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Theorems and Problems in Functional Analysis

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Preface

Even the simplest mathematical abstraction of the phenomena of reality—the real line—can be regarded from different points of view by different mathematical disciplines. For example, the algebraic approach to the study of the real line involves describing its properties as a set to whose elements we can apply "operations," and obtaining an algebraic model of it on the basis of these properties, without regard for the topological properties. On the other hand, we can focus on the topology of the real line and construct a formal model of it by singling out its "continuity" as a basis for the model. Analysis regards the line, and the functions on it, in the unity of the whole system of their algebraic and topological properties, with the fundamental deductions about them obtained by using the interplay between the algebraic and topological structures.

The same picture is observed at higher stages of abstraction. Algebra studies linear spaces, groups, rings, modules, and so on. Topology studies structures of a different kind on arbitrary sets, structures that give mathematical meaning to the concepts of a limit, continuity, a neighborhood, and so on. Functional analysis takes up topological linear spaces, topological groups, normed rings, modules of representations of topological groups in topological linear spaces, and so on. Thus, the basic object of study in functional analysis consists of objects equipped with compatible algebraic and topological structures.

The functional analysis course first given by A. N. Kolmogorov in the Mechanics–Mathematics Department of Moscow University traditionally included the theory of measures and the Lebesgue integral and dealt mainly with the classical areas of functional analysis. The present book is the result of an attempt to generalize and systematize the experience of the authors in instructing this course at Moscow University, and it has the following aims:

(1) To present the necessary theoretical material for the Analysis III course within the scope of a university mathematics program.
(2) To provide instructors giving lectures and conducting exercises in Analysis III (or functional analysis), as well as students studying this subject, with a book that is a natural combination of a textbook and a problem book with fairly detailed hints for solving the problems.
(3) To expose the reader to certain elements of the apparatus used in solving the problems of modern functional analysis (categories, functors, cohomology spaces, group characters, and so on).
(4) To provide the reader with a text suitable for independent study of the classical chapters of functional analysis and for mastering the techniques of solving the corresponding problems.

The book is divided into three closely connected parts: *Theory*, *Problems*, and *Hints* for solving the problems. The corresponding divisions of each of the three parts are combined under a common heading. The chapters are subdivided into sections, and the sections into subsections (except Chapter I). The system of division of the book into subsections can be recommended as a rough scheme for distributing the material in seminar sessions, with the material of each subsection covered in one to three sessions, depending on the direction of the course, the level of preparation of the students, and their interests. For each subsection there are 23 problems of varying difficulty; those that make up a necessary minimum are marked by small circles, and the complicated ones by asterisks. The few very difficult problems are marked by two asterisks. The solutions of the asterisked problems can be gone over by the instructor or presented by the students as topics of individual reports. On the other hand, it is reasonable to recommend the problems with circles for written control assignments.

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PART I

THEORY

Chapter I

Concepts from Set Theory and Topology

§1. Relations. The Axiom of Choice and Zorn's Lemma

Let X be a set, and R a subset of $X \times X$. Points x and y in X are said to be *in the relation* R, denoted xRy, if $(x, y) \in R$.

EXAMPLES OF RELATIONS. (1) The relation of equality:

$$R = \Delta_X = \{(x, x) \mid x \in X\}.$$

(2) The order relation on the real line:

$$R = \{(x, y) \mid x \geq y\}.$$

(3) The relation of linear dependence in a linear space L over a field K:

$$R = \{(x, y) \mid y = 0 \text{ or } x = \lambda y, \lambda \in K\}.$$

A relation R is called an *equivalence relation* if it has the following properties:

(1) *reflexivity*: $(x, x) \in R \ \forall x \in X$ (or $R \supset \Delta_X$);
(2) *symmetry*: $(x, y) \in R \Rightarrow (y, x) \in R$ (or $R' = R$, where R' denotes the transposed relation: $R' = \{(x, y) \mid (y, x) \in R\}$);
(3) *transitivity*: $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$ (or $R \circ R \subset R$, where the symbol $\circ$ denotes composition of relations: $R_1 \circ R_2 = \{(x, z) \mid \exists y : (x, y) \in R_1 \text{ and } (y, z) \in R_2\}$).

Suppose that R is an equivalence relation. Then we write $x \sim y$ instead of xRy and say that x is *equivalent* to y. Let $R(x)$ denote the set of all elements in X that are equivalent to x. It follows from the properties (1), (2), and (3) that the subsets of the form $R(x)$ exhaust all of X and pairwise either are disjoint or coincide. These subsets are called *equivalence classes*. The collection of equivalence classes is denoted by $X_{(R)}$ and is called the *quotient set* of X by the relation R.

EXAMPLES. (1) The projective space $P(L)$ associated with a given linear space L.

(2) The quotient space L_1/L_2 of a linear space L_1 by a subspace L_2 (for $x, y \in L_1$ we say that $x \sim y$ if $x - y \in L_2$).

(3) The collection of residual classes modulo n.

(4) The collection of positive rational numbers or the collection of all integers as equivalence classes of pairs of natural numbers.

A relation R on a set X is said to be a *partial-order relation* if it has the following properties:

(1) *transitivity*: $(R \circ R \subset R)$;
(2) *antisymmetry*: $(R \cap R' \subset \Delta_X)$.

Instead of xRy we usually write $x \geq y$ (or $y \leq x$) and say that "x follows y." If, moreover,

(3) $R \cup R' = X \times X$ (i.e., any two elements are comparable),

then R is called a *total order relation.*

In addition to the symbol $\geq$ for a relation R, we shall use the symbol $>$ for the relation $\tilde{R} = R \backslash \Delta_X$. Thus, the expression $x > y$ (read "x strictly follows y" or "x is strictly greater than y") means that $x \geq y$ and $x \neq y$.

EXAMPLES. (1) The usual order relation on the real line.

(2) The inclusion relation for subsets of a given set (this relation is denoted by the symbol $\subset$), which is a partial, but not total, order relation.

(3) The relation of divisibility for integers (usually denoted by the symbol $|$), also a partial-order relation.

A subset Y of a partially ordered set X is said to be *bounded above* (respectively, *below*) if it has a *majorant* (resp., *minorant*), i.e., an element $x \in X$ such that $y \leq x$ (resp., $y \geq x$) for all $y \in Y$.

An element x_0 is called a *maximal* (resp., *minimal*) element if $y \geq x_0$ (resp., $y \leq x_0$) implies $y = x_0$, and is called the *largest* (resp., *smallest*) element if $x_0 \geq x$ (resp., $x_0 \leq x$) for all x in X.

A set X with a partial-order relation R is called a *directed set* if R has the property that $R \circ R' = X \times X$ (in other words, for any x and y in X there is an element z following x and y). If (X, R) is a directed set and M is an arbitrary set, then a mapping from X into M is called a *direction* or *net* in M. This concept is a generalization of the concept of a sequence, to which it reduces if X is the series of natural numbers with the usual order relation.

The concept of a limit is defined for a net in a topological (in particular, in a metric) space X: A point $x \in X$ is a limit of a net $\{x_\alpha\}_{\alpha \in A}$ if for any neighborhood V of x there is an $\alpha(V) \in A$ such that $x_\alpha \in V$ for all $\alpha \geq \alpha(V)$.

In courses of higher mathematics it is often said that the concept of a *set* "is so general that it is difficult to give any definition for it," and then to restrict oneself to indicating some synonyms: class, collection, family, and so on. In fact, there is a rigorous theory of sets in which this concept is precisely defined (not, of course, by reducing to other simpler or more general concepts, but by describing the properties enjoyed by all sets). It turns out that not all "classes," "collections," or "families" are sets. (For example, the concept of the set of all sets is contradictory, as is well known.) Nevertheless, there are consistent theories with a plentiful supply of sets.

For most branches of mathematics it suffices that the supply of sets under consideration contains at least one infinite set and admits the following operations:

(1) union $\bigcup_{\alpha \in A} X_\alpha$;
(2) intersection $\bigcap_{\alpha \in A} X_\alpha$;
(3) difference $X \backslash Y$;
(4) construction of the *set of mappings from X into Y*, denoted by Y^X;
(5) product $\prod_{\alpha \in A} X_\alpha$.

Here X, Y, A, and all the X_α, $\alpha \in A$, are sets, and we assume that the result of the operation is also a set.

The last operation merits a more detailed discussion. Suppose that A is a set and that a nonempty set X_α is assigned to each element $\alpha \in A$. By definition, an element of the set $\prod_{\alpha \in A} X_\alpha$ is a mapping $\alpha \mapsto x_\alpha$ of A into $\bigcup_{\alpha \in A} X_\alpha$ such that $x_\alpha \in X_\alpha$ for all $\alpha \in A$. If A is an infinite set, then the existence of such a mapping is not obvious (and, as is now known, cannot be derived from its existence for finite sets and from the other natural axioms). Therefore, the assertion that $\prod_{\alpha \in A} X_\alpha$ is not empty when the X_α are not empty is taken as an independent axiom. It has received the name *the axiom of choice*, or *Zermelo's axiom*. We give two statements equivalent to the axiom of choice.

Zorn's Lemma. *If every totally ordered subset of a partially ordered set X is bounded above (below), then X contains at least one maximal (minimal) element x_0.*

Zermelo's Theorem. *Every set can be well ordered, i.e., it is possible to introduce an order relation on it such that any subset contains a smallest element.*

Both these assertions are, in essence, a generalization of the familiar principle of mathematical induction, and they replace this principle in the cases when we have to deal with uncountable sets.

To the reader wishing to acquaint himself in greater detail with the foundations of set theory we recommend the summary of results in [6], as well as the author's preface to this book.

§2. Completions

Definition. A sequence $\{x_n\}$ in a metric space is called a *Cauchy sequence* if the distance $d(x_n, x_m)$ tends to zero as n and m tend to infinity. A metric space X is said to be *complete* if every Cauchy sequence has a limit in X.

Complete spaces have an important property: the theorem on shrinking balls and the contraction mapping principle hold in them (see, for example, [18]). However, it is frequently necessary to deal with spaces that are not complete. There is a remarkable construction enabling us to make any incomplete space into a corresponding complete space by adjoining the "missing" points.

Definition. Let X be a metric space. A *completion* of X is defined to be a metric space Y having the following properties:

(1) Y is a complete space;
(2) Y contains a subset Y_0 isometric to X;
(3) Y_0 is dense in Y (i.e., the closure of Y_0 coincides with Y, or, in other words, each point of Y is a limit point of Y_0).

EXAMPLE. The set **R** of real numbers is a completion of the set **Q** of rational numbers with the usual distance.

Theorem 1. *Each metric space X admits a completion Y. Any two completions Y' and Y'' of X are isometric by an isometry leaving the points of X fixed.*

PROOF. The proof consists in an explicit construction of the completion. Let d denote the distance on X. The set of all Cauchy sequences of points in X is denoted by F. If $x = \{x_n\}$ and $y = \{y_n\}$ are two points in F, then $d(x_n, y_n)$ is a Cauchy sequence of numbers, since $|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n, y_m)$. Hence, this sequence has a limit, which we denote by $d(x, y)$. The quantity $d(x, y)$ has almost all the properties of a distance. Indeed, the inequalities $d(x, y) \geq 0$, $d(x, y) \leq d(x, z) + d(y, z)$ and the equalities $d(x, x) = 0$, $d(x, y) = d(y, x)$ are easy to get by passing to the limit from the corresponding inequalities and equalities with x_n, y_n, and z_n in place of x, y, and z. Only the separation property fails: $d(x, y) = 0$ does not necessarily imply that $x = y$.

We introduce the relation $R = \{(x, y) | d(x, y) = 0\}$ on F. The properties of $d(x, y)$ given above imply that R is an equivalence relation. Let $Y = F_{(R)}$, and define a distance on Y by setting $d(R(x), R(y)) = d(x, y)$. The verification that this defines a distance unambiguously is left to the reader.

Let us now show that Y is a completion of X. To do this we consider the mapping $\varphi: X \to Y$ that assigns to each point x the class $\varphi(x)$ containing the constant (hence, Cauchy) sequence $\bar{x} = (x, x, x, \ldots, x, \ldots)$. It is clear that φ is an isometry. Let Y_0 be the image of X under the mapping φ. Next, let y be any element of Y, and let $\{x_n\} \in F$ be some sequence in the class y. Then

$$\lim_{n\to\infty} d(\varphi(x_n), y) = \lim_{n\to\infty} \lim_{m\to\infty} d(x_n, x_m) = 0.$$

This means that y is the limit of the sequence $\{\varphi(x_n)\}$ and, hence, is a limit point of Y_0.

We show that Y is complete. Let $\{y_n\}$ be a Cauchy sequence of elements of Y. Since Y_0 is dense in Y, there is a sequence $\{\varphi(x_n)\}$ of elements of Y_0 such that $d(\varphi(x_n), y_n) \to 0$. It is clear that $\{y_n\}$ and $\{\varphi(x_n)\}$ converge or diverge simultaneously. But $\{x_n\}$ is a Cauchy sequence of elements of X since $d(x_n, x_m) = d(\varphi(x_n), \varphi(x_m)) \leq d(\varphi(x_n), y_n) + d(y_n, y_m) + d(y_m, \varphi(x_m))$, so $\{\varphi(x_n)\}$ has as its limit the class y of the sequence $\{x_n\}$. Indeed,

$$\lim_{n\to\infty} d(\varphi(x_n), y) = \lim_{n\to\infty} \lim_{m\to\infty} d(x_n, x_m) = 0.$$

Suppose now that Y' and Y'' are two completions of X, and $\varphi': X \to Y'_0$ and $\varphi'': X \to Y''_0$ are the corresponding isometric mappings. We consider the mapping $\psi_0 = \varphi' \circ (\varphi'')^{-1}$ from Y''_0 to Y'_0. It is an isometry and, hence, carries Cauchy sequences to Cauchy sequences. Since Y' and Y'' are complete, Cauchy sequences in Y'_0 (Y''_0) converge in Y' (Y''). This allows us to extend the isometry $\psi_0: Y''_0 \to Y'_0$ to an isometry $\psi: Y'' \to Y'$ by setting

$$\psi\left(\lim_{n\to\infty} \varphi''(x_n)\right) = \lim_{n\to\infty} \psi_0\left(\varphi''(x_n)\right). \qquad \square$$

The completion is often constructed in practice by another means.

Theorem 2. *Let M be a complete metric space, and X a subset of M. Then X is complete if and only if it is closed in M. In particular, the closure of X in M can be taken as its completion.*

(See Problem 31 concerning a proof.)

Example. The completion of the interval (a, b) with respect to the usual distance is the segment $[a, b]$: the closure of (a, b) in $\mathbf{R}$.

§3. Categories and Functors

It is convenient to get many of the definitions and constructions used in mathematics from a small number of general concepts that have, in recent years, formed a special area of study: category theory. We acquaint the reader with the elements of this theory.

We say that we have a *category* K if we are given a class (generally speaking, not a set; see §1) $\mathrm{Ob}(K)$ of *objects of the category* and, for each pair of objects A, B, a set $\mathrm{Mor}(A, B)$ of *morphisms of the category* from A to B. Moreover, morphisms can be multiplied, i.e., we are given a mapping $\mathrm{Mor}(A, B) \times \mathrm{Mor}(B, C) \to \mathrm{Mor}(A, C)$: the image of a pair of morphisms $f \in \mathrm{Mor}(A, B)$ and $g \in \mathrm{Mor}(B, C)$ belongs to $\mathrm{Mor}(A, C)$ and is denoted by $g \circ f$. It is assumed to have the usual property of a composition of mappings: $h \circ (g \circ f) = (h \circ g) \circ f$ for $f \in \mathrm{Mor}(A, B)$, $g \in \mathrm{Mor}(B, C)$, $h \in \mathrm{Mor}(C, D)$. Moreover, $\mathrm{Mor}(A, A)$ contains a so-called *identity morphism*, denoted by 1_A and having the properties that $1_A \circ f = f$ and $g \circ 1_A = g$ for all $f \in \mathrm{Mor}(B, A)$, $g \in \mathrm{Mor}(A, B)$. For clarity the objects of a category are frequently denoted by points and the morphisms by arrows connecting these points.

Examples. (1) The category of sets (the objects are sets, and the morphisms mappings of sets).

(2) The category of groups (resp., rings, algebras) (the objects are groups (resp. rings, algebras), and the morphisms homomorphisms).

(3) The category of topological spaces (the objects are topological spaces, and the morphisms continuous mappings).

(4) The category of linear spaces over a given field K (the objects are linear spaces over K, and the morphisms linear operators).

Two objects A and B of a category K are said to be *isomorphic* if there are morphisms $f \in \mathrm{Mor}(A, B)$ and $g \in \mathrm{Mor}(B, A)$ such that $f \circ g = 1_B$, $g \circ f = 1_A$.

An object A in a category K is said to be a *universal repelling* object* if for any object B in K the set $\mathrm{Mor}(A, B)$ consists of precisely one element. (Expressed graphically: exactly one arrow goes from the point A to any other point B.)

As an exercise in the concepts introduced we show that any two universal repelling objects A and B are isomorphic (if they exist). Indeed, let f be the single morphism from A to B and g the single morphism from B to A. Then $f \circ g \in \mathrm{Mor}(B, B)$, $g \circ f \in \mathrm{Mor}(A, A)$. But $\mathrm{Mor}(B, B)$ contains the single element 1_B (since B is universal), and $\mathrm{Mor}(A, A)$ the single element 1_A (since A is universal). Hence, $f \circ g = 1_B$ and $g \circ f = 1_A$.

Let us now show that the concepts of a quotient set and a completion are particular cases of the concept of a universal object. In the first case we consider the following category K constructed from a set X and a relation R.

* For brevity we sometimes omit the word "repelling."

An object of K is a mapping φ of X to some other set Y (depending on the object) having the property that $xRy \Rightarrow \varphi(x) = \varphi(y)$. A morphism of the object $\varphi: X \to Y$ to the object $\psi: X \to Z$ is defined to be a mapping $\chi: Y \to Z$ for which the following diagram is commutative:

$$\begin{tikzcd} & Y \arrow[d, "\chi"] \\ X \arrow[ur, "\varphi"] \arrow[r, "\psi"'] & Z \end{tikzcd} \tag{1}$$

This means that $\chi \circ \varphi = \psi$. (In general, the *commutativity of a diagram* composed of objects and morphisms in some category means that for any path from one point of the diagram to another point along the arrows of this diagram the product of the corresponding morphisms depends only on the initial and final objects, and not on the choice of paths. In the example above, there are two paths from X to Z, hence the condition $\chi \circ \varphi = \psi$.)

We verify that the canonical projection $p: X \to X_{(R)}$ is a universal object in the category K. Let $\varphi: X \to Y$ be an object in K. Consider the diagram

$$\begin{tikzcd} & X_{(R)} \arrow[d, dashed] \\ X \arrow[ur, "p"] \arrow[r, "\varphi"'] & Y \end{tikzcd}$$

It is easy to see that the commutativity condition uniquely determines the mapping corresponding to the dashed arrow. This means that $\mathrm{Mor}(p, \varphi)$ consists of a single element. Therefore, $p: X \to X_{(R)}$ is a universal object.

Let us now analyze the construction of a completion. Suppose that X is a metric space. We consider the category K whose objects are isometric mappings $\varphi: X \to Y$, where Y is a complete metric space (depending on the object). A morphism from $\varphi: X \to Y$ to $\psi: X \to Z$ is defined to be an isometric mapping $\chi: Y \to Z$ for which diagram (1) is commutative.

We verify that the canonical imbedding φ of X into its completion Y is a universal object. Indeed, for any object $\psi: X \to Z$ the diagram

$$\begin{tikzcd} & Y \\ X \arrow[ur, "\varphi"] \arrow[r, "\psi"'] & Z \end{tikzcd} \tag{2}$$

can be completed in a unique way to a commutative diagram of the form (1). The desired mapping χ is defined on the subset $\varphi(X)$ by the formula $\chi = \psi \circ \varphi^{-1}$ (commutativity condition), and then is extended by continuity:

$$\chi(\lim y_n) = \lim \chi(y_n).$$

We remark that the uniqueness of the completion (to within isomorphism) now follows from the general theorem proved above on isomorphism of universal objects.

Another basic concept in category theory is that of a functor.

Definition. A *covariant functor* from a category K_1 to a category K_2 is defined to be a mapping F that assigns to each object A in K_1 an object $F(A)$ in K_2, and to each morphism φ in $\mathrm{Mor}(A, B)$ a morphism $F(\varphi)$ in $\mathrm{Mor}(F(A), F(B))$ in such a way that:

(1) $F(1_A) = 1_{F(A)}$;
(2) $F(\varphi \circ \psi) = F(\varphi) \circ F(\psi)$.

It is also common to encounter category mappings F that assign to each morphism $\varphi \in \mathrm{Mor}(A, B)$ a morphism $F(\varphi) \in \mathrm{Mor}(F(B), F(A))$ in such a way that instead of the condition (2) we have

(2′) $F(\varphi \circ \psi) = F(\psi) \circ F(\varphi)$.

They are called *contravariant functors.*

Examples. (1) The passage from a metric space to its completion is a covariant functor from the category of metric spaces to the category of complete metric spaces (the morphisms in both categories are isometric mappings).

(2) The passage from a linear space L over a field K to the dual space L' (the space of K-linear functionals on L) is a contravariant functor from the category of linear spaces over K to itself.

(3) For any category K and any $A \in \mathrm{Ob}\, K$ the mapping $B \mapsto \mathrm{Mor}(B, A)$ (resp., $B \mapsto \mathrm{Mor}(A, B)$) extends to a covariant (resp., contravariant) functor $\mathrm{Mor}(\cdot, A)$ (resp., $\mathrm{Mor}(A, \cdot)$) from K to the category of sets. For this it is necessary to assign to a morphism $\varphi \in \mathrm{Mor}(A_1, A_2)$ the mapping of $\mathrm{Mor}(\cdot, A_1)$ to $\mathrm{Mor}(\cdot, A_2)$ (resp., $\mathrm{Mor}(A_2, \cdot)$ to $\mathrm{Mor}(A_1, \cdot)$) consisting of left (right) multiplication by φ.

The collection of covariant functors from K_1 to K_2 itself forms a category $\mathrm{Cov}(K_1, K_2)$. The morphisms in it are the so-called *functorial morphisms* or *natural transformations of functors*. They are defined as follows: Let F_1 and F_2 be functors from K_1 to K_2. A morphism φ from F_1 to F_2 is defined to be a class of mappings $\varphi(A) \in \mathrm{Mor}_{K_2}(F_1(A), F_2(A))$ (where A runs through $\mathrm{Ob}(K_1)$) such that for any $\psi \in \mathrm{Mor}_{K_1}(A, B)$ the following diagram is commutative:

$$\begin{array}{ccc} F_1(A) & \xrightarrow{F_1(\psi)} & F_1(B) \\ {\scriptstyle\varphi(A)}\downarrow & & \downarrow{\scriptstyle\varphi(B)} \\ F_2(A) & \xrightarrow{F_2(\psi)} & F_2(B) \end{array}$$

The category $\mathrm{Cont}(K_1, K_2)$ of contravariant functors from K_1 to K_2 is defined analogously.

It suffices to prove many assertions about functors only in the case of covariant functors, in view of the following general device. For any category K we define the *dual category* K^0, for which $\mathrm{Ob}(K^0) = \mathrm{Ob}(K)$, $\mathrm{Mor}_{K^0}(A, B) = \mathrm{Mor}_K(B, A)$, and the product $f \circ g$ in K^0 is the product $g \circ f$ in K. It is sometimes said that K^0 is obtained from K by *reversing the arrows*. It is clear that a contravariant functor from K_1 to K_2 is the same as a covariant functor from K_1^0 to K_2 (or from K_1 to K_2^0).

Two categories K_1 and K_2 are said to be *equivalent* if there exist covariant functors $F\colon K_1 \to K_2$, $G\colon K_2 \to K_1$ such that the functors $F \circ G$ and $G \circ F$ are isomorphic to the identity functors in the respective categories $\mathrm{Cov}(K_2, K_2)$ and $\mathrm{Cov}(K_1, K_1)$.

Example. The three famous theorems of Sophus Lie, along with the theorem of Élie Cartan, say essentially that three categories are equivalent: the category of simply connected Lie groups, the category of local Lie groups, and the category of real Lie algebras.

A simpler *example*: The category of discrete topological spaces (in which all subsets are open) is equivalent to the category of sets.

The next result is a good illustration of the concept of equivalence of categories.

Theorem 3. *If all the objects in the category K are isomorphic, then K is equivalent to the category K_0 consisting of a single object $A_0 \in \mathrm{Ob}(K)$ and all the morphisms in* $\mathrm{Mor}_K(A_0, A_0)$.

An example of such a category is the category of n-dimensional linear spaces over a given field, or the category of all groups of p elements (p a prime number).

Proof. For each object $A \in \mathrm{Ob}\, K$ we fix an isomorphism $\alpha(A)\colon A \to A_0$, and we construct a functor $F\colon K \to K_0$ as follows: $F(A) = A_0$ for all $A \in \mathrm{Ob}\, K$; for $\beta \in \mathrm{Mor}(A, B)$ we set $F(\beta) = \alpha(B) \circ \beta \circ \alpha(A)^{-1} \in \mathrm{Mor}(A_0, A_0)$. Furthermore, let G denote the functor imbedding K_0 into K. It is clear that $F \circ G = 1_{K_0}$ and $G \circ F = F$. We show that F and 1_K are isomorphic as objects in $\mathrm{Cov}(K, K)$. To do this we must define functorial morphisms $\varphi\colon F \to 1_K$ and $\psi\colon 1_K \to F$ so that $\varphi \circ \psi = \psi \circ \varphi = 1$; by the definition of factorial morphism, this means the following two conditions must hold:

(1) for any β the following diagrams must be commutative:

$$\begin{tikzcd} A \arrow[r, "\beta"] & B \\ A_0 \arrow[u, "\varphi(A)"] \arrow[r, "F(\beta)"] & A_0 \arrow[u, "\varphi(B)"'] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "\psi(A)"'] & B \arrow[d, "\psi(B)"] \\ A_0 \arrow[r, "F(\beta)"] & A_0 \end{tikzcd}$$

(2) $\varphi(A) \circ \psi(A) = 1_A$ and $\psi(A) \circ \varphi(A) = 1_{A_0}$. This can clearly be achieved by setting $\varphi(A) = \alpha(A)^{-1}$ and $\psi(A) = \alpha(A)$. $\square$

Chapter II

Theory of Measures and Integrals

§1. Measure Theory

1. Algebras of Sets

Let X be a set. We let $P(X)$ denote the collection of all subsets of X.

Definition. A *ring of subsets* of X is defined to be a nonempty family $R \subset P(X)$ that is closed under the operations of union, intersection, and difference.

This implies that R is also closed under the operation of symmetric difference $A \triangle B = (A \cup B) \backslash (A \cap B) = (A \backslash B) \cup (B \backslash A)$.

Refinements of the Ring Structure. (1) An *algebra of sets* is defined to be a ring $R \subset P(X)$ containing the whole set X as an element.

(2) A *σ-ring* is defined to be a ring R that is closed under the operation of countable union.

(3) A *δ-ring* is defined to be a ring that is closed under the operation of countable intersection.

(4) A *σ-algebra* (resp., *δ-algebra*) is defined to be a ring that is both an algebra and a σ-ring (resp., δ-ring).

Relaxation of the Ring Structure. A *half-ring* is defined to be a family $S \subset P(X)$ that is closed under intersection and has the property that if $A, B \in S$, then there exist $C_1, \ldots, C_n \in S$ such that $A \backslash B = C_1 \amalg C_2 \amalg \cdots \amalg C_n$ (the symbol $\amalg$ denotes a *disjoint union*, i.e., a union of disjoint sets).

EXAMPLES. (1) the collection S of all half-open intervals of the form $[a, b)$ on the real line is a half-ring, but not a ring.

(2) If $R_1 \subset P(X)$ and $R_2 \subset P(Y)$ are rings of sets, then the family

$$R_1 \times R_2 = \{A \times B \in P(X \times Y) | A \in R_1, B \in R_2\}$$

is a half-ring (but, generally speaking, not a ring). The same is true if R_1 and R_2 are half-rings (see Problems 79 and 80).

If S is any family of subsets of X, then there is a minimal ring (resp., σ-ring) among those containing S. It is denoted by $R(S)$ (resp., $R_\sigma(S)$) and is called the *ring* (resp., *σ-ring*) *generated by* S (see Problem 76).

Let A be a subset of X. The *characteristic function* of A is the function χ_A on X defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

It is convenient to regard the values of a characteristic function not as numbers, but as residues modulo 2. In this case we have the equalities (see Problem 86)

(1) $\chi_{A_1 \cap A_2} = \chi_{A_1} \cdot \chi_{A_2}$,
(2) $\chi_{A_1 \cup A_2} = \chi_{A_1} + \chi_{A_2} - \chi_{A_1} \cdot \chi_{A_2}$,
(3) $\chi_{A_1 \Delta A_2} = \chi_{A_1} + \chi_{A_2}$,
(4) $\chi_{A_1 \setminus A_2} = \chi_{A_1} - \chi_{A_1} \cdot \chi_{A_2}$.

Let X be a topological space, and $U \subset P(X)$ the family of open subsets of X. The elements of $R_\sigma(U)$ are called the *Borel subsets* of X.

EXAMPLE. The set of all rational points of the segment $[0, 1]$ is a Borel set on the real line.

2. Extension of a Measure

Definition. A *measure* on a half-ring $S \subset P(X)$ is defined to be a real non-negative function μ on S having the additivity property

$$\mu(A \amalg B) = \mu(A) + \mu(B).$$

A measure μ is said to be *countably additive* (or σ-additive) if it has the property

$$\mu\left(\coprod_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k). \tag{1}$$

(More precisely: if all the A_k and $A = \coprod_{k=1}^{\infty} A_k$ belong to S, then the right-hand side of (1) converges, and its sum is equal to the left-hand side.)

EXAMPLES. (1) Let x_0 be a fixed point of X. For any set $A \subset X$ we define

$$\mu(A) = \begin{cases} 1 & \text{if } x_0 \in A, \\ 0 & \text{if } x_0 \notin A. \end{cases}$$

This is a σ-additive measure on $P(X)$.

(2) Let S be the half-ring of half-open intervals of the form $[a, b)$ in $\mathbf{R}$, considered above. Define $\mu([a, b)) = b - a$. Then μ is a measure on S. It is shown below that this measure is countably additive.

Theorem 1. *Every measure μ' on a half-ring S extends uniquely to a measure μ on the ring $R(S)$. If the original measure is countably additive, then so is its extension.*

PROOF. We make use of the fact that each set $A \in R(S)$ has the form $A = \coprod_{k=1}^{n} A_k$, $A_k \in S$ (see Problem 77). Therefore, $\mu(A)$ (if it is defined) must equal $\sum_{k=1}^{n} \mu'(A_k)$. We show that this does define a measure μ on $R(S)$.

(1) *Correctness of the definition.* Suppose that $A_k, B_l \in S$, $\coprod_{l=1}^{m} B_l = \coprod_{k=1}^{n} A_k = A$. Let $C_{kl} = A_k \cap B_l$. It is then clear that $A_k = \coprod_l C_{kl}$, $B_l = \coprod_k C_{kl}$. Therefore, $\sum_k \mu'(A_k) = \sum_{k,l} \mu'(C_{kl}) = \sum_l \mu'(B_l)$.

(2) *Additivity.* Let $A = \coprod_{k=1}^{n} A_k$, $A_k \in R(S)$. We set $A_k = \coprod_{l=1}^{N(k)} C_{kl}$, where $C_{kl} \in S$. Then $A = \coprod_{k,l} C_{kl}$ and $\mu(A) = \sum_{k,l} \mu'(C_{kl}) = \sum_k \sum_{l=1}^{N(k)} \mu'(C_{kl}) = \sum_{k=1}^{n} \mu(A_k)$. It remains to check the σ-additivity of the extended measure μ if the original measure is σ-additive. Let $A = \coprod_{k=1}^{\infty} A_k$, $A, A_k \in R(S)$. Then $A = \coprod_{i=1}^{n} B_i$, $A_k = \coprod_{l=1}^{N(k)} B_{kl}$, where $B_i, B_{kl} \in S$. Let $C_{ikl} = B_i \cap B_{kl}$. We have

$$\mu(A) = \sum_i \mu'(B_i) = \sum_i \sum_{k,l} \mu'(C_{ikl}) = \sum_k \sum_{i,l} \mu'(C_{ikl}) = \sum_k \mu(A_k).$$

(We have used the relations $B_i = \coprod_{k,l} C_{ikl}$, $A_k = \coprod_{i,l} C_{ikl}$, and the fact that we can interchange the orders of summation in series with nonnegative terms.) □

An important property of countably additive measures is *countable monotonicity*: if $A, A_k \in S$ and $A \subset \bigcup A_k$, then $\mu(A) \leq \sum \mu(A_k)$.

It turns out that a countably additive measure defined on a half-ring S extends not only to the ring $R(S)$ and even to the σ-ring $R_\sigma(S)$, but also to a much broader collection of so-called measurable sets.

Definition. Suppose that X is a set, $S \subset P(X)$ is a half-ring, and μ is a σ-additive measure on S. For any $A \in P(X)$ we define the *outer measure* $\mu^*(A)$ by

$$\mu^*(A) = \inf \sum_{k=1}^{\infty} \mu(A_k), \qquad A \subset \bigcup_{k=1}^{\infty} A_k, \qquad A_k \in S.$$

A set $A \in P(X)$ is said to be *Lebesgue measurable* with respect to μ if for any $\varepsilon > 0$ there is a set $B \in R(S)$ such that $\mu^*(A \triangle B) < \varepsilon$. (In what follows, the sets that are Lebesgue measurable with respect to μ will for brevity be simply called *measurable* sets, or *μ-measurable* sets if it is necessary to make clear which measure μ is meant.)

Suppose now that the ring $R(S)$ is an algebra (i.e., contains the maximal element X). Then the outer measure of any set is finite, and the following theorem holds:

Lebesgue's Theorem. *The collection $L(S, \mu)$ of measurable sets forms a σ-algebra, and μ^* is a σ-additive measure on it.*

Proof. Let us first show that the outer measure μ^* coincides with the original measure μ' (resp., with its extension μ) on sets in the original half-ring S (resp., $R(S)$). Indeed, if $A \in R(S)$ and $A \subset \bigcup_{k=1}^{\infty} A_k$, then $\mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k)$ because μ is countably monotone. Passing to the infimum over all coverings, we get from this that $\mu(A) \leq \mu^*(A)$. On the other hand, since $A \in R(S)$, we can represent it in the form $A = \coprod_{k=1}^{n} B_k$, where $B_k \in S$. Therefore, $\mu^*(A) \leq \sum_{k=1}^{n} \mu(B_k) = \mu(A)$. We remark that a simple consequence of the definition of an outer measure is its countable monotonicity: if $A \subset \bigcup_{k=1}^{\infty} A_k$, then $\mu^*(A) \leq \sum_{k=1}^{\infty} \mu^*(A_k)$.

The rest of the proof can be briefly sketched as follows. The elements of the ring $R(S)$ form an incomplete metric space on which the measure μ is a uniformly continuous function. The measurable sets are the points of the completion, and the measure μ^* is the extension by continuity of the function μ (see Problems 98, 99).

Let us proceed to the details. We define a distance between sets A and B by the formula $d(A, B) = \mu^*(A \triangle B)$. In checking the usual properties of a distance it is useful to observe that the set $A \triangle B$ itself can be regarded as a certain "distance" between A and B taking as its values not numbers, but sets. The triangle axiom for this distance has the form (see Problem 70)

$$A \triangle B \subset (A \triangle C) \cup (B \triangle C).$$

This and the monotonicity of μ^* imply the usual triangle axiom: $d(A, B) \leq d(A, C) + d(B, C)$. The properties $d(A, B) = d(B, A)$ and $d(A, B) \geq 0$ are obvious. The final property $d(A, B) = 0 \Rightarrow A = B$ is not satisfied, generally speaking. The standard way of getting around this is to declare the sets A and B to be equivalent if $d(A, B) = 0$. The function d can be carried over to the equivalence classes of sets and has there all the properties of a distance.

The definition of a measurable set can now be formulated as follows: A set A is *measurable* if it can be approximated to any degree of accuracy by sets B in $R(S)$. In other words, the collection $L(S)$ of measurable sets coincides with the closure of $R(S)$ in the metric space constructed. It can be shown (see Problem 99) that the space $P(X)$ (more precisely, the corresponding quotient space of equivalence classes of sets) is complete. Therefore, $L(S)$ can also be regarded as the completion of $R(S)$.

First of all we verify that $L(S)$ is an algebra. Suppose that A_1 and A_2 are measurable, i.e., for any $\varepsilon > 0$ there exist sets B_1 and B_2 in $R(S)$ such that $\mu^*(A_1 \triangle B_1) < \varepsilon$ and $\mu^*(A_2 \triangle B_2) < \varepsilon$. Then (see Problem 71) we have the estimates

$$\mu^*((A_1 \cup A_2) \triangle (B_1 \cup B_2)) < 2\varepsilon,$$

$$\mu^*((A_1 \cap A_2) \triangle (B_1 \cap B_2)) < 2\varepsilon,$$

$$\mu^*((A_1 \setminus A_2) \triangle (B_1 \setminus B_2)) < 2\varepsilon,$$

which proves the measurability of the sets $A_1 \cup A_2$, $A_1 \cap A_2$, and $A_1 \setminus A_2$.

Let us now show that $L(S)$ is a σ-algebra. Suppose that $A_k \in L(S)$ and $A = \bigcup_{k=1}^{\infty} A_k$. For any $\varepsilon > 0$ there exist sets $B_k \in R(S)$ such that $\mu^*(A_k \triangle B_k) < \varepsilon/2^k$. Let $B = \bigcup_{k=1}^{\infty} B_k$. The inclusion $(\bigcup_{k=1}^{\infty} A_k) \triangle (\bigcup_{k=1}^{\infty} B_k) \subset \bigcup_{k=1}^{\infty} (A_k \triangle B_k)$ implies that $\mu^*(A \triangle B) \leq \sum_{k=1}^{\infty} \varepsilon/2^k = \varepsilon$. Next, let $B_k' = B_k \setminus (B_1 \cup B_2 \cup \cdots \cup B_{k-1})$ for $k > 1$, and $B_1' = B_1$. Then $B_k' \in R(S)$ and $B = \coprod_{k=1}^{\infty} B_k'$. Since the series $\sum_k \mu(B_k')$ converges (its partial sums are bounded by the number $\mu(X)$), there is a number N such that $\sum_{k=N+1}^{\infty} \mu(B_k') < \varepsilon$. Let $B' = \coprod_{k=1}^{N} B_k'$. Then $B' \in R(S)$, and $\mu^*(B \triangle B') < \varepsilon$. From this, $\mu^*(A \triangle B') \leq \mu^*(A \triangle B) + \mu^*(B \triangle B') < 2\varepsilon$, which proves that A is measurable.

We verify that μ^* is a countably additive measure on $L(S)$.

Lemma. $|\mu^*(A) - \mu^*(B)| \leq \mu^*(A \triangle B)$.

In other words, the function μ^* is uniformly continuous with respect to the distance $d(A, B) = \mu^*(A \triangle B)$.

The proof follows from the monotonicity of μ^* and the inclusions $A \subset B \cup (A \triangle B)$, $B \subset A \cup (A \triangle B)$.

Suppose now that $A_1, A_2 \in L(S)$ and $A = A_1 \amalg A_2$. For any $\varepsilon > 0$ we choose B_1 and B_2 in $R(S)$ in such a way that $d(A_i, B_i) < \varepsilon$, $i = 1, 2$. Then $d(A, B_1 \cup B_2) < 2\varepsilon$. Therefore, $|\mu^*(A) - \mu^*(B_1 \cup B_2)| < 2\varepsilon$. On the other hand, $\mu^*(B_1 \cup B_2) = \mu(B_1 \cup B_2) = \mu(B_1) + \mu(B_2) - \mu(B_1 \cap B_2)$. But $\mu(B_1 \cap B_2) = d(B_1 \cap B_2, \varnothing) = d(B_1 \cap B_2, A_1 \cap A_2) < 2\varepsilon$, hence, $|\mu^*(B_1 \cup B_2) - \mu(B_1) - \mu(B_2)| < 2\varepsilon$. Combining all these inequalities with the original ones, we get that $|\mu^*(A) - \mu^*(A_1) - \mu^*(A_2)| < 6\varepsilon$. Since this is true for all $\varepsilon > 0$, we have that $\mu^*(A) = \mu^*(A_1) + \mu^*(A_2)$, i.e., μ^* is additive on $L(S)$.

Finally, let us show that μ^* is countably additive on $L(S)$. Suppose that $A = \coprod_{k=1}^{\infty} A_k$. The inequality $\mu^*(A) \leq \sum_{k=1}^{\infty} \mu^*(A_k)$ follows from the countable monotonicity of μ^*. The inequality $\mu^*(A) \geq \sum_{k=1}^{\infty} \mu^*(A_k)$ is obtained by passing to the limit in the inequality $\mu^*(A) \geq \sum_{k=1}^{N} \mu^*(A_k)$, which follows from the finite additivity and monotonicity of μ^*. □

In a number of cases the condition $X \in R(S)$ turns out to be too strong. Let us consider the weaker condition $X \in R_\sigma(S)$. Then $X = \coprod_{k=1}^{\infty} X_n$, where $X_n \in S$; thus, the whole space is a countable union of sets in the half-ring. The measure μ is said to be *σ-finite* in this case.

Definition. A set A is said to be *Lebesgue measurable* with respect to a σ-finite measure μ if all the sets $A \cap X_i$ $(i = 1, 2, \ldots)$ are measurable. The *measure* of A is defined to be the sum of the series $\sum_{i=1}^{\infty} \mu^*(A \cap X_i)$ if it converges, and $+\infty$ otherwise.

It is not hard to see that, as before, the measurable sets form a σ-algebra, and the measure μ^* defined above is σ-additive (with the obvious stipulation that both sides of the equality $\mu(\coprod A_k) = \sum \mu(A_k)$ can be infinite).

The variable-sign analog of a measure is a signed measure.

Definition. Let X be a set and $R \subset P(X)$ a σ-ring. A real (resp., complex) function ν on R is called a *signed measure* (resp., *complex measure*) if it is countably additive in the following sense: For any $A_k \in R$ the assumption that $A = \coprod_{k=1}^{\infty} A_k$ belongs to R implies that the series $\sum \nu(A_k)$ converges absolutely and its sum is $\nu(A)$ (cf. Problem 131).

Example. Any linear combination of σ-additive measures on R with real (resp., complex) coefficients is a signed measure (resp., complex measure).

It turns out that the converse assertion also holds.

Theorem 2. *Every signed measure (resp., complex measure) ν can be written in the form $\nu = \mu_1 - \mu_2$ (resp., $\nu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$), where the μ_k are countably additive measures.*

Definition. The *variation of a signed measure* ν on a set A is defined to be

$$|\nu|(A) = \sup \sum_k |\nu(A_k)|, \qquad A = \coprod_{k=1}^{\infty} A_k.$$

Example. If $\nu = \mu_1 - \mu_2$, then $|\nu|(A) \leq \mu_1(A) + \mu_2(A)$. This inequality becomes an equality if μ_1 and μ_2 are *disjoint* on A (i.e., there exists a decomposition $A = A_1 \amalg A_2$ such that $\mu_2(A_1) = \mu_1(A_2) = 0$).

Theorem 3. *The function $|\nu|$ is a countably additive measure on R (see Problem 138).*

3. Constructions of Measures

Let us consider the half-ring S consisting of all the half-open intervals of the form $[a, b)$ on the real line. The collection of all measures on this half-ring admits a simple description. Namely, to each measure μ on S we assign the function F_μ on $\mathbf{R}$ described by the formula

$$F_\mu(t) = \begin{cases} \mu([0, t)) & \text{for } t > 0, \\ 0 & \text{for } t = 0, \\ -\mu([t, 0)) & \text{for } t < 0. \end{cases}$$

Obviously, F_μ is a nondecreasing function. Conversely, if F is a nondecreasing function on $\mathbf{R}$, then we can define a measure μ_F on S by setting $\mu_F([a, b)) = F(b) - F(a)$. (It is left to the reader to check that μ_F is additive.) The correspondence between measures on S and nondecreasing functions on $\mathbf{R}$ becomes one-to-one if we consider only functions with the additional property that $F(0) = 0$.

Theorem 4. *A measure μ on S is countably additive if and only if the corresponding function F on $\mathbf{R}$ is left-continuous, i.e., $F(t - 0) = F(t)$ for all $t \in \mathbf{R}$.*

PROOF. The necessity follows from the relation $F(t) - F(t - 0) = \lim_{\varepsilon \to 0} \mu([t - \varepsilon, t)) = 0$ (cf. Problem 97). To prove the sufficiency we take $[a, b) = \coprod_{k=1}^{\infty} [a_k, b_k)$. Then $[a, b) \supset \coprod_{k=1}^{N} [a_k, b_k)$ for any N, and, consequently, $\mu([a, b)) \geq \sum_{k=1}^{N} \mu([a_k, b_k))$, because μ is additive and monotone. Letting N go to infinity, we get that $\mu([a, b)) \geq \sum_{k=1}^{\infty} \mu([a_k, b_k))$. Let us prove the reverse inequality. For any $\varepsilon > 0$ we take positive numbers δ, $\delta_1, \delta_2, \ldots$ such that $F(b) - F(b - \delta) < \varepsilon$, $F(a_k) - F(a_k - \delta_k) < \varepsilon/2^k$. This can be done, since F is left-continuous. We now observe that the segment $[a, b - \delta]$ can be entirely covered by the intervals $(a_k - \delta_k, b_k)$. Hence, there is an N such that the first N intervals already cover this segment. From this, $\mu([a, b - \delta)) \leq \sum_{k=1}^{N} \mu([a_k - \delta_k, b_k))$, since $[a, b - \delta) \subset [a, b - \delta] \subset \bigcup_{k=1}^{N} (a_k - \delta_k, b_k) \subset \bigcup_{k=1}^{N} [a_k - \delta_k, b_k)$. Combined with the inequalities written above, this gives $\mu([a, b)) \leq \sum_{k=1}^{N} \mu([a_k, b_k)) + 2\varepsilon$. $\square$

We give an important *example*: $F(t) \equiv t$. In this case the measure on S coincides with the usual length, and its extension μ^* to $L(S)$ is called the *Lebesgue measure* on $\mathbf{R}$.

A second *example*: Let $[t]$ be the integral part of t, i.e., the largest integer not exceeding t. We set $F(t) = -[-t]$. It is easy to see that $F(t)$ is left-continuous. The corresponding measure can be extended to all subsets of $\mathbf{R}$ by the formula

$$\mu(A) = \text{the number of integers in } A.$$

We now describe all the signed measures defined on $R_\sigma(S)$, where S is the half-ring of half-intervals considered above. For each signed measure ν we set

$$F_\nu(t) = \begin{cases} \nu([0, t)) & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ -\nu([t, 0)) & \text{if } t < 0. \end{cases} \tag{2}$$

The following concept is needed for describing the set of functions F_ν on $\mathbf{R}$ corresponding to signed measures ν.

Definition. The *variation* of a function f on $[a, b]$ is defined to be

$$\operatorname{Var}_a^b f = \sup \sum_{k=1}^{n-1} |f(\xi_k) - f(\xi_{k+1})|,$$

$$a = \xi_1 \le \xi_2 \le \cdots \le \xi_n = b,$$

where the supremum is taken over all finite sets of points $\xi_1, \ldots, \xi_n$ on $[a, b]$. The set of functions of bounded variation is denoted by $BV[a, b]$.

Theorem 5. *A real function f on $[a, b]$ has bounded variation if and only if it can be represented as a difference of two monotone functions.*

Proof. For a monotone function the variation coincides with the increment: $\operatorname{Var}_a^b f = |f(b) - f(a)|$ (see Problem 208). Therefore, all the monotone functions and their linear combinations have bounded variation. Conversely, suppose that $\operatorname{Var}_a^b f < \infty$. Let $\varphi(t) = \operatorname{Var}_a^t f$. It is clear that φ is a nondecreasing function. Moreover, taking the simplest set of points $\xi_1 = a$, $\xi_2 = b$, we see that $\operatorname{Var}_a^b f \ge |f(b) - f(a)|$. From this, $\varphi(t_1) - \varphi(t_2) = \operatorname{Var}_{t_1}^{t_2} f \ge |f(t_1) - f(t_2)|$ for $t_1 > t_2$. This means that $\psi(t) = \varphi(t) - f(t)$ is also nondecreasing. Therefore, $f(t) = \varphi(t) - \psi(t)$, the difference of two monotone functions.

Theorem 6. *A function F on $\mathbf{R}$ corresponds to some signed measure ν according to the formula* (2) *if and only if it satisfies the conditions*:

(1) $F(0) = 0$;
(2) *F is left-continuous*;
(3) *F has bounded variation on any closed interval.*

Proof. The sufficiency follows from Theorems 4 and 5. Indeed, let F be represented as a difference of two nondecreasing functions F_+ and F_-. It is clear that if F satisfies the conditions (1) and (2), then F_+ and F_- can be chosen so that they also satisfy these conditions.

By Theorem 4, F_+ and F_- correspond to certain countably additive measures μ_+ and μ_-. Then $F = F_+ - F_-$ corresponds to the signed measure $\mu_+ - \mu_-$.

The necessity of condition (1) is obvious, and the necessity of (2) is proved just as in Theorem 4. Let us show that (3) is necessary. To do this we observe that, by the definition of the variation of the signed measure v, we have that $\mathrm{Var}_a^b F_v = |v|([a, b))$. Thus, the statement we need is a particular case of the general theorem on finiteness of the variation of a signed measure (see Problem 138). □

Let X and Y be two sets, $S \subset P(X)$ and $T \subset P(Y)$ two half-rings, and μ and v measures on S and T. We consider the half-ring $S \times T \subset P(X \times Y)$ consisting of the sets of the form $A \times B$, $A \in S$, $B \in T$ (see Problem 79), and define on it a function $\mu \times v$ by setting $(\mu \times v)(A \times B) = \mu(A) \cdot v(B)$. It is easy to see that this function is additive.

Theorem 7. *If μ and v are countably additive measures, then so is $\mu \times v$.*

(We postpone proving this theorem until we have constructed a theory of integration (see Ch. II, §3.3).)

It is clear that an analogous theorem holds for any product of finitely many measures. It turns out that under slight additional restrictions it is true also for an infinite product.

Suppose that we have some index set A, and for each $\alpha \in A$ a nonempty set X_α, a half-ring $S_\alpha \subset P(X_\alpha)$, and a countably additive measure μ_α on S_α. It is assumed that for all but finitely many $\alpha \in A$ the half-ring S_α contains X_α and $\mu_\alpha(X_\alpha) = 1$. We set $X = \prod_{\alpha \in A} X_\alpha$ and define the *cylindrical* subsets $Y \subset X$ to be those of the form

$$Y = \prod_{\alpha \in A_0} Y_\alpha \times \prod_{\alpha \in A \setminus A_0} X_\alpha, \tag{3}$$

where A_0 is any finite subset of A containing all the indices α for which $\mu(X_\alpha) \neq 1$, and Y_α is any set in S_α. For a Y of the form (3) we set

$$\mu(Y) = \prod_{\alpha \in A_0} \mu_\alpha(Y_\alpha).$$

Theorem 8. *The cylindrical sets form a half-ring S, and μ is a countably additive measure on it.*

For a proof see [28].

There is one important case in which the countable additivity of the product measure (and even of more general measures) can be established very simply.

Lemma. *Suppose that $A = \mathbf{N}$ and that all the X_n $(n \in \mathbf{N})$ are finite sets. In this case the relation $Y = \coprod_k Y_k$ for nonempty cylindrical sets is possible only if the union is finite.*

Corollary. *Every additive measure on the half-ring S is countably additive.*

PROOF OF THE LEMMA. We introduce in $X = \prod_{n=1}^{\infty} X_n$ a metric such that all the cylindrical sets are open, closed, and compact. The lemma will then follow from a finite covering lemma. The desired metric can be defined as follows. We write an element $x \in X$ as a sequence $\{x_n\}$, $x_n \in X_n$. Let

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1/k, & \text{where } k \text{ is the smallest integer such that } x_k \neq y_k, \text{ otherwise.} \end{cases}$$

It is clear that the closed ball of radius $1/k$ with center at x coincides with the open ball of radius $1/(k + 1)$ and is a cylindrical set. This implies that cylindrical sets are open and closed. That they are compact follows from the compactness of X, which, in turn, is easily proved by producing a finite ε-net for any $\varepsilon > 0$. (Regarding compactness, see Ch. III, §2.2.)

EXAMPLE. On the set X of infinite decimal fractions of the form $x = 0.x_1x_2x_3\ldots$ we can define a measure in the way described above by setting $X_k = \{0, 1, 2, \ldots, 9\}$, $S_k = P(X_k)$, $\mu_k(Y) = (\text{card } Y)/10$, where card Y is the number of elements in the set Y. It can be shown that the Lebesgue extension of this measure is essentially the same as the Lebesgue measure on $[0, 1]$.

A very interesting example of a measure stems from physical applications (Brownian motion). In the space $C[a, b]$ of continuous functions on $[a, b]$ let us consider the subsets of the form

$$X(t_1, \ldots, t_n; \Delta_1, \ldots, \Delta_n) = \{x \in C[a, b] \mid x(t_k) \in \Delta_k\},$$

where $t_1 \leq t_2 \leq \cdots \leq t_n$ are points in $[a, b]$, and $\Delta_1, \ldots, \Delta_n$ are intervals on the real line. It is easy to show that these sets form a half-ring S. It turns out that the formula

$$\mu(X(t_1, \ldots, t_n; \Delta_1, \ldots, \Delta_n)) = \pi^{(1-n)/2} \prod_{k=1}^{n-1} (t_{k+1} - t_k)^{-1/2}$$
$$\times \int_{\Delta_1} \cdots \int_{\Delta_n} \exp\left\{ -\sum_{k=1}^{n-1} \frac{(\tau_{k+1} - \tau_k)^2}{(t_{k+1} - t_k)} \right\} d\tau_1 \cdots d\tau_n$$

defines a countably additive measure on S. For $n = 1$ this formula should be understood as follows:

$$\mu(X(t, \Delta)) = \int_{\Delta} d\tau = |\Delta|.$$

The Lebesgue extension of the measure μ is called the *Wiener measure* on $C[a, b]$. This measure has many interesting properties. We remark, for example, that the functions differentiable at one or more points of $[a, b]$ form a set of measure zero.

The Wiener measure (more precisely, the measure μ_0 associated with it; see Problem 204) is interpreted physically as the probability that a particle performing a random walk on the line is located in the intervals $\Delta_1, \Delta_2, \ldots, \Delta_n$ at the respective times $t_1, t_2, \ldots, t_n$. Accordingly, with probability 1 the graph of this particle's motion is a nowhere differentiable continuous function on $[a, b]$.

§2. Measurable Functions

1. Properties of Measurable Functions

Suppose that X is a set and $\mathfrak{A} \subset P(X)$ a σ-algebra. A real function f on X is said to be $\mathfrak{A}$*-measurable* if for any $c \in \mathbf{R}$ the set

$$E_c(f) = \{x \in X : f(x) < c\}$$

(a so-called *Lebesgue set* of the function f) belongs to $\mathfrak{A}$. It can be shown (see Problem 139) that the sign $<$ in this definition can be replaced by any of the signs $\leq$, $>$, $\geq$ (but not by the sign $=$).

A complex-valued function $f(x) = u(x) + iv(x)$ is said to be $\mathfrak{A}$*-measurable* if its real part $u(x)$ and imaginary part $v(x)$ are $\mathfrak{A}$-measurable. More generally: A vector-valued function $\xi(x)$ with values in a finite-dimensional real linear space L is said to be $\mathfrak{A}$*-measurable* if for some basis $e_1, \ldots, e_n$ in L all the coefficients $\xi_i(x)$ in the expansion $\xi(x) = \xi_1(x)e_1 + \cdots + \xi_n(x)e_n$ are $\mathfrak{A}$-measurable functions. This definition does not depend on the choice of a basis in L (see Problem 161).

When we are dealing with a space X and a measure μ defined on a σ-algebra $\mathfrak{A} \subset P(X)$, we frequently say "$\mu$*-measurable*" or simply *measurable* instead of "$\mathfrak{A}$-measurable" for a function. One of the basic properties of measurable functions is:

Theorem 9. *The set of measurable functions forms an algebra that is closed under convergence almost everywhere. (A property is said to hold almost everywhere if it holds except on a set of measure zero.)*

Proof. If f is a measurable function, then so are the functions λf, $|f|$, and f^2, by virtue of the following general lemma, which is a special case of the assertion in Problem 143.

Lemma. *Let f be a measurable function and g a continuous function. Then the composition $g(f(x))$ is measurable.*

Suppose that f_1 and f_2 are measurable functions. Let us show that the sum $f_1 + f_2$ is measurable. To do this we observe that the Lebesgue set $E_c(f_1 + f_2)$ can be represented as a countable union of measurable sets:

$$E_c(f_1 + f_2) = \bigcup_{r \in \mathbf{Q}} (E_r(f_1) \cap E_{c-r}(f_2)),$$

where $\mathbf{Q}$ is the set of rational numbers.

Next, the measurability of the product $f_1 f_2$ follows from the foregoing and the identity $f_1 f_2 = (f_1 + f_2)^2/4 - (f_1 - f_2)^2/4$. Similarly, the identity $\max(f_1, f_2) = (f_1 + f_2)/2 + |f_1 - f_2|/2$ shows that the maximum of two (and, hence, of any finite number) of measurable functions is measurable. Let $\{f_n\}$ be a nonincreasing sequence of measurable functions, and f its limit. Then $E_c(f)$ is the union of the sets $E_c(f_n)$, and, consequently, is measurable. Thus, the algebra of measurable functions is closed under monotone limits. But it is well known that any limit can be replaced by two monotone limits. Namely,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{k\to\infty} \max\{f_n(x), f_{n+1}(x), \ldots, f_{n+k}(x)\}.$$

Finally, if f_1 is measurable and f_2 coincides almost everywhere with it, then f_2 is also measurable. $\square$

2. Convergence of Measurable Functions

We can define several different types of convergence for measurable functions. The most common are the following three types:

(1) *Uniform convergence* on the set X is denoted by $f_n \rightrightarrows f$ and means that $\sup_{x\in X} |f_n(x) - f(x)| \to 0$ as $n \to \infty$.

(2) *Convergence almost everywhere* (with respect to a measure μ) is denoted by $f_n \xrightarrow{\text{a.e.}} f$ and means that

$$f_n(x) \to f(x) \quad \text{as } n \to \infty$$

for every point x outside some subset of measure zero.

(3) *Convergence in measure* is denoted by $f_n \xrightarrow{\mu} f$ and means that for any $\varepsilon > 0$ the measure of the set $A_n(\varepsilon) = \{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}$ converges to zero as $n \to \infty$.

There are connections between these types of convergence. It is clear that uniform convergence implies convergence almost everywhere and convergence in measure.

Theorem 10. *If a sequence f_n converges to f almost everywhere on X and $\mu(X) < \infty$, then $f_n \xrightarrow{\mu} f$.*

Proof. Let $A_n(\varepsilon) = \{x \in X: |f_n(x) - f(x)| \geq \varepsilon\}$, and $B_n(\varepsilon) = \bigcup_{k \geq n} A_k(\varepsilon)$. It is clear that $B_1(\varepsilon) \supset B_2(\varepsilon) \supset \cdots \supset B_n(\varepsilon) \supset \cdots$. Let $B(\varepsilon) = \bigcap_{n=1}^{\infty} B_n(\varepsilon)$. If $x \in B(\varepsilon)$, then x belongs to $A_n(\varepsilon)$ for arbitrarily large numbers n. This implies that $f_n(x)$ does not converge to $f(x)$ as $n \to \infty$. Hence, the set $B(\varepsilon)$ has measure zero. But $\mu(B(\varepsilon)) = \lim_{n\to\infty} \mu(B_n(\varepsilon))$ (see Problem 97). Since $\mu(A_n(\varepsilon)) \leq \mu(B_n(\varepsilon))$, we see that $\mu(A_n(\varepsilon)) \to 0$ as $n \to \infty$, i.e., $f_n \xrightarrow{\mu} f$. □

Thus, uniform convergence implies convergence almost everywhere, and convergence almost everywhere implies (on a set of finite measure) convergence in measure. The converse assertions are false (see Problems 162, 163, 168). An interesting and important fact is that these assertions become true if we "retouch" the sequence $\{f_n\}$ or the set X. Namely, we have the following results.

Egorov's Theorem. *If $f_n \xrightarrow{\text{a.e.}} f$ on X and $\mu(X) < \infty$, then for any $\sigma > 0$ there is a subset $E_\sigma \subset X$ such that $\mu(E_\sigma) < \sigma$ and $f_n \rightrightarrows f$ outside E_σ.*

Theorem 11. *If $f_n \xrightarrow{\mu} f$ on X, then there is a subsequence $\{f_{n_k}\}$ such that*

$$f_{n_k} \xrightarrow{\text{a.e.}} f \quad \text{on } X \text{ as } k \to \infty.$$

Proof of Egorov's Theorem. We use the notation $A_n(\varepsilon)$ and $B_n(\varepsilon)$ from the proof of Theorem 10. It has been seen that $\mu(B_n(\varepsilon)) \to 0$ as $n \to \infty$, for any $\varepsilon > 0$. Therefore, for any k there is a number $N(k)$ such that $\mu(B_{N(k)}(1/k)) < \sigma/2^k$. Let E_σ be the set $\bigcup_{k=1}^{\infty} B_{N(k)}(1/k)$. Then $\mu(E_\sigma) < \sigma$, and $|f_n(x) - f(x)| < 1/k$ outside E_σ for $n > N(k)$.

Proof of Theorem 11. Using the same notation as above, we choose for each k a number n_k such that $\mu(B_{n_k}(1/k)) < 1/2^k$. Let us show that the sequence n_k works. Indeed, the set of points x at which $f_{n_k}(x)$ does not converge to $f(x)$ as $k \to \infty$ is contained in $\overline{\lim}_{k\to\infty} B_{n_k}(1/k)$ and, consequently, has measure zero.

§3. Integrals

1. The Lebesgue Integral

Let X be a set, $\mathfrak{A} \subset P(X)$ a σ-algebra, and μ a countably additive measure on $\mathfrak{A}$. A measurable (real or complex) function f on X is called a *simple* function if it takes no more than countably many values. Such a function can be represented as a countable linear combination of characteristic functions:

$$f(x) = \sum_{k=1}^{\infty} c_k \chi_{A_k}(x), \tag{4}$$

and it can be assumed that all the A_k are measurable, and $X = \coprod_{k=1}^{\infty} A_k$.

Remark. If the values c_k are pairwise distinct, then the measurability of the A_k follows from that of f. Moreover, under this condition the representation of f in the form (4) is unique. However, we prefer not to require the c_k to be distinct.

A function f of the form (4) is said to be *integrable* on X if the following series converges:

$$\sum_{k=1}^{\infty} |c_k| \mu(A_k). \tag{5}$$

This definition, as well as the definition of the integral given below, does not depend on the choice of the representation (4) (see Problem 189).

Suppose that $A \in \mathfrak{A}$; we define the *integral* of $f(x)$ over the set A by the formula

$$\int_A f(x)\, d\mu(x) = \sum_{k=1}^{\infty} c_k \mu(A \cap A_k). \tag{6}$$

The convergence of this series follows from the integrability of f.

The collection of all integrable simple functions on X will be denoted by $S(X, \mu)$, or simply $S(X)$ if it is clear from the context which measure μ is meant. When it is necessary to emphasize that we are considering real or complex functions, we shall use the notation $S^{\mathbf{R}}(X)$, $S^{\mathbf{C}}(X)$.

Theorem 12. (1) *The set $S(X)$ is a linear space.*

(2) *The correspondence $f \mapsto \int_A f(x)\, d\mu$ is a linear functional on $S(X)$.*

(3) *The correspondence $A \mapsto \int_A f(x)\, d\mu$ is a signed measure on A.*

(4) *The quantity $d_1(f, g) = \int_X |f(x) - g(x)|\, d\mu(x)$ has all the properties of a distance except, possibly, the separation property.*

(5) $|\int_A f\, d\mu - \int_A g\, d\mu| \le d_1(f, g)$ *for all $f, g \in S(x)$ and all $A \in \mathfrak{A}$.*

Proof. The assertions (1), (2), (4), and (5) are direct consequences of the definition (cf. Problem 185). Let us prove (3). If f is the characteristic function of some set $B \in \mathfrak{A}$, then the correspondence $A \to \int_A f\, d\mu = \mu(A \cap B)$ is a countably additive measure and, consequently, a signed measure on $\mathfrak{A}$. By Problem 123, the analogous assertion is true for finite linear combinations of characteristic functions, and for their limits in the sense of the distance d_1. □

Henceforth, we identify functions coinciding almost everywhere, as a rule, without special mention. After this identification the set $S(X)$ becomes a metric space with the distance d_1. It is easy to see that this space may not be complete (see Problem 190). We shall see later that the completion $S(X, \mu)$ admits an explicit realization as a certain space of functions on X (more precisely, as equivalence classes of such functions).

Definition. A function f on X is said to be *integrable with respect to the measure* μ if there exists a sequence $\{f_n\} \subset S(X, \mu)$ such that:

(1) $\{f_n\}$ is a Cauchy sequence in $S(X, \mu)$ in the sense of the distance d_1;
(2) $f_n \to f$ almost everywhere on X with respect to μ.

It is clear that if f is an integrable function and g is equivalent to f, then g is also integrable (it suffices to consider the same sequence $\{f_n\}$). The space of equivalence classes of integrable functions is denoted by $L_1(X, \mu)$.

We show that if $\mu(X) < \infty$, then every bounded measurable function f belongs to $L_1(X, \mu)$. Indeed, let $E_{kn}(f) = \{x \in X: k/n \le f(x) < (k+1)/n\}$, and $f_n(x) = \sum_{k=-\infty}^{\infty} (k/n)\chi_{E_{kn}}(x)$. (This sum is actually finite, since f is bounded.) It is clear that $|f_n(x) - f(x)| < 1/n$, so that $f_n \rightrightarrows f$, and, *a fortiori*, $f_n \xrightarrow{\text{a.e.}} f$. Moreover, $\{f_n\}$ is Cauchy, since (by part (c) of Problem 185) $d_1(f_n, f_m) = \int_X |f_n(x) - f_m(x)|\, d\mu \le (1/n + 1/m)\mu(X)$.

This result can be generalized. A measurable function f on X is said to be *essentially bounded* if there is a constant C such that $|f(x)| \le C$ almost everywhere on X.

The smallest such constant (show that it exists!) is called the *essential supremum* of the function $|f|$ and denoted by $\operatorname{ess\,sup}|f(x)|$. The quantity $d_\infty(f, g) = \operatorname{ess\,sup}|f(x) - g(x)|$ has all the properties of a distance except the separation property. The corresponding quotient space of equivalence classes of essentially bounded functions is denoted by $L_\infty(X, \mu)$. This space is complete with respect to the distance d_∞. The integrability of an essentially bounded function on a set of finite measure can be proved just as the integrability of a bounded function. Therefore, for $\mu(X) < \infty$ we have the inclusion

$$L_\infty(X, \mu) \subset L_1(X, \mu).$$

We now construct an integral for integrable functions, by means of the following.

Lemma. *If $\{f_n\}$ and $\{g_n\}$ are two Cauchy sequences in $S(X, \mu)$ that converge almost everywhere to the same function $h \in L_1(X, \mu)$, then $d_1(f_n, g_n) \to 0$ as $n \to \infty$.*

PROOF. Let $\varphi_n = f_n - g_n$. Then $\{\varphi_n\}$ is a Cauchy sequence in $S(X, \mu)$, and $\varphi_n \xrightarrow{\text{a.e.}} 0$. We show that $\int_X |\varphi_n|\, d\mu \to 0$. Suppose not. Then there exist a number $\delta > 0$ and a subsequence $\{n_k\}$ such that $\int_X |\varphi_{n_k}|\, d\mu \ge \delta$ for all k.

Relabelling the functions and multiplying them all by δ^{-1}, we may assume that $\int_X |\varphi_n| \, d\mu \geq 1$ for all n.

Let us now choose a so-called "rapidly converging" subsequence $\{\varphi_{n_k}\}$ of $\{\varphi_n\}$ having the property

$$d_1(\varphi_{n_k}, \varphi_{n_{k+1}}) \leq 1/2^k.$$

To do this it suffices to take n_k to be a number beginning with which any two terms of the sequence $\{\varphi_n\}$ are at a distance from each other not greater than $1/2^k$. Relabelling again, we can assume that $d_1(\varphi_n, \varphi_{n+1}) \leq 1/2^{n+2}$. Recall now that φ_1 is a simple function of the form (4):

$$\varphi_1(x) = \sum_{k=1}^{\infty} c_k \chi_{A_k}(x).$$

Since $\sum_{k=1}^{\infty} |c_k| \mu(A_k) = \int_X |\varphi_1(x)| \, d\mu(x) \geq 1$, there exists a number N such that $\sum_{k=1}^{N} |c_k| \mu(A_k) \geq 3/4$. Let $A = \coprod_{k=1}^{N} A_k$, and $C = \max_{1 \leq k \leq N} |c_k| = \max_{x \in A} |\varphi_1(x)|$. By Egorov's theorem, the set A contains a subset E such that $\mu(E) < 1/(4C)$ and $\varphi_n \rightrightarrows 0$ on $B = A \setminus E$. Then $\int_B |\varphi_n| \, d\mu \to 0$. On the other hand,

$$\int_B |\varphi_1| \, d\mu \geq \int_A |\varphi_1| \, d\mu - \int_E |\varphi_1| \, d\mu \geq \frac{3}{4} - \frac{1}{4C} \cdot C = \frac{1}{2},$$

$$\int_B |\varphi_1| \, d\mu - \int_B |\varphi_{n+1}| \, d\mu \leq d_1(\varphi_n, \varphi_{n+1}) \leq \frac{1}{2^{n+2}}.$$

Therefore,

$$\int_B |\varphi_n| \, d\mu \geq \int_B |\varphi_1| \, d\mu - \sum_{k=1}^{n-1} \left(\int_B |\varphi_k| \, d\mu - \int_B |\varphi_{k+1}| \, d\mu \right)$$
$$\geq \frac{1}{2} - \sum_{k=1}^{n-1} \frac{1}{2^{n+2}} \geq \frac{1}{4}.$$

This contradiction proves the lemma. $\square$

Corollary. *The functional $I_A(f) = \int_A f(x) \, d\mu(x)$, defined for any $A \in \mathfrak{A}$ on the space $S(X, \mu)$, can be extended by continuity to a functional on the space $L_1(X, \mu)$.*

Indeed, if $f \in L_1(X, \mu)$ and $\{f_n\}$ is a Cauchy sequence in $S(X, \mu)$ converging to f almost everywhere, then we can set

$$I_A(f) = \lim_{n \to \infty} \int_A f_n(x) \, d\mu(x).$$

By the preceding lemma, the limit on the right-hand side does not depend on the choice of the sequence $\{f_n\}$.

The functional so constructed is called the *Lebesgue integral* of the integrable function f over the set A, and is denoted by $\int_A f(x) \, d\mu(x)$.

Theorem 13. (1) *$L_1(X, \mu)$ is a linear space.*

(2) *For any set $A \in \mathfrak{A}$ the correspondence $f \mapsto \int_A f(x)\,d\mu(x)$ is a linear functional on $L_1(X, \mu)$.*

(3) *For any function $f \in L_1(X, \mu)$ the quantity $\nu(A) = \int_A f(x)\,d\mu(x)$ is a signed measure on $\mathfrak{A}$.*

(4) *The quantity $d_1(f, g) = \int_X |f - g|\,d\mu$ has all the properties of a distance on $L_1(X, \mu)$.*

Proof. Let f and g be integrable functions, and $\{f_n\}$ and $\{g_n\}$ Cauchy sequences of simple functions converging to them almost everywhere. Then the sequence $\{\alpha f_n + \beta g_n\}$ is Cauchy and converges almost everywhere to $\alpha f + \beta g$. This proves (1).

(2) follows from the fact that

$$\int_A (\alpha f + \beta g)\,d\mu = \lim_n \int_A (\alpha f_n + \beta g_n)\,d\mu = \alpha \lim_n \int_A f_n\,d\mu + \beta \lim_n \int_A g_n\,d\mu$$
$$= \alpha \int_A f\,d\mu + \beta \int_A g\,d\mu.$$

(3) can be proved just as the analogous assertion in Theorem 12. Finally, (4) can be derived by passing to the limit from the corresponding properties of d_1 in the space $S(X, \mu)$. □

Remark 1. We show later that $L_1(X, \mu)$ is complete with respect to the distance d_1. In many problems in functional analysis $L_1(X, \mu)$ arises in a natural way as the completion of some class of functions with respect to the integral metric d_1. The fact that the points of this space can be realized as equivalence classes of integrable functions plays a subordinate role.

Remark 2. The extension of the integral from the simple functions to the integrable functions can be carried out in many ways (in this connection, see Problems 195–197).

2. Functions of Bounded Variation and the Lebesgue–Stieltjes Integral

Let φ and f be two real functions on $[a, b]$. For any partition $T = (t_0 = a < t_1 < \cdots < t_{n-1} < t_n = b)$ of $[a, b]$ and any n-tuple of points $\xi = (\xi_1, \ldots, \xi_n)$ satisfying the conditions $t_{i-1} \le \xi_i \le t_i$, $i = 1, 2, \ldots, n$, we form the *Riemann–Stieltjes integral sum*

$$S(T, \xi, f, \varphi) = \sum_{i=1}^{n} f(\xi_i)[\varphi(t_i) - \varphi(t_{i-1})]. \tag{7}$$

The quantity $\lambda(T) = \max_{1 \le i \le n}(t_i - t_{i-1})$ is called the *diameter* of the partition T. If

$$\lim_{\lambda(T) \to 0} S(T, \xi, f, \varphi) \tag{8}$$

exists, it is called the *Riemann–Stieltjes integral* and denoted by

$$\int_a^b f(t)\,d\varphi(t).$$

The classical theorem of Stieltjes states that this integral exists if φ has bounded variation on $[a, b]$ and f is continuous there. We remark that in the particular case when $\varphi(t) = t$ the Riemann–Stieltjes integral becomes the ordinary Riemann integral.

The condition that φ be of bounded variation is essential: otherwise, the expression (7) is unbounded even for piecewise constant functions f taking the values ± 1. Therefore, in what follows we consider only functions of bounded variation as φ. Furthermore, if the functions f and φ have a common point of discontinuity, then it is easy to see that the limit (8) does not exist. But if f is continuous at a point of discontinuity of φ, then the value of φ at this point does not influence the value of (7) (see Problem 223). Hence, we can assume that φ is left-continuous from now on. To each such φ there corresponds a signed measure ν (see Theorem 6).

The integral of f with respect to this signed measure is called a *Lebesgue–Stieltjes integral.* We denote it by $\int_a^b f(x)\,d\nu(x)$.

Theorem 14. *The Riemann–Stieltjes integral $\int_a^b f(x)\,d\varphi(x)$ exists if and only if f is bounded and almost everywhere continuous on $[a, b]$ with respect to the measure $|\nu|$. In this case the Lebesgue–Stieltjes integral $\int_a^b f(x)\,d\nu(x)$ is also defined and its value coincides with the value of the Riemann–Stieltjes integral.*

PROOF. *Necessity.* We represent φ as the sum of a continuous function φ_0 and a jump function φ_1 (see Problem 215). Then the signed measure ν also is the sum of the signed measures ν_0 and ν_1. It has already been pointed out that f must be continuous at the points of discontinuity of φ. Therefore, the set Ω_f of points of discontinuity of f has $|\nu_1|$-measure zero. We show that it has $|\nu_0|$-measure zero. Let $\omega_f[\alpha, \beta]$ denote the oscillation of f on $[\alpha, \beta]$, and $\omega_f(x) = \lim_{\varepsilon \to 0} \omega_f[x - \varepsilon, x + \varepsilon]$ the oscillation of f at x. The set of points where the oscillation of f is not less than δ is denoted by $\Omega_f(\delta)$:

$$\Omega_f(\delta) = \{x \in [a, b] : \omega_f(x) \geq \delta\}.$$

It is clear that $\Omega_f = \bigcup_{\delta > 0} \Omega_f(\delta) = \bigcup_{n=1}^{\infty} \Omega_f(1/n)$. Therefore, it suffices to prove that each set $\Omega_f(\delta)$ ($\delta > 0$) has $|\nu_0|$-measure zero. Let $\varepsilon > 0$ be given. There is a partition $T = (t_0 = a < t_1 < \cdots < t_n = b)$ fine enough that

$$\sum_{i=1}^{n} |\varphi_0(t_i) - \varphi_0(t_{i-1})| > |\nu_0|([a, b]) - \varepsilon.$$

Then

$$\sum_{i=1}^{n} \{|\nu_0|([t_{i-1}, t_i]) - |\varphi_0(t_i) - \varphi_0(t_{i-1})|\} < \varepsilon. \tag{9}$$

By refining the partition T if necessary, we can assume that

$$\sum_{i=1}^{n} \omega_f[t_{i-1}, t_i] |\varphi_0(t_i) - \varphi_0(t_{i-1})| < \varepsilon\delta, \tag{10}$$

because f is an integrable function, and the expression on the left-hand side of (10) can be represented in the form

$$\sup_{\xi_1, \xi_2} [S(T, \xi_1, f, \varphi) - S(T, \xi_2, f, \varphi)].$$

Consider the terms of the sum (10) corresponding to closed intervals $[t_{i-1}, t_i]$ that contain points of $\Omega_f(\delta)$ in their interiors. For these terms $\omega_f[t_{i-1}, t_i]$ is bounded below by δ. Therefore, (10) implies that

$$\sum{}' |\varphi(t_i) - \varphi(t_{i-1})| < \varepsilon,$$

where the sum extends over the indicated segments of the partition.

It follows from (9) that $\sum' |\nu_0|([t_{i-1}, t_i]) < 2\varepsilon$, i.e., the total $|\nu_0|$-measure of these closed intervals is less than 2ε.

But these closed intervals cover all of $\Omega_f(\delta)$ except, perhaps, for finitely many points that are points of T. Since the $|\nu_0|$-measure of a point is zero (φ_0 is continuous), we see that $|\nu_0(\Omega_f(\delta))| < 2\varepsilon$.

Sufficiency. Let the signed measure ν be represented in the form $\mu_+ - \mu_-$, where μ_+ and μ_- are finite measures on $[a, b]$ and $|\nu| = \mu_+ + \mu_-$.

If $|\nu|(\Omega_f) = 0$, then $\mu_+(\Omega_f) = \mu_-(\Omega_f) = 0$. On the other hand, if φ is μ_+-integrable and μ_--integrable, then it is also ν-integrable. This shows that in proving the sufficiency we can restrict ourselves to the case of a positive signed measure (or a monotone function φ). In this case, as in the usual theory of the Riemann integral, we can use the Darboux criterion: f is integrable if the infimum over T of the expressions on the left-hand side of (10) is zero.

Let $\varepsilon > 0$. We let M be the oscillation of f on $[a, b]$, and V the variation of φ on $[a, b]$. For any $\delta > 0$ the set $\Omega_f(\delta)$ has measure zero. Moreover, this set is closed and, consequently, compact. Hence, it can be covered by finitely many segments with total μ-measure $\leq \varepsilon/(2M)$. The oscillation of f is less than δ at all points of the remainder of $[a, b]$. It is easy to show (analogously to the theorem on the uniform continuity of a function continuous on a segment) that there is a partition of this set into finitely many segments such that the oscillation of f on each is $\leq \delta$. Take δ to be $\varepsilon/(2V)$. Then for the partition T constructed the left-hand side of (10) is bounded by the constant $M(\varepsilon/(2M)) + (\varepsilon/(2V))V = \varepsilon$. $\square$

For computing the Riemann–Stieltjes and Lebesgue–Stieltjes integrals in practice, the properties described in Problems 220, 223, 224, and 225 are convenient.

3. Properties of the Lebesgue Integral

Some important properties of the Lebesgue integral have already been described above (see Theorem 13); we shall use these results repeatedly. Here we consider properties of the integral involving limits, product measures, and differentiation of signed measures.

Theorem 15 (Lebesgue Dominated Convergence Theorem). *Let $\{f_n\}$ be a sequence of μ-integrable functions on a set X that are bounded in modulus by a fixed nonnegative μ-integrable function φ on X and that converge μ-almost everywhere on X to a function f.*

Then f is μ-integrable on X, and

$$\lim_{n\to\infty}\int_A f_n\,d\mu = \int_A f\,d\mu$$

for any μ-measurable set A.

Proof. For any measurable set A let $\nu(A) = \int_A \varphi\,d\mu(x)$. By Theorem 13, ν is a finite measure on X.

Lemma. *If g is measurable and bounded on X, then $f = \varphi g$ is μ-integrable, and $\int_A f(x)\,d\mu(x) = \int_A g(x)\,d\nu(x)$ for any μ-measurable set A.*

Proof. Let us consider the set M of functions g on X such that the assertion of the lemma is true. Obviously, M contains all the characteristic functions of measurable sets. Indeed, if $g = \chi_B$, then

$$\int_A f\,d\mu = \int_A \varphi\chi_B\,d\mu = \int_{A\cap B}\varphi\,d\mu = \nu(A\cap B) = \int_A g\,d\nu.$$

Hence, M contains also the finite linear combinations of such functions. Suppose now that g is any bounded measurable function on X. Let $g_-(x) = [ng(x)]/n$, and $g_+(x) = g_-(x) + 1/n$. Then $g_\pm(x) \in M$, and $g_-(x) \le g(x) \le g_+(x)$. Therefore, $\int_A g_-\,d\nu = \int_A \varphi g_-(x)\,d\mu(x) \le \int_A \varphi g\,d\mu \le \int_A \varphi g_+\,d\mu = \int_A g_+\,d\nu$. The first and last terms of this chain converge to $\int_A g\,d\nu$ as $n\to\infty$. Hence, $g\in M$, and the lemma is proved.

We return to the proof of Theorem 15. Let

$$g_n(x) = \begin{cases} f_n(x)/\varphi(x) & \text{if } \varphi(x)\neq 0,\\ 0 & \text{if } \varphi(x) = 0,\end{cases}$$

$$g(x) = \begin{cases} f(x)/\varphi(x) & \text{if } \varphi(x)\neq 0,\\ 0 & \text{if } \varphi(x) = 0.\end{cases}$$

By the assumption of the theorem, the functions $g_n(x)$ have the properties $|g_n(x)| \le 1$, $g_n(x) \xrightarrow{\text{a.e.}} g$. We must show that $\lim_{n\to\infty}\int_A f_n(x)\,d\mu = \int_A f\,d\mu$. By the lemma, this is equivalent to the assertion that $\lim_{n\to\infty}\int_A g_n(x)\,d\nu = \int_A g\,d\nu$. Thus, we have reduced the proof of the Lebesgue theorem to the particular case when the measure of X is finite and the relevant functions are bounded in modulus by a single constant. In this case the theorem is an easy consequence of Egorov's theorem. □

Theorem 16 (B. Levi's Theorem on Monotone Convergence). *Let $\{f_n\}$ be a monotonically increasing sequence of μ-integrable functions on X, and let $f(x) = \lim_{n\to\infty} f_n(x)$ (the value $+\infty$ is allowed).*

(1) *If the integrals $\int_X f_n(x)\,d\mu(x)$ are collectively bounded, then $f(x)$ is integrable and $\int_X f(x)\,d\mu(x) = \lim_{n\to\infty} \int_X f_n(x)\,d\mu(x)$.*

(2) *If $\lim_{n\to\infty} \int_X f_n(x)\,d\mu(x) = +\infty$, then $f(x)$ is not integrable.*

Proof. (1) Subtracting f_1 from all the f_n and from f, we see that we can assume $f_n \geq 0$ and $f \geq 0$. Let E be the set where $f(x)$ takes the value $+\infty$. Then $E = \bigcap_N \bigcup_n E_{Nn}$, where $E_{Nn} = \{x \in X\colon f_n(x) \geq N\}$. We have that

$$\int_{E_{Nn}} f_n(x)\,d\mu \geq N \cdot \mu(E_{Nn}).$$

Since $\int_X f_n(x)\,d\mu \leq C$ for all n, $\mu(E_{Nn}) \leq C/N$; from this,

$$\mu(E) = \lim_{N\to\infty} \lim_{n\to\infty} \mu(E_{Nn}) = 0.$$

Accordingly, f is finite almost everywhere. Let A be a set of finite measure on which $f(x)$ is bounded above. Then

$$\int_A f(x)\,d\mu = \lim_{n\to\infty} \int_A f_n(x)\,d\mu \leq \lim_{n\to\infty} \int_X f_n(x)\,d\mu(x) \leq C,$$

which, by the assertion of Problem 197, proves that f is integrable. The rest of (1) and (2) follow from the Lebesgue dominated convergence theorem. □

Fatou's Lemma. *If a sequence $\{f_n\}$ of μ-integrable nonnegative functions is such that:*

(1) *$\int_X f_n(x)\,d\mu \leq C$ for all n, and*

(2) *$f_n(x) \to f(x)$ almost everywhere on X, then f is a μ-integrable function, and $\int_X f\,d\mu \leq C$.*

Proof. We replace the limit $f_n \to f$ by two monotone limits. Namely, let

$$g_{kn}(x) = \min\{f_n(x), f_{n+1}(x), \ldots, f_{n+k}(x)\},$$
$$g_n(x) = \lim_{k\to\infty} g_{kn}(x).$$

Then $\lim_{n\to\infty} g_n(x) = f(x)$ for almost all x. The monotonicity of the integral implies that $\int_X g_n\,d\mu \leq C$, and thus Theorem 16 implies that f is integrable and $\int_X f\,d\mu \leq C$. □

We are now able to prove the completeness of $L_1(X, \mu)$, as promised above:

Theorem 17. *$L_1(X, \mu)$ is complete.*

PROOF. Let $\{f_n\}$ be a Cauchy sequence in $L_1(X, \mu)$. Passing, if necessary, to a subsequence, we can assume that $\{f_n\}$ has the property that $d_1(f_n, f_{n+1}) < 1/2^n$. Let $\varphi_1 = f_1$, and $\varphi_n = f_n - f_{n-1}$ for $n \geq 2$. The series $\sum_{n=1}^{\infty} |\varphi_n(x)|$ converges, by Theorem 16, to an almost-everywhere finite, integrable function $\varphi(x)$. Hence, $\sum \varphi_n(x)$ converges almost everywhere to some function $f(x)$. This implies that $f_n \to f$ almost everywhere on X. Moreover, all the functions f_n are bounded in modulus by the function $\varphi(x)$, so by the dominated convergence theorem $\lim_{n\to\infty} \int_X |f_n(x) - f(x)|\, d\mu = 0$. In other words, $f_n \to f$ in the space $L_1(X, \mu)$. □

One more useful property of the Lebesgue integral is its so-called *absolute continuity*.

Theorem 18. *Let $f \in L_1(X, \mu)$. Then for any $\varepsilon > 0$ there is a $\delta > 0$ such that $|\int_A f(x)\, d\mu| < \varepsilon$ if $\mu(A) < \delta$.*

PROOF. The assertion of the theorem means essentially that the mapping I_f: $A \to \int_A f(x)\, d\mu$ from the metric space $L(X)$ of measurable sets (see Problem 100) to $\mathbf{R}$ is continuous. If χ is the characteristic function of a measurable subset of finite measure in X, then I_χ is obviously continuous. The same is true for I_f if f is a linear combination of the characteristic functions. Furthermore, if $f_n \to f$ in $L_1(X, \mu)$, then $I_{f_n} \to I_f$ uniformly on $L(X)$. The theorem now follows from the familiar fact that a uniform limit of continuous functions is continuous. □

Let us now return to the proof of Theorem 7 in §1.3 on product measures. Let (X, S, μ) and (Y, T, ν) be as in the statement of this theorem. For each set $C = A \times B$ in the half-ring $S \times T$ we define $f_C(x) = \chi_A(x)\nu(B)$. It is clear that $(\mu \times \nu)(C) = \mu(A) \times \nu(B) = \int_X f_C(x)\, d\mu$. If C can be represented in the form $C = \coprod_{k=1}^{\infty} C_k$, $C_k \in S \times T$, then the countable additivity of ν implies that $f_C(x) = \sum_k f_{C_k}(x)$. By the Lebesgue dominated convergence theorem, this gives us that

$$\int_X f_C(x)\, d\mu(x) = \sum_{k=1}^{\infty} \int_X f_{C_k}(x)\, d\mu(x),$$

and hence,

$$(\mu \times \nu)(C) = \sum_{k=1}^{\infty} (\mu \times \nu)(C_k).$$

That is, $\mu \times \nu$ is also countably additive. □

We now study in more detail the mapping $C \mapsto f_C$ defined above. Let us extend it to the ring $R(S \times T)$ by the formula

$$f_{\coprod_{k=1}^{n} C_k} = \sum_{k=1}^{n} f_{C_k},$$

It is easy to verify that

$$\|f_{C_1} - f_{C_2}\|_{L_1(X,\mu)} \le (\mu \times \nu)(C_1 \triangle C_2).$$

(Indeed, if $C_1 = A_1 \amalg B$ and $C_2 = A_2 \amalg B$, where $A_1 \cap A_2 = \varnothing$, then $f_{C_1} - f_{C_2} = f_{A_1} - f_{A_2}$ and $f_{C_1 \triangle C_2} = f_{A_1} + f_{A_2}$.)

Therefore, the correspondence $C \mapsto f_C$ extends to a mapping of the whole σ-algebra $L(X \times Y)$ of $(\mu \times \nu)$-measurable sets into $L_1(X, \mu)$ by the formula $f_{\lim_n C_n} = \lim_n f_{C_n}$ (the first limit is taken in $L(X \times Y)$ and the second in $L_1(X, \mu)$).

Lemma. *Let $C \in L(X \times Y)$. For almost all $x \in X$ the set $C_x \subset Y$ defined by $C_x = \{y \in Y : (x, y) \in C\}$ is measurable with respect to the measure ν and $\nu(C_x) = f_C(x)$.*

Proof. For elementary sets (i.e., sets in the ring $R(S \times T)$) this is true by the definition of f_C. Next, if $\{C^{(n)}\}$ is a monotone sequence of sets, then $\nu(\lim_n C_x^{(n)}) = \lim_n \nu(C_x^{(n)})$, because ν is countably additive. Therefore, the property $\nu(C_x) = f_C(x)$ is preserved under monotone limits. But every measurable set C can, to within a set of measure zero, be obtained from elementary sets by two monotone limits. Indeed, let C_n be an elementary set approximating C to within 2^{-n} with respect to the measure $\mu \times \nu$. Let $\tilde{C} = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} C_{n+k}$. Then $(\mu \times \nu)(C \setminus \bigcup_{k=1}^{\infty} C_{n+k}) = 0$ and $(\mu \times \nu)(\bigcup_{k=1}^{\infty} C_{n+k} \setminus C) \le 2^{1-n}$. This implies that $(\mu \times \nu)(\tilde{C} \triangle C) = 0$.

Hence, $f_{\tilde{C}}$ and f_C coincide almost everywhere, and, consequently, $f_C(x) = f_{\tilde{C}}(x) = \nu(\tilde{C}_x) = \nu(C_x)$ for almost all $x \in X$. □

Theorem 19. *Let μ and ν be σ-finite measures, C a $(\mu \times \nu)$-measurable subset of $X \times Y$, and $C_x = \{y \in Y : (x, y) \in C\}$. Then for μ-almost every $x \in X$ the set C_x is ν-measurable, the function $f_C(x) = \nu(C_x)$ is μ-measurable, and*

$$(\mu \times \nu)(C) = \int_X f_C(x)\, d\mu(x), \tag{11}$$

where both sides may equal $+\infty$.

Proof. If C has finite measure, then the theorem follows from the lemma proved above and from the fact that the equality (11) is preserved under passage to the limit (on the left in the space $L(X \times Y)$, and on the right in the space $L_1(X, \mu)$).

If C has infinite measure, then there is an increasing family of subsets of finite measure $C_n \subset C$ such that $\bigcup C_n = C$ and $(\mu \times \nu)(C_n) \to \infty$. Then $f_C(x) = \lim_n f_{C_n}(x)$ and

$$\int_X f_{C_n}(x)\, d\mu = (\mu \times \nu)(C_n) \to \infty.$$

Therefore, f_C is measurable and not integrable. □

This theorem justifies, in particular, the well-known method for computing the area of planar figures (resp., the volumes of spatial bodies) by integrating the lengths (resp., areas) of their sections.

Remark 1. Since the spaces (X, μ) and (Y, ν) appear symmetrically in the condition of Theorem 19, the conclusion of the theorem remains true if they are interchanged. Thus,

$$(\mu \times \nu)(C) = \int_Y \mu(C'_y)\, d\nu(y), \tag{11'}$$

where $C'_y = \{x \in X : (x, y) \in C\}$. This implies also that

$$\int_X \nu(C_x)\, d\mu(x) = \int_Y \mu(C'_y)\, d\nu(y). \tag{12}$$

Remark 2. An analogous theorem holds for a product of finitely many spaces. In the case of three spaces (X, μ), (Y, ν), (Z, λ) it has the form

$$(\mu \times \nu \times \lambda)(C) = \int_{X \times Y} \lambda(C_{x,y})\, d(\mu \times \nu)(x, y) = \int_Z (\mu \times \nu)(C_z)\, d\lambda(z), \tag{13}$$

where

$$C_{x,y} = \{z \in Z : (x, y, z) \in C\}, \qquad C_z = \{(x, y) \in X \times Y : (x, y, z) \in C\}.$$

Fubini's Theorem. *Suppose that $f(x, y)$ is an integrable function on the product of the spaces (X, μ) and (Y, ν). Then:*

(1) *for μ-almost all $x \in X$ the function $f(x, y)$ is integrable on Y, and its integral over Y is an integrable function on X;*
(2) *for ν-almost all $y \in Y$ the function $f(x, y)$ is integrable on X, and its integral over X is an integrable function on Y;*
(3) *we have the equalities*

$$\begin{aligned} \int_{X \times Y} f(x, y)\, d(\mu \times \nu)(x, y) &= \int_X \left(\int_Y f(x, y)\, d\nu(y) \right) d\mu(x) \\ &= \int_Y \left(\int_X f(x, y)\, d\mu(x) \right) d\nu(y); \end{aligned} \tag{14}$$

(4) *for nonnegative $(\mu \times \nu)$-measurable functions the existence of one of the repeated integrals in* (14) *implies that f is integrable on $X \times Y$.*

Proof. The decomposition $f = f_+ - f_-$ reduces us to the case of a nonnegative function. Consider the product of the spaces (X, μ), (Y, ν), $(\mathbf{R}, \lambda)$, where $\lambda = dz$ is the usual Lebesgue measure, along with the set $C \subset X \times Y \times \mathbf{R}$ defined by

$$C = \{(x, y, z) \in X \times Y \times \mathbf{R} : 0 \leq z \leq f(x, y)\}.$$

We apply the relation (13) to this case, obtaining

$$C_{x,y} = \{z \in \mathbf{R}: 0 \le z \le f(x, y)\}; \qquad \lambda(C_{x,y}) = f(x, y),$$

$$C_x = \{(y, z) \in Y \times \mathbf{R}: 0 \le z \le f(x, y)\}; \qquad (\nu \times \lambda)(C_x) = \int_Y f(x, y)\, d\nu(y).$$

All the assertions of the theorem follow directly from this.

We remark that the integrability conditions on f in (1), (2), and (3) and the nonnegativity condition in (4) are essential (cf. Problems 239–242).

Definition. Let X be a set, $\mathfrak{A} \subset P(X)$ a σ-algebra, and μ a σ-finite measure and ν a signed measure on $\mathfrak{A}$. The signed measure ν is said to be *absolutely continuous* with respect to μ if $\mu(A) = 0$ implies $\nu(A) = 0$. Two signed measures ν_1 and ν_2 are said to be *equivalent* if $|\nu_1|(A) = 0$ if and only if $|\nu_2|(A) = 0$.

The Radon–Nikodým Theorem. *Every finite signed measure ν that is absolutely continuous with respect to μ has the form*

$$\nu(A) = \int_A f(x)\, d\mu(x), \tag{15}$$

where f is a function in $L_1(X, \mu)$. This f (as an element of $L_1(X, \mu)$) is uniquely determined by ν.

Proof. For any rational number r define $\nu_r = \nu - r\mu$. By the result of Problem 136, X can be represented in the form $A_r^+ \amalg A_r^-$, with ν_r nonnegative on A_r^+ and nonpositive on A_r^-. Let $A_c = \bigcup_{r>c} A_r^+$ for any real number c. It is clear that $\{A_c\}$ is a decreasing family of measurable sets: $A_{c_1} \subset A_{c_2}$ for $c_1 > c_2$. We show that $A_{-\infty} = \bigcap_c A_c$ and the complement of $A_\infty = \bigcup_c A_c$ have measure zero. Indeed, if $A \subset A_{-\infty}$, then $\nu_r(A) \ge 0$ for all r, and this is possible only if $\mu(A) = 0$. And if $A \subset X \setminus A_\infty$, then $\nu_r(A) \le 0$ for all r, which again is possible only if $\mu(A) = 0$. Furthermore, by construction the family $\{A_c\}$ is right-continuous: $A_c = \bigcap_{\varepsilon>0} A_{c+\varepsilon}$. Therefore, there exists a function f on X such that $A_c = \{x \in X: f(x) > c\}$. All the sets A_c are measurable by construction, so f is measurable.

Suppose now that E is any set of finite measure. Then (see Problem 187)

$$\begin{aligned}\int_E f(x)\, d\mu &= \lim_{n\to\infty} \sum_k \frac{k}{n} \mu(E \cap A_{k/n} \setminus A_{(k+1)/n}) \\ &= \lim_{n\to\infty} \sum_k \frac{k+1}{n} \mu(E \cap A_{k/n} \setminus A_{(k+1)/n}).\end{aligned}$$

On the other hand, by the definition of A_c,

$$\begin{aligned}\frac{k}{n} \mu(E \cap A_{k/n} \setminus A_{(k+1)/n}) &\le \nu(E \cap A_{k/n} \setminus A_{(k+1)/n}) \\ &\le \frac{k+1}{n} \mu(E \cap A_{k/n} \setminus A_{(k+1)/n}).\end{aligned}$$

This proves (15) for sets of finite measure. The finiteness of ν now implies (see Problem 134) the integrability of f and the validity of (15) in the general case. The uniqueness of f (as an element of $L_1(X, \mu)$) follows from the result of Problem 192. □

Corollary. *If μ is a measure on X and ν is a finite signed measure that is absolutely continuous with respect to μ, then for any $\varepsilon > 0$ there is a $\delta > 0$ such that $\mu(A) < \delta$ implies $|\nu|(A) < \varepsilon$.*

Indeed, by the Radon–Nikodým theorem, there is a function $f \in L_1(X, \mu)$ such that $\nu(A) = \int_A f\, d\mu$. Then $|\nu|(A) = \int_A |f|\, d\mu$, and the statement follows from Theorem 18.

Chapter III
Linear Topological Spaces and Linear Operators

§1. General Theory

1. Topology, Convexity, and Seminorms

We shall consider linear spaces L over the fields $\mathbf{R}$ and $\mathbf{C}$. In cases when a statement does not depend on the choice of field, we write K instead of $\mathbf{R}$ or $\mathbf{C}$. If A and B are two subsets of L and λ and μ are two numbers in K, then $\lambda A + \mu B$ denotes the set of elements $z \in L$ of the form $\lambda x + \mu y$, where $x \in A$, $y \in B$.

Definitions. The *segment* (resp., *interval*) in L with endpoints x and y is defined to be the set of points $z \in L$ of the form $z = \tau x + (1 - \tau)y$, $0 \leq \tau \leq 1$ (resp., $0 < \tau < 1$).

A set $E \subset L$ is said to be *convex* if it contains the segment between any two of its points. A set $E \subset L$ is said to be *balanced* if $\alpha E \subset E$ for any $\alpha \in K$, $|\alpha| \leq 1$.

A set $E \subset L$ is said to be *absorbing* if $\bigcup_{\lambda \in K} \lambda E = L$.

A *seminorm* is defined to be a function p on L taking nonnegative values (the value $p(x) = \infty$ is allowed) and having the properties:

(1) $p(\lambda x) = |\lambda| p(x)$, $\lambda \in K$, $x \in L$ (homogeneity);
(2) $p(x + y) \leq p(x) + p(y)$ (subadditivity).

It follows easily from (1) and (2) that $p(0) = 0$.

A *norm* is defined to be a seminorm that takes a finite nonzero value for any nonzero $x \in L$.

The *unit ball* for a seminorm p is defined to be the set $B_p = \{x \in L : p(x) \leq 1\}$.

The *Minkowski functional* of a set $B \subset L$ is defined to be the function $p_B(x) = \inf\{\lambda : \lambda > 0,\ x \in \lambda B\}$ (if $x \notin \lambda B$ for all $\lambda > 0$, then let $p_B(x) = +\infty$).

It turns out that the correspondences $p \mapsto B_p$ and $B \mapsto p_B$ are almost inverse to each other when p runs through the set of seminorms on L, and B through the set of convex balanced sets. The qualification "almost" stems from the fact that different sets B can have the same Minkowski functional p_B (see Problems 267, 268).

Theorem 1. (1) *If p is a seminorm, then its unit ball B_p is a convex balanced set.*

(2) *If p is a norm, then B_p is an absorbing set that does not contain the whole of any line (i.e., one-dimensional subspace) in L.*

(3) *If B is a convex balanced set, then p_B is a seminorm.*

(4) *If B is a convex balanced absorbing set that does not contain a line, then p_B is a norm.*

(5) *For any seminorm p, $p_{B_p} = p$.*

Proof. Assertions (1) and (2) follow directly from the definitions. Let us prove (3). The homogeneity of p_B follows from B being balanced. We now show that the subadditivity of p_B follows from the convexity of B: Let $x, y \in L$. If $p_B(x)$ or $p_B(y)$ is infinite, there is nothing to prove. If $p_B(x) = 0$, then $\lambda x \in B$ for all $\lambda > 0$. Therefore, if $y/\lambda \in B$, then $((1 - \varepsilon)/\lambda)(y + x) = (1 - \varepsilon)(y/\lambda) + \varepsilon((1 - \varepsilon)/\varepsilon\lambda)x \in B$ for $\varepsilon \in (0, 1)$. Taking the infimum over λ and letting ε go to zero shows $p_B(y + x) \leq p_B(y) = p_B(y) + p_B(x)$. It remains to investigate the case when $p_B(x)$ and $p_B(y)$ are nonzero. Consider the vectors $x_\varepsilon = ((1 - \varepsilon)/p_B(x))x$ and $y_\varepsilon = ((1 - \varepsilon)/p_B(y))y$. By the definition of the Minkowski functional, $x_\varepsilon, y_\varepsilon \in B$ for $\varepsilon > 0$. Since B is convex, it follows that $\tau x_\varepsilon + (1 - \tau)y_\varepsilon \in B$ for $0 \leq \tau \leq 1$. In particular, for $\tau = p_B(x)/(p_B(x) + p_B(y))$ we get $((1 - \varepsilon)/(p_B(x) + p_B(y)))(x + y) \in B$. Hence, $p_B(x + y) \leq (p_B(x) + p_B(y))/(1 - \varepsilon)$ for $\varepsilon > 0$, which implies that p_B is subadditive. The assertions (4) and (5) of the theorem can easily be checked. $\square$

Definition. A *linear topological space* (briefly, an LTS) is defined to be a linear space L over the field K, endowed with a topology in which the operations of addition and multiplication by a number are continuous.

Example 1. Let p be a finite seminorm in L; we take the collection of open balls $\mathring{B}_p(x, r) = \{y \in L : p(x - y) < r\}$ as a base for a topology in L.

Let us verify that this gives a linear topological space. Indeed, suppose that $x, y \in L$ and U is a neighborhood of the point $x + y \in L$. By the definition of the topology, U contains a ball of the form $\mathring{B}_p(x + y, r)$, $r > 0$. Let $U_1 = \mathring{B}_p(x, r/2)$, $U_2 = \mathring{B}_p(y, r/2)$. Then U_1 and U_2 are open, and $U_1 + U_2 \subset \mathring{B}_p(x + y, r) \subset U$, since p is subadditive. This proves that the addition operation is continuous.

Suppose now that $x \in L$, $\lambda \in K$, and U is a neighborhood of the point $\lambda x \in L$. Then U contains a ball of the form $\mathring{B}_p(\lambda x, r)$, $r > 0$. Let $V_\varepsilon = \{\mu \in K: |\lambda - \mu| < \varepsilon\}$, $U_\delta = \mathring{B}_p(x, \delta)$. If $\mu \in V_\varepsilon$, $y \in U_\delta$, then

$$p(\lambda x - \mu y) \leq p(\lambda x - \mu x) + p(\mu x - \mu y) \leq \varepsilon p(x) + (|\lambda| + \varepsilon)\delta.$$

It is clear that for sufficiently small positive numbers ε and δ the latter expression will be less than r. Therefore, $V_\varepsilon U_\delta \subset \mathring{B}_p(\lambda x, r) \subset U$, and we have proved that multiplication is continuous.

Example 2. Let $\{p_\alpha\}_{\alpha \in A}$ be an arbitrary family of finite seminorms in L. We take the collection of balls $\mathring{B}_{p_\alpha}(x, r)$, $\alpha \in A$, $x \in L$, $r > 0$, and their finite intersections as a base for a topology in L. As in Example 1, it can be verified that this topology turns L into a linear topological space. Such an LTS is said to be *polynormed*.

Remark. Each finite seminorm in L defines a mapping of L into the set $\mathbf{R}^+$ consisting of all the real nonnegative numbers. The topology defined above is the weakest topology for which all the seminorms of the family $\{p_\alpha\}_{\alpha \in A}$ are continuous. The continuity of a seminorm p is equivalent to the openness of the ball $\mathring{B}_p(0, 1)$.

Definition. A *locally convex* linear topological space (briefly, an LCS) is defined to be an LTS whose topology has a base consisting of convex sets.

It is clear that the LTS's in Examples 1 and 2 are LCS's. It turns out that the LTS in Example 2 is the most general example of an LCS. Namely, we have the

Theorem 2. *In every* LCS L *the topology can be defined by means of a family of seminorms* $\{p_\alpha\}_{\alpha \in A}$. *The collection of all continuous seminorms on* L *can be taken as such a family.*

The proof is based on the

Lemma. *Every neighborhood of zero in an* LCS L *contains an open convex balanced set.*

Proof of the Lemma. Let U be arbitrary, and $\tilde{U} \subset U$ an open convex neighborhood of zero in L. Since the operation of multiplication by a number is continuous in L, there exists a number $\varepsilon > 0$ and an open neighborhood of zero $V \subset L$ such that $B_\varepsilon \cdot V \subset \tilde{U}$, where $B_\varepsilon = \{\lambda \in K: |\lambda| < \varepsilon\}$. Let W be the convex hull of $B_\varepsilon V$, i.e., the collection of all vectors of the form $\sum_{k=1}^N \tau_k x_k$, where $x_k \in B_\varepsilon V$, and the coefficients τ_k satisfy the constraints

$$0 \leq \tau_k \leq 1, \qquad \sum_{k=1}^N \tau_k = 1$$

(cf. Problem 269). Then W is open, convex, balanced, and contained in $\tilde{U} \subset U$. □

Proof of Theorem 2. By the lemma and assertion (3) of Theorem 1, there is a nonempty family of continuous seminorms defined on L that has the property described in Example 2. Let us introduce in L the topology of a polynormed space by taking $\{p_\alpha\}_{\alpha \in A}$ to be the family of all continuous seminorms on L. It is clear that this topology is weaker than the original one, since all the balls $\mathring{B}_{p_\alpha}(x, r)$ are open in the original topology. On the other hand, each neighborhood of zero in the original topology contains a convex balanced open set W and, hence, the ball $\mathring{B}_{p_W}(0, 1)$. Thus, the topology of the polynormed space is stronger than the original one. □

One usually considers only *separated* or *Hausdorff* LTS's, i.e., those in which any two distinct points have disjoint neighborhoods.

Theorem 3. *Every* LTS L *contains a unique subspace* L_0 *such that*:

(1) *any nonempty neighborhood of a point* $x \in L$ *contains the set* $x + L_0$,

(2) *the quotient space* L/L_0, *endowed with the natural quotient topology, is separated.*

Proof. Let L_0 be the intersection of all nonempty neighborhoods of zero. From the continuity of the operations of addition and multiplication by a number it follows that L_0 is a subspace and that (1) holds. Further, if x and y are two distinct points of the quotient space L/L_0, then there exists a neighborhood U of zero in L/L_0 that does not contain $x - y$. From the continuity of the subtraction operation it follows that there is a neighborhood V of zero such that $V - V \subset U$. Then $x + V$ and $y + V$ are disjoint neighborhoods of the points x and y. □

Example. In a polynormed space $(L, \{p_\alpha\}_{\alpha \in A})$, L_0 coincides with the set where all the seminorms p_α are zero.

Different sets of seminorms may induce the same topology in an LCS, in which case the sets of seminorms are said to be *equivalent*. It can be shown that two sets of seminorms are equivalent if and only if any seminorm in either set is majorized by some finite linear combination of seminorms in the other set.

As a rule, only separated polynormed spaces will be considered in what follows. Among them the spaces $(L, \{p_\alpha\}_{\alpha \in A})$ for which the family A is finite or countable are especially important. If A is finite, then the set $\{p_\alpha\}$ of seminorms can be replaced by the single seminorm $p = \sum_{\alpha \in A} p_\alpha$, which gives the same topology as the whole set $\{p_\alpha\}_{\alpha \in A}$. The space L is separated if and only if the seminorm p is a norm. Such LTS's, in which the topology can be determined by a norm, are said to be *normable*, or *normed* if a norm p giving the topology is fixed. If A is countable, then, generally speaking, L is not normable. However, every countably normed LTS is *metrizable*, i.e., the topology in it is given by some metric. Moreover, this metric can be assumed to be invariant under translations.† The following function is such a metric:

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x - y)}{1 + p_n(x - y)}.$$

2. Dual Spaces

Linear topological spaces form a category whose morphisms are continuous linear mappings. The collection of all continuous linear mappings from the LTS L_1 to the LTS L_2 is denoted by $\mathscr{L}(L_1, L_2)$. It is clear that $\mathscr{L}(L_1, L_2)$ is a linear space over K. The case $L_2 = K$ is of special interest.

Let an LTS L be given. The space $\mathscr{L}(L, K)$ is called the *dual space* of L. It is usually denoted by L', and its elements are called continuous linear functionals on L.

If L is a normed space, then L' can also be equipped with a norm, by the formula

$$\|f\|_{L'} = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|_L} \tag{1}$$

(see Problem 282), where $\|x\|_L$ denotes the norm of the element $x \in L$.

If L is a polynormed space, then L' can be equipped with the family $\{p'_M\}$ of seminorms, where

$$p'_M(f) = \sup_{x \in M} |f(x)|, \tag{2}$$

and M runs through all *bounded* subsets of L (see Problem 270).

The space L' with the topology corresponding to the norm (1) or the system of seminorms (2) is called the *strong dual* of L.

The *weak topology* is defined for any LTS L by the system of seminorms $p_f(x) = |f(x)|$, where f runs through L'; the notation $x_n \rightharpoonup x$ is used for weak convergence of sequences.

The *weak-*topology* is defined for the dual LTS L' by the system of seminorms $p_x(f) = |f(x)|$, where x runs through L.

A *Banach space* is defined to be a complete normed space.

Theorem 4. *For any normed space (L, p) the dual space (L', p') is complete.*‡

PROOF. Let $\{f_n\}$ be a Cauchy sequence of elements in L'. For each $x \in L$ the numerical sequence $f_n(x)$ is Cauchy and, consequently, has a limit,

† A linear space equipped with a translation-invariant metric is usually called a linear metric space. We also hold to this terminology.

‡ Here and subsequently, unless a statement to the contrary is made, the dual space is understood to be the strong dual space.

which we denote by $f(x)$. Let us show that $f \in L'$. The linearity of f is obtained by passing to the limit from the linearity of the f_n. The continuity of f is equivalent to its boundedness on the unit ball B_p (see Problem 281), and the latter follows from the fact that $\{f_n\}$ is Cauchy and from the estimate

$$|f(x)| \le p'(f)p(x). \tag{3}$$

Finally, the assertion that $p'(f_n - f) \to 0$ is obtained by passing to the limit as $m \to \infty$ in the inequality $p'(f_n - f_m) < \varepsilon$, which is valid for sufficiently large n and m, because $\{f_n\}$ is a Cauchy sequence. □

Any normed space L has a natural imbedding into the second dual space L'' (see Problem 303). If this imbedding is an isomorphism onto all of L'', then L is said to be a *reflexive* space. The weak and weak-* topologies coincide for such spaces.

The correspondence $L \to L'$ can be extended to a contravariant functor in the category $\mathscr{B}$ of all Banach spaces (the morphisms are continuous linear operators). Namely, we can assign to each operator $A \in \mathscr{L}(L_1, L_2)$ the *adjoint operator* A' acting from L_2' to L_1' according to the formula

$$A'f(x) = f(Ax), \quad \text{where } f \in L_2', x \in L_1.$$

The *norm of an operator* $A \in \mathscr{L}(L_1, L_2)$ is defined to be the number $\|A\| = \sup_{x \ne 0} \|Ax\|_{L_2}/\|x\|_{L_1}$.

Theorem 5. *The norm of A' coincides with the norm of A.*

Proof. Let us compute the norm of A'. By definition,

$$\begin{aligned}\|A'\| &= \sup \frac{\|A'f\|}{\|f\|} = \sup_{\|f\| \le 1} \|A'f\| = \sup_{\substack{\|f\| \le 1 \\ \|x\| \le 1}} |(A'f)(x)| = \sup_{\substack{\|f\| \le 1 \\ \|x\| \le 1}} |f(Ax)| \\ &= \sup_{\substack{\|f\| \le 1 \\ \|x\| \le 1}} |F_{Ax}(f)| = \sup_{\|x\| \le 1} \|F_{Ax}\| = \sup_{\|x\| \le 1} \|Ax\| = \|A\|.\end{aligned}$$

(We have made use of the equality $\|F_{Ax}\| = \|Ax\|$, which is equivalent to the assertion of Problem 303, where $F_y \in (L_2')'$ is the image of $y \in L_2$ under the natural imbedding $L \to (L')'$.) □

3. The Hahn–Banach Theorem

If the LTS's L_1 and L_2 are separated and finite-dimensional, then every linear mapping from L_1 to L_2 is continuous (see Problem 300), and, consequently, $\dim \mathscr{L}(L_1, L_2) = (\dim L_1) \cdot (\dim L_2)$. In particular, $\dim L' = \dim L$. This is not so in the infinite-dimensional case. It is known (see Problem 321) that there are separated infinite-dimensional LTS's L for which $L' = \{0\}$. It turns out that such unpleasant situations do not arise in LCS's.

Hahn–Banach Theorem. *Let p be a seminorm on L, L_0 a subspace of L, and f_0 a linear functional on L_0 having the property that $|f_0(x_0)| \le p(x_0)$ for all $x_0 \in L_0$. Then there exists a linear functional f on L that coincides with f_0 on L_0 and has the property that $|f(x)| \le p(x)$ for all $x \in L$.*

Proof. Let us consider the collection $\mathscr{L}$ of all pairs (L_1, f_1), where L_1 is a subspace of L containing L_0, and f_1 is a linear functional on L_1 having the property that $|f_1(x_1)| \le p(x_1)$ for $x_1 \in L_1$ and coinciding with f_0 on L_0. Note that $\mathscr{L}$ is not empty, since it contains the pair (L_0, f_0). We define a partial order on $\mathscr{L}$ by setting $(L_1, f_1) < (L_2, f_2)$ if $L_1 \subset L_2$ and the restriction of f_2 to L_1 coincides with f_1. The set $\mathscr{L}$ satisfies the conditions of Zorn's lemma: Each ordered subset $\{(L_\alpha, f_\alpha)\}$, $\alpha \in A$, has a majorant, namely, $(\bigcup L_\alpha, f)$, where f is the functional coinciding with f_α on L_α. Hence, $\mathscr{L}$ contains a maximal element $(\tilde{L}, \tilde{f})$.

Suppose that $\tilde{L} \neq L$. Let $x \in L$ be an element not in $\tilde{L}$. We construct an extension $\tilde{f}_1$ of the functional $\tilde{f}$ to the space $\tilde{L}_1 = \tilde{L} + Kx$ by setting $\tilde{f}_1(\tilde{x} + \lambda x) = \tilde{f}(\tilde{x}) + \lambda c$. The condition that must be satisfied by the constant $c \in K$ in order that the extension have the property $|\tilde{f}_1(x)| \le p(x)$, $x \in \tilde{L}_1$, is:

$$|f(\tilde{x}) + \lambda c| \le p(\tilde{x} + \lambda x), \qquad \tilde{x} \in \tilde{L}, \lambda \in K.$$

Replacing $\tilde{x}$ by $-\lambda y$ and dividing both sides of the inequality by $|\lambda|$, we arrive at the equivalent condition

$$|c - f(y)| \le p(x - y), \qquad y \in \tilde{L}.$$

We first analyze the case $K = \mathbf{R}$. In this case it must be shown that the family of closed intervals $[f(y) - p(x - y), f(y) + p(x - y)]$, $y \in \tilde{L}$, has a common point. To do this it suffices to show that the left-hand endpoint of any one of them lies to the left of the right-hand endpoint of any other. The desired point will then be the supremum of all the left-hand endpoints.

Thus, it remains to check the inequality

$$f(y_1) - p(x - y_1) \le f(y_2) + p(x - y_2), \qquad y_1, y_2 \in \tilde{L}.$$

But this follows at once from the inequality

$$f(y_1) - f(y_2) \le p(y_1 - y_2) \le p(y_1 - x) + p(x - y_2).$$

This argument can be carried over directly to the complex case by using *Helly's theorem* (see Problem 320). We shall give a simpler, more roundabout, way here. Regard $\tilde{L}_1$ as a real space. Then it can be obtained from $\tilde{L}$ by successively adjoining $\mathbf{R} \cdot x$ and $\mathbf{R} \cdot ix$. Successive extensions of the functional $\tilde{f}$ lead us to a real-linear functional φ that coincides with $\tilde{f}$ on $\tilde{L}$ and has the property that $|\varphi(x)| \le p(x)$ for $x \in \tilde{L}_1$. It is clear that the functional $\psi(x) = -i\varphi(ix)$ has the same properties. Finally, setting $\tilde{f}_1(x) = (\varphi(x) + \psi(x))/2$, we get the desired extension: $|\tilde{f}_1(x)| \le (|\varphi(x)| + |\psi(x)|)/2 \le p(x)$, and $\tilde{f}_1(ix) = (\varphi(ix) - i\varphi(-x))/2 = i\tilde{f}_1(x)$. Thus, we have constructed a pair $(\tilde{L}_1, \tilde{f}_1)$ coming after $(\tilde{L}, \tilde{f})$, which contradicts the maximality of the latter, so $\tilde{L}$ must, in fact, equal L. □

Corollary 1. *On any polynormed space L there are sufficiently many continuous linear functionals to separate any two points.*

Indeed, if $x, y \in L$ and $x \neq y$, then, by the lemma in §1, there exists a convex balanced neighborhood U of zero that does not contain $x - y$. Let $p = p_U$, $L_0 = K(x - y)$, and $f_0(x - y) = 1$. By the Hahn–Banach theorem, there exists a functional $f \in L'$ such that $f(x) - f(y) = 1$ and $|f(x)| \le p_U(x)$.

Corollary 2. *For any normed space (L, p) and any vector $x \in L$, $x \neq 0$, there exists a nonzero functional $f \in (L', p')$ such that*

$$f(x) = p'(f)p(x). \tag{3'}$$

Corollary 3. *For any normed space (L, p) the natural imbedding of L into L'' (carrying $x \in L$ into the functional $x(f) = f(x)$) is isometric* (cf. Problems 303, 304).

