Mathematical Logic (M345P65)
Problem Sheet 4

[1] In each of the following formulas, indicate which of the occurrences of the variables x_1 and x_2 are bound and which are free:
(a) $(\forall x_2)(R_2(x_1, x_2) \to R_2(x_2, c_1))$;
(b) $(R_1(x_2) \to (\forall x_1)(\forall x_2)R_3(x_1, x_2, c_1))$;
(c) $((\forall x_1)R_1(f(x_1, x_2)) \to (\forall x_2)R_2(f(x_1, x_2), x_1))$.

Decide whether the term $f(x_1, x_1)$ is free for x_2 in each of the above formulas (explain briefly your answer).

[2] For the formula $\phi(x_2)$ given by $((\exists x_1)R(x_1, f(x_1, x_2)) \to (\forall x_1)R(x_1, x_2))$ (in a particular language $\mathcal{L}$) give an example of a term t which is not free for x_2 in $\phi(x_2)$. Find an $\mathcal{L}$-structure $\mathcal{A}$ in which $(\forall x_2)\phi(x_2)$ is true and a valuation v in $\mathcal{A}$ which does not satisfy $\phi(t)$.

[3] (a) Show (by giving an argument involving valuations) that for any formula ϕ the following formula is logically valid:

$$((\exists x_1)(\forall x_2)\phi \to (\forall x_2)(\exists x_1)\phi).$$

(b) Give an example of a formula ϕ and an interpretation where the following is false:

$$((\forall x_1)(\exists x_2)\phi \to (\exists x_2)(\forall x_1)\phi).$$

[4] Let $\mathcal{L}$ be a first-order language with a binary relation symbol R. A *strict partial order* is an $\mathcal{L}$-structure which is a model of the closed formula ϕ:

$$(\forall x_1)(\forall x_2)(\forall x_3)((\neg R(x_1, x_1)) \wedge ((R(x_1, x_2) \wedge R(x_2, x_3)) \to R(x_1, x_3))).$$

(So in a model of this formula, the interpretation of R behaves like $<$.)

(a) Show that in any model of ϕ the formula ψ given by:

$$(\forall x_1)(\forall x_2)(R(x_1, x_2) \to (\neg R(x_2, x_1)))$$

is true.

(b) Write down an $\mathcal{L}$-formula χ which has a model and is such that any $\mathcal{L}$-structure which is a model of χ is infinite.

[5] Suppose $\mathcal{L}$ is a first-order language and $\phi(x_1)$ is an $\mathcal{L}$-formula with a free variable x_1 and possibly other free variables. Under what circumstances is the formula

$$((\forall x_1)\phi(x_1) \to (\forall x_2)\phi(x_2))$$

logically valid? Justify your answer.

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