Imperial College London

Department of Mathematics

M3S8 Time Series

Question	Examiner's Comments
Q 1	Parts (a) and (b) were generally well done, with nearly all getting more than half of the true/false questions right. Part c was generally done quite well. Part (d) was problematic with a surprising number of basic errors, such as the covariance of two summation expressions being written incorrectly. Part (e): quite poorly done even though the bits required had been worked out in (c) and (d).
Q 2	The best answered question. (a)(i) was well done. (a)(ii) should have been easy to derive given (a)(i) but most students missed the simple approach and proceeded to derive s_0, s_1, s_2 from scratch. While often done correctly this took a LOT of time and effort. Parts (b)-(d) were generally well done using links between polynomial roots and parameters and showed good understanding of these structures.
Q 3	Part 3(a)(i) was standard material and well done. Part (a)(ii) was unseen, and many students did not attempt it, but those who did often did well. Of those who didn't, summations were often thoroughly mistreated. Part (b) had elements in common with previously seen coursework, and was generally quite well done.
Q 4	This question proved problematic. (a)(i) The idea of setting up a variance to show positive semidefiniteness was done in an early lecture, but forgotten by many. (a)(ii) Showing joint stationarity was quite well done although notation was often very poor. (iv) Surprisingly many defined the coherence incorrectly even though I had emphasized its importance in class. Quite a few got its value of unity correct, often by guesswork. (v) It was generally underappreciated that linear correlation being perfect implies a simple relationship between the variables which should be exploited here. Part (b) was generally well done with only (b)(iii) causing difficulties to some.

Imperial College London

Department of Mathematics

M45S8 Time Series

Question	Examiner's Comments
Q 1	Parts (a) and (b) were generally well done, with nearly all getting more than half of the true/false questions right. Part c was generally done quite well. Part (d) was problematic with a surprising number of basic errors, such as the covariance of two summation expressions being written incorrectly. Part (e): quite poorly done even though the bits required had been worked out in (c) and (d).
Q 2	The best answered question. (a)(i) was well done. (a)(ii) should have been easy to derive given (a)(i) but most students missed the simple approach and proceeded to derive s_0, s_1, s_2 from scratch. While often done correctly this took a LOT of time and effort. Parts (b)-(d) were generally well done using links between polynomial roots and parameters and showed good understanding of these structures.
Q 3	Part 3(a)(i) was standard material and well done. Part (a)(ii) was unseen, and many students did not attempt it, but those who did often did well. Of those who didn't, summations were often thoroughly mistreated. Part (b) had elements in common with previously seen coursework, and was generally quite well done.
Q 4	This question proved problematic. (a)(i) The idea of setting up a variance to show positive semidefiniteness was done in an early lecture, but forgotten by many. (a)(ii) Showing joint stationarity was quite well done although notation was often very poor. (iv) Surprisingly many defined the coherence incorrectly even though I had emphasized its importance in class. Quite a few got its value of unity correct, often by guesswork. (v) It was generally underappreciated that linear correlation being perfect implies a simple relationship between the variables which should be exploited here. Part (b) was generally well done with only (b)(iii) causing difficulties to some.
Q 5	The integration in (a) was mostly well done with odd/even functions being recognized. (b)(i) caused surprising difficulties in showing m and finding \ell. Many completed (b)(ii) quite well and it was clear that (b)(iii)-(v) were often not attempted due to time running out.

Imperial College
London

M4/5S8

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2018

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Time Series

Date: Thursday, 10 May 2018

Time: 10:00 AM - 12:30 PM

Time Allowed: 2.5 hours

This paper has 5 questions.

Candidates should use ONE main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Each question carries equal weight.
- Calculators may not be used.

Note: Throughout this paper $\{\epsilon_t\}$ is a sequence of uncorrelated random variables (white noise) having zero mean and variance σ_ϵ^2, unless stated otherwise. The unqualified term "stationary" will always be taken to mean second-order stationary. All processes are real-valued unless stated otherwise. The sample interval is unity unless stated otherwise. B denotes the backward shift operator. The autocovariance sequence for a stationary process is denoted by $\{s_\tau\}$.

1. (a) What is meant by saying that a stochastic process is stationary?

   (b) Are the following statements true or false?:

      (i) a strictly stationary time series $\{X_t\}$ which is not Gaussian/normal is second-order stationary;

      (ii) a time series which is the sum of of a seasonal component of period $s = 12$, a linear trend, and a stationary process, can be made stationary by applying the operator $(1 - B - B^{12} + B^{13})$;

      (iii) a time series $\{X_t\}$ with general linear process form $X_t = G(B)\epsilon_t$, will be invertible if the z-polynomial $G(z)$ is analytic for $|z| \leq 1$;

      (iv) a process with a purely discrete spectrum (or line spectrum) has an autocovariance sequence $\{s_\tau\}$ such that $s_\tau \to 0$ as $|\tau| \to \infty$;

      (v) as more tapering is performed with direct spectral estimators, sidelobe leakage decreases.

   (c) The zero mean, stationary, AR(1) process $Y_t = \phi Y_{t-1} + \epsilon_t$ may be written in the form $Y_t = \sum_{j=0}^{\infty} \phi^j \epsilon_{t-j}$. By applying the same steps used in the derivation of the Yule-Walker equations, and utilising this result, show that

$$s_{Y,\tau} = \frac{\phi^{|\tau|}}{1-\phi^2}\sigma_\epsilon^2,$$

   where $s_{Y,\tau} = \text{cov}\{Y_t, Y_{t+\tau}\}$.

   (d) Show that the zero mean and stationary ARMA(p,q) process

$$X_t = \phi_{1,p}X_{t-1} + \cdots + \phi_{p,p}X_{t-p} + \epsilon_t - \theta_{1,q}\epsilon_{t-1} - \cdots - \theta_{q,q}\epsilon_{t-q}$$

   can be decomposed as

$$X_t = -\sum_{j=0}^{q} \theta_{j,q} Y_{t-j} \quad \text{where} \quad Y_t = \sum_{k=1}^{p} \phi_{k,p} Y_{t-k} + \epsilon_t \quad \text{and} \quad \theta_{0,q} = -1.$$

   Hence express $s_{X,\tau} = \text{cov}\{X_t, X_{t+\tau}\}$ in terms of $\{s_{Y,\tau}\}$.