Remark. The properties of the seminorm p were not fully used in the proof of the Hahn–Banach theorem. Namely, it can be shown that all the arguments remain valid if p is required to be subadditive and positive-homogeneous: $p(\lambda x) = \lambda p(x)$ for $\lambda \ge 0$.

Moreover, the conditions of the theorem can be weakened by requiring that $f_0(x) \le p(x)$ (i.e., the functional is only bounded above); here the functional $f(x)$ is guaranteed to satisfy the one-sided estimate $f(x) \le p(x)$.

We now give a geometric interpretation of this extended version of the Hahn–Banach theorem:

Definition. Let L and M be linear spaces. The pre-image of a point under a linear mapping $A: L \to M$ is called a *linear manifold* in the linear space L. If the image of L in M under the mapping A has dimension n, then we say that the linear manifold $A^{-1}(x)$, $x \in A(L)$, has *codimension n*. A manifold of codimension 1 is called a *hyperplane*. Thus, hyperplanes are level sets of linear functionals.

Theorem 6 (Geometric Form of the Hahn–Banach Theorem). *Let $K = \mathbf{R}$. If U is an open convex set in the* LTS *L and S is a linear manifold disjoint from U, then there exists a hyperplane T containing S and disjoint from U.*

Proof. Without loss of generality, U contains zero. Let L_0 be the subspace generated by S, and f_0 the linear functional on L_0 defined by the equality

$f_0(x) = 1$ for $x \in S$ (f_0 is well defined, since S generates L_0 and does not contain zero). Since S does not intersect U, f_0 has the property that $f_0(x) \leq p_U(x)$ for $x \in L_0$. The Minkowski functional p_U is subadditive and positive-homogeneous: $p_U(\lambda x) = \lambda p_U(x)$ for $\lambda > 0$. By the remark made above, these properties suffice for the Hahn–Banach theorem to be valid in the real case. Extend f_0 to a functional f on the whole space having the property that $f(x) \leq p_U(x)$, and set $T = \{x \in L, f(x) = 1\}$. Then T is the desired hyperplane. Indeed, $f(x) \leq 1$ on U, and $f(x) < 1$ for $x \in U$ because U is open.

Corollary. *Let U_1 and U_2 be disjoint convex sets in an* LTS *L, one of which is open. Then there exists a hyperplane T separating U_1 and U_2.*

Indeed, let $U = U_1 - U_2$, $S = \{0\}$. Then U is an open convex set, and S is a linear manifold disjoint from U. Let T be a hyperplane containing S and disjoint from U. Since T contains zero, it is determined by an equation $f(x) = 0$, where f is a linear functional on L. Since U is convex and disjoint from T, f takes values of a single sign on U.

Suppose for definiteness that $f(x) > 0$ for $x \in U$. Recalling the definition of U, we see that $f(x_1) > f(x_2)$ for all $x_1 \in U_1, x_2 \in U_2$. Let $c = \sup_{x_2 \in U_2} f(x_2)$. Then the hyperplane $T = \{x: f(x) = c\}$ separates U_1 and U_2.

Remark. The requirement that U_1 or U_2 be open in the statement of the corollary is essential (see Problem 301). See Problem 302 for another theorem on separation of convex sets.

4. Banach Spaces

Among all LTS's, Banach spaces are the most convenient to work with, and therefore the most frequently encountered (especially in problems connected with applications). Three fundamental principles of linear functional analysis are valid for them, of which one was discussed above (the Hahn–Banach theorem). The other two will be presented in §2. One can also prove for Banach spaces the inverse function theorem and the implicit function theorem, which form a basis for many results in nonlinear functional analysis. We present here a few elementary properties and constructions in Banach spaces. More detailed information can be found in the books [18], [27], and [8*].

Let us begin with finite-dimensional spaces. The most interesting example of a Banach space is the space $l_2(n, K)$, $K = \mathbf{R}$ or $\mathbf{C}$. It consists of all the vectors $x = (x_1, \ldots, x_n) \in K^n$, with the norm defined by the formula

$$\|x\| = \sqrt{\sum_{n=1}^{n} |x_n|^2}. \tag{4}$$

The space $l_p(n, K)$ is a natural generalization of the former space. It also consists of the vectors $x \in K^n$, but the norm is given by the formula

$$\|x\|_p = \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p}. \tag{5}$$

Passing to the limit as $p \to \infty$ in (5), it is not hard to show that $\|x\|_p$ approaches

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|. \tag{5'}$$

The expression on the right-hand side of (5) turns out to be a norm when p is in $[1, \infty]$, including the endpoints (cf. Problem 323). The proof of this assertion is not obvious and requires some preparation.

Call two numbers p and q in $[1, \infty]$ *conjugate* if one of the following equivalent conditions holds:

(1) $1/p + 1/q = 1$;
(2) $(p - 1)(q - 1) = 1$;
(3) $p + q = pq$.

Lemma (Hölder's Inequality). *Suppose that p and q are conjugate numbers in* $[1, \infty]$. *Then for any $x, y \in K^n$*

$$\left|\sum_{i=1}^{n} x_i y_i\right| \leq \|x\|_p \|y\|_q. \tag{6}$$

PROOF. If one of p or q is equal to ∞, then (6) is obvious. Let us consider the case when p and q are finite. We need the following auxiliary result: If $a \geq 0$ and $b \geq 0$, and p and q are conjugate, then $ab \leq a^p/p + b^q/q$. This inequality is easily proved analytically by computing the partial derivatives of the function $\varphi(x, y) = xy - x^p/p - y^q/q$. The geometric meaning of the inequality is clear from Fig. 1. It is obvious from the same picture that the inequality becomes an equality when $a = b^{q-1}$ (or $b = a^{p-1}$).

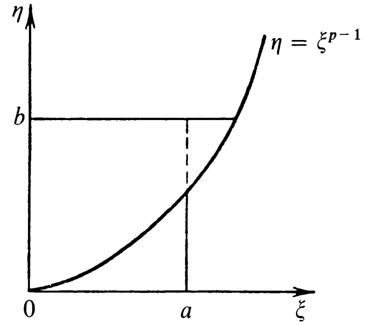


Figure 1

Since both sides of the inequality (6) are homogeneous in x and y, it suffices to consider the case $\|x\|_p = 1 = \|y\|_q$. (If one of the vectors x or y is equal to 0, then the inequality becomes obvious.) Let $a = |x_i|$, $b = |y_i|$ in the auxiliary inequality. The result is $|x_i y_i| \leq |x_i|^p/p + |y_i|^q/q$. Summing over i from 1 to n, we get

$$\sum_{i=1}^{n} |x_i y_i| \leq \frac{\|x\|_p^p}{p} + \frac{\|y\|_q^q}{q} = 1. \qquad \square$$

Remark. The following useful supplement to the Hölder inequality is clear from the proof of the lemma: For each nonzero $x \in K^n$ there is a nonzero $y \in K^n$ for which the Hölder inequality becomes an equality.

This implies the formula

$$\|x\|_p = \sup_{\|y\|_q \leq 1} \left| \sum_{i=1}^{n} x_i y_i \right|. \tag{7}$$

Indeed, the right-hand side of (7) does not exceed the left-hand side, by Hölder's inequality, and it attains this quantity by the remark made above.

Let X be a compact topological space. The linear space of continuous functions on X is denoted by $C(X)$. It is easy to see that this space becomes normed if we set

$$\|f\| = \sup_{x \in X} |f(x)|, \qquad f \in C(X).$$

Theorem 7. *Suppose that p and q are conjugate numbers. The expression $\|x\|_p$ is a norm when $p \in [1, \infty]$. The space $l_p(n, K)'$ is isomorphic to $l_q(n, K)$.*

Proof. Let $Y_q = \{y \in K^n, \|y\|_q \leq 1\}$, a compact subset of K^n. We construct a mapping from K^n to $C(Y_q)$ by the formula $x \mapsto f_x$, where $f_x(y) = \sum_{i=1}^n x_i y_i$. It follows from (7) that $\|x\|_p = \|f_x\|_{C(Y_q)}$. But $C(Y_q)$ is a normed space. Hence, $\|x\|_p$ is a norm in K^n. The second assertion of the theorem follows directly from (7) and the definition of the norm in the dual space. $\square$

The space $l_p(n, K)$ can be generalized. Let $l_p(K)$ be the space of sequences in K with the property that $\sum_{i=1}^\infty |x_i|^p < \infty$. (For $p = \infty$ let $l_\infty(K)$ be the space of bounded sequences in K.) We define

$$\|\{x_n\}\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} \quad \text{for } 1 \leq p < \infty,$$

$$\|\{x_n\}\|_\infty = \sup_n |x_n| \quad \text{for } p = \infty.$$

It can be shown (see Problems 324, 325) that the $l_p(K)$ are Banach spaces for any $p \in [1, \infty]$.

We now give several constructions of new Banach spaces from given ones:

(1) The completion of any normed space is a Banach space.
(2) If L is a Banach space and L_0 is a closed subspace of it, then L_0 itself is a Banach space.
(3) Let L_0 be a closed subspace of a Banach space L, and $L_1 = L/L_0$ the corresponding quotient space. Then L_1 is a Banach space with respect to the norm $\|x\|_{L_1} = \inf_{y \in x} \|y\|_L$ (see Problems 327, 328).
(4) Let L_1 and L_2 be two Banach spaces, and $L_1 \otimes L_2$ their algebraic *tensor product*, which is defined as follows (see also Problem 61): Denote the collection of all formal linear combinations of the symbols $x \square y$, $x \in L_1$, $y \in L_2$, by $L_1 \square L_2$, and let $L_1 \circ L_2$ be the subspace of $L_1 \square L_2$ generated by the expressions of the form

   (a) $(x_1 + x_2) \square y - x_1 \square y - x_2 \square y$,
   (b) $x \square (y_1 + y_2) - x \square y_1 - x \square y_2$,
   (c) $\lambda x \square \mu y - \lambda\mu(x \square y)$; $\lambda, \mu \in K$.

   Then $L_1 \otimes L_2 = (L_1 \square L_2)/(L_1 \circ L_2)$. For $x \in L_1$, $y \in L_2$ the equivalence class containing $x \square y$ is denoted by $x \otimes y$.

Remark. It is not true that each element of $L_1 \otimes L_2$ has the form $x \otimes y$ (see, for example, Problem 341).

If L_1 is finite-dimensional and has a basis $e_1, \ldots, e_n$, then it is easy to see that each element $a \in L_1 \otimes L_2$ can be expressed uniquely in the form $\sum_{k=1}^{n} e_k \otimes y_k$, where $y_k \in L_2$. If L_2 is also finite-dimensional and has a basis $f_1, \ldots, f_m$, then the elements $e_i \otimes f_j$, $1 \leq i \leq n$, $1 \leq j \leq m$, form a basis in $L_1 \otimes L_2$. A norm can be introduced in $L_1 \otimes L_2$ in various ways. It is natural to require that this norm p have the property that $p(x \otimes y) = p_1(x)p_2(y)$, where $x \in L_1$, $y \in L_2$, and p_1 and p_2 are the norms in the respective spaces L_1 and L_2. Such norms are called *cross-norms*.

The space $L_1' \otimes L_2'$ can be naturally mapped into $(L_1 \otimes L_2)'$: To the element $f_1 \otimes f_2 \in L_1' \otimes L_2'$ there corresponds the linear functional f on $L_1 \otimes L_2$ defined by $f(x \otimes y) = f_1(x) \cdot f_2(y)$. We require that the norm p in $L_1 \otimes L_2$ satisfy the conditions $p'(f) = p_1'(f_1)p_2'(f_2)$, where p', p_1', p_2' are the norms in the respective spaces $(L_1 \otimes L_2)'$, L_1', L_2'. Such norms are called *uniform cross-norms*. It turns out that among all the cross-norms there is a largest one, denoted by $p_1 \hat{\otimes} p_2$, and among the uniform cross-norms there is a smallest one, denoted by $p_1 \underset{\vee}{\otimes} p_2$. The completions of $L_1 \otimes L_2$ in these norms are denoted by $L_1 \hat{\otimes} L_2$ and $L_1 \underset{\vee}{\otimes} L_2$, respectively.

(5) If L_1 and L_2 are Banach spaces, then their direct sum $L_1 \oplus L_2$ can be equipped with a norm by the formula

$$\|x_1 \oplus x_2\| = \|x_1\|_1 + \|x_2\|_2.$$

The resulting space is also a Banach space. We remark that a topologically equivalent (but not isometric) space can be obtained by setting

$$\|x_1 \oplus x_2\| = \max(\|x_1\|_1, \|x_2\|_2).$$

§2. Linear Operators

1. The Space of Linear Operators

The linear topological spaces over a given field $K(K = \mathbf{R}$ or $\mathbf{C})$ form a category $\mathscr{L}_K$ whose morphisms are continuous linear mappings, usually called *continuous linear operators.* If L_1 and L_2 are two LTS's over K, then the collection of all continuous linear operators from L_1 to L_2 is denoted by $\mathscr{L}(L_1, L_2)$ (see Ch. III, §1.2). It is clear that $\mathscr{L}(L_1, L_2)$ is a linear space over K; if $L_1 = L_2 = L$, then $\mathscr{L}(L_1, L_2)$, frequently denoted by End L, is even an algebra over K.

The space $\mathscr{L}(L_1, L_2)$ can be endowed with various topologies. The following three are the most common.

1. *The Weak Operator Topology.* A base of neighborhoods of zero† in this topology is formed by the sets

$$U(x, f) = \{A \in \mathscr{L}(L_1, L_2)\colon |f(A(x))| < 1\}, \qquad x \in L_1, f \in L_2'.$$

It is easy to check that a sequence $\{A_n\}$ converges to A in the weak operator topology if and only if the sequence $\{A_n(x)\}$ converges to $A(x)$ in the weak topology of L_2 for any $x \in L_1$. This is written as follows: $A_n \rightharpoonup A$, or $A =$ w-lim A_n.

2. *The Strong Operator Topology.* A base of neighborhoods of zero in this topology is formed by the sets $U(x, V) = \{A \in \mathscr{L}(L_1, L_2)\colon Ax \in V\}$, where $x \in L_1$, and V is a neighborhood of zero in L_2. It is clear that the strong convergence of A_n to A is equivalent to the convergence of $A_n x$ to Ax in L_2 for any $x \in L_1$. This is written as follows: $A_n \to A$, or $A =$ s-lim A_n.

3. *The Uniform Topology.* Let L_1 and L_2 be normed spaces with norms p_1 and p_2. Then we can introduce a norm p in $\mathscr{L}(L_1, L_2)$ by the formula

$$p(A) = \sup_{x \neq 0} \frac{p_2(Ax)}{p_1(x)}.$$

† To define the topology in an LTS it suffices to specify a base of neighborhoods of zero. Any family of subsets containing zero whose translates form a base for the topology can be taken as such a system.

A base of neighborhoods of zero is formed by the sets

$$U(\varepsilon) = \{A \in \mathscr{L}(L_1, L_2) \colon p(A) < \varepsilon\}, \qquad \varepsilon > 0.$$

The convergence of a sequence A_n to A in the uniform topology (i.e., $p(A_n - A) \to 0$ as $n \to \infty$) is indicated by writing $A_n \Rightarrow A$, or $A = \text{u-lim}_{n\to\infty} A_n$.

A simple argument shows that the weak operator topology is weaker than the strong operator topology, and that the latter is weaker than the uniform topology. In the finite-dimensional case (i.e., when $\dim L_1 < \infty$, $\dim L_2 < \infty$) all three topologies coincide; this is no longer so in the infinite-dimensional case (cf. Problems 346–348).

There is a certain inconsistency between this (commonly accepted) terminology and the concepts of the weak and strong topologies in $L' = \mathscr{L}(L, K)$ introduced earlier (and also commonly accepted). Namely, if the linear functionals $f \in L'$ are regarded as operators from L to K, then the weak and strong operator topologies correspond to the weak-* topology in L', while the uniform operator topology corresponds to the strong topology in L'.

Linear functional analysis is based mainly on three cornerstone theorems—all associated with the name Stefan Banach. One of these, the Hahn–Banach theorem, was examined in §1. Here we present the other two.

Banach–Steinhaus Theorem. *Suppose that L_1 is a complete linear metric space, L_2 is a normed space, and $\{A_\gamma\}_{\gamma\in\Gamma}$ is a family of continuous linear operators from L_1 to L_2. If the set $\{A_\gamma x\}_{\gamma\in\Gamma}$ is bounded in L_2 (i.e., $\|A_\gamma x\| \le C(x)$ for all $\gamma \in \Gamma$) for each $x \in L_1$, then the family $\{A_\gamma\}$ is uniformly bounded on some ball with center at zero in L_1 (i.e., $\|A_\gamma x\| \le C$ for all $\gamma \in \Gamma$ and all x in a ball $B(0, r) = \{x \in L_1 : d(0, x) \le r\}$).*

Corollary. *Under the conditions of the theorem the family $\{A_\gamma\}$ is equicontinuous: For all $\varepsilon > 0$ there is a $\delta > 0$ such that $\|A_\gamma(x_1) - A_\gamma(x_2)\| < \varepsilon$ for all $\gamma \in \Gamma$ if $d(x_1, x_2) < \delta$.*

PROOF OF THE COROLLARY. Suppose that $\|A_\gamma x\| \le c$ on the ball $B(0, r)$. Choose δ small enough so that $B(0, \delta)$ is contained in the set $(\varepsilon/c)B(0, r)$. (This is possible because multiplication by a number is continuous in L_1.) If $d(x_1, x_2) < \delta$, then

$$\begin{aligned}\|A_\gamma(x_1) - A_\gamma(x_2)\| &= \|A_\gamma(x_1 - x_2)\| = (\varepsilon/c)\|A_\gamma((c/\varepsilon)(x_1 - x_2))\| \\ &\le (\varepsilon/c)\cdot c = \varepsilon,\end{aligned}$$

since

$$\frac{c(x_1 - x_2)}{\varepsilon} \in \frac{c}{\varepsilon} B(0, \delta) \subset B(0, r).$$

PROOF OF THE BANACH–STEINHAUS THEOREM. Suppose that the family $\{A_\gamma\}$ is unbounded on any ball of the form $B(0, r)$, $r > 0$. Then it is unbounded on any ball whatsoever. Indeed, let $B(x_1, r_1)$ be a ball in L_1. The translation invariance of the metric implies that $B(x_1, r_1) = x_1 + B(0, r_1)$. Since the family $\{A_\gamma\}$ is bounded on the vector x_1 and unbounded on the ball $B(0, r_1)$, it is unbounded on the ball $B(x_1, r_1)$.

A sequence of balls $B(x_n, r_n)$ and a sequence $\{\gamma_n\}$ of indices will now be constructed with the following properties:

(1) $B(x_{n+1}, r_{n+1}) \subset B(x_n, r_n)$;
(2) $r_{n+1} \leq r_n/2$;
(3) $\|A_{\gamma_n}(x)\| \geq n$ for all $x \in B(x_n, r_n)$.

We set $x_0 = 0$, $r_0 = 1$ and make use of the fact that the family $\{A_\gamma\}$ is unbounded on $B(x_0, r_0)$. Hence, there is an index γ_1 and an element $x_1 \in B(x_0, r_0)$ such that $\|A_{\gamma_1} x_1\| > 1$. By the continuity of A_{γ_1}, there is a number r_1 such that $\|A_{\gamma_1} x\| \geq 1$ for all $x \in B(x_1, r_1)$. Making r_1 smaller if necessary, we may assume that $r_1 < r_0/2$ and $B(x_1, r_1) \subset B(x_0, r_0)$. Suppose that the balls $B(x_k, r_k)$ and the indices γ_k have already been chosen for $k \leq n - 1$. Since $\{A_\gamma\}$ is unbounded on $B(x_{n-1}, r_{n-1})$, there is an index γ_n and an element $x_n \in B(x_{n-1}, r_{n-1})$ such that $\|A_{\gamma_n}(x_n)\| > n$. By the continuity of A_{γ_n}, there is a number r_n such that $\|A_{\gamma_n} x\| \geq n$ for all $x \in B(x_n, r_n)$. Making r_n smaller if necessary, we may assume that $r_n < r_{n-1}/2$, and $B(x_n, r_n) \subset B(x_{n-1}, r_{n-1})$.

Let us now use the completeness of L_1. Take a point x common to all the balls $B(x_n, r_n)$. Such a point exists, by the theorem on nested balls (see [1]). Then $\|A_{\gamma_n}(x)\| > n$ for any n, which contradicts the boundedness of the family $\{A_\gamma\}$ on the vector x. □

An important consequence of the Banach–Steinhaus theorem is the weak (sequential) completeness of $\mathscr{L}(L_1, L_2)$ when L_1 is a complete linear metric space and L_2 is a Banach space. In particular,

Theorem 8. *If L is a complete linear metric space, then the dual space L' is weakly complete.*†

Proof. Let $\{f_n\}$ be a weakly Cauchy sequence in L'. This means that for any $x \in L$ the numerical sequence $\{f_n(x)\}$ is Cauchy and, hence, has a limit, which we denote by $f(x)$. To prove the theorem it must be shown that $f \in L'$. The linearity of f is obtained by passing to the limit from the linearity of the f_n. The continuity of f follows from the corollary to the Banach–Steinhaus theorem. Indeed, the family $\{f_n\}$ of mappings from L to K is bounded on each vector $x \in L$. Hence, it is equicontinuous. Therefore, for any $\varepsilon > 0$ there is a $\delta > 0$ such that $|f_n(x)| < \varepsilon$ on the ball $B(0, \delta)$. Passing to the limit as $n \to \infty$, we get that $|f(x)| \leq \varepsilon$ on $B(0, \delta)$, which proves that f is continuous. □

† Here we have in mind completeness in the sense of weak operator convergence, i.e., weak-* convergence in the dual space.

Another useful consequence of the Banach–Steinhaus theorem is:

Theorem 9. *In a normed space L every weakly bounded set X (i.e., set such that $|f(x)| \leq c(f)$ for any $x \in X$ and $f \in L'$) is bounded.*

Proof. Let us regard the elements $x \in X$ as linear functionals on L'. By assumption, the family X is bounded on each $f \in L'$. Since L' is complete, the Banach–Steinhaus theorem implies that X is bounded on some ball $B(0, r)$ in L', i.e., $|f(x)| \leq c$ for all $x \in X$ and all $f \in B(0, r)$. By Corollary 3 to the Hahn–Banach theorem, it follows from this that $\|x\| \leq c/r$ for all $x \in X$, i.e., X is bounded.

The third fundamental principle of linear functional analysis is the

Banach Inverse Mapping Theorem. *Suppose that L_1 and L_2 are complete linear metric spaces, and A is a continuous linear operator mapping L_1 one-to-one onto L_2. Then the inverse operator $A^{-1}: L_2 \to L_1$ is continuous.*

Proof. It must be shown that for any $\varepsilon > 0$ the image of a ball $B(0, r) \subset L_1$ under A contains a neighborhood of zero in L_2. We make use of the fact that for any $\varepsilon > 0$ the union $\bigcup_{n=1}^{\infty} n\mathring{B}(0, \varepsilon)$ contains L_1, Therefore, the union of the images $X_n = A(n\mathring{B}(0, \varepsilon))$ contains L_2. Let $\overline{X}_n$ be the closure of X_n. Then some $\overline{X}_n$ contains the whole of some ball of positive radius in L_2. For, otherwise, the complement Y_n of each $\overline{X}_n$ in L_2 would be a dense open set. By Problem 32, the intersection $\bigcap_{n=1}^{\infty} Y_n$ would be dense; but it is actually empty. Thus, there exist $n \in \mathbf{N}$, $x_0 \in L_2$, and $r_0 > 0$ such that $\overline{X}_n \supset \mathring{B}(x_0, r_0)$. This means that the closure of the image of $\mathring{B}(0, \varepsilon)$ contains the ball $B(x_0/n, r_0/n)$. Let ε be small enough that $\mathring{B}(0, \varepsilon) - \mathring{B}(0, \varepsilon) \subset \mathring{B}(0, r)$. There is such a number, because $(x, y) \mapsto x - y$ is a continuous mapping. Hence the closure of the image of $\mathring{B}(0, \varepsilon)$ contains the ball $\mathring{B}(x_0/n, r_0/n)$. Therefore, the closure of the image of $\mathring{B}(0, r)$ contains the set $\mathring{B}(x_0/n, r_0/n) - \mathring{B}(x_0/n, r_0/n)$, which, in turn, contains $\mathring{B}(0, r_0/n)$.

Thus, the image of any ball $\mathring{B}(0, r) \subset L_1$, $r > 0$, is dense in some ball of the form $\mathring{B}(0, \rho) \subset L_2$. Suppose that the image of $\mathring{B}(0, r/2^n) \subset L_1$ is dense in $\mathring{B}(0, \rho_n) \subset L_2$, $n = 1, 2, \ldots$. Without loss of generality it can be assumed that $\rho_n \to 0$. Let us show that the image of $\mathring{B}(0, r)$ contains $\mathring{B}(0, \rho_1)$. Suppose that $y \in \mathring{B}(0, \rho_1)$. Since the image of $\mathring{B}(0, r/2)$ is dense in $\mathring{B}(0, \rho_1)$, there is a vector $x_1 \in \mathring{B}(0, r/2)$ such that $d(y, Ax_1) < \rho_2$. Next, since the image of $\mathring{B}(0, r/4)$ is dense in $\mathring{B}(0, \rho_2)$, there is a vector $x_2 \in \mathring{B}(0, r/4)$ such that $d(y - Ax_1, Ax_2) < \rho_3$, and so on. The series $\sum x_n$ converges in L_1 to some vector $x \in \mathring{B}(0, r)$. We have $d(y, Ax) = \lim_{n\to\infty} d(y, \sum Ax_n) = 0$, so $y = Ax$. □

This theorem is frequently employed in the following situation. Suppose that two norms p_1 and p_2 are given in the space L, and that, moreover, $p_2 \leq cp_1$ and L is complete in each norm. Then the norms p_1 and p_2 are equivalent, i.e., $p_1 \leq c'p_2$ for some constant c'. (For a proof it suffices to consider the identity operator from (L, p_1) into (L, p_2).)

A similar argument can be applied to two countable families of seminorms turning L into a complete metric space; if one system of seminorms majorizes the other, then they determine the same topology. This fact is widely used in the theory of generalized functions.

2. Compact Sets and Compact Operators

A set A in a topological space X is said to be *compact* if any covering of A by a system of open sets contains a finite subcovering; A is said to be *pre-compact* if its closure is compact.

The following definition is more convenient in the case when X is a metric space. A set $A \subset X$ is said to be compact if any sequence $\{a_n\}$ of elements of A contains a subsequence $\{a_{n_k}\}$ that converges to some element $a \in A$. For a proof of the equivalence of these definitions we refer the reader to any detailed text on analysis, for example, [42]. (For a more general result, see Problem 369.)

As a rule, we shall consider sets lying in complete metric spaces in what follows. A very useful pre-compactness criterion of Hausdorff in terms of ε-nets is valid for them (a set A is called an *ε-net* for a set B if for each $b \in B$ there is a point $a \in A$ whose distance from b is not greater than ε).

Theorem 10 (Hausdorff Criterion). *Suppose that X is a complete metric space, and A is a subset of X. Then A is pre-compact if and only if A has a finite ε-net for any finite $\varepsilon > 0$.*

PROOF. *Necessity.* Suppose that A does not have a finite ε-net for some $\varepsilon > 0$. We choose an arbitrary point $a_1 \in A$ and construct inductively a sequence $\{a_n\}$ having the property that $d(a_i, a_j) \geq \varepsilon$ for $i \neq j$. Suppose that the block $a_1, \dots, a_n$ has already been constructed. Since this block is not an ε-net for A, there exists a point $a \in A$ whose distance from every a_i, $1 \leq i \leq n$, is at least ε. This point is taken to be a_{n+1}. The so-constructed sequence $\{a_n\}$ obviously does not contain a subsequence converging to an element of $\bar{A}$, which contradicts the pre-compactness of A.

Sufficiency. Suppose that A has a finite ε-net for all $\varepsilon > 0$, and let $\{a_n\}$ be a sequence in A. If $\{b_1, \dots, b_m\}$ is a finite 1-net for A, then the whole set A is covered by the balls $\mathring{B}(b_i, 1)$, $i = 1, \dots, m$. Hence, one of these balls contains infinitely many terms of the sequence $\{a_n\}$. We choose such a ball and take a finite covering of it by balls of radius 1/2. (Such a covering exists, because A has a finite (1/2)-net.) At least one of them contains infinitely many terms of the sequence $\{a_n\}$. We cover this ball by finitely many balls of radius 1/4, and so on. As a result, a shrinking system of balls B_n of radius 2^{-n}

is obtained, and each of them contains infinitely many terms of $\{a_n\}$. Let $\{a_{n_k}\}$ be a subsequence such that $a_{n_k} \in B_k$. This subsequence is Cauchy and, consequently, converges to some point in the closure of A. □

Corollary. *In a finite-dimensional normed space L, pre-compactness is equivalent to boundedness.*

Proof. If A is pre-compact, then $\bar{A}$ is a compact and, hence, bounded set. Thus, A is also bounded. Conversely, suppose that A is bounded. We construct explicitly a finite ε-net for A. Let $x_1, \ldots, x_n$ be the coordinates in L. Since A is bounded, there is a number C such that $|x_i| \leq C$, $1 \leq i \leq n$, for all points $x \in A$ (cf. Problem 256). Let R be the radius of the smallest ball in L containing the unit cube $\{x \in L\colon |x_i| \leq 1, 1 \leq i \leq n\}$. We choose a number M so large that $R/M < \varepsilon$. The set of points of the form $(k_1/M, \ldots, k_n/M)$ can be taken as the desired ε-net, where the k_i are integers between $-MC$ and $+MC$.

Remark. It is easy to see that the number of elements in the ε-net constructed equals $(2MC)^n$, i.e., it has order $O(\varepsilon^{-n})$ as $\varepsilon \to 0$. The exponent n here brings to mind the dimension of the space L in which our compact set lies. The asymptotic behavior of the number $N(\varepsilon)$ of elements in a minimal ε-net for a pre-compact set A as $\varepsilon \to 0$ is an important and interesting characteristic of A. In particular, if $N(\varepsilon) \sim C \cdot \varepsilon^{-\gamma}$, then A is said to have *approximation dimension* γ. It can be shown that there are pre-compact sets of any approximation dimension between 0 and n in an n-dimensional normed space.

In infinite-dimensional spaces boundedness is, as a rule, no longer sufficient for a set to be pre-compact.

Theorem 11. *Let L be an infinite-dimensional normed linear space. Then the unit ball $B = \{x \in L\colon \|x\| < 1\}$ of L is not a pre-compact set.*

Proof. Suppose that B were pre-compact. Then it could be covered by finitely many balls $B_1, \ldots, B_N$ of radius $r < 1$. Consider an n-dimensional subspace L_n of L containing the centers of these balls. Such a subspace certainly exists for $n \geq N$. Let $\tilde{B}, \tilde{B}_1, \ldots, \tilde{B}_N$ be the intersections of the balls $B, B_1, \ldots, B_N$ with L_n. It is clear that the sets $\tilde{B}, \tilde{B}_i$ are balls in L_n of radii 1 and r, respectively. Let μ be the Lebesgue measure in L_n, normalized by the condition $\mu(\tilde{B}) = 1$. Then $\mu(\tilde{B}_i) = r^n$. Since B is contained in the union of the B_i, $1 \leq i \leq N$, the inequality $Nr^n \geq 1$ holds. But this is impossible for $r < 1$ and sufficiently large n. □

The following result provides a measure of compensation.

Theorem 12. *Every weakly bounded subset of a reflexive normed space L is weakly pre-compact.*

We prove this theorem under the assumption that L' is separable, i.e., contains a countable dense subset $\{f_n\}$. In this case the weak topology is metrizable on each weakly bounded set $X \subset L$. Indeed, if X is weakly bounded, then it is also strongly bounded, by Theorem 9. Hence, X is contained in a ball of radius r. Let a countable family of seminorms be defined by $p_n(x) = |f_n(x)|$. We show that this family determines the weak topology on X. The base taken for the weak topology in X consists of sets of the form

$$U(x, f) = \{y \in X \mid f(x - y)| < 1\}, \qquad x \in L, f \in L',$$

along with their finite intersections.

Let f_i be a functional in the dense subset $\{f_n\}$ such that $\|f - f_i\| < 1/(4r)$. Then the subset of $y \in X$ for which $p_i(y - x) < 1/2$ is contained in $U(x, f)$, since $|f(x - y)| = |f_i(x - y) + (f - f_i)(x - y)| < 1/2 + (1/(4r)) \cdot 2r = 1$.

Let us now show that X is weakly pre-compact. To do this we verify that each sequence $\{x_n\} \subset X$ contains a subsequence converging weakly in the closure of X. Since the numerical sequence $f_i(x_n)$ is bounded for each i, the standard diagonal process can be used to choose a subsequence converging in each seminorm p_i. Since the family $\{p_i\}$ determines the weak topology on X, this subsequence converges weakly. □

A very interesting and beautiful area of linear functional analysis is the theory of compact convex sets. Here we present only the most striking and useful result from this theory: the Krein–Mil'man theorem on extreme points.

A point x in a convex subset K of an LTS is said to be *extreme* (or extremal) if it is not the midpoint of a segment lying entirely in K. For example, the extreme points of a closed ball in a Euclidean space are precisely the points of the sphere bounding it; the extreme points of a closed cube are its vertices; an open set has no extreme points.

Theorem 13 (Krein–Mil'man). *Let L be an* LCS, *K a compact convex subset of L, and E the collection of extreme points of K. Then K coincides with the closure of the convex hull of E.*

See Problems 371–375 for a proof.

One way this theorem can be used is to show that various Banach spaces are not isomorphic (cf. Problems 376–378); another application is De Branges' elegant proof of the Stone–Weierstrass theorem in [27].

Compactness criteria for sets in diverse concrete spaces will be given below, but one case, the space $C(X)$, will be examined here, since it is used in the general theory of normed spaces.

Theorem 14 (Arzelà–Ascoli). *Let $C(X)$ be the normed space of real continuous functions on a compact metric space X, with the norm $\|f\| = \max_x |f(x)|$. A family $A \subset C(X)$ is pre-compact if and only if it is*

(1) *uniformly bounded (i.e., there is a constant C such that $|f(x)| \leq C$ for $f \in A$), and*
(2) *equicontinuous (i.e., for all $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $f \in A$ if $d(x, y) < \delta$).*

PROOF. *Necessity.* Suppose that A is pre-compact. Then it admits a finite $(\varepsilon/3)$-net $f_1, \ldots, f_N$. Each f_i is continuous and, consequently, bounded on X, which implies that A is uniformly bounded. Each f_i is uniformly continuous on X. Therefore, there is a $\delta_i > 0$ such that $|f_i(x) - f_i(y)| < \varepsilon/3$ if $d(x, y) < \delta_i$. The standard $\varepsilon/3$ argument applied to the quadruple $f(x), f_i(x), f_i(y), f(y)$ now shows the required property of equicontinuity for $\delta = \min(\delta_1, \ldots, \delta_N)$.

Sufficiency. We construct explicitly a finite ε-net for A if it is known that A is uniformly bounded and equicontinuous. Suppose that δ is chosen so that $|f(x) - f(y)| < \varepsilon/3$ for all $f \in A$ if $d(x, y) < \delta$. There is a δ-net $S = \{x_1, \ldots, x_n\}$ on X. Let us regard the restriction of a function $f \in A$ to S as a vector in the space $l_\infty(n, \mathbf{R})$. The image of A in $l_\infty(n, \mathbf{R})$ is a bounded set $\tilde{A}$. Hence, $\tilde{A}$ is pre-compact and has a finite $(\varepsilon/3)$-net $\tilde{f}_1, \ldots, \tilde{f}_N$. We show that the set of functions $f_1, \ldots, f_N$ is an ε-net in A. Let $f \in A$. The restriction $\tilde{f}$ of f to S is within $\varepsilon/3$ of some $\tilde{f}_i$ in the metric of $l_\infty(n, \mathbf{R})$. We estimate the distance between f and f_i in $C(X)$. Let x be any point in X and $x_k \in S$ the element of S closest to it. Then $d(x, x_k) < \delta$. Therefore, $|f(x) - f(x_k)| < \varepsilon/3$ and $|f_i(x) - f_i(x_k)| < \varepsilon/3$. Moreover, $|f(x_k) - f_i(x_k)| < \varepsilon/3$, by the choice of f_i. Hence, $\|f - f_i\| < \varepsilon$. □

For a more general result see Problem 379.

Definition. An operator A acting from a normed space L_1 to a normed space L_2 is said to be *compact* (or *completely continuous*) if it carries any bounded set into a pre-compact set.

This concept was introduced by Hilbert for studying integral operators. Any bounded operator of *finite rank* (i.e., an operator with a finite-dimensional range) is a compact operator, since every bounded subset of a finite-dimensional space is pre-compact. The collection of all compact operators from L_1 to L_2 will be denoted by $\mathscr{K}(L_1, L_2)$.

Theorem 15. (1) *$\mathscr{K}(L_1, L_2)$ is a norm-closed subspace of $\mathscr{L}(L_1, L_2)$.*

(2) *If $A \in \mathscr{L}(L_0, L_1)$, $B \in \mathscr{K}(L_1, L_2)$, and $C \in \mathscr{L}(L_2, L_3)$, then $C \circ B \circ A \in \mathscr{K}(L_0, L_3)$; in particular, $\mathscr{K}(L, L)$ is an ideal in $\mathscr{L}(L, L)$.*

(3) *If $A \in \mathscr{K}(L_1, L_2)$, then the adjoint operator $A' \colon L_2' \to L_1'$ is in $\mathscr{K}(L_2', L_1')$.*

PROOF. (1) Suppose that A and B are compact operators from L_1 to L_2, and X is a bounded subset of L_1. The sets AX and BX are pre-compact. Therefore, so is the set $\alpha AX + \beta BX$. Hence, $\alpha A + \beta B \in \mathscr{K}(L_1, L_2)$. Next, suppose that $A_n \in \mathscr{K}(L_1, L_2)$ and $A_n \Rightarrow A$ as $n \to \infty$. We show that AX is a pre-compact set. Let $\varepsilon > 0$. Choose n such that $\|A - A_n\| < \varepsilon/(2R)$, where $R = \sup_{x \in X} \|x\|$. Then $A_n X$ is an $(\varepsilon/2)$-net for AX, since $\|Ax - A_n x\| < \varepsilon/2$ for $x \in X$. By assumption, $A_n X$ is pre-compact and, consequently, has a finite $(\varepsilon/2)$-net S. Clearly, S is an ε-net for AX.

(2) Let X be a bounded set in L_0. Then AX is a bounded set in L_1, and $B \circ AX$ is a pre-compact set in L_2. Finally, $C \circ B \circ AX$ is a pre-compact set in L_3, because if S is a finite $(\varepsilon/\|C\|)$-net for $B \circ AX$, then CS is a finite ε-net for $C \circ B \circ AX$.

(3) Suppose that $A \in \mathscr{K}(L_1, L_2)$, and M is a bounded set in L_2'. We show that $A'M \subset L_1'$ is a pre-compact set. To do this we construct an isometric imbedding of this set into a certain space $C(X)$. Namely, X can be taken to be the closure of the set AB, where B is the unit ball in L_1 (AB is pre-compact, since A is a compact operator and B is a bounded set). To a functional $f \in A'M$ we assign a function $\tilde{f}(x)$ on X by the formula $\tilde{f}(x) = g(x)$, where $g \in M$ is chosen so that the equality $f = A'g$ holds. (This property does not, generally speaking, determine g uniquely, but if $f = A'g_1 = A'g_2$, then the values $g_1(x)$ and $g_2(x)$ coincide for points $x \in AB$. Thus, the correspondence $f \mapsto \tilde{f}$ is unambiguous, and is obviously an imbedding.) We show that the correspondence is isometric. Indeed,

$$\begin{aligned} \|\tilde{f}\|_{C(X)} &= \max_{x \in X} |\tilde{f}(x)| = \sup_{x \in AB} |g(x)| = \sup_{y \in B} |g(Ay)| \\ &= \sup_{y \in B} |A'g(y)| = \sup_{y \in B} |f(y)| = \|f\|_{L_1'}. \end{aligned}$$

It remains to verify that the functions $\tilde{f}$ corresponding to functionals $f \in A'M$ form a uniformly bounded and equicontinuous family. This follows from the estimates

$$|\tilde{f}(x)| \le \|f\|_{L_1'} \le \|A\| \, \|g\|_{L_2'} \le \|A\| \cdot \operatorname{diam} M,$$

$$|\tilde{f}(x) - \tilde{f}(y)| \le \|g\|_{L_2} \|x - y\|_{L_2} \le \operatorname{diam} M \cdot \|x - y\|_{L_2},$$

where $\operatorname{diam} M = \sup_{g \in M} \|g\|_{L_2'}$, □

The next theorem describes one of the most useful properties of compact operators.

Theorem 16. *Compact operators carry weakly convergent sequences into strongly convergent sequences.*

PROOF. Let $A\colon L_1 \to L_2$ be a compact operator, and suppose that $x_n \rightharpoonup x$ in L_1. By Theorem 9, the sequence $\{x_n\}$ is bounded in the norm. Therefore, $\{Ax_n\}$ is pre-compact. Hence, there is a subsequence $\{Ax_{n_k}\}$ that converges to some vector $y \in L_2$. We show that the whole sequence $\{Ax_n\}$ also converges to y. For otherwise there would be an $\varepsilon > 0$ and an infinite subsequence $\{Ax_{m_k}\}$ having the property that $\|Ax_{m_k} - y\| \geq \varepsilon$. Since $\{Ax_{m_k}\}$ is a pre-compact set, it has a subsequence converging to some vector $z \in L_2$. It can be assumed that $\{Ax_{m_k}\}$ is this subsequence. Clearly, $\|y - z\| \geq \varepsilon$. By Corollary 1 to the Hahn–Banach theorem, there exists a linear functional $f \in L_2'$ such that $f(y) \neq f(z)$. Let $\varphi = A'f \in L_1'$. Then $\varphi(x_{n_k}) = f(Ax_{n_k}) \to f(y)$, $\varphi(x_{m_k}) = f(Ax_{m_k}) \to f(z)$, which contradicts the convergence of the sequence $\varphi(x_n)$. □

Remark. In reflexive spaces with a separable dual this property characterizes compact operators: Every bounded operator carrying weakly convergent sequences into strongly convergent sequences is compact. For in this case the weak topology is metrizable on bounded sets and, therefore, is determined there by convergence of sequences. Thus, our operator is continuous if L_1 is equipped with the weak topology and L_2 with the strong topology. Since bounded sets are weakly pre-compact, their images are strongly pre-compact.

3. The Theory of Fredholm Operators

Let L_1 and L_2 be Banach spaces and $T \in \mathscr{L}(L_1, L_2)$. The equation

$$T(x) = y, \qquad x \in L_1, \qquad y \in L_2, \tag{8}$$

is a natural generalization of a system of linear algebraic equations to the infinite-dimensional case. It turns out that under certain additional assumptions the theory of such systems is almost completely analogous to the finite-dimensional theory. However, there are some differences. Besides more complicated proofs, the infinite-dimensional situation gives rise to a new concept: the index of a linear operator. We need certain preliminaries in order to introduce this concept. Let ker T denote the *kernel* of the operator T, i.e., the collection of all solutions of the equation

$$Tx = 0, \qquad x \in L_1. \tag{9}$$

The *range* (or image) of T, i.e., the collection of $y \in L_2$ for which Eq. (8) can be solved, is denoted by im T. It is clear that ker T is a closed subspace (as the pre-image of a point under a continuous mapping). However, the set im T is not always closed (see Problem 392). Along with T we shall consider the adjoint operator $T' \in \mathscr{L}(L_2', L_1')$ and the corresponding equations

$$T'g = f, \qquad g \in L_2', \qquad f \in L_1', \tag{10}$$

$$T'g = 0, \qquad g \in L_2'. \tag{11}$$

If im T and im T' are closed subspaces, then they determine the Banach spaces

$$\operatorname{coker} T = L_2/\operatorname{im} T \quad \text{and} \quad \operatorname{coker} T' = L_1'/\operatorname{im} T'.$$

They are called the *cokernels* of T and T', respectively. Let

$$\alpha(T) = \dim \ker T, \qquad \beta(T) = \dim \operatorname{coker} T,$$
$$i(T) = \alpha(T) - \beta(T).$$

An operator T is called a *Fredholm* operator if the numbers $\alpha(T)$ and $\beta(T)$ are finite. In this case the number $i(T)$ is called the *index* of T.

In the finite-dimensional case, when $\dim L_1 = N_1$ and $\dim L_2 = N_2$, it is easy to check that

$$N_1 - \alpha(T) = N_2 - \beta(T) = \operatorname{rank} T,$$
$$N_2 - \alpha(T') = N_1 - \beta(T') = \operatorname{rank} T', \tag{12}$$

which, combined with the equality $\operatorname{rank} T = \operatorname{rank} T'$ (the rank theorem for matrices), yield the relations

$$\alpha(T) = \beta(T'), \qquad \beta(T) = \alpha(T'), \qquad i(T) = -i(T'). \tag{13}$$

The purpose of this section is to prove the relations (13) in the infinite-dimensional case (for Fredholm operators) and to give convenient criteria for the computation of the index and for the solvability of Eqs. (8)–(11).

Suppose that we are given a sequence of linear spaces and linear operators:

$$\cdots \to L_{k-1} \xrightarrow{T_k} L_k \xrightarrow{T_{k+1}} L_{k+1} \to \cdots \tag{14}$$

This sequence is said to be *exact at the term* L_k if $\operatorname{im} T_k = \ker T_{k+1}$. The sequence (14) is said to be *exact* if it is exact at each term. It is clear that exactness at the term L_k implies that $T_{k+1} \circ T_k = 0$. The latter property has come to be called *semi-exactness* (if also $\operatorname{im} T_k$ is closed). If the sequence (14) is semi-exact at the term L_k, then $\operatorname{im} T_k \subset \ker T_{k+1}$. The quotient space $H_k = \ker T_{k+1}/\operatorname{im} T_k$ is a measure of the "deviation from exactness" at L_k. It is called the kth *cohomology space* of the sequence (14). If $H_k = \{0\}$ for all k, then (14) is an exact sequence.

We shall be interested in the case when all the L_k are Banach spaces, and the T_k are continuous operators. In this case the main result is

Theorem 17. *Suppose that* (14) *is an exact sequence of Banach spaces and continuous operators. Then the dual sequence*

$$\cdots \leftarrow L'_{k-1} \xleftarrow{T'_k} L'_k \xleftarrow{T'_{k+1}} L'_{k+1} \leftarrow \cdots \tag{15}$$

is also exact.

Proof. Let us first consider a particular case. Namely, suppose that (14) has the form

$$0 \to L_1 \xrightarrow{T} L_2 \to 0, \tag{16}$$

where 0 denotes the trivial (zero-dimensional) space, and, correspondingly, (15) has the form

$$0 \leftarrow L'_1 \xleftarrow{T'} L'_2 \leftarrow 0. \tag{17}$$

The exactness of (16) means that $\ker T = \{0\}$ and $\operatorname{im} T = L_2$, i.e., T is an isomorphism of L_1 and L_2 as linear (but not Banach!) spaces. By the Banach inverse mapping theorem, T^{-1} is continuous, and, consequently, T realizes a topological isomorphism (a linear homeomorphism) of the Banach spaces L_1 and L_2. Therefore, T' is a topological isomorphism of L_2' and L_1'. This implies the exactness of (17). Thus, the validity of Theorem 17 in this simplest particular case follows from Banach's theorem. It can be verified that the converse is also true: The theorem of Banach is a consequence of the case of Theorem 17 considered.

Let us return to the general case. It is clear that the dual sequence is semi-exact, since $T_k' \circ T_{k+1}' = (T_{k+1} \circ T_k)' = 0$. It remains to prove that $\operatorname{im} T_{k+1}' \supset \ker T_k'$. Let $f \in \ker T_k'$. This means that the functional $f \in L_k'$ vanishes on $\operatorname{im} T_k = \ker T_{k+1}$. Hence, it determines a linear functional F_0 on the subspace $\operatorname{im} T_{k+1} \subset L_{k+1}$ according to the formula $F_0(T_{k+1}(x)) = f(x)$. There are two norms on the space $\operatorname{im} T_{k+1}$: one is induced from L_{k+1}, and the other is carried over from $L_k/\ker T_{k+1}$ by means of T_{k+1}. Since T_{k+1} is bounded, the first norm is majorized by the second. By the Banach theorem, these norms must be equivalent. Hence, F_0 is continuous in the topology of L_{k+1} and, by the Hahn–Banach theorem, admits continuous extension to a functional $F \in L_{k+1}'$. Clearly $T_{k+1}'F = f$, i.e. $f \in \operatorname{im} T_{k+1}'$. □

This theorem has a converse and can be generalized (see Problems 397, 398).

Theorem 18. *Let T be a Fredholm operator in $\mathscr{L}(L_1, L_2)$. Then $T' \in \mathscr{L}(L_2', L_1')$ is also a Fredholm operator, and Eqs.* (13) *hold.*

Proof. By the definition of the spaces $\ker T$ and $\operatorname{coker} T$, the sequence

$$0 \to \ker T \xrightarrow{i} L_1 \xrightarrow{T} L_2 \xrightarrow{p} \operatorname{coker} T \to 0.$$

is exact, where i is the imbedding and p is the natural projection. By Theorem 17, this implies the exactness of the sequence

$$0 \leftarrow (\ker T)' \xleftarrow{i'} L_1' \xleftarrow{T'} L_2' \xleftarrow{p'} (\operatorname{coker} T)' \leftarrow 0.$$

But this means that we have isomorphisms

$$\ker T' \approx (\operatorname{coker} T)', \qquad \operatorname{coker} T' \approx (\ker T)'. \tag{18}$$

Thus, T' is Fredholm, and Eqs. (12) and (13) are valid. □

Any invertible operator $T \in \mathscr{L}(L_1, L_2)$ (i.e., such that there is an $S \in \mathscr{L}(L_2, L_1)$ with the property that $T \circ S = 1_{L_2}$, $S \circ T = 1_{L_1}$) is a Fredholm operator. In this case $\alpha(T) = \beta(T) = 0$. It turns out that all Fredholm operators are close in a certain sense to being invertible operators.

An operator $T \in \mathscr{L}(L_1, L_2)$ is said to be *almost invertible* if there exist operators S_1 and S_2 in $\mathscr{L}(L_2, L_1)$ such that

$$S_1 \circ T = 1_{L_1} + K_1, \qquad T \circ S_2 = 1_{L_2} + K_2, \tag{19}$$

where $K_1 \in \operatorname{End} L_1$ and $K_2 \in \operatorname{End} L_2$ are compact operators.

Theorem 19 (S. M. Nikol'skii). *Every Fredholm operator is almost invertible. Moreover, S_1 and S_2 can be chosen in such a way that K_1 and K_2 are operators of finite rank.*

PROOF. We show that there exist a closed subspace $M \subset L_1$ and a finite-dimensional subspace $N \subset L_2$ such that $L_1 = \ker T \oplus M$ and $L_2 = \operatorname{im} T \oplus N$. Choose a basis $x_1, \ldots, x_{\alpha(T)}$ in $\ker T$ and the dual basis $f_1, f_2, \ldots, f_{\alpha(T)}$ in $(\ker T)'$. (Duality of these bases means that $f_i(x_j) = \delta_{ij}$ for† $i, j = 1, \ldots, \alpha(T)$.) By the Hahn–Banach theorem, the functionals f_i can be extended to continuous linear functionals $F_i \in L_1'$; let $M = \bigcap_{i=1}^{\alpha(T)} \ker F_i$.

Then each $x \in L_1$ has a unique representation in the form

$$x = \sum_{i=1}^{\alpha(T)} c_i x_i + y, \qquad \text{where } y \in M. \tag{20}$$

Indeed, applying F_i to both sides of the equality, we see that $c_i = F_i(x)$. Conversely, for this choice of the coefficients c_i the vector y must lie in M. Hence, $L_1 = \ker T \oplus M$.

Suppose now that $\tilde{z}_1, \ldots, \tilde{z}_{\beta(T)}$ is a basis in $\operatorname{coker} T = L_2/\operatorname{im} T$, and let z_i be a representative of the class $\tilde{z}_i$. The linear span of the vectors z_i, $1 \le i \le \beta(T)$, is denoted by N. Let z be any vector in L_2, and $\tilde{z}$ its image in $\operatorname{coker} T$. Then it has a unique expansion $\tilde{z} = \sum_{i=1}^{\beta(T)} c_i \tilde{z}_i$, and this implies the existence and uniqueness of a representation of z in the form

$$z = \sum_{i=1}^{\beta(T)} c_i z_i + t, \qquad \text{where } t \in \operatorname{im} T. \tag{21}$$

Hence, $L_2 = \operatorname{im} T \oplus N$. By Banach's theorem, the operator $T|_M$ realizes a topological isomorphism of M and $\operatorname{im} T$. Let $\check{T}$ be the operator inverse to $T|_M$, and define $S \in \mathscr{L}(L_2, L_1)$ by

$$S(z) = \check{T}(t), \tag{22}$$

where $z \in L_2$ and $t \in \operatorname{im} T$ are connected by Eq. (21). The operators ST and TS can easily be computed in explicit form with the help of the expansions (20) and (21):

$$S \circ T(x) = S \circ T\left(\sum_{i=1}^{\alpha(T)} c_i x_i + y\right) = S \circ T(y) = y,$$

$$T \circ S(z) = T \circ S\left(\sum_{j=1}^{\beta(T)} c_j z_j + t\right) = T \circ \check{T}(t) = t.$$

† As usual, δ_{ij} denotes the Kronecker symbol: $\delta_{ii} = 1$, $\delta_{ij} = 0$ for $i \neq j$.

From this,

$$K_1 = ST - 1_{L_1}: x \mapsto -\sum_{i=1}^{\alpha(T)} F_i(x)x_i, \qquad K_2 = TS - 1_{L_2}: z \mapsto -\sum_{j=1}^{\beta(T)} c_j z_j,$$

where the c_j are the coordinates of the vector $\tilde{z}$ in the basis $\{\tilde{z}_i\}$. □

We remark that the operators K_1 and K_2 constructed in the proof of the theorem are projections (onto ker T, parallel to M and onto N, parallel to im T, respectively).

Theorem 20 (F. Riesz). *If $K \in \operatorname{End} L$ is a compact operator, then $T = 1 - K$ is a Fredholm operator.*

PROOF. The space ker T consists of all the vectors x for which $K(x) = x$. Therefore, K is simultaneously compact and the identity operator in ker T. This implies that ker T is finite-dimensional (cf. Problem 381).

As in the proof of Theorem 19, let $M \subset L$ be a closed algebraic complement to ker T. The operator T maps M one-to-one onto im T. Let $\check{T}$ be the inverse mapping. We claim that $\check{T}$ is bounded (i.e., $\|\check{T}(y)\|/\|y\| \leq c < \infty$ for all $y \in \operatorname{im} T$). Otherwise, there would be a sequence of unit vectors $\{x_n\} \subset M$ such that $y_n = T(x_n) \to 0$. But $T(x_n) = x_n - K(x_n)$. The sequence $\{x_n\}$ is bounded, so, the sequence $\{K(x_n)\}$ is pre-compact. Passing, if necessary, to a subsequence, we can assume that $\{K(x_n)\}$ has a limit point x. Since $T(x_n) \to 0$, it follows that $x_n \to x$. The vector x lies in M (since M is closed), has unit norm (as the limit of the x_n), and has the property that $T(x) = 0$ (since T is continuous and $T(x_n) \to 0$). Hence, $x \in M \cap \ker T$, which is impossible.

Now suppose that $y_n \in \operatorname{im} T$ and $y_n \to y$. Then the sequence y_n is Cauchy, and, hence, so is the sequence $x_n = \check{T}(y_n)$. Since M is complete (see Theorem 2 in Ch. I), $x = \lim_{n\to\infty} x_n$ exists. Then $T(x) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} y_n = y$, which shows that im T is closed. It remains to show that $\beta(T) < \infty$, i.e., coker T is finite-dimensional. This follows from the relations (18) and the already-proved assertion about the finite dimensionality of the kernel, applied to the operator T'. (We use here the theorem that K' is compact if K is.) □

From the foregoing we now derive the

Fredholm Operator Criterion. *$T \in \mathscr{L}(L_1, L_2)$ is a Fredholm operator if and only if it is almost invertible.*

PROOF. The "only if" part of the criterion was proven as Theorem 19.

Suppose that T is almost invertible, i.e., Eqs. (19) hold. Then $\ker T \subset \ker S_1 \circ T = \ker(1 + K_1)$. The latter space is finite-dimensional, by the Riesz theorem. Further, $\operatorname{im} T \supset \operatorname{im} T \circ S_2 = \operatorname{im}(1 + K_2)$. The latter space is closed and has finite codimension in L_2, by the Riesz theorem. Then im T

has the same properties (as the pre-image of im T/im $T \circ S_2 \subset L_2$/im $T \circ S_2$), and so is Fredholm. □

Denote the set of all Fredholm operators from L_1 to L_2 by $\mathscr{F}(L_1, L_2)$.

Theorem 21. *The set $\mathscr{F}(L_1, L_2)$ is open in $\mathscr{L}(L_1, L_2)$ (with respect to the uniform topology) and invariant under translations by elements of $\mathscr{K}(L_1, L_2)$.*

Proof. Let $T \in \mathscr{F}(L_1, L_2)$. Then T is almost invertible and the relations (19) hold. Suppose that the norm of an $A \in \mathscr{L}(L_1, L_2)$ is strictly less than each of the numbers $\|S_1\|^{-1}$, $\|S_2\|^{-1}$. Then the operators $1 + S_1A$ and $1 + AS_2$ are invertible. (If $\|B\| < 1$, then the sum of the convergent series $\sum_{k=0}^{\infty}(-B)^k$ can be taken as $(1 + B)^{-1}$.) Therefore,

$$\begin{aligned}(1 + S_1A)^{-1}S_1(T + A) &= (1 + S_1A)^{-1}(1 + K_1 + S_1A)\\ &= 1 + (1 + S_1A)^{-1}K_1 = 1 + \tilde{K}_1,\\ (T + A)S_2(1 + AS_2)^{-1} &= (1 + K_2 + AS_2)^{-1}\\ &= 1 + K_2(1 + AS_2)^{-1} = 1 + K_2,\end{aligned}$$

which proves that $T + A$ is almost invertible. Hence $\mathscr{F}(L_1, L_2)$ contains a neighborhood of the point T.

Suppose now that $T \in \mathscr{F}(L_1, L_2)$, $K \in \mathscr{K}(L_1, L_2)$. The equalities

$$\begin{aligned}S_1(T + K) &= 1 + K_1 + S_1K = 1 + \tilde{K}_1,\\ (T + K)S_2 &= 1 + K_2 + KS_2 = 1 + K_2\end{aligned}$$

show that $T + K$ is almost invertible. (We have used the fact that the product of a compact operator by a bounded operator is compact.) □

Theorem 22. *The function i (the index) is locally constant on $\mathscr{F}(L_1, L_2)$, does not vary under translations by elements of $\mathscr{K}(L_1, L_2)$, and has the property that $i(AB) = i(A)i(B)$, where $A \in \mathscr{F}(L_0, L_2)$, $B \in \mathscr{F}(L_1, L_0)$.*

Proof. Let us first prove a useful technical lemma that sometimes allows us to compute the index of an operator explicitly. Suppose that the spaces L_1 and L_2 are decomposed into direct sums of closed subspaces, $L_i = M_i \oplus N_i$, $i = 1, 2$. Then each operator $T \in \mathscr{L}(L_1, L_2)$ can be written in the form of a 2×2 operator matrix: $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A \in \mathscr{L}(N_1, N_2)$, $B \in \mathscr{L}(M_1, N_2)$, $C \in \mathscr{L}(N_1, M_2)$, $D \in \mathscr{L}(M_1, M_2)$.

Lemma. *If T is a Fredholm operator decomposed as above, and D is invertible, then $i(T) = i(A - BD^{-1}C)$.*

Proof of the Lemma. Observe that multiplication of T from the left or from the right by an invertible operator does not change the numbers $\alpha(T)$ and $\beta(T)$, nor, therefore, the index of T. Hence,

$$i(T) = i\begin{pmatrix} A & B \\ C & D \end{pmatrix} = i\left[\begin{pmatrix} 1 & -BD^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -D^{-1}C & 1 \end{pmatrix}\right]$$
$$= i\begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix}.$$

If T has the form $\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ (written naturally as $T_1 \oplus T_2$), then $\ker T = \ker T_1 \oplus \ker T_2$, and $\operatorname{coker} T = \operatorname{coker} T_1 \oplus \operatorname{coker} T_2$. From this, $i(T) = i(T_1) + i(T_2)$. In our case $i(T) = i(A - BD^{-1}C) + i(D) = i(A - BD^{-1}C)$. □

Suppose now that $T_0 \in \mathscr{F}(L_1, L_2)$. We set $N_1 = \ker T_0$, $M_2 = \operatorname{im} T_0$ and construct a closed subspace $M_1 \subset L_1$ and a finite-dimensional subspace $N_2 \subset L_2$ as in the proof of Theorem 19. Then T_0 has the form $\begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}$, where D_0 is invertible. Any operator T sufficiently close to T_0 in the norm has the form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where D is invertible. By the above lemma, $i(T)$ depends only on the dimensions of N_1 and N_2: $i(T) = i(A - BD^{-1}C) = \dim N_1 - \dim N_2$ (see (12)), which shows that i is locally constant.

Let us prove the second assertion of the theorem. Suppose that $T \in \mathscr{F}(L_1, L_2)$, $K \in \mathscr{K}(L_1, L_2)$. The function $\varphi(t) = i(T + tK)$ is defined on the whole real line (since $\mathscr{F}(L_1, L_2)$ is invariant under translations by elements of $\mathscr{K}(L_1, L_2)$). By the assertion already proved, this function is locally constant, and, consequently, constant on any connected set, in particular, on the line. Hence, $i(T) = \varphi(0) = \varphi(1) = i(T + K)$.

We prove the third assertion. To do this we consider the auxiliary operator $A \oplus B$ acting from $L_0 \oplus L_1$ to $L_2 \oplus L_0$. As mentioned above, $i(A \oplus B) = i(A) + i(B)$. Further, for sufficiently small ε,

$$i\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = i\begin{pmatrix} A & 0 \\ \varepsilon 1_{L_0} & B \end{pmatrix}$$
$$= i\left[\begin{pmatrix} 1_{L_2} & -\varepsilon^{-1}A \\ 0 & 1_{L_0} \end{pmatrix}\begin{pmatrix} A & 0 \\ \varepsilon 1_{L_0} & B \end{pmatrix}\begin{pmatrix} \varepsilon^{-1}1_{L_0} & B \\ 0 & -\varepsilon 1_{L_0} \end{pmatrix}\right]$$
$$= i\begin{pmatrix} 0 & AB \\ 1 & 0 \end{pmatrix} = i(AB). \qquad \square$$

The Fredholm Alternative. *Let L be a Banach space, K a compact operator in L, and λ a nonzero number. Let us consider the four equations*

(1) $Kx - \lambda x = y$,
(2) $Kx - \lambda x = 0$,
(3) $K'f - \lambda f = g$,
(4) $K'f - \lambda f = 0$,

where $x, y \in L, f, g \in L'$. Then either

(a) *equations (2) and (4) have only the trivial solution, and equations (1) and (3) have a unique solution for any right-hand side, which furthermore depends continuously on the right-hand side, or*
(b) *equation (2) has a finite-dimensional space of solutions $L_1 \subset L$, equation (4) has a finite-dimensional space of solutions $L_2 \subset L'$, and $\dim L_1 = \dim L_2$, in which case equation (1) is solvable for precisely those $y \in L$ such that $f(y) = 0$ for all $f \in L_2$, and equation (3) is solvable for precisely those $g \in L'$ such that $g(x) = 0$ for all $x \in L_1$.*

PROOF. The operator $\lambda 1$ is invertible, and, consequently, is a Fredholm operator with zero index. The operator $T = K - \lambda 1$ has these same properties, by Theorem 22. The first case of the alternative holds when $\alpha(T) = 0$. Then $\beta(T) = \alpha(T) + i(T) = 0$, so that $\alpha(T') = \beta(T') = 0$. Therefore, $\ker T = \ker T' = \{0\}$, $\operatorname{im} T = L$, $\operatorname{im} T' = L'$. The continuous dependence on the right-hand side for solutions of Equations (1) and (3) (i.e., the continuity of T^{-1} and $(T')^{-1}$) follows from the Banach theorem.

The second case of the alternative is characterized by the inequality $\alpha(T) \neq 0$. Then, since T is Fredholm, $\alpha(T) < \infty$. By the formulas (13), $-i(T) = i(T') = 0$, therefore, $\beta(T) = \alpha(T)$, $\alpha(T') = \beta(T') = \alpha(T)$. Moreover, $\ker T' = (\operatorname{im} T)^{\perp}$ and $\operatorname{im} T' = (\ker T)^{\perp}$.† □

We remark that the Fredholm alternative implies the following spectral property of compact operators: if $\lambda \neq 0$ is a point of the spectrum (i.e., the operator $K - \lambda 1$ is not invertible), then λ is an eigenvalue of finite multiplicity.

§3. Function Spaces and Generalized Functions

1. Spaces of Integrable Functions

Let X be a set with a measure μ. We denote by $L_p(X, \mu)$, $1 \leq p < \infty$, the collection of equivalence classes of μ-measurable functions whose pth powers are integrable. For $f \in L_p(X, \mu)$ let

$$\|f\|_p = \left(\int_X |f(x)|^p \, d\mu\right)^{1/p}.$$

† If L is an LTS, L' is the dual space, and X is a subset of L, then $X^{\perp}$ denotes the collection of all $f \in L'$ such that $f(x) = 0$ for all $x \in X$.

(Here we do not distinguish between an equivalence class $f \in L_p(X, \mu)$ and a particular function $f(x) \in f$.) It is not *a priori* clear that $L_p(X, \mu)$ is a normed linear space for $p \neq 1$. This fact is a consequence of the Minkowski inequality: $\|f + g\|_p \leq \|f\|_p + \|g\|_p$, which, in turn, follows from the Hölder inequality: $|\int_X fg\, d\mu| \leq \|f\|_p \|g\|_q$ if $1/p + 1/q = 1$ (cf. Problems 416, 417). In the case of a σ-finite atomic measure these inequalities become the corresponding inequalities for sequences.

We introduce in addition the space $L_\infty(X, \mu)$ of essentially bounded μ-measurable functions (cf. §3.1 in Ch. II).

Theorem 23. *The $L_p(X, \mu)$ are Banach spaces for $1 \leq p \leq \infty$. The space dual to $L_p(X, \mu)$ is isomorphic to $L_q(X, \mu)$ for $1 \leq p < \infty$, where $1/p + 1/q = 1$.*

Proof. Let F be a continuous linear functional on $L_p(X, \mu)$. If $A \subset X$ is a set of finite measure, then its characteristic function χ_A belongs to $L_p(X, \mu)$. We set $v(A) = F(\chi_A)$ and verify that v is a signed measure on the algebra of measurable subsets of X. The additivity of v follows from the linearity of F. The absolute continuity of v with respect to μ follows from the estimate $|v(A)| \leq \|F\|_{L_p'} \|\chi_A\|_{L_p} = \|F\|_{L_p'} \mu(A)^{1/p}$. By the Radon–Nikodým theorem (see Ch. II, §3.3), there exists a μ-measurable function g on X such that $U(A) = \int_A g(x)\, d\mu(x)$ for any set A of finite measure. Let us show that $g \in L_q(X, \mu)$. To do this we observe that, by the remark after Hölder's inequality (see §1.4 and Problem 416), for any μ-measurable function g we have the equality

$$\|g\|_q = \sup_{\|f\|_p \leq 1} \left| \int_X fg\, d\mu \right|. \tag{23}$$

Clearly, it suffices to take the supremum on the right-hand side of (23) over the simple functions f in $L_p(X, \mu)$. But, for a function of the form $f(x) = \sum_{k=1}^N c_k \chi_{E_k}(x)$,

$$\int_X f(x)g(x)\, d\mu = \sum_{k=1}^N c_k \int_{E_k} g(x)\, d\mu = \sum_{k=1}^N c_k v(E_k) = F(f).$$

Therefore, the right-hand side of (23) is bounded by the number $\|F\|_{L_p'}$. Hence, $g \in L_q(X, \mu)$. By Hölder's inequality, the expression $F_g(f) = \int_X f(x)g(x)\, d\mu(x)$ is a continuous linear functional on $L_p(X, \mu)$. Since F and F_g coincide on the simple functions, they coincide everywhere on $L_p(X, \mu)$. Applying the relation (23) once more, we see that $\|g\|_q = \|F\|_{L_p'}$. This proves the isomorphism $L_p'(X, \mu) = L_q(X, \mu)$. The completeness of $L_p(X, \mu)$ in the case $1 < p \leq \infty$ now follows from the general theorem on completeness of a dual space (see §1.1 in Ch. III). The completeness of $L_1(X, \mu)$ was proved in Ch. II, §3.

The assertion in Theorem 23 that $L_p'(X, \mu)$ is isomorphic to $L_q(X, \mu)$ ceases to be true for $p = \infty$. The space $L_\infty'(X, \mu)$ is not isomorphic to $L_1(X, \mu)$, except for the trivial case when it is finite-dimensional. It can be shown that an infinite-dimensional space $L_1(X, \mu)$ is not the dual of any Banach space if μ is not a purely atomic measure (cf. Problem 436). □

2. Spaces of Continuous Functions

Suppose that X is a compact space. The space $C(X)$ consists of all the continuous functions on X. The norm in $C(X)$ is defined by

$$\|f\| = \max_{x \in X} |f(x)|.$$

It is easy to check (see Problem 438) that $C(X)$ is a Banach space.

Theorem 24. *Every Banach space L is isomorphic to a closed subspace of some space $C(X)$. If L is separable, then X can be taken to be the interval* $[0, 1]$.

PROOF. Let X be the unit ball in the space L' dual to L. Then X is compact in the weak-* topology (see [13]). Each element of L can be regarded as a continuous function on X. By §1.2, the mapping of L into $C(X)$ obtained in this way is an isomorphism onto a closed subspace of $C(X)$. Suppose that L is separable. Then $X \subset L'$ is a metrizable space (see §1.1). If X is a convex compact metric space in a linear space, then there exists a continuous mapping f of $[0, 1]$ onto X (see Problem 452). We now define a mapping of L into $C[0, 1]: \varphi \mapsto \Phi(t) = [f(t)](\varphi)$. By the foregoing, this mapping is an isomorphism of L onto some closed subspace of $C[0, 1]$.

Theorem 25. *The space dual to* $C[0, 1]$ *is isomorphic to the subspace* $V[0, 1]$ *of* $BV[0, 1]$ *(the space of functions of bounded variation on* $[0, 1]$*) consisting of those functions that are left-continuous and satisfy the condition* $g(0) = 0$, *with the norm* $\|g\| = \mathrm{Var}_0^1 g$.

PROOF. Suppose that $g \in V[0, 1]$, $f \in C[0, 1]$. Let $F_g(f) = \int_0^1 f(x)\, dg(x)$. For any partition $0 = t_0 \le t_1 \le \cdots \le t_n = 1$ and any $\xi_i \in [t_{i-1}, t_i]$ we have

$$\sum_{i=1}^{n} f(\xi_i)[g(t_i) - g(t_{i-1})] \le \|f\|_{C[0,1]}\|g\|_{BV[0,1]}.$$

On the other hand, for any $\varepsilon > 0$ there is a partition T of $[0, 1]$ such that $\sum_{i=1}^{n} |g(t_i) - g(t_{i-1})| > \mathrm{Var}_0^1 g - \varepsilon$. We may assume that g is continuous at the points of T. Let $\{f_n\}$ be a sequence of continuous functions such that $|f_n(x)| \le 1$ and $f_n(x) \underset{n \to \infty}{\to} \mathrm{sgn}[g(t_k) - g(t_{k-1})]$, $x \in [t_{k-1}, t_k)$.

Then $\lim_{n \to \infty} \int_0^1 f_n(x)\, dg(x) = \sum_{x=1}^{n} |g(t_k) - g(t_{k-1})|$. Therefore, $\int_0^1 f_n\, dg(x) > \mathrm{Var}_0^1 g - \varepsilon$ for sufficiently large n. This implies that $\|F_g\|_{C'[0,1]} = \|g\|_{BV[0,1]}$.

It remains to show that any continuous linear functional F on $C[0, 1]$ has the form F_g for some $g \in V[0, 1]$. By the Banach theorem, F has an extension to a functional $\tilde{F}$ on the space $B[0, 1]$ of bounded functions, with the norm $\|f\| = \sup |f(x)|$. Let χ_a be the characteristic function of the half-open interval $[0, a)$. We set $g(a) = \langle \tilde{F}, \chi_a \rangle$. Clearly, $g(0) = 0$. Let us show that $\mathrm{Var}_0^1 g(x) < \infty$. For any partition T let $\varepsilon_T(x) = \mathrm{sgn}[g(t_k) - g(t_{k-1})]$, $x \in [t_{k-1}, t_k)$. Then $\langle \tilde{F}, \varepsilon_T \rangle = \sum_{k=1}^n |g(t_k) - g(t_{k-1})|$, from which $\mathrm{Var}_0^1 g \leq \|\tilde{F}\| = \|F\|$.

For any piecewise constant right-continuous function φ on $[0, 1]$ we have that $\langle \tilde{F}, \varphi \rangle = \int_0^1 \varphi(x)\, dg(x)$. A continuous function f can be uniformly approximated by such φ_n, for example, $\varphi_n(x) = f([nx]/n)$. Therefore, the equality $\langle \tilde{F}, f \rangle = \int_0^1 f(x)\, dg(x)$ is valid for all continuous functions f. It remains to observe that for continuous f the Stieltjes integral $\int_0^1 f(x)\, dg(x)$ does not change when $g(x)$ is replaced by $g(x - 0) = \lim_{\varepsilon \to +0} g(x - \varepsilon)$, so that g may be assumed to be left-continuous. □

Remark. Theorem 25 has the following equivalent formulation:

Every continuous linear functional on $C[0, 1]$ *has the form*

$$F_\nu(f) = \int_0^1 f(x)\, d\nu(x),$$

where ν *is a Borel signed measure on* $[0, 1]$, *and*

$$\|F_\nu\|_{C'[0,1]} = \mathrm{Var}_0^1 \nu.$$

In this form Theorem 25 can be carried over to the space of complex continuous functions (with ν replaced by a complex measure) and to a space $C(X)$, where X is any compact metric space. For an arbitrary compact space X the first assertion of the theorem remains true, but the correspondence between signed measures and continuous linear functionals ceases to be one-to-one.

3. Spaces of Smooth Functions

Let Ω be a region (i.e., an open subset) in $\mathbf{R}^n$, and $\bar{\Omega}$ the closure of Ω in $\mathbf{R}^n$. We use the following standard notation: $x = (x_1, \ldots, x_n)$ for coordinates in $\mathbf{R}^n$, $x^k = x_1^{k_1} \cdots x_n^{k_n}$, $\partial_i = \partial/\partial x_i$ is the partial derivative operator, $\partial^l = \partial_1^{l_1} \cdots \partial_n^{l_n}$, $|k| = k_1 + k_2 + \cdots + k_n$. The notation $C^r(\bar{\Omega})$ denotes the collection of functions on $\bar{\Omega}$ having partial derivatives up to order r at points of Ω and such that the functions $\partial^l f$, $|l| \leq r$, extend to bounded continuous functions on $\bar{\Omega}$. The norm in $C^r(\bar{\Omega})$ is defined by the formula

$$\|f\|_{C^r(\bar{\Omega})} = \sup_{\substack{x \in \Omega \\ |l| \leq r}} |\partial^l f(x)|.$$

Thus, convergence in $C^r(\bar{\Omega})$ means uniform convergence of the functions themselves and their partial derivatives through order r. It is not hard to verify that $C^r(\bar{\Omega})$ is a Banach space.

The spaces $C^r(\overline{\Omega})$ are convenient in problems in which the functions under consideration are required to have a specified finite smoothness (i.e., to have a specific number of continuous derivatives). However, there are problems in which the smoothness is unknown in advance or is infinite. It is natural to use spaces of infinitely differentiable functions in such problems. These spaces are locally convex, but, as a rule, are not normable, and sometimes not even metrizable.

Three types of spaces find the greatest application.

(1) The space $\mathscr{E}(\Omega)$ consists of *all the infinitely differentiable* functions on Ω. The topology in $\mathscr{E}(\Omega)$ is determined by the family of seminorms p_{Kl}, where K is a compact subset of Ω, and $l = (l_1, \ldots, l_n)$ is an arbitrary multi-index:

$$p_{Kl}(f) = \max_{x \in K} |\partial^l f(x)|. \tag{24}$$

Theorem 26. *$\mathscr{E}(\Omega)$ is a countably normable (and hence metrizable) complete space.*

Proof. Let K_m be the collection of points $x \in \Omega$ having the properties:

(1) the distance from x to the boundary of Ω (i.e., the set $\partial\Omega = \overline{\Omega} \setminus \Omega$) is not less than $1/m$;†

(2) the distance from x to 0 is not greater than m. □

It is clear that K_m is a compact set, all the points of K_m are interior to K_{m+1}, and the union of the K_m over all m exhausts the region Ω. The formula $p_m(f) = \sup\{|\partial^l f(x)| : x \in K_m, |l| \leq m\}$ defines a seminorm p_m in $\mathscr{E}(\Omega)$.

Let K be an arbitrary compact subset of Ω. The function $\delta(x) = d(x, \overline{\Omega} \setminus \Omega)$ is continuous and positive on K. Consequently, it attains a minimum $\delta_0 > 0$. The function $\Delta(x) = d(x, 0)$ is continuous on K and, hence, attains a maximum Δ_0. If the number m is chosen so that $1/m < \delta_0$, $m > \Delta_0$, then K_m will contain K. If, moreover, $m \geq |l|$, then the seminorm p_m majorizes the seminorm p_{Kl}. We have shown that the family $\{p_m\}$ of seminorms majorizes the family $\{p_{Kl}\}$. The converse is obvious: $p_m(f) \leq \sum_{l=0}^{m} p_{K_m l}(f)$.

It remains to show that $\mathscr{E}(\Omega)$ is complete. Let $\{f_n\}$ be a Cauchy sequence. This means that $\{f_n\}$ is Cauchy in any seminorm p_m. Then the restriction of $\{f_n\}$ to K_m is a Cauchy sequence in $C^m(K_m)$. Hence, there exists a function $F_m \in C^m(K_m)$ such that $f_n|_{K_m} \to F_m$ in the metric of $C^m(K_m)$. It is clear that the functions F_m agree in the sense that $F_{m+1}|_{K_m} = F_m$. Therefore, there exists a single function f coinciding with F_m on K_m. By construction, $p_m(f_n - f) \to 0$ as $n \to \infty$. This means that $f \in \mathscr{E}(\Omega)$ and $f_n \to f$ in the topology of $\mathscr{E}(\Omega)$.

Remark. $\mathscr{E}(\Omega)$ is often denoted by $C^\infty(\Omega)$. If the boundary of Ω is not empty, then $\mathscr{E}(\Omega)$ does not coincide with the intersection of the $C^k(\overline{\Omega})$ (the functions in $\mathscr{E}(\Omega)$ do not, generally speaking, extend to continuous functions on $\overline{\Omega}$).

† By the distance to the boundary we mean, as usual, the minimal distance to its points.

(2) The space $\mathscr{D}(\Omega)$ consists of the *infinitely differentiable compactly supported* functions on Ω. The *support* of a function φ is defined to be the closure of the set of points where φ is nonzero, and is denoted supp φ. Thus, $\mathscr{D}(\Omega)$ consists of those $\varphi \in \mathscr{E}(\Omega)$ for which supp φ is compact. It is easy to check that $\mathscr{D}(\Omega)$ is not closed in $\mathscr{E}(\Omega)$; hence, it is not complete in the topology of $\mathscr{E}(\Omega)$.

Suppose that K is a compact subset of Ω. Let $\mathscr{D}_K(\Omega)$ be the subspace of $\mathscr{E}(\Omega)$ consisting of those φ for which supp $\varphi \subset K$. Then $\mathscr{D}_K(\Omega)$ is a complete countably normed space with the topology induced from $\mathscr{E}(\Omega)$ (cf. Problem 463). We now define in $\mathscr{D}(\Omega)$ a topology stronger than that inherited from $\mathscr{E}(\Omega)$. Namely, a set $V \subset \mathscr{D}(\Omega)$ is regarded as open (resp., closed) if its intersection with $\mathscr{D}_K(\Omega)$ is open (resp., closed) for any compact set $K \subset \Omega$.

The topology obtained in this way can also be given by means of a family of seminorms. Let $\{K_m\}$ be the system of compact sets constructed in the proof of Theorem 26. We let the symbol α stand for a sequence $\{N_m\}$ of nonnegative integers and set

$$p_\alpha(\varphi) = \sum_{m=1}^{\infty} N_m \left(\sup_{\substack{x \in K_m \setminus K_{m-1} \\ |l| \le N_m}} |\partial^l \varphi(x)| \right) \tag{25}$$

(here K_0 is the empty set). Note that for each function $\varphi \in \mathscr{D}(\Omega)$ the series on the right-hand side of (25) contains only finitely many nonzero terms.

Theorem 27. *A sequence $\{\varphi_n\}$ converges to φ in $\mathscr{D}(\Omega)$ if and only if*

(1) *$\varphi_n \to \varphi$ in the sense of $\mathscr{E}(\Omega)$.*
(2) *all the functions φ_n (and, hence, also φ) belong to a single subspace $\mathscr{D}_K(\Omega)$.*

PROOF. The sufficiency of the conditions (1) and (2) is obvious, as is the necessity of the condition (1). Let us prove the necessity of (2). Suppose that $\{\varphi_n\}$ is such that the supports of the φ_n are not contained in any fixed compact set. Renumbering if necessary, we can assume that supp $\varphi_m \not\subset K_m$. Let x_m be a point outside K_m at which φ_m is nonzero. Consider the set V of all functions $\varphi \in \mathscr{D}(\Omega)$ satisfying the conditions $|\varphi(x_m)| < |\varphi_m(x_m)|/m$ for $m = 1, 2, 3, \ldots$. Since any compact set K contains only finitely many points x_m, the intersection of V with $\mathscr{D}_K(\Omega)$ is determined by finitely many of these conditions and, consequently, is open in $\mathscr{D}_K(\Omega)$. Hence, V is open in $\mathscr{D}(\Omega)$. Let p_V be the Minkowski functional for V. It is easy to see that V is a convex balanced set, and then p_V is a continuous seminorm in $\mathscr{D}(\Omega)$. Explicitly, p_V is given by the formula $p_V(\varphi) = \sup_m |m\varphi(x_m)/\varphi_m(x_m)|$, from which it follows that $p_V(\varphi_m) \ge m$. Therefore, the sequence $\{\varphi_m\}$ cannot converge. □

Theorem 28. *$\mathscr{D}(\Omega)$ is a complete nonmetrizable space that has the Heine–Borel property: every bounded† subset of $\mathscr{D}(\Omega)$ is pre-compact.*

† A *bounded set* in a polynormed space is defined to be a set that is bounded in each seminorm (cf. Problem 270).

Proof. If $\{\varphi_n\}$ is a Cauchy sequence, then the argument in the proof of Theorem 27 shows that this sequence lies entirely in some subspace $\mathscr{D}_K(\Omega)$. Since $\mathscr{D}_K(\Omega)$ is complete, the sequence has a limit. Suppose now that $\mathscr{D}(\Omega)$ were metrizable, and let $\{\varphi_m\}$ be a sequence for which supp $\varphi_m \not\subset K_m$. The continuity of multiplication by a number implies that for each m there is a $\delta_m > 0$ small enough that $d(0, \delta_m \varphi_m) < 1/m$. This means that the sequence $\{\delta_m \varphi_m\}$ converges to zero, which contradicts Theorem 27. Thus, $\mathscr{D}(\Omega)$ is not metrizable. Suppose, finally, that A is a bounded subset of $\mathscr{D}(\Omega)$. The argument already used above shows that $A \subset \mathscr{D}_K(\Omega)$ for some compact set $K \subset \Omega$. Since A is bounded in each seminorm p_{Kl}, all the functions in A and all their partial derivatives satisfy the conditions of the Arzelà–Ascoli theorem. This implies that A is pre-compact in $\mathscr{D}_K(\Omega)$ and hence also in $\mathscr{D}(\Omega)$. □

Remark. The notation $C_0^\infty(\Omega)$ is also used for $\mathscr{D}(\Omega)$.

(3) The space $S(\mathbf{R}^n)$ consists of the *infinitely differentiable functions on* $\mathbf{R}^n$ *that are rapidly decreasing at infinity.* The topology in $S(\mathbf{R}^n)$ is given by the countable family of seminorms

$$p_{\alpha,\beta}(f) = \sup_{x \in \mathbf{R}^n} |x^\alpha \, \partial^\beta f(x)|, \tag{26}$$

where the standard abbreviated notation has been used:

$$\alpha = (\alpha_1, \ldots, \alpha_n), \qquad \beta = (\beta_1, \ldots, \beta_n),\ x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

$$\partial^\beta = \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}} = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_x^{\beta_n}}.$$

(The space $S(\mathbf{R}^n)$ consists of all the $f \in \mathscr{E}(\mathbf{R}^n)$ such that $p_{\alpha,\beta}(f) < \infty$ for all α and β.)

Instead of the seminorms (26) it is sometimes convenient to consider the collection

$$p'_{\alpha,\beta}(f) = \int_{\mathbf{R}^n} |x^\alpha \, \partial^\beta f(x)| \, dx \tag{26'}$$

or the collection

$$p''_{\alpha,\beta}(f) = \left(\int_{\mathbf{R}^n} |x^\alpha \partial^\beta f(x)|^2 \, dx \right)^{1/2}. \tag{26''}$$

Theorem 29. *The systems of seminorms* (26), (26′), *and* (26″) *are equivalent.*

Proof. Let us first take the easiest case $n = 1$. We have the estimate

$$|x^k \partial^l f(x)| \leq \frac{1}{1 + x^2} \sup_x |(x^2 + 1) x^k \partial^l f(x)|.$$

From this,

$$p'_{kl}(f) = \int_{\mathbf{R}} |x^k \partial^l f(x)|\, dx$$
$$\leq \sup_x |(x^2 + 1)x^k \partial^l f(x)| \int_{\mathbf{R}} \frac{dx}{1 + x^2} \leq \pi(p_{k+2,l}(f) + p_{kl}(f)).$$

Similarly,

$$p''_{kl}(f) = \left(\int_{\mathbf{R}} |x^k \partial^l f(x)|^2\, dx\right)^{1/2}$$
$$\leq \left(\sup_x (1 + x^2)|x^k \partial^l f(x)|^2 \int_{\mathbf{R}} \frac{dx}{1 + x^2}\right)^{1/2}$$
$$\leq (\sqrt{\pi(p_{k+1,l}^2 + p_{kl}^2)})(f).$$

Thus, the seminorms of the system (26) majorize those of the systems (26′) and (26″). Further, applying the Cauchy–Bunyakovskii inequality to the functions

$$|x^k \partial^l f(x)|\sqrt{1 + x^2} \quad \text{and} \quad \frac{1}{\sqrt{1 + x^2}},$$

we get that

$$p'_{kl}(f)^2 = \left(\int_{\mathbf{R}} |x^k \partial^l f(x)|\, dx\right)^2$$
$$\leq \int_{\mathbf{R}} |x^k \partial^l f(x)|^2 (1 + x^2)\, dx \int \frac{dx}{1 + x^2} = \pi(p''_{k+1,l}(f)^2 + p''_{kl}(f)^2).$$

Hence, the seminorms of the system (26″) majorize those of the system (26′). It remains to estimate the seminorms of the system (26) in terms of those of the system (26′). Let us use the fact that for $f \in S(\mathbf{R})$ the functions $x^k \partial^l f(x)$ converge to zero at infinity for any k and l, and, therefore,

$$x^k \partial^l f(x) = \int_{-\infty}^{x} [t^k \partial^l f(t)]'\, dt. \tag{27}$$

From this,

$$p_{kl}(f) = \sup_x |x^k \partial^l f(x)| \leq \int_{\mathbf{R}} |[t^k \partial^l f(t)]'|\, dt$$
$$\leq k p'_{k-1,l}(f) + p'_{k,l+1}(f).$$

The case $n > 1$ differs only by technical complications: instead of the equality $\int_{\mathbf{R}} (1 + x^2)^{-1}\, dx = \pi$ it is necessary to use the estimate $\int_{\mathbf{R}^n} (1 + \|x\|^{2n})^{-1}\, dx < \infty$, and instead of the identity (27) the identity

$$\varphi(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \frac{\partial^n \varphi}{\partial x_1 \cdots \partial x_n},$$

which holds for any infinitely differentiable function which vanishes along with its derivatives at infinity. □

In its supply of functions and in its topology the space $S(\mathbf{R}^n)$ occupies an intermediate position between $\mathscr{E}(\mathbf{R}^n)$ and $\mathscr{D}(\mathbf{R}^n)$. Namely, we have the continuous imbeddings

$$\mathscr{D}(\mathbf{R}^n) \subset S(\mathbf{R}^n) \subset \mathscr{E}(\mathbf{R}^n).$$

Up to this point we have not given a single example of a function in $\mathscr{D}(\mathbf{R}^n)$ or $S(\mathbf{R}^n)$. It is not completely trivial to construct such examples. However, we have the

Theorem 30. *The space $\mathscr{D}(\mathbf{R}^n)$ is dense in $L_p(\mathbf{R}^n, dx)$ for $1 \leq p < \infty$,† in $S(\mathbf{R}^n)$, and in $\mathscr{E}(\mathbf{R}^n)$. The space $S(\mathbf{R}^n)$ is dense in $L_p(\mathbf{R}^n, dx)$ for $1 \leq p < \infty$, and in $\mathscr{E}(\mathbf{R}^n)$.*

To avoid technical complications we give a detailed proof only for the case $n = 1$. Let us begin with the construction of a nontrivial function in $\mathscr{D}(\mathbf{R})$.

Lemma 1. *The function*

$$\varphi(x) = \begin{cases} 0 & \text{for } x \geq 0, \\ e^{1/x} & \text{for } x < 0 \end{cases}$$

is infinitely differentiable on the whole line.

Proof. The assertion is obvious everywhere away from the point $x = 0$. We show that $\varphi^{(k)}(0) = 0$ for $k = 1, 2, \ldots$. To do this note that the function $(d^k/dx^k)(e^{1/x})$ has the form $P_k(x)x^{-2k}e^{1/x}$, where P_k is some polynomial of degree $\leq k$. (This can easily be established by induction.) Further,

$$\lim_{x \to -0} (P(x)/x^m)e^{1/x} = \lim_{t \to +\infty} P(-1/t)t^m e^{-t} = 0$$

for any m and any polynomial P, as is easily shown by using L'Hôpital's rule. Thus, $\lim_{\varepsilon \to 0} \varphi^{(k)}(\varepsilon) = 0$ for all k. Applying L'Hôpital's rule once more, we get the lemma.

Lemma 2. *The function*

$$\psi(x) = \begin{cases} \exp[2/(x^2 - 1)] & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geq 1 \end{cases}$$

is in $\mathscr{D}(\mathbf{R})$.

† As usual, dx denotes Lebesgue measure in $\mathbf{R}^n$.

Indeed, this function vanishes outside $[-1, 1]$ and is infinitely differentiable, since it can be written in the form $\psi(x) = \varphi(x - 1)\varphi(-x - 1)$, where φ is the function in Lemma 1.

Lemma 3. *For any $\varepsilon > 0$ let $\psi_\varepsilon(x) = (c/\varepsilon)\psi(x/\varepsilon)$, where $c^{-1} = \int_{-\infty}^{\infty} \psi(x)\,dx$. Then the function $\psi_\varepsilon(x)$ has the following properties*:

(1) $\psi_\varepsilon(x) \geq 0$;
(2) $\operatorname{supp} \psi_\varepsilon = [-\varepsilon, \varepsilon]$;
(3) $\int_{-\infty}^{\infty} \psi_\varepsilon(x)\,dx = 1$.

The proof is obvious.

We are now in a position to prove the first assertion of Theorem 30 (on the density of $\mathscr{D}(\mathbf{R})$ in $L_p(\mathbf{R}, dx)$ for $1 \leq p < \infty$). Let $f \in L_p(\mathbf{R}, dx)$. Since the integral $\int_{-\infty}^{\infty} |f|^p\,dx$ converges, there is a number N such that $\int_{-\infty}^{-N} |f|^p\,dx + \int_N^\infty |f|^p\,dx < (\varepsilon/2)^p$. Then the function

$$f_N(x) = \begin{cases} f(x) & \text{for } |x| \leq N, \\ 0 & \text{for } |x| > N \end{cases}$$

has compact support, and $\|f - f_N\|_p < \varepsilon/2$. Further, since $f_N(x)$ is continuous in the mean (see Problem 432), there exists a $\delta > 0$ such that $\int |f_N(x) - f_N(x + t)|^p\,dx < (\varepsilon/2)^p$ for $|t| < \delta$. Let us now consider the function

$$g(x) = \int_{-\infty}^{\infty} f_N(x - t)\psi_\delta(t)\,dt.$$

(This integral exists, since ψ_δ is bounded and compactly supported, and hence belongs to $L_q(\mathbf{R}, dx)$.) We estimate the distance between f_N and g in $L_p(\mathbf{R}, dx)$, using the formula $\|f\|_p = \sup_{\|h\|_q \leq 1} |\int_{\mathbf{R}} fh\,dx|$. We have

$$\begin{aligned} \|f_N - g\|_p &= \sup_{\|h\|_q \leq 1} \left| \int_{-\infty}^{\infty} (f_N - g)h\,dx \right| \\ &= \sup_{\|h\|_q \leq 1} \left| \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \psi_\delta(t) f_N(x - t)\,dt - f_N(x) \right) h(x)\,dx \right| \\ &= \sup_{\|h\|_q \leq 1} \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_\delta(t)(f_N(x - t) - f_N(x))h(x)\,dx\,dt \right|. \end{aligned}$$

(The equation $\int_{-\infty}^{\infty} \psi_\delta(t)\,dt = 1$ has been used.) The last integral admits the estimate

$$\left| \int_{-\delta}^{\delta} \psi_\delta(t) \left(\int_{-\infty}^{\infty} (f_N(x - t) - f_N(x))h(x)\,dx \right) dt \right| \leq \int_{-\infty}^{\infty} \psi_\delta(t) \cdot \frac{\varepsilon}{2}\,dt = \frac{\varepsilon}{2}$$

by the choice of δ. Thus $\|f_N - g\|_p < \varepsilon/2$, and, consequently, $\|f - g\|_p < \varepsilon$.

Let us show that $g \in \mathscr{D}(\mathbf{R})$. The fact that g is compactly supported follows from the same property for f_N and ψ_δ: It is clear that supp $g \subset$ supp $f_N +$ supp $\psi_\delta = [-N - \delta, N + \delta]$. The infinite differentiability of g follows from the identity

$$\frac{d^k}{dx^k} g(x) = \frac{d^k}{dx^k} \int_{-\infty}^{\infty} f_N(x - t)\psi_\delta(t)\, dt = \int_{-\infty}^{\infty} f_N(x - t)\psi_\delta^{(k)}(t)\, dt,$$

which is easily proved by induction (cf. §1 in Ch. IV).

We now prove that $\mathscr{D}(\mathbf{R})$ is dense in $\mathscr{E}(\mathbf{R})$. To do this we construct a function $\chi_1 \in \mathscr{D}(\mathbf{R})$ having the property that $\chi_1(x) \equiv 1$ on $[-1, 1]$. For example, the function can be taken to be an antiderivative of the function $\psi_{1/2}(x + 3/2) - \psi_{1/2}(x - 3/2)$ (Fig. 2).

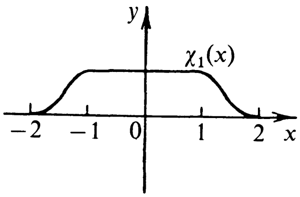


Figure 2

Let $\chi_N(x) = \chi_1(x/N)$. Then $\chi_N(x) \in \mathscr{D}(\mathbf{R})$ and $\chi_N(x) \equiv 1$ for $x \in [-N, N]$. Suppose that $f \in \mathscr{E}(\mathbf{R})$. Then $\chi_N f \in \mathscr{D}(\mathbf{R})$. We show that $\chi_N f \to f$ in $\mathscr{E}(\mathbf{R})$ as $N \to \infty$. Let K be a compact subset of the line. Then it contained is in $[-N, N]$ for sufficiently large N. Therefore, $p_{kl}(\chi_N f - f) = \sup |(\chi_N f - f)^{(l)}(x)| = 0$ for sufficiently large N. Hence, $\chi_N f \to f$ in $\mathscr{E}(\mathbf{R})$ as $N \to \infty$.

It happens that the same sequence $\chi_N f$ converges to f also in the sense of the space $S(\mathbf{R})$. The proof of this is based on the estimate

$$|x^k f^{(l)}(x)| \leq \frac{p_{k+m,l}(f)}{N^m} \quad \text{for } |x| \geq N, f \in S(\mathbf{R}),$$

which follows directly from the definition of the norm $p_{k+m,l}$. Namely, we have

$$\begin{aligned} p_{kl}(\chi_N f - f) &= \sup_{x \in \mathbf{R}} |x^k(\chi_N f - f)^{(l)}(x)| \\ &= \sup_{x \in \mathbf{R}} \left| \sum_{j=0}^{l} c_l^j x^k f^{(j)}(x)(\chi_N - 1)^{(l-j)}(x) \right| \\ &\leq \frac{1}{N} \sum_{j=0}^{l} c_l^j p_{k+1,j}(f) \sup_{x \in \mathbf{R}} |(x_N - 1)^{(l-j)}(x)|. \end{aligned}$$

Using the relation

$$|(\chi_N - 1)^{(l)}(x)| = \left| \frac{1}{N^l} (\chi_1 - 1)^{(l)}\left(\frac{x}{N}\right) \right| \leq C_l \quad \text{for } x \in \mathbf{R} \text{ and } N \geq 1,$$

we obtain

$$p_{kl}(\chi_N f - f) \leq \frac{1}{N} \sum_{j=0}^{l} c_l^j p_{K+1,j}(f) C_{l-j}.$$

The last expression converges to zero as $N \to \infty$. The remaining assertions of the theorem follow from what has already been proved. □

The Weierstrass Theorem. *Let Ω be a bounded region in $\mathbf{R}^n$. Then for any natural number k the space P_n of polynomials in n variables is dense in $C^k(\overline{\Omega})$.*

We put off the proof of the full theorem until §1 in Ch. IV, since it is based on convolution techniques. Here we give a simple proof for the important special case when $n = 1$, $k = 0$, and Ω is an interval. In this case the theorem takes the form:

Every continuous function on a segment can be uniformly approximated by polynomials.

PROOF. Clearly, every continuous function can be uniformly approximated by piecewise linear continuous functions. Further, every piecewise linear function is a linear combination of functions of the form $f(x) = |x - a|$. Finally, the function $f(x) = |x|$ on $[-N, N]$ can be expanded in a uniformly convergent series of polynomials:

$$|x| = \sqrt{N^2 - (N^2 - x^2)} = N\left[1 - \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k+2)!!}\left(1 - \frac{x^2}{N^2}\right)^{k+1}\right].$$

Corollary 1. *For any region $\Omega \subset \mathbf{R}^n$ the space P_n of polynomials in n variables is dense in $\mathscr{E}(\Omega)$.*

PROOF OF THE COROLLARY. Let $f \in \mathscr{E}(\Omega)$ and let p_{Kk} be a seminorm in $\mathscr{E}(\Omega)$. We show that for any $\varepsilon > 0$ there is a polynomial $q \in P_n$ such that $p_{Kk}(q - f) < \varepsilon$. Suppose that V is a bounded open neighborhood of the compact set K. The restriction of f to $\overline{V}$ obviously belongs to $C^k(\overline{V})$. Applying Weierstrass' theorem to this restriction, we find a polynomial $q \in P_n$ such that $\|q - f\|_{C^k(\overline{V})} < \varepsilon$. Since the norm in $C^k(\overline{V})$ majorizes the seminorm p_{Kk}, q is the desired polynomial. □

Corollary 2. *Suppose that Ω is a region in $\mathbf{R}^n$, K is a compact set in Ω, $\varphi \in \mathscr{D}_K(\Omega)$, $f \in \mathscr{D}(\Omega)$, and $f(x) \neq 0$ for $x \in K$. Then there is a sequence $\{p_k\} \subset P_n$ of polynomials such that $p_k f \to \varphi$ in $\mathscr{D}(\Omega)$.*

For the proof it suffices to choose $\{p_k\}$ in such a way that $p_k \to \varphi/f$ in $\mathscr{E}(\Omega)$.

4. Generalized Functions

The concept of a generalized function† arises naturally in diverse problems in mathematics and mathematical physics when it is desired to extend certain natural operations (differentiation, integration, solution of differential equations, Fourier transformation, etc.) to a broader domain than that in which these operations were originally defined. At first this led to paradoxical and contradictory definitions of generalized functions. For example, the famous Dirac δ-function was defined by the following properties:

$$\delta(x) = 0 \quad \text{for } x \neq 0, \qquad \delta(0) = \infty, \qquad \int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

Another definition of the same function is: $\delta(x) = (1/2)(d/dx)(\operatorname{sgn} x)$. It is clear that an ordinary function with such properties does not exist.

It turned out that these and many other properties cease to be contradictory if a generalized function is understood to be an element of a dual space L', where L is some space of "test" functions. The spaces $\mathscr{D}(\Omega)$, $\mathscr{E}(\Omega)$, or $S(\mathbf{R}^n)$ are most frequently used as L. The elements of the spaces $\mathscr{D}'(\Omega)$, $\mathscr{E}'(\Omega)$, and $S'(\mathbf{R}^n)$ have come to be called *generalized functions in the domain* Ω, *generalized functions with compact support in the domain* Ω, and *tempered distributions* in $\mathbf{R}^n$, respectively.

Let us first show how almost any ordinary function in Ω can be regarded as a generalized function.

Theorem 31. *Suppose that f is a locally integrable (i.e., integrable on each compact set) function with respect to Lebesgue measure dx in the region Ω. The correspondence $\varphi \mapsto \int_\Omega \varphi(x) f(x)\, dx$ is a continuous linear functional on $\mathscr{D}(\Omega)$. If, moreover, f vanishes outside some compact set $K \subset \Omega$, then this correspondence is a continuous linear functional on $\mathscr{E}(\Omega)$.*

PROOF. By the result in Problem 462, we must show that $\int_\Omega \varphi_n(x) f(x)\, dx \to 0$ whenever $\varphi_n \to 0$ in $\mathscr{D}(\Omega)$. But this follows from the definition of convergence in $\mathscr{D}(\Omega)$ and the Lebesgue theorem on passing to the limit under the integral sign. For a proof of the second assertion it suffices to refer to Theorem 15 in Ch. II.

Generalized functions of the type described in Theorem 31 are called *regular* generalized functions; they correspond to ordinary locally integrable functions $f(x)$. However, there are also generalized functions that are not regular. By analogy to the regular case, the value of a generalized function F on a test function φ is frequently written as $\int_\Omega F(x)\varphi(x)\, dx = \langle F, \varphi \rangle$. Of course, this expression cannot be taken literally: the corresponding integral diverges or does not make any sense at all. Suppose that Ω is a region in $\mathbf{R}^n$

† The term "distribution" is sometimes used instead.

containing the origin. The *Dirac function* $\delta(x)$ is defined as the element of the space $\mathscr{E}'(\Omega)$ given by the formula

$$\int_{\Omega} \delta(x)\varphi(x)\,dx = \varphi(0).$$

Since $\mathscr{D}(\Omega)$ is continuously imbedded in $\mathscr{E}(\Omega)$, every linear functional on $\mathscr{E}(\Omega)$ gives rise to a linear functional on $\mathscr{D}(\Omega)$ by restriction. The fact that $\mathscr{D}(\Omega)$ is dense in $\mathscr{E}(\Omega)$ implies that the natural mapping of $\mathscr{E}'(\Omega)$ into $\mathscr{D}'(\Omega)$ is an imbedding. Analogous arguments show the existence of the continuous imbeddings

$$\mathscr{E}'(\mathbf{R}^n) \subset S'(\mathbf{R}^n) \subset \mathscr{D}'(\mathbf{R}^n).$$

In other words, every generalized function with compact support is a tempered distribution, and every tempered distribution is a generalized function.

Unlike ordinary functions, generalized functions do not have definite values at a point (but then this is also a property of the now familiar elements of the spaces $L_p(X, \mu)$). Nevertheless, the expression "$F(x)$ is equal to zero in the domain $U \subset \Omega$" makes sense for a generalized function $F \in \mathscr{D}'(\Omega)$. By definition, this means that $\langle F, \varphi \rangle = 0$ for all test functions such that $\operatorname{supp} \varphi \subset U$.

Let $F \in \mathscr{D}'(\Omega)$. We say that x *does not belong to the support of* F if F vanishes in some open neighborhood of x. It is clear that the *support* of F is a closed set (since its complement is open, as follows immediately from the definition of the support). We denote it by $\operatorname{supp} F$ and let $U = \Omega\backslash\operatorname{supp} F$. Let us show that F vanishes in the domain U. Suppose that $\varphi \in \mathscr{D}(\Omega)$ and $\operatorname{supp} \varphi = K \subset U$. Each point $x \in K$ has a neighborhood V_x in which F vanishes. The covering $\{V_x\}_{x \in K}$ contains a finite subcovering $V_{x_1}, \ldots, V_{x_n}$, and there is a corresponding partition of unity $\varphi_1, \ldots, \varphi_n$ (see Problem 464). Then $\varphi = \sum_{i=1}^n \varphi\varphi_i$, and $\langle F, \varphi \rangle = \sum_{i=1}^n \langle F, \varphi\varphi_i \rangle = 0$. If V is any domain in which F is equal to zero, then all the points of V do not belong to $\operatorname{supp} F$, hence, $V \subset U$.

The expression "a generalized function on Ω with compact support" can be understood in two senses:

(1) as an element of the space $\mathscr{E}'(\Omega)$;
(2) as an element $F \in \mathscr{D}'(\Omega)$ such that $\operatorname{supp} F$ is compact.

These two concepts actually coincide. Namely, every element $F \in \mathscr{E}'(\Omega)$ determines, as remarked above, an element of $\mathscr{D}'(\Omega)$, which we denote by the same letter. It turns out that the elements obtained in this way are precisely the elements of $\mathscr{D}'(\Omega)$ that have compact support. Let us prove this.

Suppose that $F \in \mathscr{E}'(\Omega)$. Then F is continuous with respect to one of the seminorms p_{Kk} defining the topology in $\mathscr{E}(\Omega)$:

$$\langle F, \varphi \rangle \leq c p_{Kk}(\varphi) \quad \text{for } \varphi \in \mathscr{E}(\Omega).$$

It is clear that the restriction of F to $\mathscr{D}(\Omega)$ vanishes on $\Omega\backslash K$. Indeed, if $\varphi \in \mathscr{D}(\Omega)$ and $\operatorname{supp} \varphi \subset \Omega\backslash K$, then $p_{Kk}(\varphi) = 0$. Thus, $\operatorname{supp} F \subset K$, i.e., F has compact support as an element of $\mathscr{D}'(\Omega)$. Suppose now that $F \in \mathscr{D}'(\Omega)$ and $\operatorname{supp} F = K$ is compact in Ω. We construct a functional $\tilde{F}$ on $\mathscr{E}(\Omega)$ whose restriction to $\mathscr{D}(\Omega)$ coincides with F. To do this we consider some compact neighborhood V of K and construct a function $\chi_V \in \mathscr{D}(\Omega)$ having the property that $\chi_V(x) \equiv 1$ for $x \in V$ (see Problem 464). Let $\tilde{F}$ be defined by the formula $\langle \tilde{F}, f\rangle = \langle F, \chi_V f\rangle$. This definition makes sense, since $\chi_V f \in \mathscr{D}(\Omega)$ for any $f \in \mathscr{E}(\Omega)$. It is also clear that $\tilde{F} \in \mathscr{E}'(\Omega)$ (see Problem 427(a)). It remains to check that $\tilde{F}|_{\mathscr{D}(\Omega)} = F$. Let $\varphi \in \mathscr{D}(\Omega)$. Then $\langle \tilde{F}, \varphi\rangle - \langle F, \varphi\rangle = \langle F, \chi_V \varphi - \varphi\rangle = 0$, since

$$\operatorname{supp}(\chi_V \varphi - \varphi) \subset \overline{\Omega\backslash V} \subset \Omega\backslash K.$$

Theorem 32. *The spaces $\mathscr{D}'(\Omega)$, $\mathscr{E}'(\Omega)$, and $S'(\mathbf{R}^n)$ are weak-*complete. (In other words, if a sequence $\{F_n\}$ of generalized functions is such that the numerical sequence $\{\langle F_n, \varphi\rangle\}$ is Cauchy for any test function φ, then $\lim_{n\to\infty} F_n = F$ exists and is a generalized function of the same type as the F_n.)*

Proof. Let F be defined by the formula $\langle F, \varphi\rangle = \lim_{n\to\infty} \langle F_n, \varphi\rangle$. It is obvious that F is linear. We show that it is continuous. The continuity follows from the Banach–Steinhaus theorem (cf. §2.1) in the case of $\mathscr{E}(\Omega)$ and $S(\mathbf{R}^n)$, since $\mathscr{E}(\Omega)$ and $S(\mathbf{R}^n)$ are complete linear metric spaces (see Problems 480–483). Let us analyze the case of $\mathscr{D}(\Omega)$. The restrictions of the F_n to $\mathscr{D}_K(\Omega)$ converge to some element $F_K \in \mathscr{D}'_K(\Omega)$, since $\mathscr{D}_K(\Omega)$ is a complete linear metric space. We now define a functional $F \in \mathscr{D}'(\Omega)$ by setting it equal to F_K on $\mathscr{D}'_K(\Omega)$. It is easily checked that this definition is not ambiguous. (If $\operatorname{supp} \varphi \in K_1 \cap K_2$, then $\langle F_{K_1}, \varphi\rangle = \lim_{n\to\infty} \langle F_n, \varphi\rangle = \langle F_{K_2}, \varphi\rangle$.) The continuity of F now follows from Problem 462. □

Example. The generalized functions $(x \pm i0)^{-1}$. Suppose that $\varphi \in \mathscr{D}(\mathbf{R})$. Then the integral $\int_{-\infty}^{\infty} \varphi(x)(x \pm i\varepsilon)^{-1}\, dx$ converges for $\varepsilon > 0$ and has a finite limit as $\varepsilon \searrow 0$ (see Problem 499). By Theorem 32, this limit is a generalized function on the line, and we denote it by $(x \pm i0)^{-1}$.

Every generalized function with compact support in Ω is continuous with respect to one of the seminorms p_{Kl}. The smallest l for which this is true is called the *order* of the generalized function. If a generalized function $F \in \mathscr{E}'(\Omega)$ has order l and support K, then it can be extended to a continuous linear functional on $C^l(\bar{V})$, where V is any neighborhood of the compact set K.

A generalized function $F \in \mathscr{D}'(\Omega)$ is said to have *order* $\leq l$ if it can be extended to a continuous linear functional on $C^l(\Omega)$. Not every $F \in \mathscr{D}'(\Omega)$ has finite order. However, for any domain V with compact closure in Ω the restriction of F to $\mathscr{D}(V)$ has finite order.

Every tempered distribution $F \in S'(\mathbf{R}^n)$ is continuous with respect to one of the seminorms $p_{kl} = \sup\{p_{\alpha,\beta}\colon |\alpha| \le k, |\beta| \le l\}$ (see subsection 3). The *order* of F is defined to be the smallest l for which this is true. Thus, every tempered distribution has finite order. (It is not hard to verify that this definition of order is equivalent to that given above if $\Omega = \mathbf{R}^n$ and the generalized function F lies in $S'(\mathbf{R}^n) \subset \mathscr{D}'(\mathbf{R}^n)$.)

5. Operations on Generalized Functions

We show here how basic operations on ordinary functions such as multiplication by a function, differentiation, and change of variables can be carried over to generalized functions.

Let L denote one of the spaces $\mathscr{D}(\Omega)$, $\mathscr{E}(\Omega)$, $S(\mathbf{R}^n)$, let L' be the dual space, and let L_0' be some dense subspace of L' consisting of regular generalized functions. In the case $L = \mathscr{D}(\Omega)$ or $\mathscr{E}(\Omega)$ it is convenient to take $\mathscr{D}(\Omega)$ as L_0'; in the case $L = S(\mathbf{R}^n)$ we can take L_0' to be $\mathscr{D}(\mathbf{R}^n)$ or $S(\mathbf{R}^n)$. Assume that some linear operator A_0 is defined in L_0' and that this operator is continuous in the topology of L'. Then A_0 can be extended to a continuous operator A in L'. This extension is unique, because L_0' is dense in L'.

In the cases we shall need to consider, the continuity of A_0 can be established by the following useful device. Let B be a continuous operator in L, and B' the adjoint operator in L'. If the restriction of B' to L_0' coincides with A_0, then A_0 is continuous. In this case the desired extension A obviously coincides with B'.

Let us pass to specific applications of the general scheme described.

1. *Multiplication by a Function.* Let $f \in \mathscr{E}(\Omega)$; we show that the operator $M(f)$ of multiplication by f admits a continuous extension from $\mathscr{D}(\Omega)$ to $\mathscr{D}'(\Omega)$. Let $L = \mathscr{D}(\Omega) = L_0'$, $B = M(f)$. The restriction of B' to L_0' is easy to compute. Indeed, let $\varphi \in \mathscr{D}(\Omega)$, $g \in L_0'$. Then

$$\langle B'g, \varphi\rangle = \langle g, B\varphi\rangle = \langle g, f\varphi\rangle = \int_\Omega g(x)f(x)\varphi(x)\,dx.$$

Thus' B' acts on L_0' as multiplication by f. We see that this operator admits a continuous extension (namely, B') to the whole space $\mathscr{D}'(\Omega)$. Denoting this extension, as before, by $M(f)$, we get

$$\langle M(f)F, \varphi\rangle = \langle F, M(f)\varphi\rangle.$$

Example. Let us compute the product of the δ-function on the line by an infinitely differentiable function f. We have $\langle f\delta, \varphi\rangle = \langle \delta, f\varphi\rangle = f(0)\varphi(0)$. From this, $f\delta = f(0)\delta$, which agrees with our intuitive idea about the behavior of the δ-function under multiplication.

2. *Differentiation.* Recall that ∂_j denotes the partial derivative $\partial/\partial x_j$, and ∂^k the operator $\partial^{|k|}/\partial x_1^{k_1} \cdots \partial x_n^{k_n}$. Let us show that ∂^k admits a continuous extension from $\mathscr{D}(\Omega)$ to $\mathscr{D}'(\Omega)$. It suffices to consider the case of ∂_j. We let $B = -\partial_j$ and compute the restriction of B' to $\mathscr{D}(\Omega)$:

$$\langle B'f, \varphi \rangle = \langle f, B\varphi \rangle = -\int_\Omega f(x) \frac{\partial \varphi(x)}{\partial x_j} \, dx = \int_\Omega \frac{\partial f(x)}{\partial x_j} \varphi(x) \, dx.$$

Thus, B' coincides with ∂_j on $\mathscr{D}(\Omega)$. Therefore, the operator $\partial/\partial x_j$ admits a continuous extension (namely, B') to $\Omega'(\Omega)$. The explicit form of this extension (which we denote, as before, by ∂_j) is given by the formula

$$\langle \partial_j F, \varphi \rangle = -\langle F, \partial_j \varphi \rangle$$

and, in general, for any k

$$\langle \partial^k F, \varphi \rangle = (-1)^{|k|} \langle F, \partial^k \varphi \rangle.$$

Example. The generalized function $\partial^k \delta$ acts according to the formula

$$\langle \partial^k \delta, \varphi \rangle = (-1)^{|k|} \partial^k \varphi(0).$$

Change of Variables. Let $y = y(x)$ be a one-to-one infinitely differentiable mapping of Ω onto itself, and $x = x(y)$ the inverse mapping. The change-of-variables operator T: $(T\varphi)(y) = \varphi(x(y))$ is defined in the space $\mathscr{D}(\Omega)$. Let us compute the adjoint operator T' on $\mathscr{D}(\Omega)$:

$$\langle T'f, \varphi \rangle = \langle f, T\varphi \rangle = \int_\Omega f(y) \varphi(x(y)) \, dy = \int_\Omega f(y(x)) \varphi(x) \left| \frac{\partial y}{\partial x} \right| dx.$$

Thus, the operator adjoint to T is the composition of the operator T^{-1} of inverse change of variables and the operator $M(|\partial y/\partial x|)$ of multiplication by the modulus of the Jacobian of this change. From this it follows that T admits a continuous extension to $\mathscr{D}'(\Omega)$. An easy computation shows that it is given by the formula

$$\langle TF, \varphi \rangle = \langle F, M(|J|) T^{-1} \varphi \rangle,$$

where J is the Jacobian of the mapping T^{-1}.

Example. Let $a(x)$ be an infinitely differentiable one-to-one mapping of the line onto itself. Let us compute the generalized function $\delta(a(x))$:

$$\langle \delta(a(x)), \varphi \rangle = \langle \delta, \varphi(b(x)) |b'(x)| \rangle = \varphi(b(0)) |b'(0)|,$$

where $b(x)$ is the function inverse to $a(x)$. In particular, $\delta(ax + b) = |a|^{-1} \delta_{-b/a}(x)$. ($\delta_b(x)$ denotes the generalized function acting by the formula $\langle \delta_b, \varphi \rangle = \varphi(b)$.)

Remark. By construction, the operations on generalized functions defined above are continuous. Consequently, they commute with limits. In particular, a convergent series of generalized functions can be differentiated termwise any number of times.

We now show that the collection of all generalized functions with compact support arises in a natural way from the regular generalized functions through the use of the differentiation operation. Namely,

Theorem 33. *Every generalized function $F \in \mathscr{E}'(\Omega)$ can be written in the form*

$$F = \partial^k f, \tag{28}$$

where k is some multi-index, and f is a regular generalized function.

PROOF. It is convenient to take the system of seminorms determining the topology in $\mathscr{E}(\Omega)$ to be the system

$$p_{nk}(\varphi) = \int_{K_n} |\partial^k \varphi(x)| \, dx,$$

where $\{K_n\}$ is a family of compact sets exhausting the domain (see subsection 3). If a generalized function F is continuous with respect to the seminorm p_{nk}, then (by the Hahn–Banach theorem and the theorem on the general form of a linear functional on $L_1(K_n, dx)$) there exists a function $f \in L_\infty(K_n, dx)$ such that

$$\langle F, \varphi \rangle = \int_{K_n} \partial^k \varphi(x) f(x) \, dx.$$

When f is replaced by $(-1)^{|k|} f$, this equation becomes the desired relation (28). ☐

Remarks. (1) The proof given ensures only the measurability of f. By making the multi-index k larger, if necessary, it is possible to find an f that is continuous on Ω.

(2) It can be shown that outside the support of F the function f coincides with some polynomial annihilated by ∂^k.

(3) The next result is proved similarly.

Theorem 34. *Every tempered distribution $F \in S'(\mathbf{R}^n)$ admits a representation* (28) *in which f is a continuous function as well as an element of $S'(\mathbf{R}^n)$.*

The construction of the *direct product* of generalized functions plays a large role in the construction and study of generalized functions in multidimensional domains.

