Mathematical Logic (MATH6/70132;P65)
Problem Class, week 6

[1] Suppose $\mathcal{L}$ is a first-order language with relation, function and constant symbols:

$$(R_i : i \in I);\ (f_j : j \in J);\ (c_k : k \in K),$$

where R_i is of arity m_i and f_j is of arity n_j.

(a) Based on what you know to be the definition of 'isomorphism' in the case of groups, or rings (or graphs or orderings), write down a definition of what it should mean for two $\mathcal{L}$-structure $\mathcal{A}$ and $\mathcal{B}$ to be isomorphic.

(b) Suppose α is an isomorphism from the $\mathcal{L}$-structure $\mathcal{A}$ to the $\mathcal{L}$-structure $\mathcal{B}$. Suppose v is a valuation in $\mathcal{A}$. Let w be the vaulation in $\mathcal{B}$ with $w(x_i) = \alpha(v(x_i))$ for all variables x_i. Prove that for every $\mathcal{L}$-formula ϕ,

$$v \text{ satisfies } \phi \text{ in } \mathcal{A} \Leftrightarrow w \text{ satisfies } \phi \text{ in } \mathcal{B}.$$

(Use induction on the length of ϕ...)

(c) With $\mathcal{A}, \mathcal{B}$ as in (b), show that for every $\mathcal{L}$-formula ϕ:

$$\mathcal{A} \models \phi \Leftrightarrow \mathcal{B} \models \phi.$$

[2] It's difficult to do any reasoning in mathematics without using the equality symbol. In 1st-order logic, we use the following terminology to handle equality. We will say more about this during the lectures after the class.

A first-order *language with equality* $\mathcal{L}^=$ is a 1st-order language with a distinguished 2-ary relation symbol $=$. An $\mathcal{L}^=$-structure $\mathcal{A}$ is *normal* if the symbol $=$ is interpreted as equality in $\mathcal{A}$.

We write the more usual '$x_1 = x_2$' instead of '$=(x_1, x_2)$' in $\mathcal{L}^=$-formulas.

Suppose $\mathcal{L}^=$ is a language with equality which also has a 2-ary relation symbol R.

(a) Suppose $n \in \mathbb{N}$. Write down a closed $\mathcal{L}^=$-formula σ_n with the property that, for every normal $\mathcal{L}^=$-structure $\mathcal{A}$ we have $\mathcal{A} \models \sigma_n$ iff the domain of $\mathcal{A}$ has at least n elements.

(b) Suppose $n \in \mathbb{N}$. Write down a closed $\mathcal{L}^=$-formula τ_n with the property that, for every normal $\mathcal{L}^=$-structure $\mathcal{A}$ we have $\mathcal{A} \models \tau_n$ iff the domain of $\mathcal{A}$ has exactly n elements.

(c) Find a closed $\mathcal{L}^=$ formula θ with the property that for every positive $n \in \mathbb{N}$:

there is a normal $\mathcal{L}^=$ structure $\mathcal{A}$ with n elements in its domain and $\mathcal{A} \models \theta$
if and only if n is even.

(d) Suppose $=$ and R are the only relation symbols in the language and $J = K = \emptyset$. Can you construct your formula θ in (c) so that any two countably infinite normal models of θ are isomorphic? Can you find two non-isomorphic, infinite normal models of such a θ?