   (e) Using the results in parts (c) and (d), show that, for a zero mean and stationary ARMA(1,1) process, $s_{X,0} = \text{var}\{X_t\}$ is given by

$$s_{X,0} = [1 + c]\sigma_\epsilon^2,$$

   where the form of c is to be found.

2. Consider the zero mean and stationary MA(2) process $\{X_t\}$ given by

$$X_t = \epsilon_t - \theta_{1,2}\epsilon_{t-1} - \theta_{2,2}\epsilon_{t-2}.$$

(a) (i) Use the filtering approach to derive the spectral density function of $\{X_t\}$ and show it may be written as

$$S(f) = \sigma_\epsilon^2[1 + \theta_{1,2}^2 + \theta_{2,2}^2 - 2\theta_{1,2}(1 - \theta_{2,2})\cos(2\pi f) - 2\theta_{2,2}\cos(4\pi f)]$$

(ii) The spectral density function of $\{X_t\}$ is the Fourier transform of its autocovariance sequence $\{s_\tau\}$. Using just this relationship show that

$$s_0 = (1 + \theta_{1,2}^2 + \theta_{2,2}^2)\sigma_\epsilon^2; \quad s_1 = -\theta_{1,2}(1 - \theta_{2,2})\sigma_\epsilon^2; \quad s_2 = -\theta_{2,2}\sigma_\epsilon^2.$$

(b) Let $\theta_{1,2} = 1, \theta_{2,2} = -1/2$.

(i) Determine whether $\{X_t\}$ is invertible, and give the meaning of invertibility.

(ii) Find the values of the autocorrelation sequence elements ρ_1 and ρ_2.

(c) Now consider a zero mean and stationary MA(2) process $\{Y_t\}$ for which its characteristic polynomial has roots $\frac{1}{2} \pm \frac{1}{2}\mathrm{i}$.

(i) Find the parameters $\theta_{1,2}$ and $\theta_{2,2}$.

(ii) Find the autocorrelation sequence elements ρ_1 and ρ_2.

(d) Carefully explain the relationship you observe between the autocorrelation sequence of $\{X_t\}$ defined in part (b) and the autocorrelation sequence $\{Y_t\}$ defined in part (c). Is $\{Y_t\}$ invertible? What do you conclude?

3. (a) Given a sample, $X_1, \ldots, X_N$, from a zero mean stationary time series $\{X_t\}$, the periodogram spectral estimator, $\widehat{S}^{(p)}(f)$, of the spectral density function, $S(f)$, is given by

$$\widehat{S}^{(p)}(f) = \left| \sum_{t=1}^{N} \tfrac{1}{\sqrt{N}} X_t e^{-i2\pi f t} \right|^2 .$$

(i) Use the spectral representation theorem to show that the mean of the periodogram, $\widehat{S}^{(p)}(f)$, is given by

$$E\{\widehat{S}^{(p)}(f)\} = \int_{-1/2}^{1/2} \mathcal{F}(f - f') S(f') \mathrm{d}f' ,$$

where $\mathcal{F}(f)$ denotes Fejér's kernel given by $\mathcal{F}(f) = \frac{1}{N} \left| \sum_{t=1}^{N} e^{-i2\pi f t} \right|^2$.
Is the periodogram biased or unbiased if $\{X_t\}$ is white noise? Justify your answer.

(ii) Consider the case when $X_t = \epsilon_t$, where $\{\epsilon_t\}$ is Gaussian/normal distributed, i.e., the process is Gaussian/normal white noise. By writing the periodogram, $\widehat{S}^{(p)}(f_j)$, *at the Fourier frequencies* $f_j = j/N, 1 \le j < N/2$, in the form $\widehat{S}^{(p)}(f_j) = |Y_1(f_j) + iY_2(f_j)|^2$, where $Y_1(f_j), Y_2(f_j)$ are real-valued random variables, show that

$$\widehat{S}^{(p)}(f_j) \overset{\mathrm{d}}{=} \frac{\sigma_\epsilon^2}{2} \chi_2^2,$$

i.e., $\widehat{S}^{(p)}(f_j)$ is distributed as a scaled version of a chi-square random variable with 2 degrees of freedom.
You will need the following results:

$$\sum_{t=1}^{N} \cos^2(2\pi f_j t) = \sum_{t=1}^{N} \sin^2(2\pi f_j t) = \frac{N}{2}; \qquad \sum_{t=1}^{N} \cos(2\pi f_j t) \sin(2\pi f_j t) = 0.$$

(b) The autocovariance sequence $\{s_{Z,\tau}\}$ for a *complex-valued* stationary time series $\{Z_t\}$ with mean zero is defined as $s_{Z,\tau} = \mathrm{cov}\{Z_t, Z_{t+\tau}\} = E\{Z_t^* Z_{t+\tau}\}$, where * denotes complex conjugation. A second quantity, called the complementary covariance, denoted $\{r_{Z,\tau}\}$, is defined as $r_{Z,\tau} = \mathrm{cov}\{Z_t^*, Z_{t+\tau}\} = E\{Z_t Z_{t+\tau}\}$, and is the covariance sequence between $\{Z_t\}$ and its complex-conjugate. It is an important quantity in areas such as communications. If $\{r_{Z,\tau}\}$ is zero for all $\tau \in \mathbb{Z}$ then $\{Z_t\}$ is called *proper*.
Consider the time series $Z_t = X_t e^{iY_t}$. Here $\{X_t\}$ is a real-valued, zero mean, unit variance, stationary process with autocovariance $\{s_{X,\tau}\}$. $\{Y_t\}$ is a sequence of independent random variables drawn from the uniform distribution on $[-\pi, \pi]$. The sequences $\{X_t\}$ and $\{Y_t\}$ are assumed independent of each other, (i.e., the random variables $X_{t_1}, \ldots, X_{t_n}$ and $Y_{t_1'}, \ldots, Y_{t_n'}$ are mutually independent for any $n \ge 1$).