Let Ω_1 be a domain in $\mathbf{R}^m$, and Ω_2 a domain in $\mathbf{R}^n$. We set $\Omega = \Omega_1 \times \Omega_2 \subset \mathbf{R}^{m+n}$. If f_i are regular generalized functions in Ω_i, $i = 1, 2$, then it is possible to define a regular generalized function f in Ω by the formula $\langle f, \varphi \rangle = \int_\Omega f_1(x) f_2(y) \varphi(x, y) \, dx \, dy$. This f is called the *direct product* of f_1 and f_2 and denoted by $f_1 \times f_2$. By Fubini's theorem, this integral can be computed successively:

$$\langle f, \varphi \rangle = \int_{\Omega_1} f_1(x) \left(\int_{\Omega_2} f_2(y)\varphi(x, y)\, dy \right) dx$$
$$= \int_{\Omega_2} f_2(y) \left(\int_{\Omega_1} f_1(x)\varphi(x, y)\, dx \right) dy. \tag{29}$$

It turns out that this construction remains valid for any generalized functions.

Theorem 35. *Let* $\varphi \in \mathscr{D}(\Omega), f_i \in \mathscr{D}'(\Omega_i)$, $i = 1, 2$. *Then*:

(1) *the function* $\varphi_1(x) = \langle f_2, \varphi(x, \cdot)\rangle$ *belongs to* $\mathscr{D}(\Omega_1)$;
(2) *the function* $\varphi_2(y) = \langle f_1, \varphi(\cdot, y)\rangle$ *belongs to* $\mathscr{D}(\Omega_2)$;
(3) $\langle f_1, \varphi_1\rangle = \langle f_2, \varphi_2\rangle$;
(4) *the correspondence* $\varphi \to \langle f_1, \varphi_1\rangle = \langle f_2, \varphi_2\rangle$ *is a continuous linear functional on* $\mathscr{D}(\Omega)$.

Proof. Let us show that φ_1 has compact support. Suppose that $K \subset \Omega$ is the support of φ, and K_1 is the projection of K on Ω_1. Then the function $\varphi_x(y) = \varphi(x, y)$ is identically equal to zero for x outside K_1. Therefore, $\varphi_1(x) = \langle f_2, \varphi_x\rangle = 0$ for $x \notin K$, from which supp $\varphi_1 \subset K_1$. Further, the mapping $x \mapsto \varphi_x$ of Ω_1 into $\mathscr{D}(\Omega_2)$ is infinitely differentiable. This implies that φ_1 is infinitely differentiable. Thus, $\varphi_1 \in \mathscr{D}(\Omega_1)$. (2) is proved in exactly the same way.

To prove (3) we consider first the particular case when $\varphi(x, y)$ has the form $\psi_1(x)\psi_2(y)$. Obviously, $\varphi_1(x) = \psi_1(x)\langle f_2, \psi_2\rangle$, $\varphi_2(y) = \psi_2(y)\langle f_1, \psi_1\rangle$ in this case, and Eq. (29) holds. Since linear combinations of functions of the form $\psi_1(x)\psi_2(y)$ are dense in $\mathscr{D}(\Omega)$, there remains only to check that the mappings $\varphi \mapsto \langle f_1, \varphi_1\rangle$ and $\varphi \mapsto \langle f_2, \varphi_2\rangle$ are continuous in the topology of $\mathscr{D}(\Omega)$. This follows from the continuity of the mapping $\varphi \to \varphi_1$ from $\mathscr{D}(\Omega)$ into $\mathscr{D}(\Omega_1)$, which can be verified directly. □

§4. Hilbert Spaces

1. The Geometry of Hilbert Spaces

A linear space H over the field $K = \mathbf{R}$ or $\mathbf{C}$ is called a *pre-Hilbert space* if a *scalar product*† is given in it, i.e., a mapping of $H \times H$ into K denoted by

† The name "scalar product" arose from the work of Hamilton on the quaternion division ring. Each quaternion q can be represented as a sum of a "scalar part" q_0 and a "vector part" $\mathbf{q} = q_1 i + q_2 j + q_3 k$. Correspondingly, the product of two vector quaternions $\mathbf{q}$ and $\mathbf{r}$ is the sum of the scalar product $(\mathbf{q}, \mathbf{r})$ and the vector product $[\mathbf{q}, \mathbf{r}]$. (The term "inner product" is frequently used instead of "scalar product.")

(,) and having the properties:

(1) $(\lambda_1 x_1 + \lambda_2 x_2, y) = \lambda_1(x_1, y) + \lambda_2(x_2, y)$ (linearity in the first argument);
(2) $(x, y) = \overline{(y, x)}$ (*Hermitian symmetry*; the bar denotes complex conjugation);
(3) $(x, x) \geq 0$ (positive semi-definiteness).

A pre-Hilbert space H is called a *Hilbert space* if the following additional conditions hold:

(3′) $(x, x) > 0$ for $x \neq 0$;
(4) H is complete in the topology defined by the form $\|x\| = (x, x)^{1/2}$.

(The fact that this is really a norm is established below.)

One of the most important consequences of the properties of a scalar product is the

Cauchy–Bunyakovskii Inequality:

$$|(x, y)|^2 \leq (x, x)(y, y).$$

PROOF. Suppose first that (x, y) is a real number. For any $t \in \mathbf{R}$,

$$0 \leq (x + ty, x + ty) = (x, x) + 2t(x, y) + t^2(y, y).$$

Hence, this quadratic polynomial in t has a nonpositive discriminant: $(x, y)^2 - (x, x) \cdot (y, y) \leq 0$, which is the Cauchy–Bunyakovskii inequality for this case. The general case is obtained by multiplying x (or y) by a suitably chosen complex number of modulus 1. Then (x, y) becomes real without changing its absolute value, and (x, x) and (y, y) do not change. □

Corollary. $\|x\| = (x, x)^{1/2}$ *has the properties of a seminorm* (*and even of a norm, if the condition* (3′) *holds*).

Indeed,

$$\begin{aligned}\|x + y\|^2 &= (x, x) + (x, y) + (y, x) + (y, y)\\ &\leq (x, x) + 2|(x, y)| + (y, y)\\ &\leq (x, x) + 2\sqrt{(x, x) \cdot (y, y)} + (y, y) = (\|x\| + \|y\|)^2.\end{aligned}$$

The following construction can be used to "manufacture" a Hilbert space $\tilde{L}$ from any pre-Hilbert space L: Let $L_0 \subset L$ be the subset of all vectors x for which $(x, x) = 0$. The Cauchy–Bunyakovskii inequality shows that L_0 is a subspace of L. A scalar product can be defined in a natural way in the quotient space L/L_0: if $\bar{x}, \bar{y} \in L/L_0$ and $x \in \bar{x}$, $y \in \bar{y}$ are representatives of the classes $\bar{x}$ and $\bar{y}$, then we set $(\bar{x}, \bar{y}) = (x, y)$. We leave it to the reader to convince himself that this definition is unambiguous and that the scalar product obtained has the property (3′). If L/L_0 is not complete, then denote by $\tilde{L}$ its completion in the norm $\|x\| = (x, x)^{1/2}$. The Cauchy–Bunyakovskii inequality shows that the scalar product is continuous in this norm, and, consequently, can be extended to $\tilde{L}$. We omit a verification of the properties (1)–(3) for the extended scalar product.

EXAMPLES. (1) Let L be the space of sequences $\{x_n\}$ with only a finite number of nonzero terms, with the scalar product $(x, y) = \sum_{n=1}^{\infty} x_n \bar{y}_n$. In this case $L_0 = \{0\}$, $\tilde{L} = l_2(K)$.

(2) $L = C[a, b]$ with the scalar product $(f, g) = \int_a^b f(x)\overline{g(x)}\,dx$. In this case $L_0 = \{0\}$, $\tilde{L} = L_2[a, b]$.

(3) L is the space of measurable step functions on a set X with a measure μ, and the scalar product has the form $(f, g) = \int_X f(x)\overline{g(x)}\,d\mu$. Here L_0 consists of the functions equal to zero almost everywhere with respect to μ, and $\tilde{L} = L_2(X, \mu)$.

(4) L is the space of polynomials in the complex variable z, and the scalar product is given by

$$(P, Q) = \iint\limits_{|z| \le 1} P(z)\overline{Q(z)}\,dx\,dy, \qquad z = x + iy.$$

Here $L_0 = \{0\}$, and $\tilde{L}$ coincides with the collection $A^2(D)$ of all the functions analytic in the unit disk D and belonging to $L_2(D, dx\,dy)$ (see Problem 533).

In a real Hilbert space it is possible to define the *angle* $\varphi \in [0, \pi]$ between vectors x and y by the formula

$$\cos \varphi = \frac{(x, y)}{\|x\|\,\|y\|}.$$

In particular, if $(x, y) = 0$, then $\varphi = \pi/2$. In this case x and y are said to be *orthogonal*, and one writes $x \perp y$.

The angle is not defined in a complex Hilbert space, but the concept of orthogonality keeps its meaning. If S is a subset of a Hilbert space H, then $S^{\perp}$ denotes the *orthogonal complement* of S, i.e., the collection of all vectors $x \in H$ orthogonal to all vectors $y \in S$. Obviously, $S^{\perp}$ is always a closed linear subspace of H.

The next theorem describes one of the basic geometric properties of a Hilbert space.

Theorem 36. *If K is a nonempty closed convex subset of a Hilbert space H, then for any $x \in H$ there exists a unique point $y \in K$ closest to x.*

PROOF. Let $d = \inf_{y \in K} d(x, y)$, where $d(x, y)$ is the distance in the sense of the norm $\|x\| = (x, x)^{1/2}$, and suppose that y_n is a sequence of points in K

such that $\lim_{n\to\infty} d(x, y_n) = d$. Let us show that $\{y_n\}$ is a Cauchy sequence. To do this we use the *parallelogram law*:

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2, \tag{30}$$

which is valid in any pre-Hilbert space (see Problem 547). Applying this identity to the parallellogram with sides $x - y_n$ and $x - y_m$, we get

$$\|y_n - y_m\|^2 = 2\|x - y_n\|^2 + 2\|x - y_m\|^2 - \|2x - y_n - y_m\|^2.$$

Observe now that $\|2x - y_n - y_m\|^2 = 4\|x - [(y_n + y_m)/2]\|^2 \geq 4d^2$, since $(y_n + y_m)/2 \in K$. If n and m are sufficiently large, then $\|x - y_n\|^2 \leq d^2 + \varepsilon$, $\|x - y_m\|^2 \leq d^2 + \varepsilon$, from which $\|y_n - y_m\|^2 < 4(d^2 + \varepsilon) - 4d^2 = 4\varepsilon$, which shows that $\{y_n\}$ is a Cauchy sequence. Let $y = \lim y_n$. Then $d(x, y) = \lim d(x, y_n) = d$, so y is a nearest point to x. The uniqueness follows from the parallelogram law: if $d(x, y) = d(x, y') = d$, then $\|y - y'\|^2 = 2\|x - y\|^2 + 2\|x - y'\|^2 - \|2x - y - y'\|^2 \leq 4d^2 - 4d^2 = 0$. □

Remark. There is a longstanding conjecture that the converse of this theorem is true: if a set K in a Hilbert space H is such that any x in H has a unique nearest point in K, then K is closed and convex. This has been neither proved nor disproved.

Theorem 37. *Suppose that H is a Hilbert space, H_1 is a closed subspace of it, and $H_2 = H_1^\perp$. Then H is the direct sum of H_1 and H_2.*

PROOF. Let $x \in H$, and x_1 the point in H_1 closest to x. We set $x_2 = x - x_1$ and show that $x_2 \in H_2$. Indeed, let $y \in H_1$; we know that the function $f(t) = \|x - x_1 + ty\|^2$ of the real variable t has a minimum at $t = 0$, hence, $f'(0) = 0$. But

$$f'(0) = \lim_{t\to 0} \frac{\|x_2 + ty\|^2 - \|x_2\|^2}{t} = (x_2, y) + (y, x_2) = 2\mathrm{Re}(x_2, y).$$

Therefore, $\mathrm{Re}(x_2, y) = 0$. Replacing y by iy, we get $\mathrm{Im}(x_2, y) = 0$. Thus, $(x_2, y) = 0$, i.e., $x_2 \in H_2$. We have shown that H is the sum of H_1 and H_2. The fact that this sum is direct follows from the orthogonality of H_1 and H_2: if $x \in H_1 \cap H_2$, then $(x, x) = 0$, i.e., $x = 0$. The fact that H is the direct sum of two orthogonal subspaces H_1 and H_2 is expressed by writing $H = H_1 \oplus H_2$. □

Theorem 38. *Every continuous linear functional f on a Hilbert space has the form $f(x) = (x, y)$ for some $y \in H$.*

PROOF. Let $H_1 = \mathrm{Ker}\, f$ be the collection of all vectors annihilating f. If $H_1 = H$, then $f \equiv 0$, and we can set $y = 0$. If $H_1 \neq H$, then let $H_2 = H_1^\perp$. By Theorem 37, $H = H_1 \oplus H_2$. Let us show that H_2 is one-dimensional. Suppose that y_0 is a nonzero vector in H_2. Then $f(y_0) \neq 0$, since otherwise y_0 would be in H_1. For any $y_1 \in H_2$ the vector $y_1 - [f(y_1)/f(y_0)]y_0$ is in

$H_1 \cap H_2$, and, consequently, is equal to zero. This proves that $\{y_0\}$ is a basis in H_2. Now let $y = [\overline{f(y_0)}/(y_0, y_0)]y_0$ and compare f with the functional $x \mapsto (x, y)$. Both functionals vanish on H_1 and take the value $f(y_0)$ at y_0. Therefore, they coincide everywhere. □

Remark. Theorems 36–38 cease to be true in a pre-Hilbert space (see Problem 544).

A system $\{x_\alpha\}_{\alpha \in A}$ of vectors in a pre-Hilbert space is said to be *orthonormal* if

$$(x_\alpha, x_\beta) = \begin{cases} 1 & \text{for } \alpha = \beta, \\ 0 & \text{for } \alpha \neq \beta. \end{cases}$$

Bessel's Inequality. *For any orthonormal system $\{x_\alpha\}_{\alpha \in A}$ and any vector x,*

$$\sum_{\alpha \in A} |(x, x_\alpha)|^2 \leq (x, x). \tag{31}$$

(The sum of the left-hand side is understood as $\sup_{A_0} \sum_{A_0} |(x, x_\alpha)|^2$, where the supremum is taken over all finite subsets $A_0 \subset A$. It is not hard to show that this sum can be finite only if not more than countably many terms are nonzero.)

PROOF. By the definition of the sum of the left-hand side of (31), it suffices to check this inequality for a finite set A. Let H_1 be the subspace of H generated by the system $\{x_\alpha\}$, $\alpha \in A$, $H_2 = H_1^\perp$. Then $x = \sum_{\alpha \in A} c_\alpha x_\alpha + y$, where $y \in H_2$. Since $\{x_\alpha\}$ is orthonormal and H_1 and H_2 are orthogonal,

$$(x, x_\alpha) = c_\alpha, \qquad (x, x) = \sum_{\alpha \in A} |c_\alpha|^2 + (y, y),$$

and this immediately gives (31). □

An orthonormal system $\{x_\alpha\}_{\alpha \in A}$ in a Hilbert space H is said to be *complete* if its orthogonal complement consists of zero.

Parseval's Equality (A Generalization of the Pythagorean Theorem). *For any complete orthonormal system $\{x_\alpha\}_{\alpha \in A}$ and any vector x,*

$$(x, x) = \sum_{\alpha \in A} |(x, x_\alpha)|^2. \tag{32}$$

PROOF. Let A_0 be the set of all indices for which $(x, x_\alpha) \neq 0$. As already mentioned above, A_0 is countable. We number it and write $x_1, x_2, \ldots$ instead of $x_{\alpha_1}, x_{\alpha_2}, \ldots$, and $c_1, c_2, \ldots$ instead of $c_{\alpha_1}, c_{\alpha_2}, \ldots$. Let us consider the sequence of sums $S_n = \sum_{i=1}^n c_i x_i$. Since $\|S_{n+k} - S_n\|^2 = \sum_{i=n+1}^{n+k} |c_i|^2$ and the series $\sum_i |c_i|^2$ converges, the sequence $\{S_n\}$ is Cauchy. Let $S = \lim S_n$, and $y = x - S$. We show that, in fact, $y = 0$. To do this it suffices to verify that y is orthogonal to the system $\{x_\alpha\}_{\alpha \in A}$. For $\alpha \notin A_0$ this is obvious by construction, and for $\alpha \in A_0$ it follows from the equality

$$(y, x_i) = (x - S, x_i) = \lim_{n \to \infty} (x - S_n, x_i) = 0.$$

Thus, $y = 0$ and, hence, $x = S$. From this,

$$(x, x) = \lim_{n \to \infty} (S_n, S_n) = \lim_{n \to \infty} \sum_{i=1}^{n} |c_i|^2 = \sum_{\alpha \in A} |c_\alpha|^2. \qquad \square$$

In passing we have proved the

Theorem 39. *Every complete orthonormal system $\{x_\alpha\}_{\alpha \in A}$ in a Hilbert space H is a Hilbert basis, in the sense that any vector $x \in H$ can be written uniquely in the form*

$$x = \sum_{\alpha \in A} c_\alpha x_\alpha, \quad \text{where } c_\alpha = (x, x_\alpha).$$

Remark. The concept of a Hilbert basis differs in general from that of a basis in a linear space and coincides with it only in the finite-dimensional case. The difference stems from the fact that infinite linear combinations are allowed, and these do not make sense in a purely algebraic setting.

Theorem 40. *Every Hilbert space has a Hilbert basis. All bases of a given space H have the same cardinality. (This cardinality is called the Hilbert dimension of H.)*

PROOF. By using Zorn's lemma, it is easy to prove the existence of a maximal orthonormal system in H. If it were not complete, then by adjoining to this system a unit vector in its orthogonal complement, we would contradict the maximality of the system. Hence, a maximal system is complete, and, by Theorem 39, is a Hilbert basis in H.

The fact that two Hilbert bases in a finite-dimensional space have the same cardinality follows from the analogous algebraic fact, because in this case the concepts of a basis and a Hilbert basis coincide. Suppose now that H has a countable (Hilbert) basis $\{x_n\}_{n \in \mathbf{N}}$. Then H is infinite-dimensional (since the x_n are independent) and separable (i.e., has a countable dense subset; for example, the collection of all finite linear combinations of basis vectors with rational coefficients serves as one). Therefore, every other basis $\{y_\alpha\}_{\alpha \in A}$ contains infinitely many elements. If A were uncountable, then H would contain an uncountable set of disjoint balls of radius 1 (it suffices to take the balls with centers at the points $2y_\alpha$, $\alpha \in A$), which contradicts the separability of H. The case of uncountable dimension requires additional information from set theory, and we omit it.

For separable Hilbert spaces the existence of a basis can be proved by means of an *orthogonalization* process, without using Zorn's lemma. Suppose that $\{x_n\}_{n \in \mathbf{N}}$ is a countable system of vectors in H that has zero alone as its orthogonal complement (for example, a countable family of vectors dense in H, which exists because H is separable). Discarding "superfluous" vectors, we can assume that the x_n are linearly independent. Let us now define new sequences of vectors $\{y_n\}_{n\in\mathbf{N}}$, $\{z_n\}_{n\in\mathbf{N}}$ as follows:

$$
\begin{array}{ll}
y_1 = x_1, & z_1 = \dfrac{y_1}{\|y_1\|}, \\[2ex]
y_2 = x_2 - (x_2, z_1)z_1, & z_2 = \dfrac{y_2}{\|y_2\|}, \\
\dots\dots\dots\dots\dots\dots & \dots\dots\dots\dots \\
y_n = x_n - \sum\limits_{i=1}^{n-1} (x_n, z_i)z_i, & z_n = \dfrac{y_n}{\|y_n\|}, \\
\dots\dots\dots\dots\dots\dots & \dots\dots\dots\dots
\end{array}
$$

It is easy to see that the system $\{z_n\}$ is orthonormal and that the linear span of $z_1, \ldots, z_n$ coincides with that of $x_1, \ldots, x_n$. Therefore, $\{z_n\}_{\alpha\in\mathbf{N}}$ is a basis in H. Note that if the original system $\{x_n\}$ lies in some (nonclosed) subspace $H_0 \subset H$, then the system $\{z_n\}$ also lies in H_0. From this, in particular, it follows that there is a basis in every separable pre-Hilbert space.

Theorem 41. *Two Hilbert spaces are isomorphic if and only if they have the same Hilbert dimension.*

PROOF. The necessity of the condition is obvious. Let us prove the sufficiency. Suppose that H_1 and H_2 have the same dimension. This means that H_1 and H_2 have bases $\{x_\alpha\}_{\alpha\in A}$ and $\{y_\alpha\}_{\alpha\in A}$ of the same cardinality. Let the operator $U\colon H_1 \to H_2$ be defined by $U(\sum_{\alpha\in A} c_\alpha x_\alpha) = \sum_{\alpha\in A} c_\alpha y_\alpha$. Parseval's equality shows that this operator is an isometry (i.e., preserves the scalar product), and, consequently, maps H_1 onto some complete subspace $L \subset H_2$. Since L contains the basis $\{y_\alpha\}_{\alpha\in A}$, $L^\perp = \{0\}$. Therefore, $L = H_2$. □

Corollary. *All separable infinite-dimensional Hilbert spaces are isomorphic.*

2. Operators on a Hilbert Space

As we saw in subsection 1, each continuous linear functional on a Hilbert space H can be expressed as a scalar product. From this it follows that a real Hilbert space H can be identified in a natural way with its dual H': the functional $f_y(x) = (x, y)$ corresponds to the vector y. For a complex Hilbert space this relation is an anti-isomorphism, since f_y depends on y *anti-linearly*: $f_{\lambda_1 y_1 + \lambda_2 y_2} = \bar{\lambda}_1 f_{y_1} + \bar{\lambda}_2 f_{y_2}$. We introduce the *Hermitian dual* space H^* of H. The continuous antilinear functionals on H are the elements of H^*. The spaces H and H^* can be identified in a natural way: the antilinear functional $f_x^*(y) = (x, y)$ corresponds to a vector $x \in H$.

If H_1 and H_2 are Hilbert spaces, and A is a linear operator from H_1 to H_2, then it is possible to define the *Hermitian adjoint* operator A^*, which acts from $H_2^* = H_2$ into $H_1^* = H_1$ according to the formula

$$(A^*x_2, x_1) = (x_2, Ax_1), \qquad x_i \in H_i.$$

The correspondence $A \mapsto A^*$ has the following properties:

$$(\lambda_1 A_1 + \lambda_2 A_2)^* = \bar{\lambda}_1 A_1^* + \bar{\lambda}_2 A_2^*; \qquad (AB)^* = B^*A^*; \qquad (A^*)^* = A.$$

Hermitian adjoint operators are often called simply adjoints. We shall also do this, while preserving, however, the difference in the notation: the adjoint operator is denoted by A' and the Hermitian adjoint by A^*.

The following classes of operators will be singled out:

The *selfadjoint* (or *Hermitian*) operators A are characterized by the property $A^* = A$.

The *unitary* operators are characterized by the property $U^* = U^{-1}$. (This is equivalent to the conditions $U^*U = 1 = UU^*$. In the finite-dimensional case one of these equalities suffices.)

The *normal* operators N are characterized by the property $NN^* = N^*N$.

The *orthogonal projections* P are characterized by the properties $P^* = P = P^2$ (see Problem 554).

The *positive* operators A are characterized by the property that $(Ax, x) \geq 0$ for all $x \in H$. This property is expressed by $A \gg 0$. A partial order is introduced in the set End H by writing $A \gg B$ if $A - B \gg 0$.

Example. (1) The operator A of multiplication by a function $a \in L_\infty(X, \mu)$ is selfadjoint in the space $L_2(X, \mu)$ if $a(x)$ is real almost everywhere, and unitary if $|a(x)| = 1$ almost everywhere. This operator is normal for any function a, is an orthogonal projection if a takes the values 0 or 1 almost everywhere, and is positive if $a \geq 0$ almost everywhere.

(2) The one-sided shift operator T acting in $l_2(\mathbf{R})$ by the formula $T\{x_n\} = \{x_{n+1}\}$, has the property $TT^* = 1$, but it is not unitary, since $T^*T = 1 - P_1$, where P_1 is the projection onto the subspace generated by the first basis vector.

(3) Let A be the integral operator in $L_2(X, \mu)$ with kernel $K(x_1, x_2)$, i.e., $Af(x_1) = \int_X K(x_1, x_2) f(x_2)\, d\mu(x_2)$. We compute the adjoint operator A^*. In this case the basic equation $(Af, g) = (f, A^*g)$ has the form

$$\int_X \int_X K(x_1, x_2) f(x_2) \overline{g(x_1)}\, d\mu(x_1)\, d\mu(x_2) = \int_X f(x) \overline{(A^*g)(x)}\, d\mu(x),$$

from which $(A^*g)(x) = \int_X \overline{K(x_1, x) g(x_1)}\, d\mu(x_1)$. Thus, A^* is also an integral operator, with kernel $K^*(x_1, x_2) = \overline{K(x_2, x_1)}$.

Note that in the particular case when X consists of finitely many points of unit measure, we obtain the familiar relation in linear algebra between the elements of a matrix A and the Hermitian conjugate matrix A^*: $a_{ik}^* = \overline{a_{ki}}$.

Theorem 42. *A linear operator A from a Hilbert space H_1 into a Hilbert space H_2 is compact if and only if it can be approximated uniformly by operators of finite rank.*

PROOF. The sufficiency of the condition is valid in any Banach space and was established earlier (see Theorem 15). The necessity is proved as follows. Let $A \in \mathscr{K}(H_1, H_2)$, and let B_1 be the unit ball of H_1. Then AB_1 is a precompact set in H_2. Take an arbitrary $\varepsilon > 0$, let $y_1, \ldots, y_N$ be a finite ε-net for AB_1, H_0 the linear span of $y_1, \ldots, y_N$, and P_0 the orthogonal projection onto H_0 in H_2. It is clear that $P_0 A$ is an operator of finite rank, since $\operatorname{im} P_0 A \subset \operatorname{im} P_0 = H_0$. On the other hand,

$$\|A - P_0 A\| = \sup_{x \in B_1} \|Ax - P_0 Ax\| = \sup_{y \in AB_1} \|y - P_0 y\| \leq \varepsilon,$$

since

$$\|y - P_0 y\| = \|y - y_i - P_0(y - y_i)\| = \|(1 - P_0)(y - y_i)\| \leq \|y - y_i\|.$$

(The last fact follows from the fact that $1 - P_0$ is, along with P_0, an orthogonal projection and, consequently, has norm 1.) □

To each Hermitian operator A in a Hilbert space there corresponds a *Hermitian* (*quadratic*, in a real space) *form* $Q_A(x) = (Ax, x)$.

Theorem 43. *For any Hermitian operator A the equation $\sup_{\|x\|=1} |Q_A(x)| = \|A\|$ holds; if the supremum is attained at a point x_0, then x_0 is an eigenvector of A, with eigenvalue $\pm \|A\|$.*

PROOF. It is clear that $|Q_A(x)| = |(Ax, x)| \leq \|Ax\| \, \|x\| \leq \|A\|$ for $\|x\| = 1$. To get the reverse inequality we make use of the identity $Q_A(x + y) - Q_A(x - y) = 4 \operatorname{Re}(Ax, y)$, which is easily derived from the definition of Q_A. Let $\sup_{\|x\|=1} |Q_A(x)| = c$. Then our identity implies the estimate

$$4 \operatorname{Re}(Ax, y) \leq c\|x + y\|^2 + c\|x - y\|^2,$$

or, by the parallelogram law,

$$2 \operatorname{Re}(Ax, y) \leq c\|x\|^2 + c\|y\|^2.$$

Let $y = (\|x\|/\|Ax\|)Ax$ here. Then we get $2\|x\| \, \|Ax\| \leq 2c\|x\|^2$ or $\|Ax\| \leq c\|x\|$, which is what was required. Another derivation can be obtained from Problem 556.

Suppose now that $\sup |Q_A(x)|$ is attained at the point x_0. Let z be any unit vector orthogonal to x_0. The vector $x_t = x_0 \cos t + z \sin t$ coincides with x_0 for $t = 0$ and has unit length for any t. Therefore, $Q_A(x_t)$ has an extremum at $t = 0$. Hence, $(d/dt)Q_A(x_t)|_{t=0} = 0$. Computing this derivative, we get $2 \operatorname{Re}(Ax_0, z) = 0$. Replacing z by iz, we see that $(Ax_0, z) = 0$. Therefore, $Ax_0 \in (\{x_0\}^\perp)^\perp$, i.e., $Ax_0 = \lambda x_0$. Finally, $\lambda = (\lambda x_0, x_0) = (Ax_0, x_0) = \pm \|A\|$. □

Theorem 44. *If a subspace $H_1 \subset H$ is invariant under a Hermitian operator A, then so is its orthogonal complement $H_2 = H_1^\perp$.*

PROOF. Let $x_1 \in H_1$, $x_2 \in H_2$. We show that $Ax_2 \perp x_1$. This follows from the equality $(Ax_2, x_1) = (x_2, Ax_1)$. Thus, $AH_2 \subset H_2$. □

Theorem 45 (Hilbert). *Let A be a compact Hermitian operator in a Hilbert space H. There exists an orthonormal basis $\{x_\beta\}_{\beta \in B}$ consisting of eigenvectors of A. The corresponding eigenvalues $\{\lambda_\beta\}$ are real, and for any $\varepsilon > 0$ only finitely many of them lie in the domain $|\lambda| > \varepsilon$.*

PROOF. By using Zorn's lemma, it is easy to show that there exists a maximal orthonormal system consisting of eigenvectors of A. Let us show that this system is a basis in H. Otherwise, let H_0 be the orthogonal complement of this system. By Theorem 44, H_0 is invariant under A. The restriction A_0 of A to H_0 is a compact Hermitian operator without an eigenvector (by the maximality of the original system). We show that this contradicts Theorem 43. Indeed, the compactness of A_0 implies that Q_{A_0} is continuous in the weak topology on the unit ball B_1. Namely, if $x_n \rightharpoonup x$, then

$$\begin{aligned} Q_{A_0}(x_n) - Q_{A_0}(x) &= (A_0 x_n, x_n) - (A_0 x, x) \\ &= (A_0(x_n - x), x_n) + (Ax, x_n - x). \end{aligned}$$

The first factor in the first term converges strongly to zero in view of the compactness of A_0, and the second factor is bounded, since $x_n \in B_1$. Therefore, the first term converges to zero. The first factor in the second term is fixed, and the second factor converges to zero. Thus, Q_A is a weakly continuous function on B_1. Since the ball B_1 is compact in the weak topology, the function $|Q_A|$ attains its supremum on B_1 at some point $x_0 \in B_1$. By Theorem 43, x_0 is an eigenvector of A, and this gives the desired contradiction. The fact that the eigenvalues are real follows from the relations $\lambda_\beta = (Ax_\beta, x_\beta) = (x_\beta, Ax_\beta) = \bar{\lambda}_\beta$.

Let us prove the last assertion of the theorem. Denote by $B_\varepsilon \subset B$ the collection of indices β for which $|\lambda_\beta| > \varepsilon$, and let H_ε be the subspace generated by $\{x_\beta\}_{\beta \in B_\varepsilon}$. The space H_ε is invariant under A. Let A_ε be the restriction of A to H_ε. Then A_ε is a compact invertible operator, which is possible only if H_ε is finite-dimensional, so B_ε is finite. □

Remark. The proof of Hilbert's theorem can be given a constructive character: We can find the eigenvalues of A successively in order of decreasing absolute value by using Theorems 43 and 44.

It turns out that not only the maximal (in absolute value) eigenvalue of a Hermitian operator A has a variational meaning.

Courant's Theorem (Minimax Theorem). *Let A be a compact Hermitian operator in the Hilbert space H. Suppose that its nonzero eigenvalues are numbered with multiplicity taken into account in such a way that*

$$\lambda_{-1} \le \lambda_{-2} \le \cdots < 0 < \cdots \le \lambda_2 \le \lambda_1.$$

Then

$$\lambda_n = \inf_{H_{n-1}} \sup_{x \perp H_{n-1}} \frac{Q_A(x)}{\|x\|^2}, \qquad \lambda_{-n} = \sup_{H_{n-1}} \inf_{x \perp H_{n-1}} \frac{Q_A(x)}{\|x\|^2},$$

where H_{n-1} runs through all $(n-1)$-dimensional subspaces of H.

Proof. Let $\mathring{H}_{n-1}$ be the linear span of vectors $x_1, \ldots, x_{n-1}$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$. Then the maximal eigenvalue of A in $\mathring{H}_{n-1}^{\perp}$ is λ_n. Therefore, $\sup_{x \perp \mathring{H}_{n-1}} [Q_A(x)/\|x\|] = \lambda_n$. On the other hand, for each $(n-1)$-dimensional subspace H_{n-1} the space $H_{n-1}^{\perp}$ has a nonzero intersection with the space spanned by $x_1, \ldots, x_n$. Let

$$x = \sum_{k=1}^{n} c_k x_k \in H_{n-1}^{\perp}.$$

Then

$$\frac{Q_A(x)}{\|x\|^2} = \frac{\sum |c_k|^2 \lambda_k}{\sum |c_k|^2} \geq \lambda_n, \quad \text{i.e.,} \quad \sup_{x \perp H_{n-1}} \frac{Q_A(x)}{\|x\|^2} \geq \lambda_n.$$

The second formula follows from the first upon replacing A by $-A$. □

This theorem implies an interlacing principle (see Problem 566) which is useful in applications.

Chapter IV

The Fourier Transformation and Elements of Harmonic Analysis

§1. Convolutions on an Abelian Group

1. Convolutions of Test Functions

Let G be a finite group, and K some field. Denote by $K[G]$ the collection of formal linear combinations of elements of the group G with coefficients in K. The elements of $K[G]$ have the form

$$x = \sum_{g \in G} a(g)g, \quad \text{where } a(g) \in K. \tag{1}$$

The structure of an algebra over the field K can be introduced in a natural way on the set $K[G]$:

$$\begin{gathered} \lambda\left(\sum_{g \in G} a(g)g\right) = \sum_{g \in G} \lambda a(g)g, \\ \sum_{g \in G} a(g)g + \sum_{g \in G} b(g)g = \sum_{g \in G} (a(g) + b(g))g, \\ \left(\sum_{g \in G} a(g)g\right)\left(\sum_{g \in G} b(g)g\right) = \sum_{\substack{g_1 \in G \\ g_2 \in G}} a(g_1)b(g_2)g_1g_2. \end{gathered} \tag{2}$$

It is convenient to identify the element $x \in K[G]$ given by the formula (1) with the function $a(g)$ on G with values in K. With this interpretation, multiplication by a number and addition in $K[G]$ become the usual operations on functions. However, the multiplication operation differs from the usual (pointwise) multiplication. It is called the *convolution* and denoted by $*$. Its explicit form is given by the formulas

$$(a * b)(g) = \sum_{h \in G} a(gh^{-1})b(h) = \sum_{h \in G} a(h)b(h^{-1}g) = \sum_{g_1 g_2 = g} a(g_1)b(g_2). \quad (3)$$

The set $K[G]$ along with the operations introduced above in it is called the *group algebra* of G. This algebra arises naturally as a universal object in a suitable category (see Problem 579) and plays a large role in the theory of linear representations of groups.

As a rule, we shall be concerned in what follows with infinite groups G equipped with a measure μ. In this case it is natural to replace the sum in the formula (3) by an integral. More precisely, we suppose that G is an abelian *topological* group (this means that a Hausdorff topology is given in G with respect to which the group operations $(g_1, g_2) \to g_1 g_2$ and $g \to g^{-1}$ are continuous) and that a Borel measure μ† is given on G that is invariant under translations and passage to inverse elements. The group operation in G will be denoted by $+$. Then the invariance property of μ can be expressed in the form

$$\mu(X + \alpha) = \mu(X), \qquad \mu(-X) = \mu(X) \quad (4)$$

for any Borel set $X \subset G$ and any $\alpha \in G$. It is known that such a measure μ exists if and only if G is locally compact,‡ and in this case the invariant measure is uniquely determined to within a numerical multiple.

Basic Examples. (1) $G = \mathbf{R}^n$, with ordinary vector addition as the group operation and the usual Lebesgue measure in $\mathbf{R}^n$ as μ:

$$d\mu(x) = dx_1\, dx_2 \cdots dx_n.$$

(2) $\mathbf{Z}^n$, the n-dimensional integer lattice in $\mathbf{R}^n$ consisting of the vectors with integer coordinates. The group operation is addition, and the invariant measure μ has the form $\mu(X) = \operatorname{card} X$ (the number of points in the set X).

(3) $\mathbf{T}^n$, the n-dimensional torus. We shall consider two realizations of $\mathbf{T}^n$: either as the subset of $\mathbf{C}^n$ consisting of the vectors $z = (z_1, \ldots, z_n)$ with $|z_k| = 1, 1 \leq k \leq n$, and with the operation of coordinatewise multiplication, or as the quotient group $\mathbf{R}^n/\mathbf{Z}^n$, whose elements can be given by vectors $t \in \mathbf{R}^n$ with the condition $t_k \in [0, 1)$ and with the operation of addition

† More precisely, μ is defined on the σ-algebra of σ-compact Borel sets and is σ-finite on each such set. In what follows we consider only σ-compact groups.

‡ A topological space is said to be locally compact if any point of it has a pre-compact neighborhood.

modulo 1. The correspondence between the realizations is established by the formula $z_k = e^{2\pi i t_k}$, $1 \leq k \leq n$. The invariant measure μ is the usual Lebesgue measure in the coordinates $t_1, \ldots, t_n$. We remark that this group is compact and that $\mu(\mathbf{T}^n) = 1$.

The *convolution* of functions f_1 and f_2 on an abelian group G with invariant measure μ is defined by the formulas

$$(f_1 * f_2)(x) = \int_G f_1(x - y) f_2(y)\, d\mu(y) = \int_G f_1(y) f_2(x - y)\, d\mu(y), \tag{5}$$

which form the exact analog of (3) (and become (3) if G is finite and $\mu(X) = \operatorname{card}(X)$.

Theorem 1. *If $f_1, f_2 \in L_1(G, \mu)$, then the integral (5) exists for almost all $x \in G$, the function $f_1 * f_2$ belongs to $L_1(G, \mu)$, and $\|f_1 * f_2\| \leq \|f_1\|\, \|f_2\|$.*

Proof. If $f_1, f_2 \in L_1(G, \mu)$, then, by Fubini's theorem, the function $\varphi(x, y) = f_1(x) f_2(y)$ is in $L_1(G \times G, \mu \times \mu)$, and $\|\varphi\| = \|f_1\|\, \|f_2\|$.

Let us now consider the transformation τ of the space $G \times G$ that takes the point (x, y) into $(x + y, y)$. This transformation is measurable (carries Borel sets into Borel sets) and preserves the measure $\mu \times \mu$. Indeed, if $X = A \times B \subset G \times G$ is an elementary measurable set, then

$$\begin{aligned}
\mu \times \mu(\tau(X)) &= \int_{G \times G} \chi_{\tau(X)}(x, y)\, d\mu(x)\, d\mu(y) = \int_{G \times G} \chi_X(x - y, y)\, d\mu(x)\, d\mu(y) \\
&= \int_G \left(\int_G \chi_X(x - y, y)\, d\mu(x) \right) d\mu(y) \\
&= \int_B \mu(A + y)\, d\mu(y) = \mu(A)\mu(B) = \mu \times \mu(X).
\end{aligned}$$

From this it follows that τ generates an isometric transformation T of $L_1(G \times G, \mu \times \mu)$ according to the formula.

$$T\varphi(x, y) = \varphi(\tau^{-1}(x, y)) = \varphi(x - y, y).$$

Applying this result to the function $\varphi(x, y) = f_1(x) f_2(y)$, we get the statement of the theorem. □

Remark. The inequality just proven implies that the convolution operation is continuous in the space $L_1(G, \mu)$.

Theorem 2. *The convolution operation is commutative, associative, and distributive over addition.*

Proof. The last statement follows at once from the linearity of the integral. The first two can be proved by a suitable change of variables that preserves the measure μ or $\mu \times \mu$. Namely,

$$\begin{aligned} f_1 * f_2(x) &= \int_G f_1(x - y) f_2(y)\, d\mu(y) \\ &= \int_G f_1(-y) f_2(x + y)\, d\mu(y) = \int_G f_2(x - y) f_1(y)\, d\mu(y) \\ &= (f_2 * f_1)(x), \end{aligned}$$

$$\begin{aligned} ((f_1 * f_2) * f_3(x) &= \int_G (f_1 * f_2)(x - y) f_3(y)\, d\mu(y) \\ &= \int_G \int_G f_1(x - y - z) f_2(z) f_3(y)\, d\mu(z)\, d\mu(y) \\ &= \int_G \int_G f_1(x - z) f_2(z - y) f_3(y)\, d\mu(z)\, d\mu(y) \\ &= (f_1 * (f_2 * f_3))(x). \end{aligned}$$

Suppose now that $T(a)$ denotes the translation operator $(T(a)f)(x) = f(x + a)$ on G. It is clear that $T(a)$ is a linear isometric operator in $L_1(G, \mu)$. □

Theorem 3. *The convolution operation commutes with translations:*

$$T(a)(f_1 * f_2) = T(a)f_1 * f_2 = f_1 * T(a)f_2. \tag{6}$$

Proof. We have

$$\begin{aligned} T(a)(f_1 * f_2)(x) = (f_1 * f_2)(x + a) &= \int_G f_1(x + a - y) f_2(y)\, d\mu(y) \\ &= \int_G T(a) f_1(x - y) f_2(y)\, d\mu(y) = (T(a)f_1 * f_2)(x). \end{aligned}$$

The second equality follows from the first and the commutativity of the convolution. □

Henceforth, we shall use the notation $S(f)$ for the operator of convolution with f: $S(f_1)f_2 = f_1 * f_2$. The assertions of Theorem 2 can be formulated as the identities

$$S(f_1)S(f_2) = S(f_2)S(f_1) = S(f_1 * f_2), \tag{7}$$

and the assertion of Theorem 3 as the identity

$$T(a)S(f) = S(f)T(a) = S(T(a)f). \tag{8}$$

Theorem 4. *If $\varphi \in \mathscr{D}(\mathbf{R}^n)$, then $S(\varphi)$ is a continuous operator from $L_1(\mathbf{R}^n, dx)$ to $\mathscr{E}(\mathbf{R}^n)$ and from $\mathscr{D}(\mathbf{R}^n)$ to $\mathscr{D}(\mathbf{R}^n)$.*

PROOF. Let $\varphi \in \mathscr{D}(\mathbf{R}^n)$, $f \in L_1(\mathbf{R}^n, dx)$. We show that $S(\varphi)f$ is an infinitely differentiable function and that

$$\partial^k S(\varphi)f = S(\partial^k \varphi)f. \tag{9}$$

It is clearly sufficient to verify this in the case of the partial derivative $\partial_j = \partial/\partial x_j$. Let e_j be the jth basis vector in $\mathbf{R}^n$. The operator ∂_j can be written in the form $\lim_{t\to 0} [(T(te_j) - 1)/t]$. By Theorem 3, the last operator commutes with $S(f)$. Therefore,

$$\begin{aligned}\partial_j S(\varphi)f(x) = \partial_j S(f)\varphi(x) &= \lim_{t\to 0} \frac{T(te_j) - 1}{t} S(f)\varphi(x)\\ &= \lim_{t\to 0} S(f)\frac{T(te_j) - 1}{t}\varphi(x) = S(f)\,\partial_j\varphi.\end{aligned}$$

The last equality follows from the fact that the function $[(T(te_j) - 1)/t]\varphi$ converges uniformly to $\partial_j\varphi$ for $\varphi \in \mathscr{D}(\mathbf{R}^n)$, and the operator $S(f)$ preserves uniform convergence. Thus, Eq. (9) and the infinite differentiability of $S(\varphi)f$ are proved. Let us verify that $S(\varphi)\colon L_1(\mathbf{R}^n, dx) \to \mathscr{E}(\mathbf{R}^n)$ is a continuous operator. For any seminorm p_{Kk} on $\mathscr{E}(\mathbf{R}^n)$ we have

$$\begin{aligned}p_{K,k}(S(\varphi)f) = \sup_K(\partial^k S(\varphi)f(x)) &\le \sup_{\mathbf{R}^n}|S(\partial^k\varphi)f(x)|\\ &\le \sup_{\mathbf{R}^n}|\partial^k\varphi(x)|\ \|f\|_1,\end{aligned}$$

as claimed.

To prove the last assertion we verify that for $\varphi_1, \varphi_2 \in \mathscr{D}(\mathbf{R}^n)$

$$\operatorname{supp}(\varphi_1 * \varphi_2) \subset \operatorname{supp}\varphi_1 + \operatorname{supp}\varphi_2, \tag{10}$$

where supp denotes the support, and the $+$ on the right-hand side is the arithmetic sum of sets: $X + Y = \{x + y \mid x \in X,\ y \in Y\}$. Indeed, if $x \notin \operatorname{supp}\varphi_1 + \operatorname{supp}\varphi_2$, then the vector $x - y \notin \operatorname{supp}\varphi_2$, for any $y \in \operatorname{supp}\varphi_1$. Therefore, the integrand in the integral (5) defining $\varphi_1 * \varphi_2(x)$ is identically equal to zero. Hence, $\varphi_1 * \varphi_2$ vanishes outside $\operatorname{supp}\varphi_1 + \operatorname{supp}\varphi_2$. The latter set is compact and, consequently, contains $\operatorname{supp}(\varphi_1 * \varphi_2)$. We have proved that $S(\varphi)$ carries $\mathscr{D}(\mathbf{R}^n)$ into $\mathscr{D}(\mathbf{R}^n)$. It suffices to verify the continuity of $S(f)$ on the subspaces $\mathscr{D}_K(\mathbf{R}^n)$, where this property can be established by the same calculation as above, with account taken of the fact that $S(\varphi)$ carries $\mathscr{D}_K(\mathbf{R}^n)$ into $\mathscr{D}_{K_1}(\mathbf{R}^n)$, where $K_1 = K + \operatorname{supp}\varphi$. □

2. Convolutions of Generalized Functions

The definition of the convolution can be generalized to the case when one or both of the factors are not ordinary but generalized functions.

Let $F \in \mathscr{D}'(\mathbf{R}^n)$, $\varphi \in \mathscr{D}(\mathbf{R}^n)$. The convolution $F * \varphi$ can be defined in two ways.

First way. The operator $S(\varphi)$, as we know (see Theorem 4), is a continuous operator in the space $\mathscr{D}(\mathbf{R}^n)$. Let us compute the action of the adjoint operator $S(\varphi)'$ on a regular generalized function $f(x)$. We have

$$\langle S(\varphi)'f, \psi\rangle = \langle f, S(\varphi)\psi\rangle = \int_{\mathbf{R}^n} f(x)\varphi * \psi(x)\,dx$$
$$= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} f(x)\varphi(x - y)\psi(y)\,dy\,dx. \tag{11}$$

The notation $\check{\varphi}(x) = \varphi(-x)$ will be used. Then the last expression can be transformed as follows:

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} f(x)\check{\varphi}(y - x)\psi(y)\,dy\,dx = \int_{\mathbf{R}^n} f * \check{\varphi}(y)\psi(y)\,dy = \langle S(\check{\varphi})f, \psi\rangle.$$

Thus, we have established the equality of the operators $S(\varphi)'$ and $S(\check{\varphi})$ on the regular generalized functions. From this it follows that $S(\varphi)$ admits a continuous extension to $\mathscr{D}'(\mathbf{R}^n)$, namely, the operator $S(\check{\varphi})'$. This is the first definition of the convolution. We write it as the formula

$$\langle F * \varphi, \psi\rangle \overset{\text{def}}{=} \langle F, \check{\varphi} * \psi\rangle. \tag{12}$$

Second way. The integral $\int_{\mathbf{R}^n} \varphi(x - y)F(y)\,dy$ defining the convolution in the case of ordinary functions can be defined in the case $F \in \mathscr{D}'(\mathbf{R}^n)$ as the value of the functional F on the test function $\psi(y) = \varphi(x - y)$. Using the notation introduced earlier, we can express this definition by the formula

$$F * \varphi(x) = \langle F, T(-x)\check{\varphi}\rangle. \tag{13}$$

Thus, the second definition of the convolution expresses $F * \varphi$ as an ordinary function on $\mathbf{R}^n$. It turns out that the two definitions actually coincide. Namely,

Theorem 5. *If $F \in \mathscr{D}'(\mathbf{R}^n)$, $\varphi \in \mathscr{D}(\mathbf{R}^n)$, then the generalized function $F * \varphi$ defined by* (12) *is regular and infinitely differentiable, and it can be computed at a point $x \in \mathbf{R}^n$ by the formula* (13).

Proof. Note first of all that the element $T(-x)\check{\varphi} \in \mathscr{D}(\mathbf{R}^n)$ depends continuously on $x \in \mathbf{R}^n$. (If $x_n \to x$ in $\mathbf{R}^n$, then $T(-x_n)\check{\varphi} \to T(-x)\check{\varphi}$ in the topology of $\mathscr{D}(\mathbf{R}^n)$.) Therefore, the right-hand side of Eq. (13) is a continuous function. We consider it as a regular generalized function on $\mathbf{R}^n$ and show that it coincides with the generalized function (12). To do this we must check for any $\psi \in \mathscr{D}(\mathbf{R}^n)$ the equality

$$\langle F, \check{\varphi} * \psi\rangle = \int_{\mathbf{R}^n} \langle F, T(-x)\check{\varphi}\rangle\psi(x)\,dx$$

or

$$\int_{\mathbf{R}^n} F(y)\left(\int_{\mathbf{R}^n} \varphi(x - y)\psi(x)\,dx\right)dy = \int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^n} F(y)\varphi(x - y)\,dy\right)\psi(x)\,dx.$$

We prove the more general equality

$$\int_{\mathbf{R}^n} F(y)\left(\int_{\mathbf{R}^m} \alpha(x, y)\, dx\right) dy = \int_{\mathbf{R}^m} \left(\int_{\mathbf{R}^n} F(y)\alpha(x, y)\, dy\right) dx, \tag{14}$$

where $F \in \mathscr{D}'(\mathbf{R}^n)$, $\alpha \in \mathscr{D}(\mathbf{R}^{n+m})$. To do this, observe that the set of $\alpha \in \mathscr{D}(\mathbf{R}^{m+n})$ for which the equality (14) holds forms a closed linear subspace. This subspace contains all the functions α of the form $\alpha(x, y) = \beta(x)\gamma(y)$, $\beta \in \mathscr{D}(\mathbf{R}^n)$, $\gamma \in \mathscr{D}(\mathbf{R}^m)$, and, consequently, coincides with $\mathscr{D}(\mathbf{R}^{m+n})$.

Finally, the infinite differentiability of $F * \varphi$ follows from the infinite differentiability of the mapping $x \mapsto T(-x)\check{\varphi}$, which can be directly checked. □

The convolution $F * \varphi$ can be defined in a similar way when φ and F belong to other spaces of test and generalized functions (see Problems 604, 605).

Suppose now that $F \in \mathscr{D}'(\mathbf{R}^n)$, $f \in \mathscr{E}'(\mathbf{R}^n)$. We show that the convolution $F * f$ can be defined and is an element of $\mathscr{D}'(\mathbf{R}^n)$. We already know that the operator $S(F)$ of convolution with $F \in \mathscr{D}'(\mathbf{R}^n)$ carries $\mathscr{D}(\mathbf{R}^n)$ into $\mathscr{E}(\mathbf{R}^n)$. It can be shown (see Problem 604) that this operator is continuous. Further, the operation φ and the relation (12) can be carried over to generalized functions. From this it follows that $S(F)$ has a continuous extension (namely, $S(\check{F})'$) to the space $\mathscr{E}'(\mathbf{R}^n)$ and maps it into $\mathscr{D}'(\mathbf{R}^n)$. This is the desired definition of the convolution. We write it as the formula

$$\langle F * f, \varphi\rangle = \langle f, \check{F} * \varphi\rangle. \tag{15}$$

Remark. In this definition the factors F and f play a nonsymmetric role. It would be possible to construct the extension of the operator $S(f)\colon \mathscr{D}(\mathbf{R}^n) \to \mathscr{D}'(\mathbf{R}^n)$ to a continuous operator on $\mathscr{D}'(\mathbf{R}^n)$ (namely, $S(\check{f})'$) and to define $F * f$ as the result of applying this extended operator to F. We would get the formula

$$\langle F * f, \varphi\rangle = \langle F, \check{f} * \varphi\rangle. \tag{15$'$}$$

It can be shown (cf. the proof of Theorem 5) that the formulas (15) and (15′) define the same generalized function.

We give one more useful formula for the convolution of the generalized functions F and f:

$$\langle F * f, \varphi\rangle = \langle F \times f, \mathring{\varphi}\rangle, \tag{16}$$

where $F \times f$ is the direct product of F and f (see §4 in Ch. III), i.e., the generalized function on $\mathbf{R}^{2n}$ given by one of the equivalent formulas

$$\begin{aligned}\langle F \times f, \alpha\rangle &= \int_{\mathbf{R}^n} F(x)\left(\int_{\mathbf{R}^n} f(y)\alpha(x, y)\, dy\right) dx \\ &= \int_{\mathbf{R}^n} f(y)\, dy\left(\int_{\mathbf{R}^n} F(x)\alpha(x, y)\, dx\right),\end{aligned} \tag{17}$$

and $\overset{\circ}{\varphi}$ denotes the function $\varphi(x+y)$. The proof of this formula is left to the reader (it reduces to a change of variables in the integrals containing the generalized functions).

We now give a "multiplication table" for the convolution operation in the basic function spaces:

f \ φ	$\mathscr{D}(\mathbf{R}^n)$	$S(\mathbf{R}^n)$	$\mathscr{E}(\mathbf{R}^n)$	$\mathscr{E}'(\mathbf{R}^n)$	$S'(\mathbf{R}^n)$	$\mathscr{D}'(\mathbf{R}^n)$
$\mathscr{D}(\mathbf{R}^n)$	$\mathscr{D}(\mathbf{R}^n)$	$S(\mathbf{R}^n)$	$\mathscr{E}(\mathbf{R}^n)$	$\mathscr{D}(\mathbf{R}^n)$	$P\mathscr{E}(\mathbf{R}^n)$	$\mathscr{E}(\mathbf{R}^n)$
$S(\mathbf{R}^n)$	$S(\mathbf{R}^n)$	$S(\mathbf{R}^n)$	—	$S(\mathbf{R}^n)$	$P\mathscr{E}(\mathbf{R}^n)$	—
$\mathscr{E}(\mathbf{R}^n)$	$\mathscr{E}(\mathbf{R}^n)$	—	—	$\mathscr{E}(\mathbf{R}^n)$	—	—
$\mathscr{E}'(\mathbf{R}^n)$	$\mathscr{D}(\mathbf{R}^n)$	$S(\mathbf{R}^n)$	$\mathscr{E}(\mathbf{R}^n)$	$\mathscr{E}'(\mathbf{R}^n)$	$S'(\mathbf{R}^n)$	$\mathscr{D}'(\mathbf{R}^n)$
$S'(\mathbf{R}^n)$	$P\mathscr{E}(\mathbf{R}^n)$	$P\mathscr{E}(\mathbf{R}^n)$	—	$S'(\mathbf{R}^n)$	—	—
$\mathscr{D}'(\mathbf{R}^n)$	$\mathscr{E}(\mathbf{R}^n)$	—	—	$\mathscr{D}'(\mathbf{R}^n)$	—	—

The dash indicates that the corresponding convolution operation is not defined. $P\mathscr{E}(\mathbf{R}^n)$ denotes the subspace of $\mathscr{E}(\mathbf{R}^n)$ consisting of the functions of not greater than polynomial growth. To remember this table it is useful to bear in mind the following rule: A convolution is defined if at least one of the factors is compactly supported, smooth if at least one of the factors is smooth, and compactly supported if both factors are.

Theorem 6. *Convolution operators commute with each other (when their composition makes sense), with translation operators, and with differentiation operators.*

PROOF. Note that the first assertion of the theorem implies the two others, because translation operators and differentiation operators are particular cases of convolution operators. Namely,

$$T(a)\varphi = \delta_a * \varphi, \tag{18}$$

$$\partial^k \varphi = \partial^k \delta * \varphi. \tag{19}$$

For test functions φ Eq. (18) follows from (13):

$$\delta_a * \varphi(x) = \int_{\mathbf{R}^n} \delta_a(x-y)\varphi(y)\,dy = \varphi(x+a) = T(a)\varphi(x).$$

For generalized functions φ with compact support (18) follows from (15) and the equalities $\check{\delta}_a = \delta_{-a}$, $T(a)' = T(-a)$, which can be checked directly.

Equation (19) can be proved by induction on the number $|k|$. The fundamental lemma in this proof is the relation $\partial_j \varphi = \partial_j \delta * \varphi$, which is proved just as (18).

Thus, all that remains is to check that $S(f_1)S(f_2) = S(f_2)S(f_1)$ in the cases when at least one of f_1 and f_2 has compact support. We suppose for definiteness that $f_1 = f \in \mathscr{E}'(\mathbf{R}^n)$, $f_2 = F \in \mathscr{D}'(\mathbf{R}^n)$, and we must verify the commutativity of the diagram

$$\begin{tikzcd} \mathscr{E}'(\mathbf{R}^n) \arrow[r, "S(F)"] \arrow[d, "S(f)"'] & \mathscr{D}'(\mathbf{R}^n) \arrow[d, "S(f)"] \\ \mathscr{E}'(\mathbf{R}^n) \arrow[r, "S(F)"] & \mathscr{D}'(\mathbf{R}^n) \end{tikzcd}$$

By the definition of the action of a convolution operator on a generalized function, this is equivalent to the commutativity of the diagram

$$\begin{tikzcd} \mathscr{D}(\mathbf{R}^n) \arrow[r, "S(F)"] \arrow[d, "S(f)"'] & \mathscr{E}(\mathbf{R}^n) \arrow[d, "S(f)"] \\ \mathscr{D}(\mathbf{R}^n) \arrow[r, "S(F)"] & \mathscr{E}(\mathbf{R}^n) \end{tikzcd},$$

which follows from Eq. (16). □

This theorem on the commutativity of convolution operators admits the following interesting and useful generalization.

Theorem 7. *Let A be a continuous operator from $\mathscr{D}(\mathbf{R}^n)$ to $\mathscr{E}(\mathbf{R}^n)$. The following properties are equivalent for A:*

(1) *A commutes with translations;*
(2) *A commutes with differentiation operators;*
(3) *A is a convolution operator with some generalized function $f \in \mathscr{D}'(\mathbf{R}^n)$.*

Proof. According to Theorem 6, the property (3) implies (1) and (2). Let us show that (2) implies (1). Suppose that $\varphi \in \mathscr{D}(\mathbf{R}^n)$, $a \in \mathbf{R}^n$, $t \in \mathbf{R}$. We consider the function of a real variable $t \mapsto T(-ta)AT(ta)\varphi$ with values in $\mathscr{E}(\mathbf{R}^n)$. It is easy to compute the derivative of this function with respect to t by taking account of the equality

$$\frac{d}{dt}T(ta) = D_a T(ta) = T(ta)D_a, \tag{20}$$

which is valid for translation operators in $\mathscr{D}(\mathbf{R}^n)$ and $\mathscr{E}(\mathbf{R}^n)$ (D_a denotes the derivative along the vector a). This derivative has the form

$$[-T(-ta)D_a AT(ta) + T(-ta)AD_a T(ta)]\varphi,$$

which is equal to zero, because A and D_a commute. Therefore, the function is a constant function. Equating its values for $t = 1$ and $t = 0$ yields

$$T(-a)AT(a)\varphi = A\varphi,$$

which implies (1). Let us now derive (3) from (1). To do this, note that the correspondence $\varphi \mapsto A\varphi(0)$ is a continuous linear functional on $\mathscr{D}(\mathbf{R}^n)$; denote it by f. It can now be concluded from (1) that $A\varphi(x) = T(x)A\varphi(0) = AT(x)\varphi(0) = \langle f, T(x)\varphi\rangle = \check{f} * \varphi(x)$. Thus, $A = S(\check{f})$. □

Remark. The statement of the theorem and its proof can be carried over without change to the case when the operator A acts from $S'(\mathbf{R}^n)$ or $\mathscr{E}'(\mathbf{R}^n)$ to $S'(\mathbf{R}^n)$.

Here the property (3) is formulated as follows: $A = S(f)$, where $f \in S'(\mathbf{R}^n)$ or $\in \mathscr{E}'(\mathbf{R}^n)$, respectively. It is still true, although somewhat more complicated to prove, that every operator from $\mathscr{D}(\mathbf{R}^n)$ to itself that commutes with translations has the form $S(f)$, where $f \in \mathscr{E}'(\mathbf{R}^n)$. Accordingly, in all these cases the translation operators form a maximal commutative family. This property shows that the family of convolution operators resembles the family of operators of multiplication by functions. We shall see below that this similarity is not accidental: The two families pass into each other under a certain transformation of the space.

§2. The Fourier Transformation

1. Characters on an Abelian Group

Let G be an abelian group. A *character* on this group is a homomorphism of G into the group $\mathbf{T}$, i.e., a complex-valued function χ on G equal to 1 in absolute value and such that

$$\chi(x + y) = \chi(x)\chi(y). \tag{21}$$

If G is a topological group, then, as a rule, the term "character" will be mean "continuous character." All the characters under discussion will be assumed without special mention to be continuous. If χ_1 and χ_2 are characters on G, then $\chi_1\chi_2$ is also a character, and if χ is a character, then $\chi^{-1} = \bar{\chi}$ is also a character.† Thus, the collection of all characters on G forms a group with respect to the operation of ordinary multiplication of functions. This group is denoted by $\hat{G}$ and called the group *dual* to G. This $\hat{G}$ becomes a topological group if the convergence $\chi_n \to \chi$ is defined to be uniform convergence on every compact set $K \subset G$.

Example. Let $G = \mathbf{Z}$, the group of integers. It is clear that each character $\chi \in \hat{G}$ is determined by its value on the generating element $1 \in \mathbf{Z}$ (do not

† As usual, the bar denotes complex conjugation.

confuse 1 with the identity of the group, which is 0). Indeed, (21) implies that

$$\chi(n) = [\chi(1)]^n \quad \text{for all } n \in \mathbf{Z}. \tag{22}$$

The value $\chi(1)$ can be any number $z \in \mathbf{T}$. The set $\hat{\mathbf{Z}}$ itself can thereby be identified with the circle $\mathbf{T}$.

Theorem 8. *The groups* $\hat{\mathbf{Z}}$ *and* $\mathbf{T}$ *are topologically isomorphic.*

PROOF. We have already seen that the set $\hat{\mathbf{Z}}$ can be identified in a natural way with $\mathbf{T}$. Let us show that this correspondence is an isomorphism of topological groups. Let χ_z be the character defined by the condition $\chi_z(1) = z$, $z \in \mathbf{T}$. The equality $\chi_{z_1}\chi_{z_2} = \chi_{z_1 z_2}$ shows that the correspondence $z \to \chi_z$ is an isomorphism of the groups $\mathbf{T}$ and $\hat{\mathbf{Z}}$. It remains to check that this correspondence is a homeomorphism. Since $\mathbf{Z}$ is a discrete group, each compact set in $\mathbf{Z}$ consists of a finite number of points. Hence, the convergence in $\hat{\mathbf{Z}}$ is pointwise convergence. Equation (22) shows that $\chi_{z_n} \to \chi_z$ if and only if $\chi_{z_n}(1) \to \chi_z(1)$, i.e., $z_n \to z$. □

Theorem 9. *The group* $\hat{\mathbf{T}}$ *is isomorphic to* $\mathbf{Z}$.

PROOF. To each $n \in \mathbf{Z}$ there corresponds a character χ_n on $\mathbf{T}$ given by

$$\chi_n(z) = z^n, \qquad z \in \mathbf{T}. \tag{23}$$

We prove below (see also Problem 630) that $\mathbf{T}$ does not have other characters besides those given by (23). Therefore, the correspondence $n \to \chi_n$ establishes an equivalence of the sets $\mathbf{Z}$ and $\hat{\mathbf{T}}$. The equation $\chi_n \chi_m = \chi_{n+m}$ shows that this equivalence is a group isomorphism. That this correspondence is a homeomorphism of the topological spaces follows from the fact that the set $\hat{\mathbf{T}}$ is discrete, which in turn is a consequence of the more general result in Problem 625. □

We thus see that the groups $\mathbf{Z}$ and $\mathbf{T}$ are dual to each other. This fact is a special case of the following result:

Pontryagin Duality Principle. *For any abelian locally compact topological group* G *the natural mapping of* G *into* $\hat{\hat{G}}$ *that assigns to an element* $g \in G$ *the character* f_g *on* $\hat{G}$ *given by*

$$f_g(\chi) = \chi(g), \qquad \chi \in \hat{G}, \tag{24}$$

is an isomorphism of topological groups.

We mention that this principle does not always hold for general topological groups (see Problem 631).

Theorem 10. *The group* $\hat{\mathbf{R}}$ *is isomorphic to* $\mathbf{R}$.

PROOF. To each $\lambda \in \mathbf{R}$ let us assign the character $\chi_\lambda \in \mathbf{R}$ given by

$$\chi_\lambda(x) = e^{2\pi i \lambda x}. \tag{25}$$

We show that the formula (25) gives all the characters on the group $\mathbf{R}$. Let $\chi \in \mathbf{R}$. Suppose first that χ is a differentiable function. Then by differentiating with respect to y and setting $y = 0$, we get from (21) that $\chi'(x) = c\chi(x)$, where $c = \chi'(0)$. This differential equation has a unique solution satisfying the initial condition $\chi(0) = 1$, namely, $\chi(x) = e^{cx}$. The condition $|\chi(x)| \equiv 1$ implies that c is purely imaginary. Hence χ coincides with one of the characters (25).

We now drop the assumption that χ is differentiable. One procedure is to regard χ as an element of $\mathscr{D}'(\mathbf{R})$ (see Problem 627). Another is to use a smoothing technique. Let $\varphi \in \mathscr{D}(\mathbf{R})$. Then $\chi * \varphi$ is an infinitely differentiable function. On the other hand, for a suitable function φ (for example, for a term sufficiently far out in a sequence of δ-shaped functions), $\chi * \varphi$ is proportional to χ (see Problem 628) with a nonzero constant of proportionality. From this it follows that every character on $\mathbf{R}$ is an infinitely differentiable function.

Accordingly, we have established a one-to-one correspondence between $\mathbf{R}$ and $\hat{\mathbf{R}}$: $\lambda \to \chi_\lambda$. The equation $\chi_\lambda \chi_\mu = \chi_{\lambda+\mu}$ shows that this correspondence is a group isomorphism. Let us prove that it is a homeomorphism. If $\lambda_n \to \lambda$, then $\chi_{\lambda_n} \to \chi_\lambda$ uniformly on any compact subset of $\mathbf{R}$, as follows from the explicit form of χ_λ. Conversely, suppose that $\chi_{\lambda_n} \to \chi_\lambda$ uniformly on any segment. Then $\chi_{\lambda_n - \lambda}(x) \to 1$ uniformly on $[0, 1]$. But

$$\begin{aligned}\sup_{[0,1]} |\chi_{\lambda_n - \lambda}(x) - 1| &= \sup_{[0,1]} 2|\sin \pi(\lambda_n - \lambda)x| \\ &= \begin{cases} 2 & \text{for } |\lambda_n - \lambda \geq 1/2, \\ 2 \sin \pi|\lambda_n - \lambda| & \text{for } |\lambda_n - \lambda| \leq 1/2. \end{cases}\end{aligned}$$

which implies that $\lambda_n \to \lambda$. □

Remark 1. Although the group $\mathbf{R}$ is dual to itself, there is no canonical isomorphism $\mathbf{R} \to \hat{\mathbf{R}}$ (unlike the isomorphism $\mathbf{R} \to \hat{\hat{\mathbf{R}}}$, which is canonical). Our choice of the correspondence $\lambda \to e^{2\pi i \lambda x}$ is dictated by the convenience of the definition of the Fourier transform in $L_2(\mathbf{R})$ and the Poisson formula (see below). Other correspondences are frequently used: $\lambda \mapsto e^{i\lambda x}$ or $\lambda \mapsto e^{-i\lambda x}$.

Remark 2. We can now prove the statement about the characters on $\mathbf{T}$, omitted in the proof of Theorem 9. Let $\chi \in \hat{\mathbf{T}}$. Consider the function $\chi_1(t) = \chi(e^{2\pi i t})$. It is clear that χ_1 is a character on the group $\mathbf{R}$. Hence, $\chi_1(t) = e^{2\pi i \lambda t}$ for some $\lambda \in \mathbf{R}$. Since $\chi_1(1) = \chi(1) = 1$, the number λ must be an integer. From this, $\chi_1(t) = e^{2\pi i n t}$, i.e., $\chi(z) = z^n$.

We leave it to the reader to prove the following general statement, which contains Theorems 8–10 as special cases.

Theorem 11. *Let $G = \mathbf{R}^n \times \mathbf{Z}^k \times \mathbf{T}^l$ be the direct product of the groups. Then the dual group $\hat{G}$ is isomorphic to $\mathbf{R}^n \times \mathbf{T}^k \times \mathbf{Z}^l$.*

Let G be a locally compact abelian group with invariant measure μ. For any function $f \in L_1(G, \mu)$ we define its *Fourier transform* $\tilde{f}$ by the formula

$$\tilde{f}(\chi) = \int_G f(x)\overline{\chi(x)}\, d\mu(x), \qquad \chi \in \hat{G}. \tag{26}$$

Thus, Fourier transformation carries functions on G into functions on the dual group $\hat{G}$.

EXAMPLES. (1) If $G = \mathbf{Z}$, then a function $f \in L_1(\mathbf{Z})$ is a summable two-sided sequence $\{c_n\}_{n \in \mathbf{Z}}$. The Fourier transformation assigns to this sequence a function $f(Z) = \sum_{n \in \mathbf{Z}} c_n z^n$ on $\mathbf{T}$, sometimes called the *generating function* of the sequence $\{c_n\}_{n \in \mathbf{Z}}$.

(2) If $G = \mathbf{T}, f \in L_1(\mathbf{T}, dt)$, then the Fourier transform of f is none other than the sequence of *Fourier coefficients* of f: $c_n = \tilde{f}(\chi_n) = \int_0^1 f(t)e^{-2\pi i n t}\, dt$.

(3) If $G = \mathbf{R}, f \in L_1(\mathbf{R}, dx)$, then the Fourier transform $\tilde{f}$ is also a function on $\mathbf{R}$, called a *Fourier integral*: $\tilde{f}(\lambda) = \int_{\mathbf{R}} f(x)e^{-2\pi i \lambda x}\, dx$.

The Fourier transformation effects several important correspondences among the operations of convolution, multiplication, and translation:

Theorem 12. *Let G be a locally compact abelian group with invariant measure μ. The Fourier transformation carries $L_1(G, \mu)$ into the space of continuous bounded functions on $\hat{G}$. Under this transformation a convolution of functions passes into ordinary multiplication,*

$$(f_1 * f_2)\tilde{}\,(\chi) = \tilde{f}_1(\chi) \cdot \tilde{f}_2(\chi), \tag{27}$$

the translation operator $T(x)$, $x \in G$, passes into the operator of multiplication by the character $f_x \in \hat{G}$ (see (24)):

$$(T(x)f)\tilde{}\,(\chi) = f_x \tilde{f}(\chi), \tag{28}$$

and the operator of multiplication by a character $\chi \in \hat{G}$ passes into the translation operator $T(\chi^{-1})$ (in the multiplicative notation for the group operation in $\hat{G}$):

$$(M(\chi)f)\tilde{}\,(\chi_1) = T(\chi^{-1})\tilde{f}(\chi_1) = \tilde{f}(\chi\chi_1^{-1}). \tag{29}$$

PROOF. Let $\chi_n \to \chi$ in $\hat{G}$ and $f \in L_1(G, \mu)$. For any $\varepsilon > 0$ there is a compact set $K \subset G$ such that $\int_{G \setminus K} |f|\, d\mu < \varepsilon$. By the definition of the topology in $\hat{G}$, the functions $\chi_n(x)$ converge uniformly on K to $\chi(x)$. Therefore, beginning with some $n(\varepsilon)$ the inequality $|\chi_n(x) - \chi(x)| < \varepsilon$ holds for $x \in K$. From this,

$$|\tilde{f}(\chi_n) - \tilde{f}(\chi)| \le \int_K |f(x)|\, |\chi_n(x) - \chi(x)|\, d\mu(x) + \int_{G \setminus K} |f(x)(\chi_n(x) - \chi(x))|\, d\mu(x) \le \varepsilon\|f\| + 2\varepsilon,$$

which shows that $\tilde{f}$ is continuous. The boundedness of $\tilde{f}$ follows directly from an estimate of the integral (26). The equalities (27), (28), and (29) can be proved by direct computation using a change of variables. □

2. Fourier Series

The expansion of a function in a Fourier series is the most studied case of the Fourier transformation on an abelian group (the group $\mathbf{T}$ in this case). The investigation of Fourier series makes up a separate and broad area in the theory of functions. Here we restrict ourselves only to the basic facts that are most important for applications.

In what follows, the general notation $\tilde{f}(\chi)$ for the Fourier transform of a function f on $\mathbf{T}$ will be replaced by the more traditional notation

$$c_n = \tilde{f}(\chi_n) = \int_0^1 f(t)e^{-2\pi int}\,dt.$$

Theorem 13. *The Fourier transformation is a unitary operator from $L_2(\mathbf{T}, dt)$ to $l_2(\mathbf{Z})$.*

Proof. By Problem 531, the functions $\chi_n(t) = e^{2\pi int}$ form an orthonormal basis in $L_2(\mathbf{T}, dt)$. If e_n denotes the two-sided sequence containing a 1 in the nth place and 0's elsewhere, then it is precisely the Fourier transform of the function χ_n. Since $\{e_n\}_{n\in\mathbf{Z}}$ is an orthonormal basis in $l_2(\mathbf{Z})$, the Fourier transformation is unitary (see Problem 561). □

This theorem is a particular case of the following fact:

Suppose that G is a compact abelian group and $\hat{G}$ is the discrete group dual to it (see Problem 625). *It is assumed that the invariant measures μ and $\hat{\mu}$ on these groups are normalized by the conditions*:

$$\mu(G) = 1, \qquad \hat{\mu}(X) = \text{card}\, X \quad \text{for } X \subset \hat{G}.$$

Then the Fourier transformation is a unitary operator from $L_2(G, \mu)$ to $L_2(\hat{G}, \hat{\mu})$.

There are many results on the connection between the smoothness of a function f on $\mathbf{T}$ and the rapidity of decrease of its Fourier coefficients (see, for example, Problems 645–648). The basis for deriving them is

Theorem 14. *The differentiation operator is carried under the Fourier transformation into the operator of multiplication by the sequence $\{2\pi in\}_{n\in\mathbf{Z}}$.*

Proof. Let $f \in C^1(\mathbf{T})$; then the Fourier coefficients c_n^1 of the derivative of f can be computed by integrating by parts:

$$c_n^1 = \int_0^1 f'(t)e^{-2\pi int}\,dt = e^{-2\pi int}f(t)\Big|_0^1 - \int_0^1 f(t)\,de^{-2\pi int} = 2\pi inc_n,$$

where $c_n = \int_0^1 f(t)e^{-2\pi int}\,dt$ are the Fourier coefficients of f. □

Let us now consider the Fourier transformation of generalized functions on $\mathbf{T}$. Suppose that $f \in \mathscr{E}'(\mathbf{T})$. It is natural to call the sequence of coefficients $c_n = \langle f, e^{-2\pi int} \rangle$ the *Fourier transform of the generalized function f.*

A two-sided sequence $\{c_n\}_{n \in \mathbf{Z}}$ will be said to be *slowly increasing* (or a *sequence of temperate growth*) if $c_n = O(n^k)$ for some k. The space of all slowly increasing sequences is denoted by $P(\mathbf{Z})$.

Theorem 15. *The Fourier transformation defines an isomorphism of the spaces $\mathscr{E}'(\mathbf{T})$ and $P(\mathbf{Z})$ under which a convolution passes into an ordinary product, the translation operator $T(t)$ passes into the operator of multiplication by the sequence $\{e^{2\pi int}\}$, differentiation with respect to t passes into the operator of multiplication by the sequence $\{2\pi in\}$, and multiplication by the character $e^{2\pi ikt}$ passes into translation by k.*

PROOF. These properties have already been proved for smooth regular generalized functions. They remain true for arbitrary generalized functions because every generalized function on $\mathbf{T}$ is a finite sum of derivatives (of some order) of regular functions. □

Let us now consider the question of recovering a function f from its Fourier transform $\{c_n\}_{n \in \mathbf{Z}}$. If $f \in L_2(\mathbf{T}, dt)$, then, as we know,

$$f(t) = \sum_{n \in \mathbf{Z}} c_n e^{2\pi int}, \tag{30}$$

where the series on the right-hand side converges in the sense of the space $L_2(\mathbf{T}, dt)$. If the coefficients c_k decrease rapidly enough, then this series converges in a stronger sense. For example, if $c_n = O(n^{-k})$ for all $k > 0$, then the series converges in the topology of $\mathscr{E}(\mathbf{T})$. The question of convergence of Fourier series occupies a large place in the theory of functions. We shall not dwell on the (sometimes difficult) problems arising in this connection.

Comparing formula (30) with the definition of the Fourier transformation on the group $\mathbf{Z}$:

$$\{c_n\} \to \sum_{n \in \mathbf{Z}} c_n e^{-2\pi int},$$

we see that these transformations are obtained one from the other by composition with the following transformations on $\mathbf{T}$ or $\mathbf{Z}$:

$$\{\check{c_n}\} = \{c_{-n}\} \quad \text{or} \quad \check{f}(t) = f(-t).$$

The term *inverse Fourier transformation* will be used for the composition of the usual (or *direct*) Fourier transformation with the transformation $^\vee$ ("check"). It turns out that a general principle holds: the direct and inverse Fourier transformations are mutually inverse operators. The precise formulation of this principle, specifying the classes of functions on G and $\hat{G}$ corresponding to each other, depends on the specific group G.

3. The Fourier Integral

The Fourier transform of a function on the real line is given by the *Fourier integral*:

$$\tilde{f}(\lambda) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i\lambda x}\,dx. \tag{31}$$

In this case the inverse Fourier transform takes the form:

$$\check{\tilde{f}}(\lambda) = \int_{-\infty}^{\infty} f(x)e^{2\pi i\lambda x}\,dx. \tag{32}$$

Theorem 16. *The direct and inverse Fourier transformations are mutually inverse transformations of the space* $S(\mathbf{R})$.

Proof. We show that $S(\mathbf{R})$ is carried into itself under the action of the Fourier transformation (both the direct and the inverse). Indeed, since a function $f \in S(\mathbf{R})$ and all its derivatives are integrable and converge to zero at infinity, integration by parts gives the equality

$$(\tilde{f}^{(k)})(\lambda) = (2\pi i\lambda)^k\tilde{f}(\lambda). \tag{33}$$

Further, since the functions $x^kf(x)$ are integrable for any k, differentiation of the formula (31) with respect to λ gives the equality

$$\tilde{f}^{(k)} = [f(-2\pi ix)^k]^\sim. \tag{34}$$

Denoting the Fourier transformation operator by F, we can rewrite (33) and (34) in the form of the commutation relations

$$FD^k = M^kF, \qquad F(-M)^k = D^kF, \tag{35}$$

$$FDF^{-1} = M, \qquad FMF^{-1} = -D, \tag{36}$$

where D denotes the differentiation operator d/dx, and M is the operator of multiplication by $2\pi ix$. The system of seminorms defining the topology in $S(\mathbf{R})$ has the form

$$p_{kl}(f) = \sup_{x\in\mathbf{R}} |x^kf^{(l)}(x)|.$$

By Theorem 29 in Ch. III, this system is equivalent to the system

$$p'_{kl}(f) = \int_{\mathbf{R}} |x^kf^{(l)}(x)|\,dx.$$

Let us estimate the seminorm $p_{kl}(\tilde{f})$ in terms of the seminorms $p'_{mn}(f)$. To do this, we observe that $p_{kl}(f) = (2\pi)^{-k}p_{00}(M^kD^lf)$, $p'_{kl}(f) = (2\pi)^{-k}p'_{00}(M^kD^lf)$. Further, the estimate of the integral (31) gives $p_{00}(\tilde{f}) \leq p'_{00}(f)$. From this,

$$p_{kl}(\tilde{f}) = (2\pi)^{-k}p_{00}(M^kD^l\tilde{f}) = (2\pi)^{-k}p_{00}((D^kM^lf)^\sim).$$

In order to get the required estimate the operators D^k and M^l must be permuted. By Lebnitz' formula,

$$D^k M^l = \sum_{j=0}^{\min(k,l)} \frac{k!l!}{j!(k-j)!(l-j)!}(2\pi i)^j M^{l-j} D^{k-j}. \tag{37}$$

We have proved that F is a continuous mapping. The continuity of the inverse Fourier transformation $\check{F}$ can be proved in exactly the same way by using the commutation relations

$$\check{F}D^k = (-M)^k \check{F}, \qquad \check{F}M^k = D^k \check{F} \tag{35'}$$

or

$$\check{F}D(\check{F})^{-1} = -M, \qquad \check{F}M(\check{F})^{-1} = D. \tag{36'}$$

Let us now consider the composition of F and $\check{F}$. The relations (36) and (36′) imply immediately that this composition commutes with the operators D and M.

Lemma. *Every continuous operator in $S(\mathbf{R})$ that commutes with M is an operator of multiplication by some function.*

PROOF. Suppose that the operator A in $S(\mathbf{R})$ commutes with M. Then it commutes with the operator of multiplication by any polynomial. We show that for any function $\varphi \in S(\mathbf{R})$ and any point $a \in \mathbf{R}$ the value of $A\varphi(a)$ depends on φ only through $\varphi(a)$. Indeed, if $\varphi_1(a) = \varphi_2(a)$, then the difference $\varphi_1(x) - \varphi_2(x)$ vanishes at a, and, hence, has the form $(x - a)\psi(x)$, where $\psi \in S(\mathbf{R})$. Therefore, $A\varphi_1(x) = A[\varphi_2 + (x - a)\psi] = A\varphi_2(x) + (x - a)A\psi(x)$, since A commutes with multiplication by $x - a$. From this, $A\varphi_1(a) = A\varphi_2(a)$, as claimed. Thus, for any $a \in \mathbf{R}$ there is a number $f(a)$ such that $A\varphi(a) = f(a)\varphi(a)$ for all $\varphi \in S(\mathbf{R})$. It follows that $A = M(f)$. □

Remark. The proof given does not allow any conconclusions about the structure of f. But we know that $f\varphi$ is in $S(\mathbf{R})$ for any $\varphi \in S(\mathbf{R})$. This implies that f is infinitely differentiable.

Let us return to the proof of the theorem. By virtue of the lemma proved, the operators $F\check{F}$ and $\check{F}F$ are multiplication operators. Moreover,

$$DM(f) - M(f)D = M(f'),$$

by Leibnitz' rule. Therefore, an operator of the form $M(f)$ commutes with D only if $f' = 0$, i.e., $f = \text{const}$. This shows that $F\check{F}$ and $\check{F}F$ are scalar operators: $F\check{F} = C_1$, $\check{F}F = C_2$. There remains only to verify that $C_1 = C_2 = 1$, which follows, for example, from an explicit calculation of the Fourier transformation of some function in $S(\mathbf{R})$ (see Problem 668). □

Theorem 17. *There exists a unitary operator on $L_2(\mathbf{R}, dx)$ whose restriction to $L_2(\mathbf{R}, dx) \cap L_1(\mathbf{R}, dx)$ coincides with the Fourier transformation. (This operator will be denoted in what follows by the letter F, just like the direct Fourier transformation.)*

In particular, for any $f \in S(\mathbf{R})$ we have the *Plancherel formula*

$$\|f\|_{L_2(\mathbf{R}, dx)} = \|\tilde{f}\|_{L_2(\mathbf{R}, d\lambda)}. \tag{38}$$

Proof. From Theorem 16 and the general properties of the Fourier transformation (see subsection 1) it follows that the Fourier transformation takes multiplication into convolution:

$$(f_1 f_2)^{\sim} = \tilde{f}_1 * \tilde{f}_2. \tag{39}$$

Let us apply this equation to the particular case when $f_1(x) = f(x), f_2(x) = \overline{f(x)}$. We have $\tilde{\bar{f}}(\lambda) = \int_{\mathbf{R}} \bar{f}(x) e^{-2\pi i\lambda x}\, dx = \overline{\tilde{f}(-\lambda)}$. Therefore, (39) gives

$$\int_{\mathbf{R}} |f(x)|^2 e^{-2\pi i \lambda x}\, dx = \int_{\mathbf{R}} \tilde{f}(\lambda - \mu)\overline{\tilde{f}(-\mu)}\, d\mu.$$

If we set $\lambda = 0$ here and replace $-\mu$ by λ, we get (38). Since $S(\mathbf{R})$ is dense in $L_2(\mathbf{R}, dx)$, (38) implies that F has a unique unitary extension from $S(\mathbf{R})$ to $L_2(\mathbf{R}, dx)$. There remains to verify that this extension is given by the integral (31) on the subspace $L = L_2(\mathbf{R}, dx) \cap L_1(\mathbf{R}, dx)$. Suppose that $\varphi \in L$, $\varphi_n \in S(\mathbf{R})$, and $\varphi_n \to \varphi$ in the sense of the norms in $L_2(\mathbf{R}, dx)$ and $L_1(\mathbf{R}, dx)$. Then $\tilde{\varphi}_n \to \tilde{\varphi}$ uniformly, and $\tilde{\varphi}_n \to F\varphi$ in the sense of $L_2(\mathbf{R}, d\lambda)$. The latter implies that some subsequence $\tilde{\varphi}_{n_k}$ converges to $F\varphi$ almost everywhere. From this, $F\varphi = \tilde{\varphi}$. □

Remark. Theorem 17 is a particular case of a general assertion: If G is a locally compact abelian group with invariant measure μ, then there exist an invariant measure $\hat{\mu}$ on the dual group $\hat{G}$ and a unitary operator $F: L_2(G, \mu) \to L_2(\hat{G}, \hat{\mu})$ whose restriction to $L_2(G, \mu) \cap L_1(G, \mu)$ coincides with the Fourier transformation.

As in the case of the circle, the smoothness of a function f on the line is connected with the decrease of its Fourier transform $\tilde{f}$ at infinity. Since $\mathbf{R}$ is a self-dual group, the dual assertion holds: The decrease of f at infinity is connected with the smoothness of $\tilde{f}$. Here is one variant of a precise formulation.

Theorem 18. *If a function f and all its derivatives up to order k are integrable over $\mathbf{R}$, then $\tilde{f}$ satisfies the estimate*

$$|\tilde{f}(\lambda)| = O(1 + |\lambda|)^{-k}.$$

If $(1 + |x|)^k f(x)$ is an integrable function on $\mathbf{R}$, then $\tilde{f}$ has bounded continuous derivatives up to order k.

The theorem follows directly from the commutation relations (35). (We leave it to the reader to convince himself that under the hypotheses of Theorem 18 the derivation of these relations remains valid.)

The requirement of rapid decrease at infinity in the case $G = \mathbf{R}$ can be formulated in various ways. The strongest condition of this kind is that f be compactly supported. It turns out that this condition implies that $\tilde{f}$ is analytic on the complex plane $\mathbf{C}$, which contains the line $\mathbf{R}$.

Paley–Wiener Theorem. *The Fourier transformation of the space $L_2(-a, a)$ (regarded as a subspace of $L_2(\mathbf{R}, dx)$) consists of all the entire functions $g(\lambda)$ having the following properties:*

(1) $|g(\lambda)| < Ce^{2\pi a|\mathrm{Im}\,\lambda|}$, *where the constant C depends on the function g;*
(2) *the restriction of $g(\lambda)$ to $\mathbf{R}$ is in $L_2(\mathbf{R}, d\lambda)$.*

PROOF. The integral (31) defining $\tilde{f}$ for $f \in L_2(-a, a)$ converges for all $\lambda \in \mathbf{C}$. A direct computation shows that $\tilde{f}(\lambda)$ is differentiable with respect to the complex variable λ, and, consequently, is an entire function.

The necessity of the condition (1) follows from the estimate

$$|\tilde{f}(\lambda)| = \left| \int_{-a}^{a} f(x)e^{-2\pi i\lambda x}\,dx \right| \le \int_{-a}^{a} |f(x)|e^{2\pi \mathrm{Re}(i\lambda x)}\,dx.$$

The necessity of (2) follows from Theorem 17.

Let us show that these conditions are sufficient. Since the restriction of g to $\mathbf{R}$ belongs to $L_2(\mathbf{R}, d\lambda)$, there exists a function $f \in L_2(\mathbf{R}, dx)$ such that $g = \tilde{f}$. We first assume that $g(\lambda)$ decreases rapidly enough at infinity: $|g(\lambda)| \le Ce^{2\pi a|\mathrm{Im}\,\lambda|}/(1 + |\lambda|^2)$. Then $g(\lambda)$ is integrable on $\mathbf{R}$, and f can be expressed in terms of g by the inverse Fourier transformation:

$$f(x) = \int_{-\infty}^{\infty} g(\lambda)e^{2\pi i\lambda x}\,d\lambda.$$

By Cauchy's lemma, the contour of integration can be translated to the complex domain: $f(x) = \int_{-\infty}^{\infty} g(\lambda + ib)e^{2\pi i(\lambda + ib)x}\,d\lambda$. Let $x > a$. Then the integrand admits the estimate $Ce^{2\pi b(a-x)}/(1 + |\lambda|^2)$ for $b > 0$; consequently, the integral of interest to us converges to zero. Since the value of the integral does not really depend on b, we see that $f(x) = 0$ for $x > a$. Similarly, letting b go to $-\infty$, we get that $f(x) = 0$ for $x < -a$. Finally, the condition on the rate of decrease of $g(\lambda)$ at infinity is not necessary. Suppose that $\varphi \in \mathcal{D}(\mathbf{R})$ and $\mathrm{supp}\,\varphi \subset [-\varepsilon, \varepsilon]$. Then the function $\tilde{\varphi}(\lambda)$ decreases rapidly at infinity and satisfies the estimate $|\tilde{\varphi}(\lambda)| \le Ce^{2\pi\varepsilon|\mathrm{Im}\,\lambda|}$, by the part of the theorem already proved. The arguments used above are applicable to the function $g_1(\lambda) = g(\lambda)\tilde{\varphi}(\lambda)$ with a replaced by $a + \varepsilon$. Hence, the function $f_1(\varepsilon) = f * \varphi$ belongs to $L_2(-a - \varepsilon, a + \varepsilon)$. This easily implies that $f \in L_2(-a, a)$. $\square$

4. Fourier Transformation of Generalized Functions

Let $f \in \mathscr{E}'(\mathbf{R})$. The integral defining the Fourier transform of a function f can be regarded as the value of the generalized function f on the character $\bar{\chi}_\lambda \in \mathscr{E}(\mathbf{R})$:

$$\tilde{f}(\lambda) = \langle f, \bar{\chi}_\lambda \rangle. \tag{40}$$

A broader definition of the Fourier transformation can be obtained by the scheme described in §3.5 of Ch. III. If $f \in L_1(\mathbf{R}, dx)$, $\varphi \in S(\mathbf{R})$, then

$$\langle \tilde{f}, \varphi \rangle = \int_{\mathbf{R}} \int_{\mathbf{R}} e^{-2\pi i \lambda x} f(x) \varphi(\lambda)\, dx\, d\lambda = \langle f, \tilde{\varphi} \rangle.$$

The right-hand side of this equation makes sense for $f \in S'(\mathbf{R})$. Therefore, we define the *Fourier transformation in the space* $S'(\mathbf{R})$ by the formula

$$\langle \tilde{f}, \varphi \rangle = \langle f, \tilde{\varphi} \rangle, \qquad \varphi \in S(\mathbf{R}). \tag{41}$$

In the case $f \in \mathscr{E}'(\mathbf{R})$ this definition is equivalent to the one given above by formula (40).

Theorem 19. *The direct and inverse Fourier transformations are continuous one-to-one operators in* $S'(\mathbf{R})$.

This follows at once from formula (41), which shows that the Fourier transformation in $S'(\mathbf{R})$ is the operator adjoint to the Fourier transformation in $S(\mathbf{R})$.

The Fourier transformation of generalized functions keeps the basic properties of the same transformation of ordinary functions: It carries convolution into ordinary multiplication, a translation into multiplication by a character, multiplication by a character into a translation, and a differential operator with constant coefficients into the operator of multiplication by a polynomial. See the problems of the appropriate section for more precise formulations.

Remark. The Fourier transform of a generalized function in $\mathscr{D}'(\mathbf{R})$ can also be defined by the formula (41). However, the image of this mapping will no longer consist of generalized functions in the classes familiar to us. It can be shown that the image of $\mathscr{D}(\mathbf{R})$ under the Fourier transformation is the space of analytic functions $g(\lambda)$ satisfying for any N and suitable C and R the estimate

$$|g(\lambda)| < C(1 + |\lambda|)^{-N} e^{R|\mathrm{Im}\,\lambda|}.$$

The Fourier transforms of the generalized functions in $\mathscr{D}'(\mathbf{R})$ are linear functionals on this space.

EXAMPLE. The regular generalized function $f(x) = e^{x^2}$ has as its Fourier transform the linear functional

$$\langle \tilde{f}, g \rangle = \frac{i}{\pi} \int_C e^{\pi^2 \lambda^2} g(\lambda)\, d\lambda,$$

where the integral is taken over some contour C in the complex plane of λ.

We can prove some useful properties of generalized functions by a purely algebraic method, using the commutation relations of the Fourier transformation with the operators of translation and multiplication by a character. The following *Poisson summation formula* is an example.

Theorem 20. *For any function $\varphi \in S(\mathbf{R})$,*

$$\sum_{k \in \mathbf{Z}} \varphi(k) = \sum_{k \in \mathbf{Z}} \tilde{\varphi}(k). \tag{42}$$

PROOF. Equation (42) is equivalent to the assertion $\tilde{f} = f$ for the particular generalized function $f = \sum_{k \in \mathbf{Z}} \delta(x - k)$. This f has the properties: (1) $T(1)f = f$; (2) $M(1)f = f$, where $T(1)$ is translation by 1, and $M(1)$ is the operator of multiplication by $\chi_1 = e^{2\pi i x}$. Properties (1) and (2) pass into each other under the Fourier transformation (Problem 708). Let us show that f is determined by these properties to within a constant factor. Indeed, property (2) has the easy consequence that the support of f is contained in $\mathbf{Z}$ and that f has zero order at each point of the support. From this, $f = \sum_{k \in \mathbf{Z}} c_k \delta(x - k)$. It now follows from the property (1) that all the coefficients c_k are equal, and $f = c \sum_{k \in \mathbf{Z}} \delta(x - k)$. Because both f and $\tilde{f}$ obey properties (1) and (2), we must have $\tilde{f} = cf$. It remains to compute the constant c. The equation $\tilde{\tilde{f}} = f$ shows that $c^2 = 1$. Since $S(\mathbf{R})$ contains positive functions whose Fourier transforms are positive, the case $c = -1$ is impossible. $\square$

Remark 1. The condition $\varphi \in S(\mathbf{R})$ can be weakened substantially. It suffices, for example, that φ and $\tilde{\varphi}$ admit an estimate of the form $O(|x|^{-1-\varepsilon})$.

Remark 2. The Poisson formula is a particular case of a more general assertion. Let

$$0 \to G_0 \to G \to G_1 \to 0$$

be an exact sequence of locally compact abelian groups, and

$$0 \leftarrow \hat{G}_0 \leftarrow \hat{G} \leftarrow \hat{G}_1 \leftarrow 0$$

the dual sequence. Then for a suitable choice of invariant measures μ_0 on G_0 and $\hat{\mu}_1$ on $\hat{G}_1$ we have

$$\int_{G_0} f(x)\, d\mu_0(x) = \int_{\hat{G}_1} \tilde{f}(\chi)\, d\mu_1(\chi).$$

Chapter V
The Spectral Theory of Operators

§1. The Functional Calculus

1. Functions of Operators in a Finite-Dimensional Space

Linear operators can be regarded as generalizations of numbers. These two concepts coincide in the one-dimensional case. Differences already appear in the two-dimensional case, the main one being that the multiplication operation for operators is not commutative. Nevertheless, many properties of numbers are preserved in passing to operators in multi-dimensional spaces. One of these is the possibility of substituting operators as arguments in various functions. The study of functions of operator-valued arguments makes up the subject of *functional operator calculus.* We restrict ourselves to functions of a single variable. In this case difficulties associated with noncommutativity do not arise.† We shall consider functions of an operator in a finite-dimensional space in this subsection.

Let A be a linear operator in a finite-dimensional linear space L over the field $K = \mathbf{R}$ or $\mathbf{C}$. Polynomials are the simplest functions in which it is possible to substitute the operator A as an argument. If $p(x) = \sum_{k=0}^{n} c_k x^k$, then it is natural to define $p(A)$ by

$$p(A) = \sum_{k=0}^{n} c_k A^k, \tag{1}$$

† See the book [13*] by Maslov to become familiar with one variant of noncommutative functional calculus. Another variant is the theory of representations of matrix groups.

with the stipulation that $A^0 = 1$ (the identity operator). This last condition is necessary in order that we have the natural equalities

$$(\lambda_1 p_1 + \lambda_2 p_2)(A) = \lambda_1 p_1(A) + \lambda_2 p_2(A), \qquad (p_1 p_2)(A) = p_1(A)p_2(A), \tag{2}$$

which express the fact that the mapping $p \mapsto p(A)$ is a homomorphism of the algebra of polynomials into the algebra of operators.

The next (broader) class of functions is the set of rational functions. If $r(x) = p(x)/q(x)$, where p and q are polynomials, then we define $r(A)$ by the formula

$$r(A) = p(A)q(A)^{-1} = q(A)^{-1}p(A). \tag{3}$$

This definition makes sense only if $q(A)$ is an invertible operator. In the finite-dimensional case the invertibility of $q(A)$ is equivalent to the condition $\det q(A) \neq 0$.

The most important example—the *resolvent* of A—is defined by the formula

$$R_\lambda(A) = (A - \lambda 1)^{-1} = r_\lambda(A), \tag{4}$$

where $r_\lambda(x) = 1/(x - \lambda)$.

The correspondence $r \mapsto r(A)$ is a homomorphism of a certain subalgebra of the field $K(x)$ of rational functions into the algebra of operators. It can be shown that this subalgebra consists of all the rational functions whose poles lie outside the spectrum (i.e., the set of eigenvalues) of A.

Up to this point we have not used topology in our considerations. Let us now assume that L is a normed space and that the set of operators on L is thereby also endowed with a norm. Then we can define an entire function of A by the formula

$$f(A) = \sum_{k=0}^{\infty} c_k A^k, \tag{5}$$

where $f(x) = \sum_{k=0}^{\infty} c_k x_k$. Indeed, if f is an entire function, then the numerical series $\sum_{k=0}^{\infty} c_k \|A\|^k$ converges. From this it follows that the sequence of partial sums of the series (5) is Cauchy and, consequently, has a limit in the space of operators on L.

EXAMPLE. Let us compute e^{tA}, where A is the operator of rotation by 90° in $\mathbf{R}^2$, and t is a real parameter. Since $A^2 = -1$, the series for e^{tA} takes the form

$$e^{tA} = 1 + tA - \frac{t^2}{2!}\cdot 1 - \frac{t^3 A}{3!} + \frac{t^4}{4!}\cdot 1 + \frac{t^5}{5!}A - \ldots,$$

from which $e^{tA} = \cos t \cdot 1 + \sin t \cdot A$. This equation can be written in matrix form as

$$\exp\begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Let us now assume that the space of polynomials or entire functions is equipped with some norm with respect to which the mapping $f \mapsto f(A)$ is continuous. Then this mapping extends by continuity to the completion in this norm of the space of polynomials or entire functions.

EXAMPLE. Suppose that the operator A is given by a diagonal matrix with real numbers belonging to $[a, b]$ on the diagonal. Then the correspondence $f \mapsto f(A)$ is continuous in the norm of $C[a, b]$, and, consequently, it is possible to define $f(A)$ for any continuous function on $[a, b]$.

In the finite-dimensional setting the question of what functions of A make sense can be completely solved as follows. Let $\lambda_1, \dots, \lambda_k$ be a tuple of complex numbers, and $n_1, \dots, n_k$ a tuple of natural numbers. In the space of polynomials we define the seminorm

$$p_{\lambda_1, \dots, \lambda_k; n_1, \dots, n_k}(f) = \max_{1 \le j \le k} \max_{0 \le m \le n_j} |f^{(m)}(\lambda_j)|. \tag{6}$$

If A is an operator in a finite-dimensional complex linear space L, then p_A denotes the following seminorm in the space of polynomials:

$$p_A(f) = \|f(A)\|.$$

Theorem 1. *Let A be an operator in a finite-dimensional complex linear space, $\lambda_1, \dots, \lambda_k$ the roots of the polynomial of minimal degree annihilating A, and $n_1, \dots, n_k$ the multiplicities of these roots. Then the seminorms $p_{\lambda_1, \dots, \lambda_k; n_1, \dots, n_k}$ and p_A are equivalent.*

PROOF. Let L be the space of polynomials, and L_0 the subspace of polynomials having roots of multiplicity $\ge n_j$ at the points λ_j, $j = 1, 2, \dots, k$. Both the seminorms in the theorem annihilate the space L_0 and generate norms in the quotient space L/L_0. Since the latter is finite-dimensional, any two norms in it are equivalent. □

Corollary 1. *If A is an operator with simple eigenvalues, then $f(A)$ makes sense for any continuous function f.*

Corollary 2. *If A is an operator in an n-dimensional space, then $f(A)$ makes sense for any $(n-1)$-fold differentiable continuous function.*

2. Functions of Bounded Selfadjoint Operators

Let A be a bounded selfadjoint operator on a Hilbert space H over the field $K = \mathbf{R}$ or $\mathbf{C}$. The goal of this subsection is to define $f(A)$ for any Borel function f on the segment $[-a, a]$, where $a = \|A\|$.

Recall that the *spectrum* of A is defined to be the set $\sigma(A) \subset \mathbf{C}$ of those $\lambda \in \mathbf{C}$ for which $A - \lambda 1$ is not invertible. The complement of $\sigma(A)$ in $\mathbf{C}$ is called the *resolvent set* and denoted by $\rho(A)$. Thus, the resolvent operator of A, $r_\lambda(A) = (A - \lambda 1)^{-1}$, is defined for $\lambda \in \rho(A)$.

Theorem 2. *The spectrum of a bounded operator is a nonempty compact subset of* $\mathbf{C}$.

PROOF. If $|\lambda| > \|A\|$, then $A - \lambda 1$ has an inverse operator $r_\lambda(A)$ given by the series

$$r_\lambda(A) = \sum_{k=0}^{\infty} \lambda^{-k-1} A^k. \tag{7}$$

Therefore, $\sigma(A)$ is contained in the disk $|\lambda| \leq \|A\|$. Further, if the resolvent $r_{\lambda_0}(A) = B$ exists at some point $\lambda_0 \in \mathbf{C}$, then the following series converges in a disk of radius $\|B\|^{-1}$:

$$r_\lambda(A) = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k B^{k+1}. \tag{8}$$

This shows that the resolvent set $\rho(A)$ is open, and, consequently, $\sigma(A)$ is compact. We have established in passing that the norm of the resolvent admits the lower estimate

$$\|r_\lambda(A)\| \geq d(\lambda, \sigma(A))^{-1} = \left(\min_{\mu \in \sigma(A)} (\lambda - \mu)\right)^{-1}. \tag{9}$$

There remains to prove that the spectrum is not empty. Suppose it were; then the resolvent $r_\lambda(A)$ would be an entire function of $\lambda \in \mathbf{C}$. Formula (7) shows that $\|r_\lambda(A)\| \to 0$ as $\lambda \to \infty$. Hence, $\|r_\lambda(A)\|$ is bounded on the whole complex plane. For any vectors x and y in H the quantity $(r_\lambda(A)x, y)$ is an entire function of λ that converges to zero at infinity. By Liouville's theorem, such a function is identically equal to zero. Hence, $r_\lambda(A) = 0$, which is impossible. □

Let us now study how the spectrum of A changes when a transformation is performed on A.

Theorem 3. (1) *Let $r(x)$ be a rational function without poles on the spectrum of the operator A. Then the operator $r(A)$ is defined, and its spectrum is described by the formula*

$$\sigma(r(A)) = r(\sigma(A)) = \{r(\lambda) \colon \lambda \in \sigma(A)\}. \tag{10}$$

(2) *The spectrum of the adjoint operator A^* is connected with that of A by the relation*

$$\sigma(A^*) = \overline{\sigma(A)} = \{\bar{\lambda} \colon \lambda \in \sigma(A)\}. \tag{11}$$

PROOF. Let $r(x) = p(x)q(x)^{-1}$, where p and q are relatively prime polynomials. If $\mu_1, \ldots, \mu_m$ are the roots of $p(x)$ and $\lambda_1, \ldots, \lambda_n$ are the roots of $q(x)$ (counting multiplicities), then $r(x) = c \prod_{i=1}^m (x - \alpha_i) \prod_{j=1}^n (x - \beta_j)^{-1}$.

From this, $r(A) = c\prod_{i=1}^{m}(A - \alpha_i 1)\prod_{j=1}^{n} r_{\beta_j}(A)$. Suppose that $\lambda \in \sigma(A)$, and $\mu = r(\lambda)$. Since the rational function $r(x) - \mu$ vanishes at $x = \lambda$, its numerator has λ as a root and, consequently, contains the factor $x - \lambda$. Therefore, the operator $r(A) - \mu 1$ contains the noninvertible operator $A - \lambda 1$ as a factor and, consequently, is itself noninvertible. Hence, $\mu \in \sigma(r(A))$, which establishes the inclusion $r(\sigma(A)) \subset \sigma(r(A))$.

Conversely, let $\mu \in \sigma(r(A))$. We represent the rational function $r(x) - \mu$ as a product $c\prod_{i=1}^{m}(x - \alpha_i) \times \prod_{j=1}^{n}(x - \beta_j)^{-1}$. If all the α_i were in $\rho(A)$, then the operator $r(A) - \mu 1$ would have the inverse operator

$$c^{-1}\prod_{i=1}^{m} r_{\alpha_i}(A)\prod_{j=1}^{n}(A - \beta_j \cdot 1),$$

which is not true. Hence, one of the numbers α_i is in the spectrum of A. But then $r(\alpha_i) - \mu = 0$, i.e., $\mu \in r(\sigma(A))$. Assertion (2) follows from the equation $(A^{-1})^* = (A^*)^{-1}$, which yields the relation

$$r_\lambda(A^*) = r_{\bar{\lambda}}(A)^*.$$

Theorem 4. *Suppose that A is a selfadjoint operator. Then*

(1) *The spectrum of A lies in the segment $[-\|A\|, \|A\|]$;*
(2) *for any rational function r with poles outside $\sigma(A)$,*

$$\|r(A)\| = \max_{\lambda \in \sigma(A)} |r(\lambda)|. \tag{12}$$

PROOF. Let λ be a nonreal number. We show that $r_\lambda(A)$ exists. The operator $A - \lambda 1$ does not have a kernel, since $\lambda = (Ax, x) = (x, Ax) = \bar{\lambda}$, for any unit vector x in $\ker(A - \lambda 1)$, which is impossible. Further, the range of $A - \lambda 1$ is dense in H, since $\operatorname{im}(A - \lambda 1)^\perp = \ker(A - \lambda 1)^* = \ker(A - \bar{\lambda}1) = 0$. We show that the operator $(A - \lambda 1)^{-1}$, which is defined on $\operatorname{im}(A - \lambda 1)$, is bounded. Let

$$\lambda = \alpha + i\beta, \alpha, \beta \in \mathbf{R}.$$

Then

$$\begin{aligned}\|(A - \lambda 1)x\|^2 &= ((A - \alpha 1)x - i\beta x, (A - \alpha 1)x - i\beta x)\\ &= \|(A - \alpha 1)x\|^2 + \beta^2\|x\|^2 \geq \beta^2\|x\|^2.\end{aligned}$$

From this, $\|A - \lambda 1\|^{-1} \leq \beta^{-1}$, and $r_\lambda(A)$ is obtained as an extension by continuity. (Actually, it can be shown that the estimate obtained implies that $\operatorname{im}(A - \lambda 1) = H$.)

Thus, $\sigma(A)$ lies on the real axis. Assertion (1) now follows from the fact that $\sigma(A)$ lies in the disk of radius $\|A\|$ (see the proof of Theorem 2).

The proof of the assertion (2) begins with the case $r(x) \equiv x$. Then (2) is reduced to the equation

$$\|A\| = \sup\{|\lambda| : \lambda \in \sigma(A)\}. \tag{13}$$

The quantity on the right-hand side is called the *spectral radius* of A and denoted by $r(A)$.

Lemma. *For any bounded operator A,*

$$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}; \tag{14}$$

if A is selfadjoint, then $r(A) = \|A\|$.

Proof. The existence of the limit on the right-hand side of (14) follows from the general properties of subadditive sequences (see Problem 725). Next, the expansion of the resolvent in a Laurent series in a neighborhood of infinity is given by formula (7). Hadamard's formula connects the radius of convergence of this series (which is obviously equal to $r(A)$) with the coefficients by the desired relation (14). (Here we apply the Hadamard formula to an operator-valued analytic function. It is easy to verify that the usual derivation of this formula carries over in entirety to this case.) For any operator A,

$$\|A^*A\| = \sup_{\|x\|=\|y\|=1} (A^*Ax, y) = \sup_{\substack{x\neq 0\\ y\neq 0}} \frac{(Ax, Ay)}{\|x\|\,\|y\|} = \|A\|^2.$$

In particular, $\|A^2\| = \|A\|^2$ for selfadjoint A. Therefore, $\|A^{2^n}\| = \|A\|^{2^n}$, and, consequently, $\lim_{n\to\infty} \|A^n\|^{1/n} = \|A\|$. □

We return to the proof of assertion (2) in the general case. Let $B = r(A)$. Then $B^* = \bar{r}(A)$ ($\bar{r}$ denotes the function $\bar{r}(z) = \overline{r(\bar{z})}$; it is easy to see that $\bar{r}$ is the rational function whose coefficients are the complex conjugates of the coefficients of r). Applying the assertion already proved to the selfadjoint operator $B^*B = |r|^2(A)$, we get

$$\|B\|^2 = \|B^*B\| = r(B^*B) = \sup_{\mu\in\sigma(B^*B)} (\mu) = \sup_{\lambda\in\sigma(A)} |r(\lambda)|^2$$

(the last equality is by virtue of assertion (1) of Theorem 3). □

Corollary. *For any selfadjoint operator A in a Hilbert space H there exists a unique continuous homomorphism φ of the algebra $C[-a, a]$, where $a = \|A\|$, into the algebra of bounded operators on H with the following properties:*

(1) $\varphi(1) = 1$ (*here the* 1 *on the left-hand side is the function identically equal to* 1, *and that on the right-hand side is the identity operator on H*);
(2) $\varphi(\bar{f}) = \varphi(f)^*$;
(3) $\varphi(x) = A$;
(4) $\|\varphi(f)\| \le \|f\|_{C[-a,a]}$.

This homomorphism has also the property:

(5) *if $AB = BA$, then $\varphi(f)B = B\varphi(f)$.* (*See Problems* 748, 749 *for another derivation of this assertion.*)

We now extend the homomorphism φ to the algebra $B[-\alpha, \alpha]$ of bounded Borel functions on the segment $[-\alpha, \alpha]$.

Theorem 5. *Let A be a bounded selfadjoint operator on a Hilbert space H. There exists a unique homomorphism φ of the algebra $B[-a, a]$ of bounded Borel functions on $[-a, a]$, where $a = \|A\|$, having the following properties:*

(1) $\varphi(1) = 1$;
(2) $\varphi(x) = A$;
(3) *if $f_n(x) \leq C$ and $f_n(t) \to f(t)$ for each $t \in [-a, a]$, then $\varphi(f_n) \to \varphi(f)$ in the strong operator topology.*

The homomorphism φ has also the properties:

(4) $\varphi(\bar{f}) = \varphi(f)^*$;
(5) $\|\varphi(f)\| \leq \sup |f(t)|, t \in [-a, a]$;
(6) $\varphi(f)B = B\varphi(f)$ *for any operator B that commutes with A.*

Proof. The homomorphism φ is already defined on the set $C[-a, a]$. The collection of Borel functions is obtained from the collection of continuous functions by passing to pointwise limits; this implies that the desired homomorphism φ is unique. Let us prove that it exists. Suppose that $x, y \in H$. Define the linear functional $F_{x,y}$ on $C[-a, a]$ by the formula

$$F_{x,y}(f) = (\varphi(f)x, y). \tag{15}$$

Since $|F_{x,y}(f)| \leq \|\varphi(f)\| \ \|x\| \ \|y\| \leq \|f\|_{C[-a,a]}\|x\| \ \|y\|$, $F_{x,y}$ is a continuous functional whose norm does not exceed $\|x\| \ \|y\|$. Hence, there exists a signed Borel measure $\nu_{x,y}$ on $[-a, a]$ such that

$$F_{x,y}(f) = \int_{-a}^{a} f(t)\, d\nu_{x,y}(t), \tag{16}$$

and $\operatorname{Var}_{-a}^{a} \nu \leq \|x\| \ \|y\|$. Suppose now that f is a bounded Borel function on $[-a, a]$. The quantity $Bf(x, y) = \int_{-a}^{a} f(t)\, d\nu_{x,y}(t)$ depends linearly on x and antilinearly on y, and satisfies the estimate $|Bf(x, y)| \leq \sup_{t \in [-a,a]} |f(t)| \, \|x\| \ \|y\|$. From this it follows that there is a bounded operator $\varphi(f)$ such that $Bf(x, y) = (\varphi(f)x, y)$, and $\|\varphi(f)\| \leq \sup |f(t)|$, $t \in [-a, a]$. Suppose that $f_n(t) \to f(t)$ for $t \in [-a, a]$. Then for any x and y in H

$$(\varphi(f_n)x, y) = \int_{-a} f_n(t)\, d\nu_{x,y}(t) \to \int_{-a}^{a} f(t)\, d\nu_{x,y}(t) = (\varphi(f)x, y).$$

Thus, $\varphi(f_n)$ converges weakly to $\varphi(f)$. This implies that φ is a homomorphism. Indeed, the equations

$$\varphi(\lambda f + \mu g) = \lambda\varphi(f) + \mu\varphi(g),$$
$$\varphi(fg) = \varphi(f)\varphi(g)$$

hold when f and g are in $C[-a, a]$, and are preserved under passage to pointwise limits. The same argument shows that φ has the property (4). We are now in a position to prove property (3). Suppose that $f_n(t) \to f(t)$ for all $t \in [-a, a]$, and $|f_n(t)| \leq C$. Then $|f_n - f|^2(t) \to 0$ for all $t \in [-a, a]$. Therefore, $\varphi(|f_n - f|^2) \to 0$. From this, $\|\varphi(f_n - f)x\|^2 = (\varphi(f_n - f)^*\varphi(f_n - f)x, x) = (\varphi(|f_n - f|^2)x, x) \to 0$. ☐

Basic example Let $H = L_2(X, \mu)$, and A the operator of multiplication by a function $a \in L_\infty(X, \mu)$. In this case $\varphi(f)$ is the operator of multiplication by the function $f(a(x))$ (verify this by going through the construction of $\varphi(f)$ for this example). It will be explained below how the general case can be reduced to the situation of this example.

Theorem 6. *Let $A_1, \ldots, A_n$ be pairwise commuting bounded selfadjoint operators in a Hilbert space H, and T the parallelepiped in $\mathbf{R}^n$ defined by the conditions $|t_i| \leq \|A_i\|$, $i = 1, 2, \ldots, n$. There exists a unique homomorphism φ of the algebra $B(T)$ of bounded Borel functions on T into the algebra of bounded operators on H having the properties:*

(1) $\varphi(1) = 1$;
(2) $\varphi(t_i) = A_i$;
(3) *if $|f_k(t)| \leq C$ and $f_k(t) \to f(t)$ for all $t \in T$, then $\varphi(f_k) \to \varphi(f)$ in the strong operator topology.*

Moreover, φ has the properties:

(4) $\varphi(\bar{f}) = \varphi(f)^*$;
(5) $\|\varphi(f)\| \leq \sup_{t \in T} |f(t)|$;
(6) *$\varphi(f)B = B\varphi(f)$ for any operator B commuting with $A_1, \ldots, A_n$.*

Proof. Let $B_k(T)$ be the subalgebra of $B(T)$ consisting of the functions depending only on the coordinate t_k. Then the restriction of φ to $B_k(T)$ coincides with the homomorphism φ_k corresponding by Theorem 5 to the operator A_k. Let $B_0(T)$ be the subalgebra of step functions on T, i.e., functions of the form $f(t) = \sum c_{k_1} \cdots c_{k_n} \chi_{E_1}(t_1) \cdots \chi_{E_n}(t_n)$. If the desired homomorphism φ exists, then, by the above, its value on the step function f must be given by the formula

$$\varphi(f) = \sum c_{k_1 \ldots k_n} \varphi_1(\chi_{E_1}) \cdots \varphi_n(\chi_{E_n}). \tag{17}$$

This implies uniqueness of φ on $B_0(T)$, and, hence, on $B(T)$, by the property (3).

Let us prove its existence. Define φ on $B_0(T)$ by the formula (17). Since the $A_1, \ldots, A_n$ commute pairwise, the operators $\varphi_1(f_1), \varphi_2(f_2), \ldots, \varphi_n(f_n)$ also commute pairwise (by Theorem 5, part (6)). Therefore, φ is a homomorphism. Next, φ carries positive functions into positive operators, since if $f \geq 0$, then $f = g^2$ for some real function $g \in B_0(T)$, so that $\varphi(f) = \varphi(g)^2 \geq 0$. This implies property (5) (see Problem 749). Therefore, the homomorphism φ can be extended to the algebra $C(T)$ of continuous functions on T, while maintaining property (5). The derivation of the theorem from this proceeds just as the derivation of Theorem 5 from the corollary to Theorem 4. □

We mention a useful

Corollary. *Let A be a normal operator in a Hilbert space H, and T the square in the complex plane with center at zero and sides $2\|A\|$. There exists a unique homomorphism* *of the algebra $B(T)$ of bounded Borel functions on T into the algebra of operators in H having the properties:*

(1) $\varphi(1) = 1$;
(2) $\varphi(x + iy) = A$;
(3) $\varphi(\bar{f}) = \varphi(f)^*$;
(4) *if* $|f_n| \leq C$ *and* $f_n(t) \to f(t)$ *for* $t \in T$, *then* $\varphi(f_n) \to \varphi(f)$ *in the strong operator topology.*

This homomorphism has also the properties:

(5) $\|\varphi(f)\| \leq \sup |f(t)|$;
(6) $\varphi(f)B = B\varphi(f)$ *for any operator* B *commuting with* A *and* A^*.

Indeed, if A is normal, then $A = B + iC$, where B and C are bounded selfadjoint operators whose norms do not exceed that of A. Conditions (2) and (3) imply the conditions $\varphi(x) = B$, $\varphi(y) = C$. The corollary now follows from Theorem 6, applied to the operators B and C.

3. Unbounded Selfadjoint Operators

In applications one frequently encounters operators A defined not on the whole Hilbert space H, but only on some nonclosed dense subspace $\mathscr{D}_A \subset H$, and unbounded on this subspace. For such an operator it is possible to define an adjoint operator A^*, which also may not be defined everywhere and may be unbounded. Namely, the domain of A^* is defined to be the (nonclosed, in general) subspace $\mathscr{D}_{A^*}$ consisting of the vectors $y \in H$ for which the linear functional $x \mapsto (Ax, y)$ is bounded on $\mathscr{D}_A$. In this case it can be extended uniquely to a linear functional on H and can be written in the form $x \mapsto (x, z)$, $z \in H$. We set $A^*y = z$. Thus, the equality

$$(Ax, y) = (x, A^*y), \tag{18}$$

which is the definition of A^* for bounded operators A, is now valid only for $x \in \mathscr{D}_A$ and $y \in \mathscr{D}_{A^*}$.

