① 2.4 The Formal System $K_{\mathcal{L}}$

(2.4.1) Def. Suppose $\mathcal{L}$ is a 1^{st} order language. The formal system $K_{\mathcal{L}}$ has formulas the $\mathcal{L}$-formulas and:

Axioms For ϕ, ψ, χ $\mathcal{L}$-formulas

A1 $(\phi \to (\psi \to \phi))$

A2 $((\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi)))$

A3 $(((\neg\psi) \to (\neg\phi)) \to (\phi \to \psi))$

K1 $((\forall x_i)\phi(x_i) \to \phi(t))$

where t is a term free for x_i in ϕ [ϕ can have other free variables]

K2 $((\forall x_i)(\phi \to \psi) \to (\phi \to (\forall x_i)\psi))$

if x_i is not free in ϕ.

Deduction rules: MP From ϕ and $(\phi \to \psi)$ deduce ψ

Gen (Generalisation) from ϕ deduce $(\forall x_i)\phi$.

A **proof** in $K_{\mathcal{L}}$ is a finite sequence of $\mathcal{L}$-fmlas, each of which is an axiom or deduced from previous fmlas. using a deduction rule. A **theorem** of $K_{\mathcal{L}}$ is the last fmla. in some proof.

Write $\vdash_{K_{\mathcal{L}}} \phi$ if ϕ is a theorem of $K_{\mathcal{L}}$

(or $\vdash \phi$).

(2.4.2) **Def**. Suppose Σ is a set of $\mathcal{L}$-fmlas. and ψ is an $\mathcal{L}$-fmla.

A **deduction** of ψ from Σ is a finite sequence of fmlas, ending with ψ, each of which is an axiom, an elt. of Σ or obtained from fmlas earlier in the deduction using MP or Gen with the restriction that when Gen is applied it does not involve a variable occurring freely in Σ.

Write $\Sigma \vdash_{K_{\mathcal{L}}} \psi$ in this case.

(Say ψ is a *consequence* of Σ.)

(2.4.3) Remarks

(1) If Σ consists of closed fmlas, don't have to worry about the ~~Gen~~ restriction.

(2) Without the restriction would have

" $\{\phi\} \vdash (\forall x_i)\phi$ "

– not sensible.

(3) Should have: if $\Sigma' \subseteq \Sigma$ and $\Sigma' \vdash \phi$ then $\Sigma \vdash \phi$.

So ought to modify the defn. to allow for this.

(2.4.4) Theorem. Suppose ϕ is an $\mathcal{L}$-fmla which is a substitution instance of a propositional tautology χ.

Then $\vdash_{K_{\mathcal{L}}} \phi$.

Eg $((\neg(\neg\phi_1)) \to \phi_1)$

(for a $\mathcal{L}$-fmla ϕ_1)

is a subst. instance of the prop. taut. $((\neg(\neg p_1)) \to p_1)$.

Pf: Let $p_1, \ldots, p_n$ be the prop. vars. in χ & we obtain ϕ by substituting $\psi_1, \ldots, \psi_n$ in place of $p_1, \ldots, p_n$ in χ.

By the Completeness Thm for L (1.3.11), there is a pf in L of χ: $\chi_1, \ldots, \chi_r$ where χ is χ_r.

If we substitute $\psi_1, \ldots, \psi_n$ in $\chi_1, \ldots, \chi_r$ in place of $p_1, \ldots, p_n$ we obtain a pf. of ϕ in $K_{\mathcal{L}}$. // . #.

(2.4.5) Thm. (Soundness of $K_{\mathcal{L}}$) 4

If $\vdash_{K_{\mathcal{L}}} \phi$ then

$\models \phi$ (i.e. ϕ is logically valid).

Pf: Like the pf. for L (1.3.1)

- Show that the axioms are logically valid +
- the deduction rules preserve logical validity.

= A1, A2, A3 are sub. instances of prop. tauts. so are logically valid by (2.2.14).

K1 is logically valid, by 2.3.6.

<u>K2</u> $((\forall x_i)(\phi \to \psi) \to (\phi \to (\forall x_i)\psi))$

(x_i not free in ϕ).

Let v be a valuation (in an $\mathcal{L}$-str. $\mathcal{A}$) with

$v[(\phi \to (\forall x_i)\psi] = F$.

then $v[\phi] = T$ & $v[(\forall x_i)\psi] = F$.

So there is v' x_i-equiv. to v

with $v'[\psi] = F$.

x_i is not free in ϕ, so

as v, v' are x_i-equiv.

v, v' agree on the <u>free</u> variables

in ϕ. thus

$$v'[\phi] \underset{2.3.3}{=} v[\phi] = T$$

So $v'[(\phi \to \psi)] = F$.

thus

$v[(\forall x_i)(\phi \to \psi)] = F$

(as v, v' are x_i-equiv.).

So $v[K2] = T$,

as req'd. //.

<u>MP</u>: If $\models \phi$ and

$\models (\phi \to \psi)$

then $\models \psi$. // Ex.

<u>Gen</u>: If $\models \phi$

then $\models (\forall x_i)\phi$. // Easy Ex.

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