-pass filter.

    This filter will suppress low frequencies and pass through high frequencies. Therefore, it is a high band-pass filter (B).

33. Consider the process

$$X_t = \frac{1}{2}X_{t-1} + \epsilon_t + \frac{1}{2}\epsilon_{t-1} + \frac{1}{4}\epsilon_{t-2},$$

where $\{\epsilon_t\}$ is a white noise process with variance σ_ϵ^2. What is its spectral density function?

A.
$$S(f) = \sigma_\epsilon^2 \; \frac{21 + 8\cos(2\pi f) + 20\cos(4\pi f)}{16 + 20\cos(2\pi f)}$$

B.
$$S(f) = \sigma_\epsilon^2 \; \frac{20 + 16\cos(2\pi f)}{20 + 18\cos(2\pi f) + 8\cos(4\pi f)}$$

C.
$$S(f) = \sigma_\epsilon^2 \; \frac{20 + 18\cos(2\pi f) + 8\cos(4\pi f)}{20 + 16\cos(2\pi f)}$$

D.
$$S(f) = \sigma_\epsilon^2 \; \frac{20 + 16\cos(2\pi f)}{21 + 20\cos(2\pi f) + 8\cos(4\pi f)}$$

E.
$$S(f) = \sigma_\epsilon^2 \; \frac{21 + 20\cos(2\pi f) + 8\cos(4\pi f)}{20 - 16\cos(2\pi f)}$$

Rearranging this process gives

$$X_t - \frac{1}{2}X_{t-1} = \epsilon_t + \frac{1}{2}\epsilon_{t-1} + \frac{1}{4}\epsilon_{t-2},$$

Giving $\Phi(z) = 1 - \frac{1}{2}z$ *and* $\Theta(z) = 1 + \frac{1}{2}z + \frac{1}{4}z^2$. *Therefore, from the notes*

$$\begin{aligned}
S(f) &= \sigma_\epsilon^2 \frac{|1 + \frac{1}{2}\mathrm{e}^{-\mathrm{i}2\pi f} + \frac{1}{4}\mathrm{e}^{-\mathrm{i}4\pi f}|^2}{|1 - \frac{1}{2}\mathrm{e}^{-\mathrm{i}2\pi f}|^2} \\
&= \sigma_\epsilon^2 \frac{1 + \frac{1}{2}\mathrm{e}^{\mathrm{i}2\pi f} + \frac{1}{4}\mathrm{e}^{\mathrm{i}4\pi f} + \frac{1}{2}\mathrm{e}^{-\mathrm{i}2\pi f} + \frac{1}{4} + \frac{1}{8}\mathrm{e}^{\mathrm{i}2\pi f} + \frac{1}{4}\mathrm{e}^{-\mathrm{i}4\pi f} + \frac{1}{8}\mathrm{e}^{-\mathrm{i}2\pi f} + \frac{1}{16}}{1 - \frac{1}{2}\mathrm{e}^{\mathrm{i}2\pi f} - \frac{1}{2}\mathrm{e}^{-\mathrm{i}2\pi f} + \frac{1}{4}} \\
&= \sigma_\epsilon^2 \; \frac{\frac{21}{16} + \frac{5}{4}\cos(2\pi f) + \frac{1}{2}\cos(4\pi f)}{\frac{5}{4} - \cos(2\pi f)} = \sigma_\epsilon^2 \; \frac{21 + 20\cos(2\pi f) + 8\cos(4\pi f)}{20 - 16\cos(2\pi f)}.
\end{aligned}$$

So the answer is E.

34. Let $\Phi(B)X_t = \epsilon_t$ be an AR(2) process where $\Phi(z)$ has complex conjugate roots and $\{X_t\}$ has has spectral density function

$$S(f) = \frac{\sigma_\epsilon^2}{[1-\cos(2\pi(0.125-f))+0.25][1-\cos(2\pi(0.125+f))+0.25]}.$$

Expressing $\{X_t\}$ in the form $X_t = \phi_{1,2}X_{t-1} + \phi_{2,2}X_{t-2} + \epsilon_t$, what are the parameters $\phi_{1,2}$ and $\phi_{2,2}$.