(i) Find the form of the sequence $\{s_{Z,\tau}\}$. [Express the values in integers, to be found.]

(ii) Determine if $\{Z_t\}$ is proper.

4. (a) Consider the bivariate white noise process

$$\boldsymbol{X}_t = \begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix} = \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix} = \boldsymbol{\epsilon}_t$$

where $E\{\boldsymbol{\epsilon}_t\} = 0$ and $E\{\boldsymbol{\epsilon}_s \boldsymbol{\epsilon}_t^T\} = \Sigma$ if $s = t$ and zero otherwise. (Here ^{T} denotes transpose.)

(i) Show, for any nonzero real numbers a_1, a_2, that $\sum_{j=1}^{2} \sum_{k=1}^{2} \sigma_{jk} a_j a_k \geq 0$, where σ_{jk} is the (j, k)th element of Σ, i.e., that Σ is positive semidefinite.

(ii) Show that $\{X_{1,t}\}$ and $\{X_{2,t}\}$ are jointly stationary stochastic processes.

Now assume $\Sigma = \Sigma_1 \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

(iii) Verify that Σ_1 is positive semidefinite.

(iv) Find the coherence $\gamma^2_{X_1, X_2}(f)$ for $|f| \leq 1/2$.

(v) Are $\{X_{1,t}\}$ and $\{X_{2,t}\}$ in fact identical? Justify your answer.

(b) (i) Suppose the stationary process $\{X_t\}$ can be written as a one-sided linear process, $X_t = \sum_{k=0}^{\infty} \psi_k \epsilon_{t-k}$, with $\psi_0 = 1$. We wish to construct the l-step ahead forecast $X_t(l) = \sum_{k=0}^{\infty} \delta_k \varepsilon_{t-k}$. Show that the linear least squares predictor, which minimizes $E\{(X_{t+l} - X_t(l))^2\}$, corresponds to setting $\delta_k = \psi_{k+l}$, $k \geq 0$.

Now assume a zero mean stationary AR(1) process, $X_t = \phi X_{t-1} + \epsilon_t$, and linear least squares prediction for which the l-step prediction variance is $\sigma^2(l) = \sigma_\epsilon^2 \sum_{k=0}^{l-1} \psi_k^2$.

(ii) Find the resulting 2-step prediction variance $\sigma^2(2)$ in terms of σ_ϵ^2 and ϕ.

(iii) From the course notes on forecasting, we know that for linear least squares prediction, the l-step ahead forecast $X_t(l)$ of X_{t+l} may be obtained by setting future innovations to zero. Calculate the 2-step prediction variance again, this time by evaluating $E\{(X_{t+2} - X_t(2))^2\}$.
Hint: Recall the required result in Q1(c).

5.

(a) Consider a *continuous* parameter real-valued stationary process $\{X(t)\}$ with a Lorenzian spectral density function (SDF) given by

$$S(f) = \frac{2L\sigma^2}{1+(2\pi f L)^2}, \qquad f \in \mathbb{R},$$

where $\sigma^2 > 0$ is the process variance, and $L > 0$ is a real-valued parameter. Show, with full justification, that the autocovariance function for $\{X(t)\}$ is given by

$$s(\tau) = \sigma^2 \mathrm{e}^{-|\tau|/L}, \qquad \tau \in \mathbb{R}.$$

Hint: for $m \in \mathbb{R}$,

$$\int_0^\infty \frac{\cos(mx)}{1+x^2}\mathrm{d}x = \frac{\pi}{2}\mathrm{e}^{-|m|}.$$

(b) Consider a *continuous* parameter real-valued stationary process $\{X(t)\}$ with a spectral density function (SDF) given by

$$S(f) = \begin{cases} C, & |f| \le 2; \\ 0, & |f| > 2, \end{cases}$$

where $C > 0$ is a real-valued constant (a process with the above SDF is known as band-limited white noise). For a given sampling interval $\Delta t > 0$, define the associated discrete time process by $X_t = X(t\,\Delta t)$, $t \in \mathbb{Z}$.

(i) Starting from the standard aliasing formula, (which you are not required to prove), explain why the SDF $S_{X_t}(f;\Delta t)$ of $\{X_t\}$, for $f \in [0, f_N]$, can be written in terms of $S(f)$ given above, without error, as

$$S_{X_t}(f;\Delta t) = S(f) + \sum_{k=1}^{m} S(f+\tfrac{k}{\Delta t}) + \sum_{k=1}^{\ell} S(f-\tfrac{k}{\Delta t}) = \sum_{k=-\ell}^{m} S\left(f+\tfrac{k}{\Delta t}\right), \qquad \text{for} \quad f \in [0, f_N],$$

where ℓ and m are suitable integers and $f_N = 1/(2\Delta t)$ is the Nyquist frequency.
Show that $m = \lfloor 2\Delta t \rfloor$, where $\lfloor x \rfloor$ is the largest integer less than or equal to x.
Express ℓ in the form $\ell = \lfloor y \rfloor$, where y is to be found.

(ii) Determine the SDFs for $\{X_t\}$ for $f \in [0, f_N]$, when $\Delta t = \frac{2}{3}$, $\Delta t = \frac{1}{3}$ and when $\Delta t = \frac{1}{5}$.

(iii) What is $S_{X_t}(f;\Delta t)$ for $f \in [-f_N, 0)$? How can $S_{X_t}(f;\Delta t)$ be found for $f \notin [-f_N, f_N]$?

(iv) Verify that each of the integrals over $[-f_N, f_N]$ for the SDFs in (b)(ii) is the same as the integral of $S(\cdot)$ over $f \in \mathbb{R}$.

