Kenneth DeMason, March 18, 2021

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1.2.1. A family of sets $\mathcal{R} \subset \mathcal{P}(X)$ is called a **ring** if it is closed under finite unions and differences (i.e., if $E_1, ..., E_n \in \mathcal{R}$, then $\bigcup_{i=1}^n \in \mathcal{R}$ and if $E, F \in \mathcal{R}$ then $E \setminus F \in \mathcal{R}$). A ring that is closed under countable unions is called a **σ-ring**.

a) Rings (resp. σ-rings) are closed under finite (resp. countable) intersections.
b) If $\mathcal{R}$ is a ring (resp. σ-ring), then $\mathcal{R}$ is an algebra (resp. σ-algebra) iff $X \in \mathcal{R}$.
c) If $\mathcal{R}$ is a σ-ring, then $\{E \subset X \mid E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$ is a σ-algebra.
d) If $\mathcal{R}$ is a σ-ring, then $\{E \subset X \mid E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$ is a σ-algebra.

Solution:

a) Let $\{E\}_{i=1}^n \subset \mathcal{R}$. Observe that

$$\bigcap_{i=1}^n E_i = \left(\bigcup_{i=1}^n E_i\right) \setminus \left(\bigcup_{(i,j)\subset\{1,\ldots,n\}^2} (E_i \setminus E_j)\right) = \left(\bigcup_{i=1}^n E_i\right) \setminus \left(\bigcup_{\{i,j\}\subset\{1,\ldots,n\}^2} (E_i \Delta E_j)\right)$$

The same is true for infinite families by replacing n with ∞.

$$\bigcap_{i=1}^n E_i = \left(\bigcup_{i=1}^n E_i\right) \setminus \left(\bigcup_{(i,j)\subset\{1,\ldots,n\}^2} (E_i \setminus E_j)\right)$$

Perhaps another, better way to do this is to note that $E_1 \cap E_2 = E_1 \setminus (E_1 \setminus E_2)$ and extend this inductively.

b) We need only check that a ring $\mathcal{R}$ is closed under complements iff $X \in \mathcal{R}$. Suppose that $\mathcal{R}$ is closed under complements. Then if $E \in \mathcal{R}$, $E^c \in \mathcal{R}$ and hence their union $E \cup E^c \in \mathcal{R}$. Now suppose $X \in \mathcal{R}$. Since $\mathcal{R}$ is closed under differences, $E^c = X \setminus E \in \mathcal{R}$.

This same logic works whether or not $\mathcal{R}$ is a σ-ring or just a ring. The only difference is if countable unions are allowed or not; this is the distinction between a σ-algebra and an algebra.

c) First we check $\mathscr{M} := \{E \subset X \mid E \in \mathcal{R} \text{ or } E \in \mathcal{R}^c\}$ is closed under complements. Let $E \in \mathscr{M}$. Then $E \in \mathcal{R}$ or $E \in \mathcal{R}^c$. Suppose the former; then $(E^c)^c$ is countable, and hence $E^c \in \mathscr{M}$. Similarly, if E^c is countable rather than E, then $E^c \in \mathscr{M}$.
Now we show $\mathscr{M}$ is closed under countable unions. Let $\{E_j\}_{j=1}^\infty \subset \mathscr{M}$. Then for each E_j either $E_j \in \mathcal{R}$ or $E_j \in \mathcal{R}^c$. Suppose all $E_j \in \mathcal{R}$. Then, since $\mathcal{R}$ is a σ-ring, the union $\bigcup_{j=1}^\infty E_j \in \mathcal{R}$. Hence, $\bigcup_{j=1}^\infty E_j \in \mathscr{M}$. Otherwise, set $G = \bigcup_{j=1, E_j\in\mathcal{R}}^\infty E_j$ and $B = \bigcap_{j=1, E_j\in\mathcal{R}^c}^\infty E_j^c$. Hence,

$$\bigcup_{j=1}^\infty E_j = \left(\bigcup_{j=1, E_j\in\mathcal{R}}^\infty E_j\right) \cup \left(\bigcup_{j=1, E_j\in\mathcal{R}^c}^\infty E_j\right) = G \cup B^c$$

By part a), it follows that $B \in \mathcal{R}$; clearly, $G \in \mathcal{R}$ since it falls under the first case. Now, observe that $(B \setminus G)^c = (B \cap G^c)^c = G \cup B^c$. So $\left(\bigcup_{j=1}^\infty E_j\right)^c = B \setminus G$. But both G and B are in $\mathcal{R}$ so that their difference $B \setminus G$ is. Hence, $\left(\bigcup_{j=1}^\infty E_j\right)^c$ is in $\mathcal{R}$, and $\bigcup_{j=1}^\infty E_j \in \mathscr{M}$.

As an example of this, the σ-algebra of countable or co-countable sets is such a σ-algebra with the ring $\mathcal{R}$ if countable subsets of X.

d) Define $\mathscr{M} := \{E \subset X \mid E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$. Let $\{E_j\}_{j=1}^\infty \subset \mathscr{M}$. Then for $F \in \mathcal{R}$,

$$\left(\bigcup_{j=1}^\infty E_j\right) \cap F = \bigcup_{j=1}^\infty (E_j \cap F) \in \mathcal{R}$$

since each $E_j \cap F \in \mathcal{R}$ by hypothesis. Hence $\mathscr{M}$ is closed under countable unions.