A. $\phi_{1,2} = 1/\sqrt{2}$, $\phi_{2,2} = -1/4$

B. $\phi_{1,2} = 1/4$, $\phi_{2,2} = 1/\sqrt{2}$

C. $\phi_{1,2} = -1/\sqrt{2}$, $\phi_{2,2} = -1/\sqrt{2}$

D. $\phi_{1,2} = -1/4$, $\phi_{2,2} = 1/\sqrt{2}$

E. $\phi_{1,2} = 1/\sqrt{2}$, $\phi_{2,2} = 1/4$

To obtain the parameters $\phi_{1,2}$ and $\phi_{2,2}$ we recognise

$$\begin{aligned}\Phi(z) &= (1 - r\mathrm{e}^{\mathrm{i}2\pi f'}z)(1 - r\mathrm{e}^{-\mathrm{i}2\pi f'}z)\\ &= 1 - r(\mathrm{e}^{\mathrm{i}2\pi f'} + \mathrm{e}^{-\mathrm{i}2\pi f'})z + r^2z^2\\ &= 1 - 2r\cos(2\pi f')z + r^2z^2.\end{aligned}$$

Therefore, $\phi_{1,2} = 2r\cos(2\pi f')$ and $\phi_{2,2} = -r^2$. From the notes, we know that AR(2) processes with roots of this type have spectral density

$$S_X(f) = \frac{\sigma_\epsilon^2}{(1-2r\cos(2\pi(f'+f))+r^2)(1-2r\cos(2\pi(f'-f)+r^2)}.$$

We therefore identify $f' = 0.125$ and $r^2 = 0.25$. Substituting the identified values of r and f', we get $\phi_{1,2} = \cos(\pi/4) = 1/\sqrt{2}$ and $\phi_{2,2} = -1/4$. Therefore $X_t = \frac{1}{\sqrt{2}}X_{t-1} - \frac{1}{4}X_{t-2} + \epsilon_t$, and the answer is A.

Chapter 4.1

35. An estimator $\hat{\theta}$ of a parameter θ has $\text{MSE}\{\hat{\theta}\} = 3$ and $\text{Var}\{\hat{\theta}\} = 3$. What is $E\{\hat{\theta}\}$?

    A. 0.

    B. 3.

    C. $-\theta$.

    D. $\theta + 3$.

    E. θ.

    Using $\text{MSE}\{\hat{\theta}\} = \text{bias}^2\{\hat{\theta}\} + \text{Var}\{\hat{\theta}\}$ *gives* $\text{bias}^2\{\hat{\theta}\} = 0$. *Therefore* $\text{bias}\{\hat{\theta}\} = E\{\hat{\theta}\} - \theta$, *implies* $0 = E\{\hat{\theta}\} - \theta$ *which gives* $E\{\hat{\theta}\} = \theta$. *So the answer is E.*

36. An estimator $\hat{\theta}$ of a parameter θ has $E\{\hat{\theta}\} = \theta + c$, and $\text{Var}\{\hat{\theta}\} = c^2/2$. What is the mean square error of $\hat{\theta}$?

A. $c^2/2$.

B. $c + c^2/2$.

C. $3c^2/2$.

D. $\theta + c + c^2/2$.

E. $(\theta + c)^2 + c^2/2$.

We have $\text{bias}\{\hat{\theta}\} = E\{\hat{\theta}\} - \theta = c$ *and* $\text{Var}\{\hat{\theta}\} = c^2/2$. *Therefore* $\text{MSE}\{\hat{\theta}\} = \text{bias}^2\{\hat{\theta}\} + \text{Var}\{\hat{\theta}\} = c^2 + c^2/2 = 3c^2/2$. *So the answer is C.*

37. Let X_1, X_2 be a portion of an AR(1) process $X_t = \frac{1}{2}X_{t-1} + \epsilon_t$, where $\{\epsilon_t\}$ is a white noise process with zero mean and $\sigma_\epsilon^2 = 1$. What is $\mathrm{Var}\{\bar{X}\}$, where $\bar{X} = \frac{1}{2}(X_1 + X_2)$?