(v) The terms 'red noise' and 'blue noise' are used to describe spectra with certain dominant frequencies. Classify the SDFs in (b)(ii) as 'red noise' or 'blue noise'.

Course: M3S8/M4S8/M5S8
Setter: Walden
Checker: McCoy
Editor: Walden
External: Jennison
Date: March 16, 2018

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2018

M3S8/M4S8/M5S8

Time Series [FINAL SOLUTIONS]

Setter's signature	Checker's signature	Editor's signature
....	....	....

Note: In the mark scheme the following categorization has been adopted
Routine (A), Sound (B), Borderline (C), Challenging (D). seen ⇓

1. (a) $\{X_t\}$ is second-order stationary if $E\{X_t\}$ is a finite constant for all t, var$\{X_t\}$ is a finite constant for all t, and cov$\{X_t, X_{t+\tau}\}$, is a finite quantity depending only on τ and not on t. 4 A

   (b) (i) TRUE, a strictly stationary process always has first and second joint moments which are time invariant, whatever its distributional structure. sim. seen ⇓

   (ii) TRUE, we need to apply the operators $(1-B)$ and $(1-B^{12})$ one after the other, which is equivalent to the stated operation.

   (iii) FALSE, for invertibility (rather than stationarity) we need that $G^{-1}(B)X_t = \epsilon_t$, is well-defined, i.e., that $G^{-1}(z)$ is analytic for $|z| \leq 1$;

   (iv) FALSE, the stated property holds for purely continuous spectra; for a purely discrete spectrum the ACVS does not damp down;

   (v) TRUE, the main purpose of tapering is the reduction of sidelobe leakage, and the more tapering, the bigger the decrease. 5 B

   sim. seen ⇓

   (c) Firstly multiply the defining equation through by $Y_{t-\tau}$ for $\tau > 0$,

$$\begin{aligned} Y_t Y_{t-\tau} &= \phi Y_{t-1} Y_{t-\tau} + \epsilon_t Y_{t-\tau} \\ \Rightarrow Y_t Y_{t-\tau} &= \phi Y_{t-1} Y_{t-\tau} + \epsilon_t \sum_{j=0}^{\infty} \phi^j \epsilon_{t-\tau-j} \\ \Rightarrow s_{Y,\tau} &= \phi s_{Y,\tau-1} \Rightarrow s_{Y,\tau} = \phi^\tau s_{Y,0}, \end{aligned}$$

   since the expectation of the rightmost term is zero. Next multiply the defining equation by Y_t

$$\begin{aligned} Y_t^2 &= \phi Y_{t-1} Y_t + \epsilon_t \sum_{j=0}^{\infty} \phi^j \epsilon_{t-j} \\ \Rightarrow s_{Y,0} &= \phi s_{Y,1} + \sigma_\epsilon^2 = \phi^2 s_{Y,0} + \sigma_\epsilon^2 \Rightarrow s_{Y,0} = \frac{\sigma_\epsilon^2}{1-\phi^2}, \end{aligned}$$

   which, combined with the fact that $s_{Y,\tau}$ is symmetric, gives the required result. 4 A

   (d) We have

$$X_t = -\sum_{j=0}^{q} \theta_{j,q} Y_{t-j} \quad \text{with} \quad Y_{t-j} = \sum_{k=1}^{p} \phi_{k,p} Y_{t-j-k} + \epsilon_{t-j}.$$

   So

$$\begin{aligned} X_t &= -\sum_{k=1}^{p}\sum_{j=0}^{q} \phi_{k,p}\theta_{j,q} Y_{t-j-k} - \sum_{j=0}^{q} \theta_{j,q}\epsilon_{t-j} = \sum_{k=1}^{p} \phi_{k,p}\left(-\sum_{j=0}^{q} \theta_{j,q} Y_{t-j-k}\right) - \sum_{j=0}^{q} \theta_{j,q}\epsilon_{t-j} \\ &= \sum_{k=1}^{p} \phi_{k,p} X_{t-k} - \sum_{j=0}^{q} \theta_{j,q}\epsilon_{t-j}, \quad \text{which is of the required form.} \end{aligned}$$

2 C

Next,

$$\begin{aligned}
s_{X,\tau} &= \mathrm{cov}\{X_t, X_{t+\tau}\} = \mathrm{cov}\{-\sum_{j=0}^{q}\theta_{j,q}Y_{t-j}, -\sum_{k=0}^{q}\theta_{k,q}Y_{t+\tau-k}\} \\
&= \sum_{j=0}^{q}\sum_{k=0}^{q}\theta_{j,q}\theta_{k,q}\,\mathrm{cov}\{Y_{t-j}, Y_{t+\tau-k}\} \\
&= \sum_{j=0}^{q}\sum_{k=0}^{q}\theta_{j,q}\theta_{k,q}\,s_{Y,\tau-k+j}.
\end{aligned}$$

$\boxed{2\ \mathrm{C}}$

(e) From the last part of (d) with $q = 1$, and using the expression derived in part (c), with $s_{Y,1} = s_{Y,-1}$, we get $\boxed{\text{unseen} \Downarrow}$

$$\begin{aligned}
s_{X,0} = \mathrm{var}\{X_t\} &= (-1)^2 s_{Y,0} - \theta s_{Y,-1} - \theta s_{Y,1} + \theta^2 s_{Y,0} = (1+\theta^2)s_{Y,0} - 2\theta s_{Y,1} \\
&= \frac{(1+\theta^2)}{1-\phi^2}\sigma_\epsilon^2 - \frac{2\theta\phi}{1-\phi^2}\sigma_\epsilon^2 = \left[1 + \frac{(\theta-\phi)^2}{1-\phi^2}\right]\sigma_\epsilon^2,
\end{aligned}$$

so $c = (\theta - \phi)^2/(1-\phi^2)$. $\boxed{3\ \mathrm{D}}$

2. (a) (i) From linear filtering, input $e^{i2\pi ft}$ to the filter $L(\epsilon_t) = \epsilon_t - \theta_{1,2}\epsilon_{t-1} - \theta_{2,2}\epsilon_{t-2}$ to obtain the frequency response function $G(f)$: $\boxed{\text{seen} \Downarrow}$

$$\begin{aligned} L\{e^{i2\pi ft}\} &= e^{i2\pi ft}(1-\theta_{1,2}e^{-i2\pi f} - \theta_{2,2}e^{-i4\pi f}) \\ \Rightarrow G(f) &= (1-\theta_{1,2}e^{-i2\pi f} - \theta_{2,2}e^{-i4\pi f}) \\ \Rightarrow |G(f)|^2 &= |1-\theta_{1,2}e^{-i2\pi f} - \theta_{2,2}e^{-i4\pi f}|^2. \end{aligned}$$