It is possible to give another, more "geometric," definition of A^*. To do this we observe that every operator A (including those that are not everywhere defined and are unbounded) is determined by its *graph*, i.e., the subset $\Gamma_A \subset H \oplus H$ consisting of the vectors of the form $x \oplus Ax$, $x \in \mathscr{D}_A$. It is clear that Γ_A is a linear subspace of $H \oplus H$ that does not contain vectors of the form $0 \oplus x$, $x \neq 0$. Conversely, every linear subspace of $H \oplus H$ that does not contain vectors of the form $0 \oplus x$, $x \neq 0$, is the graph of some operator.

Let τ be the transformation of $H \oplus H$ carrying $x \oplus y$ into $(-y) \oplus x$ (rotation by 90°).

Theorem 7. *The graphs of the operators A and A^* are related by*

$$\Gamma_{A^*} = \tau(\Gamma_A)^{\perp}. \tag{19}$$

PROOF. Let $y \oplus z \in \Gamma_{A^*}$. This means that $(Ax, y) = (x, z)$ for all $x \in \mathscr{D}_A$ (see (18)). This relation is precisely the condition that the vectors $y \oplus z$ and $(-Ax) \oplus x$ be orthogonal in $H \oplus H$. □

An operator A is said to be *closed* if Γ_A is a closed subset of $H \oplus H$. An operator B is called the *closure* of an operator A if Γ_B is the closure of the subspace Γ_A. An operator B is called an *extension* of A, written $B \supset A$, if $\Gamma_B \supset \Gamma_A$. Theorem 7 yields the following

Corollary. *For any operator A, the operator A^* is closed, and the operator $(A^*)^*$, if it is defined, coincides with the closure of A.*

Remark. Not every operator admits a closure. For example, the operator A defined on the set of sequences in $l_2(\mathbf{R})$ with only a finite number of nonzero terms and carrying e_n into ne_1, $n = 1, 2, \ldots$, does not have a closure, since $0 \oplus e_1$ is a limit point of its graph. The reader can check that A^* is defined on the subspace $\{e_1\}^{\perp}$, which is not dense in $l_2(\mathbf{R})$.

An operator A is said to be *selfadjoint* if

$$\mathscr{D}_A = \mathscr{D}_{A^*} \tag{20}$$

and $A = A^*$ on $\mathscr{D}_A$.

Warning. This property is not equivalent to the weaker condition

$$(Ax, y) = (x, Ay) \quad \text{for } x, y \in \mathscr{D}_A. \tag{21}$$

Operators having the property (21) are said to be *symmetric*. Obviously, (21) is equivalent to the inclusion $A \subset A^*$.

An operator A is said to be *essentially selfadjoint* if its closure is selfadjoint.

EXAMPLES. (1) Let A be the operator of multiplication by x in $L_2(\mathbf{R}, dx)$ with domain $\mathscr{D}_A = \mathscr{D}(\mathbf{R})$. We determine $\mathscr{D}_{A^*}$. By definition, for $f \in \mathscr{D}_{A^*}$

$$\int_{-\infty}^{\infty} x\varphi(x)f(x)\,dx = \int_{-\infty}^{\infty} \varphi(x)\overline{g(x)}\,dx,$$

where

$$g = A^*f, \qquad \varphi \in \mathscr{D}(\mathbf{R}).$$

Since $\mathscr{D}(\mathbf{R})$ is dense in $L_2(\mathbf{R}, dx)$, $g(x) = xf(x)$. Thus, $\mathscr{D}_{A^*}$ consists of all $f \in L_2(\mathbf{R}, dx)$ such that $xf(x) \in L_2(\mathbf{R}, dx)$. The operator A^* is multiplication by x. Conclusion: A is symmetric, but not selfadjoint. It can be verified that A^* is selfadjoint, consequently, A is essentially selfadjoint.

(2) Let A be the operator d/dx of differentiation in $L_2(\mathbf{R}, dx)$, with domain $\mathscr{D}(\mathbf{R})$. We find $\mathscr{D}_{A^*}$. If $f \in \mathscr{D}_{A^*}$, then for $\varphi \in \mathscr{D}(\mathbf{R})$

$$\int_{-\infty}^{\infty} \varphi'(x)\overline{f(x)}\,dx = \int_{-\infty}^{\infty} \varphi(x)\overline{g(x)}\,dx, \quad \text{where } g = A^*f.$$

This equality shows that $g(x)$ is the generalized derivative of the function $-f(x)$. Thus, $\mathscr{D}_{A^*}$ consists of the functions $f \in L_2(\mathbf{R}, dx)$ whose generalized derivatives belong to $L_2(\mathbf{R}, dx)$. The operator A^* coincides with $-d/dx$.

In applications the operator A is often specified in the form of a differential expression. In this case it has a so-called *natural domain* $\mathscr{D}_A \subset L_2(\Omega, d\mu)$, namely, the collection of all generalized functions $f \in \mathscr{D}'(\Omega)$ for which f and Af are in $L_2(\Omega, d\mu)$. In the above examples the operators A^* have precisely their natural domains.

The theory of selfadjoint operators has been extensively worked out. In order to use it in applications it is necessary to know that a given operator is selfadjoint or at least essentially selfadjoint. A necessary condition for this is that the operator be symmetric, and this, as a rule, can be checked without difficulty. The following criterion is useful in investigating symmetric operators.

Theorem 8. *Let A be a symmetric operator in a Hilbert space H. Then the selfadjointness of A is equivalent to each of the conditions*:

(1) *A is closed and* $\ker(A^* \pm i1) = \{0\}$;
(2) $\operatorname{im}(A \pm i1) = H$.

The essential selfadjointness of A is equivalent to each of the conditions:

(3) $\ker(A^* \pm i1) = \{0\}$;
(4) $\operatorname{im}(A \pm i1)$ *is dense in H.*

Proof. If A is selfadjoint, then it is closed (since $A = A^*$, and A^* is closed). If $x \in \ker(A^* \pm i1)$, then $Ax = \mp ix$. From this, $(Ax, x) = \mp i\|x\|^2 = (x, Ax) = \pm i\|x\|^2$, i.e., $x = 0$. We have shown that condition (1) holds for a selfadjoint operator A. Let us show that (1) implies (2). We remark first that $\operatorname{im}(A \pm i1)^\perp = \ker(A^* \mp i1)$. Therefore, (1) implies that $\operatorname{im}(A \pm i1)$ is dense in H. We now use the fact that A is closed. Let y be any vector in H and $\{x_n\}$ a sequence in $\mathscr{D}_A$ such that $(A \pm i1)x_n \to y$. The sequence $\{(A \pm i1)x_n\}$ is Cauchy. The relation $\|(A \pm i1)x\|^2 = (Ax \pm ix, Ax \pm ix) = \|Ax\|^2 + \|x\|^2 \geq \|x\|^2$ shows that $\{x_n\}$ is also a Cauchy sequence. Let $x = \lim_{n\to\infty} x_n$. Then $x \oplus y \in \Gamma_{A \pm i1}$, hence, $y \in \operatorname{im}(A \pm i1)$.

We now show that (2) implies that A is selfadjoint. Since $A \subset A^*$, it is only necessary to verify the inclusion $\mathscr{D}_{A^*} \subset \mathscr{D}_A$. Let $y \in \mathscr{D}_{A^*}$. Since $\operatorname{im}(A \pm i1) = H$, there exist vectors $x_\pm \in \mathscr{D}_A$ such that $(A \pm i1)x_\pm = (A^* \pm i1)y$. The operator A^* is defined and coincides with A on $\mathscr{D}_A$. Therefore, $(A^* \pm i1)(y - x_\pm) = 0$. But $\ker(A^* \pm i1) = \operatorname{im}(A \mp i1)^\perp = 0$. Hence, $y = x_\pm$ and is in $\mathscr{D}_A$.

Let us now assume that A is essentially selfadjoint. Then A^* coincides with the closure of A (Problem 764) and, consequently, is selfadjoint. This implies (3), by the part of the theorem already proved. Next, (4) implies (3) (Problem 778). We now infer from the condition (3) that A is essentially selfadjoint. Since A is symmetric, $A \subset A^*$. This shows that A admits a closure $\bar{A}$ and that $\bar{A} \subset A^*$ (since A^* is a closed operator). Further, it follows from Theorem 7 that $\bar{A}^* = A^*$. Hence, the condition (1) holds for $\bar{A}$. Therefore, $\bar{A}$ is selfadjoint, and A is essentially selfadjoint. □

Example. Let $A = i(d/dx)$, $H = L_2(\mathbf{R}, dx)$, $\mathscr{D}_A = \mathscr{D}(\mathbf{R})$. In this case the symmetry of A follows from the equality

$$\int_{-\infty}^{\infty} \varphi'(x)\overline{\psi(x)}\, dx = -\int_{-\infty}^{\infty} \varphi(x)\overline{\psi'(x)}\, dx$$

for $\varphi, \psi \in \mathscr{D}(\mathbf{R})$. As we saw above, the adjoint operator A^* is defined on the set $\mathscr{D}_{A^*} \subset L_2(\mathbf{R}, dx)$ consisting of the functions having generalized derivative in $L_2(\mathbf{R}, dx)$, and it is equal to id/dx. We determine $\ker(A^* \pm i1)$. If $(A^* \pm i1)f = 0$, i.e., $i(f' \pm f) = 0$, then $f = ce^{\pm x}$ (Problem 523). Since these functions do not lie in $L_2(\mathbf{R}, dx)$, we get that $\ker(A^* \pm i1) = \{0\}$. Hence, A is essentially selfadjoint.

Suppose that A is unbounded, and B is bounded. They are said to commute if B carries $\mathscr{D}_A$ into itself and $BAx = ABx$ for $x \in \mathscr{D}_A$. For a selfadjoint operator A it can be shown that this condition is equivalent to the commutativity of B with either of the bounded operators $(A \pm i1)^{-1}$.

§2. Spectral Decomposition of Operators

1. Reduction of an Operator to the Form of Multiplication by a Function

In this part we show that every selfadjoint operator is unitarily equivalent to a multiplication operator. More precisely,

Theorem 9. *Let A be a selfadjoint operator in a Hilbert space H. There exist a space X with a measure μ, a measurable function a on X, and a unitary operator U from H to $L_2(X, \mu)$ such that the following diagram commutes:*

$$\begin{tikzcd} \mathscr{D}_A \arrow[r, "U"] \arrow[d, "A"'] & \mathscr{D}_{M(a)} \arrow[d, "M(a)"] \\ H \arrow[r, "U"] & L_2(X, \mu) \end{tikzcd} \tag{22}$$

(Here $M(a)$ is the operator of multiplication by the function a in $L_2(X, \mu)$, with domain $\mathscr{D}_{M(a)}$ consisting of the functions $f \in L_2(X, \mu)$ for which $af \in L_2(X, \mu)$.)

We carry out the proof of this theorem in several steps. Suppose first that A is bounded and that it has a *cyclic vector* in H, i.e., a vector ξ such that H itself is the only closed subspace of H containing ξ and invariant under A (see Problem 715). In this case A will be called an *operator with simple spectrum* (cf. Problem 716).

Theorem 10. *Suppose that A is a bounded selfadjoint operator with simple spectrum. Then the assertion of Theorem 9 holds for A, with $X = [-\|A\|, \|A\|]$, $a(x) = x$, $(U\xi)(x) \equiv 1$.*

Proof. Suppose that the theorem has been proved and that μ is the desired measure on X. Let f be a bounded Borel function on X. Then it follows from Theorem 5 that the operator $f(A)$ in H corresponds to the operator of multiplication by $f(x)$ in $L_2(X, \mu)$, i.e.,

$$Uf(A) = M(f)U. \tag{23}$$

Indeed, (23) reduces to (22) for $f(x) = x$, and the general case can be obtained by going through the construction of the operator $f(A)$.

This implies the equality

$$(f(A)\xi, \xi)_H = (Uf(A)\xi, U\xi)_{L_2(X,\mu)} = (f, 1)_{L_2(X,\mu)}. \tag{24}$$

The left-hand side of (24) can be computed from the knowledge only of the operator A and the vector ξ. We thereby reproduce the measure μ. Namely, if E is a measurable subset of X and χ_E is the characteristic function of E, then

$$\mu(E) = (\chi_E, 1)_{L_2(X,\mu)} = (\chi_E(A)\xi, \xi)_H. \tag{25}$$

Suppose now that A is a selfadjoint operator with simple spectrum in the space H, and let ξ be a cyclic vector for A. We define the function μ on the Borel subsets of the segment $X = [-\|A\|, \|A\|]$ by the formula (25). Let us show that μ is a countably additive measure. First of all, the equalities $\chi_E = \chi_E^2 = \bar{\chi}_E$ imply that the operator $\chi_E(A)$ is an orthogonal projection. Therefore, $(\chi_E(A)\xi, \xi) = \|\chi_E(A)\xi\|^2 \geq 0$. Further, if $E = \coprod_{k=1}^{\infty} E_k$, then $\chi_E = \sum_{k=1}^{\infty} \chi_{E_k}$ (the series converges at each point). Therefore, $\chi_E(A) = \sum_{k=1}^{\infty} \chi_{E_k}(A)$ (in the sense of strong convergence), hence, $\mu(A) = (\chi_E(A)\xi, \xi) = (\sum_{k=1}^{\infty} \chi_{E_k}(A)\xi, \xi) = \sum_{k=1}^{\infty} \mu(E_k)$. We now define an operator U from H to $L_2(X, \mu)$ on vectors of the form $f(A)\xi$ as follows:

$$Uf(A)\xi = f. \tag{26}$$

Let us verify that this operator is isometric, i.e.,

$$(Uf_1(A)\xi, Uf_2(A)\xi)_{L_2(X,\mu)} = (f_1(A)\xi, f_2(A)\xi)_H. \tag{27}$$

If $f_1 = \chi_{E_1}$, $f_2 = \chi_{E_2}$, then (27) reduces to the equality $(\chi_{E_1}, \chi_{E_2})_{L_2(X,\mu)} = (\chi_{E_1}(A)\xi, \chi_{E_2}(A)\xi)_H = (\chi_{E_1}(A)\chi_{E_2}(A)\xi, \xi)_H$, which holds because of the definition of μ. By linearity, Eq. (27) is true for step functions f_1 and f_2. Finally, this equality is preserved under bounded pointwise limits. This proves that U is an isometry. Since the vectors of the form $f(A)\xi$ are dense in H (they form a linear space containing ξ and invariant under A), and the functions f are dense in $L_2(X, \mu)$, the operator U can be extended to a unitary operator from H onto $L_2(X, \mu)$. Finally, denoting $xf(x)$ by $g(x)$, we have by definition that

$$UAf(A)\xi = Ug(A)\xi = g = M(x)f = M(x)Uf(A)\xi.$$

From this, $UA = M(x)U$ on vectors of the form $f(A)\xi$, hence, everywhere. □

We now eliminate the requirement that A have a cyclic vector.

Theorem 11. *Suppose that A is a bounded selfadjoint operator on the Hilbert space H. There exists a family $\{H_\beta\}_{\beta\in B}$ of subspaces of H with the properties:*

(1) *$H_\beta \perp H_{\beta'}$ for $\beta \neq \beta'$;*
(2) *$\sum_{\beta\in B} H_\beta = H$;*
(3) *$AH_\beta \subset H_\beta$ for all $\beta \in B$;*
(4) *the restriction of A to H_β has a cyclic vector for each $\beta \in B$.*

The proof follows at once from Zorn's lemma, applied to the collection of all systems $\{H_\beta\}$ having the properties (1), (3), and (4). We observe also that if H is separable, then B is at most countable. Theorems 10 and 11 imply that Theorem 9 holds for any bounded selfadjoint operator A, with the following refinement: the set X can be taken to be a union of copies of the segment $[-\|A\|, \|A\|]$, and the function $a(x)$ to be the coordinate function x.

We leave it to the reader to derive a more general result (following the scheme above):

Theorem 12. *Let $A_1, \ldots, A_n$ be bounded pairwise commuting selfadjoint operators on a Hilbert space H. There exists a realization of H as a space $L_2(X, \mu)$ such that all the operators $A_1, \ldots, A_n$ simultaneously become operators of multiplication by real functions $a_1(x), \ldots, a_n(x) \in L_\infty(X, \mu)$.*

Corollary. *Let A be a bounded normal (for example, unitary) operator on a complex Hilbert space H. There exists a realization of H as a space $L_2(X, \mu)$ such that A becomes the operator of multiplication by a complex-valued function $a \in L_\infty(X, \mu)$ ($|a(x)| = 1$ almost everywhere on X in the case of a unitary operator).*

Indeed, A has the form $B + iC$, where B and C are commuting selfadjoint bounded operators. Applying Theorem 12 to them, we get the necessary statement. If A is unitary, then $AA^* = 1$, therefore, $|a(x)|^2 = 1$ almost everywhere.

Let us now analyze the case of an unbounded selfadjoint operator A in H. We construct the auxiliary operator $V = (A + i1)(A - i1)^{-1}$. For this, note that $\operatorname{im}(A - i1) = H$, by Theorem 8. Hence, for each $x \in H$ there is a $y \in \mathscr{D}_A$ such that $x = Ay - iy$. We set $Vx = Ay + iy$. The image of $A + i1$ coincides with H by the same theorem, therefore, $\operatorname{im} V = H$. Finally, the equations $\|Ay - iy\|^2 = (Ay - iy, Ay - iy) = (Ay, Ay) + (y, y) = \|Ay + iy\|^2$ show that V is an isometry. Thus, V is unitary. By the corollary to Theorem 12, there exists a realization of H as a space $L_2(X, \mu)$ such that V becomes the operator of multiplication by a measurable function $v(x)$ with the property that $|v(x)| = 1$ almost everywhere on X.

Let us now show that the values of $v(x)$ are different from 1 almost everywhere. Otherwise V would have an eigenvector x_0 with eigenvalue 1 (in our realization, any function differing from 0 only on the set where $v(x) = 1$ is such a vector). The equality $Vx_0 = x_0$ is equivalent to the equality $Ay_0 - iy_0 = Ay_0 + iy_0$, from which $y_0 = 0$, hence, $x_0 = 0$. Accordingly, $v(x)$ differs from 1 almost everywhere. Then there exists a measurable real-valued function $a(x)$ such that $v(x) = [a(x) + i]/[a(x) - i]$ almost everywhere; it suffices to set $a(x) = i\{[v(x) + 1]/[v(x) - 1]\}$. Finally, we prove that $\mathscr{D}_A = \mathscr{D}_{(V-1)^{-1}}$ and $A = i(V + 1)(V - 1)^{-1}$. From this it will follow that A is the operator of multiplication by $a(x)$ in our realization.

By the definition of V, $(V + 1)x = Vx + x = Ay + iy + Ay - iy = 2Ay$, $(V - 1)x = Vx - x = Ay + iy - Ay + iy = 2iy$. From this, $\mathscr{D}_{(V-1)^{-1}} = \operatorname{im}(V - 1) = \mathscr{D}_A$, and $Ay = (1/2)(V + 1)x = i(V + 1)(V - 1)y$. $\square$

The following generalization of the results in §1.2 is a further consequence of this theorem.

Theorem 13. *Let A be a selfadjoint (perhaps unbounded) operator in a Hilbert space H. There exists a unique homomorphism φ of the algebra $B(\mathbf{R})$ of bounded Borel functions on $\mathbf{R}$ into the algebra* End H *having the properties*:

(1) $\varphi(1) = 1$;
(2) $\varphi[(t + i)/(t - i)) = (A + i1)(A - i1)^{-1}$;
(3) *if* $|f_n(t)| \leq C$ *and* $f_n(t) \to f(t)$ *for all* $t \in \mathbf{R}$, *then* $\varphi(f_n)$ *converges strongly to* $\varphi(f)$.

Moreover, φ has the properties:

(4) $\varphi(\bar{f}) = \varphi(f)^*$;
(5) $\|\varphi(f)\| \leq \sup_{t\in\mathbf{R}} |f(t)|$.

Proof. It will be assumed that H is realized as a space $L_2(X, \mu)$ such that A is multiplication by a measurable real-valued function $a(x)$. Then the operator $V = (A + i1)(A - i1)^{-1}$ is multiplication by $v(x) = [a(x) + i]/[a(x) - i]$. The desired homomorphism φ is defined by the formula $\varphi(f) = M(f, a)$, where $M(f, a)$ is the operator of multiplication by the function

$f(a(x))$. The properties (1)–(5) can be checked without difficulty. Let us show that φ is unique. It follows from (2) that

$$\varphi\left(\frac{t-i}{t+i}\right) = \varphi\left(\left(\frac{t+i}{t-i}\right)^{-1}\right) = V^{-1} = V^*.$$

From this,

$$\varphi\left(\frac{t^2-1}{t^2+1}\right) = \operatorname{Re} V, \qquad \varphi\left(\frac{2t}{t^2+1}\right) = \operatorname{Im} V,$$

where

$$\operatorname{Re} V = \frac{V+V^*}{2}, \qquad \operatorname{Im} V = \frac{V-V^*}{2i}.$$

This means that φ is uniquely determined on the rational functions of the form

$$P\left(\frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1}\right),$$

where P is a polynomial in two variables. By using the Weierstrass theorem it can be proved that such functions can be used to approximate uniformly any continuous function on the line for which finite limits as $t \to \pm\infty$ exist and are equal.† Finally, by successively taking bounded pointwise limits of such functions, we can obtain any function in $B(\mathbf{R})$. □

2. The Spectral Theorem

Many results in measure theory (see Ch. II) can be carried over to the case when so-called projection-valued measures are considered in place of ordinary measures.

Definition. Suppose that X is a set, B is a σ-algebra of subsets of X, and H is a Hilbert space. A mapping $\lambda: B \to \operatorname{End} H$ is called a *projection-valued measure* on (X, B) with values in End H if:

(1) $\lambda(E) = \lambda(E)^*$ for any $E \in B$;
(2) $\lambda(E_1 \cap E_2) = \lambda(E_1)\lambda(E_2)$ for any $E_1, E_2 \in B$;
(3) $\lambda(E_1 \cup E_2) = \lambda(E_1) + \lambda_2(E_2)$ for any disjoint $E_1, E_2 \in B$;
(4) if $E_n \in B$ and $\lim_{n\to\infty} E_n = E$ exists (see Problem 81), then $s\text{-}\lim_{n\to\infty} \lambda(E_n)$ exists and equals $\lambda(E)$.

† It suffices to use the change of variable

$$\frac{t^2-1}{t^2+1} \to \cos\alpha, \qquad \frac{2t}{t^2+1} \to \sin\alpha.$$

EXAMPLE. Let (X, B, μ) be a space with an ordinary σ-additive measure μ. We set $H = L_2(X, \mu)$, $\lambda(E) = M(\chi_E)$, the operator of multiplication by the characteristic function of a set $E \subset B$. The properties (1)–(3) are obvious here, and (4) follows from the Lebesgue theorem on dominated convergence.

We shall discuss here some properties of projection-valued measures that follow from the definition given above. Condition (2) implies that the operators $\lambda(E)$, $E \in B$, commute pairwise. Further, condition (1) and the identity $\lambda(E)^2 = \lambda(E)$ (which follows from (2)) show that $\lambda(E)$ is an orthogonal projection.

Let H_E be the subspace $\lambda(E)H$ onto which $\lambda(E)$ projects H. Property (2) has the following geometric meaning: $H_{E_1 \cap E_2} = H_{E_1} \cap H_{E_2}$. The more general assertion $\lambda(E_1 \cup E_2) = \lambda(E_1) + \lambda(E_2) - \lambda(E_1 \cap E_2)$ follows easily from (3). Geometrically, this means that $H_{E_1 \cup E_2} = H_{E_1} + H_{E_2}$, and that $H_{E_1} \perp H_{E_2}$ if E_1 and E_2 are disjoint. Finally, it follows from (3) that $\lambda(\varnothing) = 0$ and $\lambda(X) = 1$, i.e., $H_\varnothing = \{0\}$, $H_X = H$.

A projection-valued measure λ can be used to make a whole family of ordinary measures and signed measures. Namely, let ξ and η be two vectors in H. Then the mapping $\lambda_{\xi\eta}\colon B \to \mathbf{C}$ defined by the formula

$$\lambda_{\xi\eta}(E) = (\lambda(E)\xi, \eta), \tag{28}$$

is a complex measure on B. If $\xi = \eta$, then we simply write λ_ξ instead of $\lambda_{\xi\xi}$. The identity

$$\lambda_{\xi\eta} = \tfrac{1}{4}(\lambda_{\xi+\eta} - \lambda_{\xi-\eta} + i\lambda_{\xi-i\eta} - i\lambda_{\xi+i\eta}) \tag{29}$$

shows that the complex measures $\lambda_{\xi\eta}$ and, consequently, the projection-valued measure λ itself can be recovered from the collection of ordinary measures $\{\lambda_\xi\}_{\xi \in H}$.

As in the case of an ordinary measure, a projection-valued measure λ can be used to define an integral. Let f be a B-measurable bounded numerical function on X. A *Lebesgue integral sum* for f is defined to be an expression of the form

$$S_n(f, \lambda) = \sum_{k \in \mathbf{Z}} \frac{k}{2^n} \lambda\left(\left\{x \in X : \frac{k}{2^n} \le f(x) < \frac{k+1}{2^n}\right\}\right). \tag{30}$$

It is easy to verify that $S_{n_1}(f) \ge S_{n_2}(f)$ if $n_1 < n_2$ (i.e., the difference $S_{n_1}(f) - S_{n_2}(f)$ is a positive operator). Moreover, the sequence $\{S_n(f)\}$ is bounded below by the operator $\inf_{x \in X} f(x) \cdot 1$. Then $s\text{-}\lim_{n \to \infty} S_n(f)$ exists (see Problem 557), and it is called the *integral of f with respect to the projection-valued measure* λ and denoted by $\int_X f(x)\, d\lambda(x)$. The properties of the usual Lebesgue integral imply that for any vector $\xi \in H$ we have

$$\left(\int_X f(x)\, d\lambda(x)\xi, \xi\right) = \int_X f(x)\, d\lambda_\xi(x).$$

From this and the identity (29) there follows the more general equality

$$\left(\int_X f(x)\,d\lambda(x)\xi, \eta\right) = \int_X f(x)\,d\lambda_{\xi\eta}(x). \tag{31}$$

Finally, it is possible to define the integral $A = \int_X f(x)\,d\lambda(x)$ for an unbounded B-measurable function f on X. Namely, let $\mathscr{D}_A$ be the collection of vectors $\xi \in H$ for which the integral $\int_X |f(x)^2\,d\lambda_\xi(x)$ converges. (It can be derived from (29) that $\mathscr{D}_A$ is a linear subspace of H.)

For $\xi \in \mathscr{D}_A$ we define the operator A by

$$(A\xi, \eta) = \int_X f(x)\,d\lambda_{\xi\eta}(x). \tag{32}$$

The convergence of this integral follows from the inequality

$$|\lambda_{\xi\eta}(E)|^2 \leq \lambda_\xi(E)\lambda_\eta(E), \tag{33}$$

which is a particular case of the Cauchy–Bunyakovskii inequality. Namely, the Cauchy–Bunyakovskii inequality and (33) give us for an integral sum of the integral (32) that

$$|S_n(f, \lambda_{\xi\eta})|^2 \leq S_n(|x|^2, \lambda_\xi)S_n(1, \lambda\eta),$$

and from this $|\int_X f(x)\,d\lambda_{\xi\eta}(x)|^2 \leq \int_X |f(x)|^2\|\eta\|^2\,d\lambda$. Thus, $\|A\xi\|_H \leq \|f\|_{L_2(X,\lambda_\xi)}$.

Note that if f is a real function, then, by the definition of A, the expression $(A\xi, \xi)$ is real for $\xi \in \mathscr{D}_A$. Thus, A is a symmetric operator. In fact, with the domain $\mathscr{D}_A$ introduced above this operator is selfadjoint. This follows from Theorem 8 and the explicit construction of the operators $(A \pm i1)^{-1}$: These operators can be defined by the integrals $\int_X [d\lambda(x)]/[f(x) \pm i]$.

We can now state the main result in this section.

Theorem 14. *Let A be a selfadjoint (not necessarily bounded) operator in a Hilbert space H. There exists a unique projection-valued Borel measure λ on* **R** *with values in* End H *having the property that*

$$f(A) = \int_{-\infty}^{\infty} f(x)\,d\lambda(x) \tag{34}$$

for any bounded Borel function f on **R**.

Moreover,

$$A = \int_{-\infty}^{\infty} x\,d\lambda(x). \tag{35}$$

Proof. The uniqueness of λ follows at once from (34) with $f = \chi_E$, where E is a Borel set on the line. The proof of existence is obvious if we pass to the realization of H in which A is multiplication by a function a. Indeed, let $\lambda(E)$ be the operator of multiplication by the characteristic function of the set $a^{-1}(E)$. Then the equality (34) becomes an identity for a step function f

(verify!). The general case of (34), along with (35), is proved by passing to the limit.

The projection-valued measure λ is called the *spectral measure* of the operator A, and the equality (35) is the *spectral decomposition* of this operator.

As a consequence of Theorem 14 we get the definition of any Borel function (including unbounded ones) of any selfadjoint operator A: They are defined by the integral (34) with the stipulations indicated above. It can be checked that $B = f(A)$ is always a closed operator, and is normal in the sense that BB^* and B^*B have a common domain and coincide on it. An example of the use of this construction is the description of one-parameter groups of unitary operators. □

Definition. A collection $\{V(t)\}_{t\in\mathbf{R}}$ of unitary operators on a Hilbert space H is called a *one-parameter group* if the following conditions hold:

(1) $V(t)V(s) = V(t + s)$ for $t, s \in \mathbf{R}$;
(2) the mapping $t \mapsto V(t)$ is continuous in the weak operator topology.

Stone's Theorem. *Every one-parameter group of unitary operators on H has the form*

$$V(t) = e^{itA}, \tag{36}$$

where A is a selfadjoint operator in H.

PROOF. Observe first that the formula (36) really defines a one-parameter group; this is at once clear if we pass to the realization of H in which A is the operator of multiplication by a function; moreover, in this realization it is easy to verify the equality

$$\frac{d}{dt} V(t)\xi = iV(t)A\xi = iAV(t)\xi \quad \text{for } \xi \in \mathscr{D}_A. \tag{37}$$

Suppose now that a one-parameter group $\{V(t)\}_{t\in\mathbf{R}}$ is given. We define $\mathscr{D}_A$ as the collection of vectors $\xi \in H$ for which the function $t \mapsto V(t)\xi$ is differentiable, and we define an operator A for $\xi \in \mathscr{D}_A$ by the formula $A_\xi = -i(d/dt)V(t)\xi|_{t=0}$. Let us show that $\mathscr{D}_A$ is dense in H. Note first that $t \mapsto V(t)$ is a strongly continuous mapping (see Problem 563). This follows from the equality $\|(V(t_1) - V(t_2))\xi\|^2 = 2\|\xi\|^2 - 2\operatorname{Re}(V(t_1 - t_2)\xi, \xi)$ and the weak continuity of $V(t)$. Suppose now that $\{\varphi_n\} \subset \mathscr{D}(\mathbf{R})$ is a sequence approximating δ. Then for any $\xi \in H$ the sequence

$$\xi_n = \int_{-\infty}^{\infty} \varphi_n(\tau)V(\tau)\xi \, d\tau$$

converges to ξ. We claim that $\xi_n \in \mathscr{D}_A$. Indeed,

$$V(t)\xi_n = \int_{-\infty}^{\infty} \varphi_n(\tau)V(t + \tau)\xi \, d\tau = \int_{-\infty}^{\infty} \varphi_n(\tau - t)V(\tau)\xi \, d\tau.$$

From this, $(d/dt)V(t)\xi_n = -\int_{-\infty}^{\infty} \varphi_n'(\tau - t)V(\tau)\xi\, d\tau$. (The differentiability of the integral with respect to the parameter t can be proved as in the usual analysis.)

Let us now verify that $V(t)$ preserves the subspace $\mathscr{D}_A$ and that (37) holds. To do this it suffices to observe that the vectors $V(t + \tau)\xi$ and $V(\tau)V(t)\xi$ coincide. Differentiating them with respect to τ and setting $\tau = 0$, we get the desired relation.

The symmetry of A is demonstrated by differentiating the equation $(V(t)\xi, V(t)\xi) \equiv 1$ with respect to t at $t = 0$. Finally, we verify that A is essentially selfadjoint. Let $\eta \in \ker(A^* \pm i1)$. Then $((A \mp i1)\xi, \eta) = (\xi, (A^* \pm i1)\eta) = 0$ for any $\xi \in \mathscr{D}_A$. Therefore, the function $f(t) \equiv (V(t)\xi, \eta)$ satisfies the differential equation $f'(t) \pm f(t) = 0$. From this, $f(t) = ce^{\mp t}$. But f is bounded, since $V(t)$ is a unitary operator. Hence, $c = 0$, and $(V(t)\xi, \eta) = 0$ for all $\xi \in \mathscr{D}_A$. Since $\mathscr{D}_A$ is dense in H, $\eta = 0$. Thus, $\ker(A^* \pm i1) = 0$, and, consequently, A is essentially selfadjoint. Let $\bar{A}$ be its selfadjoint closure. Let us now compare the operators $V(t)$ and $\bar{V}(t) = e^{it\bar{A}}$. Suppose that $\xi \in \mathscr{D}_A \subset \mathscr{D}_{\bar{A}}$. We consider the function $f(t) = (V(-t)\bar{V}(t)\xi, \eta)$. Differentiating this function with respect to t and using Eqs. (37), we get

$$f'(t) = -(V(-t)iA\bar{V}(t)\xi, \eta) + (V(-t)iA\overline{V(t)}\xi, \eta) = 0.$$

From this, $f(t) \equiv (\xi, \eta)$, which gives $V(-t)\bar{V}(t)\xi \equiv \xi$, i.e., $\bar{V}(t)\xi = V(t)\xi$. Since $V(t)$ and $\bar{V}(t)$ are unitary and $\mathscr{D}_A$ is dense in H, $\bar{V}(t) = V(t)$. $\square$

PART II

PROBLEMS

Chapter I

Concepts from Set Theory and Topology

§1. Relations. The Axiom of Choice and Zorn's Lemma

1°. Which of the following relations are equivalence relations?

(a) The relation of equality of two numbers;

(b) the relation of similarity of two triangles;

(c) the order relation on the real line;

(d) the relation of linear dependence in a linear space L of dimension $n > 1$;

(e) the relation of linear dependence in the set $L^* = L \setminus \{0\}$, where L is a linear space.

2. Let us call two positive functions f_1 and f_2 on $[0, 1]$ equivalent if

$$0 < \varliminf_{x \to 0} \frac{f_1(x)}{f_2(x)}, \qquad \varlimsup_{x \to 0} \frac{f_1(x)}{f_2(x)} < \infty.$$

Verify that this is actually an equivalence relation and that the corresponding quotient set is uncountable.

3. Let the condition $\lim_{x \to 0} [f_1(x)/f_2(x)] = \infty$ define the relation $f_1 > f_2$ for positive functions on $[0, 1]$. Verify that this is a partial-order relation and prove that any countable subset is bounded.

4°. Let X and Y be partially ordered sets. Define a relation $(x_1, y_1) \geq (x_2, y_2)$ on the product $X \times Y$ by the conditions $x_1 \geq x_2$ and $y_1 \geq y_2$. Prove that this is a partial-order relation. Is it a total order relation if X and Y are totally ordered sets?

5. (a) Let $(X_\alpha)_{\alpha \in A}$ be a family of partially ordered sets. Introduce a relation $\geq$ on their direct product $\prod_{\alpha \in A} X_\alpha$ by setting $(x_\alpha) \geq (y_\alpha)$ if $x_\alpha \geq y_\alpha$ for all $\alpha \in A$. Prove that this is a partial-order relation; with this relation $\prod_{\alpha \in A} X_\alpha$ is called the *product of the partially ordered sets.*

(b) Under the conditions of (a) suppose that a point $x_\alpha \in X_\alpha$ is distinguished in each X_α. The product of the pairs $(X_\alpha; x_\alpha)$ is defined to be the subset $\prod_{\alpha \in A} (X_\alpha; x_\alpha) \subset \prod_{\alpha \in A} X_\alpha$ consisting of those tuples (y_α) such that y_α differs from x_α for at most finitely many indices α. Introduce the structure of a partially ordered set with a distinguished point in $\prod_{\alpha \in A} (X_\alpha; x_\alpha)$.

6. Prove that the set of natural numbers, partially ordered by the relation of divisibility and with distinguished point 1 (see Problem 5), is isomorphic to the product of countably many copies of the natural numbers with the usual order relation and distinguished point 0.

7°. Let $P(X)$ be the set of all subsets of X with the partial order relation determined by inclusion. Prove that if $X = \coprod_{\alpha \in A} X_\alpha$ (disjoint union), then $P(X)$ is isomorphic to the product $\prod_{\alpha \in A} P(X_\alpha)$ as a partially ordered set.

8. Let X be a partially ordered set. Suppose that the partial order on X has the property that $M(x) = \{y \in X : y < x\}$ is a finite set for all $x \in X$. For any function $f(x)$ on X define

$$F(x) = \sum_{y \leq x} f(y).$$

Prove that $f(x)$ can be recovered from $F(x)$ by a formula of the form

$$f(x) = \sum_{y \leq x} \mu(x, y) F(y).$$

The function μ is uniquely determined and is called the *Möbius function* for the partially ordered set X.

9. Let (X_α, x_α) be a family of partially ordered sets with distinguished points. Suppose that each X_α satisfies the condition of Problem 8, i.e., it has a Möbius function μ_α; assume that for all but perhaps finitely many α the point x_α is a minimal element of X_α. Prove that the partially ordered set $\prod_{\alpha \in A} (X_\alpha; x_\alpha)$ has a Möbius function μ and that it is given by the formula

$$\mu((y_\alpha), (z_\alpha)) = \prod_{\alpha \in A} \mu_\alpha(y_\alpha, z_\alpha).$$

10. Find the Möbius functions for the following partially ordered sets:

(a) the natural numbers with the usual order relation;

(b) the natural numbers with the relation of divisibility;

(c) the collection of finite subsets of a given set X, with the relation of inclusion;

(d)** the collection of subspaces of a linear n-dimensional space over a finite field, with the relation of inclusion.

11*. Express the following quantities in terms of the Möbius function from Problem 10:

(a) the *Euler function* $\varphi(n)$, which is equal to the number of positive integers less than n and relatively prime to n;

(b)** the number $P(n, q)$ of irreducible polynomials of degree n with coefficients in the finite field F_q and with leading coefficient 1;

(c) the limit $C(N)/N^2$, $N \to \infty$, where $C(N)$ is the number of irreducible fractions of the form p/q, $1 \leq p \leq N$, $1 \leq q \leq N$.

12. Let $\Phi_n(t) = \prod_{d \mid n} (t^{n/d} - 1)^{\mu(d)}$, where μ is the Möbius function in Problem 10(b). Prove that:

(a) $\Phi_n(t)$ is a polynomial of degree $\varphi(n)$ with integer coefficients (see Problem 11(a));

(b) $\prod_{d \mid n} \Phi_d(t) = t^n - 1$;

(c)* the polynomials $\Phi_n(t)$ are irreducible and mutually prime over the field $\mathbf{Q}$ of rational numbers.

13. Let A be a well-ordered set, and let X_α be a nonempty totally ordered set for each $\alpha \in A$. Define the *lexicographical order* in the set $X = \prod_{\alpha \in A} X_\alpha$ by setting $x > y$ if $x_{\alpha_0} > y_{\alpha_0}$, where α_0 is the smallest element of A for which $x_\alpha \neq y_\alpha$. Prove that this really is a total-order relation.

14. Suppose that $\mathbf{R}^n$ is totally ordered in such a way that

(a) $x_1 \geq y_1$ and $x_2 \geq y_2 \Rightarrow x_1 + x_2 \geq y_1 + y_2$;

(b) $x \geq y$ and $\lambda \geq 0$ implies that $\lambda x \geq \lambda y$ for $\lambda \in \mathbf{R}$;

(c) $x \geq y$ and $y \geq x$ implies that $x = y$.

Prove that $\mathbf{R}^n$ is isomorphic (as a totally ordered space) to the product of n lines (with the usual order), endowed with the lexicographical order (see Problem 13).

15*. Two well-ordered countable sets are said to be *equivalent* if there is a monotonic one-to-one correspondence between them. Let $\mathscr{M}$ be the corresponding set of equivalence classes. Define a partial order relation on $\mathscr{M}$ by setting $\mu \geq \nu$ if the classes μ and ν have representatives M and N such that M is equivalent to an initial segment of N (i.e., a set of the form $N(n_0) = \{n \in N: n \leq n_0\}$).

Prove that:

(a) $\mathscr{M}$ contains a smallest element μ_0;

(b) any two elements of $\mathscr{M}$ are comparable;

(c) $\mathscr{M}$ is well ordered;

(d) $\mathscr{M}$ is uncountable;

(e)** any uncountable set contains a subset of the same cardinality as $\mathscr{M}$.

16*. Let $\mathscr{M}$ be the well-ordered set described in Problem 15. Let $\mathfrak{A} = \mathscr{M} \times [0, 1)$ and define the lexicographical order on $\mathfrak{A}$: If $a = (\mu, x)$, $b = (\nu, y)$, then $a \geq b$ means that either $\mu \geq \nu$ and $\mu \neq \nu$, or $\mu = \nu$ and $x \geq y$.

Prove that any initial segment (see Problem 15) of $\mathfrak{A}$ is equivalent (as a totally ordered set) to the half-interval $[0, 1)$, though the set $\mathfrak{A}$ itself is not equivalent to this half-interval.

17*. Let $a_0 = (\mu_0, 0)$ be a minimal point of the set $\mathfrak{A}$ in Problem 16. Define a topology on $\mathfrak{A}_0 = \mathfrak{A} \setminus \{a_0\}$ by taking the "intervals" $(a, b) = \{c \in \mathfrak{A}_0 : a < c < b\}$ as a base of open sets. Prove that:

(a) each point $a \in \mathfrak{A}_0$ has a neighborhood that is homeomorphic to an ordinary interval;

(b) the topological space $\mathfrak{A}_0$ is connected and is not homeomorphic to an ordinary interval.

This space $\mathfrak{A}_0$ is called the *Aleksandrov line* and is an example of a one-dimensional manifold not having a countable base of open sets.

18°. Prove that in the collection of disks contained in a given square on the plane there is a maximal element but not a largest element relative to inclusion.

19. Using Zorn's lemma, prove that every linear space has a basis.

20. Using Zermelo's theorem, show that for any two sets A and B there is either a one-to-one mapping from A onto a subset of B or a one-to-one mapping from B onto a subset of A.

21*. Derive Zermelo's theorem from Zorn's lemma.

22. (a) Prove that any partial order relation R on a finite set X is contained in some total order relation $\tilde{R}$.

(b) Is this true for infinite sets?

23*. Prove that the field of complex numbers is isomorphic to the algebraic closure of the field of rational functions, with rational coefficients, of a continuum of algebraically independent variables.

§2. Completions

24. (a) Prove that the theorem on shrinking balls is valid in a complete space X. [Let $\{B_n\}$ be a sequence of closed balls in X such that: (1) $B_1 \supset B_2 \supset \cdots \supset B_n \supset \cdots$; (2) the radii of the balls B_n tend to zero as $n \to \infty$. Show that $\bigcap_{n=1}^{\infty} B_n$ consists of exactly one point.]

(b) Prove that if the theorem on shrinking balls holds in a metric space X, then it is complete.

25°. Prove that every uniformly continuous function on a metric space X can be uniquely extended to a continuous function on its completion, and that this extension is uniformly continuous.

26. Prove that the following metric spaces are not complete, and construct their completions:

(a) the line $\mathbf{R}$ with the distance $d(x, y) = |\arctan x - \arctan y|$;

(b) the line $\mathbf{R}$ with the distance $d(x, y) = |e^x - e^y|$.

27°. Define a distance on the set of segments on the line by the formula $d([a, b], [c, d]) = |a - c| + |b - d|$. Prove that the resulting metric space is not complete, and find its completion.

28. Define the distance on the set $\{\Delta\}$ of segments on the line to be the length of the symmetric difference, i.e.:

$$d(\Delta_1, \Delta_2) = |\Delta_1| + |\Delta_2| - 2|\Delta_1 \cap \Delta_2|.$$

Prove that this metric space is not complete and find its completion.

29°. Prove that the space $B(X)$ of all bounded functions on a set X is complete with respect to the distance

$$d(f, g) = \sup |f(x) - g(x)|, \qquad x \in X.$$

30°. Let X be a bounded metric space. Prove that the correspondence $x \mapsto d(x, \cdot)$ is an isometric mapping of X into $B(X)$ (see Problem 29).

31°. Let X be a subset of a complete metric space Y. Prove that:

(a) X is complete if and only if it is closed;

(b) the completion of X is equal to its closure in Y.

(c) Use Problems 29, 30, and parts (a) and (b) to derive the completion theorem for bounded spaces.

32. Let X be a complete metric space, and Y_i open dense subsets of X. Prove that $\bigcap_{i=1}^{\infty} Y_i$ is dense in X.

33. Prove that the space of polynomials is not complete with respect to the distances:

(a) $d(P, Q) = \max_{x \in [0, 1]} |P(x) - Q(x)|$;

(b) $d(P, Q) = \int_0^1 |P(x) - Q(x)|\, dx$;

(c) $d(P, Q) = \sum_i |c_i|$ if $P(x) - Q(x) = \sum c_i x^i$.

34. Let X be a metric space that has a finite ε-net for any $\varepsilon > 0$. Prove that the completion of X is compact.

35. Let the distance on the set $C(X, Y)$ of continuous mappings of a metric space X into a bounded complete metric space Y be defined by the formula

$$d(f_1, f_2) = \sup_{x \in X} d_Y(f_1(x), f_2(x)).$$

Prove that $C(X, Y)$ is complete.

36. Let X be a bounded complete metric space with metric d_X, and G the collection of all homeomorphisms of X onto itself. Define a distance on G by the formula

$$d(f_1, f_2) = \sup[d_X(f_1(x), f_2(x)) + d_X(f_1^{-1}(x), f_2^{-1}(x))].$$

Prove that G is a complete metric space.

37°. Let p be a prime number. Define the *p-adic norm* on the set of rational numbers by the equation

$$\|r\|_p = p^{-k} \quad \text{if } r = p^k \frac{m}{n},$$

where m and n are integers relatively prime to p. Prove:

(a) $\|r_1 r_2\|_p = \|r_1\|_p \|r_2\|_p$;

(b) $\|r_1 + r_2\|_p \leq \max\{\|r_1\|_p, \|r_2\|_p\}$;

(c) if $\|r_1\|_p < \|r_2\|_p$, then $\|r_1 + r_2\|_p = \|r_2\|_p$.

38*. Prove that the set $\mathbf{Q}$ of rational numbers is a metric space when equipped with the distance $d_p(r_1, r_2) = \|r_1 - r_2\|_p$ (see Problem 37). Let $\mathbf{Q}_p$ be its completion. Prove that all the arithmetical operations in $\mathbf{Q}$ extend by continuity to $\mathbf{Q}_p$. The field so obtained is called the *field of p-adic numbers* (regarding the noncompleteness of $\mathbf{Q}$ see Problem 39).

39*. Prove that each element of the field $\mathbf{Q}_p$ (see Problem 38) can be uniquely expressed as a p-ary fraction of the form

$$\ldots a_2 a_1 a_0 . a_{-1} a_{-2} \ldots a_{-k},$$

where $0 \leq a_i \leq p - 1$; there are only finitely many symbols after the separation point but infinitely many before it. In other words, each element $x \in \mathbf{Q}_p$ is the sum of a convergent series $\sum_{i=-k}^{+\infty} a_i p^i$. Prove that $\mathbf{Q}_p \neq \mathbf{Q}$.

40. Prove that the following equalities hold in the field $\mathbf{Q}_5$ (see Problem 38):

$$2 + 3 = \ldots 00010; \qquad 2 - 3 = \ldots 444;$$

$$2 \cdot 3 = \ldots 00011; \qquad 2 : 3 = \ldots 3131314.$$

41. Prove that the square root of the number -1 $(= \ldots 44)$ exists in the field $\mathbf{Q}_5$ of 5-adic numbers, and find its last three places. How many such roots are there?

42*. Let $\mathbf{Z}_p$ be the closure of the ring of integers $\mathbf{Z}$ in $\mathbf{Q}_p$ (the set of *p-adic integers*). Prove that $\mathbf{Z}_p$ is a compact set. Construct a homeomorphism of $\mathbf{Z}_p$ onto the Cantor set.

43*. Prove that $\lim_{n\to\infty} x^{p^n}$ exists for any $x \in \mathbf{Z}_p$ (see Problem 42). Denote this limit by $\operatorname{sgn}_p x$. Prove that the resulting function (the *p-adic signum*) takes exactly p distinct values: 0 and the $(p - 1)$-roots of 1, of which there are $p - 1$. Prove the identity

$$\operatorname{sgn}_p(xy) = \operatorname{sgn}_p x \cdot \operatorname{sgn}_p y.$$

44*. Find the domain of convergence in the field $\mathbf{Q}_p$ of the series

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \text{and} \quad \sum \frac{x^k}{k}(-1)^{k-1}.$$

45*. Prove that there is no total order relation in $\mathbf{Q}_p$ having the following properties:

(a) if $x > 0$ and $y > 0$, then $x + y > 0$;

(b) if $x > 0$ and $y > 0$, then $xy > 0$;

(c) if $x_n > 0$ and $x = \lim_{n\to\infty} x_n$ exists, then $x \geq 0$.

46. Define a distance on the set $\mathbf{N}$ of positive integers by setting $d(m, n) = 1/k$ if the last k places in the decimal expressions for the numbers m and n coincide.

(a) Prove that the resulting metric space is not complete and that its completion is isomorphic (as a ring) to the direct product of the rings $\mathbf{Z}_2$ and $\mathbf{Z}_5$.

(b) Prove that for any natural number k there are exactly four k-place terminations

$$\ldots 000\,000$$

$$\ldots 000\,001$$

$$\ldots 890\,625$$

$$\ldots 109\,376.$$

that are reproduced under multiplication. (This means that if the numbers N_1 and N_2 terminate in the indicated k digits, then the product $N_1 N_2$ terminates in the same k digits.)

§3. Categories and Functors

47°. Prove that the category of nonempty subsets of a given set X (with the morphisms being the imbeddings) does not have a universal repelling object, but that the dual category has one.

48. Construct a contravariant functor from the category of all subsets of a given set (again the morphisms are the imbeddings) into itself.

49°. Does there exist a universal repelling object in: the category of groups, the category of linear spaces over a given field, the categories dual to the categories of groups and linear spaces?

50°. Let G_1 be the category of abelian groups with a distinguished generator (the morphisms are the group homomorphisms preserving the distinguished generator.) Determine universal objects in G_1 and in G_1^0.

51. Let G_2 be the category of groups with two distinguished generators (the morphisms are the group homomorphisms preserving the distinguished generators). Prove that G_2 has a universal object. This object is called the *free group* with two generators.

52. Let AG_2 be the subcategory of G_2 (see Problem 51) whose objects are the abelian groups with two distinguished generators. Prove that there is a universal object in AG_2. This object is called the *free abelian group* with two generators.

53*. Let $A_n(K)$ be the category of associative algebras with n distinguished generators over a field K. Prove that there is a universal object in $A_n(K)$. This universal object is the so-called *tensor algebra* over an n-dimensional linear space over the field K.

54*. Prove that there exists a universal object for the subcategory $CA_n(K)$ of $A_n(K)$ consisting of the commutative algebras.

55*. Prove that there exists a universal object for the category $LA_n(K)$ of Lie algebras with n distinguished generators over the field K. The corresponding object is called the *free Lie algebra* with n generators.

56*. Let $\mathfrak{G}$ be a Lie algebra over a field K of characteristic zero. We consider the category $K(\mathfrak{G})$ whose objects are those linear mappings φ from $\mathfrak{G}$ into associative algebras (depending on the object) having the property that

$$\varphi([x, y]) = \varphi(x)\varphi(y) - \varphi(y)\varphi(x).$$

A morphism of an object $\varphi\colon \mathfrak{G} \to A$ to an object $\psi\colon \mathfrak{G} \to B$ is defined to be a homomorphism $\chi\colon A \to B$ which makes the following diagram commute:

$$\begin{tikzcd} & A \arrow[d, "\chi"] \\ \mathfrak{G} \arrow[ur, "\varphi"] \arrow[r, "\psi"'] & B \end{tikzcd}$$

Prove that $K(\mathfrak{G})$ has a universal object

$$\varphi_0\colon \mathfrak{G} \to U(\mathfrak{G}).$$

The algebra $U(\mathfrak{G})$ is called the *associative hull* or *enveloping algebra* of $\mathfrak{G}$.

57*. Prove that the associative hull (see Problem 56) of the free Lie algebra with n generators is isomorphic to the tensor algebra over an n-dimensional space.

58*. Let $\{X_\alpha\}_{\alpha\in A}$ be a family of objects in the category K.

We consider the category $\tilde{K}$ whose objects are the sets of morphisms $\varphi_\alpha \in \operatorname{Mor}(X_\alpha, Y)$, $\alpha \in A$ (Y is some object in K depending on the object in $\tilde{K}$). The morphisms in $\tilde{K}$ are the sets of commutative diagrams of the form

$$\begin{tikzcd} & Y \arrow[d, "\chi"] \\ X_\alpha \arrow[ur, "\varphi_\alpha"] \arrow[r, "\psi_\alpha"'] & Z \end{tikzcd}$$

If $\tilde{K}$ has a universal repelling object, then the corresponding object in K is called the *sum* of the objects and denoted by $\coprod_{\alpha\in A} X_\alpha$. The morphisms $i_\alpha\colon X_\alpha \to \coprod_{\alpha\in A} X_\alpha$ are called the *canonical imbeddings* of the terms in the sum.

Prove that in the category of sets and in the category of linear spaces over a given field, the sum of any family of objects is defined.

59. The definition of the *product* of a family $\{X_\alpha\}_{\alpha\in A}$ of objects in a category K is obtained from the definition of the sum (see Problem 58) by reversing the arrows. Namely, the product $\prod_{\alpha\in A} X_\alpha$ is defined to be the sum of the objects X_α in the dual category $\tilde{K}^\circ$. The morphisms $p_\alpha: \prod_{\alpha\in A} X_\alpha \to X_\alpha$ are called the *canonical projections* of the product onto the factors.

Prove that in the categories of sets and linear spaces over a given field, the product of any family of objects is defined.

60. Prove that the sum $\coprod_{k=1}^n L_k$ and the product $\prod_{k=1}^n L_k$ of finitely many objects in the category of linear spaces over a given field are isomorphic.

61. Let L_1 and L_2 be two linear spaces over a field K. We consider the category whose objects are bilinear mappings $\varphi: L_1 \times L_2 \to L$, where L is a linear space (depending on the object φ). A morphism from an object $\varphi: L_1 \times L_2 \to L$ to an object $\psi: L_1 \times L_2 \to M$ is defined to be a linear mapping $\chi: L_1 \to M$ for which the diagram

$$\begin{tikzcd} & L \arrow[d, "\chi"] \\ L_1 \times L_2 \arrow[ur, "\varphi"] \arrow[r, "\psi"'] & M \end{tikzcd}$$

commutes.

Prove that the resulting category has a universal repelling object $\pi: L_1 \times L_2 \to L_1 \otimes_K L_2$. The linear space $L_1 \otimes L_2$ is called the *tensor product* of the spaces L_1 and L_2 over the field K.

62*. Let G_1 and G_2 be finite abelian groups. We consider the category of all mappings

$$\varphi: G_1 \times G_2 \to G,$$

where G is a finite abelian group (depending on the object), that are homomorphisms in each variable. The morphisms are the commutative diagrams of the form

$$\begin{tikzcd} & G \arrow[d, "\chi"] \\ G_1 \times G_2 \arrow[ur, "\varphi"] \arrow[r, "\varphi'"'] & G' \end{tikzcd},$$

where χ is a homomorphism. Prove that this category has a universal object

$$G_1 \times G_2 \to \mathrm{Tor}(G_1, G_2)$$

(the so-called *torsion product* of the two groups). Compute $\mathrm{Tor}(C_m, C_n)$, where C_m is the cyclic group of order m.

63*. Suppose that A is a directed set and K is a category. Assume that for each $\alpha \in A$ an object $X_\alpha \in \operatorname{Ob} K$ is singled out, that for each pair $\alpha < \beta$ a morphism $\varphi_{\alpha\beta} \in \operatorname{Mor}(X_\alpha, X_\beta)$ is chosen, and that for any triple $\alpha < \beta < \gamma$ the diagram

$$\begin{tikzcd}
 & X_\beta \arrow[d, "\varphi_{\beta\gamma}"] \\
X_\alpha \arrow[ur, "\varphi_{\alpha\beta}"] \arrow[r, "\varphi_{\alpha\gamma}"'] & X_\gamma
\end{tikzcd}$$

commutes.

Consider the category K_A whose objects are the families $\{\varphi_\alpha : X_\alpha \to X\}_{\alpha \in A}$ of morphisms compatible with $\varphi_{\alpha\beta}$, where X is an object in K (depending on the family), and define a morphism from $\{\varphi_\alpha : X_\alpha \to X\}_{\alpha \in A}$ to $\{\psi_\alpha : X_\alpha \to \}_{\alpha \in A}$ to be a morphism $\chi \in \operatorname{Mor}(X, Y)$ such that the diagram

$$\begin{tikzcd}
 & X \arrow[d, "\chi"] \\
X_\alpha \arrow[ur, "\varphi_\alpha"] \arrow[r, "\psi_\alpha"'] & Y
\end{tikzcd}$$

commutes for any $\alpha \in A$. A universal object in the category K_A (if it exists) is called the *inductive limit* of the family $\{X_\alpha\}_{\alpha \in A}$. The dual concept of the *projective limit* is defined as the universal object in $(K_A)^\circ$.

Prove that:

(a) the additive group of the field of rational numbers is the inductive limit of a countable family of groups of integers;

(b)** the ring $\mathbf{Z}_p$ of p-adic integers (see Problem 42) is the projective limit of the rings of residue classes modulo p^n.

64°. Each complex linear space can be regarded as a real space, and each complex linear mapping as a real linear mapping. Prove that the correspondence described is a covariant functor from the category $L(\mathbf{C})$ of linear spaces over $\mathbf{C}$ to the category $L(\mathbf{R})$ of linear spaces over $\mathbf{R}$.

65. Prove that the mapping $L \to L \otimes_{\mathbf{R}} \mathbf{C}$ (the tensor product over $\mathbf{R}$ in the sense of Problem 61) generates a covariant functor from $L(\mathbf{R})$ to $L(\mathbf{C})$.

66. Prove that the categories $L(\mathbf{R})$ and $L(\mathbf{C})$ (see Problem 64) are not equivalent.

67. Prove that the category of all finite-dimensional linear spaces over a field K is equivalent to some subcategory of itself with a countable number of objects.

68. Prove that the category of finite groups is equivalent to some subcategory of itself with a countable number of objects.

69°. Let G be a group and K a field. Consider the collection $K[G]$ of formal linear combinations of elements of G with coefficients in K. $K[G]$

is an algebra over K with respect to the natural operations of addition, multiplication, and multiplication by elements of K.

Prove that:

(a) the correspondence $G \to K[G]$ is a covariant functor from the category of groups into the category of K-algebras;

(b) $K[G]$ is a universal object in the category of multiplicative mappings of the group G into K-algebras.

Chapter II

Theory of Measures and Integrals

§1. Measure Theory

1. Algebras of Sets

70°. Prove that the symmetric difference operation satisfies the condition

$$A \triangle B \subset (A \triangle C) \cup (B \triangle C)$$

(the analog of the triangle inequality for the set-valued "distance" $d(A, B) = A \triangle B$).

71. Prove that:

(a) $(A_1 \cup A_2) \triangle (B_1 \cup B_2) \subset (A_1 \triangle B_1) \cup (A_2 \triangle B_2)$;

(b) $(A_1 \cap A_2) \triangle (B_1 \cap B_2) \subset (A_1 \triangle B_1) \cap (A_2 \triangle B_2)$;

(c) $(A_1 \setminus A_2) \triangle (B_1 \setminus B_2) \subset (A_1 \triangle B_1) \setminus (A_2 \triangle B_2)$.

(These inclusions mean that the operations of union, intersection, and complementation are continuous with respect to the "distance" of Problem 70.)

72°. Show that a system of sets that is closed under the union and intersection operations is not necessarily a ring.

73. Prove that a system of sets that is closed under the union and difference operations is a ring.

74°. Prove that the set of all intervals (open, closed, and half-open) on the line is a half-ring, but not a ring.

75°. Let $X = \{a, b\}$ be a set with two elements, and $\mathscr{P}(X)$ the collection of all subset of X.

(a) Give an example of a half-ring of elements of $\mathscr{P}(X)$ that is not a ring.

(b) Describe the half-rings that can be constructed from elements of $\mathscr{P}(X)$.

(c) Describe the rings that can be constructed from elements of $\mathscr{P}(X)$.

(d) Describe the algebras that can be constructed from elements of $\mathscr{P}(X)$.

76°. Show that for any nonempty system S of sets there is exactly one *minimal ring* $R(S)$, i.e., a ring $R(S)$ of sets such that $S \subset R(S)$, and $R(S) \subset R$ for any ring R containing S.

77°. Prove that the minimal ring of a half-ring S is the system of sets of the form $A = \coprod_{k=1}^{n} A_k$, $A_k \in S$.

78°. Prove that any σ-algebra is a δ-algebra and, conversely, any δ-algebra is a σ-algebra.

79. Prove that a direct product of half-rings is a half-ring.

80. Prove that a direct product of rings may not be a ring.

81. The *limit supremum of a sequence of sets* E_n is defined to be the set $\overline{\lim}_n E_n = \bigcap_n (\bigcup_{k \geq n} E_k)$, i.e., the collection of points belonging to infinitely many sets E_n. The *limit infimum of a sequence of sets* E_n is defined to be the set $\underline{\lim}_n E_n = \bigcup_n (\bigcap_{k \geq n} E_k)$. Prove that $\underline{\lim}_n E_n \subseteq \overline{\lim}_n E_n$. For any sequence of sets E_n. If the limits supremum and infimum are equal, then their common value is called the *limit of the sequence of sets* E_n.

82°. Give an example of a sequence of sets E_n for which $\underline{\lim}_n E_n \neq \overline{\lim}_n E_n$.

83. Let X be a set and $\{E_n\}$ a sequence of sets such that $E_n \subset X$ for any n. Prove the formula

$$X \setminus \overline{\lim_n} E_n = \underline{\lim_n}(X \setminus E_n).$$

84. Let $\{E_n\}$ be a sequence of sets, and $\{\chi_n\}$ the sequence of their characteristic functions. Prove that the function $\overline{\lim}_n \chi_n$ is the characteristic function of the set $\overline{\lim}_n E_n$, and the function $\underline{\lim}_n \chi_n$ is the characteristic function of the set $\underline{\lim}_n E_n$.

85. Prove that the limit of a sequence of sets E_n exists if and only if the sequence of characteristic functions of the E_n has a limit.

86°. Let A be a system of sets and $\tilde{A}$ the collection of characteristic functions of the sets in A. Prove that A is a ring of sets if and only if $\tilde{A}$ is an algebraic ring (with respect to addition and multiplication modulo 2).

87*. The *Borel sets* on the line are defined to be the sets obtained from intervals by countably many operations of union, intersection, and difference. Prove that the collection of Borel sets has the cardinality of the continuum.

88*. Suppose that ten sets are given. How many new sets can be constructed from these ten sets by (multiple) application of the operations of intersection, union, difference, and symmetric difference?

89°. Let $f\colon A \to B$ be a mapping of sets, $\mathscr{A}$ a system of subsets of A, and $\mathscr{B}$ a system of subsets of B. Define

$$f(\mathscr{A}) = \{f(X) \subset \mathscr{B} : X \in \mathscr{A}\},$$
$$f^{-1}(\mathscr{B}) = \{f^{-1}(Y) \in \mathscr{A} : Y \in \mathscr{B}\}.$$

Prove that $f^{-1}(\mathscr{B})$ is a ring if $\mathscr{B}$ is.

90. In the notation of Problem 89 show that $f(\mathscr{A})$ is not necessarily a ring if $\mathscr{A}$ is a ring.

91. In the notation of Problem 89 show that $f^{-1}(\mathscr{B})$ is a σ-algebra if $\mathscr{B}$ is.

92. In the notation of Problems 76 and 89 prove that

$$R(f^{-1}(\mathscr{B})) = f^{-1}(R(\mathscr{B})).$$

2. Extension of a Measure

93. The *inner measure* of a set $A \subset [0, 1]$ is defined to be the number $\mu_*(A) = 1 - \mu^*([0, 1] \setminus A)$, where μ^* is the outer measure of A. Prove that

$$\mu^*(A) \geq \mu_*(A).$$

94. In the notation of Problem 93 prove that a set $A \subset [0, 1]$ is Lebesgue measurable if and only if

$$\mu_*(A) = \mu_*(A).$$

95. Prove that the cardinality of the set of Lebesgue measurable subsets of $[0, 1]$ is greater than that of the continuum.

96. Let μ be Lebesgue measure on $[0, 1]$, and introduce an equivalence relation on the space of Lebesgue measurable subsets of $[0, 1]$ by setting $A \sim B$ if $\mu(A \triangle B) = 0$. Prove that the collection of equivalence classes has the cardinality of the continuum.

97. Let μ be a measure on S. Prove that the following conditions are equivalent if S is a ring, and may not be equivalent if S is a half-ring.

(a) *Countable additivity*: $\mu(\coprod_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \mu(A_k)$.

(b) *Upper semicontinuity*: if $A_1 \supset A_2 \supset A_3 \supset \ldots$ and $A = \bigcap_{k=1}^{\infty} A_k$, then $\mu(A) = \lim \mu(A_k)$.

(c) *Lower semicontinuity*: if $A_1 \subset A_2 \subset A_3 \subset \ldots$ and $A = \bigcup_{k=1}^{\infty} A_k$, then $\mu(A) = \lim \mu(A_k)$.

(d) *Continuity*: $\mu(\lim_{n\to\infty} A_k) = \lim_{n\to\infty} \mu(A_k)$.

98. Let μ be a countably additive measure on a half-ring $S \subset P(X)$, and μ^* the corresponding outer measure on $P(X)$. Prove that the relation $\mu^*(A \triangle B) = 0$ is an equivalence relation and that the function $d(\tilde{A}, \tilde{B}) = \mu^*(A \triangle B)$ defines a distance on the corresponding quotient set $\mathscr{M}$. (Here $\tilde{A}$ and $\tilde{B}$ are the equivalence classes containing the sets A and B.)

99*. Prove that the metric space $\mathscr{M}$ in Problem 98 is complete.

100. Let R and L be the subspaces of $\mathscr{M}$ (see Problem 98) consisting of the classes of elementary (i.e., belonging to $R(S)$) and measurable sets, respectively. Prove that L coincides with the closure of R.

101*. Let S be the half-ring of intervals of the form $[a, b)$ on $[0, 1]$, and $\mathscr{M}$ the space constructed in Problem 98. Prove that $\mathscr{M}$ is connected and not compact.

102*. Let S_1 be the half-ring of half-open intervals of the form $[a, b)$ on the unit interval, and S_2 the half-ring of rectangles of the form $[a, b) \times [c, d)$ on the unit square.

(a) Prove that the corresponding spaces L_1 and L_2 are (see Problem 100) isometric.

(b) are the spaces S_1 and S_2 isometric?

(c) Are R_1 and R_2 isometric?

103°. Let $\{E_n\}$ be a sequence of Lebesgue measurable subsets of the line. Are the limits supremum and infimum of the sequence $\{E_n\}$ measurable sets (see Problem 81)?

104. Let A_n be a sequence of measurable sets with $\sum \mu(A_n) < \infty$. Prove that $\mu(\overline{\lim}\, A_n) = 0$.

105°. Prove that the Borel sets are Lebesgue measurable (see Problem 87).

106. Prove that every Lebesgue measurable set on the line is the union of a Borel set and a set of measure zero.

107. Suppose that X is the unit square in the plane and S is the half-ring of rectangles in X of the form

$$T_{ab} = \{a \le x < b, 0 \le y \le 1\}.$$

Define $m(T_{ab}) = b - a$. Describe the explicit form of the Lebesgue extension of the measure m.

108. Under the conditions and with the notation of Problem 107 prove that the set $\tilde{T} = \{0 \le x \le 1, y = 1/2\}$ is not measurable, and find its outer measure.

109*. Find the Lebesgue measure of the subset of $[0, 1]$ consisting of the numbers in whose decimal expressions the digit 2 is encountered before the digit 3.

110. Find the Lebesgue measure of the subset of the unit square in the plane consisting of the points (x, y) such that $|\sin x| < 1/2$ and $\cos(x + y)$ is irrational.

111. Find the Lebesgue measure of the subset of the unit square in the plane consisting of the points whose cartesian and polar coordinates are irrational.

112*. In the unit square in the plane take the system (not a half-ring) of vertical and horizontal rectangles with length or width equal to unity, and assign to each rectangle the measure equal to its area. Find at least two different extensions of the measure to the algebra generated by this system of rectangles.

113*. Suppose that a measure μ is defined on a half-ring X with unit, and let μ^* be its outer measure. A set $A \subset X$ is said to be *measurable in the Carathéodory sense* if for any subset $Z \subset X$

$$\mu^*(Z) = \mu^*(Z \cap A) + \mu^*(Z \setminus A).$$

Prove that A is Lebesgue measurable if and only it is measurable in the Carathéodory sense.

114*. Let μ be a σ-additive measure defined on a half-ring. A set A is called a *set of σ-uniqueness* for μ if:

(1) there exists a σ-additive extension λ of μ that is defined on A;
(2) for every two such σ-additive extensions λ_1 and λ_2,

$$\lambda_1(A) = \lambda_2(A).$$

(a) Prove that each set A measurable in the Lebesgue sense is a set of σ-uniqueness for the original measure μ.

(b) Prove that the collection of sets measurable in the Lebesgue sense exhausts the collection of sets of σ-uniqueness for the original measure μ.

115*. Suppose that each of the sets X_n, $n = 1, 2, 3, \ldots$, consists of the digits $0, 1, 2, \ldots, 9$. Define a measure μ_n on X_n by setting $\mu_n(Y) = (1/10)\operatorname{card} Y$. Let the measure μ on $X = \prod X_n$ be the product of the measures μ_n. Prove that μ passes into the usual Lebesgue measure on $[0, 1]$ under the mapping $\{x_n\} \mapsto 0.x_1x_2x_3\ldots$ (infinite decimal fraction) of the set X into $[0, 1]$.

3. Constructions of Measures

116*. Construct a subset of the line that is not Lebesgue measurable.

117*. Construct a subset of the plane that is not Lebesgue measurable.

118*. Construct a Lebesgue measurable set in the plane whose projections on the coordinate axes are not measurable.

119.** Let μ be Lebesgue measure, and X a measurable subset of $[0, 1]$. A point $x \in X$ is called a *point of density of the set* X if

$$\lim_{\varepsilon \to 0} \frac{\mu\{X \cap (x - \varepsilon, x + \varepsilon)\}}{2\varepsilon} = 1.$$

Prove that almost all points of X are points of density.

120. Describe all the subsets E of $[0, 1]$ whose characteristic functions $\chi_E(x)$ are Riemann integrable.

121°. Let X be a space with a σ-additive measure. Prove that the subsets of zero measure in X form a σ-ring.

122°. Prove that the countable sets on the line have Lebesgue measure zero. Give an example of an uncountable subset of the line having Lebesgue measure zero.

123°. Prove that the set of all signed measures on a σ-algebra $\mathfrak{A}$ is a linear space that is complete with respect to the distance

$$d(\nu_1, \nu_2) = \sup_{A \in \mathfrak{A}} |\nu_1(A) - \nu_2(A)|.$$

124. For any subset M of $\mathbf{R}^n$ let $M - M$ denote the set

$$M - M = \{x - y : x \in M, y \in M\}.$$

Prove that if M is measurable and has positive Lebesgue measure, then $M - M$ contains a neighborhood of zero in $\mathbf{R}^n$.

125°. Let $X = \{x_1, x_2, \ldots, x_n, \ldots\}$ be a countable set, and assign to each element x_i a number $p_i \geq 0$ in such a way that $\sum_{n=1}^{\infty} p_n = 1$. For any subset $A \subset X$ let $m(A) = \sum_{n \in N_A} p_n$, where $N_A = \{i : x_i \in A\}$.

Prove that m is a σ-additive measure on the algebra of all subsets of X.

126*. Give an example of a finitely additive measure that is not σ-additive.

127.** Prove that the Wiener measure is countably additive.

128*. Compute the Wiener measure of the set of functions $f \in C[a, b]$ having the properties: $f(a) < 0, f(b) > 0$.

129. Define a measure μ on $[0, 1]$ by the formula

$$\mu([\alpha, \beta)) = \log_2 \frac{1 + \beta}{1 + \alpha}.$$

Prove that this measure is preserved under the transformation $x \mapsto \{1/x\}$, where $\{\cdot\}$ denotes the fractional part of a number.

130. Each real number $x \in [0, 1]$ can be expanded in a continued fraction

$$x = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cdots}}$$

(the finite fractions correspond to the rational numbers, and the infinite ones to the irrational numbers).

(a) Prove that in terms of the sequence $\{n_k\}$ the transformation in Problem 129 has the form $\{n_k\} \mapsto \{n_{k+1}\}$.

(b) Compute the measure in the space of sequences that corresponds to the measure μ in Problem 129.

131°. Prove that the condition of absolute convergence of the series $\sum \nu(A_n)$ in the definition of a signed measure can be replaced by the condition of simple convergence.

132. Compute the variation of the complex measure $\nu = \mu_1 + i\mu_2$ on a set A if:

(a) it is known that the measures μ_1 and μ_2 are disjoint on A;

(b) $\mu_1 = \mu_2$ on A.

133. Let ν be a complex measure on a σ-algebra $\mathfrak{A} \subset P(X)$. Prove that the real and imaginary parts of ν are signed measures on $\mathfrak{A}$.

134*. Let ν be a signed measure on a σ-algebra $\mathfrak{A} \subset P(X)$. Prove that $\sup_{A \in \mathfrak{A}} \nu(A) < +\infty$, $\inf_{A \in \mathfrak{A}} \nu(A) > -\infty$.

135*. Prove that supremum and infimum in Problem 134 are attained on some sets A_+ and A_- in $\mathfrak{A}$.

136. In the notation of Problem 134 prove that the function ν (resp., $-\nu$) is a σ-additive measure on $\mathfrak{A} \cap P(A_+)$ (resp., on $\mathfrak{A} \cap P(A_-)$).

137. In the notation of Problem 134 prove that $\nu(A) = \nu(A \cap A_+) + \nu(A \cap A_-)$ for every $A \in \mathfrak{A}$.

138. Prove that the variation $|\nu|$ of a signed measure ν is finite and σ-additive.

§2. Measurable Functions

1. Properties of Measurable Functions

139°. Let X be a measure space and f a real-valued function defined on X. Prove that the following properties of f are equivalent:

(a) for any $a \in \mathbf{R}$ the set $\{x \in X: f(x) > a\}$ is measurable;

(b) for any $a \in \mathbf{R}$ the set $\{x \in X: f(x) \geq a\}$ is measurable;

(c) for any $a \in \mathbf{R}$ the set $\{x \in X: f(x) < a\}$ is measurable;

(d) for any $a \in \mathbf{R}$ the set $\{x \in X: f(x) \leq a\}$ is measurable.

140. Prove that under the conditions of Problem 139 any of the conditions (a)–(d) is equivalent to the condition

(e) $f^{-1}(B)$ is a measurable set for any Borel set $B \subset \mathbf{R}$.

141°. Suppose that f is measurable and does not vanish. Prove that $1/f$ is a measurable function.

142°. Prove that $|f|$ is a measurable function if f is.

143. Let $f(t_1, t_2, \ldots, t_n)$ be a continuous real-valued function defined on a real n-dimensional space, and $g_1(x), \ldots, g_n(x)$ measurable functions. Prove that $h(x) = f(g_1(x) \ldots g_n(x))$ is a measurable function.

144*. Suppose that $g(x)$ is a measurable function defined on the real line, and f is a continuous real function. Show that $h(x) = g(f(x))$ need not be a measurable function.

145. Let $f(x)$ be a real function. Describe the numbers n for which the measurability of $[f(x)]^n$ implies that of $f(x)$.

146. Let $f(x)$ be a function differentiable everywhere on $[0, 1]$. Prove that $f'(x)$ is Lebesgue measurable.

147. Let $f(x)$ be the Cantor one-to-one mapping of $[0, 1]$ onto the square: $f(x) = (y_1, y_2)$ with $y_1 = 0.x_1x_3\ldots$ and $y_2 = 0.x_2x_4\ldots$ for $x = 0.x_1x_2x_3\ldots$. Prove that $f(x)$ carries any measurable subset of the interval into a measurable subset of the square and preserves the value of the measure.

148. A function $f(x)$ defined on the real line is called a *Borel function* if $\{x \in R: f(x) < a\}$ is a Borel set for any $a \in \mathbf{R}$ (see Problem 106). Prove that any measurable function becomes a Borel function after suitable modification on a set of measure zero.

149*. *The Luzin C-property.* Let μ be Lebesgue measure on $[0, 1]$, and f a measurable function that is finite almost everywhere on this segment. Prove that for any $\varepsilon > 0$ there is a closed set $F \subset [0, 1]$ such that the restriction of f to F is continuous and $\mu(F) > 1 - \varepsilon$.

150*. Let f be a measurable function defined on the real line. A point $x \in \mathbf{R}$ is called a *Lebesgue point* of f if there is a Lebesgue measurable subset $X \subset \mathbf{R}$ that contains x, has x as a point of density, and is such that the restriction $f|_X$ is continuous at x. Prove that almost every point of the line is a Lebesgue point for f.

151. Let $x = 0.n_1n_2n_3\ldots$ and $y = 0.m_1m_2m_3\ldots$ be the decimal expressions for numbers x and y belonging to $[0, 1]$. Set $f(x, y) = k$ if $n_k = m_k$ and $n_i \neq m_i$ for $i < k$; set $f(x, y) = \infty$ if $n_k \neq m_k$ for all k. Prove that f is Lebesgue measurable and almost everywhere finite.

152. Let $f(x)$ be a continuous function defined on $[a, b]$, and $n(c)$ the number of solutions of the equation $f(x) = c$. Prove that $n(c)$ is Lebesgue measurable.

153°. Let $f_n(x)$ be a sequence of measurable functions. Prove that the functions $\sup_n f_n(x)$ and $\inf_n f_n(x)$ are measurable.

154. In the notation of Problem 153 prove that the functions $\overline{\lim}_{n\to\infty} f_n(x)$ and $\underline{\lim}_{n\to\infty} f_n(x)$ are measurable.

155. Let $\{f_n\}$ be a sequence of measurable functions. Prove that the set of all points x where $\lim_{n\to\infty} f_n(x)$ exists is measurable.

156°. Let f be a measurable function. Prove that its positive part $f^+ = \max(f, 0)$ and its negative part $f^- = -\min(f, 0)$ are measurable functions.

157. Real functions f and g that are measurable with respect to the respective measures μ and ν are said to be *equimeasurable* if $\mu\{x: f(x) < c\} = \nu\{y: g(y) < c\}$ for any $c > 0$.

Prove that if f is measurable with respect to a measure μ, then there exists a left-continuous nondecreasing function g on the segment $[0, \mu(X)]$ that is equimeasurable with f.

158. Under the conditions of Problem 157 prove that $g(x)$ is unique.

159. A complex-valued function $f(x) = u(x) + iv(x)$ is said to be measurable if its real part $u(x)$ and imaginary part $v(x)$ are measurable. Prove that the modulus and argument of $f(x)$ are measurable.

160. Prove that a complex-valued function $f(x)$ is measurable if and only if all the sets of the form $A_{r,z} = \{x: |f(x) - z| \leq r\}$, are measurable, where $z \in \mathbf{C}, r \geq 0$.

161. A vector-valued function f with values in a finite-dimensional space V is said to be *measurable* if the coordinates of $f(x)$ with respect to some basis in V are measurable. Prove that this definition does not depend on the choice of the basis.

2. Convergence of Measurable Functions

162°. Prove that the sequence $f_n(x) = nx/(n^2 + x^2)$ converges to zero everywhere on $\mathbf{R}$, but not uniformly.

163°. Investigate the convergence and uniform convergence of the sequence $f_n(x) = x^n$ on $[0, 1]$.

164°. Prove that two continuous functions on a segment are equivalent with respect to Lebesgue measure only if they are identically equal.

165. Construct on a closed interval a Lebesgue-measurable function that is not equivalent to any continuous function.

166. Show that if $f_n \xrightarrow{\text{a.e.}} f$ and $f_n \xrightarrow{\text{a.e.}} g$, then f is equivalent to g.

167°. Let $f_n(x) = n \sin x/(1 + n^2 \sin^2 x)$ on $[0, \pi]$. For a given $\delta > 0$ find explicitly the *Egorov set* E_δ on which the sequence f_n converges uniformly.

168*. Let the set of rational numbers in $[0, 1]$ be enumerated, with the kth number r_k in the form of an irreducible fraction $r_k = p_k/q_k$. Define $f_k(x) = \exp\{-(p_k - xq_k)^2\}$. Prove that $f_k \to 0$ in Lebesgue measure on $[0, 1]$, and that $\lim_{n\to\infty} f_k(x)$ does not exist at any point of the segment.

169. Under the conditions of the preceding problem determine explicitly a subsequence converging to zero almost everywhere.

170. Define functions $f_i^{(k)}$ on $[0, 1]$ by setting

$$f_i^{(k)}(x) = \begin{cases} 1 & \text{for } (i-1)/k \leq x < i/k, \quad i = 1, 2, \ldots, k, k = 1, 2, \ldots, \\ 0 & \text{otherwise,} \end{cases}$$

and let $g_n(x) = f_i^{(k)}(x)$, where i and k are determined by the condition $n = k(k - 1)/2 + i$. Prove that $g_n \to 0$ in measure, but $\lim g_n(x)$ does not exist at any point.

171. Suppose that $f_n \xrightarrow{\mu} h$ and $f_n \xrightarrow{\mu} g$. Prove that h and g are equivalent with respect to the measure μ.

172*. *Luzin's theorem.* Prove that a real function on $[a, b]$ is Lebesgue-measurable if and only if for any $\varepsilon > 0$ there is a continuous function that differs from f on a set of measure $\leq \varepsilon$.

173*. It follows from Luzin's theorem (Problem 172) that every measurable function f on $[a, b]$ is a limit almost everywhere of a sequence $\{f_n\}$ of continuous functions. Is it always possible to choose this sequence to be monotone?

174*. The *Dirichlet function*

$$\psi(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ 1 & \text{if } x \text{ is rational} \end{cases}$$

can be obtained from continuous functions by the iterated limit

$$\psi(x) = \lim_{n \to \infty} \lim_{m \to \infty} [\cos(2\pi n!x)]^m.$$

Is it possible to obtain it from continuous functions by a single limit?

175. Prove that a simple function (i.e., a function taking not more than countably many values) is measurable if and only if all its *level sets*

$$L_c(f) = \{x \in X : f(x) = c\}$$

are measurable.

Is this true for arbitrary functions?

176. Prove that every measurable function can be represented as a uniform limit of measurable simple functions.

177. Define the function $f(x)$ on $[0, 1]$ as follows. If $x = 0.n_1 n_2 n_3 \ldots$ is the decimal expansion of x, then $f(x) = \max_i n_i$. Prove that $f(x)$ is measurable and almost everywhere constant.

178. Under the conditions of Problem 177 show that the function $f(x) = \overline{\lim}_{i \to \infty} n_i$ is defined everywhere and constant almost everywhere.

179. Let μ be Wiener measure on $X = C[a, b]$. Define a function f on X by setting $f(x) = \int_a^b x(t)\,dt$. Prove that f is μ-measurable.

180. Do the same for the function $f(x) = \int_a^b \varphi(x(t), t)\,dt$, where $\varphi(x, y)$ is a continuous function of two variables.

181. Do the same for the function $f(x) = \max_{t \in [a, b]} x(t)$.

182*. Let X be the set of p-adic integers (see Problem 42), S the algebra of subsets of X that are simultaneously open and closed in X, and $\mathfrak{A} = R_\sigma(S)$. Prove that every continuous function on X is $\mathfrak{A}$-measurable.

183*. Under the conditions of Problem 182 prove that every set $A \in S$ is a union of finitely many balls. Define a measure μ on S by setting the measure of a ball equal to its radius (for balls of radius p^{-k}, $k = 0, 1, 2, \ldots$).

Prove that the measure μ is countably additive and that every $\mathfrak{A}$-measurable function is μ-equivalent to some continuous function.

184*. Prove that the measure μ in Problem 183 has the properties:

(a) $\mu(X) = 1$;

(b) $\mu(A + x) = \mu(A)$ for all $x \in X$.

Prove that every measure on $\mathfrak{A}$ having the properties (a) and (b) coincides with μ.

§3. Integrals

1. The Lebesgue Integral

185°. Prove that if f and g are integrable simple functions, then

(a) $\int_A (f(x) + g(x))\, d\mu = \int_A f(x)\, d\mu + \int_A g(x)\, d\mu$;

(b) $\int_A \alpha f(x)\, d\mu = \alpha \int_A f(x)\, d\mu$ ($\alpha = \text{const}$);

(c) if $|f(x)| \leq M$ almost everywhere on A and $\mu(A) < \infty$, then $|\int_A f(x)\, d\mu| \leq M\mu(A)$.

186°. Compute the Lebesgue integral over the interval $(0, \infty)$ of the functions:

(a) $f(x) = e^{-[x]}$,

(b) $f(x) = 1/([x + 1][x + 2])$,

(c) $f(x) = 1/[x]!$, where $[x]$ denotes the integral part of the number x.

187. Suppose that $\mu(X) < \infty$ and that f is an integrable function on X. Prove that the Lebesgue integral $\int_X f(x)\, d\mu$ can be computed by the formula

$$\int_X f(x)\, d\mu = \lim_{\lambda(T) \to 0} \sum_k \xi_k \mu(\{x \in X : t_k \leq f \leq t_{k+1}\}), \tag{1}$$

where $T = \{t_k\}$ is a partition of the real axis, $\lambda(T) = \sup_k |t_k - t_{k+1}|$ is the diameter of the partition T, and $\{\xi_k\}$ is any set of points satisfying the condition $\xi_k \in [t_k, t_{k+1}]$. The expression in (1) is called a *Lebesgue sum.*

188. Prove that the statement of Problem 187 remains true in the case $\mu(X) = \infty$ if it is required in addition that $\xi_k = 0$ for those k such that $[t_k, t_{k+1}]$ contains the point 0.

189. Suppose that the measurable simple function f is represented in two ways as a linear combination of characteristic functions of disjoint sets:

$$f(x) = \sum_k c_k \chi_{A_k}(x) = \sum_l d_l \chi_{B_l}(x).$$

Prove that $\sum_k c_k \mu(A_k) = \sum_l d_l \mu(B_l)$ in the case when one of these series converges absolutely.

190°. Let f_n be the simple function on $[0, 1]$ defined by the formula $f_n(x) = (1/n)[nx]$, where $[x]$ denotes the integral part of the number x. Prove that the sequence $\{f_n\}$ is Cauchy, but does not have a limit in the space $S[0, 1]$ of simple integrable functions with the distance $d_1(f, g) = \int_0^1 |f - g|\, dx$.

191. For what values of the parameters α and β is the function $f(x) = x^\alpha \sin x^\beta$ on $(0, 1]$

(a) Lebesgue integrable?

(b) improperly Riemann integrable?

192°. Prove that the integral of a nonnegative integrable function f over a set A is:

(a) nonnegative;

(b) equal to zero only if $f(x) = 0$ almost everywhere on A.

193. Let φ be a monotonically increasing smooth function on $[a, b]$, and ψ the inverse of this function on $[\varphi(a), \varphi(b)]$. Regarding the integral as a limit of Lebesgue sums, prove the identity

$$\int_a^b \varphi(x)\, dx = \int_{\varphi(a)}^{\varphi(b)} y\psi'(y)\, dy.$$

194. Prove that the Lebesgue integral of a nonnegative function $f(x)$ over $[a, b]$ coincides with the Lebesgue measure of the planar set determined by the inequalities $a \le x \le b$, $0 \le y \le f(x)$.

195. Prove that a nonnegative measurable function f is integrable on A if and only if, for all the simple functions g not exceeding f, the integrals $\int_A g(x)\, d\mu(x)$ are bounded by a common constant.

196. For any real function f let $f_+(x) = (f(x) + |f(x)|)/2$, $f_-(x) = (|f(x)| - f(x))/2$. Prove that f is integrable if and only if f_+ and f_- are integrable.

197. Prove that a measurable nonnegative function f is integrable if and only if $\sup \int_A f(x)\, d\mu(x) < \infty$, where the supremum runs over all sets A of finite measure on which f is bounded above.

198. Let $\mu(X) < \infty$. Prove that a nonnegative measurable function f on X is integrable if and only if the series

$$\sum_{n=0}^{\infty} 2^n \mu\{x \in X : f(x) \ge 2^n\}$$

converges.

199. Prove that a nonnegative bounded function on a set X of infinite measure is integrable if and only if the series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \mu\left(\left\{x \in X : f(x) > \frac{1}{2^n}\right\}\right)$$

converges.

200°. Compute the Lebesgue integral over $[0, \pi/2]$ of the following functions $f(x)$:

(a) $f(x) = \sin x$;

(b) $f(x) = \begin{cases} \sin x & \text{if } x \text{ is rational,} \\ \cos x & \text{if } x \text{ is irrational;} \end{cases}$

(c) $f(x) = \begin{cases} \sin x & \text{if } \cos x \text{ is rational,} \\ \sin^2 x & \text{if } \cos x \text{ is irrational.} \end{cases}$

Compute the Lebesgue integral of the function

(d) $h(x, y) = \begin{cases} 1 & \text{if } xy \text{ is irrational,} \\ 0 & \text{if } xy \text{ is rational} \end{cases}$

over the square $0 \leq x \leq 1, 0 \leq y \leq 1$.

201*. Prove that a bounded function is Riemann integrable on $[a, b]$ if and only if it is almost everywhere continuous.

202*. Prove that the Lebesgue integral of the function

$$f(x_1, \ldots, x_n) = \exp\{-\sum a_{ij} x_i x_j\}$$

is finite if and only if the symmetric matrix $A = \|a_{ij}\|$ is positive-definite. Prove that in this case the integral is equal to $\det(\pi \cdot A^{-1})$.

203*. Compute the integral with respect to Wiener measure on $C[0, 1]$ of the function

$$F(x) = \exp\left\{-ax^2(0) - b^2 \int_0^1 x^2(t)\, dt\right\}.$$

204.** Let $C_0[0, 1]$ be the space of continuous functions $x(t)$ on $[0, 1]$ satisfying the additional condition $x(0) = 0$. Prove that $C[0, 1]$ can be identified with the product $\mathbf{R} \times C_0[0, 1]$, and that the Wiener measure μ passes into $\mu_1 \times \mu_0$, where μ_1 is the usual Lebesgue measure on $\mathbf{R}$ and μ_0 is a certain measure on $C_0[0, 1]$.

205.** Let μ_0 be the measure constructed in Problem 204. Compute the integrals:

(a) $\int_{C_0[0, 1]} d\mu_0(x)$;

(b) $\int_{C_0[0, 1]} [\int_0^1 x(t)\, dt]\, d\mu_0(x)$;

(c) $\int_{C_0[0, 1]} [\int_0^1 x^2(t)\, dt]\, d\mu_0(x)$.

206. Suppose that f is a bounded measurable function on the set X, and that there exist constants $A > 0$ and $\alpha < 1$ such that $\mu\{x \in X : |f(x)| > \varepsilon\} < A/\varepsilon^\alpha$ for $\varepsilon > 0$. Prove that f is integrable with respect to μ.

207. (a) Prove that for almost all real numbers x the continued fraction expansion in Problem 130 leads to an unbounded sequence $\{n_k\}$.

(b) Let $\{a_k\}$ be a sequence of positive numbers. Consider the set $M(\{a_k\})$ of real numbers x in $[0, 1]$ whose continued fraction expansions have the property that $n_k \leq a_k$ for all k. Under what condition on $\{a_k\}$ is the measure of the set $M(\{a_k\})$ equal to zero?

2. Functions of Bounded Variation and the Lebesgue–Stieltjes Integral

208°. Establish the following properties of the total variation:

(a) for any constant α and function f of bounded variation $\mathrm{Var}_a^b(\alpha f) = |\alpha| \mathrm{Var}_a^b(f)$;

(b) if f and g are functions of bounded variation, then so is $f + g$, and

$$\mathrm{Var}_a^b(f + g) \leq \mathrm{Var}_a^b(f) + \mathrm{Var}_a^b(g);$$

(c) if $a < b < c$ and f is a function of bounded variation on $[a, c]$, then

$$\mathrm{Var}_a^b(f) + \mathrm{Var}_b^c(f) = \mathrm{Var}_a^c(f);$$

(d) if f is a monotone function, then

$$\mathrm{Var}_a^b(f) = |f(a) - f(b)|.$$

209. Check whether the following functions on $[0, 1]$ are of bounded variation:

(a) $f(x) = \begin{cases} x^2 \sin 1/x & \text{for } 0 < x \leq 1, \\ 0 & \text{for } x = 0; \end{cases}$

(b) $f(x) = \begin{cases} x \sin 1/x & \text{for } 0 < x \leq 1, \\ 0 & \text{for } x = 0. \end{cases}$

210. Prove that the set of points of discontinuity of a function of bounded variation on a segment is not more than countable and consists only of points of discontinuity of the first kind.

211. Prove that a function of bounded variation on a closed interval is Lebesgue measurable.

212°. Prove that a function with a bounded derivative on a closed interval is of bounded variation.

213*. Suppose that f has a Riemann integrable derivative on $[a, b]$. Prove the formula

$$\mathrm{Var}_a^b(f) = \int_a^b |f'(x)|\, dx.$$

214°. Find the following variations:

$$\mathrm{Var}_0^{50}(e^x), \qquad \mathrm{Var}_1^2(\ln x), \qquad \mathrm{Var}_0^{4\pi}(\cos x), \qquad \mathrm{Var}_{-1}^1(x - x^3).$$

215. Let Φ be a left-continuous function of bounded variation on $[a, b]$. Prove that Φ has a unique representation as a sum $\Phi = \Phi_0 + \Phi_1$, where Φ_0 is a continuous function of bounded variation, and Φ_1 is a so-called *jump function*: $\Phi_1(x) = \sum_{k=1} c_k \theta(x - a_k)$, where $\{a_k\}$ is a finite or countable subset of $[a, b]$, $\theta(x)$ is the *Heaviside function*, defined by

$$\theta(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ 1 & \text{for } x > 0, \end{cases}$$

and $\{c_k\}$ is a numerical sequence satisfying the condition $\sum_k |c_k| < \infty$.

216. Prove that:

(a) the product of two functions of bounded variation is a function of bounded variation;

(b) if $f(x) \geq \alpha > 0$ and f is a function of bounded variation, then $1/f$ is also a function of bounded variation.

217. Determine whether the function $\varphi(f)$ has bounded variation on $[0, 1]$ if f has bounded variation on $[0, 1]$ and φ

(a) is continuous on the whole real line;

(b) has bounded variation on the whole real line.

218°. Let E be a subset of $[0, 1]$, and χ the characteristic function of E. Prove that χ has bounded variation if and only if the boundary of E is a finite set.

219*. Let f and g be two continuous functions with bounded variation on $[a, b]$. Prove that the set $(f(x), g(x))$, $x \in [a, b]$, cannot fill the square. Is this true without the requirement of bounded variation?

220°. Prove the following properties of the Riemann–Stieltjes integral:

(a) if Φ is a function of bounded variation and f is integrable with respect to Φ, then

$$\left| \int_a^b f(x)\, d\Phi(x) \right| \leq \sup |f(x)| \operatorname{Var}_a^b(\Phi);$$

(b) if Φ_1 and Φ_2 are functions of bounded variation and f is integrable with respect to Φ_1 and Φ_2, then it is also integrable with respect to $\Phi = \Phi_1 + \Phi_2$, and

$$\int_a^b f(x)\, d\Phi(x) = \int_a^b f(x)\, d\Phi_1(x) + \int_a^b f(x)\, d\Phi_2(x).$$

221. Suppose that the function Φ has bounded variation on $[a, b]$ and is discontinuous at $c \in (a, b)$, while the function f is Riemann–Stieltjes integrable with respect to Φ. Prove that f is continuous at the point c.

222. Prove that if Φ is a function of bounded variation on $[a, b]$ that differs from zero only at a finite or countable number of points lying in (a, b), then for any function f continuous on $[a, b]$,

$$\int_a^b f(x)\, d\Phi(x) = 0.$$

223. Prove that if f is continuous, then the Riemann–Stieltjes integral $\int_a^b f(x)\, d\Phi(x)$ does not depend on the values taken by Φ at points of discontinuity lying in (a, b).

224. Prove the formula for integration by parts for the Stieltjes integral:

$$\int_a^b f(x)\, dg(x) = f(x)g(x)\Big|_a^b - \int_a^b g(x)\, df(x).$$

225. Suppose that $f(x)$ is continuous on $[a, b]$ and that $g(x)$ has a Riemann integrable derivative $g'(x)$ defined everywhere except at finitely many points $c_1, \ldots, c_k$. Prove that under these conditions the Riemann–Stieltjes integral $\int_a^b f\,dg$ exists and that it can be expressed by the formula

$$\int_a^b f\,dg = \int_a^b fg'\,dx + f(a)[g(a+0) - g(a)] + f(b)[g(b) - g(b-0)]$$
$$+ \sum_{m=1}^{k} f(c_m)[g(c_m + 0) - g(c_m - 0)].$$

226°. Let μ_φ be the measure generated by a monotone continuous function φ. Prove that the Lebesgue integral $\int_{[a,b]} x\,d\mu_\varphi$ is equal to the Stieltjes integral $\int_a^b x\,d\varphi(x)$, and compute it.

227°. Compute the Riemann–Stieltjes integrals

$$I_1 = \int_{-1}^{3} x\,dg(x); \qquad g(x) = \begin{cases} 0, & x = -1, \\ 1, & x \in (-1, 2), \\ -1, & x \in [2, 3], \end{cases}$$

$$I_2 = \int_0^2 x^2\,dg(x), \qquad g(x) = \begin{cases} -1, & x \in [0, 1/2), \\ 0, & x \in [1/2, 3/2), \\ 2, & x = 3/2, \\ -2, & x \in (3/2, 2]. \end{cases}$$

228°. Compute the integrals

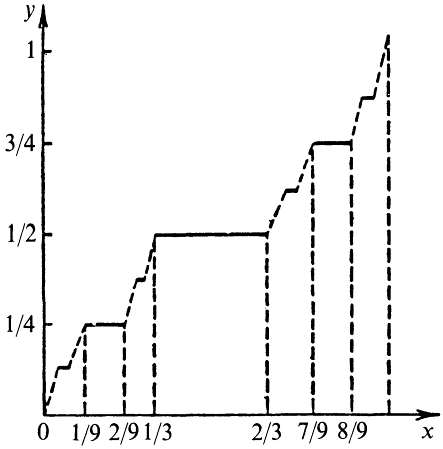
$$I_1 = \int_{-2}^{2} x\,dg(x), \qquad I_2 = 2\int_{-2}^{2} x^2\,dg(x), \qquad I_3 = \int_{-2}^{2} (x^3 + 1)\,dg(x),$$

where

$$g(x) = \begin{cases} x + 2, & x \in [-2, -1], \\ 2, & x \in (-1, 0), \\ x^2 + 3, & x \in [0, 2]. \end{cases}$$

229°. Let $f(x)$ be a continuous function on $[0, 1]$. The *Banach indicatrix* $N_f(y)$ of f is defined to be the number of roots of the equation $f(x) = y$ (if it is infinite, then set $N_f(y) = \infty$). Prove that $N_f(y)$ is a Lebesgue measurable function of y (see Problem 152) and that $\int_{-\infty}^{\infty} N_t(y)\,dy = \mathrm{Var}_0^1(f)$, if at least one side of this equation is finite.

230*. Let $\varphi(x)$ be the *Cantor staircase*, i.e., the continuous monotone function on $[0, 1]$ constant on each interval in the complement of the Cantor set and taking the values $1/2^k, 3/2^k, 5/2^k, \ldots, (2^k - 1)/2^k$ on the intervals of rank k.



Compute the integrals:
(a) $\int_0^1 x^k \, d\varphi(x)$;
(b) $\int_0^1 e^x \, d\varphi(x)$;
(c) $\int_0^1 \sin \pi x \, d\varphi(x)$.

3. Properties of the Lebesgue Integral

231°. (a) Prove that convergence in $L_1(X, \mu)$ implies convergence in measure.

(b) Is the converse true?

232°. Suppose that the sequence $f_n \in L_1(X, \mu)$ converges uniformly to the function $f(x)$. Prove that $f_n \to f$ in $L_1(X, \mu)$ if $\mu(X) < \infty$. Is this true in the case $\mu(X) = \infty$?

233. Construct a sequence of functions $f_n \in L_1[0, 1]$ having the properties:
(a) $f_n(x) \to 0$ for all $x \in [0, 1]$;
(b) $\int_0^1 |f_n(x)| \, dx \leq C$ for all n;
(c) the sequence $\{f_n\}$ does not have a limit in $L_1[0, 1]$.

234°. Let X be a set of finite μ measure. For any measurable functions f and g let

$$\rho(f, g) = \int_X \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} \, d\mu(x).$$

Prove that ρ has all the properties of a distance except the separation property, and that the corresponding metric space $M[0, 1]$ consists of classes of equivalent functions.

235. Prove that convergence in $M[0, 1]$ (see Problem 234) coincides with convergence in measure, and that the space $M[0, 1]$ is complete in the metric $\rho(f, g)$.

236. Prove the set of characteristic functions of measurable subsets is closed in $L_1(X, \mu)$.

237. Let $\{f_n\}$ be a sequence of nonnegative integrable functions converging almost everywhere to an integrable function f. Prove that if $\int_X f_n \, d\mu \to \int_X f \, d\mu$ as $n \to \infty$, then $f_n \to f$ in the space $L_1(X, \mu)$.

238*. Suppose that $f \in L_1(X, \mu)$ and $\mu(X) = 1$. Prove that there is a monotone function $g(t) \in L_1[0, 1]$ such that for any $t \in [0, 1]$,

$$\inf_{\mu(A)=t} \int_A f(x)\, d\mu(x) = \int_0^t g(\tau)\, d\tau, \qquad \sup_{\mu(A)=t} \int_A f(x)\, d\mu(x) = \int_{1-t}^1 g(\tau)\, d\tau.$$

239. Prove that the iterated integrals corresponding to the double integral

$$\int_0^\infty \int_0^\infty e^{-xy} \sin x \sin y \, dx\, dy$$

both exist and their values coincide. Does the double integral exist?

240. Prove that the iterated integrals corresponding to the double integral

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2}\, dx\, dy$$

both exist, but their values do not coincide.

241. Prove that the iterated integrals corresponding to the double integral

$$\int_{-1}^1 \int_{-1}^1 \frac{xy}{(x^2 + y^2)^2}\, dx\, dy$$

both exist and their values coincide, but the double integral does not exist.

242. Let

$$f(x, y) = \begin{cases} 2^{2n} & \text{for } 1/2^n \le x \le 1/2^{n-1},\ 1/2^n \le y < 1/2^{n-1}, \\ -2^{2n+1} & \text{for } 1/2^{n+1} \le x \le 1/2^n,\ 1/2^n \le y < 1/2^{n-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that

$$\int_0^1 \left(\int_0^1 f(x, y)\, dy \right) dx \neq \int_0^1 \left(\int_0^1 f(x, y)\, dx \right) dy.$$

243.** Let μ be a nonzero Borel measure on the set of real numbers that has the following property: for any $t \in \mathbf{R}$ the measure μ_t defined by the formula $\mu_t(A) = \mu(A + t)$ is equivalent to μ. (Such measures are said to be *quasi-invariant under translations.*) Prove that μ is equivalent to Lebesgue measure.

244*. Let μ be a measure on X, and f_1, f_2 two μ-integrable real functions on X. Define signed measures $\nu_i = f_i \mu$ by the formula $\nu_i(A) = \int_A f_i \, d\mu$, $i = 1, 2$. Prove that ν_1 and ν_2 are equivalent if and only if $\mu(N_1 \triangle N_2) = 0$, where $N_i = \{x \in X;\ f_i(x) \neq 0\}$.

245*. Let μ be a σ-finite measure on X, and ν a measure defined on the same σ-algebra and absolutely continuous with respect to μ (i.e., $\mu(A) = 0 \Rightarrow \nu(A) = 0$)). Prove that there exists a nonnegative μ-measurable function ρ having the property that $\nu(A) = \int_A \rho(x)\,d\mu(x)$ for any measurable set A (both sides of the equation may simultaneously take the value $+\infty$).

246*. Prove that there does not exist a Lebesgue-measurable subset A of the real line $\mathbf{R}$ such that for any interval Δ

$$\mu(A \cap \Delta) = \tfrac{1}{2}\mu(\Delta).$$

247*. Let $f \in L_1(a, b]$. Prove that the function $F(x) = \int_a^x f(x)\,d\mu(x)$ is differentiable almost everywhere and $F'(x) = f(x)$ for almost all $x \in [a, b]$.

248.** A real function F on $[a, b]$ is said to be *absolutely continuous* if for any $\varepsilon > 0$ there is a $\delta > 0$ such that $\sum_{i=1}^n |F(a_i) - F(b_i)| < \varepsilon$ for any family of intervals $\{\Delta_i\}$, $\Delta_i = (a_i, b_i)$, $1 \le i \le n$, with sum of lengths $< \delta$. Prove that:

(a) an absolutely continuous function F is differentiable almost everywhere;

(b) the derivative $f(x) = F'(x)$ is integrable on $[a, b]$;

(c) the Newton–Leibnitz formula holds,

$$F(b) - F(a) = \int_a^b f(x)\,d\mu(x).$$

249. Prove that the following sets are dense in $L_1[0, 1]$:

(a) the set $S(0, 1)$ of piecewise constant functions with finitely many discontinuities;

(b) the set of continuous piecewise linear functions with finitely many bends;

(c) the set of polynomials $P(x) = \sum_{k=0}^N a_k x^k$;

(d) the set of trigonometric polynomials

$$T(x) = \sum_{k=-N}^{N} c_k e^{2\pi i k x}.$$

250. Prove that the following sets are dense in $L_1(\mathbf{R})$:

(a) the piecewise constant functions of compact support;

(b) the continuous compactly supported functions;

(c)* the functions of the form $P(x)e^{-x^2}$, where P is a polynomial.

251. Let $f \in L_1(\mathbf{R})$. Prove that $\int_{\mathbf{R}} |f(x + e) - f(x)|\,dx \to 0$ as $\varepsilon \to 0$. In other words, translation is a continuous operation in $L_1(\mathbf{R})$.

252. The *convolution* of functions f_1 and f_2 on the line is defined to be the function f given by the formula

$$f(x) = \int_{-\infty}^{\infty} f_1(t) f_2(x - t)\,dt.$$

Prove that if f_1 and f_2 belong to $L_1(\mathbf{R})$, then the integrand is integrable for almost all x, and that the convolution f also belongs to $L_1(\mathbf{R})$.

253. Prove that if the integral of an integrable function f over any interval is equal to zero, then f is equal to zero almost everywhere.

Chapter III

Linear Topological Spaces and Linear Operators

§1. General Theory

1. Topology, Convexity, and Seminorms

254°. Prove that the topology in an LTS is determined by specifying the system of neighborhoods of zero.

255. Prove that in any LTS a closed set X and a point $x \notin X$ have disjoint neighborhoods.

256. Suppose that two norms $p_1(x)$ and $p_2(x)$ are given in a finite-dimensional linear space. Prove that there is a positive constant C such that

$$p_1(x) \leq Cp_2(x), \qquad p_2(x) \leq Cp_1(x).$$

257. Prove that there is only one topology in a finite-dimensional linear space L with respect to which L is a Hausdorff LTS.

258°. Prove that if two norms $p(x)$ and $q(x)$ majorize one another ($C^{-1}p(x) \leq q(x) \leq Cp(x)$, $C > 0$), then the systems of open balls $\mathring{B}_p$ and $\mathring{B}_q$ determine the same topology.

259°. (a) Prove that if A is an open set and B is arbitrary, then $A + B$ is an open set.

(b) Prove that if A is a closed set and B is compact, then $A + B$ is a closed set.

260. Give an example of closed sets A and B for which $A + B$ is not closed.

261°. Let $A_1, \ldots, A_n$ be convex sets and $\lambda_1, \ldots, \lambda_n$ fixed numbers. Prove that $A = \sum_{i=1}^{n} \lambda_i A_i$ is a convex set.

262°. Prove that the intersection of any family of convex sets is convex.

263. Let A be any bounded set in the plane and B the unit disk $x^2 + y^2 < 1$. Prove that for any positive numbers α and β the set $\alpha A + \beta B$ is Lebesgue-measurable, and if A is convex, then

$$\mu(\alpha A + \beta B) = S\alpha^2 + L\alpha\beta + \pi\beta^2.$$

What is the meaning of the coefficients S and L?

264*. Let $A_1, \ldots, A_k$ be convex bounded sets in $\mathbf{R}^n$. Prove that $\mu(\alpha_1 A_1 + \cdots + \alpha_k A_k)$ is a homogeneous polynomial of degree n in the variables $\alpha_1, \ldots, \alpha_k$.

265. Let L be a linear space over the field K. Prove that among all Hausdorff topologies turning L into an LCS there is a strongest, called the *convex core topology*, and that it has the following properties:

(a) every linear functional (i.e., linear mapping $f: L \to K$) is continuous;

(b) a basis of neighborhoods of zero in L consists of all the convex sets containing zero and intersecting each line passing through zero in an interval of positive length.

266. Prove that a set B is the unit ball for some seminorm p if and only if it is convex, balanced, and closed in the convex core topology (see Problem 265).

267. Prove that among all the convex sets B containing 0 and having a given Minkowski functional p there is a largest B_1 and a smallest B_0 (with respect to inclusion).

268. Under the conditions of Problem 267 show that B_1 is the closure of B_0 in the convex core topology.

269. Let A be any set in a linear space L, $c_1(A)$ the intersection of all the convex sets in L containing A, and $c_2(A)$ the collection of all vectors of the form $x = \sum_{i=1}^n \tau_i x_i$, where $x_i \in A$ and the coefficients τ_i are nonnegative and have the property that $\sum_{i=1}^n \tau_i = 1$. Prove that $c_1(A) = c_2(A)$. This set is called the *convex hull* of A and denoted by $c(A)$.

270. A set M in an LTS L is said to be *bounded* if for each neighborhood U of zero there is an $\varepsilon > 0$ such that $\varepsilon M \subset U$. Prove that in a polynormed space $(L, \{p_\alpha\}_{\alpha \in A})$ a set M is bounded if and only if it is bounded in each of the seminorms p_α, $\alpha \in A$.

271. Prove that in every LCS the convex hull of a bounded set is bounded. Give an example showing that this property may fail to hold in an arbitrary LTS.

272. Let $\mathbf{R}^\infty$ be the space of all sequences of real numbers with the topology of coordinatewise convergence (cf. Problem 276). Prove that there are no nonempty open bounded sets in $\mathbf{R}^\infty$.

273. In the space $C(\mathbf{R})$ of all continuous real functions on the line introduce the countable system of seminorms $p_N(f) = \max_{|x| \le N} |f(x)|$ and set

$$d(f, g) = \sum_{N=1}^{\infty} 2^{-N} \frac{p_N(f - g)}{1 + p_N(f - g)}.$$

For what $r > 0$ is the ball of radius r a convex set in $C(\mathbf{R})$?

274. In the space $C(\mathbf{R})$ of all continuous real functions on the line define a distance by the formula

$$d(f, g) = \sup_{x \in \mathbf{R}} \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|}.$$

Is $C(\mathbf{R})$ an LTS with the topology determined by this distance?

275°. Prove that the subspace $BC(\mathbf{R})$ of the space $C(\mathbf{R})$ in the preceding problem consisting of all the bounded continuous functions on the line is a normed LTS.

276. Let $\mathbf{R}^\infty$ denote the space of all sequences of real numbers with the distance

$$d(\{x_n\}, \{y_n\}) = \sum_{n=1}^{\infty} 2^{-n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}.$$

Prove that:

(a) $\mathbf{R}^\infty$ is locally convex;

(b) $\mathbf{R}^\infty$ is countably normable;

(c) $\mathbf{R}^\infty$ is not normable.

2. Dual Spaces

277°. Prove that if a linear functional is continuous at some point of a linear topological space, then it is continuous at each point of this space.

278. Let L be a linear topological space. Prove that:

(a) a linear functional f on L is continuous if and only if there exist an open set $U \subset L$ and a number t such that t does not belong to the set $f(U)$;

(b) a linear functional f on L is continuous if and only if its null subspace $\operatorname{Ker} f = \{x: f(x) = 0\}$ is closed in L.

279. Prove that if L is an infinite-dimensional normed space, then there is a discontinuous linear functional on it.

280. Suppose that the linear topological space L has a defining system of neighborhoods of zero whose cardinality does not exceed the dimension of L. Prove that there is a discontinuous linear functional on L.

281°. (a) Prove that a linear functional f on a normed space L is continuous if and only if it is bounded on the unit ball of L.

(b) Prove that for a linear functional f to be continuous on a linear topological space L it is necessary, and if L satisfies the first axiom of countability also sufficient, that it be bounded on each bounded set.

282°. Prove that the quantity $\|f\| = \sup_{x \neq 0} |f(x)|/\|x\|$ has the properties of a norm in L'.

283. Compute the norms of the following functionals on the space $C[a, b]$:

(a) $I(x) = \int_a^b x(t)\,dt$,

(b) $I_y(x) = \int_a^b x(t)y(t)\,dt$,

(c) $F(x) = \sum_{i=1}^n \lambda_i x(t_i)$,

where y is a fixed element of $C[a, b]$, $t_1, \ldots, t_n$ is a tuple of distinct points of $[a, b]$, and $\lambda_1, , \ldots, \lambda_n$ is a tuple of real numbers.

284. Prove that the norm of a functional $f \in L'$ is the reciprocal of the distance from zero to the hyperplane $f(x) = 1$.

285°. Prove that any finite-dimensional normed linear space L is reflexive.

286. Prove that a closed subspace of a reflexive space is reflexive.

287. Prove that the space c_0 of all sequences of real numbers converging to zero, with the norm $p(\{x_n\}) = \max |x_n|$, is not reflexive.

288°. Let L be an infinite-dimensional normed space. Prove that the weak topology in L does not coincide with the strong topology.

289. Let $L = l_1(\mathbf{R})$ be the space of sequences of real numbers with the norm $p_1(\{x_n\}) = \sum_{n=1}^\infty |x_n|$. Prove that weak convergence coincides with strong convergence for sequences in L.

290. A hyperplane P is called a *support* of a convex set K as it has a common point with K and the whole of K is located on one side of P. Prove that it is natural to label the support planes of the unit ball in L with points of the unit sphere in L'.

291. Suppose that the unit ball B of the normed space L is a convex polyhedron. Construct a natural correspondence between the k-dimensional faces of B and the $(n - k)$-dimensional faces of the unit ball B' in L'.

292. A convex set M in an LTS L for which the set $\{x \in M \mid \forall y \in L \; \exists \, \varepsilon(y) \in \mathbf{R}: x + ty \in M \text{ for } |t| < \varepsilon(y)\}$ is nonempty is called a *convex body*. Convex bodies B and B' in $\mathbf{R}^n$ are said to be *dual* if their Minkowski functionals determine structures of dual normed spaces in $\mathbf{R}^n$. (We identify the vector $(a_1, \ldots, a_n)$ with the functional $(x_1, \ldots, x_n) \mapsto \sum_{i=1}^n a_i x_i$.)

Prove that the section of B by a k-dimensional plane P is dual to the projection of B' on this plane.

293. Let c be the space of all real sequences $\{x_n\}$ for which $\lim_{n\to\infty} x_n$ exists, endowed with the norm $\|\{x_n\}\| = \sup |x_n|$, and let c_0 be the subspace of sequences converging to zero. Prove that the spaces c' and c_0' are isomorphic to the space $l_1(\mathbf{R})$, but that the spaces c and c_0 are not isomorphic to each other.

294. Prove the isomorphism $l_p(K') = l_q(K)$, where $p \in (1, \infty)$, $q = p/(p - 1)$, $K = \mathbf{R}$ or $\mathbf{C}$.

295. Prove that $l_\infty(K)'$ is not isomorphic to $l_1(K)$, for $K = \mathbf{R}$ or $\mathbf{C}$.

296. Let L_1 and L_2 be normed spaces whose elements can be expressed as column vectors of lengths n_1 and n_2, respectively. An operator $A \in \mathscr{L}(L_1, L_2)$

can then be expressed as a matrix with n_1 columns and n_2 rows, so that the action of A on an $x \in L_1$ amounts to multiplication of x from the left by A according to the rules of matrix multiplication. Show that the spaces L_1' and L_2' can be identified with the spaces of row vectors of lengths n_1 and n_2, respectively, so that the action of the adjoint operator A' amounts to multiplication from the right by the matrix of A.

297. (a) Prove that the space $C[0, 1]$ cannot be isometrically imbedded in $l_p(\mathbf{R})$ for $1 \leq p < \infty$.

(b) Construct an isometric imbedding of $C[0, 1]$ in $l_\infty(\mathbf{R})$.

298.** (a) Prove that a Banach space L is reflexive if and only if its unit ball is weakly compact.

(b) Let L be a reflexive Banach space, and L_0 a closed subspace of it. Prove that L_0 and $L_1 = L/L_0$ are reflexive.

299*. Prove that there does not exist a normed space whose dual space is $C[a, b]$.

3. The Hahn–Banach Theorem

300. Prove that if an LTS L_1 is Hausdorff and finite-dimensional, then every linear mapping $L_1 \to L_2$, where L_2 is an arbitrary LTS, is continuous.

301. Let P be the space of all polynomials in x with real coefficients, and U_+ (resp., U_-) the subset of polynomials with positive (resp., negative) leading coefficient. Prove that the sets U_+ and U_- are convex, but cannot be separated by a hyperplane.

302. Prove that disjoint convex closed sets A and B can be *strictly separated by a hyperplane* if one is compact. (This means that there exist a continuous linear functional f and constants $c_1 < c_2$ such that $f(x) \leq c_1$ on A and $f(x) \geq c_2$ on B.)

303°. Let L be a normed linear space with norm p, L' the dual space with norm p', and L'' the space dual to L', with norm p''. Assign to each $x \in L$ an element $F_x \in L''$ according to the formula $F_x(f) = f(x)$ for $f \in L'$. Prove that $p''(F_x) = p(x)$.

304°. Let L be a finite-dimensional normed space. Prove that L and L'' are isomorphic (i.e., there exists a linear isometric mapping of L onto L'').

305. Prove that every normed linear space is isometric to a subspace of some space of the form $C(X)$, where $C(X)$ is the space of continuous functions on a compact set X, with the norm $\|f\| = \max_{x \in X} |f(x)|$.

306*. Prove the following theorem of Riesz: On a real normed space L there exists a linear functional f with norm ≤ 1 taking the values $c_1, \ldots, c_n$ on the respective elements $x_1, \ldots, x_n$ if and only if for any real numbers $\lambda_1, \ldots, \lambda_n$

$$|\lambda_1 c_1 + \cdots + \lambda_n c_n| \leq \|\lambda_1 x_1 + \cdots + \lambda_n x_n\|.$$

307. Construct an isometric imbedding of $l_p(2, \mathbf{R})$ into $C[0, 1]$ for $p = 1, 2, \infty$.

308. Prove that there exists an isometric imbedding of $l_p(n, \mathbf{R})$ into $l_\infty(\mathbf{R})$.

309. Let $l_\infty(\mathbf{R})$ be the space of bounded real sequences $\{x_n\}$, $n = 1, 2, \ldots$. Prove that there exists a linear functional $\mathrm{LIM} \in l_\infty(\mathbf{R})'$ having the following properties:

(1) $\sup x_n \geq \mathrm{LIM}\{x_n\} \geq \inf x_n$;
(2) if $\lim_{n\to\infty} x_n = a$ exists, then $\mathrm{LIM}\{x_n\} = a$;
(3) $\mathrm{LIM}\{x_{n+1}\} = \mathrm{LIM}\{x_n\}$.

