(2.4.6) Cor. (Consistency of $K_{\mathcal{L}}$)

There is no formula ϕ with $\vdash_{K_{\mathcal{L}}} \phi$ and $\vdash_{K_{\mathcal{L}}} (\neg\phi)$.

Pf: Follows from 2.4.5 as ϕ, $(\neg\phi)$ cannot both be logically valid. //. #.

(2.4.7) Ex. Suppose Σ is a set of $\mathcal{L}$-fmlas and ψ is an $\mathcal{L}$-fmla with $\Sigma \vdash_{K_{\mathcal{L}}} \psi$. Then for every valuation v with $v(\Sigma) = T$ (i.e. $v[\sigma] = T$ for all $\sigma \in \Sigma$) we have $v[\psi] = T$. //.

(2.4.8) Thm. (Deduction Thm.)

Suppose $\mathcal{L}$ is a 1st order language, Σ is a set of $\mathcal{L}$-fmlas and ϕ, ψ are $\mathcal{L}$-formulas. If

$$\Sigma \cup \{\phi\} \vdash_{K_{\mathcal{L}}} \psi$$

then

$$\Sigma \vdash_{K_{\mathcal{L}}} (\phi \to \psi).$$

Pf: Like the DT for L (1.2.5). By induction on the length of the deduction of ψ from $\Sigma \cup \{\phi\}$.

Base case: If there is a one-step deduction, argue as in 1.2.5.

<u>Inductive step</u> Suppose ψ follows from earlier formulas in the deduction using MP or Gen

<u>MP</u> Exactly as in 1.2.5.

<u>Gen</u> Suppose ψ is obtained by using Gen. So ψ is $(\forall x_i)\theta$ and

$$\Sigma \cup \{\phi\} \vdash \theta$$

<u>and</u> x_i is not free in any formula in $\Sigma \cup \{\phi\}$.

By ind. hypothesis

$$\Sigma \vdash (\phi \to \theta)$$

Apply Gen. to this (x_i not free in Σ).

So $\Sigma \vdash (\forall x_i)(\phi \to \theta)$

Use axiom K2 (noting x_i not free in ϕ), MP to get

$$\Sigma \vdash (\phi \to (\forall x_i)\theta).$$

ie $\Sigma \vdash (\phi \to \psi)$ //.

this finishes the ind. step. #

2.5 Gödel's Completeness thm.

(2.5.1) **Def.** A set Σ of $\mathcal{L}$-formulas is <u>consistent</u> if there is no $\mathcal{L}$-fmla ϕ with

$\Sigma \vdash_{K_\mathcal{L}} \phi$ and $\Sigma \vdash_{K_\mathcal{L}} (\neg\phi)$.

[2.4.6 : "ϕ is consistent" or $K_\mathcal{L}$ is consistent.].

<u>Rk</u>: If Σ is inconsistent then $\Sigma \vdash \chi$ for any $\mathcal{L}$-fmla χ.

[as with L.].

<u>Recall</u>: A <u>closed</u> $\mathcal{L}$-fmla ϕ is one with <u>no free variables</u>.

//

<u>Show</u> if Σ is a <u>consistent</u> set of closed $\mathcal{L}$-fmlas then there is an $\mathcal{L}$-structure $\mathcal{A}$ with $\mathcal{A} \models \Sigma$ (i.e. $\mathcal{A} \models \sigma$ for all $\sigma \in \Sigma$.).

//

<u>Simplification</u> Assume $\mathcal{L}$ is <u>countable</u>

i.e. the variables are $x_0, x_1, x_2, \ldots$ and there are countably many relation, function and constant symbols.

So we can list the $\mathcal{L}$-formulas (or any subset of the $\mathcal{L}$-fmlas) as a list indexed by $\mathbb{N}$.

Eg. enumerate the closed $\mathcal{L}$-fmlas as $\psi_0, \psi_1, \psi_2, \ldots$

($\{\psi_i : i \in \mathbb{N}\}$).

(2.5.2) Proposition Suppose Σ is a consistent set of closed $\mathcal{L}$-fmlas and ϕ a closed $\mathcal{L}$-fmla.

1 (like 1.3.7) If

$$\Sigma \nvdash_{K_\mathcal{L}} \phi \quad \text{then}$$

$$\Sigma \cup \{(\neg\phi)\} \text{ is}$$

consistent.

2 (Lindenbaum Lemma, (like 1.3.8)).

There is a consistent set $\Sigma^* \supseteq \Sigma$ of closed $\mathcal{L}$-fmlas such that for every closed $\mathcal{L}$-fmla ψ either

$$\Sigma^* \vdash \psi$$

$$\text{or} \quad \Sigma^* \vdash (\neg\psi).$$

Pf: 1 As in 1.3.7.

(Use DT & $\vdash_{K_\mathcal{L}} ((\neg\phi)\to\phi)\to\phi)$)

2 Uses 1 and the enumeration $(\psi_i : i \in \mathbb{N})$ of the closed $\mathcal{L}$-fmlas. #.

(2.5.3) Thm. (Model Existence Thm.)
Suppose Σ is a consistent set of closed $\mathcal{L}$-fmlas. Then there is an $\mathcal{L}$-str. $\mathcal{A}$ with $\mathcal{A} \models \Sigma$.

Pf: Hard bit: later. #.

Notation: $\Sigma \models \phi$

means "if $\mathcal{A} \models \Sigma$ then $\mathcal{A} \models \phi$."

(2.5.4) Thm. Let Σ be a set of closed $\mathcal{L}$-fmlas and ϕ a closed $\mathcal{L}$-fmla. If $\Sigma \models \phi$, then $\Sigma \vdash_{K_{\mathcal{L}}} \phi$.

Pf: If Σ is inconsistent then $\Sigma \vdash \psi$ for every ψ. So assume Σ is consistent.

Suppose $\Sigma \nvdash \phi$. By 2.5.2 (1) $\Sigma \cup \{(\neg\phi)\}$ is consistent.

So by 2.5.3, there is an $\mathcal{L}$-str. $\mathcal{A}$ with
$$\mathcal{A} \models \Sigma \cup \{(\neg\phi)\}.$$
So $\mathcal{A} \models \Sigma$ and $\mathcal{A} \models (\neg\phi)$.

This contradicts $\Sigma \models \phi$.

#.

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(2.5.5) <u>Thm.</u> ($\underline{\mathcal{L} \text{ countable}}$)
(Gödel Completeness Thm. for $K_{\mathcal{L}}$) If ϕ is an $\mathcal{L}$-fmla with $\models \phi$, then ϕ is a theorem of $K_{\mathcal{L}}$ (i.e. $\vdash_{K_{\mathcal{L}}} \phi$).

<u>Pf</u>: If ϕ is closed, then this follows from 2.5.4 (by taking $\Sigma = \emptyset$).

Suppose ϕ has its free variables amongst $x_0, \ldots, x_n$ & ~~and~~ consider ψ:
$(\forall x_0) \ldots (\forall x_n) \phi$.
(this is closed).

As $\models \phi$ we have $\models \psi$:

So $\vdash_{K_{\mathcal{L}}} \psi$ (by the closed case!)

i.e. $\vdash_{K_{\mathcal{L}}} (\forall x_0) \ldots (\forall x_n) \phi$ ... (*)

If θ is any fmla. then $((\forall x_i)\theta \to \theta)$ is an axiom of type K1. (with t being x_i)

Using (*), these axioms and MP we obtain $\vdash_{K_{\mathcal{L}}} \phi$.

#