The output spectrum is the input spectrum times $|G(f)|^2$:

$$S(f) = |G(f)|^2 S_\epsilon(f) = \sigma_\epsilon^2|1-\theta_{1,2}e^{-i2\pi f} - \theta_{2,2}e^{-i4\pi f}|^2.$$

Then

$$\begin{aligned} S(f) &= \sigma_\epsilon^2[1+\theta_{1,2}^2+\theta_{2,2}^2 - \theta_{1,2}\left(e^{i2\pi f}+e^{-i2\pi f}\right) - \theta_{2,2}\left(e^{i4\pi f}+e^{-i4\pi f}\right) \\ &\quad + \theta_{1,2}\theta_{2,2}\left(e^{i2\pi f}+e^{-i2\pi f}\right) \\ &= \sigma_\epsilon^2[1+\theta_{1,2}^2+\theta_{2,2}^2 - 2\theta_{1,2}(1-\theta_{2,2})\cos(2\pi f) - 2\theta_{2,2}\cos(4\pi f)] \qquad (*) \end{aligned}$$

$\boxed{4\text{ A}}$

(ii) Since the process is an MA(2) its autocovariance cuts-off at $|\tau| = 2$, so

$$\begin{aligned} S(f) = \sum_{\tau=-\infty}^{\infty} s_\tau e^{-i2\pi f} &= s_0 + s_1\left(e^{i2\pi f}+e^{-i2\pi f}\right) + s_2\left(e^{i4\pi f}+e^{-i4\pi f}\right) \\ &= s_0 + 2s_1\cos(2\pi f) + 2s_2\cos(4\pi f) \qquad (**) \end{aligned}$$

A comparison of (*) and (**) gives the required expressions. $\boxed{4\text{ A}}$

$\boxed{\text{sim. seen} \Downarrow}$

(b) (i) The characteristic polynomial is $\Theta(z) = 1 - z + \frac{1}{2}z^2$ which has roots $z_1, z_2 = 1 \pm i$, both of which have modulus greater than one, so $\{X_t\}$ is invertible. This means that it can be rewritten as a well-defined autoregressive process. $\boxed{2\text{ A}}$

(ii) Now

$$\rho_1 = -\frac{\theta_{1,2}(1-\theta_{2,2})}{1+\theta_{1,2}^2+\theta_{2,2}^2} \text{ and } \rho_2 = -\frac{\theta_{2,2}}{1+\theta_{1,2}^2+\theta_{2,2}^2}$$

Putting $\theta_{1,2} = 1, \theta_{2,2} = -1/2$ gives $\rho_1 = -2/3$ and $\rho_2 = 2/9$. $\boxed{2\text{ A}}$

$\boxed{\text{unseen} \Downarrow}$

(c) (i) Write the characteristic polynomial in root form, (roots are $1/a, 1/b$):

$$1 - \theta_{1,2}z - \theta_{2,2}z^2 = (1-az)(1-bz) = 1 - (a+b)z + abz^2$$

so $\theta_{1,2} = (a+b)$ and $\theta_{2,2} = -ab$. From the stated roots we have

$$a = \frac{2}{1+i}, \qquad b = \frac{2}{1-i}$$

and therefore

$$\theta_{1,2} = a+b = \frac{2(1-i)+2(1+i)}{(1+i)(1-i)} = \frac{4}{2} = 2; \quad \theta_{2,2} = -ab = -\frac{4}{2} = -2.$$

$\boxed{4\text{ D}}$

(ii) Putting these values into ρ_1 and ρ_2 as given in (b)(ii) above, we again get $\rho_1 = -2/3$ and $\rho_2 = 2/9$, as for $\{X_t\}$. $\boxed{1\text{ C}}$

(d) We have met the idea that inverting the roots of the characteristic polynomial of a MA does not change its autocorrelation sequence. If we invert the roots specified in (c) for $\{Y_t\}$ we get

$$z_1 = \frac{2}{1+\mathrm{i}} = 1 - \mathrm{i}; \quad z_1 = \frac{2}{1-\mathrm{i}} = 1 + \mathrm{i},$$

which are the roots for $\{X_t\}$. So indeed the two MA processes here have roots which are inverses, and the equal autocorrelations follow. The process $\{Y_t\}$ has roots inside the unit circle and so, unlike $\{X_t\}$, it is not invertible. The two processes have the same autocorrelation sequence, but different invertibility properties. $\boxed{3\text{ C}}$

3. (a) (i) Let $J(f) \equiv (1/\sqrt{N})\sum_{t=1}^{N} X_t e^{-i2\pi ft}$. By the spectral representation theorem $X_t = \int_{-1/2}^{1/2} e^{i2\pi f't}\, dZ(f')$, where $\{Z(\cdot)\}$ is a process with orthogonal increments, and $E\{dZ(f)\} = 0$. Thus seen ⇓

$$
\begin{aligned}
J(f) &= (1/\sqrt{N})\sum_{t=1}^{N}\left(\int_{-1/2}^{1/2} e^{i2\pi f't}\, dZ(f')\right) e^{-i2\pi ft} \\
&= (1/\sqrt{N})\int_{-1/2}^{1/2}\sum_{t=1}^{N} e^{-i2\pi(f-f')t}\, dZ(f') \\
&= \int_{-1/2}^{1/2} F(f-f')\, dZ(f'),
\end{aligned}
$$

where $F(f) = (1/\sqrt{N})\sum_{t=1}^{N} e^{-i2\pi ft}$. Now $\widehat{S}^{(p)}(f) = |J(f)|^2$, and since $\{Z(\cdot)\}$ has orthogonal increments, and $E\{|dZ(f')|^2\} = S(f')df'$,

$$
E\{\widehat{S}^{(p)}(f)\} = \int_{-1/2}^{1/2} \mathcal{F}(f-f')S(f')\, df',
$$

where $\mathcal{F}(f) \equiv |F(f)|^2 = (1/N)\left|\sum_{t=1}^{N} e^{-i2\pi ft}\right|^2$. 4 A