310. Prove that the assertions of Problem 309 carry over to the case of two-sided sequences $\{x_n\}$, $n \in \mathbf{Z}$.

311. Let L be a normed space, and T a linear invertible operator on L having the property that

$$p(T^n x) \leq cp(x), \qquad \forall x \in L, \qquad n = 0, \pm 1, \ldots.$$

Prove that there is a norm $\tilde{p}$ in L that is equivalent to p, and with respect to which T is an isometry.

312. Let L be an LCS. Prove that it can be imbedded continuously in a product of lines $\mathbf{R}^\alpha$, where α is a sufficiently large cardinal number. (In other words, every LCS admits a coordinate description.)

313*. Let $B(\mathbf{R}^n)$ be the space of bounded real functions on $\mathbf{R}^n$, with the norm $\|f\| = \sup_{x \in \mathbf{R}^n} |f(x)|$. Prove that there exists a linear functional $\mathrm{LIM} \in B(\mathbf{R}^n)'$ having the properties:

(a) $\inf_{\mathbf{R}^n} f(x) \leq \mathrm{LIM}\, f(x) \leq \sup_{\mathbf{R}^n} f(x)$;
(b) if $\lim_{|x|\to\infty} f(x) = a$ exists, then $\mathrm{LIM}\, f(x) = a$;
(c) $\mathrm{LIM}\, f(x + y) = \mathrm{LIM}\, f(x)$ for any $y \in \mathbf{R}^n$.

314. Prove that there is a finitely additive measure on $\mathbf{R}^n$ that is defined for all subsets of $\mathbf{R}^n$, is invariant under translations, and coincides with the usual volume on parallelepipeds.

315°. Prove that a convex subset X of an LCS L is dense if and only if every linear functional $f \in L'$ equal to zero on X vanishes identically.

316. Prove that every closed convex set in a real LCS L is the intersection of some family of half-spaces of the form $f(x) \leq c$, where $f \in L'$, $c \in \mathbf{R}$.

317. Represent the unit ball of $L_p(n, \mathbf{R})$ as an intersection of countably many half-spaces.

318*. Let I^N be the *N-dimensional cube* defined in $\mathbf{R}^N$ by the inequalities $|x_i| \leq 1$, $1 \leq i \leq N$. Prove that any convex bounded subset of the plane can be approximated to any degree of accuracy by two-dimensional sections of I^N. (More precisely, for any $\varepsilon > 0$ and any convex set $V \subset \mathbf{R}^2$ there exist an N and an imbedding $\varphi\colon \mathbf{R}^2 \to \mathbf{R}^N$ such that the Minkowski functionals of the sets U and $V = \varphi^{-1}(I^N)$ are connected by the inequalities $1 - \varepsilon < p_U/p_V < 1 + \varepsilon$.)

319. Construct an isometric imbedding of $l_p(n, \mathbf{R})$ into $C[0, 1]$ for any $p \in [1, \infty]$.

320. *Helly's theorem.* Prove that if a family of convex subsets of $\mathbf{R}^n$ is such that any $n + 1$ sets of the family have a common point, then all the sets of the family have a common point.

321. Define a topology in $C[0, 1]$ by taking the sets $U_\varepsilon = \{f \in C[0, 1]: \int_0^1 \sqrt{|f(x)|}\, dx < \varepsilon\}$ as a base of neighborhoods of zero. Prove that each continuous linear functional on the resulting topological space is equal to zero.

322*. Let L be an LCS, X a set with a measure μ, and f a function on X with values in L. The function f is said to be *weakly measurable* if the numerical function $F(f(x))$ is μ-measurable for every $F \in L'$. An element $\varphi \in L$ is called a *weak integral* of f with respect to the measure μ on X if $F(f) = \int_K F(f(x))\, d\mu(x)$ for all $F \in L'$. Prove:

(a) the uniqueness of a weak integral;

(b) the existence of the weak integral in the case when L is a reflexive Banach space and $\|f\|$ is μ-integrable on X.

4. Banach Spaces

323°. Prove that $\|x\|_p = (\sum_{k=1}^n |x_k|^p)^{1/p}$ is not a norm for $p < 1$ and $n \geq 2$.

324. Prove that $l_p(K)$ is a separable Banach space for $1 \leq p < \infty$ and $K = \mathbf{R}$ or $\mathbf{C}$.

325. Prove that the space $l_\infty(K)$ of all bounded sequences $\{x_n\}$, $x_n \in K$, is a nonseparable Banach space with the norm $\|\{x_n\}\| = \sup_n |x_n|$.

326. Prove that a normed space L is a Banach space if and only if every series $\sum x_i$ for which $\sum \|x_i\| < \infty$ converges in L.

327°. Let L_0 be a closed subspace of a Banach space L. Prove that the formula $\|x\|_1 = \inf_{y - x \in L_0} \|y\|$ defines a norm on the space $L_1 = L/L_0$.

328. Prove that the space L_1 defined in Problem 327 is a Banach space.

329*. Let L be a normed space, L_0 a closed subspace of it, and $L_1 = L/L_0$, with the norm defined in Problem 327. Is it true that L is a Banach space if L_0 and L_1 are Banach spaces?

330*. Prove that every separable Banach space over the field K is a quotient space of $l_1(K)$.

331°. Prove that a normed space L is nonseparable if and only if it contains an uncountable set of pairwise disjoint balls of radius 1.

332. Prove that in a normed space the distance from a given point to an arbitrary finite-dimensional subspace is attained.

333*. Find the distance from the point $x^n \in C[-1, 1]$ to the subspace $P_{n-1} \subset C[-1, 1]$ consisting of all the polynomials of degree $< n$.

334. Let L be a Banach space and L_0 a closed subspace of L. A vector $x \in L$ is called an ε-perpendicular to L_0 if $\|x + y\| \geq (1 - \varepsilon)\|x\|$ for any $y \in L_0$. Prove that for any $\varepsilon > 0$ any proper subspace L_0 has an ε-perpendicular.

335. Use the result of Problem 334 to show that the unit ball of an infinite-dimensional Banach space is not compact.

336. Prove that the existence of a 0-perpendicular to the subspace L_0 given by the equation $f(x) = 0$, where $f \in L'$, is equivalent to the existence of $\max_{\|x\| \leq 1} |f(x)|$.

337. A subspace L_0 of a Banach space L is said to be *complemented* if there exists a closed subspace $L_1 \subset L$ such that $L = L_0 \oplus L_1$ (algebraic direct sum). Prove that:

(a) every finite-dimensional subspace is complemented;
(b) every subspace of finite codimension is complemented;
(c) the space $l_\infty(K)$ is complemented in any Banach space containing it;
(d) if L/L_0 is isometric to $l_1(K)$, then L_0 is complemented.

338*. Prove that $l_{p_1}(n, \mathbf{R})$ is isometric to $l_{p_2}(n, \mathbf{R})$ only if $p_1 = p_2$.

339. Let L_1 and L_2 be normed linear spaces with norms p_1 and p_2, respectively.

(a) Prove that the norm $p_1 \hat{\otimes} p_2$ can be defined by the formula

$$(p_1 \hat{\otimes} p_2)(x) = \inf \sum_{k=1}^{N} p_1(y_k)p_2(z_k),$$

where the infimum is over all representations of x in the form $x = \sum_{k=1}^{N} y_k \otimes z_k$, $y_k \in L_1$, $z_k \in L_2$.

(b) Prove that the norm $p_1 \underset{\vee}{\otimes} p_2$ is given by the formula

$$(p_1 \otimes p_2)(x) = \sup(f_1 \otimes f_2)(x),$$

where the supremum is over all $f_1 \in L_1', f_2 \in L_2'$ with $p_1'(f_1) \leq 1, p_2'(f_2) \leq 1$.

340*. Let L_1 and L_2 be Banach spaces, and $\mathscr{K}$ the category of bilinear mappings $A: L_1 \times L_2 \to L$ with norm ≤ 1, L a Banach space. The morphisms are defined as in Problem 61, with the extra condition $\|\varphi\| \leq 1$. Prove that the mapping A_0: $L_1 \times L_2 \to L_1 \hat{\otimes} L_2$: $(x_1, x_2) \mapsto x_1 \otimes x_2$ is a universal object in the category $\mathscr{K}$.

341*. Let $L_1 = l_2(n, \mathbf{R})$, $L_2 = l_2(m, \mathbf{R})$. The space $L_1 \otimes L_2$ can be identified with the space of $n \times m$ matrices as follows. Write the elements of L_2 as row vectors, and those of L_1 as column vectors. Then the matrix $x \cdot y$ is assigned to the element $x \otimes y$. Suppose that $a \in L_1 \otimes L_2$ corresponds to the matrix A. Prove that the norm of a in $L_1 \hat{\otimes} L_2$ can be computed by the formula $\|a\| = s_1^{1/2} + \cdots + s_k^{1/2}$, $k = \min(n, m)$, where the s_i are the eigenvalues of the matrix AA' $(A'A)$, arranged in decreasing order, if $k = n$ (resp., $k = m$).

342. Prove that under the conditions of Problem 341 the norm of $a \in L_1 \check{\otimes} L_2$ coincides with the norm of the matrix A as an operator from $l_2(m, \mathbf{R})$ to $l_2(n, \mathbf{R})$ and is equal to the number $s_1^{1/2}$.

343. Let L_1 and L_2 be Banach spaces. Show that there is a natural isometric imbedding of $L_1' \check{\otimes} L_2'$ into the space $\mathscr{L}(L_1, L_2)$.

344. Prove that $l_1(n, \mathbf{R}) \hat{\otimes} l_1(m, \mathbf{R})$ is isomorphic to $l_1(mn, \mathbf{R})$.

345. Prove that $l_\infty(n, \mathbf{R}) \check{\otimes} l_\infty(m, \mathbf{R})$ is isomorphic to $l_\infty(mn, \mathbf{R})$.

§2. Linear Operators

1. The Space of Linear Operators

346°. Suppose that the operator P_k acts in $l_2(\mathbf{R})$ according to the formula $P_k(\{x_n\}) = \{\varepsilon_{nk} x_n\}$, where $e_{nk} = 1$ for $k \le n$ and $\varepsilon_{nk} = 0$ for $k > n$. Which of the following convergences is valid as $k \to \infty$:

(a) $P_k \Rightarrow 0$;
(b) $P_k \to 0$;
(c) $P_k \rightharpoonup 0$?

347°. Let $e_1, \ldots, e_n, \ldots$ be the natural basis in the space $l_2(\mathbf{R})$. Define an operator A_n by

$$A_n e_k = \begin{cases} e_1 & \text{if } k = n, \\ 0 & \text{if } k \ne n. \end{cases}$$

Prove that $\|A_n\| = 1$, and that $A_n \rightharpoonup 0$ as $n \to \infty$.

348°. Under the conditions of Problem 347 define an operator B_n by

$$B_n e_k = \begin{cases} e_n & \text{if } k = 1, \\ 0 & \text{if } k \ne 1. \end{cases}$$

Prove that $\|B_n\| = 1$, and $B_n \rightharpoonup 0$ as $n \to \infty$, but $\text{s-lim}_{n \to \infty} B_n$ does not exist.

349°. Prove that multiplication of operators is continuous in the uniform topology: if $A_n \Rightarrow A \in \mathscr{L}(L_1, L_2)$, $B_n \Rightarrow B \in \mathscr{L}(L_0, L_1)$, then $A_n B_n \Rightarrow AB \in \mathscr{L}(L_0, L_2)$.

350°. Let L_1 and L_2 be Banach spaces. Prove that if $A_n \rightharpoonup A \in \mathscr{L}(L_1, L_2)$, then the norms of the operators A_n are collectively bounded.

351. Prove that if $A_n \to A \in \mathscr{L}(L_1, L_2)$, $B_n \to B \in \mathscr{L}(L_0, L_1)$, then $A_n B_n \to AB \in \mathscr{L}(L_0, L_2)$.

352*. Prove that multiplication of operators:

(a) is not continuous in the strong topology of the space End L if L is infinite-dimensional (a comparison with the result of Problem 351 shows that the strong topology in End L is not determined by convergent sequences);
(b) is strongly continuous on the unit ball of End L.

353. Give an example of sequences of operators $A_n \rightharpoonup 0$, $B_n \rightharpoonup 0$ such that $A_n B_n$ does not converge to 0 in the weak operator topology.

354. Suppose that a Banach space L is decomposed into an algebraic direct sum: $L = L_1 + L_2$. Prove that the operator of projection onto L_1 parallel to L_2 is bounded if and only if L_1 and L_2 are closed in L.

355. Prove that an operator P in a Banach space L is a projection onto some closed subspace L_1 parallel to a closed subspace L_2 if and only if it is bounded and satisfies the relation $P^2 = P$.

356°. Prove the inequality $\|AB\| \leq \|A\| \, \|B\|$ for $A \in \mathscr{L}(L_1, L_2)$, $B \in \mathscr{L}(L_0, L_1)$.

357. Prove that $\mathscr{L}(L_1, L_2)$ is a Banach space.

358°. Let A be the operator of multiplication by a bounded measurable function $a(x)$, acting in the space $L_p(X, \mu)$. Prove that A is bounded, and find its norm.

359°. Find the norm of the identity operator acting from $L_p[a, b]$ into $L_q[a, b]$ for $p \geq q$.

360*. For which functions $a(x)$ is the operator of multiplication by $a(x)$ a continuous operator from $L_p[0, 1]$ into $L_q[0, 1]$?

361. Let $T(t)$ be the translation operator in $L_p(\mathbf{R})$, $1 \leq p < \infty$: $T(t)f(x) = f(x + t)$.

Prove that $T(t) \to T(t_0)$ as $t \to t_0$. Is it true that $T(t) \rightrightarrows T(t_0)$ as $t \to t_0$?

362*. Let $A(t)$ be a differentiable operator-valued function on $\mathbf{R}$ with values in End L, dim $L < \infty$.

Prove that all the solutions of the differential equation $A'(t) = CA(t)$, where $C \in \text{End } L$, have the form $A(t) = e^{tC}A_0$, where $A_0 \in \text{End } L$ and $e^{tC} = \sum_{k=0}^{\infty} (t^k C^k / k!)$.

363.** Let $A(t)$ be a continuous operator-valued function on $\mathbf{R}$ with values in End L, dim $L < \infty$. Prove that all the solutions of the functional equation $A(t)A(s) = A(t + s)$ such that $A(0) = 1$ have the form $A(t) = e^{tC}$, where $C \in \text{End } L$.

364*. Prove that the assertion of Problem 363 ceases to be true in the case dim $L = \infty$.

365. Let A be a linear operator from L_1 to L_2 that takes every strongly convergent sequence into a weakly convergent sequence. Prove that A is bounded.

366*. Let A be an operator from L_1 to L_2 that is continuous in the sense of the weak topologies in L_1 and L_2. Is A continuous in the sense of the strong topologies?

367. Let $K(x, y)$ be a continuous function on the unit square in $\mathbf{R}^2$, and let the operator A act from $L_p[0, 1]$ to $L_q[0, 1]$, $1 \leq p, q < \infty$, by the formula $(Af)(x) = \int_0^1 K(x, y) f(y) \, dy$. Find the adjoint operaotr A': $L_{q'}[0, 1] \to L_{p'}[0, 1]$ where $p' = p/(p - 1)$, $q' = q/(q - 1)$.

368. Let $P: C[0, 2] \to C[0, 1]$ be the restriction operator. Find the adjoint operator $P': V[0, 1] \to V[0, 2]$.

2. Compact Sets and Compact Operators

369. Prove that the following properties of a subset A in a topological space X are equivalent:

(a) A is compact;

(b) every infinite subset of A contains a net converging to some element of A;

(c) every centered system of closed subsets of A has a nonempty intersection. (A system of sets is said to be *centered* if any finite subsystem of it has nonempty intersection, i.e., it has the finite-intersection property.)

370. Compute the approximation dimension of the *Cantor set* X. (X is the intersection of the countable collection of sets X_n, where X_n is obtained from $[0, 1]$ by discarding the 3^{n-1} intervals of the form $((3k - 2)/3^n, (3k - 1)/3^n)$, $k = 1, 2, \ldots, 3^{n-1}$.)

371°. Let K be a convex set in a linear space L. A subset $A \subset K$ is said to be *extreme* if every segment lying in K and with midpoint in A lies entirely in A. Prove that the intersection of any family of extreme subsets is either empty or an extreme subset.

372. Let K be a compact convex set. Prove that the collection of closed extreme subsets of it (see Problem 371), ordered by inclusion, has a minimal element.

373. Let K be a closed convex bounded subset of an LCS L, and A a minimal element of the family of closed extreme subsets of K (see Problem 372). Prove that A consists of a single point.

374. Prove that a compact convex set K in an LCS L has at least one extreme point.

375*. Prove the *Krein–Milman theorem*: every compact convex set K in an LCS L coincides with the closure of the convex hull of its set of extreme points.

376°. Find the extreme points of the unit ball in the space $l_p(n, \mathbf{R})$, $1 \leq p \leq \infty$.

377°. Find the extreme points of the unit ball in the spaces c and c_0 (see Problem 293).

378. Prove that neither of the spaces c and c_0 (see Problem 293) is the space dual to some normed linear space.

379. Prove the following analog of the Arzelà–Ascoli theorem. Let $B(T, X)$ be the metric space of all bounded functions on a set T taking values in a compact metric space X, with the distance $d(f, g) = \sup_{t \in T} d_K(f(t), g(t))$, where d_X is the distance in X. Then a set $M \subset B(T, X)$ is pre-compact if and only if for each $\varepsilon > 0$ there is a finite partition $T = T_1 \amalg \cdots \amalg T_n$ such that any function $f \in M$ does not vary by more than ε on each T_j.

380. Find the extreme points of the set S of doubly stochastic matrices of order n. (A matrix A is said to be *doubly stochastic* if its elements are nonnegative, and the sum of the elements in each row and each column is equal to 1.)

381°. Prove that the identity operator in an infinite-dimensional normed space is not compact.

382°. Prove that a compact operator in an infinite-dimensional normed space does not have a bounded inverse.

383. Let the operator A in $l_p(\mathbf{R})$, $1 \le p \le \infty$, be given by the formula $A\{x_n\} = \{a_n x_n\}$, where $\{a_n\}$ is a fixed bounded sequence of real numbers. Prove that A is compact if and only if $\lim_{n\to\infty} a_n = 0$.

384. Prove that the operator A acting in $C[0, 1]$ by the formula $Af(x) = x \cdot f(x)$ is not compact.

385. Let L_1 and L_2 be Banach spaces, and $A \in \mathscr{L}(L_1, L_2)$. Prove that A is compact if A' is.

386°. Let $K(x, y)$ be a continuous function on the unit square in $\mathbf{R}^2$. Prove that the operator A defined on $C[0, 1]$ by the formula $Af(x) = \int_0^1 K(x, y) f(y)\, dy$ is compact.

387. Let $K \in L_2(X \times Y, \mu \times \nu)$. Prove that the operator A acting from $L_2(Y, \nu)$ to $L_2(X, \mu)$ by the formula $Af(x) = \int_Y K(x, y) f(y)\, d\nu(y)$ is compact.

388*. Let the operator T be defined in $L_p(0, \infty)$, $p \ge 1$, by the formula $Tf(x) = (1/x) \int_0^x f(t)\, dt$.

Prove that T is bounded, but not compact. Find the norm of T.

389. Suppose that L is a reflexive space. Prove that an operator $T \in \operatorname{End} L$ that carries every weakly convergent sequence into a strongly convergent sequence is compact.

390. Let Ω be a domain in $\mathbf{R}^n$. Prove that the operator imbedding $C^{k+1}(\overline{\Omega})$ into $C^k(\overline{\Omega})$ is compact.

391. Can a compact operator A satisfy an algebraic equation $\sum_{k=0}^n c_k A^k = 0$ $(A^0 = 1)$?

3. The Theory of Fredholm Operators

392. Let A be the operator on $l_p(\mathbf{R})$ acting according to the formula $A\{x_n\} = \{a_n x_n\}$, where $\{a_n\}$ is a fixed sequence of real numbers. Under what condition on $\{a_n\}$ is the subspace $\operatorname{im} A$ closed in $l_p(\mathbf{R})$?

393°. Let T be the operator on $l_p(\mathbf{R})$ acting according to the formula $T\{x_n\} = \{x_{n+1}\}$. Find the kernels and the cokernels of the operators T^k, $k = 1, 2, \ldots$.

394*. Let P be a polyhedron in $\mathbf{R}^3$, X_k the set of oriented k-dimensional faces (the zero-dimensional faces are the vertices, the one-dimensional faces the edges, the two-dimensional faces the usual faces, and the three-dimensional face the polyhedron itself), and L_k the space of real functions on X_k. If $\Gamma \in X_{k-1}$, $\Delta \in X_k$, then it is possible to define the number $\varepsilon(\Gamma, \Delta)$ to be equal to 0 if Γ does not lie on the boundary of Δ, and ± 1 otherwise. The sign of $\varepsilon(\Gamma, \Delta)$ depends on the orientations of Γ and Δ. Let $e_1, \ldots, e_{k-1}$ be a basis giving the orientation of Γ, and $f_1, \ldots, f_k$ a basis giving the orientation of Δ and chosen in such a way that the vectors $f_1, \ldots, f_{k-1}$ lie in the plane of Γ, while f_k is transversal to Γ and directed outside Δ. Then $\varepsilon(\Gamma, \Delta)$ is equal to the sign of the determinant of the transition matrix from $e_1, \ldots, e_{k-1}$ to $f_1, \ldots, f_{k-1}$. Define an operator $d_k: L_{k-1} \to L_k$ by the formula

$$d_k f(\Delta) = \sum_{\Gamma \in X_{k-1}} \varepsilon(\Gamma, \Delta) f(\Gamma).$$

Prove that the sequence

$$0 \to L_0 \xrightarrow{d_1} L_1 \xrightarrow{d_2} L_2 \xrightarrow{d_3} L_3 \to 0$$

is semiexact, and compute its cohomology for the most simple polyhedrons (a simplex, a cube, a cube with a hole through it, a cube with an inner cavity).

395. Let $C^k(\mathbf{T})$ be the space of functions on the circle $\mathbf{T}$ that have k continuous derivatives, with the norm

$$\|f\| = \max_{t \in T} \{|f(t)|, |f'(t)|, \ldots, |f^{(k)}(t)|\}.$$

Prove that the sequence

$$0 \to C^k(\mathbf{T}) \xrightarrow{d} C^{k-1}(\mathbf{T}) \to 0$$

is semiexact, where d is the differentiation operator, and compute its cohomology.

396. Let

$$0 \to L_0 \xrightarrow{T_1} L_1 \xrightarrow{T_2} \cdots \to L_{N-1} \xrightarrow{T_n} L_n \to 0$$

be a semiexact sequence of finite-dimensional spaces with cohomology spaces H_k for $k = 0, 1, \ldots, n$.

Prove the *Euler identity*

$$\sum_{k=0}^{n} (-1)^k \dim L_k = \sum_{k=0}^{n} (-1)^k \dim H_k.$$

397.** Let

$$\cdots \to L_{k-1} \xrightarrow{T_k} L_k \xrightarrow{T_{k+1}} L_{k+1} \to \cdots$$

be a sequence of Banach spaces and continuous operators. Prove that if the dual sequence

$$\cdots \leftarrow L'_{k-1} \xleftarrow{T'_k} L'_k \xleftarrow{T'_{k+1}} L'_{k+1} \leftarrow \cdots$$

is exact, then the original sequence is also exact.

398*. Let

$$\cdots L_{k-1} \xrightarrow{T_k} L_K \xrightarrow{T_{k+1}} L_{k+1} \to \cdots$$

be a semiexact sequence of Banach spaces and continuous mappings. Prove that the dual sequence is semiexact and that the cohomology spaces of the dual sequence are dual to the cohomology spaces of the original sequence.

399°. Construct an almost inverse operator for the operator T in Problem 393.

400°. Let T be a differential operator of the form

$$T = \left(\frac{d}{dx}\right)^n + a_1(x)\left(\frac{d}{dx}\right)^{n-1} + \cdots + a_n(x),$$

acting from $C^{k+n}[0, 1]$ to $C^k[0, 1]$. Prove that T is a Fredholm operator and find its index.

401. Is the operator of multiplication by a continuous function $a(x)$ a Fredholm operator on $C[0, 1]$?

402*. Let L be the space of functions harmonic in the domain $\Omega \subset \mathbf{R}^2$ bounded by a smooth curve Γ, and continuous on its closure. Prove that the restriction operator P: $L \to C(\Gamma)$ is a Fredholm operator, and find its index.

403. Let Ω be a bounded domain in the complex plane, $H(\Omega)$ the space of functions holomorphic in Ω and continuous in $\overline{\Omega}$, and $a(z)$ a function holomorphic in some neighborhood of $\overline{\Omega}$. Prove that the operator of multiplication by $a(z)$ is Fredholm on $H(\Omega)$, and find its index.

404°. Find all the solutions of the integral equation

$$f(x) = \lambda \int_0^{\pi/2} \cos(x - y) f(y)\, dy$$

in the space $C[0, 1]$.

405°. For which functions $g \in C[0, \pi]$ is the integral equation

$$f(x) - \int_0^\pi \sin(x + y) f(y)\, dy = g(x)$$

solvable in the space $C[0, \pi]$?

406°. For which $\lambda \in \mathbf{R}$ is the equation

$$f(x) - \lambda \int_a^b e^{(x-y)} f(y)\, dy = 1$$

solvable in $L_p[a, b]$, $1 \le p < \infty$?

407*. Let $H_0 = L_2(\mathbf{R}, dx)$, and let H_1 denote the completion of the space $S(\mathbf{R})$ (see Ch. III, §3.3) in the norm $\|f\|_1^2 = \|xf\|_0^2 + \|f'\|_0^2$ (here $\|\cdot\|_0$ denotes the norm in H_0). The *creation and annihilation operators* in quantum field theory can be defined as the differential operators from H_1 to H_0 acting according to the formulas $A_\pm f = (d/dx \pm x)f$. Prove that $A_\pm$ are Fredholm, and that ind $A_\pm = \pm 1$.

408. A *Hilbert–Schmidt operator* is defined to be an integral operator

$$(A\varphi)(s) = \int_a^b K(s, t)\varphi(t)\, dt$$

acting in the space $L_2[a, b]$, and with kernel satisfying the condition $\int_a^b \int_a^b |K(s, t)|^2\, ds\, dt < \infty$. Prove that for a Hilbert–Schmidt operator

$$\|A\| \leq \sqrt{\int_a^b \int_a^b |K(s, t)|^2\, ds\, dt}.$$

409°. In the notation of Problem 408 prove that the correspondence $A \mapsto K(s, t)$ between the Hilbert–Schmidt operators and their kernels is one-to-one, to within equivalence of measurable functions.

410°. Given the Hilbert–Schmidt operator determined by the kernel $K(s, t)$ (see Problem 408), prove that the operator adjoint to it is determined by the "conjugate" kernel $\overline{K(t, s)}$.

411. A *Fredholm integral equation* (*of the second kind*) *with degenerate kernel* is defined to be an equation of the form

$$\varphi(s) = \int_a^b \left(\sum_{i=1}^n P_i(s)Q_i(t)\right)\varphi(t)\, dt + f(s),$$

where P_i, Q_i are functions in $L_2[a, b]$. Prove that the general solution of this equation has the form $\varphi(s) = \sum_{i=1}^n q_i P_i(s) + f(s)$, and that the unknown coefficients q_i can be found from a system of algebraic equations of the form

$$\sum_{j=1}^n a_{ij} q_j + b_i = q_i.$$

412*. A *Volterra equation* (of the second kind) is defined to be an integral equation

$$\varphi(s) = \int_a^s K(s, t)\varphi(t)\, dt + f(s),$$

where $K(s, t)$ is a bounded measurable function. Prove that for any $f \in L_2$ the Volterra equation has one and only one solution.

413. Prove that the product of two Hilbert–Schmidt operators with kernels $K(s, t)$, $Q(s, t)$ (see Problem 408) is an operator of the same type with kernel

$$R(s, t) = \int_a^b K(s, u)Q(u, t)\, du.$$

414. Let A be a Hilbert–Schmidt operator (see Problem 408), and suppose that its kernel satisfies the relation

$$\int_a^b \int_a^b |K(s, t)|^2 \, ds \, dt = K^2 < \infty.$$

Prove that the kernel $K_n(s, t)$ of the operator A^n satisfies the estimate

$$\int_a^b \int_a^b |K_n(s, t)|^2 \, ds \, dt \leq K^{2n}.$$

§3. Function Spaces and Generalized Functions

1. Spaces of Integrable Functions

415. Prove that for any measurable functions on a set X with measure μ the Hölder inequality is valid:

$$\left| \int_X f(x)g(x) \, d\mu(x) \right| \leq \left(\int_X |f(x)|^p \, d\mu(x) \right)^{1/p} \left(\int_X |g(x)|^q \, d\mu(x) \right)^{1/q},$$

where the numbers p and q are connected by the relation $1/p + 1/q = 1$.

416°. Prove that

$$\left(\int_X |f(x)|^p \, d\mu(x) \right)^{1/p} = \sup \left| \int_X f(x)g(x) \, d\mu(x) \right|$$

for any measurable function f on a set X with measure μ, where the supremum is taken over all functions $g(x)$ satisfying the inequality $\int_X |g(x)|^q \, d\mu(x) \leq 1$, and the numbers p and q are connected by the relation $1/p + 1/q = 1$.

417. Prove the *Minkowski integral inequality*

$$\left(\int_X |f(x) + g(x)|^p \, d\mu(x) \right)^{1/p} \leq \left(\int_X |f(x)|^p \, d\mu(x) \right)^{1/p} + \left(\int_X |g(x)|^p \, d\mu(x) \right)^{1/p}$$

for $1 < p < \infty$.

418°. Say that a measure μ on a set X has a *countable base* if there exists a countable family $\{A_n\}$ of measurable subsets of X such that for any measurable subset B there is a set A_n for which $\mu(A_n \triangle B) < \varepsilon$.

Prove that $L_1(X, \mu)$ is a separable space if and only if μ has a countable base.

419°. Prove that $L_p(X, \mu)$, $1 < p < \infty$, is a separable space if and only if $L_1(X, \mu)$ is.

420°. Prove that $L_\infty(X, \mu)$ is either finite-dimensional or nonseparable.

421°. Let $\mu(X) < \infty$. Prove that for $p \geq q \geq 1$ the space $L_p(X, \mu)$ is contained in the space $L_q(X, \mu)$.

422°. Prove that if $p \neq q$, then neither of the spaces $L_p(\mathbf{R}, dx)$, $L_q(\mathbf{R}, dx)$ is contained in the other.

423°. Let $0 < \alpha \leq \beta < \infty$. For what p does the function $f(x) = 1/(x^\alpha + x^\beta)$ belong to $L_p(\mathbf{R}_+, dx)$ ($\mathbf{R}_+$ denotes the positive half-line)?

424. Suppose that the numbers p, q, r are connected by the relation $1/p + 1/q + 1/r = 1$, and that $f \in L_p(X, \mu)$, $g \in L_q(X, \mu)$, $h \in L_r(X, \mu)$. Prove that fgh is an integrable function and that $\|fgh\|_1 \leq \|f\|_p \|g\|_q \|h\|_r$.

425. Let $1 \leq p \leq r \leq q \leq \infty$. Prove that $L_p(X, \mu) \cap L_q(X, \mu)$ is contained in $L_r(X, \mu)$, and that $\|f\|_r \leq \|f\|_p^\alpha \|f\|_q^\beta$ for every function $f \in L_p(X, \mu) \cap L_q(X, \mu)$, where $\alpha = (r^{-1} - q^{-1})/(p^{-1} - q^{-1})$, $\beta = (p^{-1} - r^{-1})/(p^{-1} - q^{-1})$.

426. Let $\mu(X) < \infty$. Prove that $L_\infty(X, \mu) \subset L_p(X, \mu)$ for all $p \geq 1$, and that $\|f\|_\infty = \lim_{p \to \infty} \|f\|_p$.

427°. Prove that the following sets of functions are dense in $L_p[a, b]$:

(a) the set $S[a, b]$ of all piecewise constant functions;

(b) the set $C[a, b]$ of all continuous functions;

(c) the set $\mathscr{P}$ of all polynomials;

(d) the set $\mathscr{P}_0$ of polynomials equal to zero at the endpoints of the segment.

428. Find the norm of the function $f(x) = x^\alpha$ in those spaces $L_p[0, 1]$, $1 \leq p \leq \infty$, to which it belongs.

429. In $L_1(\mathbf{R}, dx)$ construct:

(a) an infinite-dimensional closed subspace consisting of continuous functions;

(b) an infinite-dimensional closed subspace not containing any nonzero continuous function.

430. Let $\mu(X) < \infty$ and suppose that $V \subset L_1(X, \mu)$ is a closed infinite-dimensional subspace. Prove that V cannot be contained in $L_\infty(X, \mu)$.

431°. Prove that the set $C_0(\mathbf{R})$ of continuous compactly supported functions is dense in $L_p(\mathbf{R}, dx)$ for $1 \leq p < \infty$.

432. Prove that every function $f \in L_p(\mathbf{R}, dx)$ is *continuous in the mean*, i.e., for any $\varepsilon > 0$ there is a $\delta > 0$ such that for $|t| < \delta$

$$\int_{-\infty}^{\infty} |f(x + t) - f(x)|^p \, dx < \varepsilon.$$

433. Prove the assertion of Problem 432 for the space $L_p(\mathbf{R}^n, dx)$.

434*. Let $M \subset L_p(\mathbf{R}^n, dx)$, $1 \leq p < \infty$. Prove that M is pre-compact if and only if:

(a) there is a constant c such that $\|f\|_p \leq c$ for all $f \in M$;

(b) for any $\varepsilon > 0$ there is a number $R(\varepsilon)$ such that

$$\int_{|x| > R(\varepsilon)} |f(x)| \, dx < \varepsilon, \qquad f \in M;$$

(c) for any $\varepsilon > 0$ there is a number $\delta(\varepsilon) > 0$ such that for $|t| < \delta(\varepsilon)$

$$\int_{R^n} |f(x + t) - f(x)|^p \, dx < \varepsilon.$$

435*. Verify the isomorphism of spaces

$$L_1(X, \mu) \mathbin{\hat{\otimes}} L_1(Y, \nu) \approx L_1(X \times Y, \mu \times \nu).$$

436. Find the extreme points of the unit ball in $L_p(X, \mu)$:
(a) for $p = 1$;
(b) for $1 < p < \infty$;
(c) for $p = \infty$.

437*. Prove that the spaces $L_1[0, 1]$ and l_1 are not isomorphic.

2. Spaces of Continuous Functions

438°. Prove that $C(X)$ is a Banach space for any compact space X.

439°. Prove that $C(X)$ is a separable space if X is a compact subset of $\mathbf{R}^n$.

440. Call a linear functional F on $C(X)$ *positive* if $F(f) \geq 0$ for all non-negative functions $f \in C(X)$. Prove that every positive linear functional F is continuous, and its norm is equal to $F(1)$, where 1 is the function identically equal to 1 on X.

441. Prove that any functional $F \in C'(X)$ can be written in the form $F = F_1 - F_2$, where F_1 and F_2 are positive functionals (see Problem 440).

442*. Let X be a compact metric space, and $F \in C'(X)$ a positive functional (see Problem 440). Define $\mu(K) = \inf_{\chi_K \leq \varphi \leq 1} F(\varphi)$ for a compact set $K \subset X$, and $\mu(E) = \sup_{K \subset E} \mu(K)$ (K compact) for a Borel set $E \subset X$. Prove that μ is a countably additive measure.

443°. Compute the norms of the following functionals on $C[-1, 1]$:

(a) $F(f) = \int_0^1 f(x)\, dx$;

(b) $F(f) = \int_{-1}^1 \operatorname{sgn} x f(x)\, dx$;

(c) $F(f) = \int_{-1}^1 f(x)\, dx - f(0)$;

(d) $F_\varepsilon(f) = \dfrac{f(\varepsilon) + f(-\varepsilon) - 2f(0)}{\varepsilon^2}$;

(e) $F(f) = \sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2} f\left(\dfrac{1}{n}\right)$;

(f) $F_n(f) = \int_{-1}^1 f(x)\, dx - \dfrac{1}{2n + 1} \sum_{k=-n}^{n} f\left(\dfrac{k}{n}\right)$.

444°. Express the following functionals on $C[-1, 1]$ as Stieltjes integrals and compute their norms:

(a) $F(f) = f(0)$;
(b) $F(f) = \int_{-1}^{1} f(x)\,dx - 2f(0)$;
(c) $F(f) = \int_0^1 xf(x)\,dx$;
(d) $F(f) = \int_{-1}^{0} f(x)\,dx - 2\int_0^1 f(x)\,dx$.

445. *First theorem of Helly.* Prove that a sequence of functionals $F_n(f) = \int_0^1 f(x)\,dg_n(x)$, with $g_n \in BV[0, 1]$ has a weak-* limit functional $F(f) = \int_0^1 f(x)\,dg(x)$ with $g \in BV[0, 1]$ if and only if $g_n(x) \to g(x)$ at each point of $[0, 1]$ and the variations of the functions g_n are collectively bounded.

446*. *Second theorem of Helly.* Let $M \subset BV[0, 1]$.

Prove that if the functions in M have collectively bounded variation, then each sequence in M contains a subsequence $\{g_n(x)\}$ converging at each point of $[0, 1]$.

447. Let $\mathscr{P}$ be the subspace of polynomials in $C[0, 1]$. Which of the following linear functionals on $\mathscr{P}$ admit continuous extension to $C[0, 1]$ (p denotes the polynomial $\sum_{k=0}^{\deg p} a_k x^k$):

(a) $F_1(p) = a_0$;

(b) $F_2(p) = \sum_{k=0}^{\deg p} a_k$;

(c) $F_3(p) = \sum_{k=0}^{\deg p} (-1)^k a_k$;

(d) $F_4(p) = \sum_{k=0}^{N} c_k a_k$, where $\{c_k\}$ is a fixed vector in $\mathbf{R}^N$?

448°. Let X be a connected compact space. Prove that the unit ball in $C(X)$ has exactly two extreme points.

449*. Prove that the signed point measures $\pm\mu_x$, $x \in X$, defined by the formula $\langle \mu_x, f \rangle = f(x)$ are the extreme points of the unit ball of $C'(X)$.

450.** *Stone–Weierstrass theorem.* Let X be a compact metric space, and $A \subset C(X)$ a closed subalgebra separating points (i.e., for any two distinct points x_1 and x_2 in X there is a function $\varphi \in A$ such that $\varphi(x_1) \neq \varphi(x_2)$) and containing the function identically equal to 1. Prove that $A = C(X)$.

451*. Is the assertion of Problem 450 true for algebras not containing the unit function?

452*. Let X be an arcwise connected compact metric space. Construct a continuous mapping of $[0, 1]$ onto X.

453. Construct a continuous mapping of $[0, 1]$ onto the unit square.

454. Construct an isometric imbedding of $l_p(2, \mathbf{R})$ into $C[0, 1]$ with the aid of a continuous mapping of $[0, 1]$ onto the unit sphere of $l_p(2, \mathbf{R})$.

455. Prove that the spaces $C[0, 1] \check{\otimes} C[0, 1]$ and $C(\square)$ are isomorphic, where $\square$ is the unit square in $\mathbf{R}^2$.

456*. Prove that there is an isomorphism $C(X) \check{\otimes} C(Y) \approx C(X \times Y)$ for any compact subsets X and Y of $\mathbf{R}^n$.

457. Let $A: C(X) \to C(Y)$ be an isomorphism of Banach spaces. Prove that A has the form $(Af)(y) = a(y)f(\varphi(y))$, where a is a continuous function on Y taking the values ± 1, and φ is a homeomorphism of Y onto X.

458. Prove that the space of all functions of the form $f(x) + g(y)$, where $f, g \in C[0, 1]$, is closed in $C(\square)$ ($\square$ is the unit square in $\mathbf{R}^2$).

459.** Prove that the space $C[0, 1]$ has a *countable topological basis* $\{f_n(x)\}$, i.e., a system of functions $\{f_n(x)\}$ such that any $f \in C[0, 1]$ can be uniquely represented as a uniformly convergent series $f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$.

460*. Prove that the system of functions $\{e^{2\pi inx}\}$, $n \in \mathbf{Z}$, is not a topological basis (see Problem 459) in the space $CP[0, 1]$ of all continuous functions on $[0, 1]$ with the condition $f(0) = f(1)$.

3. Spaces of Smooth Functions

461. Let $\mathscr{D}(\mathrm{N})$ be the space of finitely nonzero sequences (i.e., sequences with only finitely many terms different from zero). For any sequence $\alpha = (\alpha_1, \alpha_3, \ldots)$ of positive numbers let the seminorm p_α in $\mathscr{D}(\mathrm{N})$ be defined by the equation

$$p_\alpha(\{x_n\}) = \sum_{k=1}^{\infty} \alpha_k |x_k|.$$

(a) Prove that the collection of seminorms p_α turns $\mathscr{D}(\mathrm{N})$ into a complete nonmetrizable LCS.

(b) Describe the convergence in this space.

(c) Prove that for any nonempty domain Ω there is a closed subspace of $\mathscr{D}(\Omega)$ homeomorphic to $\mathscr{D}(\mathrm{N})$.

462°. Let A be a linear mapping of $\mathscr{D}(\Omega)$ to a locally convex space L. Prove that the following statements are equivalent:

(a) A is a continuous mapping;

(b) A is a *bounded mapping* (i.e., carries bounded sets into bounded sets);

(c) A is *sequentially continuous* (i.e., if $\varphi_n \to 0$ as $n \to \infty$, then $\lim_{n\to\infty} A\varphi_n = 0$);

(d) the restriction of A to any subspace $\mathscr{D}_K(\Omega) \subset \mathscr{D}(\Omega)$ is continuous.

463°. Prove that $\mathscr{D}_K(\Omega)$ is closed in $\mathscr{D}(\Omega)$.

464. Let K be a compact subset of a domain $\Omega \subset \mathbf{R}^n$, and $\{U_i\}$ an open covering of K. Prove that there exist nonnegative functions $\varphi_i \in \mathscr{D}(\Omega)$, $i = 1, \ldots, N$, such that:

(1) $\operatorname{supp} \varphi_i \subset U_i$ for all i;

(2) $\sum_{i=1}^{N} \varphi_i(x) = 1$ for $x \in K$.

The collection $\{\varphi_i\}$ is called a *partition of unity* on K.

465. Prove that $\mathscr{D}(\Omega)$ is dense in $\mathscr{E}(\Omega)$ for any domain $\Omega \subset \mathbf{R}^n$.

466*. Prove that any closed subset of $\mathbf{R}^n$ is the set of zeros of some function $f \in \mathscr{E}(\mathbf{R}^n)$.

467*. Let $\{c_n\}$ be any numerical sequence. Does there exist a function $f \in \mathscr{D}(\mathbf{R})$ for which $f^{(n)}(0) = c_n$, $n = 0, 1, 2, \ldots$?

468*. Prove Theorem 30 in Ch. III for any n by determining functions ψ_δ and χ_N in $\mathscr{D}(\mathbf{R}^n)$ such that:

(a) $\psi_\delta(x) \geq 0$, $\psi_\delta(x) = 0$ for $\|x\| > \delta$, $\int_{\mathbf{R}^n} \psi_\delta(x)\, dx = 1$;

(b) $\chi_N(x) \geq 0$, $\chi_N(x) \equiv 1$ for $\|x\| \leq N$.

469°. Which of the following functions belong to $\mathscr{E}(\mathbf{R})$?

(a) $f(x) = x^k$, $k \in \mathbf{N}$;

(b) $f(x) = e^x$;

(c) $f(x) = |x|$;

(d) $f(x) = (\sin x)/x$;

(e) $f(x) = \begin{cases} 0 & \text{for } x \geq 0, \\ e^{1/x} & \text{for } x < 0; \end{cases}$

(f) $f(x) = \begin{cases} e^{-1/|x|} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$

470°. Which of the following functions belongs to $S(\mathbf{R})$?

(a) $f(x) = e^x$;

(b) $f(x) = 1/(1 + x^2)$;

(c) e^{-x^2};

(d) $x^k e^{-x^2}$, k a natural number

471°. Prove that the operations of differentiation $\partial/\partial x_i$ and of multiplication by the independent variable x_i are continuous operators in the spaces $\mathscr{D}(\mathbf{R}^n)$, $S(\mathbf{R}^n)$, $\mathscr{E}(\mathbf{R}^n)$.

472. Prove that if $f \in \mathscr{D}(\Omega)$, $g \in \mathscr{E}(\Omega)$, then $fg \in \mathscr{D}(\Omega)$. Determine whether the bilinear mapping

$$(f, g) \mapsto fg \quad \text{from } \mathscr{D}(\Omega) \times \mathscr{E}(\Omega) \text{ to } \mathscr{D}(\Omega)$$

(a) is continuous in each variable;

(b) is jointly continuous in the variables;

(c) is *sequentially continuous* in the variables jointly (i.e., if $f_n \to f$ in $\mathscr{D}(\Omega)$, and $g_n \to g$ in $\mathscr{E}(\Omega)$, then $f_n g_n \to fg$ in $\mathscr{D}(\Omega)$).

473. Prove that the sequence $f_n = nx/(n^2 + x^2)$ converges to zero in $\mathscr{E}(\mathbf{R})$.

474°. Let f be a bounded infinitely differentiable function on the line. Determine whether multiplication by f is a continuous operator

(a) in $\mathcal{D}(\mathbf{R})$;
(b) in $S(\mathbf{R})$;
(c) in $\mathcal{E}(\mathbf{R})$;
(d) from $\mathcal{D}(\mathbf{R})$ to $S(\mathbf{R})$;
(e) from $\mathcal{D}(\mathbf{R})$ to $\mathcal{E}(\mathbf{R})$;
(f) from $S(\mathbf{R})$ to $\mathcal{E}(\mathbf{R})$.

475*. (a) Let $G(\mathbf{R}^2)$ be the subspace of $\mathcal{E}(\mathbf{R}^2)$ consisting of the functions φ having the property that

$$\varphi(x + m, y + n) = e^{2\pi i m y}\varphi(x, y), \qquad m, n \in \mathbf{Z}.$$

Prove that the operator A acting according to the formula

$$Af(x, y) = \sum_{k \in \mathbf{Z}} f(x + k)e^{-2\pi i k y}$$

carries $S(\mathbf{R})$ to $G(\mathbf{R}^2)$.

(b) Construct an isomorphism between $S(\mathbf{R}^n)$ and the subspace $G(\mathbf{R}^{2n}) \subset \mathcal{E}(\mathbf{R}^{2n})$ consisting of the functions g such that

$$g(x + p, y + q) = e^{2\pi i p y} g(x, y),\ x, y \in \mathbf{R}^n,\ p, q \in \mathbf{Z}^n.$$

476.** The space $\mathcal{D}(\mathbf{T}^n)$ of infinitely differentiable functions on the n-dimensional *torus* $\mathbf{T}^n \approx \mathbf{R}^n/\mathbf{Z}^n$ is defined as the collection of functions φ on $\mathbf{T}^n$ for which the corresponding functions $\Phi(t_1, \ldots, t_n) = \varphi(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n})$ on $\mathbf{R}^n$ belong to $\mathcal{E}(\mathbf{R}^n)$. Prove the isomorphisms:

$$\mathcal{D}(\mathbf{T}^m) \hat{\otimes} \mathcal{D}(\mathbf{T}^n) \approx \mathcal{D}(\mathbf{T}^m) \underset{\pi}{\otimes} \mathcal{D}(\mathbf{T}^n) \approx \mathcal{D}(\mathbf{T}^{m+n}).$$

477°. (a) Let $f \in \mathcal{D}(\mathbf{R}^n)$. Prove that for any $y \in \mathbf{R}^n$ the net $f_t(x) = [f(x + ty) - f(x)]/t$ has a limit in $\mathcal{D}(\mathbf{R}^n)$ as $t \to 0$.

(b) Let $f \in \mathcal{E}(\mathbf{R}^n)$. Prove that for any $y \in \mathbf{R}^n$ the net $f_t(x) = [f(x + ty) - f(x)]/t$ has a limit in $\mathcal{E}(\mathbf{R}^n)$ as $t \to 0$.

478*. Let $\{\delta_k\}$ be a sequence of positive numbers for which the series $\sum_{k=1}^{\infty} \delta_k$ converges. Define a sequence of functions $\{f_n\}$ on the line by setting $f_0(x) = \operatorname{sgn}\ x$, $f_n(x) = (1/\delta_n) \int_{x-\delta_n}^{x} f_{k-1}(x)\, dx$ for $n \geq 1$. Prove that the sequence f_n converges uniformly to a function $f \in \mathcal{E}(\mathbf{R})$ with the properties:

(a) $f(x) = -1$ for $x < 0$, $f(x) = +1$ for $x > \sum_{k=1}^{\infty} \delta_k$;
(b) $|f^{(n)}(x)| \leq 2^n(\delta_1 \cdots \delta_n)^{-1}$ for all $x \in \mathbf{R}$.

479*. Suppose that a countably normed space L with system of seminorms $\{p_k\}$ has the property that every set bounded with respect to p_{k+1} is pre-compact with respect to p_k.

(a) Prove that L has the Heine–Borel property.

(b) Derive from this that the spaces $\mathcal{D}_K(\Omega)$, $\mathcal{E}(\Omega)$, $S(\mathbf{R}^n)$ have the Heine–Borel property.

480*. Let L be a complete LCS, and Ω a domain in $\mathbf{R}^n$. Denote by $\mathscr{E}(\Omega, L)$ the space of infinitely differentiable vector-valued functions on Ω with values in L. If $\{p_\alpha\}_{\alpha \in A}$ is a set of seminorms determining the topology in L, then the topology in $\mathscr{E}(\Omega, L)$ is defined by the family of seminorms $p_{Kl\alpha}$, where, for K a compact set in Ω, $l = (l_1, \dots, l_n)$ a multi-index, and $\alpha \in A$

$$p_{Kl\alpha}(\varphi) = \sup_{x \in K} p_\alpha(\partial^l \varphi(x)).$$

Prove that $\mathscr{E}(\Omega, L)$ is a complete LCS that is metrizable if L is.

481*. Let Ω_1 be a domain in $\mathbf{R}^n$, Ω_2 a domain in $\mathbf{R}^m$, and $\Omega_1 \times \Omega_2 \subset \mathbf{R}^{m+n}$ their direct product. In the notation of Problem 480 prove the isomorphisms:

$$\mathscr{E}(\Omega_1, \mathscr{E}(\Omega_2)) \approx \mathscr{E}(\Omega_1 \times \Omega_2) \approx \mathscr{E}(\Omega_2, \mathscr{E}(\Omega_1)).$$

482*. Prove the isomorphisms:

$$\mathscr{E}(\Omega_1) \hat{\otimes} \mathscr{E}(\Omega_2) \approx \mathscr{E}(\Omega_1 \times \Omega_2) \approx \mathscr{E}(\Omega_1) \underset{\vee}{\otimes} \mathscr{E}(\Omega_2).$$

483.** Formulate and prove the analogs of the statements in Problems 480–482 for the spaces $\mathscr{D}(\Omega)$ and $S(\mathbf{R}^n)$.

4. Generalized Functions

484°. Prove that the following functionals on $\mathscr{D}(\mathbf{R})$ are singular generalized functions, and find their supports:

(a) $\left(\mathscr{P}\dfrac{1}{x}, \varphi\right) = PV \displaystyle\int_{-\infty}^{\infty} \frac{\varphi(x)}{x}\, dx;$

(b) $\left(\mathscr{P}\dfrac{1}{x^2}, \varphi\right) = PV \displaystyle\int_{-\infty}^{\infty} \frac{\varphi(x) - \varphi(0)}{x^2}\, dx;$

(c) $\left(\mathscr{P}\dfrac{1}{x^3}, \varphi\right) = PV \displaystyle\int_{-\infty}^{\infty} \frac{\varphi(x) - \varphi(0) - x\varphi'(0)}{x^3}\, dx;$

(d) $\left(\mathscr{P}\dfrac{\cos kx}{x}, \varphi\right) = PV \displaystyle\int_{-\infty}^{\infty} \frac{\cos kx}{x}\, \varphi(x)\, dx$

(the symbol $PV \int_{-\infty}^{\infty}$ denotes the *Cauchy principal value* $\lim_{\varepsilon \searrow 0} (\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty})$).

485. Under the conditions of Problem 484(d) find the limit of $\mathscr{P}(\cos kx)/x$ as $k \to \infty$.

486°. Prove that the space $\mathscr{D}(\mathbf{R})$ of test functions is included in the space $\mathscr{D}'(\mathbf{R})$ of generalized functions.

487°. (a) Let $f \in L_1(\mathbf{R}, dx)$, and $f_\varepsilon(x) = \varepsilon^{-1} f(x/\varepsilon)$. Prove that $\lim_{\varepsilon \searrow 0} f_\varepsilon$ exists in $\mathscr{D}'(\mathbf{R})$ and is equal to $c \cdot \delta(x)$, where $c = \int_{-\infty}^{\infty} f(x)\, dx$.

(b) Prove the equalities

$$\lim_{\varepsilon \searrow 0} \frac{1}{\sqrt{\varepsilon}} e^{-x^2/\varepsilon} = \sqrt{\pi}\,\delta(x), \qquad \lim_{\varepsilon \searrow 0} \frac{\varepsilon}{x^2 + \varepsilon^2} = \pi\delta(x).$$

488. Prove the existence of the limits $\lim_{\varepsilon \searrow 0} 1/(x \pm i\varepsilon)$ in the space $\mathscr{D}'(\mathbf{R})$.

489. Find $\lim_{\varepsilon \searrow 0} (\sin(x/\varepsilon))/x$ in $\mathscr{D}'(\mathbf{R})$.

490°. Does $\lim_{\varepsilon \searrow 0} \sin(x/\varepsilon)$ exist in $\mathscr{D}'(\mathbf{R})$?

491°. Suppose that two locally integrable functions f and g in a domain $\Omega \subset \mathbf{R}^n$ determine the same regular generalized function (i.e., $\int_\Omega f(x)\varphi(x)\,dx = \int_\Omega g(x)\varphi(x)\,dx$ for all $\varphi \in \mathscr{D}(\Omega)$).

Prove that f and g coincide almost everywhere in Ω.

492°. Prove that the Dirac δ-function, which is defined by the formula $\int_{-\infty}^{\infty} \delta(x)\varphi(x)\,dx = \varphi(0)$, is not regular.

493. The generalized functions on the n-dimensional torus $\mathbf{T}^n$ are defined as the continuous linear functionals on the space $\mathscr{D}(\mathbf{T}^n)$ (see Problem 476). Prove that the series of regular generalized functions $\sum e^{2\pi ikt}$ (here $t \in \mathbf{R}^n$, $kt = k_1t_1 + \cdots + k_nt_n$, and the summation runs over all $k \in \mathbf{Z}^n$) converges in the sense of $\mathscr{D}'(\mathbf{T}^n)$ to the generalized function $\delta(t)$ defined by the formula $\int_{\mathbf{T}^n} \delta(t)\varphi(t)\,dt = \varphi(0)$.

494°. Prove that each generalized function on the torus $\mathbf{T}^n$ (see Problem 493) has finite order (i.e., can be extended to a continuous linear functional on the space $C^k(\mathbf{T}^n)$ of k-smooth functions on $\mathbf{T}^n$ for some k).

495. Prove that the generalized function F on the line defined by the formula $\langle F, \varphi \rangle = \sum_{k=0}^{\infty} \varphi^{(k)}(k)$ does not have finite order.

496. What is the order of the δ-function?

497. What is the order of the generalized function $\mathscr{P}(1/x)$ (see Problem 484)

(a) in the interval $(-1, 1)$?
(b) in the interval $(1, 2)$?

498. Prove the *Sokhotskii identity* $1/(x \pm i0) = \mathscr{P}(1/x) \mp \pi i\delta(x)$.

499. Prove that the functions $1/(x \pm i0)$ have order 1 in any bounded domain on the line containing 0.

500. (a) Let L be an LCS, and L' the space dual to L, endowed with the weak-* topology. Prove that every continuous linear functional $F \in (L')'$ has the form $F(f) = f(\varphi)$, where $\varphi \in L$.

(b) Prove that the regular generalized functions are weak-* dense in the spaces $\mathscr{E}'(\Omega)$, $\mathscr{D}'(\Omega)$, $S'(\mathbf{R}^n)$.

501°. What are the supports and orders of the generalized functions

(a) $\varphi \mapsto \int_{-1}^{1} |x|\varphi'(x)\,dx$,
(b) $\varphi \mapsto \int_{-1}^{1} (\operatorname{sgn} x)\varphi'(x)\,dx$?

502*. Let $\varphi \in \mathscr{D}(\mathbf{R})$.

(a) Prove that the function

$$f_\varphi(\lambda) = \Gamma(\lambda)^{-1} \int_0^\infty x^{\lambda-1}\varphi(x)\,dx,$$

which is defined for Re $\lambda > 0$, admits an analytic continuation to the left half-plane.

(b) Prove that for fixed $\lambda \in \mathbf{C}$ the correspondence $\varphi \mapsto f_\varphi(\lambda)$ is a generalized function (usually denoted by $x_+^{\lambda-1}/\Gamma(\lambda)$).

(c) Compute the generalized function defined above for the parameter values $\lambda = -n, n = 0, 1, 2, \ldots$.

503*. Let $f(x, y)$ be a real smooth function on the plane. Define a generalized function $F_c \in \mathscr{D}'(\mathbf{R}^2)$ by

$$\langle F_c, \varphi\rangle = \int_{f(x,y)\le c} \varphi(x, y)\,dx\,dy.$$

(a) Prove that if

$$\frac{dF_c}{dc} = \lim_{\varepsilon\to 0}\frac{F_{c+\varepsilon} - F_c}{\varepsilon}$$

exists, then this generalized function is concentrated on the set $f(x, y) = c$.

(b) Prove that dF_c/dc exists if the gradient of f does not vanish on the curve $f(x, y) = c$.

(c) Give an explicit expression for dF_c/dc in the form of a curvilinear integral.

504.** The generalized function $F_c \in \mathscr{D}'(\mathbf{R}^3)$ is defined by

$$\langle F_c, \varphi\rangle = \iiint_{x^2+y^2\le z^2+c} \varphi(x + y + z)\,dx\,dy\,dz.$$

Prove that dF_c/dc exists, and give an explicit expression for this generalized function in terms of surface integrals.

505*. *Kernel theorem.* Prove that every continuous linear mapping $A\colon \mathscr{D}(\Omega_1) \to \mathscr{D}'(\Omega_2)$ has the form

$$\langle A\varphi_1, \varphi_2\rangle = \int_{\Omega_1\times\Omega_2} K(x, y)\varphi_1(x)\varphi_2(y)\,dx\,dy,$$

where $K \in \mathscr{D}'(\Omega_1 \times \Omega_2)$.

506. In the conditions of Problem 505 let $\Omega_1 = \Omega_2 = \mathbf{R}$. Find explicitly the generalized function $K \in \mathscr{D}'(\mathbf{R}^2)$ if the mapping A

(a) is the natural imbedding of $\mathscr{D}(\mathbf{R})$ into $\mathscr{D}'(\mathbf{R})$;

(b) has the form $\varphi \mapsto \varphi(a) \cdot \delta_b$.

5. Operations on Generalized Functions

507°. Compute the derivatives of the following generalized functions:

(a) $\operatorname{sgn} x$;

(b) $\theta(x) = \begin{cases} 0 & \text{for } x \le 0, \\ 1 & \text{for } x > 0; \end{cases}$

(c) $[x]$, the integral part of x.

508°. Compute the second derivatives of the following generalized functions:

(a) $|x|$;

(b) $e^{-a|x|}$;

(c) $|\sin x|$;

(d) $\sin x \cdot e^{-|x+a|}$.

509°. Prove that a generalized function $F \in \mathscr{D}'(\mathbf{R})$ with the property that $F' = 0$ is a constant.

510. Prove that all the solutions of the equation $xF = 0$ among the generalized functions $F \in \mathscr{D}'(\mathbf{R})$ are proportional to the δ-function.

511*. Prove that every generalized function on the line with support at a point $a \in \mathbf{R}$ has the form $P(d/dx)\,\delta_a(x)$, where P is a polynomial.

512. Suppose that $g \in \mathscr{E}(\mathbf{R})$ determines a one-to-one mapping of the line onto itself, and let $h \in \mathscr{E}(\mathbf{R})$ be the inverse mapping. Express the function $\delta'(g(x))$ explicitly in terms of $\delta(x)$ and its derivatives.

513. A generalized function F on the line is said to be *homogeneous of degree* (λ, ε), where $\lambda \in \mathbf{C}$ and $\varepsilon = 0$ or 1, if $F(tx) = |t|^\lambda(\operatorname{sgn} t)^\varepsilon F(x)$ for $t \neq 0$.

Prove that the following functions are homogeneous, and find their degrees of homogeneity:

(a) $F(x) = |x|^\lambda$, $\operatorname{Re} \lambda > -1$;

(b) $F(x) = \operatorname{sgn} x$;

(c) $F(x) = \delta(x)$;

(d) $F(x) = \mathscr{P}(1/x)$ (see Problem 484);

(e) $F(x) = \delta'(x)$.

514. Prove that a homogeneous function of degree (λ, ε) on the line (see Problem 513) satisfies the differential equation $xF' = \lambda F$.

515.** Prove that for all $\lambda \in \mathbf{C}$ and $\varepsilon = 0, 1$ there exists a homogeneous generalized function on the line of degree (λ, ε), and that it is unique to within a numerical factor.

516*. Prove that the only generalized functions on the line that are translation invariant are the constants.

517*. Suppose that F is a generalized function on the plane that is invariant under translations along the axis Ox.

(a) Prove that there exists a generalized function f on the line such that $\langle F, \varphi\rangle = \langle f, \int_{\mathbf{R}} \varphi(x, y)\, dx\rangle$.

(b) Express F as a direct product.

518*. Suppose that F is a generalized function on the plane with support the segment $[0, 1]$ of the axis Ox.

(a) Prove that there exist a number N and generalized functions $f_0, f_1, \ldots, f_N$ on the line such that

$$\langle F, \varphi\rangle = \sum_{i=0}^{N} \left\langle f_i, \left.\frac{\partial^i \varphi}{\partial y^i}\right|_{y=0}\right\rangle.$$

(b) Formulate this assertion in terms of direct products of generalized functions.

519*. Prove that the regular generalized function $f(x) = \exp(ie^x)$ is in $S'(\mathbf{R})$, but that its derivative in the sense of generalized functions does not coincide with $f'(x) = ie^x e^{ie^x}$.

520. Solve the equation $(\sin x) \cdot f(x) = 0$ in the generalized functions on the line.

521*. Suppose that the generalized function F on $\mathbf{R}^n$ satisfies the relation $(\sum_i x_i^2 - R^2)F(x) = 0$ and is invariant under rotations of $\mathbf{R}^n$. Prove that $\langle F, \varphi\rangle = c\int_{S_R} \varphi(x)\, d\sigma(x)$, where c is a constant and σ is the rotation invariant surface element on the sphere S_R of radius R in $\mathbf{R}^n$.

522. Suppose that the generalized function F on the line has the following properties:

(a) $F(x + 1) = F(x)$;

(b) $e^{2\pi i x}F(x) = F(x)$.

Prove that $F(x) = c\sum_{k\in\mathbf{Z}} \delta(x - k)$.

523. Prove that every generalized function F on the line satisfying the equation $F'(x) = a(x)F(x) + b(x)$, a, $b \in \mathscr{E}(\mathbf{R})$, is regular (and, hence, coincides with the ordinary solution of this equation).

524.** Suppose that F is a generalized function on $\mathbf{R}^n$ such that all the partial derivatives of the form $\partial^k F/\partial x_i^k$, $1 \le i \le n$, $0 \le k \le r$, belong to $L_2(\mathbf{R}^n, dx)$. Prove that for $r > n/2 \pm l$ the function F coincides almost everywhere with a function in the class $C^l(\mathbf{R}^n)$.

525*. Under what conditions on the coefficients $\{c_n\}$ does the series $\sum_{n\in\mathbf{Z}} c_n e^{2\pi i n x}$ converge in $\mathscr{D}'(\mathbf{R})$?

526. Compute the sums:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$;

(b) $\displaystyle\sum_{n\in\mathbf{Z}} \frac{e^{inx}}{a^2 + n^2}$;

(c) $\displaystyle\sum_{n\in\mathbf{Z}} n^k e^{inx}, \qquad k = 0, 1, 2, \ldots;$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{\sin(2n + 1)x}{2n + 1}$.

527. Is it possible to define a multiplication operation in $\mathscr{D}'(\mathbf{R})$ in such a way that it is continuous in each variable and coincides with ordinary multiplication on regular functions?

528*. Compute the generalized derivatives:

(a) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \ln(x^2 + y^2)$;

(b) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(x^2 + y^2 + z^2)^{-1/2}$.

529. Let $\mathscr{P}$ be a polynomial in n variables. The generalized function $\delta(\mathscr{P}(x))$ is defined on $\mathbf{R}^n$ as the limit (if it exists) of the sequence $\varphi_k(\mathscr{P}(x))$, where $\{\varphi_k\}$ is a δ-shaped sequence in $\mathscr{D}(\mathbf{R})$. Compute $\delta(\mathscr{P}(x))$ in the following examples:

(a) $n = 1$, $\mathscr{P}(x) = x^2 + px + q$;
(b) $n = 2$, $\mathscr{P}(x, y) = x^2 + y^2 - 1$;
(c) $n = 2$, $\mathscr{P}(x, y) = x^2 - y^2$;
(d) $n = 3$, $\mathscr{P}(x, y, z) = x^2 + y^2 - z^2$.

§4. Hilbert Spaces

1. The Geometry of Hilbert Spaces

530. (a) Prove that the correspondence $L \to \tilde{L}$ constructed in §4.1 of Ch. III of the *Theory* part defines a covariant functor from the category of pre-Hilbert spaces to the category of Hilbert spaces.

(b) Show that $\tilde{L}$ is a universal object in a suitable category.

531°. Prove that the system of functions $e_n(x) = e^{2\pi inx}$, $n \in \mathbf{Z}$, is a Hilbert basis in $L_2(0, 1)$.

532*. Apply the orthogonalization process to the sequence of monomials $1, x, x^2, \ldots$ in the following Hilbert spaces:

(a) $L_2([-1, 1], dx)$; (b) $L_2\left([-1, 1], \frac{dx}{\sqrt{1 - x^2}}\right)$;

(c) $L_2([0, \infty), e^{-x}\, dx)$; (d) $L_2((-\infty, \infty), e^{-x^2}\, dx)$.

533. Apply the orthogonalization process to the sequence of monomials $1, z, z^2, \ldots$ in the pre-Hilbert spaces of polynomials with the following scalar products:

(a) $(P, Q) = \displaystyle\iint_{|z| \le R} P(z)\overline{Q(z)}\, dx\, dy$;

(b) $(P, Q) = \displaystyle\iint_{\mathbf{C}} P(z)\overline{Q(z)} e^{-|z|^2}\, dx\, dy$.

534*. Express the linear functional $F_x(f) = f(x)$ in the spaces of Problem 533 in the form of a scalar product.

535*. Prove that the completions of the pre-Hilbert spaces in Problem 533 consist of all the analytic functions in the disk of radius R that are square integrable with respect to the measure $dx\,dy$ in case (a), and of all the analytic functions on the plane that are square integrable with respect to the measure $e^{-|z|^2}\,dx\,dy$ in case (b).

536. Find the expansion coefficients with respect to the basis of Problem 531 for the following functions:

(a) $f(x) = \operatorname{sgn}(2x - 1)$;
(b) $f(x) = e^{\lambda x}$;
(c)* $f(x) = B_k(x)$, where $\{B_k\}$ is the sequence of *Bernoulli polynomials*, which is uniquely determined by the conditions
(1) $B_k'(x) = kB_{k-1}(x)$;
(2) $B_k(0) = B_k(1)$ for $k > 1$;
(3) $B_1(x) = x - (1/2)$.

537*. Prove that the orthogonal complement of the system of functions $e_n(x) = e^{2\pi inx}$, $n \in \mathbf{Z}$, in the space $L_2([a, b], dx)$
(a) is equal to zero for $|b - a| \le 1$,
(b) is different from zero for $|b - a| > 1$.

538*. Let B_0 be the space of all trigonometric polynomials (i.e., finite linear combinations of the functions $e^{i\lambda x}$, $\lambda \in \mathbf{R}$). Define in B_0 the scalar product

$$(f, g) = \lim_{A \to +\infty} \frac{1}{2A} \int_{-A}^{A} f(x)\overline{g(x)}\,dx.$$

(a) Prove that the corresponding Hilbert space B is nonseparable and has the continuum Hilbert basis $\{e^{i\lambda x}\}_{\lambda \in \mathbf{R}}$.

(b)** Prove that B contains the space of *almost periodic continuous functions*, i.e., the closure of B_0 in the uniform norm $\|f\| = \sup_{x \in \mathbf{R}} |f(x)|$, but does not coincide with it.

539*. Let μ be the measure on the line defined by the formula $\mu(A) = \operatorname{card} A$ (the number of points in A). Prove that $L_2(\mathbf{R}, \mu)$ is nonseparable and isomorphic to the space of Problem 538.

540. Prove that $L_2[a, b]$ contains complete orthonormal systems consisting of:

(a) polynomials;
(b) step functions;
(c) trigonometric polynomials;
(d) functions lying in any given dense subspace.

541. The *system of Haar functions* $\{\varphi_{mn}\}$, $m \in \mathbf{N}$, $1 \le n \le 2^m$, is defined by the formula

$$\varphi_{mn}(x) = \begin{cases} 2^{m/2} & \text{if } \dfrac{n-1}{2^m} \le x < \dfrac{n-1/2}{2^m}, \\ -2^{m/2} & \text{if } \dfrac{n-1/2}{2^m} \le x < \dfrac{n}{2^m}, \\ 0 & \text{if } x \notin \left[\dfrac{n-1}{2^m}, \dfrac{n}{2^m}\right). \end{cases}$$

Prove that this system is an orthonormal basis in $L_2(0, 1)$.

542. The *system of Rademacher functions* $\{\varphi_m(x)\}$, $m \in \mathbf{N}$, is defined by the formula $\varphi_m(x) = (-1)^{[2^m x]}$, where [ ] denotes the integral part of a number.

(a) Prove that the Rademacher system is orthonormal but not dense in $L_2(0, 1)$.

(b) Prove that the *system of Walsh functions*, defined by the formula $\varphi_{m_1 \cdots m_n}(x) = \varphi_{m_1}(x) \cdot \ldots \cdot \varphi_{m_n}(x)$, where $m_1 < m_2 < \cdots < m_n$, and the φ_{m_i} are Rademacher functions, is orthonormal and complete in $L_2(0, 1)$.

543. Find the orthogonal complements in $L_2(1, 0)$ of the following sets:

(a) the polynomials in x;

(b) the polynomials in x^2;

(c) the polynomials with zero free term;

(d) the polynomials with sum of coefficients equal to zero.

544°. Find the orthogonal complements of the following spaces in the pre-Hilbert space $C[-1, 1]$ of all continuous functions on $[-1, 1]$ with the scalar product $(f, g) = \int_{-1}^{1} f(x)\overline{g(x)}\,dx$:

(a) the functions equal to zero for $x \le 0$;

(b) the functions equal to zero at the point $x = 0$.

Is the theorem on orthogonal complements true in these cases?

545°. Compute the angles of the triangle formed by the following points in $L_2(-1, 1)$:

$$f_1(x) \equiv 0, f_2(x) \equiv 1, f_3(x) \equiv x.$$

546. Let e_t be the characteristic function of the closed interval $[0, t]$, $t \ge 0$. Find the angles between two chords of the curve e_t in $L_2(0, \infty)$:

(a) if the chords have a common endpoint and opposite directions with respect to t;

(b) if the chords have a common endpoint and the same direction with respect to t.

547. (a) Prove that in every pre-Hilbert space the *parallelogram law* is satisfied:

$$\|x - y\|^2 + \|x + y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

(b)* Prove that in every normed linear space over **R** or **C** in which this law is satisfied it is possible to introduce a scalar product such that $\|x\|^2 = (x, x)$.

548. Prove that in a complex Hilbert space the following equalities hold:
(a) $(x, y) = (1/N)\sum_{k=1}^{N} \|x + e^{2\pi ik/N}y\|^2 e^{2\pi ik/N}$ for $N \geq 3$;
(b) $(x, y) = (1/2\pi)\int_0^{2\pi} \|x + e^{i\theta}y\|^2 e^{i\theta}\, d\theta$.

549. Suppose that the family $\{x_n\}$ of vectors in a Hilbert space H has the properties that $\|x_n\| = 1$, $(x_n, x_m) = c$ for $m \neq n$. Prove that the sequence $\{x_n\}$ has a weak limit.

550. Prove that the following conditions are equivalent for an orthogonal system $\{x_n\}$ of vectors:
(a) $\sum_n x_n$ is strongly convergent;
(b) $\sum_n x_n$ is weakly convergent;
(c) $\sum_n \|x_n\|^2$ converges.

551°. Let S be any subset of a Hilbert space H. Prove that $(S^\perp)^\perp$ coincides with the closure of the linear span of S.

552. Let L be a subspace of a Hilbert space H, and f_0 a continuous linear functional on L. Prove that f_0 has a unique extension to H with the same norm.

2. Operators on a Hilbert Space

553°. (a) Prove that every operator A on a complex Hilbert space H has a unique representation in the form $A = B + iC$, where B and C are Hermitian operators. (This is sometimes written $B = \operatorname{Re} A$, $C = \operatorname{Im} A$.)
(b) Verify that A is normal if and only if $\operatorname{Re} A$ and $\operatorname{Im} A$ commute.
(c) Prove that A is unitary if and only if it is normal and $(\operatorname{Re} A)^2 + (\operatorname{Im} A)^2 = 1$.

554. Prove that:
(a) every operator P on a Hilbert space having the property that $P^2 = P^* = P$ is an orthogonal projection, i.e., a projection onto a closed subspace H_1 parallel to its orthogonal complement H_2;
(b) every operator S having the property $S^{-1} = S^* = S$ is an orthogonal reflection in some subspace.

555. Prove the equality $\|A^*A\| = \|AA^*\| = \|A\|^2 = \|A^*\|^2$ for any bounded operator A on a Hilbert space H.

556. Let A be a positive operator on a pre-Hilbert space H, and x any vector in H.
(a) Prove that the numerical sequence $a(k) = \ln(A^k x, x)$ is convex, i.e., $a[(k + l)/2] \leq [a(k) + a(l)]/2$.
(b) Prove the inequality $\|Ax\|^2 \leq Q_A(x) \cdot \|A\|$, where $Q_A(x) = (Ax, x)$.

557. Let $\{A_n\}$ be a monotone bounded sequence of operators on a Hilbert space H. Prove that $\text{s-lim}_{n\to\infty} A_n$ exists.

558. Suppose that H_1 is a closed subspace of a Hilbert space H, $H_2 = H_1^{\perp}$, P is the orthogonal projection onto H_1, and A is an operator on H. Express the following statements in the form of algebraic relations between A and P:

(a) H_1 is invariant under A;

(b) H_1 and H_2 are invariant under A.

559. A pair (L_1, L_2) of finite-dimensional subspaces of a Hilbert space H (over $\mathbf{R}$ or $\mathbf{C}$) is said to be *congruent* to a pair (M_1, M_2) if there exists a unitary operator U on H that carries L_1 to M_1 and L_2 to M_2. If such a congruence exists, then $\dim L_1 = \dim M_1$, $\dim L_2 = \dim M_2$. The latter conditions are assumed to hold in what follows. Let P_i be the orthogonal projection onto L_i, and Q_i the orthogonal projection onto M_i, $i = 1, 2$.

(a) Suppose that $\dim L_i = \dim M_i = 1$. Prove that the pairs (L_1, L_2) and (M_1, M_2) are congruent in a real space if and only if the angles between the vectors generating these spaces are equal.

(b) Express the angle between the vectors generating the spaces L_1 and L_2 in terms of the projections P_1 and P_2.

(c) Find a criterion for the congruence of two pairs of one-dimensional subspaces of a complex space.

560. In the notation of Problem 559 let $\lambda_1, \ldots, \lambda_n, \ldots$ be the eigenvalues of the operator $P_1 P_2 P_1$.

(a) Prove that the λ_i are real numbers between 0 and 1.

(b) Prove that there are not more than $k = \min(\dim L_1, \dim L_2)$ nonzero numbers λ_i. Arrange them in decreasing order and set $\varphi_i = \arccos\sqrt{\lambda_i}$, $i = 1, 2, \ldots, k$. The numbers φ_i are called the *angles between the subspaces* L_1 and L_2.

(c) If $\dim L_1 = 1$, $\dim M_1 = 1$, then there is a unique angle φ_1 between L_1 and L_2 and a unique angle ψ_1 between M_1 and M_2. Prove that the pairs (L_1, L_2) and (M_1, M_2) are congruent if and only if $\varphi_1 = \psi_1$.

(d)* Prove the following general criterion for congruence: The pair (L_1, L_2) is congruent to the pair (M_1, M_2) if and only if the angles between L_1 and L_2 are equal to the corresponding angles between M_1 and M_2.

(e) The *opening between the subspaces* L_1 and L_2 is defined to be the number $\|P_1 - P_2\|$. Express this number in terms of the angles between L_1 and L_2.

561°. Prove that for an operator $U \in \mathscr{L}(H_1, H_2)$ to be unitary it is necessary that U carry any Hilbert basis in H_1 into a basis in H_2, and sufficient that U carry some basis in H_1 into a basis in H_2.

562°. For any operator A on a Hilbert space prove the relations:

(a) $(\operatorname{im} A)^{\perp} = \ker A^*$;

(b) $(\ker A)^{\perp} = \overline{(\operatorname{im} A^*)}$

(the bar denotes closure).

563. Suppose that the sequence $\{A_n\}$ of operators on a Hilbert space H converges weakly to A and, moreover, that $\|A_n x\| \to \|Ax\|$ for any $x \in H$.

Prove that $\{A_n\}$ converges strongly to A. In particular, this means that strong and weak convergence to a unitary limit coincide for sequences of unitary operators.

564. Let $\mathscr{L}(H)$ be an algebra of bounded operators on a Hilbert space H. Prove that every automorphism of $\mathscr{L}(H)$ that commutes with the operation of taking adjoints has the form $A \mapsto UAU^{-1}$, where U is a unitary operator in H.

565*. Find all the norm-closed ideals in the algebra $\mathscr{L}(H)$ (see Problem 564) for a separable Hilbert space H.

566. Derive the following *interlacing principle* from the Courant theorem. Let A be a compact Hermitian operator on a Hilbert space H, H_1 a closed subspace of codimension 1 in H (i.e., $\dim H_1^\perp = 1$), and P the orthogonal projection onto H_1. Then the eigenvalues $\{\lambda_i\}$ of A and the eigenvalues $\{\mu_i\}$ of PAP, numbered as in Courant's theorem, satisfy

$$\lambda_{-1} \le \mu_{-1} \le \lambda_2 \le \cdots \le 0 \le \cdots \le \mu_2 \le \lambda_2 \le \mu_1 \le \lambda_1.$$

567*. (a) Prove that every positive operator A on a Hilbert space has a positive square root B, which can be obtained as the strong limit of the sequence B_n determined by the initial condition $B_0 = 0$ and the recursion formula $B_{n+1} = B_n + [(A - B_n^2)/2\sqrt{\|A\|}]$.

(b) Prove that the positive square root B of a positive operator A is unique.

568. (a) Prove that every operator A on a finite-dimensional Hilbert space H admits representations in the form $A = RU$ and $A = VS$, where R and S are positive operators, and U and V are unitary operators. These expressions are called *polar decompositions* of A. (In the case $\dim H = 1$ this reduces to the expression of a complex number a in the form $re^{i\varphi}$.)

(b) Prove that R and S are uniquely determined by A. Is this true for U and V?

(c) Verify that the one-sided shift operator T on l_2 does not admit a polar decomposition in the sense of part (a).

569. Let U be an operator on a Hilbert space H, H_1 the orthogonal complement of $\ker U$, and H_2 the closure of $\operatorname{im} U$. The operator U is called a *partial isometry* if it maps H_1 isometrically onto H_2. Express this property in terms of the orthogonal projections P_1, P_2 onto H_1, H_2.

570. Let A be an operator on a Hilbert space H. Prove that there is a unique representation of A in the form $A = RU$, where R is a positive operator and U is a partial isometry (see Problem 569) such that $\ker U = \ker A$. This representation is also called the *polar decomposition* of A (cf. Problem 568).

571. An operator A is called a *Hilbert–Schmidt operator* if the series $\sum_{\beta\in B} \|Ax_\beta\|^2$ converges for some Hilbert basis $\{x_\beta\}_{\beta\in B}$.

(a) Prove that if A is a Hilbert–Schmidt operator, then the quantity $\|A\|_2 = (\sum_{\beta \in B} \|Ax_\beta\|)^{1/2}$ does not depend on the choice of basis and defines in the space $\mathscr{L}_2(H)$ of Hilbert–Schmidt operators a norm which majorizes the usual operator norm.

(b) Prove that the norm $\|\cdot\|_2$ is generated by the scalar product $\langle A, B\rangle_{\mathscr{L}_2(H)} = \sum_{\gamma \in \Gamma} (Ay_\gamma, By_\gamma)_H$, where $\{y_\gamma\}_{\gamma \in \Gamma}$ is any Hilbert basis in H.

(c) Prove that every Hilbert–Schmidt operator is compact.

(d) Construct an isomorphism between $\mathscr{L}_2(H)$ and the Hilbert tensor product of H and H'.

(e) Prove that every Hilbert–Schmidt operator on $L_2(X, \mu)$ is an integral operator with kernel $K \in L_2(X \times X, \mu \times \mu)$.

572. An operator A in a Hilbert space H is said to be *nuclear* if it can be represented in the form $A = BC$, where B and C are Hilbert–Schmidt operators.

Prove that:

(a) if A is a nuclear operator, then for any basis $\{y_\gamma\}_{\gamma \in \Gamma}$ in H the series $\sum (Ay_\gamma, y_\gamma)$ converges, and its sum does not depend on the choice of basis (it is denoted by tr A and called the *trace of the operator* A);

(b) if A is a nuclear operator and B is a bounded operator, then AB and BA are nuclear, and $\operatorname{tr} AB = \operatorname{tr} BA$;

(c) the collection of all nuclear operators forms a Banach space $\mathscr{L}_1(H)$ with respect to the norm $\|\cdot\|_1$ defined by the formula $\|A\| = \operatorname{tr} R$, where $A = RU$ is the polar decomposition of A;

(d) the isomorphisms $\mathscr{K}(H)' \approx \mathscr{L}_1(H)$, $\mathscr{L}_1(H)' \approx \mathscr{L}(H)$ are valid, where $\mathscr{K}(H)$ is the space of compact operators on H with the usual norm.

573. Find the eigenvectors and eigenvalues of the integral operator A on $L_2[0, 1]$ given by the formula $(Af)(x) = \int_0^1 K(x, y)f(y)\,dy$ if:

(a) $K(x, y) = \cos 2\pi(x - y)$;

(b) $K(x, y) = \min(x, y)$.

574*. Let H be a Hilbert space, and X a set with measure μ. A collection of unit vectors $\{\xi_x\}_{x \in X}$ in H is called a *continuous basis* (or a coherent or overfilled system) if for any $\xi \in H$ the function $x \mapsto (\xi, \xi_x)$ is μ-measurable and

$$\|\xi\|^2 = \int_X |(\xi, \xi_x)|^2\, d\mu(x).$$

(a) Construct continuous bases in the spaces of Problem 533.

(b) Prove that the mapping $\xi \mapsto (\xi, \xi_x)$ is an isometry of H into $L_2(X, \mu)$.

(c) Prove that $\dim H = \mu(X)$.

(d)** Prove that $\operatorname{tr} A = \int_X (A\xi_x, \xi_x)\, d\mu(x)$ for a nuclear operator A.

575.** Let A be a nuclear integral operator on $L_2[0, 1]$ with continuous kernel $K(x, y)$. Prove the identity

$$\operatorname{tr} A = \int_0^1 K(x, x)\, dx.$$

Chapter IV

The Fourier Transformation and Elements of Harmonic Analysis

§1. Convolutions on an Abelian Group

1. Convolutions of Test Functions

576. Let G be a finite group, and K some field.

(a) Prove that the center of $K[G]$ consists of the functions $a \in K[G]$ having the property that $a(gh) = a(hg)$ for any $h, g \in G$.

(b) A *conjugacy class* of elements in G is defined to be a set of the form $C_h = \{ghg^{-1}, g \in G\}$. Prove that the number of distinct conjugacy classes in G is equal to the dimension of the center of $K[G]$.

(c) Suppose that the algebra $K[G]$ is commutative. Does it follow that G is an abelian group?

577. Let C_n be the cyclic group of order n.

(a) Prove that $\mathbf{C}[C_n]$ is isomorphic to the direct sum of n copies of the field $\mathbf{C}$.

(b) Is the analogous assertion true for the field $\mathbf{R}$?

578*. Prove that $\mathbf{R}[S_3]$ is isomorphic to $\mathbf{R} + \mathbf{R} + \mathrm{Mat}_2\,\mathbf{R}$.

579*. Prove that the natural imbedding of G into $K[G]$ is the universal object for the category of mappings φ of the group G into associative K-algebras with unit such that $\varphi(g_1g_2) = \varphi(g_1)\varphi(g_2)$, $\varphi(1) = 1$ (a morphism of an object $\varphi\colon G \to A$ to an object $\psi\colon G \to B$ is defined to be a homomorphism $\chi\colon A \to B$ such that the diagram

$$\begin{tikzcd} & A \arrow[d, "\chi"] \\ G \arrow[ur, "\varphi"] \arrow[dr, "\psi"'] & \\ & B \end{tikzcd}$$

commutes). Can the condition $\varphi(1) = 1$ in this definition be omitted?

580°. Compute explicitly the convolution in $L_1(\mathbf{R}, dx)$ of two characteristic functions of closed intervals.

581°. Let $G = \mathbf{R}^n$ or $\mathbf{T}^n$. Prove that the convolution of two bounded functions in $L_1(G, \mu)$ is a continuous function.

582. Compute the convolution $f_1 * f_2$ in $L_1(\mathbf{R}, dx)$ if:
(a) $f_1(x) = 1/(x^2 + a^2)$, $f_2(x) = 1/(x^2 + b^2)$;
(b) $f_1(x) = e^{-x^2/2a}$, $f_2(x) = e^{-x^2/2b}$.

583°. Suppose that the function φ is compactly supported and k times continuously differentiable in $\mathbf{R}^n$, and that the function f belongs to $L_1(\mathbf{R}^n, dx)$. Prove that the convolution $\varphi * f$ has a continuous derivatives up to order k.

584. Suppose that L is a Banach space of functions on a locally compact abelian group G with invariant measure μ, with a norm $\|\cdot\|_L$ having the properties:
(1) $\|T(a)f\|_L = \|f\|_L$, where $T(a)$ is the translation operator $T(a)f(x) = f(x + a)$;
(2) $\|T(a)f - f\|_L \to 0$ as $a \to 0$ in G.
Prove that the operator $S(\varphi)$, $\varphi \in L_1(G, \mu)$, carries L into itself and has a norm in this space not exceeding $\|\varphi\|_L$.

585. A sequence $\{f_k\}$ of functions on a topological space X with a Borel measure μ is said to be *δ-shaped* for a point $a \in X$ if:
(1) $f_k(x) \geq 0$ on X;
(2) $\int_X f_k(x)\, d\mu(x) = 1$;
(3) $\int_{X \setminus U} f_k(x)\, d\mu(x) \to 0$ for any neighborhood U of a as $k \to \infty$.
Prove that the following sequences are δ-shaped for the point $0 \in \mathbf{R}^n$:
(a)* $f_n(x) = k^n\varphi(kx)$, where φ is any Borel function on $\mathbf{R}^n$ with the properties (1) and (2);
(b) $f_n(x) = \begin{cases} c_k(1 - \|x\|^2)^k & \text{for } \|x\| \leq 1, \\ 0 & \text{for } \|x\| > 1, \end{cases}$
where $\{c_k\}$ is a suitable sequence of constants.

586. Let G be a topological abelian group with invariant measure μ, $\{f_k\}$ a δ-shaped sequence for the point $a \in G$ (see Problem 585), and L a Banach space of functions on G satisfying the conditions of Problem 584.
Prove that $S(f_k) \to T(a)$ as $k \to \infty$.

587. Suppose that the function φ on $\mathbf{R}^n$ coincides inside the ball of radius R with some polynomial and equals zero outside this ball, and that the function $\psi \in L_1(\mathbf{R}^n, dx)$ has support in the ball of radius $r < R$.
Prove that the convolution $\varphi * \psi$ has support in the ball of radius $R + r$ and coincides with some polynomial in the ball of radius $R - r$. (All these balls have center at 0.)

588*. *Weierstrass theorem.* Prove that every continuous function on $\mathbf{R}^n$ can be approximated uniformly by polynomials on any compact set.

589. Let $f \in L_1(G, \mu)$. Prove that $S(f)$ is a bounded operator in the Hilbert space $L_2(G, \mu)$. Compute the adjoint operator $S(f)^*$.

590. Let f_1 and f_2 be in $L_2(G, \mu)$. Prove that the convolution $f_1 * f_2$ is defined and belongs to $L_\infty(G, \mu)$.

591. Define an operation $*$ in the space of functions on the group G by the equation $f^*(x) = \overline{f(-x)}$.

Prove that

(a) $f_1^* * f_2^* = (f_1 * f_2)^*$ for $f_1, f_2 \in L_1(G, \mu)$;

601°. Prove the identity $f * 1 = \langle f, 1 \rangle \cdot 1$ for any generalized function $f \in \mathscr{E}'(\mathbf{R})$.

602°. Let $f_1 = \chi_{[a,b]}, f_2 = \chi_{[c,d]}$. Compute $(f_1 * f_2)$.

603. Let $\check{f}(x) = f(-x)$. Prove the identity $(f_1 * f_2)\check{} = \check{f}_1 * \check{f}_2$, where f_1 and f_2 are generalized functions, one of which has compact support.

604*. Let $f \in S'(\mathbf{R}^n)$, $\varphi \in S(\mathbf{R}^n)$. The convolution $f * \varphi$ is defined by the formula $f * \varphi = S(\check{\varphi})'f$, i.e., $\langle f * \varphi, \psi \rangle = \langle f, \varphi * \psi \rangle$). Prove that:

(a) $f * \varphi$ is a regular generalized function;

(b) $(f * \varphi)(x) = \langle f, T(-x)\check{\varphi} \rangle$;

(c) $(f * \varphi)(x)$ is not of greater than polynomial growth in $|x|$ at infinity.

605*. Let $f \in \mathscr{E}'(\mathbf{R}^n)$, $\varphi \in \mathscr{E}(\mathbf{R}^n)$. The convolution $f * \varphi$ is defined by the formula $f * \varphi = S(\check{\varphi})f$ (cf. Problem 604). Prove that:

(a) $f * \varphi$ is a regular generalized function;

(b) $(f * \varphi)(x) = \langle f, T(-x)\check{\varphi} \rangle$;

(c) $f * \varphi \in \mathscr{E}(\mathbf{R}^n)$;

(d) the operator $S(f)\colon \mathscr{E}(\mathbf{R}^n) \to \mathscr{E}(\mathbf{R}^n)$ is continuous.

606*. Let $f \in \mathscr{D}'(\mathbf{R}^n)$. Prove that the operator $S(f)\colon \mathscr{D}(\mathbf{R}^n) \to \mathscr{E}(\mathbf{R}^n)$ is continuous.

607. Let $\mathscr{E}(\mathbf{T}^n)$ be the space of infinitely differentiable functions on the torus $\mathbf{T}^n$ (with the topology of uniform convergence of all derivatives), and $\mathscr{E}'(\mathbf{T}^n)$ the dual space of generalized functions. Define a convolution operation in $\mathscr{E}'(\mathbf{T}^n)$ and prove that $\mathscr{E}'(\mathbf{T}^n) * \mathscr{E}(\mathbf{T}^n) \subset \mathscr{E}(\mathbf{T}^n)$.

608. Let $\varphi_N(t) = (\sin(2N + 1)\pi t/\sin \pi t)^2/(2N + 1)$. Prove that

$$\lim_{N \to \infty} \varphi_N(t) = \delta(t).$$

609. Prove that the space of trigonometric polynomials is dense in $\mathscr{E}(\mathbf{T})$.

610. Prove the statement analogous to that in Problem 609 for $\mathscr{E}(\mathbf{T}^n)$.

611°. Express the following difference operators as convolutions:

(a) $A_h f(x) = [f(x + h) - f(x - h)]/2h$,

(b) $B_h f(x) = [f(x + h) + f(x - h) - 2f(x)]/h^2$,

and find their limits as $h \to 0$.

612. (a) Find the kth derivative of the Bernoulli polynomial $B_k(t)$ as a generalized function on $\mathbf{T}$.

(b) Prove the identity $B_k * B_l = -[k!l!/(k + l)!]B_{k+l}$ for $k > 0, l > 0$.

613. Suppose that the generalized function f on $\mathbf{T}$ is given by the formula

$$\langle f, \varphi \rangle = \mathscr{P}\int_0^1 (\tan \pi t)\varphi(t)\, dt = \int_0^{1/2} (\tan \pi t)[\varphi(t) - \varphi(1 - t)]\, dt.$$

(a) Compute the convolution $f * e_k$, where $e_k(t) = e^{2\pi i k t}$.

(b) Compute the convolution $f * f$.

614*. Let

$$f_1(x, y) = \delta(x^2 + y^2 - r_1^2), \qquad f_2(x, y) = \delta(x^2 + y^2 - r_2^2).$$

Compute the convolution $f_1 * f_2$ in $\mathscr{E}'(\mathbf{R}^2)$.

615*. Let f_1 and f_2 be compactly supported continuous functions on the half-line $[0, \infty)$. Define $F_i = f_i(\sqrt{x^2 + y^2})$.

Prove that the convolution $F = F_1 * F_2$ also has the form $F(x, y) = f(\sqrt{x^2 + y^2})$, where f is a compactly supported continuous function on $[0, \infty)$, and give an explicit expression for f in terms of f_1 and f_2.

616*. Let $\mathscr{E}_{\pm}(\mathbf{R})$ be the subspaces of $\mathscr{E}(\mathbf{R})$ consisting of the functions with support bounded to the left or to the right, and $\mathscr{D}'_{\pm}(\mathbf{R})$ the analogous subspaces of $\mathscr{D}'(\mathbf{R})$.

(a) Verify the isomorphism $\mathscr{E}'_{\pm}(\mathbf{R}) = \mathscr{D}'_{\mp}(\mathbf{R})$ (convergence $\varphi_n \to \varphi$ in $\mathscr{E}_{\pm}(\mathbf{R})$ is defined by the conditions: the supports supp φ_n are bounded to the one side by a common constant; $\varphi_n \to \varphi$ in the sense of $\mathscr{E}(\mathbf{R})$).

(b) Define a convolution operation in $\mathscr{D}'_{+}(\mathbf{R})$.

(c) Prove that $\mathscr{E}_{\pm}(\mathbf{R}) * \mathscr{D}'_{\pm}(\mathbf{R}) \subset \mathscr{E}_{\pm}(\mathbf{R})$.

617*. Let $f_\alpha(x) = [1/\Gamma(\alpha)]x^{\alpha-1}\theta(x)$ for $\alpha > 0$.

(a) Verify that $f_\alpha \in \mathscr{D}'_{+}(\mathbf{R})$ for $\alpha > -1$.

(b) Prove the identity $f_\alpha * f_\beta = f_{\alpha+\beta}$.

(c) Prove the identity $(d/dx)f_\alpha = f_{\alpha-1}$ for $\alpha > 1$.

(d) Find the limit of f_α in $\mathscr{D}'_{+}(\mathbf{R})$ as $\alpha \to 0$.

618*. Construct a family of operators $I(\alpha)$, $\alpha \in \mathbf{R}$, in $\mathscr{D}'_{+}(\mathbf{R})$ with the properties:

(a) $I(\alpha)I(\beta) = I(\alpha + \beta)$, $I(0) = 1$;

(b) $I(1)\varphi(x) = \int_0^x \varphi(t)\,dt$ for $\varphi \in \mathscr{E}'_{+}(\mathbf{R})$;

(c) $I(-1)\varphi(x) = \varphi'(x)$ for $\varphi \in \mathscr{E}'_{+}(\mathbf{R})$;

(d) $I(\alpha)f_\beta = f_{\alpha+\beta}$ for $\beta > -1, \alpha + \beta > -1$ (the f_α are defined in Problem 617).

The operator $I(\alpha)$ is called the *operator of fractional integration of order* α (or *of fractional differentiation of order* $-\alpha$) and denoted sometimes by $(d/dx)^{-\alpha}$.

619. Compute the following integrals and derivatives of fractional order:

(a) $I(1/2)\chi_{[0,1]}(x)$;

(b) $(d/dx)^{1/2}\theta(x)$;

(c)* $I(1/2)[J_0(\sqrt{x})\theta(x)]$,

where $J_0(t) = \sum_{k=0}^{\infty}(-1)^k[t^{2k}/2^{2k}(k!)^2]$ is a *Bessel function.*

620*. Suppose that the generalized function f on $\mathbf{R}^2$ has the form

$$\langle f, \varphi \rangle = \frac{1}{2\pi}\int_0^{2\pi} \varphi(\cos t, \sin t)\,dt.$$

Prove that $f * f$ is a regular generalized function, and compute it.

621. Compute the convolution $f * f$, where f is the generalized function on $\mathbf{R}^3$ given by the formula $\langle f, \varphi \rangle = (1/4\pi) \int_S \varphi(x_1, x_2, x_3)\, d\sigma$, where S is the sphere $\|x\| = 1$ and $d\sigma$ is the area element of the sphere.

§2. The Fourier Transformation

1. Characters on an Abelian Group

622°. Find the explicit form of the characters on the cyclic group C_n of order n.

623. Prove that every finite abelian group G is isomorphic (not canonically) to its dual group $\hat{G}$.

624. A *generalized* or *nonunitary character* on a group G is defined to be a homomorphism of it into the multiplicative group of the field of complex numbers.

Prove that for a compact group G the generalized characters are all ordinary characters. Find the generalized characters of the groups: (a) $\mathbf{Z}$; (b) $\mathbf{R}$; (c) $\mathbf{C}$; (d) $\mathbf{R}^*$; (e) $\mathbf{C}^*$ (the * denotes the multiplicative group).

625*. Prove that if G is a compact group, then the dual group $\hat{G}$ is discrete.

626*. Prove that if G is a discrete group, then the dual group $\hat{G}$ is compact.

627. Let χ be a character on $\mathbf{R}$, and regard it as an element of the space $\mathscr{D}'(\mathbf{R})$. Prove that χ satisfies the differential equation $\chi' = c\chi$, where c is some constant.

628°. Let χ be a character on the group G, and $f \in L_1(G, \mu)$. Prove that $\chi * f = cf$, where

$$c = \chi * f(0) = \int_G f(x)\overline{\chi(x)}\, d\mu(x).$$

629°. Let G be a compact group with invariant measure μ normalized by the condition $\mu(G) = 1$. Prove that for any two characters $\chi_1, \chi_2 \in \hat{G}$

$$\chi_1 * \chi_2 = \begin{cases} 0 & \text{if } \chi_1 \neq \chi_2, \\ \chi_1 & \text{if } \chi_1 = \chi_2. \end{cases}$$

630. Prove that the collection of characters on the group $\mathbf{T}^n$ is exhausted by the collection of functions $e_k(t) = e^{2\pi i k t}$ (cf. Problem 592).

631*. Prove that the correspondence $G \mapsto \hat{G}$ defines a contravariant functor in the category of topological abelian groups.

632. Let L be an LTS over the field $\mathbf{R}$, and regard it as a topological abelian group. Find the group $\hat{L}$ dual to L.

633.** Let $\mathbf{Q}_p$ be the field of p-adic numbers (see Problem 38), and $\mathbf{Z}_p$ the subring of p-adic integers. Find the dual groups of the following groups: (a) $\mathbf{Q}_p$; (b) $\mathbf{Z}_p$; (c) $\mathbf{Q}_p/\mathbf{Z}_p$.

634.** Let G_0 be a closed subgroup of G, and $G_1 = G/G_0$ the corresponding factor group. This can be expressed briefly in the form of the exact sequence

$$0 \to G_0 \xrightarrow{i} G \xrightarrow{p} G_1 \to 0,$$

where 0 is the trivial group with one element. Prove that the dual sequence

$$0 \leftarrow \hat{G}_0 \xleftarrow{\hat{i}} \hat{G} \xleftarrow{\hat{p}} \hat{G}_1 \leftarrow 0$$

is also exact.

635*. Find the dual group $\hat{G}$ if $G = \mathbf{Q}/\mathbf{R}$. ($G$ can be identified in a natural way with the group of all roots of unity with the aid of the mapping $x \bmod \mathbf{Z} \to e^{2\pi i x}$.)

636. Let $G = \prod_{n=1}^{\infty} C_2$ be the group of all sequences of 0's and 1's (the group operation is addition modulo 2, and the topology is defined by coordinatewise convergence).

(a) Prove that G is compact.

(b) Prove that the dual group is isomorphic to the countable group $\sum_{n=1}^{\infty} C_2$ of all finitely nonzero sequences of 0's and 1's (the group operation is addition modulo 2, and the topology is discrete).

(c) Establish a measurable correspondence between G and the closed interval $[0, 1]$ such that the characters on G pass into the Walsh functions (see Problem 542).

637. Let α be an irrational number, and $f \in L_1(\mathbf{T}, dt)$ a function having the property $f(t + \alpha) = f(t)$ almost everywhere. Prove that f is a constant almost everywhere.

638. (a)° Let $f \in L_1(\mathbf{R}, dx)$. Show that $\tilde{f}(\lambda) \to 0$ as $\lambda \to \infty$.

(b) Suppose that the group G has the form $\mathbf{R}^n \times \mathbf{T}^m \times \mathbf{Z}^k$ and $f \in L_1(G, \mu)$. Prove that $f(x) \to 0$ as $x \to \infty$ in $\hat{G} = \mathbf{R}^n \times \mathbf{Z}^m \times \mathbf{T}^k$.

639.** Let $G = \mathbf{Q}_p^+$ be the additive group of the field of p-adic numbers. Denote by $\mathscr{D}(G)$ the space of compactly supported locally constant functions on G. Prove that the Fourier transformation carries the space $\mathscr{D}(G)$ into itself.

640. Let $S(\mathbf{Z})$ be the space of two-sided sequences $\{c_n\}$ having the property that $c_n = O(n^{-k})$ for all k. The topology in $S(\mathbf{Z})$ is defined by the family of norms

$$p_k(\{c_n\}) = \sup_n |n^k c_n|, \qquad k = 0, 1, \ldots$$

Prove that the Fourier transformation establishes an isomorphism between the linear topological spaces $\mathscr{E}(\mathbf{T})$ and $S(\mathbf{Z})$.

641. A continuous function f on a group G is said to be *positive-definite* if for any finite set $x_1, \ldots, x_n$ of elements in G the matrix A with elements

$a_{kj} = f(x_k - x_j)$ is positive-definite. Prove the following relations for a positive-definite function f:

(a) $|f(x)| \leq f(0), \quad f(x) = \overline{f(-x)}$;

(b) $|f(0)f(x-y) - f(x)\overline{f(y)}|^2 \leq (f^2(0) - |f(x)|^2)(f^2(0) - |f(y)|^2)$.

642. (a) Prove that a linear combination of characters on a group G with positive coefficients is a positive-definite function on G.

(b) Prove that the product of two positive-definite functions is a positive-definite function.

(c) Prove that if $\varphi \in L_1(G, \mu)$, then the function $\varphi * \varphi^*$ (where $\varphi^*(x) = \overline{\varphi(-x)}$) is positive-definite.

643. Let G be a finite group. Prove that f is positive-definite on G if and only if $\tilde{f}$ is nonnegative on $\hat{G}$.

644. Let $\varphi \in L_1(G, \mu)$ and $\varphi \geq 0$. Prove that $\tilde{\varphi}$ is positive-definite on $\hat{G}$.

2. Fourier Series

645°. What can be said about the Fourier coefficients of a function f on $\mathbf{T}$ if it is known that f

(a) is even ($f(t) = f(1-t)$)?

(b) is odd ($f(t) = -f(1-t)$)?

(c) is real almost everywhere on $\mathbf{T}$?

646°. Suppose that a function f on $\mathbf{T}$ has a piecewise-differentiable kth derivative. What is the maximal number l for which it is possible to ensure the estimate $|c_n| = o(|n|^{-l})$ for the Fourier coefficients of f?

647. Prove that the image of the space $C^k(\mathbf{T})$ under the Fourier transformation is contained in the set of sequences having the property $|c_n| = o(|n|^{-k})$ and contains the set of sequences having the property $|c_n| = o(n^{-k-1-\varepsilon})$, $\varepsilon > 0$.

648. Let $W^k(\mathbf{T})$ be the collection of functions on $\mathbf{T}$ for which the kth generalized derivative belongs to $L_2(\mathbf{T}, dt)$. Give a description of this space in terms of the Fourier coefficients.

649. Prove that the function $f(t) = \ln \sin \pi\{t\}$ belongs to $L_1(\mathbf{T}, dt)$, and find its Fourier transform.

650. Express the following properties of a function f in terms of the Fourier coefficients:

(a) $f(t + 1/2) = f(t)$;

(b) $f(t + 1/k) = \lambda f(t)$.

For what $k \in \mathbf{Z}$ do there exist nonzero functions having the latter property?

651*. The exact sequence $0 \to C_n \xrightarrow{i} \mathbf{T} \xrightarrow{p} \mathbf{T} \to 0$ is given by the imbedding $i\colon C_n \to \mathbf{T}$ defined by the formula $i(k \bmod n) = e^{2\pi i k/n}$ and the projection $p\colon \mathbf{T} \to \mathbf{T}$ defined by the formula $p(z) = z^n$. Describe the dual exact sequence.

652. Let the function f be integrable on $[0, 1/4]$. Extend f to $[0, 1]$ in such a way that its Fourier coefficients satisfy the relations $c_{2k} = 0$, $c_{2k-1} = -c_{1-2k}$.

653. Let $\{c_n\}$ be the Fourier coefficients of a function $f \in L_1(\mathbf{T}, dt)$. Find the Fourier coefficients $\{c_n(h)\}$ for the smoothed function (or *Steklov function*) $f_h(x) = (1/2h)\int_{x-h}^{x+h} f(\xi)\, d\xi$.

654. Find the Fourier series expansion of the function $f(t_1, t_2) = \operatorname{sgn}(\{t_1\} - \{t_2\})$ on $\mathbf{T}^2$, where $\{t\}$ denotes the fractional part of t.

655. What properties characterize the sequence $\{c_n\}$ of Fourier coefficients
(a) of a trigonometric polynomial?
(b) of a polynomial in $\{t\}$?
(c) of a polynomial in $\{t - 1/2\}$?

656*. Prove that a continuous function f on $\mathbf{T}$ is positive-definite (see Problem 641) if and only if its Fourier coefficients are nonnegative.

657. A sequence $\{c_n\}, n \in \mathbf{Z}$, is said to be *positive-definite* if $\sum_{n,m} c_{n-m} z_n \bar{z}_m \geq 0$ for any finitely nonzero sequence $\{z_n\}, n \in \mathbf{Z}$.

Prove that a positive-definite sequence is bounded and has the properties: $c_n = \bar{c}_{-n}$, $c_0 \geq |c_n|$, $|c_0 c_{m+n} - c_m c_n|^2 \leq (c_0^2 - |c_m|^2)(c_0^2 - |c_n|^2)$.

658*. Prove that every positive-definite sequence (see Problem 657) is the Fourier transform of a finite Borel measure μ on $\mathbf{T}$: $c_n = \int_0^1 e^{-2\pi i n t}\, d\mu(t)$.

659. Let U be a unitary operator on a Hilbert space H, and $\xi \in H$. Prove that the sequence $c_n = (U^n \xi, \xi)$ is positive-definite.

660. Under the conditions of Problem 659 suppose that ξ is a *cyclic vector* for U (i.e., the linear span of the vectors $U^n\xi$, $n \in \mathbf{Z}$, is dense in H). Construct an isomorphism between H and $L_2(\mathbf{T}, \mu)$, where μ is the Fourier transform of the sequence $\{c_n\}$, under which U passes into multiplication by $e^{2\pi i t}$.

661*. Suppose that f is a piecewise differentiable real function on $\mathbf{T}$, $S_n = \sum_{k=-n}^{n} c_k e^{2\pi i k t}$ is the nth partial sum of its Fourier series, and $\Gamma_n \subset \mathbf{T} \times \mathbf{R}$ is the graph of the function S_n. Find the limit set for $\{\Gamma_n\}$, i.e., the collection of all limit points of sequences $\{\gamma_n\}$, $\gamma_n \in \Gamma_n$.

662. Find the Fourier series expansions of the generalized functions:
(a) $f(t) = \delta(t)$,
(b) $f(t) = \operatorname{cotan} \pi t$.

663. Compute the sums of the following series of generalized functions:

(a) $\sum_{n \in \mathbf{Z}} e^{2\pi i n t}$;

(b) $\sum_{n \in \mathbf{Z}} |n| e^{2\pi i n t}$;

(c) $\sum_{n=1}^{\infty} (\sin \pi n \{t\})/n$.

664. Prove that a generalized function on $\mathbf{T}$ is uniquely determined by its Fourier coefficients.

665. A generalized function f on $\mathbf{T}$ is said to be *positive-definite* if $\langle f, \varphi * \varphi^* \rangle \geq 0$ for any function $\varphi \in \mathscr{E}(\mathbf{T})$. Characterize the positive-definite generalized functions in terms of their Fourier coefficients.

666. Let α be an irrational real number, and X a measurable subset of $\mathbf{T}$ that is invariant under translation by α. Prove that either $\mu(X) = 0$ or $\mu(X) = 1$, where μ is Haar measure. (This property of a transformation of a measure space is called *ergodicity*: the transformation is said to be *ergodic* if every measurable invariant subset either is of measure zero itself or has complement of measure zero.)

667*. Use the Fourier method to solve the heat conduction equation $\partial u/\partial t = \partial^2 u/\partial x^2$ on the circle $\mathbf{T}$ with initial data $u(0, x) = v(x)$ (see Problem 707).

3. The Fourier Integral

668. Compute the Fourier transforms of the following functions:

(a) $f(x) = e^{-ax^2}, a > 0$;
(b) $f(x) = 1/(x^2 + a^2)$;
(c) $f(x) = \theta(x)e^{-ax}, a > 0$;
(d) $f(x) = \chi_{[a,b]}(x)$;
(e)* $f(x) = 1/\cosh ax$;
(f)* $f(x) = x/\sinh ax$;
(g)* $f(x) = 1/\cosh^2 ax$;
(h) $f(x) = (\sin ax)(\sin bx)/x^2$.

669. Let $D_k = \partial/\partial x_k$, and M_k the operator of multiplication by x_k. Define the operators $A_k = iD_k + M_k$, $A_k^* = iD_k - M_k$, $k = 1, 2, \ldots, n$ (the so-called *creation* and *annihilation operators* in quantum field theory) in the space $S(\mathbf{R}^n)$.

(a) Prove that the system of equations $A_k f = 0$, $1 \leq k \leq n$, has a one-dimensional space of solutions in $S(\mathbf{R}^n)$.

(b)* Let $f_0 \in S(\mathbf{R}^n)$ be a basis vector in the solution space of the system $A_k f = 0$, $1 \leq k \leq n$ (the so-called *vacuum vector*). Prove that the system of functions $f_m = (A_1^*)^{m_1} \cdots (A_n^*)^{m_n} f_0$, $m \in \mathbf{N}^n$, is dense in $S(\mathbf{R}^n)$.

(c) Let $N_k = (1/4\pi)A_k^* A_k$, $N = \sum_{k=1}^n N_k$ (the so-called *occupation number operator* and *particle number operator*). Prove that the functions f_m, $m \in \mathbf{Z}^n$, are eigenfunctions for the operators N_k and N, and compute the corresponding eigenvalues.

(d)* Construct an isomorphism between the space $S(\mathbf{R}^n)$ and the space of n-fold sequences $\{c_m\}$, $m \in \mathbf{N}^n$, with the property that $|c_m| = o(|m|^{-k})$ for all $k \in \mathbf{N}$.

(e) Compute the Fourier transforms of the functions f_m, $m \in \mathbf{N}^n$.

670. Prove that every continuous operator in $S(\mathbf{R}^n)$ that commutes with the operators M_k, $1 \leq k \leq n$, (see Problem 669) is the operator of multiplication by a function.

671. Prove that every continuous operator in $S(\mathbf{R}^n)$ that commutes with the operators M_k and D_k, $1 \leq k \leq n$, (see Problem 669) is a scalar operator.

672. Prove that the direct and inverse Fourier transformations preserve the space $S(\mathbf{R}^n)$ and are mutually inverse continuous transformations in it.

673. Let $G(\mathbf{R}^{2n})$ be the space isomorphic to $S(\mathbf{R}^n)$ in Problem 667. What operator in $G(\mathbf{R}^{2n})$ corresponds to the Fourier transformation in $S(\mathbf{R}^n)$?

674°. Find the Fourier transforms of the following functions in $L_2(\mathbf{R}, dx)$, where $a, b \in \mathbf{R}$:

(a) $f(x) = 1/(x + \alpha)$, $\alpha \in \mathbf{C} \setminus \mathbf{R}$;
(b) $f(x) = (\sin ax)/x$;
(c) $f(x) = (\sin ax \sin bx)/x$;
(d) $f(x) = x/(x^2 + a^2)$;
(e) $f(x) = (\tanh ax)/x$.

675°. What can be said about the Fourier transform of a function f if it is known that f

(a) is even?
(b) is odd?
(c) is real?
(d) satisfies the condition $f(x) = \overline{f(-x)}$?

676. The functions f and g on $\mathbf{R}^n$ are connected by the relation $f(x) = g(Ax + b)$, where A is an invertible linear operator on $\mathbf{R}^n$, and $b \in \mathbf{R}^n$. How are the Fourier transforms $\tilde{f}(\lambda)$ and $\tilde{g}(\lambda)$ related?

677. Prove that if $f \in L_1(\mathbf{R}^n, dx)$ and $\tilde{f}(\lambda) \equiv 0$, then $f(x) = 0$ almost everywhere.

678. The space $H_s(\mathbf{R}^n)$ is defined for $s \geq 0$ as the space of Fourier transforms of the functions in $L_2(\mathbf{R}^n, (1 + \|\lambda\|^{s/2})\, d\lambda)$. Prove that for $s > n/2$ each function $f \in H_s(\mathbf{R}^n)$ coincides almost everywhere with some bounded continuous function.

679. Prove that the operators D_k: $H_s(\mathbf{R}^n) \to H_{s-1}(\mathbf{R}^n)$, $1 \leq k \leq n$, $s \geq 1$, are continuous (see Problems 669 and 678).

680°. Prove that the convolution of two functions in $S(\mathbf{R}^n)$ is also in $S(\mathbf{R}^n)$.

681. Prove that the convolution of functions $f_1 \in H_{s_1}(\mathbf{R}^n)$ and $f_2 \in H_{s_2}(\mathbf{R}^n)$ (see Problem 678) belongs to $BC^k(\mathbf{R}^n)$ if $s_1 + s_2 \geq k$. ($BC^k(\mathbf{R}^n)$ denotes the space of functions on $\mathbf{R}^n$ with continuous bounded derivatives up to order k, endowed with the norm

$$\|f\| = \sup_{x \in \mathbf{R}^n, |l| \le k} |f^{(l)}(x)|.$$

682. Let P be a polynomial on $\mathbf{R}$ of degree $2m$ without real roots.

(a) Prove that the Fourier transform of the function $f(x) = 1/P(x)$ is infinitely differentiable everywhere except at the point $\lambda = 0$.

(b) Prove that $\tilde{f}(\lambda)$ has one-sided derivatives of all orders at the point $\lambda = 0$.

(c) What is the order of smoothness of $\tilde{f}(\lambda)$ (the number of continuous derivatives)?

683. Let $f \in L_1(\mathbf{R}, dx)$ be a rational function. Prove that $|\tilde{f}(\lambda)| \le ce^{-\varepsilon|\lambda|}$, $\lambda \in \mathbf{R}$, for some constants $c > 0$ and $\varepsilon > 0$.

684. (a) Suppose that $f \in S(\mathbf{R})$ and $\int_{\mathbf{R}} x^n f(x)\, dx = 0$ for all $n \in \mathbf{N}$. Does it follow that $f \equiv 0$?

(b) Suppose that $\varphi \in \mathscr{D}(\mathbf{R})$ and $\int_{\mathbf{R}} x^n \varphi(x)\, dx = 0$ for all $n \ge n_0$. Does it follow that $\varphi \equiv 0$?

685*. Prove that every continuous positive-definite function f on the real line has the form $f(x) = \int_{\mathbf{R}} e^{2\pi i \lambda x}\, d\mu(\lambda)$, where μ is some finite Borel measure on $\mathbf{R}$.

686. Let $\{U(t)\}$, $t \in \mathbf{R}$, be a one-parameter group of unitary operators on a Hilbert space H (i.e., $U(t)U(s) = U(t + s)$) that is continuous with respect to t in the strong operator topology. Prove that for any vector $\xi \in H$ the function $f(t) = (U(t)\xi, \xi)$ is positive-definite.

687. Under the conditions of Problem 686 assume that the vector ξ is cyclic for $U(t)$ (i.e., the linear span of the vectors $U(t)\xi$, $t \in \mathbf{R}$, is dense in H). Construct an isomorphism between the spaces H and $L_2(\mathbf{R}, \mu)$ under which the operator $U(t)$ passes into the operator of multiplication by $e^{2\pi i \lambda t}$.

688*. *Paley–Wiener theorem.* Prove that the Fourier transforms of the functions in $\mathscr{D}(\mathbf{R})$ form the space of entire functions g of $\lambda \in \mathbf{C}$ having the property that there exists a number $a > 0$ and constants c_k such that

$$|g(\lambda)|(1 + |\lambda|)^k \le c_k e^{a|\operatorname{Im} \lambda|}.$$

689*. Let f be a continuous function on $\mathbf{R}^n$ that decreases like $O(\|x\|^{-n})$ at infinity. Then for any affine submanifold $L \subset \mathbf{R}^n$ of dimension $n - 1$ the restriction of f to L is integrable over L with respect to the natural Lebesgue measure μ_L on L.

(a) Prove that if $\int_L f(x)\, d\mu_L(x) = 0$ for all $L \subset \mathbf{R}^n$, then $f(x) \equiv 0$.

(b)** Express $f(x)$ explicitly in terms of $\varphi(L) = \int_L f(x)\, d\mu_L(x)$ in the case $n = 3$.

690.** Find a function $f \in S(\mathbf{R}^3)$ if the integrals of this function over all lines intersecting a given line $l \subset \mathbf{R}^3$ are known.

4. Fourier Transformation of Generalized Functions

691°. Compute the Fourier transforms of the following generalized functions on the line:

(a) $f(x) \equiv 1$;
(b) $f(x) = \delta^{(k)}(x)$;
(c) $f(x) = \theta(x - a)$;
(d) $f(x) = \operatorname{sgn} x$;
(e) $f(x) = x^k$;
(f) $f(x) = |x|^{2k+1}$;
(g) $f(x) = x^{2k} \operatorname{sgn} x$.

