Mathematical Logic (MATH6/70132; P65)
Problem sheet $2\frac{1}{2}$ - for problem class. Some notes on solutions.

[1] (Warm-up) Decide whether the following are true or false - give reasons.

1. Every L-formula is a theorem.
2. If ϕ is an L-formula, then one of ϕ, $(\neg\phi)$ is a theorem of L.
3. In every L-formula, the number of opening brackets (equals the number of closing brackets).
4. In every L-formula, the number of opening brackets (is equal to the number of connectives in the formula.

Solution:

FFTT. If you got either of the first two wrong you need to check your notes as a matter of urgency. The second two are proved by induction on the length of the formula, using the definition of formulas given in the notes.

[2] See the notes from the class.

[3] (An alternative formal system for propositional logic: natural deduction) The formal system $\widehat{L}$ has the same language and formulas as L, but it has only deduction rules and no axioms. The notion $\Gamma \vdash_{\widehat{L}} \phi$, where Γ is a set of $\widehat{L}$-formulas and ϕ is an $\widehat{L}$-formula, is defined by saying that it satisfies the following deduction rules:

- If $\phi \in \Gamma$ then $\Gamma \vdash_{\widehat{L}} \phi$;
- (Modus Ponens) If $\Gamma \vdash_{\widehat{L}} \phi$ and $\Gamma \vdash_{\widehat{L}} (\phi \to \psi)$, then $\Gamma \vdash_{\widehat{L}} \psi$;
- (Deduction Theorem) If $\Gamma \cup \{\phi\} \vdash_{\widehat{L}} \psi$, then $\Gamma \vdash_{\widehat{L}} (\phi \to \psi)$;
- (PBC) If $\Gamma \vdash_{\widehat{L}} ((\neg\phi) \to \psi)$ and $\Gamma \vdash_{\widehat{L}} ((\neg\phi) \to (\neg\psi))$, then $\Gamma \vdash_{\widehat{L}} \phi$;

and any instance of $\Gamma \vdash_{\widehat{L}} \phi$ arises after a finite number of applications of these rules.

We say that ϕ is a theorem of $\widehat{L}$ if $\emptyset \vdash_{\widehat{L}} \phi$.

Prove that:

(a) $(\phi \to \phi)$ is a theorem of $\widehat{L}$, for every $\widehat{L}$-formula ϕ.

(b) Every axiom of L is a theorem of $\widehat{L}$.

(c) The deduction rule (PBC) is valid with L in place of $\widehat{L}$.

(d) A formula ϕ is a theorem of $\widehat{L}$ if and only if it is a theorem of L.

Solution: (a) We have $\{\phi\} \vdash_{\widehat{L}} \phi$. So by DT, $\emptyset \vdash_{\widehat{L}} (\phi \to \phi)$.

(b) For axioms of type (A1) use two applications of DT:
1. $\{\phi, \psi\} \vdash_{\widehat{L}} \phi$
2. $\{\phi\} \vdash_{\widehat{L}} (\psi \to \phi)$ (by 1. and (DT))
3. $\emptyset \vdash_{\widehat{L}} \phi \to (\psi \to \phi)$ (by 2. and (DT)).

For axioms of type (A2) let Γ be the formulas $\{\phi, \phi \to \psi, \phi \to (\psi \to \chi)\}$. Repeated application of (MP) shows that the formulas $\phi, \phi \to \psi, \psi, \psi \to \chi, \chi$ are consequences of Γ (in $\widehat{L}$). Now use (DT) three times (as above), to take the formulas out of Γ and get an axiom (A2) as a consequence (in $\widehat{L}$) of the empty set.

For the axiom (A3): $(\neg\psi \to \neg\phi) \to (\phi \to \psi)$ take $\Gamma = \{(\neg\psi \to \neg\phi), \phi\}$. Then $\Gamma \cup \{\neg\psi\}$ has ϕ and $\neg\phi$ as consequences, so (using DT) Γ has $\neg\psi \to \neg\phi$ and $\neg\psi \to \phi$ as consequences. So by PBC, $\Gamma \vdash_{\widehat{L}} \psi$. Now apply DT twice to get that the axiom (A3) is a consequence of $\emptyset$.

(c) We use the fact that $(((\neg\phi) \to \phi) \to \phi)$ is a theorem of L (by 1.2.7) and argue as follows. By MP and A3, both ψ and $(\psi \to \phi)$ are consequences (in L) of

$$\{(\neg\phi), ((\neg\phi) \to \psi), ((\neg\phi) \to (\neg\psi))\}.$$

Thus by DT (which holds in L) we have that

$$\{((\neg\phi) \to \psi), ((\neg\phi) \to (\neg\psi))\} \vdash_L ((\neg\phi) \to \phi).$$

Using the above theorem and MP gives

$$\{((\neg\phi) \to \psi), ((\neg\phi) \to (\neg\psi))\} \vdash_L \phi.$$

(d) It follows from (b) and the fact that MP is a deduction rule in $\widehat{L}$ that every theorem of L is a theorem of $\widehat{L}$ (formally, a proof by induction on the length of the proof in L of the theorem can be given, but this is not required). Similarly, as all of the deduction rules of $\widehat{L}$ are valid in L, any theorem of $\widehat{L}$ is a theorem of L.

[4] (For fun) The following is known as Hofstadter's MU puzzle. You can look at the Wikipedia entry, but first try the problem yourself.

The formal system H has: alphabet M, I, U; formulas all (finite) strings of these symbols; one axiom MI; and the following deduction rules (where x, y are any formulas):

1. from xI deduce xIU;
2. from Mx deduce Mxx;
3. from $xIIIy$ deduce xUy;
4. from $xUUy$ deduce xy.

The problem is to decide whether MU is a theorem of H. But you could first write down some theorems of H, just to test your understanding of what a formal system is.

Solution: Use your favourite search engine....