For white noise $S(f) = \sigma_\epsilon^2$, so $E\{\widehat{S}^{(p)}(f)\} = \sigma_\epsilon^2 \int_{-1/2}^{1/2} \mathcal{F}(f-f')\, df' = \sigma_\epsilon^2$, since $\mathcal{F}$ has a period of unity and integrates to 1. So **unbiased**. 2 A

(ii) From the question we set $\widehat{S}^{(p)}(f_j) = |Y_1(f_j) + iY_2(f_j)|^2 = Y_1(f_j)^2 + Y_2^2(f_j)$ and unseen ⇓

$$
Y_1(f_j) = \frac{1}{\sqrt{N}}\sum_{t=1}^{N} \epsilon_t \cos(2\pi f_j t); \qquad Y_2(f_j) = -\frac{1}{\sqrt{N}}\sum_{t=1}^{N} \epsilon_t \sin(2\pi f_j t).
$$

Using the shorthand $Y_1 = Y_1(f_j)$ and $Y_2 = Y_2(f_j)$ we have $E\{Y_1\} = E\{Y_2\} = 0$. Since the white noise terms are uncorrelated,

$$
\mathrm{var}\{Y_1\} = \frac{\sigma_\epsilon^2}{N}\sum_{t=1}^{N}\cos^2(2\pi f_j t) = \frac{\sigma_\epsilon^2}{2} = \frac{\sigma_\epsilon^2}{N}\sum_{t=1}^{N}\sin^2(2\pi f_j t) = \mathrm{var}\{Y_2\}.
$$

Also,

$$
\begin{aligned}
\mathrm{cov}\{Y_1, Y_2\} = E\{Y_1 Y_2\} &= -\frac{1}{N}\sum_{t=1}^{N}\sum_{t'=1}^{N} E\{\epsilon_t\epsilon_{t'}\}\cos(2\pi f_j t)\sin(2\pi f_j t') \\
&= -\frac{\sigma_\epsilon^2}{N}\sum_{t=1}^{N}\cos(2\pi f_j t)\sin(2\pi f_j t) = 0.
\end{aligned}
$$

Since the process is normal, Y_1, Y_2 are jointly normal and uncorrelated and hence independent. So, as required,

$$
\frac{2}{\sigma_\epsilon^2}[Y_1^2(f_j) + Y_2^2(f_j)] \stackrel{d}{=} \chi_2^2 \Rightarrow \widehat{S}^{(p)}(f_j) \stackrel{d}{=} \frac{\sigma_\epsilon^2}{2}\chi_2^2.
$$

6 B

(b) (i) Now $Z_t = X_t e^{iY_t}$ and $Y_t \stackrel{d}{=} U[-\pi, \pi]$. $\{X_t\}, \{Y_t\}$ are independent of each other. Since $\{X_t\}$ has a zero mean, $E\{Z_t\} = E\{X_t\}E\{e^{iY_t}\} = 0$. unseen ⇓ 1 D

Then, by independence of $\{X_t\}, \{Y_t\}$,

$$s_{Z,\tau} = E\{Z_t^* Z_{t+\tau}\} = E\{X_t e^{-iY_t} \cdot X_{t+\tau} e^{iY_{t+\tau}}\} = s_{X,\tau} \cdot E\{e^{-iY_t} e^{iY_{t+\tau}}\}.$$

When $\tau = 0$, $s_{Z,0} = s_{X,0} \cdot E\{e^{-iY_t} e^{iY_t}\} = \sigma_X^2 \cdot E\{1\} = \sigma_X^2 = 1$.
When $\tau \neq 0$, $s_{Z,\tau} = s_{X,\tau} \cdot E\{e^{-iY_t}\} \cdot E\{e^{iY_{t+\tau}}\}$, by independence. But

$$E\{e^{-iY_t}\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(y)\mathrm{d}y - i\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(y)\mathrm{d}y = 0$$

and likewise for $E\{e^{iY_{t+\tau}}\}$, (identically distributed). So,

$$s_{Z,\tau} = \begin{cases} 1, & \tau = 0; \\ 0, & \tau \neq 0. \end{cases}$$

4 D

(ii) Next,

$$r_{Z,\tau} = E\{Z_t Z_{t+\tau}\} = E\{X_t e^{iY_t} \cdot X_{t+\tau} e^{iY_{t+\tau}}\} = s_{X,\tau} \cdot E\{e^{i(Y_t + Y_{t+\tau})}\}.$$

When $\tau = 0$, $r_{Z,0} = s_{X,0} \cdot E\{e^{i2Y_t}\} = 0$, since $\int_{-\pi}^{\pi} \cos(2y)\mathrm{d}y = 0$, and similarly for sin.
When $\tau \neq 0$, $r_{Z,\tau} = s_{X,\tau} \cdot E\{e^{i(Y_t+Y_{t+\tau})}\} = s_{X,\tau} \cdot E\{e^{iY_t}\}E\{e^{iY_{t+\tau}}\} = s_{X,\tau} \cdot 0 = 0$.
So $r_{Z,\tau} = 0$ for all τ, and therefore $\{Z_t\}$ is **proper**. 3 D