692. Find the Fourier transform of the generalized function $f(x) = \cos ax^2$, $a \in \mathbf{R}$.

693. Find the Fourier transforms of the generalized functions:

(a) $f(x) = \mathscr{P}(1/x)$ (see Problem 484);
(b) $f(x) = 1/(x + i0)$ (see Problem 499).

694. Find the general solution of the equation $x^n f(x) = 0$ in $\mathscr{D}'(\mathbf{R})$.

695. Find the general solution of the equation $x^n f^{(k)}(x) = 0$ in:
(a) $\mathscr{D}'(\mathbf{R})$;
(b) $\mathscr{E}'(\mathbf{R})$.

696. Find the Fourier transform of the generalized function

$$f_{\pm}(x) = \frac{1}{x^2 - r^2 \pm i0} = \lim_{\varepsilon \to 0} \frac{1}{x^2 - r^2 \pm i\varepsilon^2}.$$

697. Find the Fourier transform of the generalized function $f(x) = |x^2 - a^2|$, $a \in \mathbf{R}$.

698. Find the Fourier transforms of the generalized functions:

(a) $f(x) = \operatorname{sgn}(\sin ax)$;
(b) $f(x) = \operatorname{sgn}(\cos ax)$;
(c) $f(x) = |\sin ax|$.

699. Let f be a homogeneous generalized function of degree (λ, ε). Prove that $\tilde{f}$ is also homogeneous, and find its degree.

700*. Compute the Fourier transform of the generalized function $f_\alpha(x) = x_+^\alpha/\Gamma(\alpha + 1)$ (see Problem 502).

701. Let f be a regular generalized function on the circle $\mathbf{T}$, and F the generalized periodic function on the line connected with f by the relation $\langle F, \varphi\rangle = \int_{\mathbf{R}} f(e^{2\pi i t})\varphi(t)\, dt$.
How are the Fourier transforms of F and f connected?

702*. A continuous function f on $\mathbf{R}^n$ is said to be *quasi-periodic* with period R if its integral over any ball of radius R does not depend on the location of the center of the ball.

(a) Prove that quasi-periodicity is equivalent to ordinary periodicity for $n = 1$.

(b) Construct a nonconstant quasi-periodic function on the plane.

(c) Can a nonconstant quasi-periodic function have two different periods R_1 and R_2?

703*. Find the Fourier transform of the generalized function $f(x) = e^{-\pi(Ax,x)}$ on $\mathbf{R}^n$, where A is a symmetric matrix with positive-definite real part.

704. Find the Fourier transform of the generalized function $f(x) = e^{i\pi(Ax,x)}$ on $\mathbf{R}^n$, where A is a real symmetric nonsingular matrix.

705. Prove that the image of $\mathscr{E}'(\mathbf{R})$ under the Fourier transformation is the collection of entire functions $g(\lambda)$, $\lambda \in \mathbf{C}$, satisfying an estimate

$$|g(\lambda)| < C|1 + |\lambda||^N e^{R\cdot|\operatorname{Im}\lambda|},$$

where C, N, R are constants (depending on g). On what properties of the function being transformed do the constants R and N depend?

706. Prove that the equation $\Delta f = f$ does not have nonzero solutions in $S'(\mathbf{R}^n)$. Here $\Delta = \sum_{k=1}^{n} (\partial^2/\partial x_k^2)$ is the *Laplace operator*.)

707*. Let $u(t, x)$ be the solution of the *heat conduction equation* $\partial u/\partial t = \partial^2 u/\partial x^2$ with initial data $u(0, x) = v(x)$, $v \in L_1(\mathbf{R}, dx)$. Show that $u(t, x)$ has the form $v * f_t(x)$, and find the function f_t.

708. Let $T(a)$ be the operator of translation by the vector $a \in \mathbf{R}^n$, and $M(a)$ the operator of multiplication by $2\pi iax$ in $S'(\mathbf{R}^n)$. Derive the commutation relations

$$\mathscr{F}T(a)\mathscr{F}^{-1} = M(a), \qquad \mathscr{F}M(a)\mathscr{F}^{-1} = T(-a),$$

where $\mathscr{F}$ is the Fourier transformation.

709. Compute the following sums with the help of the Poisson formula:

(a) $\displaystyle\sum_{n\in\mathbf{Z}} \frac{1}{n^2 + a^2}$;

(b) $\displaystyle\sum_{n\in\mathbf{Z}} \frac{1}{(a + n)^2}$;

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)^3}$.

710. Find the Fourier transforms of the following generalized functions on $\mathbf{R}^n$:

(a) $\delta(\|x\|^2 - r^2)$,

(b) $\theta(r^2 - \|x\|^2)$.

711*. The generalized function $f \in S'(\mathbf{R}^3)$ is regular and depends only on the radius $r = \|x\|$, i.e., $f(x) = \varphi(\|x\|)$. Prove that its Fourier transform is given by the formula

$$\tilde{f}(\lambda) = \int_0^\infty k(r\|\lambda\|)\varphi(r)\,dr,$$

and find the function k.

712*. Prove the identity $\mathscr{F}[\theta(x)] = \pi\delta(\lambda) + i\mathscr{P}(1/\lambda)$.

713. Compute the Fourier transform of the generalized function $\mathscr{P}(1/x^3)$ (see Problem 484(c)).

Chapter V

The Spectral Theory of Operators

§1. The Functional Calculus

1. Functions of Operators in a Finite-Dimensional Space

714°. Let A be an operator in an n-dimensional space L over the field K. Prove that the operators $1, A, A^2, \dots, A^n$ are linearly dependent.

715. Prove that the following properties of an operator A on an n-dimensional space L over the field $K = \mathbf{R}$ or $\mathbf{C}$ are equivalent:

(a) the operators $1, A, A^2, \dots, A^{n-1}$ are linearly independent;

(b) there is a vector $\xi \in L$ such that $\xi, A\xi, \dots, A^{n-1}\xi$ is a basis in L;

(c) there is a vector $\xi \in L$ that is *cyclic* for A (i.e., L itself is the only subspace of L containing ξ and invariant under A).

Operators A having these properties are called *regular*.

716. Prove that a diagonal matrix determines a regular operator if and only if the elements on the diagonal are distinct.

717. Prove that the following properties of a matrix A are equivalent:

(a) A defines a regular operator;

(b) the minimal polynomial for A coincides with the characteristic polynomial;

(c) there is only one Jordan block for each eigenvalue of A.

718. Prove that the set of regular operators is open and dense in the set of all operators.

719. Let R_n be the collection of $n \times n$ matrices of the form

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ a_1 & a_2 & a_3 & a_4 & \cdots & a_n \end{pmatrix}$$

Prove that

(a) every regular operator on an n-dimensional space is represented by a matrix $A \in R_n$ in a suitable basis;

(b) each matrix $A \in R_n$ determines a regular operator;

(c) two matrices A and B in R_n are *similar* (i.e., $A = CBC^{-1}$) only if $A = B$.

720. Prove that two regular operators A and B on an n-dimensional space L over the field $K = \mathbf{R}$ or $\mathbf{C}$ are similar if and only if

$$\operatorname{tr} A^k = \operatorname{tr} B^k, \qquad k = 1, 2, \ldots, n.$$

721°. Let

$$A = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & & \lambda \end{pmatrix}$$

be a *Jordan block* of order n with eigenvalue λ. Compute the matrices:

(a) A^k, $k = 2, 3$;

(b) $p(A)$, where p is a polynomial;

(c) $f(A)$, where f is an entire function;

(d) $r(A)$, where r is a rational function without a pole at the point λ.

722. Let $\mathfrak{A}$ be an algebra over the field $K = \mathbf{C}$ or $\mathbf{R}$. An *idempotent* is defined to be an element $x \in \mathfrak{A}$ having the property $x^2 = x$. The *direct sum* of algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ is defined to be the linear space $\mathfrak{A}_1 \oplus \mathfrak{A}_2$ with componentwise multiplication. Prove that the following properties of $\mathfrak{A}$ are equivalent:

(a) $\mathfrak{A}$ is isomorphic to the direct sum of some (nonzero) algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$;

(b) $\mathfrak{A}$ contains a nontrivial (different from 0 and 1) idempotent.

Algebras not having these properties will be called *irreducible*.

723. (a) Prove that the field $\mathbf{C}$ is an irreducible algebra over $\mathbf{R}$.

(b) Prove that every irreducible algebra with unit that is singly generated over $\mathbf{C}$ is isomorphic to one of the algebras $\mathfrak{A}_n = \mathbf{C}[x]/(x^n)$ (the quotient algebra of the polynomials in x with respect to the ideal generated by x^n).

724. Prove that every finite-dimensional algebra is a direct sum of irreducible algebras.

725. A numerical sequence $\{a_n\}$ has the property that $0 \leq a_{m+n} \leq a_m + a_n$ for all m and n. Prove that $\lim_{n\to\infty}(a_n/n)$ exists and is equal to $\inf_n (a_n/n)$.

726. Let A be an operator in an n-dimensional linear space L over a field K. Denote by $\mathfrak{A}(A)$ the algebra over K generated by 1 (the identity operator) and A. Prove that $\dim \mathfrak{A}(A) \leq n$.

727. Let $K = \mathbf{C}$. Prove that the algebra $\mathfrak{A}(A)$ is irreducible if and only if A has a unique eigenvalue.

728. Let S be some set of operators in a linear space L. Let $S^!$ be the collection of operators in L that commute with all the operators in S. For what operators A does the equality $\mathfrak{A}(A)^! = \mathfrak{A}(A)$ hold?

729. Prove that every polynomial in the coefficients of the matrix A that is invariant under the similarity transformations $A \mapsto CAC^{-1}$ is a polynomial in $\operatorname{tr} A, \operatorname{tr} A^2, \ldots, \operatorname{tr} A^n$.

730*. Let A and B be 2×2 matrices. Prove that every polynomial in the coefficients of A and B that is invariant under the substitutions $A \to CAC^{-1}$, $B \to CBC^{-1}$ has the form $P(\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} A^2, \operatorname{tr} B^2, \operatorname{tr} AB)$, where P is some uniquely determined polynomial in five variables.

731.** Let A and B be $n \times n$ matrices. Prove that the algebra of those polynomials in the coefficients of A and B that are invariant under the substitutions $A \to CAC^{-1}$, $B \to CBC^{-1}$ contains no fewer than $n^2 + 1$ generators.

732. In the space of $2n \times 2n$ matrices find a subspace of dimension $1 + n^2$ consisting of pairwise commuting matrices.

733. Let A be an operator in an n-dimensional space with a single eigenvalue λ. Prove that $f(A) = \sum_{k=0}^{n-1} [f^{(k)}(\lambda)/k!](A - \lambda \cdot 1)^k$ for any function f that is $n - 1$ times differentiable at the point λ.

734*. Let A be an operator in an n-dimensional space with distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Prove the formula

$$f(A) = \sum_{k=1}^{n} f(\lambda_k) \prod_{j \neq k} \frac{A - \lambda_j \cdot 1}{\lambda_k - \lambda_j}.$$

735*. Suppose that the operator A has eigenvalues $\lambda_1, \ldots, \lambda_n$ with multiplicities $m_1, \ldots, m_n$. Prove the formula

$$f(A) = \sum_{k=1}^{n} \sum_{j=0}^{m_k - 1} f^{(j)}(\lambda_k) B_{jk}$$

and find the explicit form of the operators B_{jk}.

736*. Let K be the collection of all positive (see Problem 747) operators with trace 1 on a finite-dimensional Hilbert space H. Prove that K is a convex compact set, and find the extreme points of K.

2. Functions of Bounded Selfadjoint Operators

737. Let A be the operator of multiplication by a continuous real function $a(x)$ on the space $L_2(0, 1)$. Prove that A is selfadjoint, and find $\sigma(A)$.

738°. Find the spectrum of the operator A acting in $L_2(0, 1)$ according to the formula $Af(x) = a(x)f(x)$, where $a \in L_\infty(0, 1)$.

739°. Let $f \in L_1(\mathbf{R}, dx)$. Find the spectrum of the convolution operator $S(f)$ on the space $L_2(\mathbf{R}, dx)$.

740°. Let $f \in L_1(\mathbf{T}, dt)$. Find the spectrum of the operator $S(f)$ of convolution with f on the space $L_2(\mathbf{T}, dt)$.

741°. Prove that the spectrum of a unitary operator U lies on the unit circle.

742°. Let A be a selfadjoint operator. Prove that the operator $(A + \lambda 1)(A + \bar{\lambda} 1)^{-1}$ is unitary for nonreal λ.

743°. Suppose that the operator $(A - i1)$ is invertible and the operator $(A + i1)(A - i1)^{-1}$ is unitary. Prove that A is selfadjoint.

744°. It is known that the operator U is unitary, and $U - 1$ is invertible. Prove that the operator $A = i(U + 1)(U - 1)^{-1}$ is selfadjoint.

745. Compute the spectral radius of the *Volterra operator* A on $L_2(0, 1)$ defined by the formula

$$Af(x) = \int_0^x f(t)\, dt.$$

746. Compute explicitly the resolvent of the Volterra operator in Problem 745.

747°. An operator A on a Hilbert space H is said to be *positive* if $(Ax, x) \geq 0$ for all $x \in H$, $x \neq 0$. In this case write $A \gg 0$. Prove that a positive operator A satisfies the formula

$$\|A\| = \sup_{x \neq 0} \frac{(Ax, x)}{(x, x)}.$$

748*. Let A be a selfadjoint operator such that $a \cdot 1 \ll A \ll b \cdot 1$, and $p(x)$ a nonnegative polynomial on $[a, b]$. Prove that $p(A) \gg 0$.

749. Prove that the mapping $p \mapsto p(A)$ is continuous relative to the norm in $C[a, b]$ if $a \cdot 1 \ll A \ll b \cdot 1$.

750. Let A be a bounded selfadjoint operator. Prove that $U(t) = e^{itA}$ is a unitary operator for all $t \in \mathbf{R}$ and that

$$U(t)U(s) = U(t + s), \qquad U(t)^* = U(-t).$$

751. Prove that under the conditions of Problem 750 the operator-valued function $U(t)$ is differentiable and $U'(t) = iAU(t) = iU(t)A$.

752.** Prove that every operator-valued function $U(t)$ that is norm-continuous and satisfies the equations $U(t)U(s) = U(t+s)$, $U(t)^* = U(-t)$ has the form indicated in Problem 750.

753°. Find the polar decomposition of the operator A of multiplication by a function $a \in L_\infty(X, \mu)$ on $L_2(X, \mu)$.

754. Find the polar decomposition of the one-sided shift operator on $l_2(\mathbf{C})$.

755. Let A and B be commuting operators, and $A = RU$ the polar decomposition of A.

(a) Prove that R and U commute with B if B is unitary.

(b) Is this true in the general case?

756. Suppose that $A \gg B \gg 0$ and that B is invertible. Prove that A is invertible and $A^{-1} \ll B^{-1}$.

757*. Let T be the shift operator in $l_2(\mathbf{Z})(T\{x_n\} = \{x_{n+1}\})$. Prove that there exists a unique selfadjoint operator A with the following properties:

(1) $T = e^{iA}$;

(2) $\|A\| \leq \pi$.

758. Let H_1 and H_2 be subspaces of H, and P_1 and P_2 the corresponding orthogonal projections. Prove that $\lim_{n\to\infty} (P_1P_2)^n$ exists and is equal to the orthogonal projection onto $H_1 \cap H_2$.

759. Let A be the operator on $L_2[(0, \infty), dx]$ given by the formula $Af(x) = \int_0^\infty [f(y)/(x+y)]\, dy$. Prove that A commutes with the dilation operators $L(a)\colon f(x) \mapsto f(ax)$.

3. Unbounded Selfadjoint Operators

760°. In the notation of Theorem 7 in Ch. V prove that $\tau(\Gamma_A)^\perp$ is the graph of some operator if and only if D_A is dense in H.

761°. Suppose that the operators A and A^* are densely defined (i.e., D_A and D_{A^*} are dense in H). Prove that $(A^*)^*$ coincides with the closure of A.

762°. Let A be the operator d/dx in $L_2(\mathbf{R}, dx)$ with domain:

(a) $D_A = \mathscr{D}(\mathbf{R})$;

(b) $D_A = \{\varphi \in \mathscr{D}(\mathbf{R}); \varphi(0) = 0\}$;

(c) $D_A = \{\varphi \in \mathscr{D}'(\mathbf{R}); \varphi \in L_2(\mathbf{R}, dx), \varphi' \in L_2(\mathbf{R}, dx)\}$, the *natural domain.*

Find A^* and D_{A^*} in these cases.

763°. Let $A = d/dx$ act in $L_2((0, \infty), dx)$ with the domain:

(a) $D_A = \{\varphi \in \mathscr{D}(\mathbf{R}), \operatorname{supp} \varphi \subset (0, \infty)\}$;

(b) $D_A = \{\varphi \in \mathscr{E}(0, \infty), \varphi \in L_2(0, \infty), \varphi' \in L_2(0, \infty)\}$.

Find A^* and D_{A^*}.

764°. Prove that each of the following conditions is equivalent to the essential selfadjointness of an operator A:

(a) A^* is selfadjoint;

(b) $\bar{A} = A^*$.

765°. In which of the following cases is the operator $A = id/dx$ in $H = L_2(0, 1)$ symmetric, essentially selfadjoint, or selfadjoint:

(a) $D_A = C^1[0, 1]$,

(b) $D_A = \{\varphi \in C^1[0, 1], \varphi(0) = \varphi(1)\}$,

(c) $D_A = \{\varphi \in C^1[0, 1], \varphi(0) = \varphi(1) = 0\}$?

766. The operator $A = id/dx$ in $L_2(0, 1)$ is defined in the domain $D_A \subset C^1[0, 1]$ given by the boundary condition $\varphi(0) = \lambda\varphi(1)$, $\lambda \in \mathbf{C}$.

(a) Find A^* and D_{A^*}.

(b) For which λ is A essentially selfadjoint?

767°. Determine whether the Laplace operator $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ is symmetric in $L_2(\mathbf{R}^2, dx)$ when:

(a) $D_\Delta = S(\mathbf{R}^2)$;

(b) $D_\Delta = \mathscr{D}(\mathbf{R}^2)$; D_Δ is the natural domain.

768*. Show that every symmetric operator A for which $D_A = H$ is bounded.

769. Let A be a selfadjoint operator.

(a) Prove that the operator $(A + i1)(A - i1)^{-1} = U$ is unitary.

(b) Prove that $\ker(U - 1) = \{0\}$.

770. Let U be a unitary operator for which $\ker(U - 1) = \{0\}$. Prove that the operator $A = i(U + 1)(U - 1)^{-1}$ with domain $D_A = \operatorname{im}(U - 1)$ is selfadjoint.

771*. Compute the operator A in Problem 770 if U is the operator of shift by 1 in $l_2(\mathbf{Z})$.

772. Suppose that $H = l_2(\mathbf{C})$, D_A consists of all the finitely nonzero sequences with zero sum, and A is the operator given by the matrix $A = (a_{jk})$, where $a_{jk} = i \operatorname{sgn}(j - k)$.

(a) Is A symmetric?

(b) Is A essentially selfadjoint?

773. Let A_i, $i = 1, 2$, be the operator of multiplication by x in $L_2(\mathbf{R}, dx)$ with domains D_{A_i}. Suppose that A_1 and A_2 are essentially selfadjoint. It is possible that the subspaces D_{A_1} and D_{A_2} have zero intersection?

774. Suppose that the operator A is densely defined and positive (see Problem 747). Prove that A is essentially selfadjoint if and only if $\ker(A^* + 1) = \{0\}$.

775*. Prove that for any closed densely defined operator A the operator $T = A^*A + 1$ with domain $D_\tau = \{x \in D_A; Ax \in D_{A^*}\}$ is selfadjoint.

776*. Suppose that H_1 and H_2 are Hilbert spaces, and $H_1 \otimes H_2$ their Hilbert tensor product. Prove that if A_1 and A_2 are selfadjoint operators in H_1 and H_2, respectively, then the operator $A_1 \otimes 1 + 1 \otimes A_2$, with domain $D_{A_1} \otimes D_{A_2}$, is essentially selfadjoint in H.

777°. Prove that the spectrum of a selfadjoint operator lies on the real axis.

778°. (a) Prove that $(\operatorname{im} A)^{\perp} = \ker A^*$ for a densely defined operator A in a Hilbert space.

(b) Is the relation $(\ker A)^{\perp} = \overline{\operatorname{im} A^*}$ true in this case?

779. Prove that a selfadjoint operator A does not have a symmetric extension different from A itself.

780. Let A be a symmetric operator. Prove that $(A + i)(A - i)^{-1}$ extends to an isometry U from $\overline{\operatorname{im}(A - i)}$ to $\overline{\operatorname{im}(A + i)}$.

781. Let A be a closed symmetric operator. Prove that the space $\operatorname{im}(A + i1)$ is closed in H.

782. Prove that a closed symmetric operator A admits a selfadjoint extension if and only if $\dim \ker(A^* - i1) = \dim \ker(A^* + i1)$.

§2. Spectral Decomposition of Operators

1. Reduction of an Operator to the Form of Multiplication by a Function

783°. Let A be a selfadjoint operator in a finite-dimensional space. Reduce it to the form of multiplication by a function.

784°. Let $H = L_2[-1, 1]$, and suppose that A is the operator of multiplication by the function $a(x) = x$. Determine which of the following functions are cyclic vectors for A in H:

(a) $f(x) = 1$;

(b) $f(x) = \operatorname{sgn} x$;

(c) $f(x) = \theta(x)$.

785. The operator A consists in multiplication by the function $a(x) = x^2$ in $H = L_2[-1, 1]$. Prove that:

(a) A does not have cyclic vectors in H;

(b) H can be represented as the union of two subspaces having cyclic vectors for A.

786. Prove that if A is an operator with simple spectrum in an infinite-dimensional Hilbert space H, then the operators $1, A, \ldots, A^n$ are linearly independent for any n.

787*. Suppose that B and C are bounded selfadjoint operators that commute. Prove that there exists a bounded selfadjoint operator A such that B and C are functions of A.

788. Let A be a selfadjoint operator with simple spectrum. Prove that every bounded operator B that commutes with A is a function of A.

789. Prove that $\|f(A)\| = \sup_{t \in \sigma(A)} |f(t)|$ for any bounded Borel function of a selfadjoint operator A.

790. Determine which of the following operators have simple spectra:
(a) multiplication by x^2 in $L_2(0, 1)$;
(b) multiplication by x^2 in $L_2(-1, 1)$;
(c) multiplication by x^2 in $L_2([0, 1] \times [0, 1])$.

791. Reduce the convolution operator $S(f), f \in L_1(\mathbf{R}, dx)$, to the form of multiplication by a function. For what functions is this operator selfadjoint?

792. Can the convolution operator $S(f), f \in L_1(\mathbf{R}, dx)$, be unitary?

793*. Let G be a locally compact abelian group with invariant measure μ. Determine conditions on a function $f \in L_1(G, \mu)$ under which the convolution operator $S(f)$ is:
(a) selfadjoint;
(b) unitary;
(c) compact.

794. Let A be the integral operator on $L_2(0, 1)$ given by the formula $Af(x) = \int_0^1 \min(x, y) f(y)\, dy$. Reduce this operator to the form of multiplication by a function.

795. Suppose that the operator A in $l_2(\mathbf{Z})$ is given by the formula $(Ac)_n = c_{n-1} - 2c_n + c_{n+1}$. Reduce this operator to the form of multiplication by a function and find its spectrum.

796. Reduce the unbounded operator $A = id/dx$ in $L_2(\mathbf{R}, dx)$ with natural domain to the form of multiplication by a function.

797. Let A be an unbounded selfadjoint operator in H, and Γ_A the graph of A in $H \oplus H$. Prove that the orthogonal complement of Γ_A in $H \oplus H$ is $\tau(\Gamma_A)$.

798. Under the conditions of Problem 797 denote the projection of a vector $x \oplus 0$ on Γ_A by $(y \oplus Ay)$ and its projection on $\Gamma_A^\perp$ by $(-Az) \oplus z$. Prove that:
(a) the correspondences $x \mapsto y$ and $x \mapsto z$ are bounded operators on H (denote them by B and C, respectively);
(b) $C = -AB, (1 + A^2) B = 1$.

799. Suppose that the operator A in $L_2(\mathbf{R}, dx)$ coincides with id/dx on the natural domain $D_A = \{\varphi \in L_2(\mathbf{R}, dx),\ \varphi' \in L_2(\mathbf{R}, dx)\}$. Compute $f(A)$ explicitly if:
(a) $f(t) = e^{iat}, a \in \mathbf{R}$;
(b) $f(t) = (\sin at)/it, a \in \mathbf{R}$;
(c) $f(t) = 1/(a^2 + t^2), A \in \mathbf{R}$.

800 Find the operator $f(A)$ explicitly if $f(t) = e^{at^2}$, $a > 0$, and $A = d/dx$ in $L_2(\mathbf{R}, dx)$ with domain $D_A = \{\varphi \in L^2(\mathbf{R}, dx),\ \varphi' \in L^2(\mathbf{R}, dx)\}$.

801. Prove that the finite-difference operator $\Delta_h \varphi(x) \equiv (1/h)[\varphi(x + h) - \varphi(x)]$ is a function of the differentiation operator.

802. Reduce to the form of multiplication by a function the operator A acting in $L_2((0, \infty), dx)$ according to the formula $Af(x) = \int_0^\infty [f(y)/(x + y)]\, dy$ (cf. Problem 759), and find its spectrum.

803*. Reduce to the form of multiplication by a function the operator A_R acting in $L_2(\mathbf{R}^2)$ according to the formula

$$A_R f(x, y) = \frac{1}{2\pi} \int_0^{2\pi} f(x + R\cos\varphi, y + R\sin\varphi)\, d\varphi.$$

804. The operator A_R acts in $L_2(\mathbf{R}^3)$ according to the formula

$$A_R f(y) = \frac{1}{4\pi R^2} \int_{S_R} f(x + y)\, d\sigma(y),$$

where $d\sigma(y)$ is the area element on the sphere S_R of radius R with center at the origin of coordinates. Reduce this operator to the form of multiplication by a function.

805*. Let f be a continuous function on the circle. Give an explicit expression for the operator $f(F)$, where F is the Fourier operator $F\varphi(y) = \int_{\mathbf{R}} e^{-2\pi ixy}\varphi(y)\, dy$.

2. The Spectral Theorem

806. Let λ be a projection-valued measure on $[a, b]$ with values in End H, and f a continuous function on $[a, b]$. For each partition $T = \{a = t_0 \le t_1 \le \cdots \le t_n = b\}$ and each tuple of points $\xi = \{\xi_k\}$, $\xi_k \in [t_k, t_{k+1}]$, define the *Riemann integral sum*

$$S(f, T, \xi) = \sum_{k=0}^{n} f(\xi_k)\lambda([t_k, t_{k+1}]).$$

(a) Prove that the integral sums $S(f, T, \xi)$ tend in norm to some operator as the diameter $\delta(T) = \max_k (t_{k+1} - t_k)$ of the partition tends to zero. This operator is called the *Riemann integral* of f with respect to the measure λ, and denoted by

$$R\int_a^b f(x)\, d\lambda(x).$$

(b) Prove that the Riemann integral

$$R\int_a^b f(x)\, d\lambda(x)$$

coincides with the Lebesgue integral

$$\int_a^b f(x)\, d\lambda(x)$$

defined in §2.2 of Ch. V.

807°. Prove the following properties of the integral of bounded functions with respect to a projection-valued measure:

(a) $\int_X [\alpha_1 f_1(x) + \alpha_2 f_2(x)]\, d\lambda(x) = \alpha_1 \int_X f_1(x)\, d\lambda(x) + \alpha_2 \int_X f_2(x)\, d\lambda(x)$;

(b) $\int_X (f_1 f_2)(x)\, d\lambda(x) = \int_X f_1(x)\, d\lambda(x) \int_X f_2(x)\, d\lambda(x)$;

(c) $\left\| \int_X f(x)\, d\lambda(x) \right\| \le \sup_{x \in X} |f(x)|$;

(d) $\left(\int_X f(x)\, d\lambda(x) \right)^* = \int_X \overline{f(x)}\, d\lambda(x)$;

(e) if $f_n(x) \to f(x)$, $|f_n(x)| \le C$ for all $x \in X$, then $\int_X f_n(x)\, d\lambda(x) \to \int_X f(x)\, d\lambda(x)$ strongly.

808. (a) Prove that the property (3) in the definition of a projection-valued measure can be replaced by the weaker condition (3′) $\lambda(X \setminus E) = 1 - \lambda(E)$ and the normalizing condition $\lambda(\varnothing) = 0$.

(b) Prove that condition (2) in the definition of a projection-valued measure follows from conditions (1), (3) and the normalizing conditions $\lambda(\varnothing) = 0$, $\lambda(X) = 1$.

809. Suppose that $H_1 = L_2[0, 1]$, $H_2 = L_2([0, 1] \times [0, 1])$. Define projection-valued measures λ_1 and λ_2 with values in End H_1 and End H_2, respectively, by setting $\lambda_i(E) = M(\chi_E(x))$. Does there exist an isomorphism $U: H_1 \to H_2$ under which λ_1 passes into λ_2?

810. Suppose that X is a set with a σ-algebra B, H is a Hilbert space, and for each $\xi \in H$ a finite measure μ_ξ is given on (X, B) such that:

(1) $\mu_\xi(X) = \|\xi\|^2$;

(2) $\mu_{\xi-\eta} + \mu_{\xi+\eta} = 2\mu_\xi + 2\mu_\eta$.

Does there exist a projection-valued measure λ on X with values in End H such that $\mu_\xi = \lambda_\xi$ for all $\xi \in H$?

811*. Let X be a compact metric space, and H a Hilbert space. A *representation of the algebra* $C(X)$ in H is defined to be a mapping $\varphi: C(X) \to \text{End}\, H$ having the following properties:

(1) φ is an algebra homomorphism;

(2) $\varphi(\bar{f}) = \varphi(f)^*$;

(3) $\varphi(1) = 1$ (the 1 on the left is the function on X and that on the right is the identity operator on H).

Prove that there exists a unique projection-valued measure λ on X with values in End H such that $\varphi(f) = \int_X f(x)\, d\lambda(x)$.

812. Let A be a selfadjoint operator in a Hilbert space H, λ its spectral measure, and E a Borel set on the line. Denote by H_E the subspace $\lambda(E)H$. Prove that:

(a) H_E is invariant under A;

(b) if E is bounded, then $A|_{H_E}$ is a bounded operator;
(c) if E is closed, then $\sigma(A|_{H_E}) \subset E$.

813. Prove that the spectrum of a selfadjoint operator A consists of precisely those points $a \in \mathbf{R}$ such that $\lambda((a-\varepsilon, a+\varepsilon)) \neq 0$ for any $\varepsilon > 0$. (Here λ is the spectral measure of A.)

814. *The Weyl criterion.* Prove that a point a belongs to the spectrum of a selfadjoint bounded operator A on a Hilbert space H if and only if there exists a sequence of unit vectors $\xi_n \in H$ such that $\|A\xi_n - a\xi_n\| \to 0$ as $n \to \infty$.

815. The definition of the *essential spectrum* of a bounded selfadjoint operator A is obtained from the Weyl criterion (see Problem 814) by imposing the additional condition that the sequence $\{\xi_n\}$ be orthonormal. Prove that if B is a selfadjoint compact operator, then the essential spectra of A and $A + B$ coincide.

816. *Stone's formula.* Suppose that A is a bounded selfadjoint operator with spectral measure λ. Prove the equality

$$\operatorname*{s-lim}_{\varepsilon\to 0} \frac{\varepsilon}{\pi} \int_a^b [(a-\lambda)^2 + \varepsilon^2 1]^{-1}\, d\lambda$$
$$= \frac{1}{2}\lambda(\{a\}) + \lambda((a,b)) + \frac{1}{2}\lambda(\{b\}) = \frac{1}{2}\lambda([a,b]) + \frac{1}{2}\lambda((a,b)).$$

817. Suppose that U is a unitary operator in a Hilbert space H. Prove that there exists a unique Borel projection-valued measure λ on the circle $\mathbf{T}$ with values in End H such that

$$f(U) = \int_0^1 f(e^{2\pi i t})\, d\lambda(t)$$

for any bounded Borel function f on $\mathbf{T}$.

818. *Von Neumann's ergodic theorem.* Let U be a unitary operator on a Hilbert space H. Prove that $\operatorname{s-lim}_{N\to\infty} (1/N)\sum_{k=1}^N U^k$ exists and is equal to the orthogonal projection onto $\ker(U - 1)$.

819. Let A be any bounded operator, and f an analytic function in a domain Ω containing $\sigma(A)$. Define the operator $f(A)$ by

$$f(A) = \frac{i}{2\pi}\int_C f(\lambda)(A - \lambda 1)^{-1}\, d\lambda,$$

where C is any contour in Ω encompassing $\sigma(A)$.
(a) Prove that the correspondence $f \mapsto f(A)$ is an algebra homomorphism.
(b) Prove that for a normal operator A this definition of $f(A)$ coincides with that in §1.2 of Ch. V.

820*. Prove that any family of pairwise commuting bounded selfadjoint operators on a separable Hilbert space can be simultaneously reduced to the form of multiplication by a function.

821*. Let A be a selfadjoint operator in a Hilbert space H, λ its spectral measure, and f a Borel function on $\mathbf{R}$. Define $f(A)$ by the formula

$$(f(A)\xi, \eta) = \int_{-\infty}^{\infty} f(x)\, d\lambda_{\xi_\eta}(x)$$

on the subspace $D_A \subset H$ consisting of the vectors $\xi \in H$ for which

$$\int_{-\infty}^{\infty} |f(x)|^2\, d\lambda_\xi(x) < \infty.$$

Prove that:

(a) the operator $B = f(A)$ is closed and densely defined;

(b) the operators BB^* and B^*B have a common dense domain and coincide on this domain.

822. Find explicitly the spectral measure for the operator $A = id/dx$ in $L_2(\mathbf{R}, dx)$ with the natural domain

$$D_A = \{\varphi \in L_2(\mathbf{R}, dx) \colon \varphi' \in L_2(\mathbf{R}, dx)\}.$$

823. Find explicitly the spectral measure for the selfadjoint extension of the operator $A = d^2/dx^2$ in $D_A = \mathscr{D}(\mathbf{R}) \subset L_2(\mathbf{R}, dx)$.

824*. Find explicitly the spectral measure for the selfadjoint extension of the Laplace operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$ in $D_A = S(\mathbf{R}^2) \subset L_2(\mathbf{R}^2, dx)$.

825. Find the operator A in the representation $V(t) = e^{itA}$ for the one-parameter group in $L_2(\mathbf{R}, dx)$ defined by $V(t)f(\tau) = f(t + \tau)$.

826. Suppose that the one-parameter group $U(t)$ of unitary operators in H has the property $U(1) = 1$. Prove that $U(t) = e^{2\pi itA}$, where $\sigma(A) \subset \mathbf{Z}$.

827. Does there exist a one-parameter group $V(t)$ such that $V(1) = U$ for any given unitary operator U?

828*. Find the spectral decomposition of the selfadjoint extension of the operator $A = -(d^2/dx^2) + x^2$ with initial domain $D(A) = S(\mathbf{R})$.

PART III

HINTS

Chapter I

Concepts from Set Theory and Topology

§1. Relations. The Axiom of Choice and Zorn's Lemma

1. Answers: (a), (b), (e) are equivalence relations, (c) and (d) are not.

2. The definition can be reformulated: f_1 and f_2 are equivalent if there exist positive numbers a, b, and ε such that $a < f_1(x)/f_2(x) < b$ for $0 < x < \varepsilon$. This implies that it is an equivalence relation. The uncountability of the quotient set follows from the fact that all the functions x^α for $\alpha \geq 0$ are pairwise nonequivalent.

3. Let $f_1, f_2, f_3, \ldots$ be a sequence of positive functions. Define functions $\underline{f}$ and $\bar{f}$ by setting $\underline{f}(x) = (1/k)\min_{1 \leq i \leq k} f_i(x)$, $\bar{f}(x) = k \max_{1 \leq i \leq k} f_i(x)$ for $1/(k+1) < x \leq 1/k$. Then $\underline{f} < f_i < \bar{f}$ for all $i = 1, 2, \ldots$.

4. Example: X and Y are the set of natural numbers with the usual total order relation. Then the points $(1, 2)$ and $(2, 1)$ are not comparable in $X \times Y$.

5. (a) Follows immediately from the definition of the partial-order relation.

(b) The order relation in $\prod_{\alpha \in A} (X_\alpha, x_\alpha)$ is induced by the order relation in the larger set $\prod_{\alpha \in A} X_\alpha$. The distinguished point is (x_α), $\alpha \in A$.

6. Use the decomposition into prime factors.

7. Each subset $Y \subset X$ is equal to $\coprod Y_\alpha$, where $Y_\alpha = Y \cap X_\alpha$. Moreover, $Y = \coprod_{\alpha \in A} Y_\alpha \supset Z = \coprod_{\alpha \in A} Z_\alpha \Leftrightarrow Y_\alpha \supset Z_\alpha$ for all $\alpha \in A$.

8. The function $\mu(x, y)$ is found from the following system of equations: $\mu(x, x) = 1$ for all $x \in X$; $\sum_{y \leq z \leq x} \mu(z, y) = 0$ if $y < x$.

9. Use the hint for Problem 8.

10. (a) $\mu(x, x) = 1$, $\mu(x, x-1) = -1$, and $\mu(x, y) = 0$ in the remaining cases.

(b) Use (a) and Problems 6 and 9. Answer: $\mu(x, y) = \mu_0(x/y)$, where μ_0 is the *classical Möbius function* ($\mu_0(1) = 1$, $\mu_0(p_1 \cdot \ldots \cdot p_k) = (-1)^k$, where $p_1, \ldots, p_k$ are distinct prime numbers, and $\mu_0(n) = 0$ in the remaining cases).

(c) Use Problems 7 and 9. Answer: $\mu(A, B) = (-1)^{|A|-|B|}$, where $B < A < X$, and $|A|$ and $|B|$ are the numbers of elements in A and B, respectively.

(d) Suppose that the ground field consists of q elements. Then $\mu(A, B) = (-1)^d \cdot q^{d(d-1)/2}$, where $d = \dim A - \dim B$. The plan of the proof is as follows. Let $\binom{n}{k}_q$ be the number of k-dimensional subspaces in an n-dimensional space over a field of q elements. The properties of the coefficients $\binom{n}{k}_q$ are analogous to those of the usual binomial coefficients (which they become when $q = 1$). In particular,

$$\binom{n+1}{k+1}_q = q^{n-k} \cdot \binom{n}{k}_q + \binom{n}{k+1}_q \tag{1}$$

(divide all the $(k+1)$-dimensional subspaces of an $(n+1)$-dimensional space into two subsets: those that are contained in a given hyperplane and those that are not),

$$\sum_{k=0}^{n} q^{\frac{k(k-1)}{2}} \binom{n}{k}_q \cdot t^k = (1+t)(1+q^t) \cdots (1+q^{n-1}t). \tag{2}$$

(This formula is derived from (1) just as the usual Newton binomial formula is derived from the basic property of the binomial coefficients.) The required equality $\sum_{B \subset C \subset A} \mu(C, B) = 0$ (see the hint for Problem 8) follows from Eq. (2) when t is set $= -1$.

11. (a) Verify that $\sum_{d|n} \varphi(d) = n$ ($\varphi(d)$ is equal to the number of positive integers $m \leq n$ with GCD $(m, n) = n/d$). From this it follows that $\phi(n) = \sum_{d|n} \mu(n, d)d = \sum_{d|n} \mu_0(n/d)d$ (see the hint for Problem 10(b).

(b) To each irreducible polynomial P of degree d and with leading coefficient 1 assign the series $f_P(X) = 1 + X^d + X^{2d} + \cdots$. Prove that the coefficient of X^n in $\prod_P f_P(X)$ (the product over all irreducible polynomials) is equal to the number of the polynomials of degree n with leading coefficient 1, i.e., q^n. From this it follows that

$$\prod_P f_P(x) = 1 + qX + q^2X^2 + \cdots = \frac{1}{1 - qX}.$$

But

$$\prod_P f_P(X) = \prod_{n \geq 1} \left(\frac{1}{1 - X^n}\right)^{P(n, q)},$$

so that

$$\prod_{n \geq 1} (1 - X^n)^{-P(n, q)} = (1 - qX)^{-1}.$$

Taking the logarithmic derivative of both sides of this equation and expanding it in powers of X, we get that $\sum_{d|n} dP(q, d) = q^n$. This implies that

$$P(n, q) = \frac{1}{n} \sum_{d|n} \mu(n, d)q = \frac{1}{n} \sum_{d|n} \mu_0 \left(\frac{n}{d}\right) q^d.$$

(c) Verify that $C(N) = 2\sum_{1 \le k \le N} \varphi(k) - 1$. Using the equality in Problem 11(a), derive from this that

$$C(N) = \sum_{1 \le d \le N} \mu_0(d) \left[\frac{N}{d}\right]\left(\left[\frac{N}{d}\right] + 1\right) = 1,$$

where $[N/d]$ is the integral part of the number N/d. This implies that $\lim_{N \to \infty}(C(N)/N^2) = \sum_{d \ge 1} (\mu_0(d)/d^2)$, which is the desired expression.

We remark that the computations can be continued: from the basic property of the Möbius function (see the hint for Problem 8) it follows easily that $\sum_{d \ge 1} (\mu_0(d)/d^2) = (\sum_{n \ge 1} (1/n^2))^{-1}$; it is known that this number is equal to $6/\pi^2$.

12. All the assertions follow easily from the fact that $\Phi_n(t) = \prod_\varepsilon (t - \varepsilon)$, where ε runs through the primitive nth roots of unity. Use the definition of the Möbius function for the proof of this equality (Problem 8).

13. Verify directly.

14. Let O^+ be the interior of the set of positive points (i.e., the collection of points $x \in \mathbf{R}^n$ such that every point y in some neighborhood of x has the property that $y \ge 0$). Similarly, O^- denotes the interior of the set of negative points. Prove that O^+ and O^- are nonempty convex sets. (Use the fact that in $\mathbf{R}^n$ we can choose bases of positive vectors and bases of negative vectors.) Then show that the set $\Gamma = \overline{O^+} \cap \overline{O^-}$ (the bar denotes closure) is a linear subspace of $\mathbf{R}^n$ which separates $\mathbf{R}^n$ into two parts. (Any segment joining a point in O^+ to a point in O^- intersects Γ.) This implies that Γ is a hyperplane. Then use induction on n.

15. (a) Let μ_0 be the equivalence class containing the natural sequence $\mathbf{N}$ with the natural order. If M is any well-ordered countable set, then it contains a smallest element m_1, the set $M \setminus \{m_1\}$ contains a smallest element m_2, and so on. Let m_∞ be the smallest element of the set $M \setminus \{m_i\}_{i=1}^{\infty}$ (if this set is empty, then M is equivalent to the natural sequence). It is clear that the segment $M(m_\infty)$ of M is equivalent to $\mathbf{N}$. Hence, the class M is larger than μ_0, and μ_0 is a minimal element.

(b) Let M and L be two well-ordered countable sets. Say that an element $m \in M$ is admissible if the segment $M(m)$ is equivalent to some segment in L. If all the elements of M are admissible, then it is possible to construct a monotone mapping φ of M onto some segment in L by the following rule: If φ has been defined for all $m < m_1$, then $\varphi(m_1)$ is defined as the smallest element of L that is not in the set $\varphi(M(m_1))$. If L is exhausted by one of the sets $\varphi(M(m))$, then L and M are comparable, and $L < M$. Suppose now that there are nonadmissible elements, and let m_0 be the smallest of them. Analyze separately the cases when m_0 has a predecessor and when it does not. Prove that in the first case $M > L$, while the second case is actually impossible.

(c) Suppose that $\mathscr{M}_0 \subset \mathscr{M}$. Choose $\mu \in \mathscr{M}_0$ and a representative M of the class μ. If M_1 is a representative of the class $\mu_1 \in \mathscr{M}_0$, then either $\mu_1 \geq \mu$ or M_1 is equivalent to a segment $M(m_1)$. We have obtained a monotone mapping $\mu_1 \mapsto m_1$ of that part of $\mathscr{M}_0$ "to the left" of μ onto a subset of M. Since M is well-ordered, this subset has a minimal element. Hence, the minimal element is also in $\mathscr{M}_0$.

(d) Suppose that $\mathscr{M}$ is countable. Choose a representative M_i in each class $\mu_i \in \mathscr{M}$, and let $M = \bigcup_{i=1}^{\infty} M_i$. We introduce an equivalence relation in M. If $\mu_i < \mu_j$, then M_i is mapped onto a segment in M_j, and we say that a point $m \in M_i$ is equivalent to its image in M_j. Verify that the so-constructed relation is indeed an equivalence relation, that the corresponding quotient set $\tilde{M}$ is countable and well-ordered, and that its class $\tilde{\mu}$ is larger than all the classes $\mu \in \mathscr{M}$, which is impossible.

(e) Use Zermelo's theorem.

16. Say that a $\mu \in \mathscr{M}$ is admissible if the segment of $\mathfrak{A}$ determined by the element $(\mu, 0)$ is equivalent to the half-interval $[0, 1)$. Arguing as in the proof of 15(b), show that all the elements of $\mathscr{M}$ are admissible.

The second assertion of the problem follows from the uncountability of $\mathscr{M}$ (see Problem 15(d)) and the countability of a base for the topology on $[0, 1)$.

17. See the hint for Problem 16.

18. The maximal elements are the disks tangent to at least two sides of the square. Among them there is no largest one.

19. Apply Zorn's lemma to the set, ordered by inclusion, of all linearly independent systems of vectors in the given space.

20. Arguing as in the proof of Problem 15(b), prove that any two well-ordered sets are comparable (i.e., one is equivalent to a segment of the other).

21. Consider the partially ordered set whose elements are subsets of a given set X with well-orderings on them. Show that the conditions of Zorn's lemma hold, and that the maximal element is the whole set X with a well-ordering.

22. (a) Since X is finite, it contains a minimal element (i.e., one which does not follow strictly any other element), say x_1. Similarly, $X\backslash\{x_1\}$ contains a minimal element, say x_2, and so on. The desired total order relation has the form

$$x_1 < x_2 < \cdots < x_n.$$

(b) Yes. One way of proving this is to consider the partially ordered collection of partial order relations on X and apply Zorn's lemma to it. Another way: prove that every partially ordered set X can be monotonically imbedded in $P(X)$ (to each $x \in X$ assign the subset $\mu(x) = \{y \in X : y \leq x\}$); then prove that there exists a total order relation on $P(X)$ which contains the inclusion relation (totally order X by Zermelo's theorem and apply Problem 13 to $P(X) = \prod_X \{0, 1\}$).

23. Use Zorn's lemma to prove that there exists a transcendence basis for $\mathbf{C}$ over $\mathbf{Q}$, i.e., a maximal family Ω of complex numbers that are algebraically independent over $\mathbf{Q}$. Prove that $\mathbf{C}$ is the algebraic closure of the field of rational functions of the variables $z \in \Omega$ with coefficients in $\mathbf{Q}$. Finally, show that the family Ω has the cardinality of a continuum (if the cardinality of Ω is less than that of a continuum, then so is the cardinality of $\mathbf{C}$).

§2. Completions

24. (a) Let $\{x_n\}$ be the sequence of centers of the balls B_n. Prove that this is a Cauchy sequence, and that its limit is the desired intersection point of all the balls.

(b) Let $x_1, x_2, x_3, \ldots$ be a Cauchy sequence. By definition, there exists a sequence of indices $n_1 < n_2 < n_3 < \cdots$ such that all the points x_n for $n > n_k$ lie in the closed ball B_k of radius $1/2^k$ with center at one of these points. Consider the ball $\bar{B}_k$ concentric with B_k and with double its radius. Verify that the sequence $\bar{B}_k$ is shrinking and that its intersection is $\lim_{n\to\infty} x_n$.

25. Let f be a uniformly continuous function on X, $x_1, x_2, \ldots$ a Cauchy sequence, and x the point of the completion corresponding to it. Then $f(x) \stackrel{\text{def}}{=} \lim_{n\to\infty} f(x_n)$ (the limit exists because of the uniform continuity of f and the completeness of the real line $\mathbf{R}$).

26. (a) The mapping $x \mapsto \arctan x$ maps the line, with the distance introduced, isometrically onto the interval $(-\pi/2, \pi/2)$ with the usual distance. Therefore, the completion of $\mathbf{R}$ is isometric to the segment $[-\pi/2, \pi/2]$.

(b) Similarly to (a), the mapping $x \mapsto e^x$ is an isometry of our space onto the ray $(0, \infty)$. The completion is isometric to the half-line $[0, \infty)$.

27. The completion is obtained by adding the "singleton segments" $[a, a]$, $a \in \mathbf{R}$.

28. First of all, any sequence of closed intervals with lengths converging to 0 is a Cauchy sequence: to all of them there corresponds a single supplementary point in the completion, whose distance from any closed interval Δ is equal to $|\Delta|$. Prove that the completion is obtained by adjoining this unique point. To do this, prove that any Cauchy sequence of closed intervals with lengths not converging to 0 contains a subsequence for which all the intersections $\Delta_i \cap \Delta_j$ are nonempty. Use the fact that for intersecting closed intervals the distance coincides with that defined in Problem 27.

29. Prove that for any Cauchy sequence $\{f_n\}$ the limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all $x \in X$ and that the sequence $\{f_n\}$ converges uniformly to f. Use the estimate $|f(x) - f(y)| \le |f(x) - f_n(x)| + f_n(x) - f_n(y)| + |f_n(y) - f(y)|$ to prove that f is bounded.

30. Use the triangle inequality.

31. (a), (b) Any Cauchy sequence in X converges to a point in the closure of X. Conversely, any closure point of X is the limit of some Cauchy sequence in X.

(c) The completion of X can be taken to be the closure in $B(X)$ of its image under the isometric imbedding of Problem 30.

32. Prove first that the intersection is nonempty by using the theorem on shrinking balls (see Problem 24). Then apply this to the case when the role of X is played by the closure of some open subset of X.

33. The sequence $\{P_k\}$, where $P_k = \sum_{i=0}^{k} (x/2)^i$, is a nonconvergent Cauchy sequence for all three distances.

We remark that the proof of this assertion, which may seem almost obvious, frequently presents difficulties for students.

The point is that the equality $\sum_{i=0}^{\infty} (x/2)^i = 2/(2 - x)$ and the fact that the right-hand side is not a polynomial (incidentally, this still has to be proved; the simplest way is to use the fact that all the derivatives of the function $2/(2 - x)$ are nonzero) are not yet enough to imply the divergence of the given sequence in the sense of the metrics (a), (b), and (c). A correct proof can be arrived at by contradiction. Show that if the given sequence has some polynomial $P(x) = \sum c_i x^i$ as its limit, then this polynomial has the properties that:

$P(x) = 2/(2 - x)$ for $x \in [0, 1]$ in case (a);

$\int_0^1 |P(x) - 2/(2 - x)|\, dx = 0$ in case (b);

$c^i = 1/2^i$ for all $i = 0, 1, \ldots$ in case (c).

In case (a) argue as indicated above; case (b) can be reduced to (a) by using the continuity of the function $P(x) - 2/(2 - x)$ on the interval $[0, 1]$.

34. Verify that an ε-net for X is an ε-net also for the completion of X; therefore, it can be assumed that X is complete. Prove that any sequence in X has a Cauchy subsequence.

35. See the hint for Problem 29.

36. Let (f_n) be a Cauchy sequence in G. Then (f_n^{-1}) is also a Cauchy sequence. By Problem 35, (f_n) and (f_n^{-1}) converge uniformly to continuous mappings f and g. To prove that f and g are mutual inverses, use the estimate

$$d_X(fg(x), x) \le d_X(fg(x) - ff_n^{-1}(x)) + d_X(ff_n^{-1}(x) - f_n f_n^{-1}(x)).$$

37. Follows directly from the definition.

38. Let (x_n) and (y_n) be Cauchy sequences in $\mathbf{Q}$ with respect to the distance d_p. Prove that $(x_n + y_n)$ and $(x_n y_n)$ are also Cauchy sequences; if $x_n \not\to 0$, i.e., $\|x_n\|_p > a > 0$ for sufficiently large n, then $(1/x_n)$ is a Cauchy sequence (use the estimate

$$\|xy - x'y'\|_p \leq \|x\|_p \|y - y'\|_p + \|y'\|_p \|x - x'\|_p$$

and the equality $\|1/x - 1/x'\|_p = \|x - x'\|_p / \|x\|_p \|x'\|_p$). As for the fact that $\mathbf{Q}$ is not complete, see the hint for Problem 39.

39. Prove that the series $\sum_{i=-k}^{+\infty} a_i p^i$, where $0 \leq a_i \leq p - 1$, converges in $\mathbf{Q}_p$ (this follows from the fact that the partial sums form a Cauchy sequence). Let $x = \sum_{i=-k}^{+\infty} a_i p^i$, $y = \sum_{i=-l}^{\infty} b_i p^i$, and let i be the smallest index for which $a_i \neq b_i$. Prove that $d_p(x, y) = p^{-i}$ (in particular, the expansion in a series of powers of p is unique). Derive from this that the sequence

$$\left(x_n = \sum_{i=-k_n}^{+\infty} a_i^{(n)} p^i\right)$$

is Cauchy if and only if for any index i the sequence $a_i^{(n)}$ stabilizes for large n. This implies that the set of elements representable in the form of a sum of this kind is closed. Then prove that any rational number r can be represented in such a form (it suffices to consider the case $r = m/n$, where m and n are integers, and n is relatively prime to p; use induction to construct integers $m_0, m_1, m_2, \ldots$ and $a_0, a_1, a_2, \ldots$ $(0 \leq a_i \leq p - 1)$ such that $m_0 = m$, $m_i - a_i n = m_{i+1} p$ for $i \geq 0$; then $r = \sum_{i=0}^{\infty} a_i p^i$ is the desired expansion). The rest follows from Problem 31.

Prove that a number $x \in \mathbf{Q}_p$ is rational if and only if the corresponding fraction is periodic (recall how the analogous assertion is proved for the real numbers). This implies that $\mathbf{Q}_p \neq \mathbf{Q}$, i.e., that $\mathbf{Q}$ is not complete in the p-adic metric.

40. Last equality. In the notation of the hints for the preceding problem we have $n = 3$, $m = m_0 = 2$. The number a_0 is chosen in such a way that $2 - 3a_0$ is a multiple of 5, i.e., $a_0 = 4$, $m_1 = -2$. Further,

$$-2 - 3a_1 = 5m_2 \Rightarrow a_1 = 1, m_2 = -1, -1 - 3a_2 = 5m_3 \Rightarrow a_2 = 3,$$
$$m_3 = -2.$$

Since $m_1 = m_3$, the numbers a_i then repeat periodically, i.e., $a_3 = 1$, $a_4 = 3$, $a_5 = 1$, $a_6 = 3$, and so on. One should check directly that the sum $4 + 5 + 3 \cdot 5^2 + 5^3 + 3 \cdot 5^4 + \cdots$ is equal to 213 (use the formula for the sum of a geometric progression).

41. Write the number $\sqrt{-1}$ in the form $\sum_{i=0}^{\infty} a_i 5^i$, where $0 \leq a_i \leq 4$. Find the numbers a_i inductively from the congruences $a_0^2 \equiv -1\ (5)$, $a_0^2 + 10a_0 a_1 \equiv -1\ (5^2)$, $(a_0 + 5a_1)^2 + 50a_0 a_2 \equiv -1\ (5^3), \ldots, (a_0 + 5a_1 + \cdots + 5^{k-2} a_{k-2})^2 + 2 \cdot 5^{k-1} a_0 a_{k-1} \equiv -1\ (5^k)$. This system has two solutions:

$$a_0 = 2, a_1 = 1, a_2 = 2, \ldots \quad \text{and} \quad a_0 = 3, a_1 = 3, a_2 = 2, \ldots.$$

42. It is most simple to use the Hausdorff criterion (§2.2 of Ch. III) and to observe that the numbers $1, 2, \ldots, p^k$ form a p^{-k}-net in $\mathbf{Z}_p$. The second assertion follows from the fact that both sets are homeomorphic to the product of a countable number of p-point sets.

In the case when $p = 2$ the desired correspondence can be constructed explicitly as follows. A number $\ldots a_2 a_1 a_0 \in \mathbf{Z}_2$ is placed into correspondence with the real number in $[0, 1]$ whose ternary expansion has the form $0 \cdot b_0 b_1 b_2 \ldots$, where $b_k = 2a_k$.

43. Use induction on n to prove that $x^{p^n} = x^{p^n} + p^n \cdot u_n$, where $u_n \in \mathbf{Z}_p$ (for $n = 1$ use the "little" theorem of Fermat). From this it follows that $\mathrm{sgn}_p(x) = \lim_{n\to\infty} x^{p^n} = \lim_{n\to\infty} (x + p_1 u_1 + p^2 u_2 + \cdots + p^n u_n)$ exists and $\|\mathrm{sgn}_p x - x\|_p \le p^{-1}$. Conclude from this inequality that the numbers $\mathrm{sgn}_p a$ are distinct for $a = 0, 1, \ldots, p - 1$, i.e., sgn_p takes $\ge p$ values. On the other hand, conclude from the definition of sgn_p that $(\mathrm{sgn}_p x)^p = \mathrm{sgn}_p x$ for all $x \in \mathbf{Z}_{pn}$, and use the fact that the equation $y^p = y$ cannot have more than p roots in the field $\mathbf{Q}_p$.

44. Derive from Problem 37(b) that a series in $\mathbf{Q}_p$ converges if and only if its general term converges to zero. Use the estimates

$$\|k\|_p \ge p^{-\log_p k}, \qquad \|k!\|_p = p^{-([k/p]+[k/p^2]+\cdots)} \ge p^{-[k/(p-1)]}.$$

Conclude from them that the domain of convergence of the series $\sum_{k=0}^{\infty} (-1)^{k-1} x^k/k$ is $\{x \in \mathbf{Q}_p \mid \|x\| < 1\} = p\mathbf{Z}_p$; the domain of convergence of the series $\sum_{k=0}^{\infty} x^k/k!$ is the same for $p \ne 2$. But if $p = 2$, then the latter domain of convergence is

$$\{x \in \mathbf{Q}_2 \mid \|x\| < \tfrac{1}{2}\} = 4\mathbf{Z}_2.$$

45. The number -1 is the limit in $\mathbf{Q}_p$ of the sequence of natural numbers.

46. (a) Prove that a sequence of natural numbers is Cauchy with respect to the distance d if and only if it is Cauchy with respect to the 2-adic and 5-adic distances. The mapping obtained from the completion of $\mathbf{N}$ with respect to d into $\mathbf{Z}_2 \times \mathbf{Z}_5$ is the desired isomorphism.

(b) By the isomorphism in (a), the infinite "terminations" reproduced under multiplication correspond to the solutions of the equation $x^2 = x$ in the ring $\mathbf{Z}_2 \times \mathbf{Z}_5$. This equation has four solutions: (0, 0), (0, 1), (1, 0), and (1, 1).

§3. Categories and Functors

47. The universal repelling object in the dual category is the set X.

48. Assign to each set its complement.

49. The answer to all the questions is "yes."

50. The universal repelling object in G_1 is the group of integers $\mathbf{Z}$, and that in G_1^0 is the identity group.

51. The universal property follows easily from any of the familiar constructions of a free group. We present one construction of the free group F_2 with generators a and b.

Let C_a and C_b be infinite cyclic groups with generators a and b. The elements of F_2 are *words* $(x_1, x_2, \ldots, x_n)$, where x_k belongs to one of the groups C_a or C_b for $k = 1, 2, \ldots, n$, any two successive terms belong to different groups, and no term is the identity element of its group; the number n is called the *length* of the word. The length of a word can be equal to 0, i.e., F_2 contains the empty word $\varnothing$. Multiplication of words is defined with the help of induction on the length. Let $\varnothing \cdot \varnothing = \varnothing$, $\varnothing \cdot (x_1, \ldots, x_n) = (x_1, \ldots, x_n) \cdot \varnothing = (x_1, \ldots, x_n)$ (i.e., $\varnothing$ is the identity element of F_2). The product $(x_1, \ldots, x_n)(y_1, \ldots, y_m)$ is defined separately in three cases.

(1) If x_n and y_1 are in different groups, then

$$(x_1, \ldots, x_n)(y_1, \ldots, y_m) = (x_1, \ldots, x_n, y_1, \ldots, y_m).$$

(2) If x_n and y_1 are in the same group and $x_n \neq y_1^{-1}$, then

$$(x_1, \ldots, x_n)(y_1, \ldots, y_m) = (x_1, \ldots, x_{n-1}, x_n y_1, y_2, \ldots, y_m).$$

(3) If $x_n = y_1^{-1}$, then

$$(x_1, \ldots, x_n)(y_1, \ldots, y_m) = (x_1, \ldots, x_{n-1})(y_2, \ldots, y_m)$$

(the product on the right-hand side is defined due to the induction hypothesis).

Verify that F_2 with this multiplication is a group with the two generators a and b and that it is the desired universal object.

52. The free abelian group with generators a and b can be defined as the direct product of the infinite cyclic groups C_a and C_b. Another way of constructing it is to take the factor group of the free group with two generators (see Problem 51) by its commutator subgroup.

53. We present one construction of the universal object. Consider a vector space A_n over K with the basis e_I, where I runs through the finite sequences $(k_1, \ldots, k_N)$, $k_i \in \{1, 2, \ldots, n\}$; if we are considering algebras with unit, then the empty sequence $I = \varnothing$ is allowed. A multiplication in A_n that turns it into a K-algebra is determined by the rule $e_I \cdot e_{I'} = e_{II'}$, where II' is obtained by writing I' after I. Verify that A_n is an associative K-algebra with the n distinguished generators $e_{(1)}, e_{(2)}, \ldots, e_{(n)}$ and that this is the universal object.

54. The universal object in $CA_n(K)$ is the quotient algebra of the universal object in $A_n(K)$ (see Problem 53) by the two-sided ideal spanned by the elements of the form $xy - yx$.

55. We present a construction of the free Lie algebra with n generators $e_1, \ldots, e_n$. Use induction to define a family of sets E_n, $n \geq 1$, by letting $E_1 = \{e_1, \ldots, e_n\}$ and then $E_n = \bigcup_{k+l=n} E_k \times E_l$ for $n \geq 2$. Let $M = \bigcup_n E_n$ and define a multiplication $M \times M \to M$ by means of the mappings

$E_k \times E_l \to E_{k+l} \subset M$ (the arrow is the canonical inclusion following from the definition of E_{k+l}). Let $K[M]$ be the vector space over K with basis M; the multiplication introduced on M turns $K[M]$ into a K-algebra. The free Lie algebra with n generators can be defined as the quotient algebra of $K[M]$ by the two-sided ideal spanned by the elements of the form $a \cdot a$ and $(ab)c + (bc)a + (ca)b$. Verify the universal property.

We remark that the universal objects in Problems 53 and 54 can be obtained by an analogous construction, i.e., by taking the quotient algebra of $K[M]$ by a suitable two-sided ideal.

56. Define $V(\mathfrak{G})$ as the quotient of the tensor algebra of the space $\mathfrak{G}$ by the two-sided ideal spanned by the elements of the form $x \circ y - y \circ x - [x, y]$, $x, y \in \mathfrak{G}$. Prove that $V(\mathfrak{G})$ is universal, starting from the fact that the tensor algebra is universal (see Problem 53).

57. Let $\mathfrak{G}$ be the free Lie algebra with n generators. Using the universal property of $\mathfrak{G}$ (Problem 55) and the universal property of $V(\mathfrak{G})$ (Problem 56), prove that $V(\mathfrak{G})$ is the universal object in the category $A_n(K)$ (see Problem 53).

58. The sum in the category of sets is the disjoint union; in the category of linear spaces it is the direct sum ($\coprod_{\alpha \in A} V_\alpha$ is the subspace of the Cartesian product $\prod_{\alpha \in A} V_\alpha$ consisting of the vectors for which only finitely many components are not zero).

59. The products in the categories of sets and linear spaces are the usual Cartesian products.

60. See the hints for Problems 58 and 59.

61. Let $L_1 \boxtimes L_2$ denote the linear subspace over K consisting of formal linear combinations of symbols of the form $a \boxtimes b$, where $a \in L_1$, $b \in L_2$. Let $L_1 \circ L_2$ be the subspace of $L_1 \boxtimes L_2$ generated by expressions of the form $(\lambda_1 a_1 + \lambda_2 a_2) \boxtimes b - \lambda_1(a_1 \boxtimes b) - \lambda_2(a_2 \boxtimes b)$ and $a \boxtimes (\mu_1 b_1 + \mu_2 b_2) - \mu_1(a \boxtimes b_1) - \mu_2(a \boxtimes b_2)$. The quotient space $L_1 \boxtimes L_2/L_1 \circ L_2$ is denoted by $L_1 \otimes L_2$, and the image in it of an element $a \boxtimes b \in L_1 \boxtimes L_2$ by $a \otimes b$. Verify that the mapping of $L_1 \times L_2$ to $L_1 \otimes L_2$ carrying (a, b) into $a \otimes b$ is the desired universal object.

62. Let d be the greatest common divisor of the numbers m and n. Verify that C_d with the canonical morphism $C_m \times C_n \to C_d$ carrying (a mod m, b mod n) into ab mod d is a universal object (and, consequently, $\mathrm{Tor}_a(C_m, C_n) = C_d$). In the general case use the fact that any finite abelian group is a direct sum of cyclic groups, and the fact that the functor Tor is additive in each argument.

63. (a) Let A be the set of natural numbers and make A into a directed set by means of divisibility ($\alpha \leq \beta$ if $\alpha | \beta$). Let $X_\alpha = \mathbf{Z}$ for all $\alpha \in A$, and let $\varphi_{\alpha\beta}$ be multiplication by β/α for $\alpha < \beta$. Verify that the inductive limit of this family is isomorphic to the additive group **Q** (the morphisms $\varphi_\alpha: X_\alpha \to \mathbf{Q}$ are given by the formulas $\varphi_\alpha(k) = k/\alpha$).

(b) Prove that the imbedding $\mathbf{Z} \to \mathbf{Z}_p$ induces an isomorphism

$$\mathbf{Z}/p^n\mathbf{Z} \xrightarrow{\sim} \mathbf{Z}_p/p^n\mathbf{Z}_p.$$

64. Follows directly from the definition.

65. In the notation of the hint for Problem 61 the structure of a vector space over **C** in $L \otimes_{\mathbf{R}} \mathbf{C}$ is determined by the formula $z \cdot (e_{(a,w)} + L'') = e_{(a,zw)} + L''$, where $a \in L$, $z, w \in \mathbf{C}$, and zw is the product of complex numbers.

66. Use the fact that a functor F realizing equivalence of categories determines an isomorphism of the semigroup of automorphisms $\mathrm{Aut}(A)$ onto $\mathrm{Aut}(F(A))$ and the fact that a semigroup of real numbers is not isomorphic to any semigroup of matrices with complex coefficients.

67. The category of spaces K^n, $n = 0, 1, 2, \ldots$, can be taken to be this subcategory.

68. The category of finite groups that can be realized as groups of transformations of the sets $\{1, 2, \ldots, n\}$ for some n can be taken as this subcategory.

69. Follows directly from the definitions.

Chapter II

Theory of Measures and Integrals

§1. Measure Theory

1. Algebras of Sets

70. Follows from the fact that $(A \triangle B) = (A \setminus B) \cup (B \setminus A)$, $(A \setminus B) \subset (A \setminus C) \cup (C \setminus B)$, $(B \setminus A) \subset (C \setminus A) \cup (B \setminus C)$.

71. The simplest way to see the given assertions is to note that the sets A_1 and B_1 coincide outside $A_1 \triangle B_1$, while the sets A_2 and B_2 coincide outside $A_2 \triangle B_2$. Therefore, outside $(A_1 \triangle B_1) \cup (A_2 \triangle B_2)$ we can substitute B_1 for A_1 and B_2 for A_2 in all the formulas.

72. Consider the system consisting of a single nonempty set (for other examples see Problem 75).

73. $A \cap B = (A \cup B) \setminus ((B \setminus A) \cup (A \setminus B))$, $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

74. The union of two disjoint intervals is not an interval.

75. There are 16 families of subsets of X in all (i.e., elements of $P(P(X))$):

(1) the empty family	(9) $\{\{a\}, \{b\}\}$
(2) $\{\varnothing\}$	(10) $\{\{a\}, X\}$
(3) $\{\{a\}\}$	(11) $\{\{b\}, X\}$
(4) $\{\{b\}\}$	(12) $\{\varnothing, \{a\}, \{b\}\}$
(5) $\{X\}$	(13) $\{\varnothing, \{a\}, X\}$
(6) $\{\varnothing, \{a\}\}$	(14) $\{\varnothing, \{b\}, X\}$
(7) $\{\varnothing, \{b\}\}$	(15) $\{\{a\}, \{b\}, X\}$
(8) $\{\varnothing, X\}$	(16) $\{\varnothing, \{a\}, \{b\}, X\}$

The families (1) and (12) are half-rings, the families (2), (6), and (7) are rings, and the families (8) and (16) are algebras.

76. Consider the intersection of all rings containing S and contained in $P(X)$.

77. Let $\tilde{S}$ be the family of sets of the form $A = \coprod_{k=1}^{n} A_k$, $A_k \in S$. If $B = \coprod_{j=1}^{m} B_j$, $B_j \in S$, then $A \cap B = \coprod_{k,j} A_k \cap B_j$. Since $A_k \cap B_j$ is in S, $A \cap B$ is in $\tilde{S}$. Further, $A \backslash B = \coprod_{k=1}^{n} \bigcap_{j=1}^{m} A_k \backslash B_j$. There exists sets $C_{kj}^{l} \in S$ $(1 \leq l \leq n_{kj})$ such that $A_k \backslash B_j = \coprod_l c_{kj}^{l}$. Then

$$A \backslash B = \coprod_k \bigcap_j \coprod_l C_{kj}^l = \coprod_{k,l} \bigcap_j C_{kj}^l,$$

and $A \backslash B \in \tilde{S}$.

78. If E is the unit of the algebra, then

$$\bigcup_n A_n = E \setminus \bigcap_n (E \setminus A_n), \qquad \bigcap_n A_n = E \setminus \bigcup_n (E \setminus A_n).$$

79. Consider the product of the two half-rings S_1 and S_2 (the proof is similar for a larger number of factors). If $A \equiv A_1 \times A_2$, $B = B_1 \times B_2$, where A_i, $B_i \in S_i$ for $i = 1, 2$, then $A \cap B = (A_1 \cap B_1) \times (A_2 \cap B_2) \in S_1 \times S_2$. Let $B_1 \subset A_1$, $B_2 \subset A_2$; then there exist $B_1^{(i)} \in S$ and $B_2^{(i)} \in S_2$ such that $A_1 = B_1 \cup B_1^{(i)} \cup \cdots \cup B_1^{(k)}$, $A_2 = B_2 \cup B_2^{(1)} \cup \cdots \cup B_2^{(l)}$ and $A_1 \times A_2 = (B_1 \times B_2) \cup (\bigcup_{i=1}^{k} \bigcup_{j=1}^{l} B_1^{(i)} \times B_2^{(j)})$.

80. Let $P(X)$ be the algebra of subsets of a set of three elements (see Problem 75). $\{a, a\} \cup \{b, b\} \notin P(X) \times P(X)$.

81. $\overline{\lim}\, E_n$ is the collection of points belonging to infinitely many of the sets E_n; $\underline{\lim}\, E_n$ is the collection of points belonging to all but finitely many of the sets E_n.

82. The limit supremum of the sequence $A, B, A, B, \ldots$ is $A \cup B$, and the limit infimum is $A \cap B$.

83. $X \setminus \bigcap_n (\bigcup_{k \geq n} E_k) = \bigcup_n (X \setminus \bigcup_{k \geq n} E_k) = \bigcup_n (\bigcap_{k \geq n} (X \setminus E_k))$.

84. Consider $\chi(\overline{\lim}_n E_n)$ ($\chi(\underline{\lim}_n E_n)$ is handled similarly). It is easy to see that the condition $\chi(x_0) = 1$ (i.e., x_0 is in infinitely many of the E_n) is equivalent to the condition $\overline{\lim}_n \chi_n(x_0) = 1$.

85. From Problem 84 it follows that the conditions $\underline{\lim}_n E_n = \overline{\lim}_n E_n$ and $\underline{\lim}_n \chi_n = \overline{\lim}_n \chi_n$ are equivalent.

86. Multiplication of characteristic functions corresponds to intersection of sets, and addition modulo 2 to the symmetric difference.

87. To each $\mu \in \mathcal{M}$ (see Problem 15) assign the collection B_μ of Borel sets of the class μ: B_{μ_0} is the collection of intervals; B_μ is the collection of sets obtained from the sets of a class $< \mu$ by a single operation of countable union, countable intersection, or complementation. Prove that $B = \bigcup_{\mu \in \mathcal{M}} B_\mu$ and all the B_μ have the cardinality of the continuum.

88. Prove that it is not possible to obtain more than $2^n - 1$ nonempty disjoint subsets from the n original sets (the latter will be called primitive). Obviously, exactly 2^k distinct sets are obtained from any k primitive sets.

Consider the last example, which shows that our estimate is sharp: the original set A_i consists of all sequences of 0's and 1's of length n which have a 1 at the ith place, $i = 1, \dots, n$.

89. $f^{-1}(Y_1) \cap f^{-1}(Y_2) = f^{-1}(Y_1 \cap Y_2), f^{-1}(Y_1) \triangle f^{-1}(Y_2)$
$= f^{-1}(Y_1 \triangle Y_2)$.

90. Let $A = \{a, b, c, d\}$,
$B = \{a', b', d'\}$,
$\mathscr{A} = \{\varnothing, \{a, b\}, \{c, d\}, \{a, b, c, d\}\}$,
$f(a) = a', f(b) = f(c) = b', f(d) = d'$.

Then $f(\{a, b\}) \cap f(\{c, d\}) \notin f(\mathscr{A})$.

91. If E is the unit in $\mathscr{B}$, then $f^{-1}(E)$ is the unit in $f^{-1}(\mathscr{B}) \bigcap_{n=1}^{\infty} f^{-1}(Y_n) = f^{-1}(\bigcap_{n=1}^{\infty} Y_n)$.

92. $f^{-1}(Y_1) \cap f^{-1}(Y_2) = f^{-1}(Y_1 \cap Y_2)$,
$f^{-1}(Y_1) \triangle f^{-1}(Y_2) = f^{-1}(Y_1 \triangle Y_2)$

for any $Y_1, Y_2 \subset B$. Compare with the method of construction of the minimal ring of the system of sets described in the hints for Problem 76.

2. Extension of a Measure

93. By the subadditivity of an outer measure,

$$\mu^*(A) + \mu^*([0, 1] \setminus A) \geq \mu^*([0, 1]) = 1.$$

94. Suppose that A is measurable. For any $\varepsilon > 0$ there is a set $B \in R$ such that $\mu^*(A \triangle B) < \varepsilon$. Let $\bar{A} = X \setminus A$, $\bar{B} = X \setminus B$. Then $\mu^*(\bar{A} \triangle \bar{B}) = \mu^*(A \triangle B) < \varepsilon$. By the lemma in §1.2 of Chapter II, this implies that $\mu^*(A) < \mu(B) + \varepsilon$ and $\mu^*(\bar{A}) < \mu(\bar{B}) + \varepsilon$. Therefore, $\mu_*(A) = \mu(X) - \mu^*(\bar{A}) > \mu(X) - \mu(\bar{B}) - \varepsilon = \mu(B) - \varepsilon > \mu^*(A) - 2\varepsilon$. Since ε is arbitrary, $\mu_*(A) \geq \mu^*(A)$. By Problem 93, this gives us the equality $\mu_*(A) = \mu^*(A)$.

Conversely, suppose that $\mu_*(A) = \mu^*(A)$. Let us choose sets $B_k \in R$ such that $A \subset \bigcup_{k=1}^{\infty} B_k$ and $\sum_{k=1}^{\infty} \mu(B_k) < \mu^*(A) + \varepsilon$. Then the set $B = \bigcup_{k=1}^{\infty} B_k$ is in R and contains A, and $\mu^*(B \setminus A) < \varepsilon$. Let us now choose sets $C_k \in R$ such that $\bar{A} \subset \bigcup_{k=1}^{\infty} \bar{C}_k$ and $\sum_{k=1}^{\infty} \mu(\bar{C}_k) < \mu^*(\bar{A}) + \varepsilon$. Let $\bar{C} = \bigcup_{k=1}^{\infty} \bar{C}_k$ and $C = X \setminus \bar{C}$. We have that $C \in R$, $C \subset A$, and $\mu(C) = \mu(X) - \mu(\bar{C}) \geq \mu(X) - \mu^*(\bar{A}) - \varepsilon = \mu_*(A) - \varepsilon$. Thus, we have constructed sets B and C in R having the property that $B \supset A \supset C$, $\mu(B) < \mu^*(A) + \varepsilon$, and $\mu(C) > \mu_*(A) - \varepsilon$. It now remains to use the equality $\mu^*(A) = \mu_*(A)$ to show that each of the sets B and C approximates A with accuracy ε.

95. Any subset of the Cantor set, which has the cardinality of a continuum, is measurable (its measure is zero).

96. Each equivalence class contains a Borel set (see Problems 87, 106).

97. Obviously, (a) ⇔ (b), (a) ⇔ (c), (d) ⇒ (a). If (b) and (c) hold, then

$$\mu\left(\overline{\lim_n} A_n\right) = \mu\left(\bigcap_k \bigcup_{n\geq k} A_n\right) = \lim_k \mu\left(\bigcup_{n\geq k} A_n\right) \geq \overline{\lim_k}\, \mu(A_k),$$

$$\mu\left(\underline{\lim}_n A_n\right) = \mu\left(\bigcup_k \bigcap_{n\geq k} A_n\right) = \lim_k \mu\left(\bigcap_{n\geq k} A_n\right) \leq \underline{\lim}_k\, \mu(A_k),$$

and this yields the implication (a) ⇒ (d).

Consider the following example of a measure on the half-ring S of subsets of $[0, 1) \cap \mathbf{Q}$ that is upper and lower semicontinuous but not countably additive:

$$S = \{s_{a,b} = [a, b) \cap [0, 1) \cap \mathbf{Q}\}, \qquad \mu(s_{a,b}) = b - a.$$

98. Use the inequality $\mu^*(A \triangle C) \leq \mu^*((A \triangle B) \cup (B \triangle C)) \leq \mu^*(A \triangle B) + \mu^*(B \triangle C)$, which follows from Problem 70 and the subadditivity of μ^*.

99. Let $\{\tilde{A}_n\}$ be a Cauchy sequence of elements in $\mathcal{M}$, $A_n \in \tilde{A}_n$. Then for any $n \in \mathbf{N}$ there exists an $l(n) \in \mathbf{N}$ such that $\rho(\tilde{A}_{n'}, \tilde{A}_{n''}) < 1/2^n$ for any $n' > l(n)$, $n'' > l(n)$. Let $m(1) = l(1)$, $m(2) = \max\{m(1) + 1, l(2)\}$, $m(3) = \max\{m(2) + 1, l(3)\}$, etc. It is not hard to prove that

$$\mu\left(\overline{\lim_j} A_{m(j)} \setminus \underline{\lim}_j A_{m(j)}\right) = 0,$$

and, consequently, $\{\tilde{A}_n\}$ has a limit.

100. If B is a measurable set, then, by definition, $\forall \varepsilon > 0\ \exists A \in R(S)$ such that $\mu^*(A \triangle B) < \varepsilon$.

101. Defining the sets $A_n = \bigcup_{k=1}^{2^{n-1}} [(2k - 1)/2^n, 2k/2^n]$ produces a collection $\{\tilde{A}_n\}_{n=1,2,\ldots}$ of elements of $\mathcal{M}$ such that $\rho(\tilde{A}_l, \tilde{A}_m) = 1/2$ for any $l \neq m$, from which it follows that $\mathcal{M}$ is not compact. To prove that $\mathcal{M}$ is connected use the continuous mappings $f_{\tilde{E}}\colon [0, 1] \to \mathcal{M}$, defined by the formula $f_{\tilde{E}}(t) = \tilde{A}$, where $\tilde{A} \ni [0, t] \cap E$, $E \in \tilde{E}$.

102. (a) Delete from the square $[0, 1] \times [0, 1]$ all the points for which at least one coordinate is a dyadic rational, and consider the mapping $\varphi\colon (x, y) \mapsto z$ defined by the rule: If $x = 0.\xi_1\xi_2\ldots$ and $y = 0.\eta_1\eta_2\ldots$ are the binary expansions of the numbers x and y, then z has the binary expansion $z = 0.\xi_1\eta_1\xi_2\eta_2\ldots$. Verify that φ is isometric on the half-ring of rectangles of the form $a \leq x \leq b$, $c \leq y \leq d$ with dyadic rational parameters a, b, c, d and, consequently, can be extended to an isometry of L_2 onto L_1.

(b) They are not isometric. Consider the pairs of points in S_i that are at a distance of 1 from each other.

(c) Let φ be an isometry of R_1 onto R_2. Since the mapping $A \mapsto A \triangle B$ is an isometry (verify!) for each B, it can be assumed that φ carries $\varnothing$ into $\varnothing$. Next, the condition $A \subset B$ is equivalent to the relation $\rho(\varnothing, B) = \rho(\varnothing, A) + \rho(\varnothing, B)$. Therefore, φ preserves the inclusion relation. Since

$C \subset A \cap B$ is equivalent to $\{C \subset A$ and $C \subset B\}$, and $C \supset A \cup B$ is equivalent to $\{C \supset A$ and $C \supset B\}$, φ commutes with the operations of union and intersection. Moreover, the complement $\bar{A}$ of A is characterized by the property $\rho(\bar{A}, A) = 1$. *Conclusion*: φ is an isomorphism of the $\mathbf{Z}_2$-algebra R_1 onto the $\mathbf{Z}_2$-algebra R_2.

Let us now consider the family of rectangles $P_{t,s} = [t, 1] \times [s, 1]$ and their pre-images $Q_{t,s} = \varphi^{-1}(P_{t,s})$. The set $Q_{t,s}$ belongs to R_1 and, consequently, is the union of a finite number of nonadjacent half-intervals. Denote this number by $n(t, s)$. Two cases are possible: (1) the function $n(t, s)$ is unbounded in a neighborhood of any point of the unit square; (2) $n(t, s)$ is bounded on some open square.

In the first case it is possible to construct a sequence of rectangles $P_{t_1,s_1} \supset P_{t_2,s_2} \supset \cdots \supset P_{t_n,s_n} \supset \cdots$ such that the set $\bigcap_{k=1}^{\infty} Q_{t_k,s_k}$ is not elementary. (For this it is necessary to choose t_{k+1} and s_{k+1} close enough to t_k and s_k, respectively, that each half-interval in Q_{t_k,s_k} has no more than $1/2^k$ of its length outside $Q_{t_{k+1},s_{k+1}}$ and, moreover, $n(t_k, s_k) > k$.) This, however, contradicts the equality $\bigcap_{k=1}^{\infty} Q_{t_k,s_k} = Q_{t,s}$, where $t = \lim_{k\to\infty} t_k$ and $s = \lim_{k\to\infty} s_k$.

In the second case let $N = \sup\{n(t, s) < \infty: |t - t_0| < \varepsilon, |s - s_0| < \varepsilon\}$; it can be assumed that $n(t_0, s_0) = N$. Then, decreasing the number ε if necessary, we can assume that $n(t, s) = N$ for $t_0 \le t \le t_0 + \varepsilon$ and $s_0 \le s \le s_0 + \varepsilon$. Thus, for these values of t and s the set $Q(t, s)$ consists of N half-intervals $[a_k, b_k)$, $1 \le k \le N$. We investigate the dependence of a_k and b_k on s and t.

It is clear that a_k is nondecreasing, while b_k is nonincreasing in each argument. Moreover, the equality $P_{t,s} = P_{t_0,s} \cap P_{t,s_0}$ implies that $a_k(t, s) = \max\{a_k(t, s_0), a_k(t_0, s)\}$ and $b_k(t, s) = \min\{b_k(t, s_0), b_k(t_0, s)\}$.

Therefore, in some neighborhood (depending on k) of the point $(t_0 + \varepsilon, s_0)$ the function $a_k(t, s)$ depends only on t, and in some neighborhood of the point $(t_0, s_0 + \varepsilon)$ it depends only on s. The functions $b_k(t, s)$ must have the same property, and then so must $c_k(t, s) = b_k(t, s) - a_k(t, s)$. But this contradicts the equality

$$\sum_{k=1}^{N} c_k(t, s) = (1 - t)(1 - s).$$

103. Yes, since the measurable sets form a σ-algebra.

104. $\mu(\bigcap_k \bigcup_{n\ge k} A_n) \le \sum_{n\ge k}^{\infty} \mu(A_n)$.

105. The measurable sets form a σ-algebra.

106. Let $A \subset \mathbf{R}$ be measurable. It follows from Problem 94 that for any $\varepsilon > 0$ there exists a closed set $B_\varepsilon \subset A$ such that $\mu^*(A \setminus B) < \varepsilon$. Then $\bigcup_{n=1}^{\infty} B_{1/n}$ is the desired Borel set.

107. A subset of the square is measurable if and only if it has the form $A \times [0, 1]$, where $A \subset [0, 1]$ and is Lebesgue-measurable.

108. $\mu_*(\tilde{T}) = 0$, $\mu^*(\tilde{T}) = 1$, consequently, $\tilde{T}$ is not measurable (see Problem 94).

109. This set can be obtained in a way analogous to that used to get the Cantor set: delete from $[0, 1]$ the set $[0.3, 0.4)$; delete the eight sets of the form $[0.n_1 3, 0.n_1 4)$, where $n_1 = 0, 1, 4, 5, \ldots, 9$; and so on.

The measure of the remainder of the set is

$$1 - 0.1 - \sum_{n=1}^{\infty} 8^n \cdot 10^{-n+1} = 0.5.$$

110. The subset of the square consisting of the points (x, y) for which $\cos(x + y)$ is rational has measure 0, since it consists of countably many line segments of the form $x + y = \text{const}$. Answer: $\pi/6$.

111. Represent the complement of the subset under consideration as the union of four subsets of measure 0. Answer: 1.

112. Extend the measure to the half-ring of all rectangles contained in the square and with sides parallel to the corresponding sides of the square, and assign the measure $l/\sqrt{2}$ to each such rectangle, where l is the length of the intersection of the rectangle with some fixed diagonal of the square.

113. From the measurability of a set A in the Carathéodory sense it follows that

$$\mu(X) = \mu^*(A) + \mu^*(X \setminus A) = \mu^*(A) + \mu_*(A),$$

which implies (similarly to Problem 94) the Lebesgue measurability of A. On the other hand, for any subset $Z \subset X$ there exists a Lebesgue-measurable set Z_1 such that $X \supset Z_1 \supset Z$, $\mu(Z_1) = \mu^*(Z)$ (Z_1 is the intersection of a sequence of countable coverings of Z by elements of the half-ring such that μ(nth covering) is less than $\mu^*(Z) + 1/n$). We have

$$\mu^*(Z) \leq \mu^*(Z \cap A) + \mu^*(Z \setminus A),$$

$$\mu^*(Z) = \mu(Z_1) = \mu(Z_1 \cap A) + \mu(Z \setminus A) \geq \mu^*(Z \cap A) + \mu^*(Z \setminus A),$$

from which it follows that A is measurable in the Carathéodory sense.

114. (a) If A is Lebesgue-measurable, then for all $\varepsilon > 0$ there exists a B in the minimal ring such that $\mu^*(A \triangle B) < \varepsilon$. This implies that $\lambda_i(A \triangle B) \leq \mu^*(A \triangle B) < \varepsilon$, and, consequently, $|\lambda_i(A) - \lambda_i(B)| < \varepsilon$, where $i = 1, 2$. Since $\lambda_1(B) = \lambda_2(B)$, it follows that $|\lambda_1(A) - \lambda_2(A)| < 2\varepsilon$, which concludes the proof.

(b) Let $a = \mu_*(Y) \leq y \leq \mu^*(Y) = b$. For the Lebesgue measure μ generated by m construct a Lebesgue extension ν such that Y is ν-measurable and $\nu(Y) = y$. There exist μ-measurable sets E_1 and E_2 such that

$$E_1 \subset Y \subset E_2, \mu(E_1) = a, \mu(E_2) = b.$$

Add to the system of μ-measurable sets all the subsets of $E = E_2 \setminus E_1$ that have the form $C = A(Y \setminus E_1) \cup B(E_2 \setminus Y)$, $A \subset E$, $B \subset E$, where A, B are measurable and are uniquely determined by C to within a set of measure 0. Let

$$\nu(C) = \frac{y-a}{b-a}\mu(A) + \left(1 - \frac{y-a}{b-a}\right)\mu(B)$$

115. Let ν be Lebesgue measure on $[0, 1]$. Identify images and preimages under the mapping $f: X \to [0, 1]$ (since f is a bijection almost everywhere). If $Y = \prod_n Y_n$ and $Y_k \neq X_k$ for infinitely many indices k, then $\mu(Y) = \nu(Y) = 0$. If $Y = Y_1 \times \cdots \times Y_k \times X_{k+1} \times X_{k+2} \times \cdots$, then $\mu(Y) = 10^{-n} \prod_{i=1}^{k} \operatorname{card} Y_i = \nu(Y)$ since Y consists of $\prod_{i=1}^{k} \operatorname{card} Y_i$ intervals of length 10^{-k}. Consider now the half-ring L of sets of the form $[a_n 10^{-k}, b_k 10^{-k})$. It is easy to see that μ and ν coincide on L, and the Lebesgue extension from the half-ring L coincides with the usual Lebesgue measure.

3. Constructions of Measures

116. Define an equivalence relation on the interval $[0, 1]$ by setting $x \sim y$ if $x - y \in \mathbf{Q}$. Let A be a subset of $(0, 1]$ containing one element from each equivalence class. For $r \in (0, 1]$ let $A_r \subset (0, 1]$ be obtained from A by a translation by r modulo 1:

$$A_r \equiv ([r + A] \cup [(r-1) + A]) \cap (0, 1].$$

It is easy to see that $(0, 1]$ is the union of the collection of pairwise disjoint sets $\{A_r\}$, where $r \in \mathbf{Q} \cap (0, 1]$. Assume that A is measurable and come to a contradiction.

117. Construct an example analogous to that of Problem 116 by introducing the following equivalence relation: $(x_1, x_2) \sim (y_1, y_2)$ if

$$x_1 - y_1 \in \mathbf{Q}, \qquad x_2 - y_2 \in \mathbf{Q}.$$

118. Let $A \subset [0, 1)$ be nonmeasurable. Consider the set

$$\{A \times \{0\}\} \cup \{\{0\} \times A\} \subset [0, 1] \times [0, 1].$$

119. For hints on solving this problem without using the concept of an integral the reader is referred to the book [3], Ch. V, §6, Exercise 15. Observe also that if the integral concept is used, then the problem is not difficult, for if φ is the characteristic function of the set A and $\Phi(x) = \int_0^x \varphi(t)\,dt$, then the assertion of the problem follows easily from the fact that $\Phi'(x) = \varphi(x)$ almost everywhere.

120. Lebesgue's theorem on Riemann integrability yields a necessary and sufficient condition: the boundary of the set has measure 0.

121. A trivial check.

122. The first part of the problem is a trivial consequence of Problem 121. The Cantor set serves as an example for the second part of the problem.

123. For a Cauchy sequence $\{\nu_n\}$ let $(\lim_{n\to\infty} \nu_n)(A) = \lim_{n\to\infty} \nu_n(A)$ for any $A \in \mathfrak{A}$. The countable additivity of the set function $\lim_{n\to\infty} \nu_n$ follows from the equality

$$\lim_{n\to\infty} \sum_i \nu_n(A_i) = \sum_i \lim_{n\to\infty} \nu_n(A_i), \quad \text{where} \quad A = \bigcup_i A_i,\, A_k \cap A_l = \varnothing$$

for $k \neq l$, which follows from the uniform convergence of the series $\sum_i \nu_n(A_i)$ with respect to n.

124. It is not hard to prove directly from the definition of a measurable set that there exists a parallelepiped B such that

$$0.75\mu(B) \leq \mu(M \cap B).$$

Prove that the open parallelepiped B' with center at the point $0 \in \mathbf{R}^n$ and homothetic with coefficient 1/2 to the parallelepiped B belongs to $M - M$. The idea of the proof: if $b \in B'$, then $(b + M \cap B') \cap (M \cap B')$ is nonempty, since it has positive measure.

125. Check directly, using the properties of an absolutely convergent double series.

126. Let $X = \mathbf{Q} \cap [0, 1]$. Consider the ring of subsets of X generated by intervals, with the usual measure. X consists of a countable number of points, each with measure 0.

127. The known proofs of countable additivity for the Wiener measure reduce to establishing a correspondence between $X = C[0, 1]$ and a certain space Y with a measure ν, under which the Wiener measure goes over into ν. For example, see [6*]; in Ch. I an isomorphism is constructed between the space (X_0, μ_0) (see Problem 204) and the closed interval [0, 1] with the standard measure. A different (though related) presentation can be found in [4], Ch. IX, §6.7; here the role of Y is played by a countable product of lines, and that of ν by the countable product of the Gaussian measures $(1/\sqrt{\pi})e^{-x^2/2}\,dx$.

128. The set in the problem is a particular case of a set of the form $\chi(t_1, t_2; \Delta_1, \Delta_2)$; namely $t_1 = a$, $t_2 = b$, $\Delta_1 = (-\infty, 0)$, $\Delta_2 = (0, \infty)$. Therefore, the desired measure is

$$\frac{1}{\sqrt{\pi(b-a)}} \int_{-\infty}^{0} \int_{0}^{\infty} \exp\left[-\frac{(\sigma-\tau)^2}{2(b-a)}\right] d\sigma\, d\tau$$

$$= \frac{1}{\sqrt{\pi(b-a)}} \int_0^\infty s \cdot \exp\left[-\frac{s^2}{2(b-a)}\right] ds = \sqrt{\frac{b-a}{\pi}},$$

129. Let $f\colon x \to \{1/x\}$. We have

$$\begin{aligned}\mu f^{-1}([\alpha, \beta)) &= \sum_{n=1}^{\infty} \log_2\left(\frac{1 + [1/(\alpha + n)]}{1 + [1/(\beta + n)]}\right)\\ &= \sum_{n=1}^{\infty} [\log_2(\alpha + n + 1) + \log_2(\beta + n) - \log_2(\beta + n + 1)\\ &\quad - \log_2(\alpha + \mathrm{n})]\\ &= \log_2(1 + \beta) - \log_2(1 + \alpha) = \mu([\alpha, \beta)).\end{aligned}$$

130. (a)

$$f\left(\cfrac{1}{n_1 + \cfrac{1}{n_2 + \cdots}}\right) = \left\{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cdots}}\right\} = \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cdots}}.$$

(b) The desired measure in the product of countably many copies of the measure on $\mathbf{N}$ determined by

$$\mu(\{k\}) = \log_2[(k + 1)^2/(k(k + 2))].$$

131. If the sum of the series does not depend on the order of summation, then the series converges absolutely.

132. (a) $|\nu|(X) = \mu_1(X) + \mu_2(X)$;
(b) $|\nu|(X) = \sqrt{2}\mu_1(X)$.

133. Define the complex conjugate measure by setting $\bar{\nu}(A) = \overline{\nu(A)}$ for any $A \in \mathfrak{A}$. Then

$$\operatorname{Re} \nu = (\nu + \bar{\nu})/2, \qquad \operatorname{Im} \nu = (\nu - \bar{\nu})/2.$$

134. For any $A \in \mathfrak{A}$ let $f(A) = \sup\{|\nu(A')|: A' \subset A, A' \in \mathfrak{A}\}$. Suppose that $\sup_{A \in \mathfrak{A}}|\nu(A)| = \infty$; then there exists an $A_0 \in \mathfrak{A}, f(A_0) = \infty$. By induction construct a sequence $A_0 \supset A_1 \supset A_2 \supset \cdots$ such that $f(A_n) = \infty, |\nu(A_n)| \geq n$. (Let $B \subset A_{n-1}$ and $|\nu(B)| \geq |\nu(A_{n-1})| + n$; if $f(B) = \infty$, then set $A_n = B$, otherwise $A_n = A_{n-1} \setminus B$.)

The continuity property of a countably additive function (see Problem 97(d)) yields a contradiction: $\nu(\bigcap_{n=0}^{\infty} A_n) = \lim_{n\to\infty} \nu A_n = \infty$, while $C = \bigcap_{n=0}^{\infty} A_n \in \mathfrak{A}$, so that $\nu(C) \neq \infty$.

135. A set $E \in \mathfrak{A}$ is said to be *negative with respect to* ν if $\nu(E \cap F) \leq 0$ for any $F \in \mathfrak{A}$; a *positive set* is defined similarly. Prove the existence of a negative set A_- such that $A_+ = X \setminus A_-$ is positive; the solution of the problem follows from this.

Let $\{A_n\}$ be a sequence of negative sets such that $\lim_{n\to\infty} \nu(A_n) = a = \inf\{\nu(A)\colon A$ is negative$\}$. Then $A = \bigcup_n A_n$ is negative and $\nu(A) = a$. If $A_+ = X \setminus A_-$ is not positive, then there exists a $c_0 \subset A_+$, $\nu(c_0) < 0$. There is a smallest natural number k_1 such that there exists a $c_1 \subset c_0$, $\nu(c_1) \geq 1/k_1$. Repetition of the operation for $c_0 \setminus c_1$ yields a c_2 and a $k_2 \geq k_1$, and so on.

Prove that the set $F_0 = c_0 \setminus \bigcup_{i=1}^{\infty} c_i$ is nonempty and negative, which contradicts the definition of a. Hence, A_+ is positive.

136. Any set A_+ (resp., A_-) is positive (resp., negative).

137. If a set $E \in \mathfrak{A}$ lies in $X \setminus (A_+ \cup A_-)$, then $\nu(E) = 0$.

138. $\nu_+(E) = \nu(E \cap A_+)$, $\nu_-(E) = -\nu(E \cap A_-)$. Use Problems 134–136.

§2. Measurable Functions

1. Properties of Measurable Functions

139. The chain of equivalent statements is proved by the equations

$$\{x \in X : f(x) \geq a\} = \bigcap_{n=1}^{\infty} \left\{x \in X : f(x) > a - \frac{1}{n}\right\},$$

$$\{x \in X : f(x) < a\} = X \setminus \{x \in X : f(x) \geq a\},$$

$$\{x \in X : f(x) \leq a\} = \bigcap_{n=1}^{\infty} \left\{x \in X : f(x) < a + \frac{1}{n}\right\},$$

$$\{x \in X : f(x) > a\} = X \setminus \{x \mid f(x) \leq a\}.$$

140. Every ray is a Borel set; the smallest σ-ring containing all the rays is the ring of Borel sets.

141.

$$\left\{x \in X : \frac{1}{f(x)} > a\right\} = \begin{cases} \{x \in X : 0 < f(x) < a^{-1}\} & \text{if } a > 0, \\ \{x \in X : 0 < f(x) < \infty\} & \text{if } a = 0, \\ \{x \in X : -\infty < f(x) < a^{-1}\} \cup \{x \in X : 0 < f(x) < \infty\} & \\ & \text{if } a < 0. \end{cases}$$

142. $\{x \in X : |f(x)| < a\} = \{x \in X : f(x) < a\} \cap \{x \in X : f(x) > -a\}$.

143. Obviously, the set $\{(t_1, \ldots, t_n) : f(t_1, \ldots, t_n) > a\}$ is open and can be represented as a countable union of open parallelepipeds in $\mathbf{R}^n$ of the form $(a_k^{(1)}, b_k^{(1)}) \times \cdots \times (a_k^{(n)}, b_k^{(n)})$. Then

$$\{x \in \mathbf{R} : h(x) > a\} = \bigcup_{k=1}^{\infty} \bigcap_{i=1}^{n} \{x \in \mathbf{R} : a_k^{(i)} < x < b_k^{(i)}\}.$$

144. Construct a continuous monotonically increasing function $f: \mathbf{R} \to \mathbf{R}$ such that the pre-image of some set of measure 0 is a set X' of positive measure (use the series in the *Cantor staircase* for this). Suppose that $X \subset X'$ is a nonmeasurable set, and $\varphi(y)$ is the characteristic function of the set $f^{-1}(x)$. Then $\varphi[f(x)]$ is nonmeasurable.

145. The expression $[f(x)]^n$ makes sense for any function $f(x)$ only if n can be represented in the form $n = k/l$, where $k = 0, 1, 2, \ldots, l = 1, 3, 5, 7, \ldots$. If k is odd, then $f(x) = ([f(x)]^{k/l})^{l/k}$ is measurable, by Problem 143. For even k the following function serves as a counterexample:

$$f_A(x) = \begin{cases} 1 & \text{for } x \in A, \\ -1 & \text{for } x \notin A, \end{cases}$$

where A is a nonmeasurable set.

146. Assuming that $f(x)$ has been extended to the right of the point $x = 1$ in a differentiable manner (for example, $f(1 + \alpha) = f(1) + \alpha f'(1)$), we set $\varphi_n(x) = n[f[x + (1/n)] - f(x)]$. Then $f'(x)$ is the limit of a convergent sequence of continuous (hence, measurable) functions: $f'(x) = \lim_{n\to\infty} \varphi_n(x)$.

147. The pre-image of the square

$$D_{nml} = \left[\frac{m}{2^n}, \frac{m+1}{2^n}\right] \times \left[\frac{l}{2^n}, \frac{l+1}{2^n}\right]$$

is an interval of length 2^{-2n}. Consider the σ-ring generated by D_{nml} and $f^{-1}(D_{nml})$.

148. Set $U(a) = \{x \in \mathbf{R}: f(x) < a\}$. Each of the countable collection of sets $U((k + 1)/2^n)$ becomes Borel after some set of measure 0 is discarded (Problem 106). The union of all the discarded sets is a Borel set, and we can, for example, set $f(x) \equiv 0$ on it.

149. Prove the assertion of the problem for simple functions taking finitely many values by first proving that for any measurable set $A \subset [0, 1]$ and any $\delta > 0$ there exists a closed set $B \subset A$ such that $\mu(A \setminus B) < \delta$. For any function $f(x)$ there exists a sequence of simple functions $f_n(x)$ converging to $f(x)$. Then there exists a sequence $\{K_n\}$ of closed sets such that $\mu(K_n) > 1 - (\varepsilon/2^n)$ and $f_n(x)$ is continuous on K_n. The sequence of continuous functions $f_n(x)$ converges uniformly to $f(x)$ on the compact set $K = \bigcap_n K_n$, and the assertion of the problem follows from this.

150. Reducing the problem to the case when $f(x)$ is defined on an interval $[a, b]$, we have, by Problem 149, that for all $\varepsilon > 0$ there is a measurable set $X_\varepsilon \subset [a, b]$, $\mu(X_\varepsilon) > b - a - \varepsilon$ such that $f(x)$ is continuous on X_ε. Let $X'_\varepsilon \subset X_\varepsilon$ be the set of points of density; by Problem 119, $\mu(X'_\varepsilon) = \mu(X_\varepsilon)$. Obviously, all the points in X'_ε are Lebesgue points of f, and the problem follows from this, since ε is arbitrary.

151. The measurability of f follows from the fact that the sequence of step functions $f_k(x, y) = \min(k, f(x, y))$ converges to f almost everywhere. The measure of the set on which $f(x)$ is finite is equal to

$$10 \cdot 10^{-2} + 90(10 \cdot 10^{-4} + 90(10 \cdot 10^{-6} + \cdots)) = 1.$$

152. See the hints for Problem 229.

153. $$\left\{x\colon \sup_n f_n(x) > c\right\} = \bigcup_{n=1}^{\infty} \{x\colon f_n(x) > c\},$$

$$\{x\colon \inf_n f_n(x) < c\} = \bigcup_{n=1}^{\infty} \{x\colon f_n(x) < c\},$$

154. Follows from Problem 153, since

$$\overline{\lim_{n\to\infty}}\, f_n(x) = \inf_m \left[\sup_{n>m} f_n(x)\right],$$

$$\underline{\lim_{n\to\infty}}\, f_n(x) = \sup_m \left[\inf_{n>m} f_n(x)\right].$$

155. The desired set:

$$\bigcap_k \bigcup_n \bigcap_m \{x\colon |f_n(x) - f_{n+m}(x)| < 1/k\}.$$

156. A consequence of Problem 153.

157. If $F(c) = \mu\{x\colon f(x) \le c\}$, then

$$g(y) = \inf_{F(c)\le y} c.$$

158. See the hint for Problem 157.

159. A consequence of Problem 143.

160. On the set of complex numbers the disks and the rectangles

$$\{z \in \mathbf{C}\colon a \le \operatorname{Re} z \le b, c \le \operatorname{Im} z \le d\}$$

generate the same σ-ring of measurable sets.

161. The coordinates of the vectors in one basis depend continuously on the coordinates in another basis. Use Problem 143.

2. Convergence of Measurable Functions

162. Convergence is obvious. The convergence is not uniform, because $f_n(n) = 1/2$.

163. The sequence converges to the discontinuous function

$$f(x) = \begin{cases} 0 & \text{for } 0 \le x < 1, \\ 1 & \text{for } x = 1, \end{cases}$$

consequently, the convergence is not uniform.

164. If two continuous functions do not coincide at a certain point, then they differ everywhere in a neighborhood of that point.

165. $$f(x) = \begin{cases} x^{-1} & \text{for } 0 < x \le 1, \\ 0 & \text{for } x = 0. \end{cases}$$

166. Suppose that the sequence $\{f_n\}$ fails to converge to f on just the set F and fails to converge to g on just the set G. Then f and g can differ only on the set $F \cup G$.

167. $[\delta/3, \pi - \delta/3]$.

168. For $0 < \varepsilon < 1$ the set of points $x \in \mathbf{R}$ where $f_n(x) \geq \varepsilon$ is an interval of length $2\sqrt{\ln \varepsilon^{-1}}/q_k$. This quantity tends to zero as $k \to \infty$. Hence, $f_k \to 0$ in measure.

For any point $x \in [0, 1]$ consider a subsequence $\{r_{k_n}\}$ of the rational numbers such that $\lim_{n\to\infty} r_{k_n}$ exists and is not equal to x. Then $(p_{k_n} - xq_{k_n})^2 \to \infty$ as $n \to \infty$, hence, $f_{k_n}(x) \to 0$ as $n \to \infty$. Further, there exists a subsequence $\{r_{k'_n}\}$ for which $|p_{k'_n}/q_{k'_n} - x| \leq 1/q_{k'_n}$. For this subsequence $f_{k'_n}(x) \geq e^{-1}$. Hence, $\lim_{n\to\infty} f_n(x)$ does not exist.

169. Choose $\{k_l\}$, $l = 1, 2, \ldots$, so that the intervals on which $f_{k_l} > 1/l$ do not intersect.

170. It suffices to observe that the most natural way of enumerating the set $\{f_i^{(k)}(x)\}$ is by the condition $n = k(k-1)/2 + i$. Obviously, $\mu(f_i^{(k)}(x) \neq 0) = 1/k$, $\overline{\lim}_{n\to\infty} g_n(x) = 1$, $\underline{\lim}_{n\to\infty} g_n = 0$.

171. Use the relations

$$E(|h - g| \geq \delta) \subset E(|f_n - h| \geq \delta/2) \cup E(|f_n - g| \geq \delta/2),$$

$$E(h \neq g) = \bigcup_{n=1}^{\infty} E(|h - g| \geq 1/n).$$

172. Use Egorov's theorem.

173. Consider $\tan x$ on the segment $[-\pi/2, \pi/2]$. If $|f_1(x)| = M$, then the sequence $\{f_n(x)\}$ does not converge to $f(x)$ on at least one of the two segments $[-\pi/2, \arctan M]$ or $[\arctan M, \pi/2]$.

Remark. By using Levi's theorem, it is easy to prove that no measurable function $f(x) = f_+(x) - f_-(x)$ such that $\int_{[a,b]} f_+(x)\,dx = \int_{[a,b]} f_-(x)\,dx = +\infty$ can be represented as a monotone limit of a sequence of integrable functions.

174. Suppose that $\psi(x) = \lim_{n\to\infty} \varphi_n(x)$, where the $\varphi_n(x)$ are continuous, and come to a contradiction. Let $F_n = E(\varphi_n \leq 0.5)$ be the set of points x such that $\varphi_n(x) \leq 0.5$.

We have $E(\varphi < 0.5) = \underline{\lim}_n F_n$, which contradicts the fact that the set of irrational numbers is not the union of a countable number of closed sets (prove!).

Remark. Noncontinuous functions representable as the limit of a sequence of continuous functions are called functions of the 1*st Baire class*. The Dirichlet function belongs to the 2nd Baire class (double limit).

175. The necessity (for any measurable function) is obvious. The sufficiency follows from the fact that the pre-image of any Borel set is a union of not more than countably many level sets. Consider the following example of a nonmeasurable function f_E: $\mathbf{R} \to \mathbf{R}$, where $E \subset \mathbf{R}$ is a nonmeasurable set:

$$f_E(x) = \begin{cases} x & \text{if } x \in E, \\ -x & \text{if } x \notin E. \end{cases}$$

176. For the function $f(x)$ consider the sequence $\{f_n(x)\}$, where $f_n(x) = m/n$ when $m/n \le f(x) < (m+1)/n$, $n \in \mathbf{N}$, $m \in \mathbf{Z}$.

177. The measure of the set of numbers in $[0, 1]$ whose decimal expansion contains a 9 is equal to $10^{-1} + 9(10^{-2} + 9(10^{-3} + \cdots)) = 1$.

178. $f(x)$ is measurable, as a limit of measurable functions, and is almost everywhere equal to 9.

179–181. The functions $f(x)$ are continuous in the topology of uniform convergence, and μ is defined on the Borel sets relative to this topology.

182. It suffices to prove that any open set belongs to $\mathfrak{A}$. This follows from the facts that each ball belongs to S, the balls form a base of neighborhoods, and there are countably many different balls in X (there are countably many different radii, and there is a natural number in each ball).

183. From the fact that A is open it follows that it can be represented as the union of a collection of balls, and from this collection it is possible to choose a finite subcovering, since A is a closed subset of X and is, consequently, compact. It suffices to verify the countable additivity of μ for the half-ring of all balls.

184. It suffices to verify properties (a) and (b) for the half-ring of balls. The uniqueness of the measure follows from the fact that each ball of radius p^{-k}, $k = 0, 1, 2, \ldots$, consists of p balls of radius $p^{-(k+1)}$ whose measures coincide, by the property (b).

§3. Integrals

1. The Lebesgue Integral

185. Use the properties of absolutely convergent series.

186. (a) $$\sum_{n=0}^{\infty} \frac{1}{e^n} = \frac{e}{e-1};$$

(b) $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1;$$

(c) $$\sum_{n=0}^{\infty} \frac{1}{n!} = e.$$

187. For each partition $T = \{t_k\}$ define the upper and lower Lebesgue integral sums: $\bar{S}(T) = \sum_k t_{k+1}\mu(e_k)$, $\underline{S}(T) = \sum_k t_k \mu(e_k)$, where $e_k = \{x \in X: t_k \le f(x) < t_{k+1}\}$. Prove that if $f(x)$ is integrable, then $\bar{S}(T)$ and $\underline{S}(T)$ converge, $\underline{S}(T) \le \int_X f(x) d\mu \le \bar{S}(T)$, and $\int_X f(x) d\mu = \lim_{\lambda(T)\to 0} \underline{S}(T) = \lim_{\lambda(T)\to 0} \bar{S}(T)$ (use the estimate $\bar{S}(T) - \underline{S}(T) \le \lambda(T)\mu(X)$).

188. By using the functions

$$f_T(x) = \begin{cases} 0 & \text{if } x \text{ and } 0 \text{ belong to } [t_k, t_{k+1}], \\ f(x) & \text{if not,} \end{cases}$$

reduce the problem to Problem 187.

189. Use the properties of absolutely convergent series.

190. $S[0, 1]$ is isometrically imbedded in $L_1[0, 1]$, and there the sequence $f_n(x)$ has the limit function $f(x) \equiv x \notin S[0, 1]$.

191. (a) For $\alpha > -1 - \beta$ $(\beta > 0)$ and $\alpha > -1$ $(\beta < 0)$;
(b) for $\alpha > -1 - |\beta|$.

192. (a) Obvious;
(b) $\mu(\{x: f(x) > 0\}) \le \sum_n \mu(\{x: f(x) > 1/n\}) = 0$.

193. $\mu(\{x \in [a, b]: t_k \le \varphi(x) < t_{k+1}\}) = \psi(t_{k+1}) - \psi(t_k)$. By the mean value theorem, $\psi(t_{k+1}) - \psi(t_k) = \psi'(\xi_k)(t_{k+1} - t_k)$.

194. In the notation of the hints for Problem 187 define the measurable subset $E(T) = \bigcup_k \{(x, y): f(x) \in [t_k, t_{k+1}), 0 \le y \le t_k\}$ of the plane. The assertion of the problem follows from the fact that $\mu E(T) = \underline{S}(T)$, $\mu E \triangle E(T) \le \lambda(T)(b - a)$.

195. Use a monotonically nondecreasing sequence of simple functions that converges uniformly to $f(x)$.

196. Prove that $|f(x)|$ is integrable if $f(x)$ is. Use the equality

$$\int_A [g(x) + h(x)]\, d\mu(x) = \int_A g(x)\, d\mu(x) + \int_A h(x)\, d\mu(x).$$

197. If $f(x)$ is not integrable, then for any $c > 0$ there exists a simple function $g(x)$ such that $f(x) - 1 \le g(x) \le f(x)$, and

$$\int_X g(x)\, d\mu(x) > c.$$

Obviously, there exists a set A on which $g(x)$ takes finitely many values such that

$$\int_A f(x)\, d\mu(x) \ge \int_A g(x)\, d\mu(x) > c - 1.$$

198. Let $a_n = \mu(\{x \in X: 2^n \le f(x) < 2^{n+1}\})$. The integrability of f is equivalent to the convergence of each of the series $\sum_{n=0}^{\infty} a_n 2^n$ and $\sum_{n=0}^{\infty} a_n 2^{n+1}$. Show that the partial sums of the series appearing in the condition of the problem are between the corresponding partial sums of these series.

199. Use the method of solution of Problem 198.

200. (a) 1; (b) 1; (c) $\pi/4$; (d) 1, since the set $\{(x, y): xy \in \mathbf{Q}\}$ consists of countably many hyperbolas $xy = \text{const}$.

201. The Riemann integral is defined only for bounded functions. Take a sequence $\{P_n\}$ of partitions of $[a, b]$ such that P_{k+1} refines P_k and the diameters of the partitions converge to 0. Let $m_n(x)$ and $M_n(x)$ be the functions corresponding to the lower and upper Darboux sums for P_n, $m_n(x) \leq f(x) \leq M_n(x)$. Define $m(x) = \lim_{n\to\infty} m_n(x)$, $M(x) = \lim_{n\to\infty} M_n(x)$. Prove that $f(x)$ is Riemann-integrable if and only if

$$\int_a^b m(x)\,dx = \int_a^b M(x)\,dx.$$

By Problem 192, the last condition is equivalent to the condition $m(x) = M(x)$ almost everywhere, which is equivalent (for $x \notin P_k \forall k$) to the condition that $f(x)$ is continuous almost everywhere.

202. By a linear orthogonal change of variables the problem can be reduced to the situation when the matrix A is diagonal.

203. Replace the integral $\int_0^1 x^2(t)\,dt$ by the integral sum

$$\left(\frac{1}{n}\right) \sum_{k=1}^{n} x^2\left(\frac{k}{n}\right).$$

Then the integrand $\varphi_{abn}(x)$ is a step function and its integral can be computed by the formula

$$I_n = \pi^{-n/2} n^{n/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left\{-n \sum_{k=0}^{n-1} (\tau_{kn} - \tau_k)^2 - a\tau_0^2 - b^2 \sum_{k=1}^{n} \tau_k^2\right\} \times d\tau_0\, d\tau_1 \cdots d\tau_n.$$

By Problem 202, $In = \sqrt{(\pi/n)}(\det A)^{-1/2}$, where A is the $(n+1) \times (n+1)$ matrix

$$\begin{pmatrix}
1 + \frac{a}{n}, & -1, & 0, & 0, \ldots, & 0, & 0, & 0 \\
-1, & 2 + \frac{b^2}{n^2}, & -1, & 0, \ldots, & 0, & 0, & 0 \\
0, & 1, & 2 + \frac{b^2}{n^2}, & -1, \ldots, & 0, & 0, & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0, & 0, & 0, & 0, \ldots, & -1, & 2 + \frac{b^2}{n^2}, & -1 \\
0, & 0, & 0, & 0, \ldots, & 0, & -1, & 1 + \frac{b^2}{n^2}
\end{pmatrix}$$

To compute det A proceed as follows. Let $D_N(\lambda, \mu)$ be the determinant of the $N \times N$ matrix with the number λ on the main diagonal, the number μ on the two adjacent diagonals, and zero elsewhere. Expanding this determinant by the first row, we get the basic identity: $D_N(\lambda, \mu) = \lambda D_{N-1}(\lambda, \mu) - \mu^2 D_{N-2}(\lambda, \mu)$. This gives us by induction that $D_N(\lambda, \mu) = (x_+^{N+1} - x_-^{N+1})/(x_+ - x_-)$, where $x_\pm$ are the roots of the quadratic equation $x^2 - \lambda x + \mu^2 = 0$. If $\lambda = 2 + (b^2/n^2)$, $\mu = -1$, then $x_\pm = 1 + (b^2/2n^2) \pm (b/n)\sqrt{1 + (b^2/4n^2)}$.

For these λ, μ denote $D_N(\lambda, \mu)$ simply by D_N. Then

$$\det A = D_{n+1} + \left(\frac{a}{n} - \frac{b^2}{n^2} - 1\right)D_n + (-1)D_n + \left(\frac{a}{n} - \frac{b^2}{n^2} - 1\right)(-1)D_{n-1}$$

(expansion by the first and last rows). By using the basic identity, this expression can be reduced to the form

$$\frac{a}{n}(D_n - D_{n-1}) - \frac{b^2}{n^2} D_{n-1}.$$

Let us compute D_n:

$$\begin{aligned} D_n &= \frac{x_+^{n+1} - x_-^{n+1}}{x_+ - x_-} \\ &= \frac{\left(1 + \frac{b}{n} + \frac{b^2}{2n^2} + o(n^{-2})\right)^{n+1} - \left(1 - \frac{b}{n} + \frac{b^2}{2n^2} + o(n^{-2})\right)^{n+1}}{\frac{2b}{n} + o(n^{-2})} \\ &= \frac{\exp\left\{\left[\frac{b}{n} + o(n^{-2})\right](n+1)\right\} - \exp\left\{-\left[\frac{b}{n} + o(n^{-2})\right](n+1)\right\}}{\frac{2b}{n} + o(n^{-2})} \\ &= \frac{\exp\left\{b + \frac{b}{n} + o(n^{-1})\right\} - \exp\left\{-b - \frac{b}{n} + o(n^{-1})\right\}}{\frac{2b}{n} + o(n^{-2})} \\ &= n\frac{\sinh b}{b} + \cosh b + o(1). \end{aligned}$$

Similarly, $D_{n-1} = n(\sinh b)/b + o(1)$. From this,

$$\begin{aligned} \det A &= \frac{a}{n}(\cosh b + o(1)) + \frac{b^2}{n^2}\left(n\frac{\sinh b}{b} + o(1)\right) \\ &= \frac{a\cosh b + b\sinh b}{n} + o(n^{-1}). \end{aligned}$$

Hence,

$$I_n = \sqrt{\frac{\pi}{a\cosh b + b\sinh b}} + o(1).$$

Thus,

$$\int_{C[0,1]} \varphi_{abn}(x)\, d\mu(x) = I_n \to \sqrt{\frac{\pi}{a\cosh b + b\sinh b}} \quad \text{as } n \to \infty.$$

By Fatou's lemma, this implies that the functions $\varphi_{ab}(x) = \lim_{n\to\infty} \varphi_{abn}(x)$ are μ-integrable for $a > 0$, $b \geq 0$ or $a \geq 0$, $b > 0$. Since $\varphi_{abn}(x) \leq \varphi_{a0n}(x) = \varphi_{a0}(x)$, this implies that the Lebesgue theorem is applicable, and

$$\int \varphi_{ab}\, d\mu = \lim \int \varphi_{abn}\, d\mu = \sqrt{\frac{\pi}{a\cosh b + b\sinh b}}.$$

204. Each function $x \in C[0, 1]$ can be represented uniquely in the form $x(t) = c + y(t)$, where $c = x(0)$ is a constant, and $y \in C_0[0, 1]$. The measure μ_0 on $C_0[0, 1]$ is defined by the formula

$$\mu_0(\chi(t_1, \ldots, t_n; \Delta_1, \ldots, \Delta_n)) = \pi^{-n/2} \prod_{k=1}^{n} (t_k - t_{k-1})^{-1/2}$$
$$\times \int_{\Delta_1} \cdots \int_{\Delta_n} \exp\left\{-\sum_{k=1}^{n} \frac{(\tau_k - \tau_{k-1})^2}{(t_k - t_{k-1})}\right\} d\tau_1, \ldots, d\tau_n,$$

where $t_0 = \tau_0 = 0$, and the set $\chi(t_1, \ldots, t_n; \Delta_1, \ldots, \Delta_n)$ is defined in the same way as before.