A. $\frac{10}{9}$.

B. 1.

C. $\frac{7}{2}$.

D. $\frac{5}{4}$.

E. $\frac{8}{3}$.

$$s_\tau = s_0 \phi^{|\tau|}$$

$$s_0 = \frac{\sigma_\varepsilon^2}{1-\phi^2} = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

The AR(1) process of this type as acvs $s_\tau = \left(\frac{1}{2}\right)^{|\tau|}\frac{4}{3}$. *Therefore, using the formula in the notes*

$$\begin{aligned}\mathrm{Var}\{\bar{X}\} &= \frac{1}{2}\sum_{\tau=-1}^{1}\left(1-\frac{|\tau|}{2}\right)\left(\frac{1}{2}\right)^{|\tau|}\frac{4}{3} \\ &= \frac{1}{2}\left(\frac{1}{2}\cdot\frac{4}{6}+\frac{4}{3}+\frac{1}{2}\cdot\frac{4}{6}\right) = 1\end{aligned}$$

So the answer is B.

38. Let $X_1, X_2, ..., X_{100}$ be a portion of an MA(2) process $X_t = \epsilon_t - \frac{1}{2}\epsilon_{t-1} + \frac{1}{4}\epsilon_{t-2}$, where $\{\epsilon_t\}$ is zero mean white noise process with $\sigma_\epsilon^2 = 2$. What is $E\{\hat{s}_1^{(p)}\}$ computed from this portion?

A. $\frac{99}{100}$.

B. $-\frac{99}{100}$.

C. $\frac{100}{99}$.

D. $-\frac{99}{80}$.

E. $-\frac{80}{99}$.

From the notes we have

$$E\{\hat{s}_\tau^{(p)}\} = \left(1 - \frac{|\tau|}{N}\right) s_\tau,$$

and therefore

$$E\{\hat{s}_1^{(p)}\} = \left(1 - \frac{1}{100}\right) s_1 = \frac{99}{100} s_1.$$

We are left with finding s_1. This is an MA(2) process with $\theta_{0,2} = -1$, $\theta_{1,2} = 1/2$ and $\theta_{2,2} = -1/4$. Therefore, $s_1 = 2 \cdot \sum_{j=0}^{1} \theta_{j,2}\theta_{j+1,2} = 2(\theta_{0,2}\theta_{1,2} + \theta_{1,2}\theta_{2,2}) = 2(-1/2 - 1/8) = -5/4$. Therefore $E\{\hat{s}_1^{(p)}\} = -\frac{99}{100} \cdot \frac{5}{4} = 495/400 = -99/80$. The answer is D.

Chapter 4.2

39. Which of the following statements is FALSE

    A. The periodogram is an asymptotically ($N \to \infty$) unbiased estimator of $S(f)$.

    B. The effect of side-lobe leakage is greater for processes with large dynamic range.

    C. The periodogram for a white noise process is unbiased but the direct spectral estimator for a white noise process is biased.

    D. Tapering (with a non-rectangular taper such as those given in the notes) reduces side-lobe leakage.

    E. $\hat{s}_0^{(p)} = \int_{-1/2}^{1/2} \hat{S}^{(p)}(f)\mathrm{df}$.

A is TRUE (see notes). B. is TRUE (see notes). C. is FALSE. Following an analogous argument to that presented for the periodogram, the direct spectral estimator when applied to a white noise process is also unbiased. D is TRUE (see notes). E is TRUE because $\{\hat{s}_\tau^{(p)}\}$ *and* $\hat{S}^{(p)}(f)$ *form a Fourier transform pair, i.e.* $\hat{s}_\tau^{(p)} = \int_{-1/2}^{1/2} S(f)\mathrm{e}^{\mathrm{i}2\pi f\tau}\mathrm{df}$*, therefore* $\hat{s}_0^{(p)}$ *is as given. So the answer is C.*

40. For any taper, it can be shown that

$$\int_{-1/2}^{1/2} E\{S^{(d)}(f)\}\mathrm{df} = \mathrm{s}_0.$$

Which of the following is a true statement about the periodogram?

A. $\int_{-1/2}^{1/2} E\{S^{(p)}(f)\}$df is less than s_0.

B. $\int_{-1/2}^{1/2} E\{S^{(p)}(f)\}$df is greater than s_0.

C. $\int_{-1/2}^{1/2} E\{S^{(p)}(f)\}$df is equal to s_0.

The Periodogram is just a direct spectral estimator with a rectangular taper. Therefore the equality holds for the periodogram as well, so the answer is C.