4. (a) (i) Let $\boldsymbol{a}^T = [a_1, a_2]$. Then

sim. seen ⇓

$$\text{var}\{\boldsymbol{a}^T \boldsymbol{\epsilon}_t\} = \boldsymbol{a}^T \Sigma \boldsymbol{a} = \sum_{j=1}^{2} \sum_{k=1}^{2} \sigma_{jk} a_j a_k \geq 0.$$

1 B

(ii) Two real-valued discrete time stochastic processes $\{X_t\}$ and $\{Y_t\}$ are said to be jointly stationary stochastic processes if $\{X_t\}$ and $\{Y_t\}$ are each, separately, second-order stationary processes, and $\text{cov}\{X_t, Y_{t+\tau}\}$ is a function of τ only.

seen ⇓

2 A

For the example, for $j, k \in \{1, 2\}$,

unseen ⇓

$$\text{cov}\{X_{j,t}, X_{k,t+\tau}\} = E\{\epsilon_{j,t}\epsilon_{k,t+\tau}\} = \begin{cases} \sigma_{jk} & \tau = 0; \\ 0 & \tau \neq 0. \end{cases}$$

so they are jointly stationary.

2 B

(iii) For $\Sigma = \Sigma_1$, the eigenvalues are the solutions of

$$\det\left\{\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}\right\} = 0.$$

These are $\lambda_1, \lambda_2 = 0, 2$. So the matrix is positive semidefinite. Alternatively, observe that the principal minors/subdeterminants are all nonnegative.

1 B

(iv) The coherence is defined as

$$\gamma^2_{X_1,X_2}(f) = \frac{|S_{X_1X_2}(f)|^2}{S_{X_1}(f)S_{X_2}(f)}.$$

From part (ii) and the form of Σ_1,we know

$$s_{X_1,\tau} = s_{X_2,\tau} = s_{X_1X_2,\tau} = \begin{cases} 1 & \tau = 0; \\ 0 & \tau \neq 0. \end{cases}$$

So $S_{X_1}(f) = \sum_{\tau=-\infty}^{\infty} s_{X_1,\tau} e^{-i2\pi f\tau} = 1$, $|f| \leq 1/2$, and likewise for $S_{X_2}(f)$ and $S_{X_1X_2}(f)$. Hence, $\gamma^2_{X_1,X_2}(f) = 1$, $|f| \leq 1/2$.

2 B

(v) From (iv) we know that $\{X_{1,t}\}$ and $\{X_{2,t}\}$ are perfectly correlated at all frequencies so they are related through a linear filtering, i.e., $X_{2,t} = \sum_u g_u X_{1,t-u}$. Multiplying through by $X_{1,t}$ and taking expectations gives $1 = g_0 \cdot 1$ so $g_0 = 1$. Multiplying through by $X_{1,t+\tau}$ for $|\tau| \neq 0$ and taking expectations gives $0 = g_{-\tau} \cdot 1$, so only g_0 is non-zero. Thus $X_{2,t} = X_{1,t}$, they are identical. [Other valid justifications are fine.]

2 D

(b) (i) We want to minimize, $\boxed{\text{seen} \Downarrow}$

$$
\begin{aligned}
E\{(X_{t+l}-X_t(l))^2\} &= E\left\{\left(\sum_{k=0}^{\infty}\psi_k\epsilon_{t+l-k}-\sum_{k=0}^{\infty}\delta_k\epsilon_{t-k}\right)^2\right\} \\
&= E\left\{\left(\sum_{k=0}^{l-1}\psi_k\epsilon_{t+l-k}+\sum_{k=0}^{\infty}[\psi_{k+l}-\delta_k]\epsilon_{t-k}\right)^2\right\} \\
&= \sigma_\epsilon^2\left\{\left(\sum_{k=0}^{l-1}\psi_k^2\right)+\sum_{k=0}^{\infty}(\psi_{k+l}-\delta_k)^2\right\}.
\end{aligned}
$$

The first term is independent of the choice of the $\{\delta_k\}$ and the second term is clearly minimized by choosing $\delta_k = \psi_{k+l}, k = 0, 1, 2, \ldots$. $\boxed{4 \text{ A}}$

(ii) $X_t = (1-\phi B)^{-1}\epsilon_t$. So $\boxed{\text{unseen} \Downarrow}$

$$\Psi(z) = 1+\phi z+\phi^2 z^2+\cdots = \psi_0+\psi_1 z+\psi_2 z^2+\cdots$$

so $\psi_k = \phi^k$. Then $\sigma^2(l) = \sigma_\epsilon^2\sum_{k=0}^{l-1}\psi_k^2 \Rightarrow \sigma^2(2) = \sigma_\epsilon^2(\psi_0^2+\psi_1^2) = \sigma_\epsilon^2(1+\phi^2)$. $\boxed{2 \text{ B}}$

(iii) We set future innovations to zero: $X_t(1) = \phi X_t$ and $X_t(2) = \phi X_t(1) = \phi^2 X_t$. So

$$
\begin{aligned}
E\{(X_{t+2}-X_t(2))^2\} &= E\{X_{t+2}^2\}+E\{\phi^4 X_t^2\}-2E\{X_{t+2}\cdot\phi^2 X_t\} \\
&= \frac{\sigma_\epsilon^2}{1-\phi^2}+\phi^4\frac{\sigma_\epsilon^2}{1-\phi^2}-2\phi^2 s_2 \\
&= \frac{\sigma_\epsilon^2}{1-\phi^2}+\phi^4\frac{\sigma_\epsilon^2}{1-\phi^2}-2\phi^2\frac{\sigma_\epsilon^2\phi^2}{1-\phi^2} \\
&= \frac{\sigma_\epsilon^2(1-\phi^2)(1+\phi^2)}{1-\phi^2} = \sigma_\epsilon^2(1+\phi^2),
\end{aligned}
$$