205. (a)

$$\mu_0(C_0[0, 1])$$
$$= \mu_0(\chi(t_1, \ldots, t_n; R, \ldots, R))$$
$$= \pi^{-n/2} \prod_{k=1}^{n} (t_k - t_{k-1})^{-1/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left[-\sum_{k=1}^{n} \frac{(\tau_k - \tau_{k-1})^2}{t_k - t_{k-1}}\right]$$
$$\times\, d\tau_1, \ldots, d\tau_k.$$

Denoting $\tau_k - \tau_{k-1}$ by σ_k and $t_k - t_{k-1}$ by s_k, we get the value

$$\pi^{-n/2} \prod_{k=1}^{n} s_k^{-1/2} \prod_{k=1}^{n} \int_{-\infty}^{\infty} e^{\sigma_k^2/s_k}\, d\sigma_k = 1.$$

(b) Since the measure μ_0 does not change under the mapping $x \to -x$, but the integrand changes sign, the integral is equal to 0, if it exists. The integrability of the integrand follows from Problem 203 and the Cauchy–Bunyakovskii inequality.

(c) Let $x(t) = c + y(t)$, $y \in C_0[0, 1]$. The result of Problem 203 can be formulated as follows. If

$$\varphi_1(y) = \int_0^1 y(t)\,dt, \qquad \varphi_2(y) = \int_0^1 y^2(t)\,dt,$$

then

$$\int_R \int_{C_0[0,1]} \exp\{-ac^2 - b^2[c^2 + 2c\varphi_1(y) + \varphi_2(y)]\}\,dc\,d\mu_0(y)$$
$$= \sqrt{\frac{\pi}{a\cosh b + b\sinh b}}.$$

Take $c = t/\sqrt{a}$ here and let a go to $+\infty$. Then we come to the equation

$$\int_R \int_{C_0[0,1]} e^{-t^2 - b^2\varphi_2(y)}\,dt\,d\mu_0(y) = \sqrt{\frac{\pi}{\cosh b}}$$

or

$$\int_{C_0[0,1]} e^{-b^2\varphi_2(y)}\,d\mu_0(y) = \frac{1}{\sqrt{\cosh b}}.$$

From this it follows that the integrals $\int_{C_0[0,1]} \varphi_2^k(y)\,d\mu_0(y)$ can be found by expanding the function $1/\sqrt{\cosh b}$ in a Taylor series. In particular,

$$\int_{C_0[0,1]} \varphi_2(y)\,d\mu_0(y) = \frac{1}{4}, \qquad \int_{C_0[0,1]} \varphi_2^2(y)\,d\mu_0(y) = \frac{7}{48}.$$

206. Suppose that $|f(x)| \le M$. It is easy to see that

$$\int_X |f(x)|\,d\mu \le \sum_{n=1}^{\infty} \left(\frac{M}{2^{n-1}} - \frac{M}{2^n}\right)\mu\left\{x \in X : |f(x)| > \frac{M}{2^n}\right\}$$
$$\le AM^{1-\alpha} \sum_1^{\infty} 2^{(\alpha-1)n} < \infty,$$

which implies that $f(x)$ is integrable.

207. (a) Use the result of Problem 130.

(b) By the assertion of Problem 130,

$$\mu(M(\{a_k\})) = \prod_{k=1}^{\infty} \sum_{l=1}^{[a_k]} \log_2((l+1)^2 l^{-1}(l+2)^{-2})$$
$$= \prod_{k=1}^{\infty} \log_2(2([a_k]+1)([a_k]+2)^{-1}).$$

The general factor in this product can be written as $1 - \log_2(([a_k]+2)([a_k]+1)^{-1})$. Answer: $\sum_{k=1}^{\infty} (1/a_k) = \infty$.

2. Functions of Bounded Variation and the Lebesgue–Stieltjes Integral

208. A trivial check.

209. (a) Yes, since $f(x)$ satisfies a Lipschitz condition;
(b) no.

210, 211. Follow from the fact that the assertions of the problems are correct for monotone functions.

212. Use the mean value theorem.

213. Use the Darboux integral sums and the mean value theorem.

214. (q) $e^{50} - 1$; (b) $\ln 2$; (c) 8; (d) $8(3)^{1/2}/9$.

215. $\{a_k\}$ are points of discontinuity (see Problem 210),

$$c_k = \Phi(a_k + 0) - \Phi(a_k).$$

216. Use the estimates:

(a)
$$\begin{aligned} |f \cdot g(x_{n+1}) - f \cdot g(x_n)| &\leq |f(x_{n+1})g(x_{n+1}) - f(x_n)g(x_{n+1})| \\ &\quad + |f(x_n)g(x_{n+1}) - f(x_n)g(x_n)| \\ &\leq |f(x_{n+1}) - f(x_n)| \sup\{|g(x)|\} \\ &\quad + |g(x_{n+1}) - g(x_n)| \sup\{|f(x)|\}; \end{aligned}$$

(b)
$$\left|\frac{1}{f(x_{n+1})} - \frac{1}{f(x_n)}\right| = \frac{|f(x_n) - f(x_{n+1})|}{|f(x_{n+1})||f(x_n)|} \leq \alpha^{-2}|f(x_{n+1}) - f(x_n)|.$$

217. (a) Consider the functions

$$f(x) = x, \qquad \varphi(x) = \begin{cases} x \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

(b) No. Construct a monotone function $\varphi(x)$ such that $\varphi(1/2^n) = 1/n$ for $n = 1, 2, 3, \ldots$, $\varphi(0) = 0$, and an $f(x)$ with bounded variation on $[0, 1]$ such that $f(x_n) = 2^{-n}$, $f(y_n) = 0$ for some sequence $1 > x_1 > y_1 > x_2 > y_2 > \cdots > 0$. Then $\varphi(f)$ has unbounded variation becuase the series $\sum_{n=1}^{\infty} (1/n)$ diverges.

218. Obvious.

219. Let $\varphi: I = [a, b] \mapsto S = [0, 1] \times [0, 1]$ be the mapping onto S defined by the formula $\varphi(x) = (f(x), g(x))$. Consider the n^2 points in S of the form $(k/n, l/n)$, where $k, l = 1, \ldots, n$, and their pre-images $a \leq x_1 < \cdots < x_{n^2} \leq b$, $\varphi(x_i) \neq \varphi(x_j)$ for $i \neq j$.

We have $\mathrm{Var}_a^b[f(x)] + \mathrm{Var}_a^b[g(x)] \geq \sum_{i=1}^{n^2-1} [|f(x_{i+1}) - f(x_i)| + |g(x_{i+1}) - g(x_i)|] \geq (n^2 - 1)/n$, which contradicts the fact that $f(x)$ and $g(x)$ are both of bounded variation.

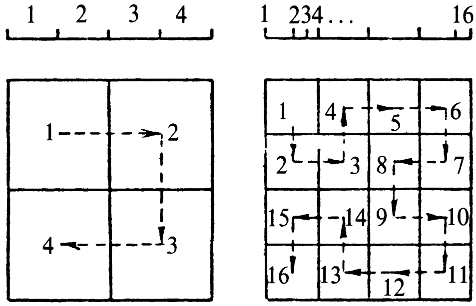


Figure 3

The assertion is not true without the requirement that f and g have bounded variation. For $n = 1, 2, 3, \ldots$ partition S and I into 4^n equal closed squares and intervals, respectively, and establish a correspondence between these squares and intervals in such a way that if a certain square corresponds to a certain interval, then its subsquares correspond to subintervals of the indicated interval (Fig. 3). If $x \in I$, then x is the point of intersection of some sequence of closed intervals to which there corresponds a sequence of nested closed squares whose point of intersection is taken as $\varphi(x)$.

220. Verify for integral sums and pass to the limit.

221. From the discontinuity of $\Phi(x)$ at the point c it follows that there exist sequences $\{a_n\}$ and $\{b_n\}$ such that $a < a_1 < a_2 < \cdots < c < \cdots < b_2 < b_1 < b$, $\lim_{n\to\infty}(b_n - a_n) = 0$, $|\Phi(b_n) - \Phi(a_n)| > K > 0\ \forall n$. From the integrability of f with respect to Φ it follows that $\lim_{n\to\infty}|\Phi(b_n) - \Phi(a_n)|\, f(x)|_{a_n}^{b_n} = 0$. Thus, $f(x)|_{a_n}^{b_n} \to 0$, which means that $f(x)$ is continuous at the point c.

222. By Problem 220(b), $\int_a^b f(x)\, d\Phi(x) = \int_a^b f(x)\, d\Phi_N(x) + \int_a^b f(x)\, d\tilde{\Phi}_N(x)$, where $\Phi_N(x)$ is nonzero at the first N points, and $\tilde{\Phi}_N(x)$ is nonzero at the remaining points. Obviously, the first integral is zero; apply the estimate in Problem 220(a) to the second integral.

223. Suppose that Ψ has bounded variation and is different from Φ only at the points of discontinuity. By Problem 210, $\Phi - \Psi$ is nonzero at not more than countably many points. Use Problems 220(b) and 222.

224. Consider the two partitions $x_0, x_1, \ldots, x_n$ and $\xi_0 = a$, $\xi_1, \ldots, \xi_{n+1} = b$, where $x_{i-1} \le \xi_i \le x_i$ for $i = 1, \ldots, n$. We have

$$\sum_{i=1}^{n} f(\xi_i)[g(x_i) - g(x_{i-1})]$$

$$= f(b)g(b) - f(a)g(a) - \sum_{i=1}^{n+1} g(x_i)[f(\xi_i) - f(\xi_{i-1})].$$

225. Form the integral sum and consider the intervals of the partition containing the points $\{c_k\}$ in their interiors.

226. See Problem 224. Answer: $b\varphi(b) - a\varphi(a) - \int_a^b \varphi(x)\, dx$.

227. $I_1 = -1 \cdot 1 + 2(-2) = -5, I_2 = 1/4 \cdot 1 + (9/4)(-2) = -17/4.$

228.

$$I_1 = \int_{-2}^{-1} x\,dx + 1 + \int_0^2 2x^2\,dx = 17/6,$$

$$I_2 = \int_{-2}^{-1} x^2\,dx + 1 + \int_0^2 2x^3\,dx = 34/3,$$

$$I_3 = \int_{-2}^{-1} (x^3 + 1)\,dx + 1 + \int_0^2 2(x^3 + 1)x\,dx = 301/20.$$

229. Break up $[0, 1]$ into 2^n equal intervals, and for $y \in [\min f(x), \max f(x)]$ define $N_n(y)$ to be the number of intervals in which there is at least one root of the equation $f(x) = y$. The functions $N_n(y)$ are measurable, since they have no more than finitely many discontinuities. Prove that $N_f(y) = \lim_{n\to\infty} N_n(y)$, and use Problem 154. The equality of the integral and the variation can be verified directly.

230. (a) Denote $\int_0^1 x^k\,d\Phi(x)$ by a_k. Use the fact that the function $\Phi(x)$ has the properties that $\Phi(x/3) = (1/2)\Phi(x)$, $\Phi(2/3 + x/3) = 1/2 + (1/2)\Phi(x)$, as follows easily from its definition. Therefore,

$$a_k = \int_0^{1/3} x^k\,d\Phi(x) + \int_{2/3}^1 x^k\,d\Phi(x) = \frac{1}{3^k}\cdot\frac{1}{2}\left[\int_0^1 y^k\,d\Phi(y) + \int_0^1 (2 + y)^k\,d\Phi(y)\right]$$

$$= \frac{a_k}{3^k} + \frac{1}{2\cdot 3^k}\sum_{s=1}^{k} C_k^s\cdot 2^s a_{k-s}.$$

From this,

$$a_k = \frac{1}{2(3^k - 1)}\sum_{s=1}^{k} C_k^s\cdot 2^s a_{k-s}.$$

Since $a_0 = 1$, we get successively that $a_1 = 1/2$, $a_2 = 3/8$, $a_3 = 5/16$, $a_4 = 87/320, \ldots.$

(b) Denote $\int_0^1 e^{ax}\,d\Phi(x)$ by $\Psi(a)$ and use the method of solution of part (a). We have

$$\Psi(a) = \int_0^{1/3} e^{ax}\,d\Phi(x) + \int_{2/3}^1 e^{ax}\,d\Phi(x)$$

$$= \frac{1}{2}\int_0^1 e^{ay/3}\,d\Phi(y) + \frac{1}{2}\int_0^1 e^{2a/3 + ay/3}\,d\Phi(y) + \frac{1}{2}(1 + e^{2a/3})\Psi\left(\frac{a}{3}\right)$$

$$= e^{2a/3}\cosh\frac{a}{3}\,\Psi\left(\frac{a}{3}\right).$$

Integrating the formula obtained, we get

$$\Psi(a) = e^{a/3}e^{a/9}\cdots e^{a\cdot 3^{-k}}\cosh\frac{a}{3}\cosh\frac{a}{9}\cdots\cosh\frac{a}{3^k}\,\Psi\left(\frac{a}{3^k}\right).$$

Obviously, $\Psi(a) \to 1$ as $a \to 0$. Therefore,

$$\Psi(a) = e^{a/2} \prod_{k=1}^{\infty} \cosh\left(\frac{a}{3^k}\right).$$

(c) From the result of part (b) it follows that

$$\int_0^1 \sin \pi x \, d\Phi(x) = \frac{1}{2i} [\Psi(\pi i) - \Psi(-\pi i)] = \prod_{k=1}^{\infty} \cos\left(\frac{\pi}{3^k}\right).$$

3. Properties of the Lebesgue Integral

231. (a) Follows from Tchebycheff's inequality:

$$\mu(\{x \in X : |f(x)| > \varepsilon\}) \leq \varepsilon^{-1} \int_X |f(x)| \, d\mu(x).$$

(b) Yes, for a sequence of bounded functions on a set of finite measure. No, in the general case. Counterexamples:

$f_n(x) = n/(n^2 + x^2)$ on $(-\infty, \infty)$; $\quad f_n(x) = n/(1 + n^2x^2)$ on $[-1, 1]$.

232. Consider the example: $X = \mathbf{R}$, μ is Lebesgue measure $f_n(x) = (1/n)\chi_{[-n^2, n^2]}$.

233. $f_n(x) = n\chi_{[0, 1/n]}$.

234. To derive the triangle inequality use the inequality $\lambda \leq (\lambda a + \mu b)/(a + b) \leq \mu$ for $\lambda \leq \mu$, $a \geq 0$, $b \geq 0$.

235. Let $\{f_n\}$ be a sequence of functions that is Cauchy in measure. Prove that $\{f_n\}$ contains a subsequence $\{f_n^1\}$ such that $\mu\{x \in X : \exists \lim_{n\to\infty} f_n^1(x)\} \geq \mu(X) - 1$. Similarly, the sequence $\{f_n^k\}$ contains a subsequence $\{f_n^{k+1}\}$ such that $\mu\{x \in X, \exists \lim_{n\to\infty} f_n^{k+1}(x)\} \geq \mu(X) - [1/(k+1)]$. It is easy to verify that the sequence $\{f_n^n\}$, $n = 1, 2, \ldots$, converge almost everywhere on X to some measurable function $f(x)$ and that $f_n(x)$ converges to $f(x)$ in measure. The convergence in measure of a Cauchy sequence in $M[0, 1]$ follows from the inequality

$$\rho(f, g) \geq \frac{\sigma}{1 + \sigma} \mu\{x \in X : |f(x) - g(x)| \geq \sigma\},$$

which holds for any $\sigma > 0$.

236. It must be shown that the $L_1(X, \mu)$-limit of a sequence of characteristic functions is equivalent to a characteristic function. Use the result of Problem 231(a) and Theorem 11 in §2 of Ch. II.

237. Let $\int_X f(x) \, d\mu = A$. For any $\varepsilon > 0$ there exists a subset E_1 of finite measure in X for which $\int_{E_1} f(x) \, d\mu > A - \varepsilon$. By the theorem on the absolute continuity of the integral, there exists a $\delta(\varepsilon) > 0$ such that $\int_E f(x)\,d\mu < \varepsilon$ for all sets E of measure $< \delta(\varepsilon)$. By Egorov's theorem, the convergence of f_n to f is uniform on some subset $E_2 \subset E_1$ such that $\mu(E_1 \setminus E_2) < \delta(\varepsilon)$. Further, there exist indices $n_1(\varepsilon)$ and $n_2(\varepsilon)$ such that $\int_{E_2} |f_N(x) - f(x)|\,dx < \varepsilon$ for $N > n_1(\varepsilon)$, and $|\int_X f_N(x)\,d\mu - A| < \varepsilon$ for $N > n_2(\varepsilon)$. Let $n(\varepsilon) = \max\{n_1(\varepsilon), n_2(\varepsilon)\}$. From all the preceding it follows that for $N > n(\varepsilon)$

$$\begin{aligned}\int_X |f_N - f|\,d\mu &= \int_{E_2} |f_N - f|\,d\mu + \int_{X\setminus E_2} f_N\,d\mu + \int_{X\setminus E_2} f\,d\mu \\ &\le \varepsilon + \int_X f_N\,d\mu - \int_{E_2} f_N\,d\mu + \int_X f\,d\mu - \int_{E_2} f\,d\mu \\ &\le \varepsilon + 1 + \varepsilon - (1 - 3\varepsilon) + 1 - (1 - 2\varepsilon) = 7\varepsilon\end{aligned}$$

238. Let $a \in \mathbf{R}$, $X_a = \{x \in X : f(x) \le a\}$, and $p(a) = \mu(X_a)$. Then $p(a)$ is a nondecreasing function on $\mathbf{R}$ such that $p(a) \to 0$ as $a \to -\infty$ and $p(a) \to 1$ as $a \to +\infty$.

Suppose first for simplicity that $p(a)$ is continuous. Then there exists an inverse function $g(t)$ defined on $(0, 1)$ and having the property that $p(g(t)) = t$. We show that $\inf_{\mu(A)=t} \int_A f(x)\,d\mu$ is attained for $A = X_{g(t)}$. Indeed, any set A of measure t is obtained by discarding some subset Y of $X_{g(t)}$ and then adding to it some subset $Z \subset X\setminus X_{g(t)}$, such that $\mu(Y) = \mu(Z)$.

But at the points of Y the function f does not exceed $g(t)$, and at the points of Z it is greater than $g(t)$. Thus, in this case the set A supplying the infimum is uniquely determined to within a set of μ-measure zero.

We now show that the function f on X and the function g on $(0, 1)$ are equimeasurable, i.e., $\mu_0(\{t \in (0, 1) : g(t) \le a\}) = \mu(\{x \in X : f(x) \le a\})$ for all $a \in \mathbf{R}$, where μ_0 denotes the Lebesgue measure on the interval $(0, 1)$. The left-hand side of this equation is equal to $p(a)$, since the inequality $g(t) \le a$ is equivalent to $t \le p(a)$. The right-hand side is equal to $p(a)$ by definition.

Since f and g are equimeasurable, $\int_0^1 \chi(g(t))\,dt = \int_X \chi(f(x))\,d\mu(x)$ for any Borel function χ. In particular, setting

$$\chi(h) = \begin{cases} 0 & \text{for } h > a, \\ h & \text{for } h \le a, \end{cases}$$

we get the desired equality.

239. No.

240. Let $f(x, y) = (x^2 + y^2)/[(x^2 + y^2)^2]$; for $x \ne 0$ we have $f(x, y) = (\partial/\partial y)(y/(x^2 + y^2))$ and $\int_0^1 f(x, y)\,dy = 1/(x^2 + 1)$. Consequently, $\int_0^1 (\int_0^1 f(x, y)\,dy)\,dx = \pi/4$. Similarly,

$$\int_0^1 \left(\int_0^1 f(x, y)\,dx\right) dy = -\frac{\pi}{4}.$$

241. Obviously, both iterated integrals are equal to 0. If the double integral exists, then it exists also on the set $\{0 \le x \le 1, 0 \le y \le 1\}$, and Fubini's theorem is applicable; but

$$\int_0^1 \frac{xy}{(x^2+y^2)^2}\,dy = \frac{1}{2x} - \frac{x}{2(x^2+1)}$$

for $x \neq 0$, and this function is not integrable on $(0, 1]$.

242.

$$\int_0^1 \left(\int_0^1 f(x, y)\,dx\right) dy = 0, \qquad \int_0^1 \left(\int_0^1 f(x, y)\,dy\right) dx = 1.$$

243. Let μ_0 be Lebesgue measure, μ a quasi-invariant measure, and X any Borel subset of the line. Consider the set Y on the plane $\mathbf{R}^2$ consisting of the pairs (x_1, x_2) for which $x_1 - x_2 \in X$. Applying Fubini's theorem to this set for the product measure $\mu \times \mu_0$, we get

$$\int_{-\infty}^{\infty} \mu_0(x_1 - X)\,d\mu(x_1) = \int_{-\infty}^{\infty} \mu(x_2 + X)\,d\mu_0(x_2).$$

From this, $\mu(x_2 + X) = 0$ for almost all x_2 if $\mu_0(x_1 - X) = \mu_0(X) = 0$, hence $\mu(X) = 0$. But if $\mu_0(X) \neq 0$, then the left-hand side is $\mu_0(X)\mu(\mathbf{R})$; hence, $\mu(x_2 + X) \not\equiv 0$, i.e., $\mu(X) \neq 0$. Thus, the measures μ and μ_0 are equivalent.

244. The variation of ν_i is equal to $|\nu_i| = |f_i|\mu$. Therefore, the conditions $|\nu_1|(A) = 0$ and $|\nu_2|(A) = 0$ are equivalent if and only if the conditions $\mu(A \cap N_1) = 0$ and $\mu(A \cap N_2) = 0$ are equivalent. The latter is true for all A if and only if $\mu(N_1 \triangle N_2) = 0$.

245. By the Radon–Nikodým theorem, for each subset A of finite μ-measure there exists a measurable function $\rho_A(x)$ such that $\nu(B) = \int_B \rho_A(x)\,d\mu(x)$ for any measurable $B \subset A$. Now let $X = \bigcup X_i$, $\mu(X_i) < \infty$, and $\rho_i = \rho_{X_i}$. It is easy to see that the functions ρ_i and ρ_j coincide almost everywhere on $X_i \cap X_j$. Therefore, there exists a measurable function ρ on X that coincides with ρ_i almost everywhere on X_i. It has the necessary property.

246. Suppose the contrary. Then $\mu(A \cap [0, 1]) = 1/2$, and there exists a covering of $A \cap [0, 1]$ by disjoint intervals $\{\Delta_n\}_{n=1,2,\dots}$ such that $\sum_n \mu(\Delta_n) < 1$. We get the contradiction:

$$\frac{1}{2} = \mu(A \cap [0, 1]) \le \sum_n \mu(A \cap \Delta_n) < \frac{1}{2}.$$

247. For bounded functions f the assertion is easy to deduce with the help of the results in Problem 149 or 150. Prove that the desired equality $F'(x) \overset{\text{a.e.}}{=} f(x)$ is preserved under monotone limits.

248. Define a signed measure ν on the half-ring of half-open intervals in $[a, b]$ by setting $\nu([\alpha, \beta)) = F(\beta) - F(\alpha)$. The absolute continuity of F implies the absolute continuity of ν with respect to Lebesgue measure (verify!). The Radon–Nikodým theorem then gives us the existence of a function $f \in L_1[a, b]$ such that $F(\beta) - F(\alpha) = \int_\alpha^\beta f(x)\,dx$. Now use the result of Problem 247.

249. (a) This is a direct consequence of the definitions. (b) Any function φ in S can be approximated by these functions. (c)–(d) The polynomials and the trigonometric polynomials are dense (even in the sense of uniform convergence) in the space of continuous functions, and the latter space is dense in $L_1[0, 1]$, by part (b).

250. (a) Follows from the definition of the metric in $L_1(\mathbf{R})$.

(b) Note that every piecewise constant compactly supported function can be approximated in the $L_1(\mathbf{R})$ metric by piecewise linear compactly supported functions, and use part (a).

(c) Consider the collection L_0 of all functions $f \in L_2(\mathbf{R})$ such that $\int_R e^{-x^2} x^k f(x)\,dx = 0$ for all nonnegative integers k. Show that the functions f are characterized by the condition:

$$\varphi_f(\lambda) = \int_{\mathbf{R}} e^{-x^2 + \lambda x} f(x)\,dx = 0$$

for all $\lambda \in \mathbf{C}$. Indeed, the integrability of the functions $e^{-x^2+\lambda x} f(x)$ and $xe^{-x^2+\lambda x} f(x)$ implies that $\varphi_f(\lambda)$ is differentiable for all $\lambda \in \mathbf{C}$. Hence, it is analytic. Since $\varphi_f^{(k)}(0) = \int_{\mathbf{R}} e^{-x^2} x^k f(x)\,dx$, the conditions $f \in L_0$ and $\varphi_f \equiv 0$ are equivalent. Let $a(\lambda)$ be any integrable function on $\mathbf{R}$. Then

$$0 = \int_{\mathbf{R}} a(\lambda)\varphi_f(a)\,d\lambda = \int_{\mathbf{R}} \int_{\mathbf{R}} a(\lambda) e^{-x^2 + i\lambda x} f(x)\,dx = \int_{\mathbf{R}} e^{-x^2} b(x) f(x)\,dx,$$

where $b(x) = \int_{\mathbf{R}} a(\lambda) e^{i\lambda x}\,d\lambda$. Show that by suitably choosing $a(x)$ it is possible to get any continuous piecewise linear function as $b(x)$. From this it follows that $e^{-x^2} f(x) = 0$ almost everywhere, and, consequently, $f(x) = 0$ almost everywhere.

Note that multiplication of $a(\lambda)$ by $e^{i\lambda t}$ leads to translation of $b(x)$ by t. Moreover, the correspondence between $a(\lambda)$ and $b(x)$ is linear. Therefore, it suffices to obtain as $b(x)$ the elementary function equal to 0 outside $[-1, 1]$ and equal to $1 - |x|$ on this segment. For this, it turns out to be sufficient to take $a(\lambda) = (2/\pi)[(\sin^2 \lambda/2)/\lambda^2]$ (cf. Problem 668(h)).

251. Prove that the set of functions for which translation is continuous is closed in $L_1(\mathbf{R})$. Verify this condition for simple functions.

252. Show that $f_1(t)f_2(x - t)$ is measurable with respect to Lebesgue measure on the (x, t)-plane, and apply the Fubini theorem for σ-finite measures. The use of Fubini's theorem can be avoided by first considering the convolution on the dense subset of L_1 consisting of the continuous compactly supported functions, then the case of a bounded function $f_1(t) \in L_1$, and, finally, by approximating $f_1(t)$ by bounded functions.

253. The condition implies that $\int_A f(x)\,dx = 0$ for any elementary set A. Since the integral is a continuous function of the set A, this statement is true for all measurable sets, in particular, for the sets $X_+ = \{x: f(x) > 0\}$ and $X_- = \{x: f(x) < 0\}$. Now use Problem 192(b).

Chapter III

Linear Topological Spaces and Linear Operators

§1. General Theory

1. Topology, Convexity, and Seminorms

254. The operations of translation by a vector and multiplication by a scalar are homeomorphisms of an LTS.

255. For any neighborhood of zero W in an LTS there is a balanced neighborhood of zero U such that $U + U \subset W$. If $x \notin X$, then there exists a neighborhood of zero W such that $x + W \cap X = \varnothing$. Consider $X + V$ and and $x + V$, where V is a balanced neighborhood of zero and $V + V \subset W$.

256. Choosing a basis $\{e_i\}$ in the given space, we can identify it with K^n. Show that any norm p in K^n is equivalent to the norm $\|x\| = \sum_{i=1}^n |x_i|$, where $x = (x_1, \ldots, x_n) \in K^n$. Indeed, $p(x) = p(\sum x_i e_i) \leq \sum_{i=1}^n |x_i| p(e_i) \leq C\|x\|$, where $C = \max_i p(e_i)$. Next, the inequality $|p(x) - p(y)| \leq p(x - y) \leq C\|x - y\|$ shows that p is continuous in K^n with respect the (usual) topology determined by the norm $\|\cdot\|$. The sphere $\|x\| = 1$ is closed and bounded, hence compact in K^n. Let $c = \min_{\|x\|=1} p(x)$. Then

$$C\|x\| \geq p(x) \geq c\|x\|. \quad \text{QED}$$

257. Prove by induction on $\dim L$ that every linear isomorphism between L and $K^{\dim L}$ (K is the field of scalars) is a homeomorphism.

258. Consider inclusions of the neighborhoods of zero.

259. (a) $A + B = \bigcup_{x \in B} (A + x)$. Use Problem 254.

(b) Suppose that $a \notin A + B$. Then for each $x \in B$ the set $x + A$ is closed, consequently (see Problem 255), there exists a balanced neighborhood of zero $U(x)$ for which $(a + U(x)) \cap (x + A) = \varnothing$. The sets $x + (1/2)U(x)$ form an open covering of B. Let $\{x_i + (1/2)U(x_i): 1 \leq i \leq n\}$ be a finite subcovering, and $V = \bigcap_{1 \leq i \leq n} (1/2)U(x_i)$. Prove that $(a + V) \cap (A + B) = \varnothing$.

260. For each $t \in \mathbf{R}$ and each integer n let $e_n(t) = e^{int}$, $f_n = e_{-n} + ne_n$, $n = 1, 2, \ldots$. Consider these functions as elements of the space $L_2(-\pi, \pi)$.

Let X_1 be the smallest closed subspace of L_2 containing $e_0, e_1, \ldots$, and X_2 the smallest closed subspace of L_2 containing $f_1, f_2, \ldots$. Show that $X_1 + X_2$ is dense in L_2, but not closed.

For example, the vector $x = \sum_{n=1}^{\infty} n^{-1}e_{-n}$ is in L_2, but not in $X_1 + X_2$.

261, 262. Consequences of the definition of a convex set.

263. See the hint for Problem 264. Answer: $S = \mu(A)$, L is the perimeter of A.

264. For greater simplicity and clarity we give a proof for the case of two convex sets in the plane. Suppose, in addition, that these convex sets A_1 and A_2 are bounded by smooth curves Γ_1 and Γ_2 that do not have rectilinear pieces. Suppose that the set $A = \alpha_1 A_1 + \alpha_2 A_2$ is bounded by the curve Γ.

We choose special parametrizations of the curves Γ_1, Γ_2, and Γ. Namely, assign to each $\tau \in [0, 2\pi)$ that point $(x_1(\tau), y_1(\tau)) \in \Gamma_1$ at which the quantity $x \cos \tau + y \sin \tau$ attains a maximum as (x, y) runs through A_1. (In other words, $(x(\tau), y(\tau))$ is the point of tangency of Γ_1 with a support line forming an angle τ with the y-axis.) The parametrization $(x_2(\tau), y_2(\tau))$ of the curve Γ_2 and the parametrization $(x(\tau), y(\tau))$ of the curve Γ are defined similarly. Now show that these parametrizations are connected by the equations

$$x(\tau) = \alpha_1 x_1(\tau) + \alpha_2 x_2(\tau),$$
$$y(\tau) = \alpha_1 y_1(\tau) + a_2 y_2(\tau).$$

Indeed,

$$\begin{aligned}
\max_{(x, y) \in A} (x \cos \tau + y \sin \tau) &= \max_{(x_i, y_i) \in A_i} [(\alpha_1 x_1 + a_2 x_2)\cos \tau \\
&\qquad + (\alpha_1 y_1 + \alpha_2 y_2)\sin \tau] \\
&= \max_{(x_i, y_i) \in A_i} [\alpha_1(x_1 \cos \tau + y_1 \sin \tau) \\
&\qquad + \alpha_2(x_2 \cos \tau + y_2 \sin \tau)] \\
&= \alpha_1 \max_{(x_1, y_1) \in A_1} (x_1 \cos \tau + y_1 \sin \tau) \\
&\qquad + \alpha_2 \max_{(x_2, y_2) \in A_2} (x_2 \cos \tau + y_2 \sin \tau),
\end{aligned}$$

which implies the desired equality.

It now remains to use the familiar formula for the area of the set A bounded by a curve Γ given in parametric form: $\mu(A) = \int_0^{2\pi} x(\tau)\, dy(\tau)$. We get that

$\mu(A) = \alpha_1^2\mu(A_1) + \alpha_2^2\mu(A_2) + 2\alpha_1\alpha_2 \cdot M(A_1, A_2)$, where $M(A_1, A_2) = \int_0^{2\pi} [(x_1 y_2' + x_2 y_1')/2]\, d\tau$. The last quantity is called the *Minkowski mixed area of the pair of sets A_1 and A_2*.

In exactly the same way it can be shown that for k convex sets $A_1, \ldots, A_k$ in the plane

$$\mu(\alpha_1 A_1 + \cdots + \alpha_k A_k) = \sum_{i,j} M(A_i, A_j)\alpha_i\alpha_j.$$

The proof can be carried out by the same scheme for sets in an n-dimensional space. The boundary ∂A of the convex set A can be parametrized by the points of the unit sphere S in $\mathbf{R}^n$, and then the following volume formula is used:

$$\mu(A) = \int_S x_1(\tau)\, dx_2(\tau) \wedge \cdots \wedge dx_n(\tau).$$

Here the concept arises of the *mixed volume $M(A_1, \ldots, A_n)$ of a collection of n convex sets in $\mathbf{R}^n$*; many geometric characteristics of convex bodies can be expressed in terms of this.

265. Prove that the convex core topology is given by the system of all seminorms on L.

(a) $p_f = |f(x)|$ is a seminorm for any linear functional f on L (not necessarily continuous).

(b) The Minkowski functional of a convex balanced absorbing set is a seminorm.

266. If $B = \{x : p_B(x) \le 1\}$, where p_B is the Minkowski functional of B, then B intersects each line passing through zero in an interval closed in the Euclidean topology of the line.

267. Consider sets $B_0 = \{x : p_B(x) < 1\}$ and $B_1 = \{x : p_B(x) \le 1\}$, where p_B is the Minkowski functional of the set B.

268. Use Problem 266.

269. $c_2(A)$ is a convex set. Then use Problem 262.

270. The collection of sets $\{p_\alpha(x) < \varepsilon\}$, $\alpha \in A$, $\varepsilon > 0$, is a subbase of the topology in the polynormed space L.

271. In every LCS there is a base of neighborhoods of zero consisting of convex sets.

272. Prove that in the topology of coordinatewise convergence $\mathbf{R}^\infty$ is an LCS with the system of seminorms p_n, $n = 0, 1, \ldots$, where $p_n(\{x_i\}_1^\infty) = |x_n|$; then use Problem 270.

273. For $r \ge 1$ the ball of radius r is a convex set, since it coincides with $C(\mathbf{R})$. For $0 < r < 1$ the ball of radius r is not a convex set. Indeed, let $n = -[\log_2 r] \ge 1$. Then the ball of radius r contains the subspace V_n of $C(\mathbf{R})$ consisting of the functions equal to zero on $[-n, n]$. Therefore, it can be convex only if it contains the linear submanifold $f + V_n$ along with any f in it. Verify that this is not so for the constant function $f \equiv r/(1 - r)$.

274. The balls $S_R = \{f \in C(\mathbf{R}), d(f, 0) \leq R\}$ are not absorbing sets for $R < 1$, consequently, $C(\mathbf{R})$ is not an LTS in this topology.

275. The function $p(f) = \sup |f(x)|$ is a norm on $BC(\mathbf{R})$.

276. Prove that the topology in $\mathbf{R}^\infty$ given by the metric $d(\{x_n\}, \{y_n\})$ is the topology of coordinatewise convergence (see Problem 272).

2. Dual Spaces

277. Follows from the linearity of the functional and the invariance of the topology of an LTS under translations.

278. (a) Reduce the problem to the case when U is a balanced neighborhood of zero.

(b) $\ker f$ is closed and, consequently, nowhere dense, therefore, there exist an $x \in L$ and a balanced neighborhood of zero V such that $(x + V) \cap \ker f = \varnothing$. Prove that $f(V) \not\ni -f(x)$, and use (a).

279. Use a Hamel basis. A *Hamel basis* is defined to be a linearly independent system $\{x_\alpha\}$ of elements in a linear space L whose linear span is L.

280. The Hahn–Banach theorem gives us that the topology on L for which all functionals are continuous coincides with the convex core topology (see Problem 265). Prove that the cardinality of a base of neighborhoods of zero for the convex core copology is 2^β, where β is the cardinality of a Hamel basis (see Problem 279) in L.

281. (a) If $|f(x)| \leq c$ on the unit ball of L, then $|f(x)| \leq c\|x\|$ for all $x \in L$, because f is homogeneous. Hence, f is continuous at zero and, consequently (Problem 277) everywhere. Conversely, if f is continuous at zero, then $|f(x)| \leq 1$ on an open ball of sufficiently small radius $r > 0$. Since f is homogeneous, this implies that $|f(x)| \leq r^{-1}$ on the unit ball.

(b) If f is continuous, then it is bounded in some neighborhood of zero and, consequently, on any bounded set. Conversely, let $V_1 \supset V_2 \supset \cdots \supset V_n \supset \cdots$ be a basis of neighborhoods of zero and suppose that the functional f is unbounded in each of these neighborhoods. Let $x_n \in V_n$ be such that $|f(x_n)| > n$. Verify that $X = \{x_n\}$ is a bounded set.

282. $\|f\| = \sup_{x \neq 0} (|f(x)|/\|x\|) = \sup_{\|x\|=1} |f(x)|$.

283. (a) $b - a$;

(b) $\int_a^b |y(t)|\, dt$;

(c) $\sum_i |\lambda_i|$.

284. Verify that the hyperplane $f(x) = 1$ does not contain points x with norm $< \|f\|^{-1}$, but does contain points with norm arbitrarily close to $\|f\|^{-1}$.

285. Use Problem 257.

286. A Banach space B is reflexive if and only if the ball $\|x\| \leq 1$ is compact in the weak topology.

287. $(c_0)' \supset l_1$ (actually, $(c_0)' = l_1$; see Problem 293), therefore, weak convergence implies coordinatewise convergence. Considering the sequence x_n $(x_{ni} = 1, i \le n, x_{ni} = 0, i > n)$, prove that the unit ball of c_0 is not compact in the weak topology.

288. A basis for the weak topology consists of sets that are unbounded in the strong topology.

289. We use the obvious fact that $(l_1)' \supseteq l_\infty$ (in fact, $(l_1)' = l_\infty$; see Problem 294). Let $\{x^{(n)}\}$ be a sequence of elements in l_1 that does not tend strongly to zero. Passing to a subsequence and multiplying by a constant, we come to the situation $\|x^{(n)}\| \ge 1$ for all n.

We say that a sequence $x \in l_1$ is concentrated on the interval $[k, l]$ to within ε if $\sum_{i=k}^{l} |x_i| \ge (1 - \varepsilon)\|x\|$.

Suppose that $x^{(n)} \rightharpoonup 0$ as $n \to \infty$. Then $x_k^{(n)} \to 0$ as $n \to \infty$ for all k. Therefore, by passing once more to a subsequence we can assume that $x^{(n)}$ is concentrated on $[k_n, l_n]$ to within 1/4 and, moreover, that these intervals do not overlap for different n. Now let $a_i = \operatorname{sgn} x_i^{(n)}$ if $i \in [k_n, l_n]$, and $a_i = 0$ otherwise. Then

$$\sum_{i=1}^{\infty} a_i x_i^{(n)} \ge \sum_{i=k_n}^{l_n} |x_i^{(n)}| - \sum_{i \notin [k_n, l_n]} |x_i^{(n)}| \ge (3/4)\|x^{(n)}\| - (1/4)\|x^{(n)}\| \ge 1/2,$$

and this contradicts the assumption that $x^{(n)} \rightharpoonup 0$.

Remark. This result shows that the topology in an LTS may not be determined by the class of convergent sequences (though every topology is determined by the class of convergent nets).

290. Prove that a support plane of the unit ball in L is given by an equation of the form $f(x) = 1$, where $f \in L'$ and $\|f\| = 1$.

291. Take a k-dimensional face and $k + 1$ vertices x_i, $i = 1, \dots, k + 1$, in it (such a vertices exist, because a convex polyhedron is the convex hull of its vertices). Assign to this k-dimensional face the set $\{f \in \beta', f(x_i) = 1, i = 1, \dots, k + 1\}$. Prove that the set obtained is an $(n - k - 1)$-dimensional face of B'.

292. Choose a basis in P and supplement it to form a basis in $\mathbf{R}^n$.

293. Verify that the unit ball of c_0 does not have extreme points, while the unit ball in c has two extreme points: $x_n \equiv 1$ and $x_n \equiv -1$. The formula $\langle \{a_n\}, \{x_n\} \rangle = \sum_{n=1}^{\infty} a_n x_n$ establishes an isomorphism between l_1 and c_0', and the formula $\langle \{a_n\}, \{x_n\} \rangle = a_1 \lim_{n\to\infty} x_n + \sum_{n=1}^{\infty} a_{n+1} x_n$ establishes an isomorphism between l_1 and c'. To compute the norm of $\{a_n\}$ in c' consider the sequences of the form

$$x_i = \begin{cases} \operatorname{sgn} a_{i+1} & \text{for } i < N, \\ \operatorname{sgn} a_1 & \text{for } i \ge N. \end{cases}$$

294. Use the Hölder inequality.

295. Consider the continuous functional $f\{(x_n\}) = \lim_{n\to\infty} x_n$ on $c \subset l_\infty$, and apply the Hahn–Banach theorem.

296. Choose bases $\tilde{e}_i$, $i = 1, \dots, n_1$, and $\tilde{f}_j$, $j = 1, \dots, n_2$, in L'_1 and L'_2 that are biorthogonal to the bases e_i, $i = 1, \dots, n_1$, and f_j, $j = 1, \dots, n_2$, in L_1 and L_2, respectively.

297. (a) Each segment in the space $l_p(\mathbf{R})$ has a unique "midpoint" (in other words, for given x and y the equality $\|x - y\| = \|x - z\| + \|z - y\|$ is attained at the unique point $z = (x + y)/2$).

(b) Let $\{x_n\}$ be a countable dense set in $[0, 1]$, and assign to a function $f \in C[0, 1]$ the sequence $\{f(x_n)\} \in l_\infty(\mathbf{R})$.

298. (a) For L reflexive and L' separable the compactness of the unit ball $B \subset L$ was proved in Theorem 12 of Ch. III. In the nonseparable case, compactness follows from Tychonoff's theorem. (Imbed B in a product of closed intervals $\prod_{f \in L'} I_f$ according to the formula $x \mapsto \{f(x)\}_{f \in L'}$ and verify that the image of B is closed, since it is distinguished by the conditions $|f(x)| \le \|f\|$.) Suppose now that B is compact in the weak topology. The canonical image of B in L'' is then compact in the weak-* topology. But in this topology the image of B is dense in the unit ball (since there does not exist a hyperplane of the form $f(x) = c$, $f \in L'$, $x \in L''$, which separates a point of the unit ball of L'' from the image of B). Therefore, $L'' = L$.

(b) Use the result in (a).

299. The unit ball in $C[a, b]$ has two extreme points: $f(x) \equiv 1$ and $f(x) \equiv -1$ (Problem 293). By the Krein–Mil'man theorem, this implies that it is not compact in any locally convex topology.

3. The Hahn–Banach Theorem

300. Use Problem 257.

301. A separating hyperplane must have the form $f(x) = 0$, $f \in P'$. Prove that $f \equiv 0$.

302. Suppose that A is compact; then there exists a convex neighborhood V of zero such that $(A + V) \cap B = \varnothing$ (see the hint for Problem 255). Apply the geometric form of the Hahn–Banach theorem to $A + V$ and B, and use once more the compactness of A.

303. Fix $x \in L$; from the Hahn–Banach theorem it follows that there exists an $x^* \in L'$, $\|x^*\| = 1$, $(x^*, x) = \|x\|$.

304. The isometric mapping $L \to L''$ constructed in Problem 303 is an isomorphism because the dimensions of L and L'' are equal.

305. Use the method of proof of Theorem 24 in Ch. III, along with Problem 303.

306. Consider the restriction of the desired functional to the subspace L_0 generated by the vectors $x_1, \dots, x_n$, and use the Hahn–Banach theorem.

307. Use polar coordinates.

308. Use the separability of $l_q(n, \mathbf{R})$.

309. Let L be the subspace of $l_\infty(\mathbf{R})$ generated by c_0 and the sequences of the form $y_n = x_{n+1} - x_n$, $\{x_n\} \in l_\infty$. Prove that the sequence $y_n \equiv 1$ does not lie in $\bar{L}$. Then use the Hahn–Banach theorem.

310. See the hints for Problem 309.

311. $\tilde{p}(x) = \sup_n p(T^n x)$, $n = 0, \pm 1, \dots$.

312. We can take α to be the cardinality of the space L'. An imbedding of L in $\mathbf{R}^\alpha$ is given by the formula $x \mapsto \{f(x)\}_{f \in L'}$.

313. Let L be the subspace of $B(\mathbf{R}^n)$ generated by all the functions f such that $\lim_{\|x\| \to \infty} f(x) = 0$ and by the functions of the form $f(x + t) - f(x)$. We compute the distance from L to the function $f \equiv 1$. This distance is clearly not greater than 1. Suppose that it is strictly less than 1. Then there exist functions $f_0, f_1 \dots, f_N \in B(\mathbf{R}^n)$, vectors $t_1, \dots, t_N \in \mathbf{R}^n$, and an $\varepsilon > 0$ such that

$$f_0(x) + \sum_{k=1}^{N} (f_k(x + t_k) - f_k(x)) \geq \varepsilon, \tag{$*$}$$

where $f_0(x) \to 0$ as $\|x\| \to \infty$.

Let us consider the additive subgroup of $\mathbf{R}^n$ generated by $t_1, \dots, t_N$. It is easy to see that it is isomorphic to $\mathbf{Z}^m$ for some $m \leq N$. Let $e_1, \dots, e_m$ be generators of this group. It can be assumed that all the e_i are positive with respect to a certain total order in $\mathbf{R}^n$ (see Problem 14). Then the relation ($*$) implies that there exist functions $\phi_0, \phi_1, \dots, \phi_m$ on $\mathbf{Z}^m$ such that $\phi_0(z) \to 0$ as $z_i \to \infty$ ($i = 1, 2, \dots, m$) and

$$\phi_0(z) + \sum_{j=1}^{m} (\phi_j(z + e_j) - \phi_j(z)) \geq \varepsilon \text{ for all } z \in \mathbf{Z}^m. \tag{$**$}$$

(Use the fact that each t_k is an integral combination of the vectors e_j, along with the relation $f(x + t_1 + t_2) - f(x) = f(x + t_1) - f(x) + g(x + t_2) - g(x)$, where $g(x) = f(x + t_1)$.)

Let T_j denote the operator of translation by e_j: $T_j f(x) = f(x + e_j)$. From ($**$) it follows that there is an M such that $\sum_{j=1}^m (T_j - 1)\varphi_j(z) \geq \varepsilon/2$ if $z_j > M$ ($1 \leq j \leq m$). The last relation leads to a contradiction. To see this, apply the operator $\prod_{j=1}^m (1 + T_j + \cdots + T_j^{n-1})$ to both sides. This gives the inequality

$$\sum_{j=1}^{m} (T_j^n - 1) \prod_{k \neq j} (1 + T_k + \cdots + T_k^{n-1})\varphi_j(z)$$
$$\geq n^m(\varepsilon/2) \text{ for } z_i > M (1 \leq i \leq m).$$

Denoting the maximal norm of the φ_j in $B(\mathbf{Z}^n)$ by C, we arrive at the inequality $2mn^{m-1}C \geq n^m(\varepsilon/2)$, which is not true for $n > 4mC/\varepsilon$. Accordingly, $d(1, L) = 1$. Therefore, the functional F defined on $L + \mathbf{R} \cdot 1$ by the formula $F(f + \lambda) = \lambda$ for $f \in L$ has norm 1. Extending it by the Hahn–Banach theorem to $B(\mathbf{R}^n)$, we get a functional LIM. This functional is called a *Banach limit*, in honor of Stefan Banach. We remark that the properties (a), (b), and (c) do not determine LIM uniquely (unlike in the case of the usual limit). See Problem 114.

314. Let $\mathbf{Z}^n$ be the lattice in $\mathbf{R}^n$ generated by the standard basis vectors. Define the measure μ by the formula

$$\mu(A) = \mathrm{LIM}\left(\sum_{z \in Z^n} \chi_{A+z}\right),$$

where LIM is the Banach limit determined in Problem 313.

315. Use the Hahn–Banach theorem.

316. See the hints for Problem 315.

317. Consider the hyperplanes $(x, f_i) = 1$, where the f_i form a countable dense subset of the unit ball in $L_q(n, \mathbf{R})(1/p + 1/q = 1)$, and use Problem 284.

318. It can be assumed that the given convex set V contains zero and that the intersection of V with each line passing through zero is a closed set. The boundary of V is given in polar coordinates (r, φ) by the equation of a convex function $r(\varphi)$, consequently, $r(\varphi)$ is a continuous function on $[0, 2\pi]$ satisfying the condition $r(0) = r(2\pi)$. Prove that for all $\varepsilon > 0$ there is a polygon $V_n(\varepsilon)$ such that $|r(\varphi) - \tilde{r}(\varphi)| < \varepsilon$, where $\tilde{r}(\varphi)$ is the boundary function for $V_n(\varepsilon)$. This $V_n(\varepsilon)$ is an intersection of half-spaces of the form $a_i x + b_i y \leq 1$. Consider the imbedding $\varphi: \mathbf{R}^2 \to \mathbf{R}^n: (x, y) \to x\bar{a} + y\bar{b}$, where $\bar{a} = (a_1, \ldots, a_n)$, $\bar{b} = (b_1, \ldots, b_n)$.

319. Construct first an isometric imbedding of $l_p(n, \mathbf{R})$ into $l_\infty(\mathbf{R})$ by using the separability of $l_p(n, \mathbf{R})$ and Problem 303, and then construct an imbedding $l_\infty(\mathbf{R}) \to C[0, 1]$.

320. First use induction on the number of sets. Let $N > n + 2$ and suppose that the assertion has been proved for $N - 1$ sets. If $X_1, \ldots, X_N$ satisfy the conditions of the theorem, then any $N - 1$ of them have a common point, by the induction hypothesis. Let $Y_i = X_i \cap X_N$, $1 \leq i \leq N - 1$. Then any $N - 2$ of the sets Y_i have a common point. Since $N - 2 \geq n + 1$, the family $\{Y_i\}$ again satisfies the conditions of the theorem. Hence, the Y_i have a common point, which is a common point for all the X_i. It remains to analyze the case $N = n + 2$.

Now use induction on the dimension. Suppose that the theorem has been proved for dimensions $<n$, and consider sets X_i, $1 \leq i \leq n + 2$, in $\mathbf{R}^n$, each $n + 1$ of which have a common point. If X_{n+2} does not have common points with the intersection $Z = \bigcap_{i=1}^{n+1} X_i$, then there exists a hyperplane L separating these two convex sets. Any n sets among the X_i, $1 \leq i \leq n + 1$,

have a point common with X_{n+2} and one common with Z. Therefore, they have a point common with L. Let $Y_i = X_i \cap L$, $1 \le i \le n+1$. Then by identifying L with $\mathbf{R}^{n-1}$ we see by the induction hypothesis that $\bigcap_{i=1}^{n+1} Y_i$ is nonempty. This contradicts the definition of L, since $\bigcap_{i=1}^{n+1} Y_i = Z \cap L$. It remains to check the theorem for $n = 0$, when it is trivial.

Remark. By topological arguments it can be shown that the convexity condition in Helly's theorem can be replaced by the condition that all the sets of the form $X_{i_1} \cap \cdots \cap X_{i_k}$ for $k \le n+1$ are homotopically equivalent to a point (i.e.,contractible to a point).

321. Show that the convex hull of U_ε coincides with the whole space $C[0, 1]$; this will imply the assertion of the problem. Let $\{\varphi_i\}_{0 \le i \le n}$ be a system of continuous functions on $[0, 1]$ with the following properties: (1) $0 \le \varphi_i(x) \le 1$; (2) $\sum_{i=0}^{n} \varphi_i(x) \equiv 1$; (3) supp $\varphi_i = [(i-1)/n, (i+1)/n]$ (for example, $\varphi_i(x) = (1 - |i - nx|)_+$). For any $f \in C[0, 1]$ we have $f = \sum_{i=0}^{n} \varphi_i f$. Therefore, f is in the convex hull of the functions $\psi_i = (n+1)\varphi_i f$ $(0 \le i \le n+1)$. Show that all the functions ψ_i are in U_ε for sufficiently large n. This follows from the estimate

$$\int_0^1 \sqrt{|\psi_i|}\, dx = \sqrt{n+1} \int_{(i-1)/n}^{(i+1)/n} \sqrt{|\varphi_i f|}\, dx \le (2\sqrt{n+1})n^{-1}\sqrt{\|f\|}.$$

322. (a) L' separates the points of L.

(b) Using the fact that

$$\left|\int_X F(f(x))\, d\mu(x)\right| \le \int_X |F(f(x))|\, d\mu(x) \le \|F\| \int_X \|f\|\, d\mu(x),$$

prove that $F(f)$ is continuous with respect to F.

4. Banach Spaces

323. $\{x\colon \|x\|_p \le 1\}$ is not a convex set for $0 < p < 1$.

324. Consider the set of finitely nonzero sequences with rational coefficients.

325. Consider the set of sequences of 0's and 1's.

326. To prove that a Cauchy sequence converges, it suffices to show that some subsequence of it converges.

327. To prove that $\|x\| = 0 \Leftrightarrow x = 0$ use the fact that L_0 is closed in L.

328. From a Cauchy sequence y_n in L_1 choose a subsequence y_{n_k} such that $\sum \|y_{n_k} - y_{n_{k+1}}\|_{L_1} < \infty$.

Let $\alpha_k = y_{n_k} - y_{n_{k+1}}$ and choose an $a_k \in \varphi^{-1}(\alpha_k)$ (φ is the natural mapping $L \to L_1$) such that $\|\alpha_k\|_L \le \|\alpha_k\|_{L_1} + 2^{-k}$. Consider $S_k = a_1 + \cdots + a_k$. Prove that $\lim y_k = \lim f(\xi_k) - y_1$.

329. Let φ be the natural mapping of L onto L_1, and z_n a Cauchy sequence in L. Then $y_n = \varphi(z_n)$ is Cauchy in L_1, consequently, $y = \lim y_n$ exists, from which $\|y - y_n\|_{L_1} \to 0$; hence, there exist $r_n \in \varphi^{-1}(y - y_n) = \varphi^{-1}(y) - \varphi^{-1}(y_n)$ such that $\|r_n\|_L \to 0$. Choose a fixed element f in $\varphi^{-1}(y)$; then $r_n = f - f_n$, $f_n \in \varphi^{-1}(y_n)$. Consequently, $f_n \to f$; but $z_n \in \varphi^{-1}(y_n)$, and therefore $f_n - z_n = x_n$ is a Cauchy sequence in L_0.

Finally, the limits $\lim z_n = f - x$, $x = \lim x_n$ exist, so L is a Banach space.

330. Let X be a separable Banach space, and $\{x_1, \ldots, x_n, \ldots\}$ a countable dense subset of the unit ball of X.

Define a mapping $l_1(K) \mapsto X$ by $A: \{\alpha_n\} \mapsto \sum_{n=1}^{\infty} \alpha_n x_n$, $\{\alpha_n\} \in l_1(K)$.

(a) Prove that A is well defined and continuous.

(b) For a given x, $\|x\| = 1$, choose the $x_n(i)$ from the requirement that $\|x - \sum_{i=1}^{k} 2^{-i+1} x_{n(i)}\| \leq 2^{-k}$. This implies that im $A = X$.

(c) Replacing the number 2 in this estimate by $3, 4, \ldots$, show that $\hat{A}: \varphi/\ker A \to X$ is an isometry.

331. The necessity is obvious. To prove sufficiency use Zorn's lemma; it suffices to show that there is an uncountable collection of disjoint balls of radius ε for some $\varepsilon > 0$.

332. Use Problem 257.

333. The problem can be reformulated as follows: among all the polynomials of degree n with leading coefficient 1 find the one with smallest norm in the sense of the space $C[-1, 1]$ (the so-called "polynomial which deviates least from zero" on $[-1, 1]$). Consider the polynomial $T_n(x) = 2^{1-n}\cos(n \arccos x)$. Verify by induction that T_n has leading coefficient 1 and norm 2^{1-n} and takes the values $\pm 2^{1-n}$ alternately at the points $x_k = \cos(k\pi/n)$, $k = 0, 1, \ldots, n$. Further, if P is another polynomial of degree n with leading coefficient 1 and it deviates from zero by less than 2^{1-n}, then the graphs of P and T_n have at least n points of intersection (since the difference $P - T_n$ changes sign in passing from x_k to x_{k+1}). But this is impossible, because $P - T_n$ is a polynomial of degree $< n$ and therefore has no more than $n - 1$ zeros.

334. Under the natural mapping $\varphi: L \to L/L_0$ the open unit ball of L goes into the open unit ball of L/L_0.

335. Choose a sequence y_n of 1/2-perpendiculars to $L(y_1, \ldots, y_{n-1})$.

336. Suppose that there is a 0-perpendicular x_1 to the subspace $L_0 = \{x_0 \in L: f(x_0) = 0\}$. This is equivalent to the assertion that $\|x_1 + x_0\| \geq \|x_1\|$ for all $x_0 \in L_0$. In other words, the distance $d(x_1, L_0)$ is attained and equals $\|x_1\|$. Any vector $x \notin L_0$ can be written in the form $x = \alpha(x_1 + x_0)$, where $\alpha \in R \setminus \{0\}$, $x_0 \in L_0$.

We have that

$$\frac{|f(x)|}{\|x\|} = \frac{|\alpha||f(x_1)|}{|\alpha|\,\|x_1 + x_0\|} \leq \frac{|f(x_1)|}{\|x_1\|}.$$

Hence, the norm of f is attained on the vector x_1. The converse assertion is derived in the same way.

337. Use the following fact: a subspace $L_0 \subset L$ is complemented if and only if it has the form $L_0 = PL$, where P is a continuous projection (see Problems 354, 355). Let us determine the explicit form of this projection in the cases indicated in the problem.

(a) Suppose that L_0 is generated by the linearly independent vectors $e_1, \ldots, e_n$. Let $f_1, \ldots, f_n$ be linear functionals in L' with the property that $f_i(e_j) = \delta_{ij}$. Then P can be taken to be the operator $x \mapsto \sum_{i=1}^n f_i(x)e_i$.

(b) Suppose that L_0 is distinguished by the system of equations $f_1(x) = 0, \ldots, f_n(x) = 0$, where $\{f_i\}$ is a set of linearly independent functionals in L'. Choose $e_1, \ldots, e_n$ such that $f_i(e_j) = \delta_{ij}$, and define the projection P by the formula $P(x) = x - \sum_{i=1}^n f_i(x)e_i$.

(c) Let $\{e_i\}$ be the family of coordinate vectors in $l_\infty(K)$, and $\{f_i\}$ a dual family of linear functionals such that $f_i(e_j) = \delta_{ij}$. Extend f_i with preservation of norm to a functional $\tilde{f}_i$ on L. The desired projection has the form $P(x) = \{\tilde{f}_i(x)\} \in l_\infty(K)$.

(d) Let e_i be the standard basis in $l_1(K)$, and $\tilde{e}_i$ a pre-image of e_i in L with the property that $\|\tilde{e}_i\| \leq 2$. The desired projection can be defined by $P(x) = x - \sum_{i=1}^\infty c_i \tilde{e}_i$, where $\{c_i\} \in l_1(K)$ is the image of x in the quotient space L/L_0.

338. The linear transformation $y \mapsto y - (2f(y)/f(x))x$ is called the reflection in the hyperplane $f = 0$ parallel to a vector x not in this hyperplane. Verify that this transformation is an isometry of $l_p(n, \mathbf{R})$ for $p \neq 2$ if and only if x is one of the standard basis vectors (to within a sign) and f is the basis functional dual to it. Therefore, the collection of vectors $\{\pm e_k\}$ must be carried into itself under any isometry of $l_{p_1}(n, \mathbf{R})$ onto $l_{p_2}(n, \mathbf{R})$. But $d_p(\pm e_k, \pm e_j) = 2^{1/p}$, hence, the desired isometry exists only if $p_1 = p_2$.

Remark. It can be proved that any isometry of l_{p_1} onto l_{p_2} must be an affine operator and, consequently, exists only if $p_1 = p_2$.

339. (a) Verify that the formula given in the problem really defines a norm, and that this norm majorizes any cross-norm.

(b) Verify that the formula given defines a norm, and this norm is majorized by any uniform cross-norm.

340. Verify that the mapping from $L_1 \times L_2$ to $L_1 \otimes L_2$ (see the hint for Problem 61) has norm ≤ 1.

341. Note that under orthogonal transformations Q_i in the spaces L_i $(i = 1, 2)$ the matrix A is carried into Q_1AQ_2, while the eigenvalues of the matrices $A'A$ and AA' do not change. This shows that it suffices to solve the problem for matrices of diagonal form: $a_{ij} = \lambda_i \delta_{ij}$, where $\lambda_i \geq 0$. In this case

$s_i = \lambda_i^2$ for $i \le \min(m, n)$, and $s_i = 0$ for the remaining i. Let $\{e_j\}$ and $\{f_j\}$ be the standard bases in L_1 and L_2. It remains to show that

$$\hat{p}\left(\sum_{j=1}^{\min(m,n)} \lambda_j e_j \otimes f_j\right) = \sum_{j=1}^{\min(m,n)} \lambda_j.$$

The fact that the norm is bounded above by the indicated number is obvious, because $\hat{p}(e_j \otimes f_j) = 1$. To get a lower estimate consider the functional $F = \sum e_j^* \otimes f_j^*$, where x^* denotes the linear functional $x^*(y) = (x, y)$. Using the Cauchy–Bunyakovskii inequality, verify that $\|F\| = 1$ (it suffices to show that $F(x \otimes y) \le \|x\| \cdot \|y\|$). We have that $F(\sum \lambda_j e_j \otimes f_j) = \sum \lambda_j$, which gives the needed estimate.

342. Use the explicit form of the norm on $L_1 \otimes L_2$.

343. The space $L_1' \otimes L_2$ can be identified with the space of finite-rank operators by the correspondence

$$x \otimes y \to A_{x,y} \in L(L_1, L_2)\colon A_{x,y}(z) = (x, z)y, \qquad x \in L_1',\ y \in L_2,\ z \in L_1.$$

Prove that the norm on $L_1 \check{\otimes} L_2$ coincides with the usual operator norm.

344. Prove that the norm on $l_1(mn, \mathbf{R})$ is a cross-norm for the norms on $l_1(n, \mathbf{R})$ and $l_1(m, \mathbf{R})$.

345. Prove that the norm on $l_\infty(mn, \mathbf{R})$ is a cross-norm for the norms on $l_\infty(n, \mathbf{R})$ and $l_\infty(m, \mathbf{R})$.

§2. Linear Operators

1. The Space of Linear Operators

346. (b) and (c) hold.

347. Let $x = (x_1, x_2, \ldots, x_n, \ldots) \in l_2(\mathbf{R})$: then $A_n x = x_n e_1 \to 0$.

348. $(y, B_n x) = (x, y_n)$, where $x = (x_1, x_2, \ldots, x_n, \ldots)$ and $y = (y_1, y_2, \ldots, y_n, \ldots) \in l_2(\mathbf{R})$.

349. $A_n B_n - AB$
$$= A_n B_n - AB_n + AB_n - AB = (A_n - A)B_n + A(B_n - B).$$

350. Use the Banach–Steinhaus theorem.

351. See Problem 349.

352. A base for the strong topology on End L is given by the set of seminorms

$$p_x(A) = \|Ax\|, \qquad A \in \operatorname{End} L, \qquad x \in L.$$

It suffices to prove the continuity of multiplication at the point (0, 0), where 0 is the zero operator in End L.

353. See Problems 347 and 348.

354. Let P be the projection onto L_1 parallel to L_2. Then $1 - P$ is the projection onto L_2 parallel to L_1. If P is continuous, then L_1 and L_2 are closed, since $L_1 = \ker(1 - P)$ and $L_2 = \ker P$. Conversely, if L_1 and L_2 are closed, then the continuity of P follows from the Banach inverse mapping theorem. (Consider the natural mapping $Q\colon L_1 \to L/L_2$ and represent P in the form $Q^{-1} \circ \pi$, where π is the natural projection of L onto L/L_2.)

355. The equality $P^2 = P$ implies that $L = L_1 + L_2$ (algebraic direct sum), where $L_1 = PL$ and $L_2 = (1 - P)L$. Now use the result of the preceding problem.

356. $\|AB\| = \sup_{\|x\| \le 1} \|ABx\| \le \sup_{\|x\| \le 1} \|A\| \|Bx\| \le \|A\| \|B\|$.

357. Modify the argument given in the proof of Theorem 4 in Ch. III.

358. $\|a\| = \operatorname{ess\,sup}_{x \in X} |a(x)|$.

359. If $p \ge q$, then

$$\left(\int_a^b |f|^q\,dx\right)^{1/q} \le (b - a)^{1/q - 1/p}\left(\int_a^b |f|^p\,dx\right)^{1/p}.$$

360. If $p \ge q$, then $a(x) \in L_\infty(0, 1)$; $a(x) \equiv 0$ for $p < q$.

361. (a) Verify that $\|f(x + t) - f(x)\|_{L_p} \to 0$ as $t \to 0$ for continuous functions with compact support.

(b) $\|T(t') - T(t'')\| = 2$ for $t' \ne t''$.

362. Verify that $A(t) = e^{tC}A_0$ really is a solution (to do this, establish the equality $(e^{tC})' = Ce^{tC}$). Then use the uniqueness theorem from the theory of ordinary differential equations.

363. The result follows from the preceding problem if $A(t)$ is differentiable. The differentiability of $A(t)$ can be derived from its continuity. Let ε be a positive number small enough that $\|A(t) - 1\| < 1$ for $|t| < \varepsilon$, and let φ be a nonnegative smooth function supported on $[-\varepsilon, \varepsilon]$ such that $\int_{-\varepsilon}^{\varepsilon} \varphi(t)\,dt = 1$. Then the operator $\Phi = \int_{-\varepsilon}^{\varepsilon} A(-s)\varphi(s)\,ds$ is invertible, because $\|1 - \Phi\| < 1$. Verify that the operator-valued function $\tilde{A}(t) = \int_{-\infty}^{\infty} A(s)\varphi(t - s)\,ds$ is differentiable.

364. $A(s)\colon L_2(\mathbf{R}) \to L_2(\mathbf{R})$, $A(s)x(t) = x(t + s)$, $x(t) \in L_2(\mathbf{R})$.

365. Suppose that $\|x_n\| \to 0$ and $\|Ax_n\| \not\to 0$; then there exist an $\varepsilon > 0$ and a subsequence $\{x_{n_k}\}$ such that $\|Ax_{n_k}\| \ge \varepsilon$. Since $\|x_{n_k}\| \to 0$, there exist $\alpha_{n_k} \to \infty$ such that $\alpha_{n_k} x_{n_k} \to 0$. Then the fact that $\|A\alpha_{n_k} x_{n_k}\| \to \infty$ implies that $\{A\alpha_{n_k} x_{n_k}\}$ does not converge weakly.

366. In Banach spaces weak and strong boundedness coincide; the continuity of an operator is equivalent to its boundedness.

367. $A'\colon L_{q'}[0, 1] \to L_{p'}[0, 1]$, $A'f(x) = \int_0^1 K(y, x)f(y)\,dy$.

368.

$$P'F(t) = \begin{cases} F(t), & 0 \le t \le 1, \\ F(l), & 1 \le t \le 2. \end{cases}$$

2. Compact Sets and Compact Operators

369. (a) $\Rightarrow$ (b). Suppose that A is compact and $\{y_a\}_{a\in I}$ is a net. If $\{y_\alpha\}$ does not have limit points, then for any $x \in A$ there exist an open set $V_x \ni x$ and an $\alpha_x \in I$ such that $y_\alpha \notin V_x$ for $\alpha > \alpha_x$. The family $\{V_x | x \in A\}$ is an open covering of S. Choosing a finite cubcovering of $\{V_x | x \in A\}$, obtain a contradiction.

(b) $\Rightarrow$ (a). Suppose that the open covering U does not have a finite subcovering, and order the set W of finite subfamilies of U by inclusion. Obtain a contradiction by considering the net $\{x_F)$, where $F = \{F_1, \ldots, F_n\} \in W$ and $x_F \notin \bigcup_{i=1}^n F_i$.

(a) $\Leftrightarrow$ (c) Let V be any system of closed subsets of A, and $U = \{A - F; F \in V\}$. Then V is centered $\Leftrightarrow U$ does not have finite subcoverings, and $\bigcap_{F\in V} F \neq \varnothing \Leftrightarrow U$ is not a covering of A.

370. The Cantor set can be covered by 2^n segments of length 3^{-n} and cannot be covered by fewer segments of that length. Therefore, $N(\varepsilon) = 2^n$ for $\varepsilon = 3^{-n}/2$. Hence, $N(\varepsilon) = O(\varepsilon^{-\log_3 2})$, and the approximation dimension is equal to $\log_3 2 \approx 0.63$.

371. The condition that A be an extreme subset of K is as follows: if $x \in K$, $y \in K$, $x \neq y$, and $(x + y)/2 \in A$, the $x \in A$ and $y \in A$.

372. Let P be the family of all compact extreme subsets of K. Partially order P by inclusion. Problem 369 gives us that every totally ordered subset of P has a lower bound (the intersection serves as a minorant). By Zorn's lemma, P contains a minimal element.

373. Suppose that A contains at least two points, and let $f \in L'$ be a linear functional separating these two points. Let $c = \max_{x\in K} f(x)$ and $B = \{x \in A : f(x) = c\}$. Prove that B is an extreme subset which is strictly contained in A.

374. Follows from Problems 372, 373.

375. Let H be the convex hull of the set of extreme points of K, and $\bar{H}$ the closure of H. If $x_0 \in K\setminus\bar{H}$, then there exists a hyperplane $f(x) = c$ separating x_0 and $\bar{H}$: $f(x) > c$ at x_0 and $f(x) \leq c$ on $\bar{H}$. Conclude from this that K contains an extreme point not in H (see the hint for Problem 373).

376. If $1 < p < \infty$, then the unit ball is a strictly convex body, and all its boundary points are extreme points. If $p = 1$, then the extreme points are $\pm e_j$, where $\{e_j\}$ is the standard basis in $l_1(n, \mathbf{R})$. If $p = \infty$, then the extreme points are the vectors of the form $\sum_{j=1}^n \varepsilon_j e_j$, where $\varepsilon_j = \pm 1$.

377. The unit ball of c_0 does not have extreme points. The unit ball of c has the two extreme points

$$(1, 1, \ldots, 1, \ldots) \quad \text{and} \quad (-1, -1, \ldots, -1, \ldots).$$

378. Use the Krein–Mil'man theorem (Problem 375).

379. Let M be a pre-compact set. Then it has an $(\varepsilon/3)$-net $\{f_i\}$, $1 \leq i \leq N$. The compact set X can be represented as the union of a finite number of subsets of diameter $<\varepsilon/3$. Therefore, for each i there exists a partition of T into finitely many subsets on which the oscillation of f_i is not greater than $\varepsilon/3$. Taking the common refinement of these partitions, we get a partition of T into subsets $\{T_j\}$, $1 \le j \le n$, such that the oscillation of f_i on T_j does not exceed $\varepsilon/3$ for all i and j. If f is now any function in M, f_i is the point of the $(\varepsilon/3)$-net closest to f, and t and s are any points in T_j, then $d_X(f(t), f(s)) \le d_X(f(t), f_i(t)) + d_X(f_i(t), f_i(s)) + d_X(f_i(s), f(s)) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$.

Suppose now that $\varepsilon > 0$ is given and that there exists a partition $T = \coprod_{j=1}^n T_j$ such that $\omega_f(T_j) < \varepsilon/4$ for all $f \in M$. (Here $\omega_f(T_j)$ is the oscillation of f on T_j.) In each set T_j choose a point t_j and consider the mapping $\varphi\colon M \to X^n\colon f \mapsto (f(t_1), \ldots, f(t_n))$. Since X is compact, so is X^n (the distance in X^n is defined by the formula $d(x, y) = \max_{1 \le i \le n} d_X(x_i, y_i)$). Hence, the image of M is a pre-compact set. Choose a finite $(\varepsilon/2)$-net $\varphi(f_1), \ldots, \varphi(f_n)$ in $\varphi(M)$. Then $f_1, \ldots, f_n$ is an ε-net for M. Indeed, if f is any function in M and $\varphi(f_i)$ is the point of the $(\varepsilon/2)$-net closest to $\varphi(f)$, then $d_X(f(t), f_i(t)) \le d_X(f(t), f(t_j)) + d_X(f(t_j), f_i(t_j)) + d_X(f_i(t_j), f_i(t)) < \varepsilon/4 + \varepsilon/2 + \varepsilon/4 = \varepsilon$ for $t \in T_j$.

380. Prove that an extreme point of S must be a matrix A with a 1 in each row and in each column. (Otherwise, A contains a 2×2 submatrix A_0 in which at least three elements are positive; consider the intersection of S with the collection of matrices which differ from A only outside A_0.) The matrices A having this property can all be obtained one from another by permutations of the rows and columns, that is, by isometric transformations of S. Hence, they are all extreme (otherwise there would not be any extreme points at all).

381. See Problem 335.

382. The compact operators form an ideal in $\mathrm{End}(L)$.

383. If $a_i \to 0$, then $\forall \varepsilon > 0\ \exists N\colon\ |a_n| < \varepsilon\ \forall n > N$. Consider $K = \{\{x_i\} \in l_p(\mathbf{R})\colon \|x_i/a_i\|_p \le 1\}$ (it can be assumed that $a_i \ne 0$) and $K_N = K \cap L(e_1, \ldots, e_N)$. Choose an ε-net $x_1, \ldots, x_m$ in K_N and prove that it is a (2ε)-net for K. The operator A is compact if and only if K is compact.

384. The operator $Af = xf$ is invertible on the subspace $L = \{f \in C[0, 1], f|_{[0, 1/2]} = 0\}$.

385. Suppose that A' is compact. Then A'' is compact. Therefore, the set $A''S''$, where S'' is the closed unit ball in L_1'', is pre-compact. The space L_2 is isometrically imbedded in L_2''. Identifying L_2 with the image of this imbedding in L_2'', we get $AS \subseteq A''S''$. Consequently, AS is pre-compact in the strong topology of L_2'' and, therefore, also in the strong topology of L_2.

386. Use the Weierstrass theorem.

387. If $\{\varphi_i\}$ and $\{\psi_j\}$ are complete orthonormal systems in $L_2(X, \mu)$ and $L_2(Y, \nu)$, then $\{\varphi_i \psi_j\}$ is a complete orthonormal system in $L_2(X \times Y, \mu \times \nu)$. Expand $K(x, y)$ in a series with respect to this system and show that the series of rank-one operators obtained in this way converges uniformly to A.

388. The operator T commutes with the dilation operators $A_\alpha f(x) = f(\alpha x)$. This suggests passing to the new variable $\tau = \ln x$. Let $\varphi(\tau) = f(e^\tau)e^{\tau/p}$. Then $f(t) = \varphi(\ln t)t^{-1/p}$, and the correspondence $f \leftrightarrow \varphi$ is an isometry between $L_p(0, \infty)$ and $L_p(\mathbf{R})$. To our operator T there corresponds an operator $\tilde{T}$ on $L_p(\mathbf{R})$:

$$\tilde{T}\varphi(\tau) = \int_{-\infty}^{\tau} \varphi(\sigma)e^{(\sigma-\tau)/q}\, d\sigma,$$

where $q = p/(p-1)$ is the number conjugate to p. To compute the norm of T we can use the properties of convolution operators (see Ch. IV) and reduce $\tilde{T}$ to the form of multiplication by a function. A more direct way is to rewrite the definition of $\tilde{T}$ in the form $\tilde{T} = \int_{-\infty}^{0} e^{\sigma/q}\, T_\sigma\, d\sigma$, where $T_\sigma \varphi(\tau) = \varphi(\sigma + \tau)$ and use the fact that $\|T_\sigma\| = 1$. Answer: $\|T\| = q$. To prove that $\tilde{T}$ is not compact, consider the family of functions $\varphi_n = n^{1/p}\chi_{[-n,n]}$ and verify that $\varphi_n \rightharpoonup 0$ in $L_p(\mathbf{R})$, but $\|\tilde{T}\varphi_n\|_{L_p(\mathbf{R})}$ does not tend to zero.

389. L is reflexive $\Leftrightarrow$ the unit ball of S' is weakly compact.

390. Use the Arzelà–Ascoli theorem.

391. If $c_0 \neq 0$, then it is impossible. If $c_0 = 0$, then it is possible. (For example, a finite-dimensional projection.)

3. The Theory of Fredholm Operators

392. There exists a $c > 0$ such that $\forall a_n \neq 0$, $|a_n| > c$. Apply the open mapping theorem.

393. im $T^k = l_p(\mathbf{R})$.

394. (a) The semi-exactness at L_0 and L_3 is obvious.

(b) Semi-exactness at L_1. Let x be an arbitrary vertex of P, and let e_x be equal to 1 at x and to zero at the remaining vertices. Then $d_1 e_x$ is equal to 1 on the edges going out from x, -1 on the edges going into x, and 0 on the remaining edges. Consider any face Δ containing x; then x belongs to two successive edges Γ_1 and Γ_2 of Δ. If Γ_1 and Γ_2 both go into x or both go out from x, then $\varepsilon(\Gamma_1, \Delta) = -\varepsilon(\Gamma_2, \Delta)$. If one edge goes out from x and the other goes into x, then $\varepsilon(\Gamma_1, \Delta) = \varepsilon(\Gamma_2, \Delta)$. From this it follows that $\alpha_2\alpha_1 = 0$.

(c) Semi-exactness at L_2. Consider any edge Γ and the function f_Γ equal to 1 on Γ and to 0 on the remaining edges. Let Δ_1 and Δ_2 be any pair of faces containing Γ. If $\varepsilon(\Gamma, \Delta_1) = \varepsilon(\Gamma, \Delta_2)$, then $\varepsilon(\Delta_1, p) = -\varepsilon(\Delta_2, p)$. If $\varepsilon(\Gamma_1, \Delta_1) = -\varepsilon(\Gamma, \Delta_2)$, then $\varepsilon(\Delta_1, p) = \varepsilon(\Delta_1, p)$. From this, $d_3 d_2 = 0$. For the cube and a simplex $H_0 = \mathbf{R}$, and $H_1 = H_2 = 0$.

395. im $C^k(\mathbf{T})$ is the subspace of $C^{k-1}(\mathbf{T})$ consisting of the functions f for which $\int_0^{2\pi} f(t)\, dt = 0$.

396. Assuming that $T_0 = T_{n+1} = 0$, we have

$$H_i \equiv \ker T_{i+1}/\mathrm{im}\ T_i, \qquad L_i/\ker T_{i+1} \equiv \mathrm{im}\ T_{i+1}, \qquad i = 0, \ldots, n,$$

from which

$$\dim H_i + \dim \mathrm{im}\ T_i = \dim \ker T_{i+1},$$
$$\dim \mathrm{im}\ T_{i+1} + \dim \ker T_{i+1} = \dim L_i.$$

397. The operator $T_{k+1} \circ T_k$ is equal to zero, since its adjoint $T'_k \circ T'_{k+1}$ is zero. Hence, the given sequence is semi-exact: $\mathrm{im}\ T_k \subset \ker T_{k+1}$. Further, if a functional $f \in L'_k$ vanishes on $\mathrm{im}\ T_k$, then $T'_k f = 0$. Therefore, $f \in \ker T'_k = \mathrm{im}\ T'_{k+1}$. Thus, f vanishes on $\ker T_{k+1}$, and hence $\ker T_{k+1} \subset \overline{\mathrm{im}\ T_k}$. It remains to verify that the image of T_k is closed. Two norms are defined in the space $\mathrm{im}\ T_k$: the norm of the space L_k and the norm of the quotient space $L_{k-1}/\ker T_k$. The space of functionals continuous with respect to the first norm consists of the restrictions to $\mathrm{im}\ T_k$ of the functionals in L'_k (Hahn–Banach theorem). It can be identified with $\mathrm{im}\ T'_k$. The functionals continuous with respect to the second norm form a subspace of L'_{k-1} which coincides with $\ker T'_{k-1}$ (since $\ker T_k = \overline{\mathrm{im}\ T_{k-1}}$). By assumption, $\mathrm{im}\ T'_k = \ker T'_{k-1}$. Consequently, the supply of weakly bounded sets with respect the two norms coincides. But weak and strong boundedness are equivalent in normed spaces. Therefore, the collections of strongly bounded sets coincide. But this means that the two norms are equivalent. The results of Problems 328 and 31 imply that $\mathrm{im}\ T_k$ is closed.

Remark. The example given in Problem 293 shows that there are exact sequences $0 \leftarrow L'_1 \leftarrow L'_2 \leftarrow 0$ which are not dual to any exact sequence of the form $0 \to L_1 \to L_2 \to 0$.

398. The semi-exactness of the dual sequence follows from the equality $T'_k \circ T'_{k+1} = (T_{k+1} \circ T_k)' = 0$.

Let f be a linear functional on the space $H_k = \ker T_{k+1}/\mathrm{im}\ T_k$, and regard it as a functional on $\ker T_{k+1}$ which equals zero on $\mathrm{im}\ T_k$. Let F be an extension of this functional to L_k. Clearly, $F \in \ker T'_k$. Moreover, if F_1 and F_2 are two such extensions, then $F_1 - F_2$ vanishes on $\ker T_{k+1}$ and, consequently, lies in $\mathrm{im}\ T'_{k+1}$ (this assertion uses the fact that $\mathrm{im}\ T_{k+1}$ is closed). Therefore, all the possible extensions form the elements of the quotient space $\ker T'_k/\mathrm{im}\ T'_{k+1}$.

399. The operator $S: \{x_n\} \to \{x_{n-1}\}$ works (set $x_0 = 0$).

400. Write T in the form $T = (d/dx)^n \circ (1 + K)$, where K is an integral operator, hence, a compact operator. Answer: $\mathrm{ind}\ T = n$. Another way is to describe explicitly the kernel and cokernel of T.

401. If $a(x)$ is nonzero everywhere, then the operator is invertible and, consequently, Fredholm. Otherwise the cokernel of the operator is infinite-dimensional.

402. $\ker P = 0$, $\mathrm{im}\ P = C(\Gamma)$, since any continuous function u on Γ can be extended in a unique way to a harmonic function in Ω.

403. By the uniqueness theorem for holomorphic functions, the kernel of the operator A of multiplication by $a(z)$ is zero.

Let $z_1, \dots, z_n$ be the zeros of $a(z)$ on Ω, and let their multiplicities be $k_1, \dots, k_n$. Then $\operatorname{im} A = \{f \in H(\Omega), f^{(j)}(z_i) = 0, j = 1, \dots, k_i; i = 1, \dots, n\}$, and $\operatorname{ind} A = -\sum_{i=1}^{n} k_i$.

404. From the explicit form of the kernel it follows that the solution must have the form $f(x) = a \cos x + b \sin x$. Substituting this expression in the equation, we get a system of linear equations in a and b: $a = \lambda((\pi/4)a + (1/2)b)$, $b = \lambda((1/2)a + (\pi/4)b)$. This system has a nonzero solution for $\lambda = 4/(\pi \pm 2)$. The eigenfunctions are proportional to $\cos x \pm \sin x$.

405. For any $g(x)$. The equation has a unique solution.

406. For $\lambda \neq 1/(b - a)$.

407. Let $\varphi_k(x) = H_k(x)e^{-x^2/2}$, where the H_k are the Hermite polynomials (see Problem 532(d)). Verify that $\{\varphi_k\}$ is an orthogonal basis in H_0 and that $A_+ \varphi_k = -2k\varphi_{k-1}$ and $A_- \varphi_k = \varphi_{k+1}$.

408. Using Fubini's theorem, prove that the function $\psi(s) = (A\varphi)(s)$ is defined almost everywhere. Applying the Cauchy–Bunyakovskii inequality, obtain the estimate

$$|\psi(s)|^2 \leq \|\varphi\|_{L_2} \cdot \int_a^b |K(s, t)|^2 \, dt.$$

Integrating the last inequality with respect to s produces the desired estimate.

409. Verify directly.

410. Use Fubini's theorem.

411. Set

$$a_{ij} = \int_a^b Q_i(t)P_j(t) \, dt, \; b_j = \int_a^b Q_i(t) f(t) \, dt.$$

412. Show that some power of the operator on the right-hand side of the equation is a contraction, and, consequently, the homogeneous equation has a unique (trivial) solution. Then use the Fredholm alternative.

413. Use Fubini's theorem and the Cauchy–Bunyakovskii inequality.

414. Use the result of Problem 413 and induction.

§3. Function Spaces and Generalized Functions

1. Spaces of Integrable Functions

415. First prove the numerical inequality $a^{1/p}b^{1/q} \leq a/p + b/q$ for $a, b \geq 0$: considering the function $\varphi(t) = t^{1/p} - t/p$, prove the inequality $\varphi(t) \leq \varphi(1)$ for $t > 0$, and substitute $t = a/b$.

416. From Hölder's inequality it follows that

$$\left|\int_X fg\,d\mu\right| \le \left(\int_X |fg|\,d\mu\right) \le \left(\int_X |f|^p\,d\mu\right)^{1/p}\left(\int_X |g|^q\,d\mu\right)^{1/q} \le \left(\int |f|^p\,d\mu\right)^{1/p}.$$

Consequently, $\sup|\int_X fg\,d\mu| \le (\int_X |f|^p\,d\mu)^{1/p}$. Select a function $g(x)$ for which the equality sign is attained.

417. Let q be connected with $p > 1$ by the relation $1/p + 1/q = 1$. Applying Hölder's inequality, we have

$$\begin{aligned}\int_X |f+g|^p\,d\mu &\le \int_X |f||f+g|^{p-1}\,d\mu + \int_X |g||f+g|^{p-1}\,d\mu\\ &\le \left(\int_X |f|^p\,d\mu\right)^{1/p}\left(\int_X |f+g|^p\,d\mu\right)^{1/q}\\ &\quad + \left(\int_X |g|^p\,d\mu\right)^{1/p}\left(\int_X |f+g|^p\,d\mu\right)^{1/q},\end{aligned}$$

from which the inequality to be proved follows immediately.

418. Let $n, k \in \mathbf{Z}$, $m \in \mathbf{N}$, and let $\chi(A)$ be the characteristic function of a set A. If $\{A_n\}$ is a base, then finite sums of the functions $f_{kmn}(x) = (k/m)\chi(A_n)$ form a dense subset of $L_1(X, \mu)$. If $\{g_n(x)\}$ is dense in $L_1(X, \mu)$, then $B_{kmn} = \{x \in X \mid (k/m) \le g_n(x) < (k+1)/m\}$ is a base.

419. The solution is analogous to that of Problem 418.

420. Prove that the distance between two characteristic functions in $L_\infty(X, \mu)$ is equal to either 0 or 1, depending on whether the corresponding sets are equivalent or not.

421. Use the facts that the function $|f|^q$ is in the space $L_{p/q}(X, \mu)$, while the function identically equal to 1 is in the dual space $L_{p/(p-q)}(X, \mu)$. In passing we get that the imbedding $L_p(X, \mu) \subset L_q(X, \mu)$ is continuous with norm $\le \mu(X)^{1/q - 1/p}$.

422. Let $q > p$, $1/p > k > 1/q$. Consider the functions

$$f_q(x) = x^{-k}\theta(x-1),$$

$$f_p(x) = x^{-k}\gamma_{[0,1]}(x).$$

423. $1/\beta < p < 1/\alpha$.

424. If $1/q + 1/r = 1/s$, then $[1/(q/s)] + [1/(r/s)] = 1$. Applying Hölder's inequality twice, we have

$$\|fgh\|_1 \le \|f\|_p\|gh\|_s \le \|f\|_p\|g\|_q\|h\|_r.$$

425. It is easy to see that $1/r = \alpha/p + \beta/q$, $\alpha + \beta = 1$. We restrict ourselves to the case $q < \infty$, $f(x) \ge 0$. Then $1 = [1/(p/(r\alpha))] + [1/(q/(r\beta))]$; apply Hölder's inequality to the product $f^{\alpha r} \cdot f^{\beta r}$.

426. Use the obvious inequality

$$(\|f\|_\infty - \varepsilon)(\mu(X'))^{1/p} \le \|f\|_p \le (\mu(X))^{1/p}\|f\|_\infty,$$

where the set $X' \subset X$ is chosen for $\varepsilon > 0$ in such a way that $|f| \ge \|f\|_\infty - \varepsilon$ for $x \in X'$, $\mu(X') \ne 0$.

427. (a) Use the results of Problems 315 and 253, along with Theorem 23.
(b), (c), and (d): see the hints for Problem 249.

428. If $\alpha \ne 0$ and $-1/p < \alpha$, then $\|x^\alpha\|_p = (p\alpha + 1)^{-1/p}$; if $1 \le p < \infty$, then $\|x^0\|_p = 1$; if $\alpha \ge 0$, then $\|x^\alpha\|_\infty = 1$.

429. (a) The subspace of polygonal curves with vertices at the points $0, \pm 1, \pm 2, \ldots$;
(b) the functions in L_1 satisfying the condition $f(x) = f([x])$.

430. Let $V \subset L_\infty(X, \mu)$. The identity mapping from $L_\infty(X, \mu)$ to $L_1(X, \mu)$ is continuous. By the Banach theorem, the inverse mapping is continuous on V. Hence, there is a constant M_1 such that $\|f\|_\infty \le M_1\|f\|_1$ for $f \in V$. This and the Cauchy–Bunyakovskii inequality gives us the estimate $\|f\|_\infty \le M_2\|f\|_2$ for $f \in V$. Let $\varphi_1, \ldots, \varphi_n$ be an orthonormal system in V with respect to the scalar product in $L_2(X, \mu)$.

If $(c_1, \ldots, c_n)$ is any unit vector in $l_2(n, \mathbf{R})$, then

$$\left\| \sum_{k=1}^n c_k \varphi_k \right\|_\infty \le M_2 \left\| \sum_{k=1}^n c_k \varphi_k \right\|_2 = M_2.$$

This implies that for almost all $x \in X$ the vector $(\varphi_1(x), \ldots, \varphi_n(x))$ has norm $\le M_2$ in $l_2(n, \mathbf{R})$. Therefore, $n = \int_X \sum_{k=1}^n |\varphi_k(x)|^2\, d\mu \le M_2\, \mu(X)$, and $\dim V < \infty$.

431. For any $\varepsilon > 0$ and any $f \in L_p(\mathbf{R}, dx)$ there exists a closed interval $[a, b]$ such that $(\int_{\mathbf{R}\setminus[a,b]} |f(x)|^p\, dx)^{1/p} < \varepsilon$. Use Problem 427.

432. Use Problem 431 to verify continuity in the mean on the space $C_0(\mathbf{R})$.

433. Verify continuity in the mean on the space $C_0(\mathbf{R}^n)$.

434. Suppose first that M consists of a single function f. Then the condition (a) is automatically satisfied, the condition (b) follows from the definition of an integrable function, and the condition (c) follows from Problem 432. Next, if M consists of finitely many functions $f_1, \ldots, f_n$, then for each function f_i the conditions (*a*), (*b*), (*c*) hold with the constants c_i, $R_i(\varepsilon)$, $\delta_i(\varepsilon)$, respectively. Let $c = \max_i c_i$, $R(\varepsilon) = \max_i R_i(\varepsilon)$, $\delta(\varepsilon) = \min_i \delta_i(\varepsilon)$. Then the conditions (a), (b), (c) hold for M. Finally, if M is any pre-compact set and $\{f_1, \ldots, f_n\}$ is an $(\varepsilon/3)$-net for it, then the conditions (a), (b), (c) hold for M with the constants $c + \varepsilon/3$, $R(2\varepsilon/3)$, $\delta(\varepsilon/3)$. This proves the necessity of the conditions (a), (b), (c).

Suppose now that these conditions hold. Consider the mapping φ_ε of M into the subspace $C[-R(\varepsilon), R(\varepsilon)] \subset L_p(\mathbf{R}, dx)$ by the formula $\varphi_\varepsilon(f)(x) = [1/\delta(\varepsilon)] \int_x^{x+\delta(\varepsilon)} f(t)\, dt$. The condition (a) implies that $\varphi_\varepsilon(M)$ is bounded in $C[-R(\varepsilon), R(\varepsilon)]$, and (b) and (c) imply that the distance between f and $\varphi_\varepsilon(f)$ in $L_p(\mathbf{R}, dx)$ does not exceed 2ε. Finally, (c) implies that the set of functions

$\varphi_\varepsilon(f)$, $f \in M$, is equicontinuous. Therefore, $\varphi_\varepsilon(M)$ is pre-compact in $C[-R(\varepsilon), R(\varepsilon)]$ and so also in $L_p(\mathbf{R}, dx)$.

If $\{\varphi_\varepsilon(f_1), \dots, \varphi_\varepsilon(f_n)\}$ is an ε-net in $\varphi_\varepsilon(M)$, then $f_1, \dots, f_n$ is a (3ε)-net in M. Since ε is arbitrary, M is pre-compact.

435. The mapping $f(x) \otimes g(x) \mapsto f(x) \cdot g(x)$ extends by continuity to a mapping of $L_1(X, \mu) \otimes L_1(Y, \nu)$ into $L_1(X \times Y, \mu \times \nu)$ that does not increase the norm. Verify that this mapping is in fact an isometry. Let $\varphi \in L_1(X \times Y, \mu \times \nu)$. Then φ can be approximated in the norm by functions of the form $\tilde{\varphi}(x, y) = \sum_{i=1}^n c_i \chi_{E_i}(x)\chi_{F_i}(y)$, where the E_i (F_i) are disjoint measurable subsets of X (Y). Without loss of generality it may be assumed that $\mu(E_i)$ and $\nu(F_i)$ are rational numbers; but then it may be assumed that these numbers are integers (multiply $\tilde{\varphi}$ by a suitable integer). Thus, our assertion reduces to the particular case when X and Y consist of finitely many points of unit measure. This means we must establish that the spaces $l_1(n, \mathbf{R}) \hat{\otimes} l_1(m, \mathbf{R})$ and $l_1(mn, \mathbf{R})$ are isomorphic. Let $e_1, \dots, e_n$ be a basis in the first space, and $f_1, \dots, f_m$ a basis in the second; then $e_i \otimes f_j$ can be taken as a basis in the tensor product. Let g_{ij} be a corresponding basis in the third space. It is necessary to prove that $\|\sum_{ij} c_{ij} e_i \otimes f_j\| = \|\sum_{ij} c_{ij} g_{ij}\|$, i.e., $\inf \sum_\alpha \sum_i |a_i^{(\alpha)}| \sum_j |b_j^{(\alpha)}| = \sum_{ij} |c_{ij}|$, where the infimum is taken over all representations of the vector $\sum_{ij} c_{ij} e_i \otimes f_j$ as a sum $\sum_\alpha \varphi_\alpha \otimes \psi_\alpha$, where $\varphi_\alpha = \sum_i a_i^{(\alpha)} e_i$, $\psi_\alpha = \sum_j b_j^{(\alpha)} f_j$. The estimate in one direction follows from the equation $e_{ij} = \sum_\alpha a_i^{(\alpha)} b_j^{(\alpha)}$. The estimate in the other direction is obtained from an examination of the specific representation in which α runs through all the pairs i, j and $\varphi_{ij} = c_{ij} e_i$, $\psi_{ij} = f_j$.

436. (a) Call a subset E of a space X with measure μ an *atom* if $\mu(E) > 0$ and any measurable subset $F \subset E$ either has measure zero or $\mu(F) = \mu(E)$. (It is easy to see that for the Borel measures μ the atoms are the points of positive measure.) Prove that the extreme points of the unit ball in $L_1(X, \mu)$ are the characteristic functions of the atoms and only these. (In particular, the unit ball of the space l_1 has extreme points, and that of $L_1[0, 1]$ does not.)

(b) All the boundary points of the ball (to prove this determine when the Minkowski inequality becomes an equality).

(c) The set of f such that $|f(x)| = 1$ for almost all x.

437. l_1 is the space dual to the space of sequences converging to zero, but $L_1[0, 1]$ is not the dual space of any Banach space, since otherwise its unit ball would have extreme points, contrary to Problem 436(a) (use the Krein–Mil'man theorem).

2. Spaces of Continuous Functions

438. To prove completeness consider the pointwise limit of a Cauchy sequence in $C(X)$.

439. The polynomials in n variables with rational coefficients form a dense subset of $C(X)$.

440. If the function g belongs to the unit ball in $C(X)$, then

$$|F(g)| = \left|F\left(\frac{g + |g|}{2}\right) - F\left(\frac{|g| - g}{2}\right)\right| \leq F(1).$$

441. If $f(x)$ is a nonnegative function, then let $G_f = \{g : 0 \leq g(x) \leq f(x)\}$. Then $F_1(f) = \sup_{g \in G_f} F(g)$. The inequalities $F_1(f) \geq F(f)$ and $F_1(f) \geq 0$ for $f \geq 0$ are obvious. The additivity of F_1 follows from the equality $G_{f_1 + f_2} = G_{f_1} + G_{f_2}$ (the inclusion $G_{f_1} + G_{f_2} \subset G_{f_1 + f_2}$ is obvious, and the reverse inclusion follows from the equality $g = gf_1/(f_1 + f_2) + gf_2/(f_1 + f_2)$).

442. Denote by E_ε the ε-neighborhood of the set E and by $\bar{E}$ its closure. We show that $\mu(\bar{K}_\varepsilon) \to \mu(K)$ as $\varepsilon \to 0$ for any compact set $K \subset X$. To do this fix $\delta > 0$ and choose a function $\varphi \in C(X)$ such that $\chi_K(x) \leq \varphi(x) \leq 1$, $F(\varphi) \leq \mu(K) - \delta$.

Let L be the set of all points $x \in X$ for which $\varphi(x) \leq 1 - \delta$. It is clear that L is compact and does not intersect K. Let d be the distance between K and L. If $\varepsilon < d$, then the function $\psi(x) = [\varphi(x)/(1 - \delta)]$ has the properties $\chi_{\bar{K}_\varepsilon}(x) \leq \psi(x) \leq 1$. Therefore, $\mu(K_\varepsilon) \leq F(\psi) = F(\varphi)/(1 - \delta) = [\mu(K) - \delta]/(1 - \delta)$. The last expression converges to $\mu(K)$ as $\delta \to 0$.

This implies the equation $\mu(K) + \mu(X \setminus K) = 1$, as well as the finite additively and the regularity of the function μ:

$$\mu(A) = \sup_{K \subset A} \mu(K) = \inf_{G \supset A} \mu(G),$$

where K denotes a compact set and G an open set.

The inequality $\mu(\coprod_{n=1}^\infty E_n) \geq \sum_{n=1}^\infty \mu(E_n)$ follows directly from the definition of $\mu(E_n)$ and the inequality $\mu(\coprod_{i=1}^n K_i) \geq \sum_{i=1}^n \mu(K_i)$, which is a consequence of the definition of $\mu(K)$.

Use the regularity of μ to derive the reverse inequality; let $K \subset E = \coprod_{n=1}^\infty E_n$ be a compact set and let $G_i \supset E_i$ be open sets such that

$$\mu(E \setminus K) < \frac{\varepsilon}{2}, \qquad \mu(G_n \setminus E_n) < \frac{\varepsilon}{2^n}.$$

The inclusion $K \subset \bigcup_{n=1}^\infty G_n$ implies the inclusion $K \subset \bigcup_{n=1}^N G_n$ for some N. The finite additivity of μ now gives us the estimate $\mu(K) \leq \sum_{n=1}^N \mu(G_n)$ and, consequently, the inequality $\mu(E) \leq \sum_{n=1}^\infty \mu(G_n) + \varepsilon$.

443. (a) 1; (b) 2; (c) 3; (d) $4/\varepsilon^2$; (e) $\sum_{n=1}^\infty (1/n^2) = \pi^2/6$; (f) 2.

444. $F(f) = \int_{-1}^1 G(x)\, df(x)$. $G(x)$ and $\|F\|$ are equal respectively to: (a) $\theta(x)$, 1; (b) $x - 2\theta(x)$, 4; (c) $x^2\theta(x)/2$, 1/2; (d) $x - 3x\theta(x)$, 3.

445. To prove the sufficiency of the condition show that any step function $S(x)$ satisfies

$$\lim_{n \to \infty} \int_0^1 S(x)\, dg_n(x) = \int_0^1 S(x)\, dg(x).$$

Approximate an arbitrary function $f(x) \in C[0, 1]$ by step functions.

446. Reduce the problem to the case when M consists of monotonically nondecreasing functions. Choose a sequence in M converging at some point, then a subsequence of it converging at another point, and so on; by the diagonal process (take the nth term from the nth subsequence) obtain a subsequence $\{\varphi_n\}$ converging at all the rational points of $[0, 1]$. Prove that $\{\varphi_n\}$ converges to some nondecreasing function $\varphi(x)$ everywhere except the points of discontinuity of $\varphi(x)$ (of which there are no more than countably many), and then use the diagonal process to choose a subsequence of $\{\varphi_n\}$ that converges also at these points.

447. (a), (b) $f(0)$ and $f(1)$ are the extensions, respectively; (c), (d) there is no extension, since any function $f \in C[0, 1]$ can be approximated by polynomials of the form $(x + 1)p_1(x)$, for which $F_3 \equiv 0$, and by polynomials of the form $p_2(x^{N+1})$, for which $F_4(f) = c_0 f(0)$. Verify that these extensions do not work.

448. $f_1(x) \equiv 1$, $f_2(x) \equiv -1$ (see Problem 299).

449. Suppose that $\mu_x = \tau\mu_1 + (1 - \tau)\mu_2$, where $\tau \in (0, 1)$, and that μ_1 and μ_2 belong to the unit ball of $C'(X)$. Let f_x be a function in $C(X)$ equal to 1 at x and taking values in $[0, 1)$ at the remaining points (for example, $f_x(y) = \max\{1 - d(x, y), 0\}$). Then $\mu_x(f_x) = \|f_x\| = 1$, $|\mu_1(f_x)| \le 1$, $|\mu_2(f_x)| \le 1$. Therefore, $\mu_1(f_x) = \mu_2(f_x) = 1$. This is possible only if $\mu_1(\{x\}) = \mu_2(\{x\}) = 1$, i.e., $\mu_1 = \mu_2 = \mu_x$. Hence, μ_x is an extreme point.

Suppose now that μ is any extreme point of the unit ball in $C'(X)$, and $f(x)$ is any continuous function on X taking values in $(0, 1)$. It is easy to see that either μ or $-\mu$ is a positive measure. Suppose for definiteness that $\mu > 0$. Let $\mu_1 = f\mu/\mu(f)$, $\mu_2 = (1 - f)\mu/[1 - \mu(f)]$. Then μ_1 and μ_2 lie in the unit ball of $C'(X)$, and

$$\mu = \mu(f)\cdot\mu_1 + [1 - \mu(f)]\mu_2.$$

Since μ is an extreme point, μ_1 and μ_2 coincide with μ. From this it follows easily that $\mu(fg) = \mu(f)\mu(g)$ for any f, $g \in C(X)$ taking values in $(0, 1)$. Since this relation is bilinear, it holds for all $f, g \in C(X)$. Let L be the kernel of the functional μ. This is a closed ideal of codimension 1 in $C(X)$. It is easy to prove that there is a point $x \in X$ at which all the functions in L vanish. (Otherwise, X can be covered by finitely many neighborhoods U_i for which there exist $f_i \in L$ such that $f_i(x) \neq 0$ on U_i. Then $f = \sum_i |f_i|^2 \in L$ and $f \neq 0$ on X, which implies that $L = C(X)$.) The condition $\operatorname{codim} L = 1$ implies that the point is unique. It is now clear that $\mu = \mu_x$.

450. *First method.* Along with any function φ in it, the algebra A contains also the function $P(\varphi)$, where P is a polynomial. The Weierstrass theorem and the fact that A is closed give us that A contains $f \circ \varphi$ for any continuous function f on the line. Using this, prove in turn that A contains the following kinds of functions:

(1) for any $x \neq y$ in X, a function φ such that $\varphi(x) = 0$, $\varphi(y) = 1$, and $0 \le \varphi(z) \le 1$ for the remaining $z \in X$;

(2) for any point $x \in X$ and any neighborhood U of it, a function φ such that $\varphi(x) = 0$ and $\varphi(z) = 1$ for all $z \in X \setminus U$;

(3) for any disjoint compact sets K_1 and K_2, a function φ which equals 0 on K_1, equals 1 on K_2, and takes values between 0 and 1 at the remaining points of X.

Using functions of the last kind, show that each $f \in C(X)$ with norm 1 can be approximated to within 2/3 by a function $\varphi \in A$ with norm 1/3.

Second method. Let L be the annihilator of A in $C(X)'$, and F an extreme point of the unit sphere in L. Prove that for any function $a \in A$ the functional $F_a(f) = F(af)$ is proportional to F (compare with the hint to Problem 449). Derive from this that F is proportional to some μ_x and, hence, is equal to zero.

451. No; consider

$$A_{x_0} = \{f(x) \mid f(x) \in C(X), f(x_0) = 0\}.$$

452. Suppose that the diameter of the set X is equal to 1 (this obviously does not restrict the generality). Since X is compact, it can be representted as the union of a finite number of compact subsets $X_1, \ldots, X_{n_1}$ of diameter 1/2. Each of the X_i ($i = 1, \ldots, n_1$) can be represented as the union of a finite number of compact sets $X_{i1}, \ldots, X_{in_2}$ of diameter 1/4, and so on. The mapping φ will be constructed by steps. First break up $[0, 1]$ into $2n_1 - 1$ equal segments $\Delta_1, \ldots, \Delta_{2n_1-1}$ and assume that $\varphi(\Delta_{2k-1}) \subset X_i$, while $\varphi(\Delta_{2k})$ is a path joining some point $x_k \in X_k$ to a point $x_{k+1} \in X_{k+1}$. (Such a path exists, because X is arcwise connected.) Break up in the segment Δ_{2k-1} into $2n_2 - 1$ equal segments $\Delta_{2k-1,i}$, $1 \le i \le 2n_2 - 1$, and assume that $\varphi(\Delta_{2k-1,2l-1}) \subset X_{kl}$, while $\varphi(\Delta_{2k-1,2l})$ is a path joining a point $x_{kl} \in X_{kl}$ to a point $x_{k,l+1} \in X_{k,l+1}$. Continuing this process defines φ on some dense subset of $[0, 1]$, and this mapping is uniformly continuous where it is defined. Therefore, it can be extended to a continuous mapping of the whole segment.

453. This problem is a particular case of Problem 452. In this case the construction can be illustrated by a sketch (Fig. 4). Here the numbers $n_1, n_2, \cdots$ are all equal to 4, and the representative $x_{i_1,\ldots,i_k}$ of the square $X_{i_1,\ldots,i_k}$ is taken to be its center; the four squares of the kth rank lying in a square of rank $k - 1$ are traversed in the clockwise direction, beginning from the lower left-hand one.

454. The mapping $t \mapsto (|\cos 2\pi t|^{2/q} \operatorname{sgn} \cos 2\pi t, |\sin 2\pi t|^{2/q} \operatorname{sgn} \sin 2\pi t)$ carries $[0, 1]$ to the unit circle in $l_p(2, \mathbf{R})$. The corresponding imbedding of $l_p(2, \mathbf{R})$ into $C[0, 1]$ has the form

$$(\alpha, \beta) \mapsto \varphi_{\alpha,\beta}(t) = \alpha |\cos 2\pi t|^{2/q} \operatorname{sgn} \cos 2\pi t + \beta |\sin 2\pi t|^{2/q} \operatorname{sgn} \sin 2\pi t.$$

Using Hölder's inequality, verify that

$$\max_{t \in [0,1]} |\varphi_{\alpha,\beta}(t)| = \sqrt[p]{|\alpha|^p + |\beta|^p}.$$

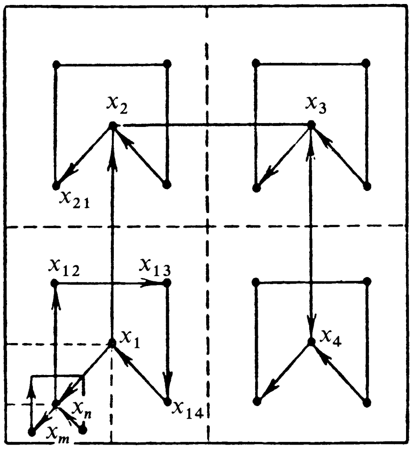


Figure 4

455. Consider the natural linear mapping φ

$$C[0, 1] \otimes C[0, 1] \to C(\square),$$

given by

$$\varphi(f \otimes g)(x, y) = f(x)g(y).$$

It is clearly injective, and the Weierstrass theorem implies that its image is dense in $C(\square)$. It remains to check that it is an isometry. By the definition of the norm in the tensor product.

$$\left\| \sum_{i=1}^{n} f_i \otimes g_i \right\| = \sup_{\|\mu\| = \|\nu\| = 1} \left| \sum \mu(f_i)\nu(g_i) \right|.$$

It suffices to take the supremum only over the extreme points of the unit ball in $C[0, 1]'$. Therefore (see Problem 449),

$$\left\| \sum_i f_i \otimes g_i \right\| = \sup_{x, y} \left| \sum_i f_i(x)g_i(y) \right|$$
$$= \sup_{x, y} \left| \varphi\left(\sum_i f_i \otimes g_i \right)(x, y) \right| = \left\| \varphi\left(\sum_i f_i \otimes g_i \right) \right\|.$$

456. Using the Stone–Weierstrass theorem, prove that $C(X) \otimes C(Y)$ is dense in $C(X \times Y)$. Complete the proof by the trivial check that the norm $p_X \underset{\vee}{\otimes} p_Y$, where p_X and p_Y are the norms in $C(X)$ and $C(Y)$, coincides with the norm in $C(X \times Y)$.

457. Use the fact that the adjoint operator A' gives an isometry of the unit ball of $C(Y)'$ onto the unit ball of $C(X)'$. Therefore, for each point $y \in Y$ there exists a point $x = \varphi(y)$ and a number $a(y) = \pm 1$ such that $A'\mu_y = a(y)\mu_x$. From this, $(Af)(y) = a(y)f(\varphi(y))$. Setting $f = \text{const}$, we see that $a \in C(Y)$. Therefore, $f \circ \varphi \in C(Y)$ for any $f \in C(X)$. From this it follows that φ is continuous. Applying this to the operator A^{-1}, we see that the inverse function is also continuous.

458. Let $F_n(x, y) = f_n(x) + g_n(y)$ be a Cauchy sequence in $C(\square)$. Then $F_n(0, y) = f_n(0) + g_n(y)$ is Cauchy in $C[0, 1]$ so $f_n(x) - f_n(0)$ is Cauchy and

$$\lim_{n\to\infty} F_n(x, y) = \lim_{n\to\infty}(f_n(x) - f_n(0)) + \lim_{n\to\infty}(g_n(y) + f_n(0)).$$

459. Let $\{r_n\}$ be a dense sequence in $[0, 1]$ with $r_0 = 0, r_1 = 1$. Consider the system $\{f_n\}$, where $f_0(x) \equiv 1, f_1(x) = x$, and $f_n(x)$ is defined for $n > 1$ as follows. Suppose that r_n belongs to (r_{s_1}, r_{s_2}), one of the $n - 1$ intervals into which the points $r_2, \ldots, r_{n-1}$ partition $[0, 1]$; then

$$f_n(0) = 0, \qquad f_n(r_{s_1}) = 0, \qquad f_n(r_n) = 1, \qquad f_n(r_{s_2}) = 0, \qquad f_n(1) = 0.$$

and the graph of $f_n(x)$ is a polygonal line with four links.

Remark. There are topological bases also in the spaces $L_p(0, 1)$ and l_p for $1 \le p < \infty$ and in a separable Hilbert space (obviously), but not in every separable Banach space.

460. Assume that for any $f \in CP[0, 1]$ there exists a trigonometric series $\sum_{k\in\mathbf{Z}} c_k(f)e^{2\pi ikx}$ which converges uniformly to f. Then the series converges also in the sense of $L_2(0, 1)$. Therefore, the numbers $c_k(f)$ are the Fourier coefficients of f. Let $S_n(f) = \sum_{k=-n}^{n} c_k(f)e^{2\pi ikx}$. It follows from our assumption that $S_n \to 1$. However, this contradicts the fact that $\|S_n\| \to \infty$. (Verify that $\|S_n\| = \int_0^1 |(\sin(2n + 1)\pi x)/\sin \pi x|\, dx$ and that this integral admits a lower estimate of order $c \cdot \ln n$.)

3. Spaces of Smooth Functions

461. (a) The nonmetrizability follows from the fact that for any sequence $\{\lambda_n\}$ of numbers not eventually zero the sequence $\{\lambda_n e_n\}$ does not converge to zero in $\mathscr{D}(\mathbf{N})$.

(b) The sequence $\{x_K^{(n)}\}$ converges to $\{x_K\}$ for $n \to \infty$ if and only if: (1) there is an N such that $x_K^{(n)} = 0$ for $K > N$ and all n; (2) $x_K^{(n)} \to x_K$ for $K = 1, 2, \ldots, N$.

(c) Let x_K be a sequence of points in Ω not having a limit point in the interior of Ω, $\{U_K\}$ a collection of disjoint neighborhoods of the points x_K, and φ_K a nonzero function with support in U_K. The desired mapping of $\mathscr{D}(\mathbf{N})$ into $\mathscr{D}(\Omega)$ can be defined by the formula

$$\{c_k\} \to \sum_{k=1}^{\infty} c_k \varphi_k.$$

462. The implication (a) $\Rightarrow$ (b) is obvious; (b) $\Rightarrow$ (c), because a sequence φ_n converging to zero converges to zero in all the seminorms; (c) $\Rightarrow$ (d), since $\mathscr{D}_K(\Omega)$ is metrizable; (d) $\Rightarrow$ (a) by the definition of the topology in $\mathscr{D}(\Omega)$.

463. $\mathscr{D}_K(\Omega)$ is the intersection of the family of closed sets $I_x = \{\varphi \in \mathscr{D}(\Omega): \varphi(x) = 0\}$, where x runs through $\Omega \setminus K$.

464. First construct a finite set of functions $\{\psi_i\}$, $1 \le i \le N$, for which $\operatorname{supp} \psi_i \subset U_i$ and $\psi = \sum_{i=1}^N \psi_i \ge \delta > 0$ on K. Now let $f \in \mathscr{E}(\mathbf{R})$ be such that $f(x) = 0$ for $x < \delta/2$ and $f(x) = 1/x$ for $x \ge \delta$. Then $\phi_i = \psi_i \cdot f(\psi(x))$ is the desired collection.

465. Use the result in Problem 464.

466. Represent $\Omega = \mathbf{R}^n \setminus K$ as a union of finitely many balls. Use the fact that any function in $\mathscr{E}(\Omega)$ satisfying the estimate $|\varphi(x)| < e^{-[1/d(x,K)]}$ can be extended (by letting it be zero on K) to a function in $\mathscr{E}(\mathbf{R}^n)$.

467. It exists.

468. Use the method of proof of Theorem 30 in Ch. III.

469. All except (c).

470. (c) and (d).

471. In the case of $\mathscr{D}(\mathbf{R}^n)$ verify that $\varphi_k \to 0$ implies that $x_i \cdot \varphi_k \to 0$ and $\partial\varphi_k/\partial x_i \to 0$; in the cases $S(\mathbf{R}^n)$ and $\mathscr{E}(\mathbf{R}^n)$ give an estimate of the corresponding seminorms.

472. (a) yes, (b) no, (c) yes.

473. Use the identity $f_n^{(k)}(x) = n^{-k} f_1(x/n)$.

474. (a) yes, (b) no, (c) yes, (d) yes, (e) yes, (f) yes.

475. (a) Use the fact that the series $\sum_{k \in \mathbf{Z}} k^m f^{(n)}(x + k)$ converges absolutely and uniformly on $[0, 1]$ if $f \in S(\mathbf{R})$. The operator A^{-1} can be given by the explicit formula $(A^{-1}g)(x) = \int_0^1 g(x, y)\, dy$.

(b) The operators $A: S(\mathbf{R}^n) \to G(\mathbf{R}^{2n})$ and $A^{-1}: G(\mathbf{R}^{2n}) \to S(\mathbf{R}^n)$ have the form:

$$(Af)(x, y) = \sum_{k \in \mathbf{Z}} f(x + k) e^{-2\pi i k y},$$

$$(A^{-1}g)(x) = \int_{\mathbf{T}^n} g(x, y)\, dy.$$

476. Let p_k be the norm in $C^k(\mathbf{T}^m)$, q_k the norm in $C^k(\mathbf{T}^n)$, and r_k the norm in $C^k(\mathbf{T}^{m+n})$. Verify that the norm $p_k \check{\otimes} q_k$ is equivalent to r_k (by using the fact that every continuous linear functional on $C^k(\mathbf{T}^m)$ has the form $\langle f, \varphi \rangle = \sum_{|i| \le k} \int_{\mathbf{T}^m} \partial^i \varphi(t)\, d\nu_i(t)$, where the ν_i are signed Borel measures with finite variation on $\mathbf{T}^m$). Then derive from the Cauchy–Bunyakovskii inequality for $l_2(\mathbf{T}^{m+n})$ that the norm $p_k \hat{\oplus} q_k$ is majorized by the norm r_s for $s > (m + n)/2 + 2k$. (More precisely, the Fourier series of an $f \in C^s(\mathbf{T}^{m+n})$ converges to f in the norm $p_k \hat{\oplus} q_k$.)

Thus, the systems of norms $\{p_k \check{\otimes} q_n\}$, $\{r_k\}$ and $\{p_k \hat{\oplus} q_n\}$ are equivalent, which proves the desired statement, because of the Weierstrass theorem on the density of the trigonometric polynomials in $\mathscr{D}(\mathbf{T}^n)$.

477. (a) It is necessary to check that for $t \in [-1, 1]$ all the functions vanish outside some compact set K not dependent on t, and that $f_t^{(l)}$ converges uniformly on K to $(\partial_y f)^{(l)}$, where l is any multi-index and ∂_y denotes the partial derivative with respect to the direction y. Use the mean value theorem.

(b) It must be verified that $f_t^{(l)}$ converges uniformly on any compact set $K \subset \mathbf{R}^n$ to $(\partial_y f)^{(l)}$.

478. The property (a) is obvious; the property (b) can be proved by induction with the use of the identity $f_n^{(r)}(x) = (1/\delta_n) \int_0^{\delta_n} f_{n-1}^{(r)}(x - t)\, dt$, which is true for $r < n$. The convergence of the sequence $f_n^{(r)}$ as $n \to \infty$ for fixed r follows from the estimate

$$|f_n^{(r)} - f_{n-1}^{(r)}| \leq \frac{2^{r+1}}{\delta_1 \cdots \delta_{r+1}} \cdot \delta_n,$$

which is true for $n \geq r + 2$.

