M4P70 Markov Processes, Solutions 1

Lecture 5, 14 Oct

Borel-Cantelli Lemmas

For A_n a sequence of events, its limit superior, or the event $A_n i.o.$ (i.o. for 'infinitely often') is

$$\{A_n\ i.o.\} := \bigcap_m \bigcup_{n>m} A_n.$$

[Compare Analysis, where a sequence of reals has $\limsup x_n := \inf_m \sup_{n>m} x_n$.]

Q1. Prove the first Borel-Cantelli Lemma:
If $\sum \mathbb{P}(A_n) < \infty$, then $\mathbb{P}(A_n\ i.o.) = 0$.

Proof. By countable additivity,

$$\begin{aligned} \mathbb{P}(A_n\ i.o.) := \mathbb{P}(\lim_m \bigcup_{n=m}^{\infty} A_n) &= \lim_m \mathbb{P}(\bigcup_{n=m}^{\infty} A_n) \\ &\leq \lim_m \sum_{n=m}^{\infty} \mathbb{P}(A_n) \\ &\to 0 \qquad (m \to \infty), \end{aligned}$$

as $\sum \mathbb{P}(A_n)$ converges. So $\mathbb{P}(A_n\ i.o.) = 0$.

Q2. Prove the second Borel-Cantelli Lemma:
If the events A_n are *independent* and $\sum \mathbb{P}(A_n) = \infty$, then $\mathbb{P}(A_n\ i.o.) = 1$.

Proof. By de Morgan's laws, with A^c for the complement of A,

$$\mathbb{P}(\bigcup_m^{\infty} A_n) = 1 - \mathbb{P}(\bigcap_m^{\infty} A_n^c).$$

By independence,

$$\mathbb{P}(\bigcap_m^{\infty} A_n^c) = \prod_m^{\infty} \mathbb{P}(A_n^c) = \prod_m^{\infty} (1 - \mathbb{P}(A_n)).$$

But $e^{-x} \geq 1 - x$ (e^{-x} is *convex* – 'curve below chord'; the curve e^{-x} and the line $1 - x$ touch each other at $(0, 1)$; the curve is elsewhere above the line). Taking logs, $-x \geq \log(1 - x)$. So

$$\log \prod_m^\infty (1 - \mathbb{P}(A_n)) = \sum_m^\infty \log(1 - \mathbb{P}(A_n)) \leq - \sum_m^\infty \mathbb{P}(A_n) = -\infty,$$

as $\sum \mathbb{P}(A_n)$ diverges. Taking exponentials,

$$\prod_m^\infty (1 - \mathbb{P}(A_n)) = 0.$$

Combining,

$$\mathbb{P}(\bigcup_m^\infty A_n) = 1,$$

as required. □

Note. The events $\limsup A_n$, $\liminf A_n$ are *tail events*: they are invariant under deletion of finitely many events A_n. The *Kolmogorov* $0 - 1$ *law* says that more generally, tail events of sequences of *independent* events have probability 0 or 1.

This remains true under various forms of *weak dependence*, e.g. in the hierarchy of *mixing conditions*, but we cannot pursue this here. NHB