as before. $\boxed{4 \text{ C}}$

5. (a) $\boxed{\text{unseen} \Downarrow}$

$$
\begin{aligned}
s(\tau) &= \int_{-\infty}^{\infty} S(f) e^{i2\pi f\tau} \, df = \int_{-\infty}^{\infty} \frac{2L\sigma^2}{1+(2\pi f L)^2} e^{i2\pi f\tau} \, df \\
&= 4L\sigma^2 \int_0^{\infty} \frac{\cos(2\pi f\tau)}{1+(2\pi f L)^2} \, df,
\end{aligned}
$$

since the imaginary part of the integral is zero by symmetry because its integrand is the product of the even function $S(\cdot)$ and the odd function $\sin(\cdot)$. For the real part, the integrand is an even function, and thus we can rewrite as twice an integral ranging from 0 to ∞. Next, make the change of variable $x = 2\pi f L$ and set $m = \tau/L$ in the definite integral in the hint, so

$$
2\pi L \int_0^{\infty} \frac{\cos(2\pi f\tau)}{1+(2\pi f L)^2} \, df = \int_0^{\infty} \frac{\cos(mx)}{1+x^2} \, dx = \frac{\pi}{2} e^{-|\tau|/L},
$$

i.e.,

$$
\int_0^{\infty} \frac{\cos(2\pi f\tau)}{1+(2\pi f L)^2} \, df = \frac{1}{4L} e^{-|\tau|/L},
$$

from which we obtain $s(\tau) = \sigma^2 e^{-|\tau|/L}$, as required. $\boxed{4}$

(b) (i) Let $S_{X_t}(f;\Delta t)$ denote the spectral density function (SDF) of $\{X_t\}$ for a given sampling interval Δt. So initially assume $f \in [0, f_N]$ in what follows. Then from the given reading material, $\boxed{\text{seen} \Downarrow}$

$$
S_{X_t}(f;\Delta t) = \sum_{k=-\infty}^{\infty} S(f+\tfrac{k}{\Delta t}) = S(f) + \sum_{k=1}^{\infty} S(f+\tfrac{k}{\Delta t}) + \sum_{k=1}^{\infty} S(f-\tfrac{k}{\Delta t}).
$$

$\boxed{1}$

For a given Δt, and the form of $S(f)$ given, the two sums on the right will only have a finite number of terms, (none if no aliasing), since $S(f)$ has finite support. $\boxed{\text{unseen} \Downarrow}$

In the first summation on the right-hand side, the SDF is non-zero only when $f + \frac{k}{\Delta t} \le 2$, i.e., when $k \le \lfloor (2-f)\,\Delta t \rfloor$, where $\lfloor x \rfloor$ is the largest integer less than or equal to x. Since $f \in [0, f_N]$, we can replace the upper limit of the summation by $m = \lfloor 2\,\Delta t \rfloor$. $\boxed{1}$

In the second summation, the SDF is non-zero only when $f - \frac{k}{\Delta t} \ge -2$, i.e., when $k \le \lfloor (2+f)\,\Delta t \rfloor$. Since $f \in [0, f_N]$, we can replace the upper limit of the summation by $\ell = \lfloor 2\,\Delta t + \frac{1}{2} \rfloor$, (so $y = 2\,\Delta t + \frac{1}{2}$). $\boxed{2}$

[Hence we have

$$
S_{\Delta t}(f) = \sum_{k=-\ell}^{m} S(f+\tfrac{k}{\Delta t}).
$$

with $\ell = \lfloor 2\,\Delta t + \frac{1}{2} \rfloor$ and $m = \lfloor 2\,\Delta t \rfloor$.]

(ii) Now specialize to the cases of interest. When $\Delta t = \frac{2}{3}$, then $f_{\mathcal{N}} = \frac{3}{4}$, and $\ell = 1, m = 1$, and

$$S_{X_t}(f; \tfrac{2}{3}) = S(f) + S(f + \tfrac{3}{2}) + S(f - \tfrac{3}{2}) = \begin{cases} 3C, & f \in [0, \frac{1}{2}]; \\ 2C, & f \in (\frac{1}{2}, \frac{3}{4}]. \end{cases}$$

When $\Delta t = \frac{1}{3}$ we have $f_{\mathcal{N}} = \frac{3}{2}$ and $\ell = 1, m = 0$. So

$$S_{X_t}(f; \tfrac{1}{3}) = S(f - 3) + S(f) = \begin{cases} C, & f \in [0, 1); \\ 2C, & f \in [1, \frac{3}{2}]; \end{cases}$$

When $\Delta t = \frac{1}{5}$, then $f_{\mathcal{N}} = \frac{5}{2}$, and $\ell = 0, m = 0$, and there is no aliasing:

$$S_{X_t}(f; \tfrac{1}{5}) = S(f) = \begin{cases} C, & f \in [0, 2]; \\ 0, & f \in (2, \frac{5}{2}]. \end{cases}$$

$\boxed{6}$

(iii) Firstly, $S_{X_t}(f; \Delta t) = S_{X_t}(-f; \Delta t)$ for $f \in [-f_{\mathcal{N}}, 0)$. $\boxed{\text{seen} \Downarrow}$
Secondly, $S_{X_t}(f; \Delta t)$ for f outside of $[-f_{\mathcal{N}}, f_{\mathcal{N}}]$ is defined by periodic extension, (period of $2f_{\mathcal{N}}$). $\boxed{3}$

(iv) The integral of $S(\cdot)$ over $f \in \mathbb{R}$ is $4C$, and the integrals over $[-f_{\mathcal{N}}, f_{\mathcal{N}}]$ of $S_{X_t}(f; \frac{2}{3}), S_{X_t}(f; \frac{1}{3})$ and $S_{X_t}(f; \frac{1}{5})$ are also $4C$. $\boxed{\text{unseen} \Downarrow}$

(v) $S_{X_t}(f; \frac{2}{3})$ is dominated by low frequencies, so red noise; $S_{X_t}(f; \frac{1}{3})$ is dominated by high frequencies, so blue noise; $S_{X_t}(f; \frac{1}{5})$, is dominated by low frequencies, so red noise. $\boxed{1}$ $\boxed{2}$