479. (a) Let M be a bounded set in L. Then it is pre-compact with respect to any seminorm p_k, since it is bounded in the seminorm p_{k+1}. As usual, a distance is introduced in L by the formula $d(f, g) = \sum_{k=1}^{\infty} 2^{-k} p_k(f - g)$. If $\{f_i\}$ is a finite (2^{-l})-net for M with respect to the seminorm $\sum_{k=1}^{l} p_k$, then it is a (2^{1-l})-net in the sense of the distance d.

(b) We analyze the case $L = \mathscr{D}_K(\Omega)$, $\Omega \subset \mathbf{R}^n$. By the Arzelà–Ascoli theorem, a set M bounded in the norm $p_{k+1}(f) = \max_{x \in K; |l| \leq k+1} |\partial^l f(x)|$ is pre-compact in the norm p_k, since the functions of the form $\partial^l f$, $|l| \leq k$, are uniformly bounded and equicontinuous in Ω.

480. If φ_n is a Cauchy sequence in $\mathscr{E}(\Omega, L)$, then for any multi-index l and any point $x \in \Omega$ the sequence $\partial^l \varphi_n(x)$ is Cauchy in L. Let $\psi_l(x) = \lim_{n \to \infty} \partial^l \varphi_n(x)$. Prove that $\psi_l(x) = \partial^l \psi_0(x)$ and that $\varphi_n \to \psi_0$ in the topology of $\mathscr{E}(\Omega, L)$. The metrizability of $\mathscr{E}(\Omega, L)$ follows from the existence of a countable set of norms. (If $\{p_j\}$ is a countable set of norms determining the topology in L and $\{K_i\}$ is a countable set of compact sets exhausting the domain Ω, then the seminorms $p_{K_i l_j}$ determine the topology in $\mathscr{E}(\Omega, L)$.)

481. Consider the mapping of $\mathscr{E}(\Omega_1 \times \Omega_2)$ into $\mathscr{E}(\Omega_1, \mathscr{E}(\Omega_2))$ given by $\varphi \to f$, where $(f(x))(y) = \varphi(x, y)$. Use the result in Problem 477.

482. Use the result in Problem 476 and the fact that the periodic functions are dense in $\mathscr{E}(\mathbf{R}^n)$.

483. Use the result in Problem 475 for the case of $S(\mathbf{R}^n)$.

4. Generalized Functions

484. All the supports coincide with $\mathbf{R}$. The continuity of the functionals follows from the theorem on the weak completeness of $\mathscr{D}(\mathbf{R})$.

485. Investigate this limit separately for even and odd functions. Answer: 0.

486. It must be checked that if the integral $\int_{\mathbf{R}} \varphi(x)\psi(x)\,dx$ is equal to zero for all $\psi \in \mathscr{D}(\mathbf{R})$, then $\varphi \in \mathscr{D}(\mathbf{R})$ is identically equal to zero.

487. (a) Prove that $\lim\langle f_\varepsilon, \varphi\rangle = 0$ for all φ having the property that $\varphi(0) = 0$.

488. Investigate separately the functions equal to zero at zero and the functions constant in a neighborhood of zero.

489. Answer: $\pi\delta(x)$.

490. It exists and is equal to 0.

491. Use the fact that $\mathscr{D}(\mathbf{R})$ is dense in $L_p(\mathbf{R}, dx)$.

492. Let $p(x)$ be a locally integrable function. For any closed interval $[a, b]$ not containing the origin there exists a sequence $\varphi_n \in \mathscr{D}(\mathbf{R})$ converging to $\chi_{[a,b]}(x)$ and having support in an interval $[a - \varepsilon, b + \varepsilon]$ that also does not contain the origin. It follows from the equation $0 = \varphi_n(0) = \int_{\mathbf{R}} \varphi_n(x)p(x)\,dx$ that $\int_a^b p(x)\,dx = 0$ for any a and b of the same sign. But the function $q(x) = \int_0^x p(t)\,d(t)$ is continuous with respect to x. From this, $q(x) = \text{const}$ and $p(x) = 0$ almost everywhere.

493. We remark that every function $\varphi \in \mathscr{D}(\mathbf{T}^n)$ can be represented by a uniformly convergent series: $\varphi(t) = \sum_{k\in\mathbf{Z}^n} c_k e^{2\pi ikt}$. Therefore, $\langle e^{2\pi ikt}, \varphi\rangle = c_{-k}$ and $\langle \sum_{k\in\mathbf{Z}^n} e^{2\pi ikt}, \varphi\rangle = \sum_{k\in\mathbf{Z}^n} c_{-k} = \varphi(0)$.

494. The norms of the spaces $C^k(\mathbf{T}^n)$ can be taken as a defining system of seminorms in $\mathscr{D}(\mathbf{T}^n)$.

495. Let $\varphi(x) = e^{2x}\cdot\omega(x) \in \mathscr{D}(\mathbf{R})$, where $\omega \in \mathscr{D}(\mathbf{R})$ is a function with support $[-1/3, 1/3]$ and identically equal to 1 on $[-1/6, 1/6]$. Consider the action of F on the translates $\varphi(x \pm k)$.

496. Verify that $\langle\delta, \varphi\rangle = -\int_{\mathbf{R}} \theta(x)\varphi'(x)\,dx$. Answer: 1.

497. (a) The order is equal to 1. Construct a sequence of functions $\varphi_n \in \mathscr{D}(\mathbf{R})$ with support on $[0, 1]$ that converge uniformly on $[0, 1]$ to zero but are such that $PV\int [\varphi_n(x)/x]\,dx$ does not converge to zero. (For example, set $\varphi_n(x) = 1/(\ln n)$ on $[1/(2n), 1/2]$.) (b) The order is equal to 0.

498. One method: Decompose $1/(x \pm i0)$ into the sum of an even and an odd component and use Problem 487(b).

499. See Problem 497 and 498.

500. (a) Use the following lemma from linear algebra. Suppose that $f_1, \ldots, f_n$ and f are linear functionals on a linear space L. If $f(x) = 0$ whenever $f_1(x) = 0, \ldots, f_n(x) = 0$ then f is a linear combination of $f_1, \ldots, f_n$.
(b) Use (a) and the Hahn–Banach theorem for LCV's.

501. (a) $[-1, 1], 0$;
(b) $\{-1, 0, 1\}, 0$.

502. (a) Use integration by parts.

(b) Use the theorem on weak-* completeness of $\mathscr{D}'(\mathbf{R})$.

(c) Use the relation

$$\frac{d}{dx}\left(\frac{x_+^{\lambda-1}}{\Gamma(\lambda)}\right) = \frac{x_+^{\lambda-2}}{\Gamma(\lambda-1)}$$

and the "initial condition" $x_+^0/\Gamma(1) = \theta(x)$.

Answer: $x_+^{(-n-1)}/\Gamma(-n) = \delta^{(n)}(x)$.

503. (a) If the support of $\varphi \in \mathscr{D}(\mathbf{R}^2)$ does not intersect the set $f(x, y) = c$, then $\langle F_c, \varphi\rangle$ is constant in a neighborhood of c. Hence, $\langle dF_c/dc, \varphi\rangle = 0$.

(b) By choosing a local system of coordinates we can reduce the problem to the case $f(x, y) = x$.

(c) Choose a local parameter t on the curve Γ given by the equation $f(x, y) = c$ in such a way that $dx \wedge dy = df \wedge dt$. Then $\langle dF_c/dc, \varphi\rangle = \int_\Gamma \varphi(t)\,dt$.

504. In the case $c \neq 0$ we can argue as in Problem 503(c). Answer: $\langle dF_c/dc, \varphi\rangle = \iint_{\Gamma_c} \varphi \cdot \omega$, where ω is the differential 2-form on the hyperboloid $x^2 + y^2 - z^2 = c$ uniquely determined by the condition $\omega \wedge d(x^2 + y^2 - z^2) = dx \wedge dy \wedge dz$. If $c = 0$, then dF_c/dc coincides with the generalized function given by the integral $(1/2) \iint \varphi[(dx \wedge dy)/z]$ over the cone $x^2 + y^2 = z^2$. (Everywhere outside the origin this coincidence can be proved as for $c \neq 0$ by passing to a local system of coordinates $x, y, c = x^2 + y^2 - z^2$. The difference between these generalized functions is concentrated at the point $(0, 0, 0)$ and is invariant under the Lorentz group. Moreover, it is easy to see that $(dF_c/dc)|_{c=0}$ is a homogeneous generalized function of degree -2. Therefore, it is equal to zero (see the hint for Problem 529).)

505. Use the isomorphism $\mathscr{L}(L_1, L_2') \approx (L_1 \hat{\otimes} L_2)'$ and the results in Problem 483.

506. (a) $K(x, y) = \delta(x - y)$; (b) $K(x, y) = \delta(x - a) \times \delta(y - b)$.

5. Operations on Generalized Functions

507. (a) $2\delta(x)$;

(b) $\delta(x)$;

(c) $\sum \delta(x - k)$, $k \in \mathbf{Z}$.

508. (a) $2\delta(x)$;

(b) $a^2 e^{-a|x|} - 2a\delta(x)$;

(c) $\sum_{k\in\mathbf{Z}} 2\delta(x - \pi k) - |\sin x|$;

(d) $-2 \sin a \cdot \delta(x + a) - 2 \cos x \cdot \operatorname{sgn}(x + a)e^{-|x+a|}$.

509. Prove that every function $\varphi \in \mathscr{D}(\mathbf{R})$ having the property that $\int_{\mathbf{R}} \varphi(x)\,dx = 0$ has the form $\varphi = \psi'$, where $\psi \in \mathscr{D}(\mathbf{R})$.

510. Prove that every function $\varphi \in \mathscr{D}(\mathbf{R})$ having the property that $\varphi(0) = 0$ has the form $\varphi(x) = x\psi(x)$, $\psi \in \mathscr{D}(\mathbf{R})$.

511. Suppose that on $[a - \varepsilon, a + \varepsilon]$ the unknown function F is the kth derivative of a continuous function f. Prove theat $f(x)$ coincides with some polynomial $P_-(x)$ on $[a - \varepsilon, a]$ and with some polynomial $P_+(x)$ on $(a, a + \varepsilon]$, with $\deg P_\pm < k$.

Let $P(x) = P_+(x) - P_-(x)$. Then

$$F(x) = \left(\frac{d}{dx}\right)^n [P(x)\theta(x - a)] = \sum_{l=1}^{k} C_k^l p^{(k-l)}(a)\delta^{(l)}(x - a).$$

512. $\delta'(g(x)) = h''(0) \operatorname{sgn} h'(0)\delta(x - h(0)) + h'(0)^2\delta'(x - h(0))$.

513. (a) $(\lambda, 0)$; (b) $(0, 1)$; (c) $(-1, 0)$; (d) $(-1, 1)$; (e) $(-2, 1)$.

514. Use the relation $\lim_{t\to 1} [F(tx) - F(x)]/(t - 1) = xF'(x)$, which can be proved by starting from the definition of $F(tx)$.

515. Use Problem 502.

516. Let $\varphi \in \mathscr{D}(\mathbf{R})$ and suppose that $\int_{\mathbf{R}} \varphi(x)\,dx = 0$. Prove that there exist $\psi_n \in \mathscr{D}(\mathbf{R})$ and $a_n \in \mathbf{R}$ such that

$$\varphi(x) = \lim_{n\to\infty} [\psi_n(x + a_n) - \psi_n(x)].$$

(For example, we can set $a_n = 1/n$, $\psi_n(x) = n \int_0^x \varphi(t)\,dt$.)

517. (a) Prove that if $\varphi \in \mathscr{D}(\mathbf{R}^2)$ has the property that $\int_{\mathbf{R}} \varphi(x, y)\,dx = 0$ for all $y \in \mathbf{R}$, then $\varphi = \partial\psi/\partial x$ for some $\psi \in \mathscr{D}(\mathbf{R}^2)$;

(b) $F = 1 \times f$.

518. (a) Generalize the method described in the hint for Problem 517.

(b) $F = \sum_{i=0}^{N} f_i \times \delta^{(i)}$.

519. The function $f'(x)$ is not a regular tempered distribution, since $|f'(x)| = e^x$ grows more rapidly than any polynomial. The procedure of integration by parts is not applicable to the integral $\int_{\mathbf{R}} f(x)\varphi'(x)\,dx$.

520. Answer: $\sum_{k\in\mathbf{Z}} c_k\delta(x - k\pi)$, where $\{c_k\}$ is any two-sided numerical sequence.

521. Let L be the subspace of $\mathscr{D}(\mathbf{R}^n)$ generated by the functions of the form

$$\left(\sum_{i=1}^{n} x_i^2 - R^2\right)\varphi(x),\ x_i\frac{\partial\varphi}{\partial x_j} - x_j\frac{\partial\varphi}{\partial x_i}, \qquad 1 \le i < j \le n, \quad \varphi \in \mathscr{D}(\mathbf{R}^n).$$

Prove that F annihilates L and that L has codimension 1 in $\mathscr{D}(\mathbf{R}^n)$. (For simplicity analyze the case $n = 2$.)

522. Use the result in Problem 520.

523. Prove that the function $\varphi = e^{-A(x)}(F(x) - B(x))$, where A and B are anti-derivatives for a and b, respectively, satisfies the equation $\varphi' = 0$.

524. Use the Fourier transformation and the Plancherel formula.

525. There exist constants C and N such that $|c_n| \le Cn^N$ for $n \ne 0$. (That is, the sequence $\ln|c_n|/(\ln n)$ is bounded above.)

526. Use the equation $\sum e^{inx} = 2\pi \sum_{n\in\mathbf{Z}} \delta(x - 2\pi n)$. Derive from this the differential equations satisfied by the desired sum in the interval $(0, 2\pi)$. Answers:

(a) $((x - \pi)/2)^2 - \pi^2/12$ for $x \in [0, 2\pi]$;

(b) $(\pi/a)[\cosh a(x - \pi)/\sinh a\pi]$ for $x \in [0, 2\pi]$;

(c) $(2\pi i)^{-k} \sum_{n\in\mathbf{Z}} \delta^{(k)}(x - 2\pi n)$;

(d) $\begin{cases} \pi/4 \operatorname{sgn} x & \text{for } x \in (-\pi, \pi), \\ 0 & \text{for } x = n\pi. \end{cases}$

527. No. (For example, the limit $\lim_{n\to\infty} \varphi_n(x)\delta(x)$, where $\{\varphi_n\}$ is a δ-shaped sequence, does not exist.)

528. (a) Use the fact that $\ln(x^2 + y^2) = \lim_{\varepsilon\to 0} \ln(x^2 + y^2 + \varepsilon^2)$ and the fact that $(\partial^2/\partial x^2 + \partial^2/\partial y^2)\ln(x^2 + y^2 + \varepsilon^2) = 4\varepsilon^2/(x^2 + y^2 + \varepsilon^2)^2$. Then generalize the result in Problem 487(a) to functions of two variables. Answer: $4\pi\delta(x, y)$.

(b) See the hint for (a). Answer: $-4\pi\delta(x, y, z)$.

529. (a) Answer: 0 if $\mathscr{P}(x)$ does not have real roots; $|\lambda - \mu|(\delta(x - \lambda) + \delta(x - \mu))$, if $\mathscr{P}(x)$ has two real roots λ and μ; it does not exist if $\mathscr{P}(x)$ has a multiple root.

(b) Pass to polar coordinates.
Answer: $(\delta(x^2 + y^2 - 1), \varphi) = (1/2) \int_0^{2\pi} \varphi(\cos\alpha, \sin\alpha)\, d\alpha$.

(c) The equality $\delta(x^2 - y^2) = (1/2|x|)[\delta(x - y) + \delta(x + y)]$ holds in the domain $\Omega = \mathbf{R}^2 \setminus \{0, 0\}$. For the proof it is convenient to pass to the new coordinates $u = x$, $v = x^2 - y^2$. The generalized function $\delta(x^2 - y^2)$ does not exist on the whole plane.

(d) In the domain $\Omega = \mathbf{R}^3 \setminus \{0, 0, 0\}$ we have the identity

$$\delta(x^2 + y^2 - z^2) = \frac{1}{2|z|}[\delta(z - \sqrt{x^2 + y^2}) + \delta(z + \sqrt{x^2 + y^2})],$$

or, in cylindrical coordinates z, r, α: $\delta(x^2 + y^2 - z^2) = (1/2r)[\delta(z - r) + \delta(z + r)]$. This formula gives a generalized function in the whole space, since the integral

$$\iint \varphi(x, y, \pm\sqrt{x^2 + y^2}) \frac{dx\, dy}{|z|} = \iint \varphi(r\cos\alpha, r\sin\alpha, \pm r)\, dr\, d\alpha$$

converges for all $\varphi \in \mathscr{D}(\mathbf{R}^3)$. If $\delta(x^2 + y^2 - z^2)$ exists, then the difference $\delta(x^2 + y^2 - z^2) - (1/2|z|)\cdot[\delta(z - \sqrt{x^2 + y^2}) + \delta(z + \sqrt{x^2 + y^2})]$ has support at the point $(0, 0, 0)$. Hence, it is a linear combination of the functions $(\partial^{k+l+m}/\partial x^k \partial y^l \partial z^m)\delta(x, y, z)$. Moreover, this difference is homogeneous of degree -2 and is invariant under *Lorentz transformations* (linear transformations of $\mathbf{R}^3$ that preserve the form $x^2 + y^2 - z^2$). Hence, it is equal to zero. Verify the existence of $\delta(x^2 + y^2 - z^2)$ first on test functions equal to 0 at (0, 0, 0), and then on a function different from 0 at (0, 0, 0).

Answer:

$$\delta(x^2 + y^2 - z^2) = \frac{1}{2|z|}(\delta(z - \sqrt{x^2 + y^2}) + \delta(z + \sqrt{x^2 + y^2})).$$

§4. Hilbert Spaces

1. The Geometry of Hilbert Spaces

530. (b) Consider the category of isometric mappings of a given pre-Hilbert space into all possible Hilbert spaces.

531. For a proof of completeness use the Weierstrass theorem and the result in Problem 247(b).

532. To within a constant factor, the following special functions are the results of orthogonalization:

(a) The *Legendre polynomials* $P_n(x) = (d/dx)^n[(1 - x^2)^n]$;

(b) the *Tchebycheff polynomials* $T_n(x) = \cos(n \arccos x)$;

(c) The *Laguerre polynomials* $L_n(x) = e^x(d/dx)^n(e^{-x}x^n)$;

(d) the *Hermite polynomials* $H_n(x) = e^{x^2}(d/dx)^n e^{-x^2}$.

533. (a) $f_k(z) = \sqrt{[(k + 1)/S]}(z/R)^k$, where $S = \pi R^2$ is the area of the disk;

(b) $f_k(z) = z^k/\sqrt{\pi \cdot k!}$.

534. Find the expansion of the desired function $g_x(z)$ in the basis of Problem 533. Answers:

(a)

$$g_x(z) = \frac{1}{\pi[R - (\bar{x}z/R)]^2};$$

(b) $g_x(z) = (1/\pi)e^{\bar{x}z}$.

535. With the help of the result in Problem 533 prove that each L_2-convergent sequence of analytic functions converges uniformly on any compact set lying in the interior of the given domain.

536. (a) $c_n = 0$ for n even, and $c_n = 2/(\pi i n)$ for n odd;

(b) $c_n = (e^\lambda - 1)/(\lambda - 2\pi i n)(\lambda \neq 2\pi i n)$;

(c) $c_n = (2\pi i n)^{-k}$ for $n \neq 0$, $c_0 = 0$.

537. (a) Imbed $L_2(a, b)$ in $L_2(0, 1)$.

(b) Prove that any function in $L_2(a, b - 1)$ can be uniquely extended to a function in the desired orthogonal complement in $L_2(a, b)$.

538. (b) Prove that the Hilbert norm can be estimated in terms of the uniform norm and that the converse is not true, as follows from examination of the sequence $f_n(x) = \sum_{k=1}^{n} (1/k)e^{i\lambda_k x}$, where $\{\lambda_k\}$ is an arbitrary sequence of real numbers.

539. Let $f_\lambda(x)$ be the function on $\mathbf{R}$ equal to 1 at the point λ and to 0 at the other points. Then $\{f_\lambda\}_{\lambda\in\mathbf{R}}$ is an orthonormal basis in $L_2(\mathbf{R}, \mu)$. The correspondence $f_\lambda \leftrightarrow e^{i\lambda x}$ establishes an isomorphism of the bases and, consequently, of the Hilbert spaces.

540. Use the orthogonalization process.

541. The completeness follows from the fact that any continuous function on $[0, 1]$ can be uniformly approximated by linear combinations of the functions φ_{mn}.

542. (a) The function $\varphi_{12} = \varphi_1\varphi_2$ is orthogonal to all the functions in the Rademacher system.

(b) The proof is similar to that in Problem 541.

543. The orthogonal complement is equal to zero in all the cases.

544. (a) The space of functions equal to zero for $x \geq 0$; (b) $\{0\}$.

545. 30°, 60°, 90°.

546. (a) 90°; (b) $\arccos (a/b)^{1/2}$, where a is the length of the shorter chord and b is the length of the longer chord.

547. (a) Verify directly.

(b) Suppose that $K = \mathbf{R}$. Define a scalar product by the formula

$$(x, y) = \frac{1}{2}(\|x + y\|^2 - \|x\|^2 - \|y\|^2).$$

The equation $(x + y, z) = (x, z) + (y, z)$ is equivalent to the relation $\|x + y + z\|^2 + \|x\|^2 + \|y\|^2 + \|z\|^2 = \|x + y\|^2 + \|y + z\|^2 + \|x + z\|^2$. This relation follows from the parallelogram law, applied to all the parallelograms that can be formed from the vertices of a three-dimensional parallelepiped. Next, induction on n leads to the equation $(nx, z) = n(x, z)$, and it implies that $(\lambda x, z) = \lambda(x, z)$ for rational λ. Since (x, y) depends continuously on x by construction, $(\lambda x, y) = \lambda(x, y)$ holds for all real λ. In the case of the complex field we can first consider the real space $H_{\mathbf{R}}$ of the Hilbert space H (i.e., the same space H, in which only the operations of addition and of multiplication by a real number are allowed). Then, by what was proved, there is a (real) scalar product $(x, y)_{\mathbf{R}}$ in $H_{\mathbf{R}}$ such that $\|x\|^2 = (x, x)_{\mathbf{R}}$. Define a scalar product in H by the formula: $(x, y) = (x, y)_{\mathbf{R}} + i(x, iy)_{\mathbf{R}}$. Verify that this expression really has the necessary properties. (Use the relation $(x, ix)_{\mathbf{R}} = (1/2)(\|x + ix\|^2 - \|x\|^2 - \|ix\|^2) = 0$, since $\|\lambda x\|^2 = |\lambda|^2\|x\|^2$.)

548. Use the identity $\|x + e^{i\theta}y\|^2 e^{i\theta} = \|x\|^2 e^{i\theta} + (x, y) + (y, x)e^{2i\theta} + \|y\|^2 e^{i\theta}$ and the relation $\sum_{k=1}^{N} e^{2\pi ik/N} = \sum_{k=1}^{N} e^{4\pi ik/N} = 0$ for $N \geq 3$.

549. Verify that the strong limit y of the sequence $y_n = (1/n) \sum_{i=1}^{n} x_i$ exists and that the vectors $z_i = x_i - y$ are orthogonal to each other and to the vector y.

550. (b) $\Rightarrow$ (c), by the corollary to the Banach–Steinhaus theorem (on boundedness of weakly convergent sequences).

551. Let $L(S)$ be the closure of the linear span of S. Then $L(S)^{\perp} = S^{\perp}$. Therefore, $(S^{\perp})^{\perp} = L(S)$, by the theorem on the orthogonal complement.

552. Represent H in the form $\bar{L} \oplus L^{\perp}$.

2. Operators on a Hilbert Space

553. (a) $\operatorname{Re} A = (1/2)(A + A^*)$, $\operatorname{Im} A = (1/2i)(A - A^*)$;

(b) $AA^* - A^*A = 2i(\operatorname{Im} A \cdot \operatorname{Re} A - \operatorname{Re} A \cdot \operatorname{Im} A)$;

(c) $VV^* = (\operatorname{Re} V)^2 + i(\operatorname{Im} V \cdot \operatorname{Re} V - \operatorname{Re} V \cdot \operatorname{Im} V) + (\operatorname{Im} V)^2$.

554. (a) Let $H_1 = PH$, $H_2 = (1 - P)H$. Verify that H_1 and H_2 are orthogonal and have sum H, and that P is the projection onto H_1 parallel to H_2.

(b) Let $P = (S + 1)/2$. Verify that P is an orthogonal projection.

555. Use the equality $\|A\| = \sup_{x,y} [(|(Ax, y)|)/(\|x\| \cdot \|y\|)]$.

556. (a) If k and l are even, then the desired inequality can be rewritten in the form $(A^{k/2}x, A^{l/2}x) \le \|A^{k/2}x\| \cdot \|A^{l/2}x\|$. But if k and l are odd, then we introduce the new scalar product $(x, y)_A = (Ax, y)$. Then the required inequality has the form $(A^{(k-1)/2}x, A^{(l-1)/2}x)_A \le \|A^{(k-1)/2}x\|_A \cdot \|A^{(l-1)/2}x\|_A$.

(b) Derive from (a) the inequality $\|Ax\|^{2(n+1)} \le (Ax, x)^n \cdot (A^{n+2}x, x)$, and then from it the desired inequality.

557. Prove that the sequence of quadratic forms $QA_n(x) = Q(A_n x, x)$ converges pointwise to some quadratic form $Q_A(x)$. Then use the inequality in Problem 556(b).

558. (a) $AP = PAP$; (b) $AP = PA$.

559. (a) It suffices to consider the case $\dim H = 2$;

(b) $\cos^2 \varphi = \operatorname{tr} P_1 P_2 P_1 = \|P_1 P_2 P_1\|$.

(c) Suppose that the unit vectors ξ_i generate L_i, and that the unit vectors η_i generate M_i, $i = 1, 2$. The equation $(\xi_1, \xi_2) = (\eta_1, \eta_2)$ is a condition for the congruence of the pairs (L_1, L_2) and (M_1, M_2), and this is equivalent to the equation $\operatorname{tr} P_1 P_2 P_1 = \operatorname{tr} Q_1 Q_2 Q_1$.

560. (a) The operators $P_1 P_2 P_1$ and $1 - P_1 P_2 P_1 = P_1(1 - P_2)P_1 + (1 - P_1)$ are positive.

(b) The rank of the operator $P_1 P_2 P_1$ does not exceed the ranks of P_1 and P_2.

(c) In solving the problem it can be assumed that $L_2 = M_2$ (if necessary, the pair (M_1, M_2) can be replaced by a congruent pair). Consider the projections of generating vectors in L_1 and M_1 on $L_2 = M_2$ and on the orthogonal complement of this space.

(d) First method: expand on the arguments in the preceding part. Second method. Call a pair (L_1, L_2) decomposable if the space H is representable in the form $H = H' \oplus H''$ and $L_i = L_i' \oplus L_i''$, where $L_i' = L_i \cap H'$, $L_i'' = L_i \cap H''$. In this case we say that the pair (L_1, L_2) is the sum of the pairs (L_1', L_2') and $L_1'', L_2'')$. Prove that every pair is a sum of indecomposable pairs, and that indecomposable pairs occur only when $\dim H = 1$ or 2. The latter is obvious from the fact that if ξ is an eigenvector of the operator $P_1 P_2 P_1$, then the space H' spanned by ξ and $P_2\xi$ is invariant under P_1 and P_2. Hence, $H'' = (H')^\perp$ also has this property. From this it follows that the original pair is decomposable if $\dim H > 2$.

(e) The opening is equal to $\sin\varphi$, where φ is the largest of the angles between L_1 and L_2.

561. (a) If U is unitary and $\{e_\alpha\}_{\alpha\in A}$ is a basis in H_1, then $\{Ue_\alpha\}_{\alpha\in A}$ is an orthonormal system in H_2. Its completeness follows from the fact that $x \perp Ue_\alpha$ implies that $U^{-1}x \perp e_\alpha$.

(b) If $\{e_\alpha\}_{\alpha\in A}$ is an orthonormal basis in H_1 and $\{Ue_\alpha\}_{\alpha\in A}$ is an orthonormal basis in H_2, then for any $x, y \in H_1$ we have

$$x = \sum_\alpha (x, e_\alpha)e_\alpha, \qquad y = \sum_\beta (y, e_\beta)e_\beta.$$

Therefore, $Ux = \sum_\alpha (x, e_\alpha)Ue_\alpha$, $Uy = \sum_\beta (y, e_\beta)Ue_\beta$, and

$$(Ux, Uy) = \sum_{\alpha,\beta} (x, e_\alpha)\overline{(y, e_\beta)}(Ue_\alpha, Ue_\beta) = \sum_\alpha (x, e_\alpha)\overline{(y, e_\alpha)} = (x, y).$$

562. (a) The condition $y \perp \operatorname{im} A$ is equivalent to the relation $(y, Ax) = 0$ for all $x \in H$, and the condition $y \in \ker A^*$ is equivalent to the relation $(A^*y, x) = 0$ for all $x \in H$. But $(y, Ax) = (A^*y, x)$.

(b) By the theorem on the orthogonal complement (cf. also Problem 551), the equality $(\ker A)^\perp = \overline{(\operatorname{im} A^*)}$ is equivalent to the equality $\ker A = (\operatorname{im} A^*)^\perp$, which was proved in part (a) (with A replaced by A^*).

563. Use the relation

$$\|(A_n - A)x\|^2 = \|A_n x\|^2 + \|Ax\|^2 - 2\operatorname{Re}(A_n x, Ax).$$

564. Let $\{x_\alpha\}$ be a basis in H, and $E_{\alpha\beta}$ the operator carrying x_β into x_α and the remaining basis vectors into zero. Verify the following relations:

(1) $E_{\alpha\beta}^* = E_{\beta\alpha}$;

(2) $E_{\alpha\beta}E_{\gamma\delta} = E_{\alpha\delta}$ if $\beta = \gamma$, otherwise $E_{\alpha\beta}E_{\gamma\delta} = 0$;

(3) if P is an orthogonal projection for which $E_\alpha P = P$, then $P = 0$ or E_α.

Prove that every set of operators having these properties is so constructed: There is a Hilbert basis $\{y_\alpha\}$ such that $E_{\alpha\beta}$ carries y_β into y_α and the remaining basis vectors into zero. Apply this assertion to the set $\sigma(E_{\alpha\beta})$, where σ is a given homomorphism.

565. If the ideal I contains at least one nonzero operator, then it contains all finite-rank operators and, hence, all compact operators. If I contains a noncompact operator, then it contains an orthogonal projection onto an infinite-dimensional space and, consequently, all operators. Answer: $\{0\}$, $\mathscr{K}(H)$, $\mathscr{L}(H)$.

566. Use the relations

$$\sup_{x\in L}\frac{(PAPx, x)}{(x, x)} = \sup_{x\in L}\frac{(APx, Px)}{(Px, Px)}\cdot\frac{(Px, Px)}{(x, x)} \le \sup_{y\in PL}\frac{(Ay, y)}{(y, y)},$$

$$u\sup_{x\in L}\frac{(PAPx, x)}{(x, x)} \ge \sup_{x\in L\cap PH}\frac{(PAPx, x)}{(x, x)} = \sup_{x\in L\cap PH}\frac{(Ax, x)}{(x, x)}.$$

567. (a) Prove the relations $\|A\|^{1/2}\cdot 1 \gg B_n \gg 0$, $B_n^2 \ll A$ by induction, and use the result of Problem 557.

(b) First prove uniqueness for the case $\ker A = 0$ by using the fact that the square root B constructed is a limit of polynomials in A and, consequently, commutes with any other square root C; this implies the equality $(B + C) \times (B - C)x = 0$, from which $(B - C)x = 0$. The general case follows from the relation $\ker C = \ker C^2$, which is true for any $C \gg 0$.

568. (b) The operators R and S satisfy the relations $R^2 = AA^*$, $S^2 = A^*A$. The operator V is uniquely determined only on $\operatorname{im} S$, and the operator U is determined modulo $\ker R$.

(c) The operators A admitting the desired expression have the property that $\dim\ker A = \dim\ker A^*$. However, $\dim\ker T \neq \dim\ker T^*$.

569. $U^*U = P_1$; $UU^* = P_2$.

570. Let $R = (AA^*)^{1/2}$ and define U on $\operatorname{im} A^*$ by the equation $UA^*x = Rx$.

571. (a) Use the fact that for any two bases $\{x_\beta\}_{\beta\in B}$ and $\{\varphi_\gamma\}_{\gamma\in\Gamma}$ we have the equation:

$$\sum_{\beta\in B}\|Ax_\beta\|^2 = \sum_{\beta\in B}\sum_{\gamma\in\Gamma}|(Ax_\beta, x_\gamma)|^2 = \sum_{\gamma\in\Gamma}\sum_{\beta\in B}|(x_\beta, A^*x_\gamma)| = \sum_{\gamma\in\Gamma}\|A^*x_\gamma\|^2.$$

(b) The convergence of the series $\sum_{\gamma\in\Gamma}(Ay_\gamma, By_\gamma)_H$ follows from the Cauchy–Bunyakovskii inequality, applied twice: once for the scalar product in H, and a second time for the scalar product in $l_2(\Gamma)$.

(c) Let Γ_0 be a finite subset of Γ, and P_{Γ_0} the projection onto the corresponding subspace of H. Estimate the norm of the difference of A and PAP in $L_2(H)$.

(d) The mapping of $H \otimes H'$ into $L_2(H)$ carries the vector $x \otimes f$ into the operator $A\colon y \to f(y)x$.

(e) Let $\{f_\beta\}_{\beta\in B}$ be a basis in $L_2(X, \mu)$. Show that A is determined by the kernel $K(x_1, x_1) = \sum_{\beta_1,\beta_2}(Af_{\beta_1}, f_{\beta_2})f_{\beta_1}(x_1)\overline{f_{\beta_2}(x_2)}$.

572. (a) Follows from the definition.

(b) Prove that right multiplication by a bounded operator $B \in L_2(H)$ is a bounded operator in $L_2(H)$; denote it by $M(B)$. Prove that $M(B)^* = M(B^*)$.

(c) Verify the equality $\|A\|_1 = \sup_{U,V} |\mathrm{tr}\, UAV|$, where U and V run through the collection of all partial isometries.

(d) Each operator $A \in \mathscr{L}_1(H)$ determines a linear functional f_A on $\mathscr{K}(H)$: $f_A(K) = \mathrm{tr}\, AK$. Each bounded operator B determines a linear functional F_B on $\mathscr{L}_1(H)$: $F_B(A) = \mathrm{tr}\, AB$. To prove that these are the full collections of functionals use the fact that the finite-rank operators form a dense subset of $\mathscr{K}(H)$ and of $\mathscr{L}_1(H)$.

573. (a) $f_1(x) = e^{2\pi i x}$, $\lambda_1 = 1/2$; $f_2(x) = e^{-2\pi i x}$, $\lambda_2 = -1/2$. The remaining eigenfunctions are any functions in the orthogonal complement of $\{f_1, f_2\}$, and the remaining eigenvalues are zero.

(b) Rewrite the basic equation for an eigenfunction in the form $\lambda f(x) = \int_0^x y f(y)\, dy + x \int_x^1 f(y)\, dy$. Prove that there are no solutions for $\lambda = 0$, and that $f(x)$ is twice differentiable and satisfies the equation $\lambda f'' + f = 0$ for $\lambda \neq 0$.

Answer: $f_n(x) = \sin \pi(n + 1/2)x$, $\lambda_n = \pi^{-2}(n + 1/2)^{-2}$, $n \in \mathbf{Z}$.

574. (a) Use the functions $g_x(z)$ constructed in Problem 534.

(c) Represent $\dim H$ in the form $\sum_k |\xi_k|^2$, where $\{\xi_k\}$ is a basis in H.

(d) Begin with the operators of rank 1.

575. Let $l_n(x) = e^{2\pi i n x}$ be a basis in $L_2[0, 1]$,

$$\mathrm{tr}_N(A) = \sum_{n=-N}^{+N} (Al_n, l_n), \quad \text{and} \quad s_k(A) = \frac{1}{k} \sum_{N=1}^{k} \mathrm{tr}_N(A).$$

Prove that $\lim_{k\to\infty} s_k(A) = \mathrm{tr}\, A$ for $A \in \mathscr{L}_2(H)$. Then verify that

$$s_k(A) = \int_0^1 \int_0^1 K(x, y) \left[\frac{\sin(2k + 1)\pi(x - y)}{\sin \pi(x - y)}\right]^2 dx\, dy$$

for the integral operator with kernel $K(x, y)$. For a continuous kernel this implies that $\lim_{k\to\infty} s_k(A) = \int_0^1 K(x, x)\, dx$.

Chapter IV

The Fourier Transformation and Elements of Harmonic Analysis

§1. Convolutions on an Abelian Group

1. Convolutions of Test Functions

576. (a) Let δ_g be the element of $K[G]$ corresponding to the function equal to 1 at the point g and to 0 at the remaining points.

Write out explicitly the condition that $a \in K[G]$ and δ_g commute.

(b) The condition $a(gh) = a(hg)$ can be rewritten in the form $a(h) = a(ghg^{-1})$.

(c) True.

577. (a) Let $\varepsilon = e^{2\pi i/n}$, and suppose that a is a generating element of the group C_n (in the additive notation). Let $e_k = (1/n)\sum_{k=1}^{n} \varepsilon^k \delta_k a$. Verify the equations $e_k * e_j = 0$ for $k \neq j$, $e_k * e_k = e_k$.

(b) True for $n = 2$. False for larger n. It can be shown that $\mathbf{R}[C_{2k}] \approx \mathbf{R} + \mathbf{R} + \underbrace{\mathbf{C} + \cdots + \mathbf{C}}_{k-1}$.

578. To each function $a(g)$ assign the numbers $a_0 = \sum_{g \in S_3} a(g)$ and $a_1 = \sum_{g \in S_3} a(g) \operatorname{sgn} g$, where $\operatorname{sgn} g$ is the parity of the permutation g: $\operatorname{sgn} n = \prod_{i<j} \{[g(i) - g(j)]/(i - j)\}$. Prove that the mappings $a \to a_0$ and $a \to a_1$ are homomorphisms of $\mathbf{R}[S_3]$ onto $\mathbf{R}$. Next, let e_1, e_2, e_3 be three vectors in the plane with sum equal to zero. To each element $g \in S_3$ there corresponds a linear transformation $T(g)$ of the plane acting according to the formula $T(g)e_i = e_{g^{-1}(i)}$. Prove that the mapping $a \mapsto \sum a(g)T(g)$ is a homomorphism of $\mathbf{R}[S_3]$ onto $\mathrm{Mat}_2\,\mathbf{R}$. Use these homomorphisms for constructing the desired isomorphism.

579. Let φ be a mapping of G into a K-algebra A with unit, with the properties that $\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$ and $\varphi(1) = 1$. Then it can be uniquely extended to a homomorphism $I: K[G] \to A$ by the formula $a \to \sum a(g)\varphi(g)$.

If the condition $\varphi(1) = 1$ is discarded, then the trivial mapping of G to the zero algebra becomes the universal object.

580. Answer: $\chi_{[a,b]} * \chi_{[c,d]}$ is a piecewise linear continuous compactly supported function on the line; the graph of this function is the polygonal line with vertices $(a + c, 0)$, $(b + c, b - a)$, $(a + d, b - a)$, $(b + d, 0)$. (We assume that $a \le b$, $c \le d$, $b - a \le d - c$.)

581. For step functions the assertion follows from the result in Problem 580. The general case is obtained from the estimate $\|f * g\|_\infty \le \|f\|_\infty * \|g\|_1$.

582. (a)

$$\pi\left(\frac{1}{a} + \frac{1}{b}\right)\frac{1}{x^2 + (a + b)^2}\,(a > 0, b > 0);$$

(b)

$$\sqrt{\frac{\pi ab}{a + b}}\exp\left(\frac{-x^2}{2(a + b)}\right)(a > 0, b > 0).$$

583. See the proof of Theorem 4 in Ch. IV.

584. Establish the equality $S(\varphi) = \int_G \varphi(g)T(-g)\,d\mu(g)$.

585. (a) Property 3 follows from the absolute continuity of the Lebesgue integral.

(b) It suffices to verify property 3 for ball neighborhoods.

586. Represent $S(f_k) - T(a)$ in the form $\int_G f_k(g)[T(g) - T(a)]\,d\mu(g)$.

587. Use the formula $\partial^k(\varphi * \psi) = (\partial^k \varphi) * \psi$.

588. Use the results in Problems 585–587.

589. Use Problem 584. Answer: $S(f)^* = \overline{S(f(-x))}$ (cf. Problem 591).

590. The existence of $f_1 * f_2$ follows from the fact that $f(x - y)$, a function of y, belongs to $L_2(G, \mu)$ for each x. The measurability follows from the definition of the integral as a limit of integral sums, and the boundedness from the Cauchy–Bunyakovskii inequality.

591. Make suitable changes of variables.

592. Prove by direct calculation. (Cf. Problem 629.)

593. Use the result in Problem 592 and prove the equality $f * e_k = (f, e_k)e_k$ for any function $f \in L_1(G, \mu)$.

594. It is possible, for example, to set $f_k(t) = \prod_{j=1}^n \varphi_k(t_j)$, where

$$\varphi_k(t) = \frac{1}{2k + 1}\sum_{s=-2k}^{2k}(2(k + 1 - |s|))e_s(t) = \frac{1}{2k + 1}\left[\frac{\sin(2k + 1)\pi t}{\sin \pi t}\right]^2.$$

595. Use Problems 586, 594.

596. (a) Answer: $B_1 = x - 1/2$, $B_2 = x^2 - x + 1/6$, $B_3 = x^3 - 3x^2/2 + x/2$, $B_4 = x^4 - 2x^3 + x^2 - 1/30$.
(b) Answer: $-B_2/2$. The general result is in Problem 612(b).

597. (a) $B(\alpha + 1, \beta + 1)\theta(x)x^{\alpha+\beta+1}$.
(b) $\begin{cases} [(e^{bx} - e^{ax})/(b - a)]\theta(x), & \text{for } a \neq b, \\ xe^{ax}\theta(x) & \text{for } a = b. \end{cases}$

598. Show that if $f \in L_p(G, \mu)$, $g \in L_q(G, \mu)$, $h \in L_s(G, \mu)$, then the function $\varphi(x, y) = f(x - y)g(y)h(x)$ belongs to $L_t(G \times G, \mu \times \mu)$, where $2/t = 1/p + 1/q + 1/s$.
Derive from this the necessary assertion by setting $1/s = 1 - (1/r)$, $t = 1$.

2. Convolutions of Generalized Functions

599. (a) δ_{a+b}; (b) $\theta(x - a)$; (c) δ; (d) 0.

600. $Lf = (\sum_{k=0}^{n} c_k\delta^{(k)}) * f$.

601. Use the identities $\langle f, 1 * \varphi\rangle = \langle f * 1, \varphi\rangle$ and $1 * \varphi = \langle 1, \varphi\rangle \cdot 1$ for $\varphi \in \mathscr{D}(\mathbf{R})$.

602. Use the identity $(f_1 * f_2)'' = f_1' * f_2'$. Answer: $\delta_{a+c} - \delta_{a+d} - \delta_{b+c} + \delta_{b+d}$.

603. It is possible to use the formula $\langle f_1 * f_2, \varphi\rangle = \langle f_1 \times f_2, \mathring{\varphi}\rangle$ and the relations $f_1^\vee \times f_2^\vee = (f_1 \times f_2)^\vee$, $\langle f^\vee, \varphi^\vee\rangle = \langle f, \varphi\rangle$.

604, 605. Use the definition of the convolution in terms of the direct product.

606. Use the definition of the topologies in $\mathscr{E}(\mathbf{R}^n)$ and $\mathscr{D}(\mathbf{R}^n)$ and the theorem on representation of f as the derivative of a regular function.

607. Set $\langle f_1 * f_2, \varphi\rangle = \int_{\mathbf{T}^n}\int_{\mathbf{T}^n} f_1(t)f_2(s)\varphi(t + s)\,dt\,ds$.

608. First method. Verify that $\{\varphi_N(t)\}$ has the properties of a δ-shaped sequence (see Problem 585). Second method. Express $\langle \varphi_N, f\rangle$ in terms of the Fourier coefficients of f.

609. Use the results of Problems 608 and 593.

610. Use the hint for Problem 609.

611. (a) $A_h = S[(\delta_{-h} - \delta_h)/2h]$;
(b) $B_h = S[(\delta_h + \delta_{-h} - 2\delta)/h^2]$.

612. (a) Answer: $k!(1 - \delta)$ for $k > 0$.
(b) Use the result of (a) and the fact that convolution commutes with differentiation.

613. (a) Use the identity $\sin 2\pi kt/\sin \pi t = \sum_{i=1}^{k} \cos(2i - 1)\pi t$.
Answer: $f * e_k = -i \operatorname{sgn} k e_k$.
(b) Use the result of (a) and Problem 609.
Answer: $f * f = 1 - \delta$.

614. Use the formula $\delta(x^2 - a^2) = (1/2|a|)(\delta(x - a) + \delta(x + a))$.
Answer:

$$(f_1 * f_2)(x, y) = \begin{cases} 1/4S & \text{if segments with lengths } r_1, r_2, (x^2 + y^2)^{1/2} \\ & \text{can form a triangle, and it has area } S, \\ 0 & \text{if } \sqrt{x^2 + y^2} \notin [|r_1 - r_2|, r_1 + r_2]. \end{cases}$$

615. Prove that f can be expressed in terms of f_1 and f_2 by the formula $f(r) = \int_0^\infty \int_0^\infty K(r_1, r_2; r) f_1(r_1) f_2(r_2)\, dr_1\, dr_2$, where $K(r_1, r_2; r)$ is a certain locally integrable function. Use the result in Problem 614 to compute K. Answer: $K(r_1, r_2; r) = 0$ if segments with lengths r_1, r_2, r cannot form a triangle; $K(r_1, r_2; r) = 1/4S$ if segments with lengths r_1, r_2, r can form a triangle, and it has area S.

616. Since $\mathscr{E}_+(\mathbf{R})$ contains $\mathscr{D}(\mathbf{R})$ as a dense subspace, every continuous linear functional on $\mathscr{E}_+(\mathbf{R})$ determines some generalized functional $f \in \mathscr{D}'(\mathbf{R})$ and is itself uniquely determined by this function. Let $\alpha \in \mathscr{E}_-(\mathbf{R})$. Then multiplication by α is a continuous operator from $\mathscr{E}_+(\mathbf{R})$ to $\mathscr{D}(\mathbf{R})$. Hence, the adjoint operator carries $\mathscr{D}'(\mathbf{R})$ into $\mathscr{E}'_+(\mathbf{R})$. From this it follows that $\mathscr{E}'_+(\mathbf{R})$ contains $\mathscr{D}'_-(\mathbf{R})$. Conversely, if $f \in \mathscr{D}'_-(\mathbf{R})$ and $\alpha \in \mathscr{E}_-(\mathbf{R})$ is a function identically equal to 1 in a neighborhood of supp f, then $f = \alpha f \in \mathscr{E}'_+(\mathbf{R})$.

(b) First method: $\langle f_1 * f_2, \varphi \rangle = \langle f_1 \times f_2, \mathring{\varphi} \rangle$, where $\mathring{\varphi}(x, y) = \varphi(x + y)$. Use here the fact that supp$(f_1 \times f_2)$ has a compact intersection with supp $\mathring{\varphi}$ if $f_1, f_2 \in \mathscr{D}'_\pm(\mathbf{R})$ and $\varphi \in \mathscr{E}_\mp(\mathbf{R})$.

Second method: Define first the convolution of $\mathscr{D}'_\pm(\mathbf{R})$ with $\mathscr{E}_\pm(\mathbf{R})$ by the formula $(f * \varphi)(x) = \langle f, T(-x)\varphi^\vee \rangle$, and then set $\langle f_1 * f_2, \varphi \rangle = \langle f_1, f_2^\vee * \varphi \rangle$.

617. (a), (b), (c) can be verified directly.

(d) Use the result in part (c), the continuous dependence of f_α on α for $\alpha > 0$, and the continuity of the differentiation operation in $\mathscr{D}'_+(\mathbf{R})$. Answer: $\lim_{\alpha \to 0} f_\alpha = \delta$.

618. Let $I(\alpha) = (d/dx)^n \int f_{\alpha+n} dx$ for $\alpha > -n$, where the f_α are the functions in Problem 617. Verify that this is independent of the choice of n (Problem 617(c)).

619. (a) Answer: $2(x/\pi)^{1/2}$ for $x \leq 1$, $2[(\sqrt{x} - \sqrt{x - 1})/\sqrt{\pi}]$ for $x \geq 1$.
(b) Answer: $(x/\pi)^{1/2}$. (c) Answer: $(1/\sqrt{\pi})\sin\sqrt{x}$.

620. Use the equality $f = (1/\pi)\delta(x^2 + y^2 - 1)$ and the result in Problem 614. Answer:

$$(f * f)(x, y) = \begin{cases} \dfrac{1}{\pi^2\sqrt{(x^2 + y^2)(4 - x^2 - y^2)}} & \text{for } x^2 + y^2 \leq 4, \\ 0 & \text{for } x^2 + y^2 > 4. \end{cases}$$

621. Use the equality $f = (1/2\pi)\delta(x^2 + y^2 + z^2 - 1)$ and the hint for Problem 614. Answer:

$$(f * f)(x, y, z) = \begin{cases} \dfrac{1}{8\pi\sqrt{x^2 + y^2 + z^2}} & \text{for } x^2 + y^2 + z^2 \leq 4, \\ 0 & \text{for } x^2 + y^2 + z^2 > 4. \end{cases}$$

§2. The Fourier Transformation

1. Characters on an Abelian Group

622. $\chi_k(l \bmod n) = e^{2\pi ikl/n}$, $k = 1, 2, \ldots, n$.

623. Use the result of Problem 622 and the fact that every finite abelian group is a direct sum of cyclic groups.

624. (a) $\chi_z(n) = n^2$, $z \in \mathbf{C}^*$;
(b) $\chi_\lambda(x) = e^{\lambda x}$, $\lambda \in \mathbf{C}$;
(c) $\chi_{v,w}(z) = e^{vz + w\bar{z}}$, $v, w \in \mathbf{C}$;
(d) $\chi_{\lambda,\varepsilon}(x) = |x|^\lambda(\operatorname{sgn} x)^\varepsilon$, $\lambda \in \mathbf{C}$, $\varepsilon = 0, 1$;
(e) $\chi_{\lambda,n}(z) = |z|^\lambda(\operatorname{sgn} z)^n$, $\lambda \in \mathbf{C}$, $n \in \mathbf{Z}$; $\operatorname{sgn} z = z/|z|$.

625. Let U_ε be the neighborhood of the character χ_0 determined by the inequality $|\chi(x) - \chi_0(x)| < \varepsilon$ for all $x \in G$. Prove that for $\varepsilon \leq 3^{1/2}$ this neighborhood does not contain points of $\hat{G}$ different from χ_0. (Use the fact that the set of complex numbers of the form $\chi(x)\overline{\chi_0(x)}$, $x \in G$ forms a subgroup of $\mathbf{T}$.)

626. Prove that $\hat{G}$ can be identified with a closed subset of the product $\prod_{g \in G} \mathbf{T}$, which is compact with respect to coordinatewise convergence (*Tychonoff's theorem*).

627. Prove that the generalized function $\chi'(x)$ belongs to the one-dimensional space generated by $\chi(x)$.

628. Make a change of variables in the integral defining the convolution.

629. Use the result of Problem 628.

630. Use the results in Problems 631 and 592.

631. To each homomorphism $\varphi: G \to H$ there corresponds a homomorphism $\hat{\varphi}: \hat{H} \to \hat{G}$ acting according to the formula

$$\hat{\varphi}(\chi)(x) = \chi(\varphi(x)), \qquad \chi \in \hat{H}, x \in G.$$

632. Answer: $\hat{L}$ coincides with the dual space L'. For a proof consider the restrictions of a character to the one-dimensional subspaces of L and prove that χ has the form $\chi(x) = e^{if(x)}$, where $f \in L'$.

633. (a) Every character $\chi \in \hat{\mathbf{Q}}_p$ has the form $\chi_\lambda(x) = e^{2\pi i\{\lambda x\}}$, where $\lambda \in \mathbf{Q}_p$ and $\{\cdot\}$ is the mapping of $\mathbf{Q}_p$ into $\mathbf{Q}_p/\mathbf{Z}_p \subset \mathbf{Q}/\mathbf{Z}$ (the "fractional part"). Answer: $\hat{\mathbf{Q}}_p = \mathbf{Q}_p$.

(b) Every character $\chi \in \hat{\mathbf{Z}}_p$ has the form $\chi_r(x) = e^{2\pi i\{rx\}}$, where r is a rational number of the form m/p^n that is determined modulo 1. Answer: $\hat{\mathbf{Z}}_p \simeq \mathbf{Q}_p/\mathbf{Z}_p$.

(c) The characters on the group $\mathbf{Q}_p/\mathbf{Z}_p$ can be identified with those on the group $\mathbf{Q}_p$ that are trivial on $\mathbf{Z}_p$. Answer: $(\mathbf{Q}_p/\mathbf{Z}_p)\hat{} \approx \mathbf{Z}_p$.

634. Exactness at the term $\hat{G}_1$ means that $\hat{p}$ is a monomorphism, i.e., each nontrivial character on $G_1 = G/G_0$ determines a nontrivial character on G. Exactness at the term G means that those and only those characters on G that are trivial on G_0 can be represented in the form $\hat{p}(\chi_1)$. Finally, exactness at the term $\hat{G}_0$ means that any character on G_0 is obtained by restriction of some character on G. This can be proved by transfinite induction (the group G_0 can be extended to G by operations of adjoining elements).

635. Use the fact that the group $\mathbf{Q}/\mathbf{Z}$ is isomorphic to the direct sum of the groups $\mathbf{Q}_p/\mathbf{Z}_p$ over all the prime numbers p (each fraction m/n can be uniquely represented as a sum of fractions whose denominators are powers of prime numbers). Answer: $(\mathbf{Q}/\mathbf{Z})\hat{} \approx \prod_p \mathbf{Z}_p$.

636. (b) Decompose the numbers of $[0, 1]$ into infinite binary fractions.

637. The Fourier transform of the function f is invariant under multiplication by the sequence $\{e^{2\pi i n\alpha}\}$.

638. Prove the required assertion for step functions.

639. Let χ be the characteristic function of the set $\mathbf{Z}_p \subset \mathbf{Q}_p$. Every element of $\mathscr{D}(G)$ is a linear combination of the form

$$\sum c_k \chi(a_k x + b_k), \quad \text{where} \quad c_k \in \mathbf{C},\ a_k, b_k \in \mathbf{Q}_p.$$

Prove that the function χ goes into itself under the identification of $\hat{\mathbf{Q}}_p$ with $\mathbf{Q}_p$ in Problem 633(a).

640. Use the equivalence of the systems of seminorms

$$p_k(f) = \sup_{t \in \mathbf{T}} |f^{(k)}(t)| \quad \text{and} \quad p_k'(f) = \int_{\mathbf{T}} |f^{(k)}(t)|\, dt.$$

641. (a) The matrix

$$\begin{pmatrix} f(0) & f(x) \\ f(-x) & f(0) \end{pmatrix}$$

is positive-definite if and only if $f(0) \geq 0, f(x) = \overline{f(-x)}$ and $f(0)^2 - |f(x)|^2 \geq 0$.

(b) Use the positivity of the matrix

$$\begin{pmatrix} f(0) & f(x) & f(x-y) \\ f(-x) & f(0) & f(-y) \\ f(y-x) & f(y) & f(0) \end{pmatrix}$$

642. (a) The positive definiteness of the matrix A means that $\sum_{k,j} a_{kj} z_k \bar{z}_j \geq 0$ for all tuples $\{z_k\} \in \mathbf{C}^n$.

(b) A componentwise product of positive-definite matrices is positive-definite. (For a proof use the fact that a positive-definite matrix is a sum of matrices of rank 1 having the same property.)

(c) Transform the expression $\sum (\varphi * \varphi^*)(x_k - x_j) z_k \bar{z}_j$ to the form $\int_G |f(x)|^2\, dx$, where $f(y) = \sum_k z_k \varphi(y - x_k)$.

643. The matrix A corresponding to the set of all elements of G is the matrix of the operator $S(f)$. Under the Fourier transformation this operator goes into the operator of multiplication by $\tilde{f}$.

644. $\sum_{k,j} \tilde{\varphi}(\chi_k \chi_j^{-1}) z_k \bar{z}_j = \int_G \varphi(x) |\sum_k z_k \chi_k(x)|^2\, d\mu(x)$.

2. Fourier Series

645. (a) $c_n = c_{-n}$;

(b) $c_n = -c_{-n}$;

(c) $c_n = \overline{c_{-n}}$.

646. Answer: $l = k + 1$. Represent f as the sum of a $(k + 1)$-times differentiable function and a linear combination of functions of the form $|t - a|$, for $k = 0$.

647. It suffices to analyze the case $k = 0$. The first assertion can be derived from the inclusion $C[\mathbf{T}] \subset L_2(\mathbf{T}, dt)$, and the second from the uniform convergence of the Fourier series.

648. Answer: $\sum_{n \in \mathbf{Z}} n^{2k} |c_n|^2 < \infty$.

649. Use the evenness of $f(t)$, the equality $f'(t) = \pi \operatorname{cotan} \pi t$ for $0 < t < 1$, and the relation $\sin 2\pi n t / \sin \pi t = 2 \sum_{k=1}^{n} \cos(2k - 1)\pi t$. Answer: $c_0 = -\ln 2$ (to compute this coefficient use the relation $I = \int_0^1 \ln \sin \pi t\, dt = \ln 2 + \int_0^1 \ln \sin \pi t/2\, dt + \int_0^1 \ln \cos(\pi t/2)\, dt = \ln 2 + 2I$); $c_n = -(1/2|n|)$ for $n \neq 0$.

650. (a) $c_{2k+1} = 0$, $k \in \mathbf{Z}$;

(b) for $\lambda = e^{2\pi i m/k}$, $m \in \mathbf{Z}$, $c_n = 0$, if $n \not\equiv m \pmod{k}$.

651. $0 \to \mathbf{Z} \xrightarrow{i} \mathbf{Z} \xrightarrow{p} C_n \to 0$, where i is multiplication by n, and p is passage to the residue classes.

652. For example, for the extended function to have the properties $f(t + (1/2)) = f(1 - t) = -f(t)$ (see Problem 645(b) and 650(a)) it is necessary to let

$$f(t) = \begin{cases} f(1/2 - t), & \text{on } [1/4, 1/2], \\ -f(t - 1/2), & \text{on } [1/2, 3/4], \\ -f(1 - t), & \text{on } [3/4, 1]. \end{cases}$$

653. $c_n(h) = c_n \cdot (\sin 2\pi h n)/2\pi h n$ for $n \neq 0$; $c_0(h) = c_0$.

654. Use the relations: $\partial f/\partial t_1 = 2\delta(t_1 - t_2) - 2\delta(t_1)$, $\partial f/\partial t_2 = -2\delta(t_1 - t_2) + 2\delta(t_2)$.

Answer: $c_{n_1 n_2} = 0$ for $n_1 n_2(n_1 + n_2) \neq 0$ and for $n_1 = n_2 = 0$; $c_{n0} = i/\pi n$; $c_{0n} = c_{n,-n} = -i/\pi n$.

655. (a) $\{c_n\}$ is a finitely nonzero sequence;

(b) $c_n = P(1/n)$ for $n \neq 0$, where P is some polynomial;

(c) $c_n = (-1)^n P(1/n)$ for $n \neq 0$, where P is a polynomial.

656. To prove the necessity derive the inequality $\int_T \int_T f(s-t)\varphi(s)\overline{\varphi(t)}\,ds\,dt \geq 0$ for any $\varphi \in C(\mathbf{T})$ from the positive definiteness of $f \in C(\mathbf{T})$. Apply this inequality to $\varphi(t) = e^{2\pi i n t}$. To prove the sufficiency use the fact that f is a limit in $C(\mathbf{T})$ of the *Cesàro means* $C_n = (1/n)\sum_{k=1}^n S_k$, where $S_k = \sum_{j=-k}^k c_j e^{2\pi i j t}$.

657. See the hint for Problem 641.

658. Consider the linear functional F on the space of trigonometric polynomials taking the value c_n on $e^{2\pi i n t}$. Prove that this functional is positive on the polynomials of the form $P(t) = |Q(t)|^2$, where Q is also a polynomial, and that every positive trigonometric polynomial can be represented in this form. (Use the symmetry principle that the roots of a polynomial $P(z)$ taking real values on the circle $|z| = 1$ are symmetric with respect to this circle: if λ is a root, then so is $\bar{\lambda}^{-1}$.) Conclude that F has a continuous extension to the space $C(\mathbf{T})$ and, consequently, can be represented by some measure μ.

659. $\sum_{n,m} c_{n-m} z^n \bar{z}^{-m} = \|\sum_n z_n U^n \xi\|^2$.

660. The desired isomorphism V carries the vector $U^n\xi \in H$ into the function $e^{2\pi i n t}$ in $L_2(\mathbf{T}, \mu)$.

661. If f is a smooth function, then $S_n \Rightarrow f$, and the limit set coincides with the graph of f. Every piecewise differentiable function f can be represented as the sum of a smooth function and a finite linear combination of functions of the form $f(t) = \{t - \alpha\}$, $\alpha \in [0, 1)$. For these functions the investigation reduces to the study of the sum $S_n = \sum_{k=1}^n (\sin 2\pi k t)/\pi k$, which converges to $1/2 - \{t\}$ for $t \in \mathbf{R}/\mathbf{Z}$. We have

$$S_n(\varepsilon_n) = 2\sum_{k=1}^n \int_0^{\varepsilon_n} \cos 2\pi k t\, dt = \int_0^{\varepsilon_n} \frac{\sin(2n+1)\pi t}{\sin \pi t}\, dt - \varepsilon_n$$
$$= \int_0^{\varepsilon_n} \frac{\sin(2n+1)\pi t}{\pi t}\, dt + O(\varepsilon_n) = \frac{1}{\pi}\int_0^{(2n+1)\pi\varepsilon_n} \frac{\sin \tau}{\tau}\, d\tau + O(\varepsilon_n).$$

Thus, the limit set contains, in addition to the graph of the function $f(t) = 1/2 - \{t\}$, a vertical segment at $t = 0$ of length $2A$, where

$$A = \sup_a \frac{1}{\pi}\int_0^a \frac{\sin \tau}{\tau}\, d\tau = \frac{1}{\pi}\int_0^\pi \frac{\sin \tau}{\tau}\, d\tau \approx 0.588.$$

(see Fig. 5).

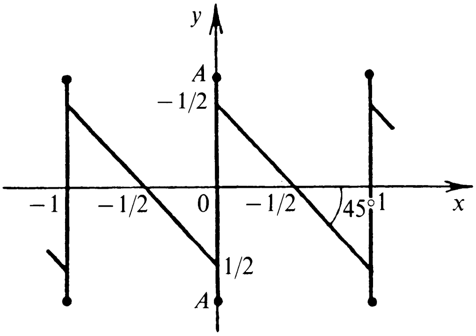


Figure 5

Answer: The limit set contains the graph of the function f and the vertical segments at the points t_k of discontinuity of f. The length of the segment is greater by a factor of $2A = 1.177$ than the size of the jump of f at t_k, and the center of the segment coincides with the point $(t_k, [f(t_k - 0) + f(t_k + 0)]/2)$. This fact has come to be called the *Gibbs phenomenon*.

662. (a) $c_n = 1$; (b) $c_n \equiv i \operatorname{sgn} n$ (see the hint for Problem 649).

663. (a) $\delta(t)$; (b) $1/(1 - \cos 2\pi t)$ (understood as the generalized derivative of $-(1/2\pi)\operatorname{cotan} \pi t$; (c) $(\pi/2)\{-t\}$.

664. Use the fact that the trigonometric polynomials form a dense subset of $\mathscr{E}(\mathbf{T})$.

665. $c_n \geq 0$ for all $n \in \mathbf{Z}$.

666. Let $\{c_k\}$ be the Fourier coefficients of the characteristic function of a set $X \subset \mathbf{T}$: $c_k = \int_X e^{-2\pi ikt}\,dt$. If $X = X + \alpha$, then

$$c_k = \int_{X+\alpha} e^{-2\pi ikt}\,dt = \int_X e^{-2\pi ik(t+\alpha)}\,dt = c_k e^{-2\pi ik\alpha}.$$

For irrational α the equality $e^{-2\pi ik\alpha} = 1$ is possible only if $k = 0$. Therefore, the characteristic function of X is almost everywhere constant.

667. Let $\alpha(t, x)$ denote the generalized solution of our equation with initial data $\alpha(0, x) = \delta(x)$. Prove that the desired solution with the initial data $u(0, x) = v(x)$ has the form

$$u(t, x) = \int_{\mathbf{T}} v(x - y)\alpha(t, y)\,dy.$$

To compute $\alpha(t, x)$, represent this function by a Fourier series with respect to x:

$$\alpha(t, x) = \sum_{k \in \mathbf{Z}} c_k(t) e^{2\pi ikx}.$$

Then the equation takes the form $c_k'(t) = -k^2 c_k(t)$, and the initial conditions are $c_k(0) = 1$. Hence, $c_k(t) = e^{-k^2 t}$. For fixed t the function

$$\alpha(t, x) = \sum_{k \in \mathbf{Z}} e^{-k^2 t + 2\pi i k x}$$

cannot be expressed in terms of elementary functions of x. It is connected in a very simple way with the so-called Weierstrass Theta-function

$$\theta(z, q) = \sum_{n \in \mathbf{Z}} q^{n^2}(-1)^n e^{2\pi i n z},$$

namely, $\alpha(t, x) = \theta(x + 1/2, e^{-t})$.

3. The Fourier Integral

668. (a) $\tilde{f}(\lambda) = \sqrt{(\pi/a)}\, e^{-(\pi^2\lambda^2/a)}$;
(b) $\tilde{f}(\lambda) = (\pi/|a|) \cdot e^{-\pi|a\lambda|}$;
(c) $\tilde{f}(\lambda) = 1/(a + 2\pi i\lambda)$;
(d) $\tilde{f}(\lambda) = e^{-\pi i\lambda(a+b)}\{[\sin \pi(b - a)\lambda]/\pi\lambda\}$.
(e) Consider the integral of the function $f(z) = e^{-2\pi i\lambda z}/\cosh az$ over the boundary of the strip $0 \le \operatorname{Im} z \le \pi/a$.
Answer:

$$\tilde{f}(\lambda) = \frac{\pi}{|a|\cosh(\pi^2\lambda/a)};$$

(f)
$$\tilde{f}(\lambda) = \frac{\pi^2}{2a^2}\frac{1}{\cosh(\pi^2\lambda/a)};$$

(g)
$$\tilde{f}(\lambda) = \frac{2\pi^2}{a^2}\frac{\lambda}{\sinh(\pi^2\lambda/a)};$$

(h) for $0 \le a \le b$

$$\tilde{f}(\lambda) = \begin{cases} \pi^2 a & \text{for } |\lambda| \le \dfrac{b-a}{2\pi}, \\[2ex] \dfrac{\pi a(a - s)}{2} & \text{for } |\lambda| = \dfrac{b+s}{2\pi},\ |s| < a, \\[2ex] 0 & \text{for } |\lambda| \ge \dfrac{b+a}{2\pi}. \end{cases}$$

669. (a) After substitution of the unknown function $f(x) \mapsto \varphi(x)e^{-\pi\|x\|^2}$ the given equations become the system $d\varphi/dx_k = 0$, $1 \le k \le n$, from which $\varphi(x) = \text{const}$.
(b) Prove the identity $e^{-\pi\|x-a\|^2} = e^{-\pi\|a\|^2/2} \sum_{m \in \mathbf{N}^n} (ia/2)^m (f_m/m!)$, where $a \in \mathbf{R}^n$, $a^m = a_1^{m_1} \cdots a_n^{m_n}$, $m! = m_1! \cdots m_n!$ (use the relation $f_m = e^{\pi\|x\|^2}\partial^m$

$\times\, e^{-2\pi\|x\|^2}$). Verify that the series on the right-hand side of the identity converges in the topology of $S(\mathbf{R}^n)$. Therefore, the smallest closed subspace $L \subset S(\mathbf{R}^n)$ containing all the functions f_m, $m \in \mathbf{N}^n$, contains all the functions of the form $\varphi_a(x) = e^{-\pi\|x-a\|^2} = f_0(x-a)$. Conclude from this that $\varphi * \varphi_a$ belongs to L for any $\varphi \in S(\mathbf{R}^n)$. This implies that the Fourier transformation of the space L contains all the functions of the form $\varphi_1 f_0$, where $\varphi \in S(\mathbf{R}^n)$. In particular, it contains the space $\mathscr{D}(\mathbf{R}^n)$, which is dense in $S(\mathbf{R}^n)$.

(c) $N_k f_m = m_k f_m$. (Use the relations: $A_k^* f_m = f_{m+\varepsilon_k}$, $A_k f_m = c_{k,m} f_{m-\varepsilon_k}$, where the ε_k are the basis vectors in $\mathbf{N}^n$. To compute the constants $c_{k,m}$ use the relation $A_k A_k^* - A_k^* A_k = 4\pi$.)

(d) To each function $f \in S(\mathbf{R}^n)$ there corresponds the sequence $c_m = \int_{\mathbf{R}^n} f(x)\overline{f_m(x)}\,dx = (f, f_m)_{L_2(\mathbf{R}^n)}$. Estimate the values of the seminorms determining the topology of $S(\mathbf{R}^n)$ on the vectors f_m by using the relations $\partial/\partial x_k = (A_k + A_k^*)/2i$, $x_k = (A_k - A_k^*)/4\pi i$.

(e) $\tilde{f}_m = i^{|m|} f_m$. (Use the relation $FA_k^*F^{-1} = F(iD_k + M_k)F^{-1} = iA_k^* = iM_k - D_k$.)

670. Show that if $f \in S(\mathbf{R}^n)$ and $f(a) = 0$, $a \in \mathbf{R}^n$, then there exist functions $\varphi_k \in S(\mathbf{R}^n)$, $1 \le k \le n$, such that $f(x) = \sum_{k=1}^n (x_k - a_k)\varphi_k(x)$. (For example, $\varphi_k(x) = \int_0^I (\partial f/\partial x_k)(a + \tau(x-a))\,d\tau$.)

671. See Problem 670 and the proof of the theorem for $n = 1$ in the main text.

672. First method: Generalize the arguments given in the corresponding subsection of the *Theory* part. Second method: Use the result of Problem 669(d) and (e).

673. The desired operator carries g(x, y) into $g(-y, x)e^{2\pi ixy}$. (Use the Poisson summation formula.)

674. (a)

$$\tilde{f}(\lambda) = \begin{cases} \dfrac{e^{2\pi i\lambda\alpha}}{2\pi i}\operatorname{sgn}(\operatorname{Im}\alpha) & \text{for } \operatorname{sgn}\lambda = \operatorname{sgn}(\operatorname{Im}\alpha), \\ 0 & \text{for } \operatorname{sgn}\lambda = -\operatorname{sgn}(\operatorname{Im}\alpha) \end{cases}$$

(b)

$$\tilde{f}(\lambda) = \begin{cases} \pi|\alpha| & \text{for } |\lambda| < a/2\pi, \\ 0 & \text{for } |\lambda| > a/2\pi \end{cases}$$

(c)

$$\tilde{f}(\lambda) = \begin{cases} \dfrac{\pi}{2i} & \text{for } \dfrac{b-a}{2n} < \lambda < \dfrac{a+b}{2n}, \\ -\dfrac{\pi}{2i} & \text{for } -\dfrac{a+b}{2n} < \lambda < \dfrac{a-b}{2n} \\ 0 & \text{otherwise} \end{cases}$$

for $0 \leq a \leq b$;

(d) $\tilde{f}(\lambda) = (1/4\pi i)e^{-2\pi|\lambda a|}\, a \operatorname{sgn} \lambda$;

(e) $\tilde{f}(\lambda) = 2 \ln \operatorname{cotan} \pi^2/2a$ (cf. Problem 668(g)).

675. (a) $\tilde{f}$ is even; (b) $\tilde{f}$ is odd; (c) $\tilde{f}(-\lambda) = \overline{\tilde{f}(\lambda)}$; (d) $\tilde{f}$ is real.

676. $\tilde{f}(\lambda) = |\det A|^{-1}\tilde{f}(A'^{-1}\lambda)e^{2\pi i\lambda A^{-1}b}$.

677. Use the relation $f = \lim_{n\to\infty} f * \varphi_n$, where $\{\varphi_n\}$ is a δ-shaped sequence in $\mathscr{D}(\mathbf{R}^n)$ (the limit is with respect to the norm of the space $L_1(\mathbf{R}^n, dx)$). Verify that under the assumptions in the problem $\tilde{f}(\lambda)$ is integrable (use the Cauchy–Bunyakovskii inequality and the fact that $(1 + \|\lambda\|^2)^{-s/2} \in L_2(\mathbf{R}^n, d\lambda)$ for $s > n/2$).

678. Prove that for $s > n/2$ the space $L_2(\mathbf{R}^n, (1 + \|\lambda\|^2)^s\, d\lambda)$ is contained in $L_1(\mathbf{R}^n, d\lambda)$. (Use the Cauchy–Bunyakovskii inequality for the functions $f(\lambda)(1 + \|\lambda\|^2)^{s/2}$ and $(1 + \|\lambda\|^2)^{-s/2}$, along with the fact that $(1 + \|\lambda\|^2)^{-s/2} \in L_2(\mathbf{R}^n, d\lambda)$ for $s > n/2$.)

679. Pass to the Fourier transform.

680. Pass to the Fourier transform.

681. Use the results of Problems 678 and 679, as well as the rule for differentiating convolutions.

682. Use the expansion of $1/P(x)$ into partial fractions of the form $1/[(x - a)^2 + b^2]$. The answer in part (c): the order of smoothness is equal to $2m - 2$.

683. Represent f as a sum of partial fractions.

684. (a) It does not follow; (b) it follows (pass to the Fourier transform).

685. Consider the functional F on $S(\mathbf{R})$ acting by the formula $\langle F, \varphi\rangle = \int_{\mathbf{R}} f(\lambda)\tilde{\varphi}(\lambda)\, d\lambda$.

Prove that F is nonnegative on nonnegative φ. (Use the fact that if $\varphi \in S(\mathbf{R})$ and $\varphi \geq 0$, then $\varphi = \psi^2$, where $\psi \in S(\mathbf{R})$.) Conclude that $\langle F, \varphi\rangle = \int_{\mathbf{R}} \varphi\, d\mu$, where μ is some measure on $\mathbf{R}$.

686. See the hint for Problem 641.

687. See the hint for Problem 641.

688. Suppose that $f \in \mathscr{D}(\mathbf{R})$, $\operatorname{supp} f \subset [-b, b]$, and $g = Ff$. Then $(2\pi i\lambda)^k g(\lambda) = F(f^{(k)})$, from which

$$g(\lambda)|\lambda|^k = \left|(2\pi)^{-k}\int_{-b}^{b} e^{-2\pi i\lambda x} f^{(k)}(x)\, dx\right| \leq (2\pi)^{-k} \sup_x f^{(k)}(x)e^{2\pi b \operatorname{Im}\lambda}.$$

Thus, g has the required properties with the constants $a = 2\pi b$ and $c_k = (2\pi)^{-k}\sup_x |f^{(k)}(x)|$. Conversely, if g satisfies the estimates $|g(\lambda)|\cdot|\lambda|^k \leq c^k e^{a|\operatorname{Im}\lambda|}$, then $g \in L_1(\mathbf{R}, d\lambda)$, and it is possible to define the continuous function $f = \check{F}g$. From the same estimates it follows that f is infinitely differentiable. Finally, if $|x| > a/(2\pi)$, then

$$|f(x)| = \left|\int_{\mathbf{R}+it\,\mathrm{sgn}\,x} e^{2\pi i\lambda x} g(\lambda)\,d\lambda\right| = \left|\int_{\mathbf{R}} e^{2\pi i\mu x - 2\pi t|x|} g(\mu + it\,\mathrm{sgn}\,x)\,d\mu\right|$$

$$\leq \int_{\mathbf{R}} e^{-2\pi t|x| + ta} \cdot \frac{c_0 + c_2}{1 + \mu^2}\,d\mu = \pi(c_0 + c_2)e^{-t(2\pi x - a)}.$$

This quantity converges to zero as $t \to \infty$. Hence, $\operatorname{supp} f \subset [-a/2\pi. a/2\pi]$.

689. (a) Let $a \in \mathbf{R}^n$, $b \in \mathbf{R}$; define $\varphi(a, b) = \int_{\mathbf{R}^n} \delta(ax - b) f(x)\,dx$. Prove the identities:

$$\int_{\mathbf{R}} \varphi(a, b)e^{-2\pi i b}\,db = \tilde{f}(a), \qquad \varphi(a, b) = |a|^{-1} \int_L f(x)\,d\mu_L(x),$$

where L is the hyperplane $ax = b$, $|a| = \sqrt{a_1^2 + \cdots + a_n^2}$. If the last integral is equal to zero for all L, then $\varphi(a, b) = 0$ for $a \neq 0$, and, hence, $\tilde{f} \equiv 0$.

(b) Let us determine $f(0)$. By the inversion formula,

$$f(0) = \int_{\mathbf{R}^3} \tilde{f}(a)\,da = \int_{\mathbf{R}^3} \left(\int_{\mathbf{R}} \varphi(a, b)e^{-2\pi i b}\,db\right) da.$$

We use the relation $\varphi(\tau a, \tau b) = |\tau|^{-1}\varphi(a, b)$, which follows from the definition of $\varphi(a, b)$ and the identity $\delta(\tau x) = |\tau|^{-1}\delta(x)$. The result is

$$f(0) = \int_{S^2} \left(\int_0^\infty \left(\int_{\mathbf{R}} \varphi(r\alpha, b)e^{-2\pi i b}\,db\right) r^2\,dr\right) d\sigma(\alpha)$$

$$= \int_{S^2} \left(\int_0^\infty \left(\int_{\mathbf{R}} \varphi(\alpha, \beta)e^{-2\pi i\beta r}\,d\beta\right) r^2\,dr\right) d\sigma(\alpha)$$

where $r = |a|$, $\alpha = a/|a| \in S^2$, $d\sigma(\alpha)$ is the area element of the sphere, and $\beta = r^{-1}b$. If $(4\pi)^{-1}\int_{S^2} \varphi(\alpha, \beta)\,d\sigma(\alpha)$ is denoted by $\psi(\beta)$, then the last expression is equal to $4\pi \int_0^\infty \tilde{\psi}(r)r^2\,dr = (1/\pi)\psi''(0)$. We remark that the geometric meaning of the quantity $\psi(\beta)$ is the mean value of the integrals of f over the planes at a distance β from the origin of coordinates. Thus, to reproduce the function f at the point x it is necessary to know its integrals only over the planes that intersect an arbitrarily small neighborhood of x. This property turns out to be valid in all odd-dimensional spaces.

690. Suppose that the given line l is the x-axis in $\mathbf{R}^3$. A line intersecting l at the point $(t, 0, 0)$ has the parametric representation $x = t + \alpha s$, $y = \beta s$, $z = \gamma s$. Let $\varphi(\alpha, \beta, \gamma, t) = \int_{\mathbf{R}} f(t + \alpha s, \beta s, \gamma s)\,ds$. The function φ is homogeneous of degree -1 in the first three variables:

$$\varphi(\alpha\tau, \beta\tau, \gamma\tau, t) = |\tau|^{-1}\varphi(\alpha, \beta, \gamma, t).$$

We regard φ as a regular generalized function and let $\tilde{\varphi}(\lambda, \mu, \nu, \tau)$ be its Fourier transform. It can be shown (cf. Problem 699) that $\tilde{\varphi}$ is regular off the line $\lambda = \mu = \nu = 0$ in $\mathbf{R}^4$ and homogeneous of degree -2 in the first three variables. We also have the identity

$$\tilde{\varphi}(\lambda, \mu, \nu, \tau) = \tilde{f}(\tau, \mu\tau\lambda^{-1}, \nu\tau\lambda^{-1})|\tau\lambda^{-2}|.$$

(To see this apply both sides of the identity to a test function $\tilde{\psi} \in S(\mathbf{R}^4)$ and use the definition of φ and the identity $\langle\tilde{\varphi}, \tilde{\psi}\rangle = \langle\varphi, \psi\rangle$.) Therefore, $\tilde{f}$ can be expressed in terms of $\tilde{\varphi}$: $\tilde{f}(a, b, c) = \tilde{\varphi}(a, b, c, a)|a|$. From this,

$$f(x, y, z) = -\frac{1}{2\pi^2}\iint \varphi(x - s, y, z, t)\frac{ds\,dt}{(s - t)^2},$$

where the integral should be understood as the value of the generalized function $|a|$ on the test function $\psi(a) = \iint \varphi(x - s, y, z, t)e^{2\pi i a(s-t)}\,ds\,dt$ (cf. Problem 691(e)).

4. Fourier Transformation of Generalized Functions

691.
(a) $\tilde{f}(\lambda) = \delta(\lambda)$;
(b) $\tilde{f}(\lambda) = (2\pi i\lambda)^k$;
(c) $\tilde{f}(\lambda) = \dfrac{e^{-2\pi i\lambda a}}{2\pi i(\lambda - i0)}$
(d) $\tilde{f}(\lambda) = \dfrac{1}{\pi i\lambda}$;
(e) $\tilde{f}(\lambda) = \left(\dfrac{i}{2\pi}\right)^k \delta^{(k)}(\lambda)$;
(f) $\tilde{f}(\lambda) = 2\dfrac{(2k+1)!}{(2\pi i\lambda)^{2k+2}}$;
(g) $\tilde{f}(\lambda) = 2\dfrac{(2k!)}{(2\pi i\lambda)^{2k+1}}$.

692. Use the fact that $\cos ax^2 = \lim_{\varepsilon\searrow 0} \cos ax^2 \cdot e^{-\varepsilon x^2}$. Answer:

$$\tilde{f}(\lambda) = \sqrt{\frac{\pi}{2|a|}}\left(\cos\frac{\pi^2\lambda^2}{a} - \sin\frac{\pi^2\lambda^2}{|a|}\right).$$

693. (a) $\tilde{f}(\lambda) = -\pi i \operatorname{sgn}\lambda$ (cf. Problem 691(d));
(b) $\tilde{f}(\lambda) = -2\pi i\theta(\lambda)$.

694. $f(x) = P(d/dx)\delta(x)$, where P is a polynomial of degree $<n$.

695. (a) $f(x) = P(d/dx)\delta(x)$, where P is a polynomial of degree $<n - k$;
(b) $f(x) = P(d/dx)\delta(x) + Q(x)\theta(x) + R(x)$, where P is a polynomial of degree $<n - k$, and Q and R are polynomials of degree $<k$.

696. $\tilde{f}_{\pm}(\lambda) = \mp(\pi i/r)e^{\mp 2\pi i|\lambda|r}$.

697. Use the fact that

$$f'''(x) = 4[\delta(x - a) - \delta(x + a)] + 4|a|[\delta'(x - a) + \delta'(x + a)].$$

Answer: $\tilde{f}(\lambda) = (2\pi a\lambda\cos 2\pi a\lambda - \sin 2\pi a\lambda)/(\pi^3\lambda^3 \operatorname{sgn} a)$.

698. (a)

$$\tilde{f}(\lambda) = \sum_{l \in \mathbf{Z}} \frac{\operatorname{sgn} a}{\pi i(2l+1)} \delta\left(\lambda - \frac{2l+1}{2\pi} a\right);$$

(b)
$$\tilde{f}(\lambda) = \sum_{l \in \mathbf{Z}} \frac{1}{\pi(2l+1)} \delta\left(\lambda - \frac{(2l+1)|a|}{2\pi}\right);$$

(c)
$$\tilde{f}(\lambda) = \frac{2}{\pi} \sum_{l \in \mathbf{Z}} \frac{\delta(\lambda - la/\pi)}{4l^2 - 1}.$$

699. Use Problem 676. Answer: $(-1, -\lambda, \varepsilon)$.

700. Answer:

$$(2\pi i\lambda + 0)^{-\alpha-1} = |2\pi\lambda|^{-\alpha-1}[\theta(\lambda)e^{(i\pi/2)(\alpha+1)} + \theta(-\lambda)e^{-i(\pi/2)(\alpha+1)}].$$

701. Answer: $F(\lambda) = \sum_{l \in \mathbf{Z}} \tilde{f}(l)\delta(\lambda - l)$.

702. (a) Use the identity

$$\frac{d}{dx}\int_x^{x+R} f(t)\,dt = f(x+R) - f(x).$$

(b) The function $e^{i\lambda x}$ is quasi-periodic with period R if $\lambda \in \mathbf{R}^n$ satisfies the condition $\int_{\|x\| \le R} e^{i\lambda x}\,dx = 0$. The latter is equivalent to the requirement that the number $R\|\lambda\|$ be a root of the equation $I_{n/2}(x) = 0$, where $I_{n/2}$ is the Bessel function (see Problem 710). It is known that this equation has countably many roots on the positive semiaxis, and the asymptotic formula $x_{mn} \approx (m + [(n-1)/4])\pi$ holds for them as $m \to \infty$. If n is odd, then the function $I_{n/2}$ can be expressed in terms of elementary functions. In particular, $I_{3/2}(x) = \sqrt{(2/\pi x)}[(\sin x)/x - \cos x]$ for $n = 3$, so that in this case the equation in λ takes the form

$$R \cdot \|\lambda\| = \tan(R \cdot \|\lambda\|).$$

(c) Yes; see the hint for (b).

703. Answer: $\tilde{f}(\lambda) = (\det A)^{-1/2} e^{-\pi((A')^{-1}\lambda, \lambda)}$, where the argument of $\det A$ is chosen by continuity on the path joining A linearly with the identity matrix.

704. Answer: $\tilde{f}(\lambda) = (\det A)^{-1/2} e^{i\pi s/4} e^{-i\pi(A^{-1}\lambda, \lambda)}$, where s is the *signature of the matrix A* (the difference between the numbers of positive and negative eigenvalues).

705. The constant R has to do with the size of the support, and the constant N with the order of the function being transformed (cf. the Paley–Wiener theorem in Problem 688).

706. Rewrite the equation in terms of the Fourier transforms.

707. Answer: $f_t(x) = (1/2\sqrt{t})e^{-\pi x^2/(4t)}$.

708. Verify these relations in the space $S(\mathbf{R}^n)$.

709. Answers: (a) π/a cotanh πa; (b) $\pi^2/\sin^2 \pi a$; (c) $\pi^3/32$.

710. (a) $\tilde{f}(\lambda) = [2\pi/(\lambda R)^{n/2-1}]I_{n/2-1}(2\pi\lambda R)$;

(b) $\tilde{f}(\lambda) = [(2\pi)^{n/2}/\lambda R]I_{n/2}(2\pi\lambda R)$, where $I_n(x)$ is the *Bessel function* defined by the integral representations

$$I_n(x) = \frac{2}{\pi}\int_0^{\pi/2} \cos 2n\theta \cos(\sin\theta)\, d\theta$$

$$= \frac{z^n}{(2n-1)!!}\cdot\frac{2}{\pi}\int_0^{\pi/2} \cos(x\cos\theta)\cos^{2n}\theta\, \mathrm{d}\theta,$$

or by the series expansion

$$I_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}\left(\frac{x}{2}\right)^{n+2k}.$$

711. $k(r) = (2\sin 2\pi r)/r$.

712. Use the Sokhotskii identity (see Problem 499).

713.

$$F\left[\mathscr{P}\frac{1}{x^3}\right] = F\left[-\frac{1}{2}\frac{d}{dx}\mathscr{P}\frac{1}{x^2}\right] = \frac{i\lambda}{2}\mathscr{P}\left[P\frac{1}{x^2}\right] = -\frac{i\pi\lambda|\lambda|}{2}.$$

Chapter V
The Spectral Theory of Operators

§1. The Functional Calculus

1. Functions of Operators in a Finite-Dimensional Space

714. Use *Cayley's identity*: $P_A(A) = 0$, where P_A is the characteristic polynomial of the matrix A.

715. (b) $\Rightarrow$ (c) $\Rightarrow$ (a) are obvious. To derive (b) from (a) consider for each vector ξ the ideal I_ξ defined by $I_\xi = \{P: P(A)\xi = 0\}$ in the ring of polynomials in one variable. The ideal I_ξ is principal (as is every ideal in the ring of polynomials in one variable), i.e., it is generated by a single polynomial P_ξ. Since $P_A \subset I_\xi$, P_ξ is a divisor of P_A. Let $P_1, \dots, P_k$ be the distinct divisors of P_A with degree $<n$ and with leading coefficient 1. Let $L_i = \ker P_i(A)$. Part (a) implies that $L_i \neq L$. Then $\bigcup_{i=1}^k L_i \neq L$. Therefore, there is a vector $\xi \in L$ which is not annihilated by any of the operators $P_i(A)$ nor, consequently, by any operator of the form $P(A)$, $\deg P < n$. This implies (b).

716. Use the formula for the Vandermonde determinant.

717. Employ the scheme (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a).

718. The condition that A is not regular can be expressed by a system of algebraic equations in the matrix coefficients of A (which reflect the linear dependence of $1, A, \dots, A^{n-1}$).

719. (a) Let ξ be a cyclic vector for A. Write A in the basis $\xi, A\xi, \dots, A^n\xi$.

(b) The first basis vector e_1 is cyclic for A.

(c) The coefficients $\{a_i\}$ are uniquely determined by the characteristic polynomial of the matrix A.

720. The coefficients of the polynomial $p(x) = x^n + a_1x^{n-1} + \cdots + a_n$ can be expressed in terms of the sums $s_k = \sum_{i=1}^n \lambda_i^k$, $k = 1, 2, \ldots, n$, of powers of its roots. (Namely, *Newton's formulas* hold: $ka_k = -\sum_{i=0}^{k-1} a_i s_{k-i}$, where $a_0 = 1$.)

721. (c) Answer:

$$f(A) = \begin{pmatrix} f(\lambda) & \dfrac{f'(\lambda)}{1!} & \dfrac{f''(\lambda)}{2!} & \cdots & \dfrac{f^{(n-1)}(\lambda)}{(n-1)!} \\ 0 & f(\lambda) & \dfrac{f'(\lambda)}{1!} & \cdots & \dfrac{f^{(n-2)}(\lambda)}{(n-2)!} \\ 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & f(\lambda) \end{pmatrix}.$$

722. If $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2$, and e_i is the unit in $\mathfrak{A}_i$, then the elements $e_1 \oplus 0$ and $0 \oplus e_2$ are nontrivial idempotents. Conversely, if e is an idempotent in $\mathfrak{A}$ different from 0 and 1, then $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2$, where $\mathfrak{A}_1 = e\mathfrak{A}e$, $\mathfrak{A}_2 = (1-e)\mathfrak{A} \times (1-e)$.

723. (a) The equation $\lambda^2 - \lambda$ has only the trivial solutions 0 and 1 in $\mathbf{C}$.

(b) Prove that a generating element x satisfies the equation $(x - \lambda \cdot 1)^n = 0$, where $\lambda \in \mathbf{C}$ and $n = \dim \mathfrak{A}$.

724. Argue by contradiction and consider an algebra of minimal dimension that is indecomposable into a sum of irreducible algebras.

725. Let $A = \inf a_n/n$. Then for any $\varepsilon > 0$ there is an n_ε such that $a_{n_\varepsilon}/n_\varepsilon < A + \varepsilon$. Represent an arbitrary N in the form $N = k \cdot n_\varepsilon + l$, where $0 \le l < n_\varepsilon$. Then $a_N/N \le (ka_{n_\varepsilon} + a_l)/(kn_\varepsilon + l)$. We have that $k \to \infty$ as $N \to \infty$. This gives us the assertion of the problem.

726. Use Problem 714.

727. Use Problem 723(b).

728. For regular operators.

729. Use Problems 718 and 720.

730. Prove that almost every pair of matrices (A, B) can be reduced to the form

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \qquad B = \begin{pmatrix} \gamma & \varepsilon \\ 1 & \delta \end{pmatrix}.$$

731. The codimension of an orbit of the action of the group $PGL(n)$ is equal to $2n^2 - (n^2 - 1) = n^2 + 1$ in this case.

732. The matrices of the form $\begin{pmatrix} \lambda \cdot 1_n & A \\ 0 & \lambda \cdot 1_n \end{pmatrix}$, where 1_n is the identity matrix of order n, commute pairwise.

733. Represent A in the form $\lambda \cdot 1 + N$, where $N^n = 0$, and verify the equality for $f(\lambda) = \lambda^k$, $k = 0, 1, \ldots, n-1$.

734. Verify that $f(A)$ can be expressed linearly in terms of the values of f at the points $\lambda_1, \ldots, \lambda_n$, and find the corresponding (matrix) coefficients.

735. Verify that $f(A)$ can be expressed linearly in terms of $f^{(j)}(\lambda_k)$, $0 \le j \le m_k - 1$. Answer: $B_{jk} = P_{jk}(A)$, where $P_{jk}(x)$ is a polynomial of degree $n - 1$ having the properties:

(1) $P_{jk}^{(s)}(\lambda_i) = 0$ for all pairs (s, i), $0 \le s \le m_i - 1$, except the pair (j, k);
(2) $P_{jk}^{(s)}(\lambda_k) = 1$.

736. The extreme points of K are the positive operators of rank 1, i.e., the orthogonal projections onto one-dimensional subspaces of H.

2. Functions of Bounded Selfadjoint Operators

737. The set $\sigma(A)$ coincides with the range of the function $a(x)$.

738. The spectrum of A is the essential range of the function $a(x)$, i.e., the numbers $\lambda \in \mathbf{C}$ such that for any neighborhood U of λ the set

$$E_U = \{x \in X : a(x) \in U\}$$

has positive measure.

739. Pass to the Fourier transform. Answer: the range of the Fourier transform of the function f.

740. Pass to the Fourier transform. Answer: the collection of Fourier coefficients of f.

741. Prove that the spectra of U and U^{-1} lie in the unit disk.

742. Verify that the mapping $(A + \bar{\lambda}1)\xi \mapsto (A + \lambda 1)\xi$ is an isometry and that $\operatorname{im}(A + \bar{\lambda}1)$ is dense.

743. $U = (A + i1)(A - i1)^{-1}$. Then

$$(A^* + i1)^{-1}(A^* - i1) = U^* = U^{-1} = (A - i1)(A + i1)^{-1},$$

from which

$$(A^* - i1)(A + i1) = (A^* + i1)(A - i1), \quad \text{and} \quad A = A^*.$$

744. Use the fact that $U + 1$ and $(U - 1)^{-1}$ commute.

745. Use the formula

$$A^n f(x) = \int_0^x \frac{(x - t)^{n-1}}{(n - 1)!} f(t)\, dt$$

and prove the inequality $\|A^n\| \le 1/(n - 1)!$. Answer: $\rho(A) = 0$.

746. Use the formula $R_\lambda(A) = -\sum_{k=0}^{\infty} \lambda^{-1-k} A^k$. Answer:

$$R_\lambda(A) f(x) = -\lambda^{-1} f(x) - \lambda^{-2} \int_0^x e^{(x-t)/\lambda} f(t)\, dt.$$

747. Use Problem 556.

748. Prove that every polynomial that is positive on $[a, b]$ can be represented in the form $P(x) = Q_1^2(x) + (x - a)Q_2^2(x) + (b - x)Q_3^2(x)$, where the Q_i are polynomials with real coefficients. *Hint*:

$$(b - x)(x - a) = (b - x)\left(\frac{x - a}{\sqrt{b - a}}\right)^2 + (x - a)\left(\frac{b - x}{\sqrt{b - a}}\right)^2.$$

749. Prove that if $\alpha \cdot 1 \leq A \leq \beta \cdot 1$, then $\|A\|$ does not exceed $\max\{|\alpha|, |\beta|\}$.

750. Use the formula $e^{itA} = \sum_{k=0}^{\infty} [(itA)^k/k!]$.

751. See the hint for Problem 750.

752. Prove that $U(t)$ is a differentiable function (see the smoothing method in the proof of Stone's theorem). Then derive the differential equation of Problem 751 and prove that it has a unique solution with the initial condition $U(0) = 1$.

753. $A = RU$, where R is the operator of multiplication by the function $|a(x)|$, and U is the operator of multiplication by the function $\operatorname{sgn}(a(x))$.

754. The one-sided shift operator T has the properties: $T^*T = 1$ and $TT^* = P$, where P is the orthogonal projection onto the orthogonal complement of the first basis vector. Answer: $T = PT$.

755. (a) $A = BRB^{-1}BUB^{-1}$ is the polar decomposition of A.

(b) False. Analyze the case when A and B are both the one-sided shift operator in Problem 754.

756. If $B = 1$, then A is invertible and $A^{-1} \ll 1$, as shown by the relations $(A^{-1}x, x) = (A^{-1/2}x, A^{-1/2}x) \leq (A \cdot A^{-1/2}x, A^{-1/2}x) = (x, x)$. The general case is obtained from this one by passing to the operator $B^{-1/2}AB^{-1/2}$.

757. Reduce T to the form of multiplication by a function.

758. Use the monotonicity of the sequence $(P_1P_2P_1)^n$.

759. Verify directly.

3. Unbounded Selfadjoint Operators

760. Prove the equivalence of the relations:

(1) $0 \oplus x \in \tau(\Gamma_A)^\perp$;

(2) $x \perp D_A$.

761. Use the fact that the operations τ and $\perp$ commute, along with the result in Problem 551.

762. (a) $A^* = -d/dx$, D_{A^*} is the natural domain;

(b) $D_{A^*} = \{f \in L_2(\mathbf{R}, dx), f' \in L_2(\mathbf{R}, dx) \oplus \mathbf{C}\delta(x)\}$;

(c) $A^* = -A$.

763. (a) $A^* = -d/dx$ with the natural domain;

(b) $A^* = -d/dx$, D_{A^*} is obtained from the natural domain by imposing the additional condition $f(0) = 0$.

764. Use the equalities $(\bar{A})^* = A^*$ and $(A^*)^* = \bar{A}$ (Problem 761).

765. (a) Not symmetric; (b) essentially selfadjoint; (c) symmetric, but not essentially selfadjoint.

766. (a) $A^* = i(d/dx)$, D_{A^*} is obtained from the natural domain by imposing the condition $f(1) = \bar{\lambda} f(0)$;

(b) $|\lambda| = 1$.

767. Answer: symmetric, in all three cases.

768. Prove that the image of the unit ball under the mapping A is weakly bounded.

769. (a) Verify the equation $\|(A - i1)x\| = \|(A + i1)x\|$.

(b) If $x \in \ker(U - 1)$ and $x = (A - i1)y$, then $x = (A + i1)y$, from which $y = 0$.

770. Check the relation $(\tau\Gamma_A)^\perp = \Gamma_A$.

771. Pass to the Fourier transform. Answer:

$$A: \{x_n\} \to \left\{ \sum_{k=1}^{\infty} (x_{n-k} - x_{n+k}) \right\}.$$

772. (a) Yes. (b) Yes. Verify that $\operatorname{im}(A \pm i1)$ contains all finitely nonzero sequences.

773. It is possible. For Example, the space $\mathscr{D}(\mathbf{R})$ and the space of step functions.

774. To prove sufficiency use the inequality in Problem 556(b) and show that $(1 + A)^{-1}$ can be extended from $\operatorname{im}(A + 1)$ to the whole of H and has norm ≤ 1.

775. Consider the projections of the vector $x \oplus 0 \in H \oplus H$ on Γ_A and on $\Gamma_A^\perp = \tau(\Gamma_{A^*})$. Prove that $(1 + A^*A)^{-1}$ is a bounded selfadjoint operator (see Problem 798).

776. Use the criterion for essential selfadjointness or Stone's theorem.

777. Prove that $(A - \lambda 1)^{-1}$ is bounded for nonreal λ (substituting $\alpha A + \beta$, $\alpha, \beta \in \mathbf{R}$, for A reduces the general case to the case $\lambda = i$).

778. (b) False. It is possible that $\ker A = 0$, but $D_{A^*} = \{0\}$. For this it suffices to take Γ_A to be any dense subspace of $H \oplus H$ that has zero intersection with $H \oplus 0$ and with $0 \oplus H$.

779. If $A \subset A_1$, then $A_1^* \subset A^*$, and these inclusions are either both proper or both not proper.

780. Verify the equality $\|(A + i1)x\| = \|(A - i1)x\|$.

781. Prove that the graphs of the operator A and the operator U in Problem 780 can be obtained one from the other by invertible linear transformations of the space $H \oplus H$.

782. To each symmetric extension of the operator A there corresponds an isometric extension of the operator U in Problem 780. To a selfadjoint operator there corresponds a unitary extension.

§2. Spectral Decomposition of Operators

1. Reduction of an Operator to the Form of Multiplication by a Function

783. Consider a space X consisting of finitely many points.

784. (a) and (b) are, (c) is not.

785. (a) Let f be any vector in H. Set $g(x) = f(-x) \cdot \operatorname{sgn} x$. Prove that the vector g is orthogonal to the cyclic subspace generated by f.

(b) The subspaces of even and odd functions are cyclic.

786. Suppose the opposite: $\sum_{k=0}^{N} c_k A^k = 0$ and $c_N \neq 0$. Then the linear span of the operators $\{A^k\}_{k=0}^{\infty}$ coincides with the linear span of the operators $\{A_k\}_{k=0}^{N-1}$. Therefore, for any vector $\xi \in H$ the space generated by the vectors $\{A^k \xi\}_{k=1}^{\infty}$ has dimension not greater than N. Hence, A cannot have a cyclic vector.

787. Use the fact that a square and a closed interval are isomorphic as measure spaces.

788. Every operator in $L_2([a, b], \mu)$ that commutes with multiplication by x is the operator of multiplication by a function of x.

789. Use the theorem on reduction of the operator A to the form of multiplication by a function $a(x)$ in a space $L_2(X, \mu)$. Prove that the set of $x \in X$ for which $a(x) \notin \sigma(A)$ has measure zero. Therefore, for almost all $x \in X$ the inequality $|f(a(x))| \leq \sup_{t \in \sigma(A)} f(t)$ holds.

790. Only (a). The multiplicity is equal to 2 in the case (b) (see Problem 785); in the case (c) the multiplicity is infinite (i.e., it is impossible to decompose H into a sum of finitely many cyclic subspaces).

791. Pass to the Fourier transforms. The condition for selfadjointness of $S(f)$ is that $\tilde{f}(\lambda)$ be a real function (or: $f(x) = \overline{f(-x)}$).

792. No, since $\tilde{f}(\lambda) \to 0$ as $\lambda \to \infty$.

793. (a) $f(g) = \overline{f(-g)}$;

(b) $|\tilde{f}(x)| \equiv 1$ (this is possible only if the group $\hat{G}$ is compact, and G is discrete);

(c) when the group G is compact.

794. Use Problem 573(b).

795. Pass to the Fourier transform. Answer: $\sigma(A) = [-1, 0]$.

796. Perform the Fourier transformation.

797. Use the relation $\tau(\Gamma_A)^{\perp} = \Gamma_{A^*}$, which is valid for any operator A and is equivalent to the definition of A^*.

798. (a) By the definition of a projection,

$$\|x\|^2 = \|y\|^2 + \|Ay\|^2 + \|Az\|^2 + \|z\|^2,$$

from which $\|B\| \leq 1$, $\|C\| \leq 1$.

(b) The equation $x \oplus 0 = (y \oplus Ay) + (-Az \oplus z)$ implies that $x = y - Az$, $z = -Ay$. Recalling that $y = Bx$, $z = Cx$, we get that $x = Bx - ACx$, $Cx = -ABx$, or $1 = B - AC$, $C = -AB$.

From this we have: $1 = B + A^2B$, i.e., $(1 + A^2)B = 1$.

799. (a) $e^{iaA}: f(x) \to f(x - a)$ (translation operator);

(b) $\dfrac{\sin aA}{ia}: f(x) \to \dfrac{f(x + a) - f(x - a)}{2a}$ (finite difference operator);

(c) $\dfrac{1}{A^2 + a^2}: f(x) \to \dfrac{1}{2a} \displaystyle\int_{\mathbf{R}} e^{-a(x-y)} f(y)\, dy$, $a > 0$.

800. $e^{aA^2}: f(x) \to (1/2\sqrt{\pi a}) \int_{\mathbf{R}} e^{-[(x-y)^2/4a]} f(y)\, dy$.

801. $\Delta_h = (e^{-ihA} - 1)/h$.

802. Prove that A is a convolution operator for the group of positive numbers with multiplication as the group operation. Make the substitutions:

$$x = e^t, \qquad y = e^s, \qquad f(x) = e^{-t/2}\varphi(t).$$

Answer: the operator of multiplication by $(1/(2 \cosh t/2))^\sim = \pi/\cosh 2\pi^2\lambda$, $\sigma(A) = [0, \pi]$ (see the hint for Problem 710).

803. After Fourier transformation A passes into the operator of multiplication by the function $a(\lambda, \mu) = I_0(2\pi\ R)$, where $\rho = \sqrt{\lambda^2 + \mu^2}$, and $I_0(z)$ is the Bessel function (see Problem 710).

804. After Fourier transformation A passes into multiplication by the function

$$a(\lambda, \mu, \nu) = \frac{\sin 2\pi\rho R}{2\pi\rho R},$$

where $\rho = \sqrt{\lambda^2 + \mu^2 + \nu^2}$.

805.

$$\begin{aligned} f(F)\colon \varphi(x) \mapsto\ & \frac{f(1) + f(i) + f(-1) + f(-i)}{4}\,\varphi(x) \\ &+ \frac{f(1) + if(i) - f(-1) - if(-i)}{4}\,\tilde{\varphi}(-x) \\ &+ \frac{f(1) - f(i) + f(-1) - f(-i)}{4}\,\varphi(-x) \\ &+ \frac{f(1) - if(i) - f(-1) + if(-i)}{4}\,\tilde{\varphi}(x). \end{aligned}$$

2. The Spectral Theorem

806. (a) Let $w(t)$ be the modulus of continuity of f (i.e., $|f(x) - f(y)| \leq w(|x - y|)$). Prove the estimate

$$\|S(f, T, \xi) - S(f, T, \eta)\| \leq w(\delta(T)).$$

(b) Use the fact that the integral sum $S(f, T, \xi)$ coincides with the Lebesgue integral of the step function

$$f_{T\xi}(x) = \sum_{k=0}^{n} f(\xi_k)\chi_{[t_k, t_{k+1}]}(x).$$

807. To prove properties (a), (c), (d) it is expendient to use the equality

$$\left(\int_X f(x)\, d\lambda(x)\xi, \eta\right) = \int_X f(x)\, d\lambda_{\xi,\eta}$$

for any $\xi, \eta \in H$. Verify the equality (b) first for characteristic functions, and then use (a). To prove (e) use (b) and Problem 563.

808. (a) Use the relation

$$X \setminus (E_1 \cup E_2) = (X \setminus E_1) \cap (X \setminus E_2).$$

(b) Consider first the case $E_1 \cap E_2 = \varnothing$. Let $\lambda(E_1) = \alpha$ and $\lambda(E_2) = \beta$. Since α, β, and $\alpha + \beta$ are orthogonal projections, $\alpha^2 = \alpha$, $\beta^2 = \beta$, and $(\alpha + \beta)^2 = \alpha + \beta$. This implies that $\alpha\beta + \beta\alpha = 0$. Multiplying this equality by α from the left, from the right, and from both sides, we get: $\alpha\beta + \alpha\beta\alpha = 0$, $\alpha\beta\alpha + \beta\alpha = 0$, and $2\alpha\beta\alpha = 0$. Thus, $\alpha\beta = \beta\alpha = 0$. To prove (2) in the general case use the relations $E_1 = (E_1 \cap E_2)\amalg(E_1\setminus E_2)$, $E_2 = (E_1 \cap E_2)\amalg(E_2\setminus E_1)$. Then $\lambda(E_1)\lambda(E_2) = [\lambda(E_1 \cap E_2) + \lambda(E_1\setminus E_2)][\lambda(E_1 \cap E_2) + \lambda(E_2\setminus E_1)]$. By what has already been proved, the last expression equals $\lambda(E_1 \cap E_2)$.

809. It exists. Use a measure-preserving mapping of the closed interval onto the square (see Problem 102(a)).

810. True. Use the fact that the correspondence $\xi \to \mu_\xi(E)$ defines a seminorm in H that satisfies the parallelogram law.

811. Apply the assertion of the preceding problem to the measure μ_ξ that represents the functional $f \to (\varphi(f)\xi, \xi)$.

812. (a) Follows from the fact that $\lambda(E)$ and A commute, which is a consequence of the construction of λ.

(b), (c) Follow from consideration of a realization in which A is an operator of multiplication by a function (cf. Problem 738).

813. Follows from Problem 812(c) (and is proved similarly).

814. If $(A - a \cdot 1)^{-1}$ exists, then $\|A\xi - a\xi\| \geq \|A - a \cdot 1\|^{-1}\|\xi\|$. And if a is a point of the spectrum, then use the result of Problem 813.

815. Use the fact that every orthonormal system converges weakly to zero.

816. Let $I_\varepsilon(t) = (\varepsilon/\pi) \int_a^b \{d\lambda/[(t-\lambda)^2 + \varepsilon^2]\}$.
Use the change of variables $\lambda \mapsto (\lambda - t)/\varepsilon$ and compute directly that

$$I_\varepsilon(t) = \begin{cases} 0 & \text{if } t \notin [a, b], \\ 1/2 & \text{if } t = a \text{ or } t = b, \\ 1 & \text{if } t \in (a, b). \end{cases}$$

817. First method: represent U in the form $\operatorname{Re} U + i \cdot \operatorname{Im} U$. Second method: use the results of Problems 769, 770.

818. Use the result of Problem 817 and the relation

$$\frac{1}{N} \sum_{k=1}^{N} e^{2\pi i k t} \to \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{for } t \neq 0 \end{cases}, \quad t \in \mathbf{T} = \mathbf{R}/\mathbf{Z},$$

as $N \to \infty$.

819. (a) For a proof of multiplicativity use the *Hilbert identity* $R_\lambda(A) \times R_\mu(A) = [R_\lambda(A) - R_\mu(A)]/(\lambda - \mu)$ and assume that the contour C traversed by the variable λ contains the contour C' traversed by the variable μ in the integral

$$-\frac{1}{4\pi^2} \int_C \int_{C'} f_1(\lambda) f_2(\mu) R_\lambda(A) R_\mu(A) \, d\lambda \, d\mu.$$

(b) Use the fact that the integral is a limit of Riemann integral sums, and that the corresponding numerical integral sums converge to the integral $(i/2\pi) \int_C [f(\lambda)/(t - \lambda)] \, d\lambda = f(t)$.

820. See the hint for Problems 306 and 787.

821. Consider first the case of a cyclic subspace.

822. Answer: $\lambda(E) = S(f_E)$, where f_E is the Fourier transform of the characteristic function of the set $(1/2\pi) \cdot E$, and $S(f)$ is the operator of convolution with f.

823. Answer: $\lambda(E) = S(f_E)$, where f_E is the Fourier transform of the function

$$g_E(\lambda) = \begin{cases} 1 & \text{if } -4\pi^2\lambda^2 \in E, \\ 0 & \text{if } -4\pi^2\lambda^2 \notin E. \end{cases}$$

824. Answer: $\lambda(E) = S(f_E)$, where f_E is the Fourier transform of the function

$$g_E(\lambda, \mu) = \begin{cases} 1 & \text{if } -4\pi^2(\lambda^2 + \mu^2) \in E, \\ 0 & \text{if } -4\pi^2(\lambda^2 + \mu^2) \notin E. \end{cases}$$

825. Answer: $A = -i(d/dx)$.

826. Use Stone's theorem and the realization of $U(t)$ as the operator of multiplication by $e^{2\pi i t}$ in a direct sum of spaces of the form $L_2(R, \mu)$.

827. Yes. Use Problem 817.

828. Verify that the function $f_0(x) = e^{-x^2/2}$ is an eigenfunction of A, and that the operator

$$B = -\frac{d}{dx} + x$$

carries an eigenfunction of A with eigenvalue λ into an eigenfunction with eigenvalue $\lambda + 2$. Answer:

$$A = \sum_{k=0}^{\infty} (2k + 1)P_k,$$

where P_k is the projection onto the subspace generated by the function $f_k = B^k f_0$.

List of Notation

The three numbers following a symbol denote the numbers of the chapter, section, and subsection, respectively, in the *Theory* part in which the meaning of this symbol is explained.

$\mathbf{C}$ is the set of complex numbers;
$\mathbf{N}$ is the set of natural numbers;
$\mathbf{Q}$ is the set of rational numbers;
$\mathbf{R}$ is the set of real numbers;
$\mathbf{Z}$ is the set of integers.

Spaces

$A^2(D)$, III.4.1, the space of analytic functions in the unit disk D;
$B(X)$, III.3.2, the space of bounded functions on the set X;
$C(X)$, III.1.4, the space of continuous functions on the set X;
$C^r(\Omega)(C^r(\bar{\Omega}))$, III.3.3, the space of r times continuously differentiable functions in Ω (respectively, that extend continuously to $\bar{\Omega}$);
$\mathscr{D}(\Omega) = C_0^\infty(\Omega)$, III.3.3, the space of compactly supported infinitely differentiable functions on Ω;
$\mathscr{D}'(\Omega)$, III.3.4, the space of generalized functions;
$\mathscr{E}(\Omega) = C^\infty(\Omega)$, III.3.3, the space of infinitely differentiable functions on Ω;
$\mathscr{E}'(\Omega)$, III.3.4, the space of generalized functions with compact support;
End $L = \mathscr{L}(L, L)$;
$\mathscr{F}(L_1, L_2)$, III.2.3, the space of Fredholm operators from the LTS L_1 to the LTS L_2;

H^*, III.4.2, the Hermitian dual space of the Hilbert space H;
$H_1 \oplus H_2$, III.4.1, direct sum of linear spaces;
$H_1 \otimes H_2$, III.1.4, tensor product of linear spaces;
$\mathscr{K}(L_1, L_2)$, III.2.2, the space of compact operators from the LTS L_1 to the LTS L_2;
$\mathscr{L}(L_1, L_2)$, III.1.2, the space of continuous linear mappings from the LTS L_1 to the LTS L_2;
L', III.1.2, the dual space of the LTS L;
$L_p(X, \mu)$, III.3.1, the space of functions on X with μ-integrable pth powers;
$L_\infty(X, \mu)$, III.3.1, the space of essentially bounded functions on X;
$l_p(n, K)$, III.1.4, the n-dimensional space over the field K with norm $\|x\|_p$;
$l_p(K)$, III.1.4, the space of sequences over the field K with norm $\|x\|_p$;
$p(\mathbf{Z})$, III.2.2, the space of slowly increasing sequences;
$P\mathscr{E}(\mathbf{R}^n)$, IV1.2, the subspace of $\mathscr{E}(\mathbf{R}^n)$ consisting of functions of not greater than polynomial growth;
$S(\mathbf{R}^n)$, III.3.3, the space of smooth rapidly decreasing functions;
$S'(\mathbf{R}^n)$, III.3.4, the space of tempered distributions;
$BV[a, b]$, II.1.3, the space of functions of bounded variation on $[a, b]$.

Convergence

$A_n \rightrightarrows A$, $A = \text{u-lim}\, A_n$, III.2.1, uniform convergence of operators;
$A_n \to A$, $A = \text{s-lim}\, A_n$, III.2.1, strong convergence of operators;
$A_n \rightharpoonup A$, $A = \text{w-lim}\, A_n$, III.2.1, weak convergence of operators;
$f_n \rightrightarrows f$, II.2.2, uniform convergence of functions;
$f_n \xrightarrow{\text{a.e.}} f$, II.2.2, convergence of functions almost everywhere;
$f_n \xrightarrow{\mu} f$, II.2.2, convergence of functions in the measure μ.

Operators

A', III.1.2, the operator adjoint to the operator A;
A^*, III.4.2, the Hermitian adjoint operator of the operator A;
$A \geqslant 0$, III.4.2, a positive operator A;
$\bar{A}$, V.I.3, the closure of the operator A;
$A_1 \oplus A_2$, III.2.3, the direct sum of the operators A_1 and A_2;
$B \supset A$, V.I.3, the operator B is an extension of the operator A;
coker A, III.2.3, the cokernel of the operator A;
D_A, III.2.3, the domain of the operator A;
∂, III.3.3, a partial derivative operator;
$i(A)$, III.2.3, the index of the operator A;
im A, III.2.3, the range of the operator A;
ker A, III.2.3, the kernel of the operator A;
$M(f)$, III.3.5, the operator of multiplication by the function f;
$r(A)$, V.1.2, the spectral radius of the operator A;

$r_\lambda(A)$, V.1.1, the resolvent of the operator A;
rank A, III.2.3, the rank of the operator A;
$S(f)$, IV.1.1, the operator of convolution with the function f;
$T(a)$, IV.1.1, the operator of translation by a;
$\rho(A)$, V.1.2, the resolvent set of the operator A;
$\sigma(A)$, V.1.2, the spectrum of the operator A.

Other Notation

$A \triangle B$, II.1.1, the symmetric difference of the sets A and B;
$A \coprod B$ (or $\coprod A_i$), II.1.1, disjoint union of sets;
$\mathring{B}(x, r)$ ($B(x, r)$), III.1.1, the open (resp., closed) ball of radius r with center at the point x in a metric space;
Card Y, II.1.3, the number of elements of the set Y;
$\mathrm{Cont}(K_1, K_2)$ ($\mathrm{Cov}(K_1, K_2)$), I.3, the contravariant (resp., covariant) functors;
diam X, III.2.2, the diameter of the set X;
ess sup f, II.3.1, the essential supremum of the function f;
$\tilde{f}$, IV.2.1, the Fourier transform of the function f;
$f_1 \times f_2$, III.3.5, the direct product of the generalized functions f_1 and f_2;
$f_1 * f_2$, IV.1.1, the convolution of the functions f_1 and f_2;
$\hat{G}$, IV.2.1, the dual group of the group G;
$K[G]$, IV.1.1, the group algebra of the group G;
K^0, I.3, the category dual to the category K;
$L(S, \mu)$, II.1.2, the collection of μ-measurable sets;
$\mathrm{Mor}(A, B)$, $\mathrm{Ob}(K)$, I.3, the morphisms and objects in the category K;
$P(X)$, II.1.1, the set of subsets of X;
$Q_A(X)$, III.4.2, the Hermitian (quadratic) form corresponding to the operator A;
$R(S)$, II.1.1, the ring generated by the family of sets S;
$R_\sigma(S)$, II.1.1, the σ-ring generated by the family of sets S;
supp φ, III.3.3, the support of the function φ;
$\mathbf{T}^n$, IV.1.1, the n-dimensional torus;
$\mathrm{Var}_a^b f$, II.1.3, the variation of the function f on the segment $[a, b]$;
$X^\perp$, III.2.3, the orthogonal complement of X;
$x \perp y$, III.4.1, the vector x is orthogonal to the vector y;
$\delta(x)$, III.3.4, the Dirac function (δ-function);
$\delta_b(x)$, III.3.5, the translated δ-function;
δ_{ij}, III.2.3, the Kronecker symbols;
$\mu(A)$, II.1.2, the measure of the set A;
$\mu^*(A)$, II.1.2, the outer measure of the set A;
$\nu(A)$, II.1.2, a signed measure;
$|\nu|(A)$, II.1.2, the variation of the signed measure ν;
$\check{\varphi}(x)$, IV.1.2, $\check{\varphi}(x) = \varphi(-x)$;
χ_A, II.1.1, the characteristic function of the set A.

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Distribution of Literature by Chapters

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Chapter II. [2], [3], [4], [18], [26], [28], [11*], [20*].

Chapter III. [1], [2], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [25], [27], [31], [32], [2*], [4*], [8*], [13*], [14*], [15*], [17*].

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