Now let $E \in \mathscr{M}$. For $F \in \mathcal{R}$ we have $E \cap F \in \mathcal{F}$. Then $E^c \cap F = F \setminus (E \cap F)$, the difference of two sets in $\mathcal{R}$. Hence $E^c \cap F \in \mathcal{R}$ and $\mathscr{M}$ is closed under complements.

1.2.2. Complete the proof of proposition 1.2.

Solution: Recall that Proposition 1.2. says that $\mathscr{B}_{\mathbb{R}}$ is generated by each of the following.

a) The open intervals: $\mathcal{E}_1 = \{(a,b) \mid a < b\}$.
b) The closed intervals: $\mathcal{E}_1 = \{[a,b] \mid a < b\}$.
c) The half-open intervals: $\mathcal{E}_3 = \{(a,b] \mid a < b\}$ or $\mathcal{E}_4 = \{[a,b) \mid a < b\}$.
d) The open rays: $\mathcal{E}_5 = \{(a,\infty) \mid a \in \mathbb{R}\}$ or $\mathcal{E}_6 = \{(-\infty,b) \mid b \in \mathbb{R}\}$.
e) The closed rays: $\mathcal{E}_7 = \{[a,\infty) \mid a \in \mathbb{R}\}$ or $\mathcal{E}_8 = \{(-\infty,b] \mid b \in \mathbb{R}\}$.

The book already proves that $\mathscr{M}(\mathcal{E}_j) \subset \mathscr{B}_{\mathbb{R}}$ for all j, since the elements of each $\mathcal{E}_j$ are Borel (either open/closed or G_δ/F_σ). As for the reverse inclusion $\mathscr{B}_{\mathbb{R}} \subset \mathscr{M}(\mathcal{E}_j)$, we immediately have this for $j = 1$ since every open set is the countable union of open intervals. It suffices then to show that $\mathscr{M}(\mathcal{E}_1) \subset \mathscr{M}(\mathcal{E}_j)$. To do this, we appeal to Lemma 1.1. and show that $\mathcal{E}_1 \subset \mathscr{M}(\mathcal{E}_j)$ for all j. So, we just need to show that an open interval can be written as a countable union/intersection of elements in each $\mathcal{E}_j$. The book does this for $j = 2$, so let us start with $j = 3$.

c) Consider an open interval (a,b). Then

$$(a,b) = \bigcup_{n=1}^{\infty} \left(a, b - \frac{1}{n}\right]$$

Since each $(a, b - 1/n] \in \mathcal{E}_3$, it follows that $(a,b) \in \mathscr{M}(\mathcal{E}_3)$. Similarly,

$$(a,b) = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b\right)$$

so that $(a,b) \in \mathscr{M}(\mathcal{E}_4)$.

d) Consider an open interval (a,b). Then

$$(a,b) = (a,\infty) \cap (-\infty,b) = (a,\infty) \cap [b,\infty)^c = (a,\infty) \cap \left(\bigcap_{n=1}^{\infty} \left(b - \frac{1}{n}, \infty\right)\right)^c$$

Since (a,∞) and all the $(b - 1/n, \infty)$ are in $\mathcal{E}_5$, and we only use the operations of countable intersection and complements to combine them, it follows that $(a,b) \in \mathscr{M}(\mathcal{E}_5)$. As for $\mathcal{E}_6$, we do something slightly different:

$$(a,b) = (-\infty,a]^c \cap (-\infty,b) = \left(\bigcap_{n=1}^{\infty} \left(-\infty, a + \frac{1}{n}\right)\right)^c \cap (-\infty,b).$$

By the same logic, $(a,b) \in \mathscr{M}(\mathcal{E}_6)$.

e) For an open interval (a,b) we can massage the form used for $\mathcal{E}_6$ and get

$$(a,b) = \left(\bigcap_{n=1}^{\infty} \left(-\infty, a + \frac{1}{n}\right)\right)^c \cap (-\infty,b) = \left(\bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, \infty\right)\right) \cap [b,\infty)^c.$$

so that $(a,b) \in \mathscr{M}(\mathcal{E}_7)$. Similarly, by massaging the form used for $\mathcal{E}_5$,

$$(a,b) = (a,\infty) \cap \left(\bigcap_{n=1}^{\infty} \left(b - \frac{1}{n}, \infty\right)\right)^c = (-\infty,a]^c \cap \left(\bigcup_{n=1}^{\infty} \left(-\infty, b - \frac{1}{n}\right]\right)$$

so that $(a,b) \in \mathscr{M}(\mathcal{E}_8)$.

1.2.3. Let $\mathscr{M}$ be an infinite σ-algebra.

a) $\mathscr{M}$ contains an infinite sequence of disjoint sets.

b) $\text{card}(\mathscr{M}) \geq \mathfrak{c}$.

Solution:

a) Let $\{E_i\}_{i=1}^{\infty} \subset \mathscr{M}$, which exists since $\mathscr{M}$ is infinite. Now define $F_1 = E_1$ and define F_n by

$$F_n = E_n \setminus \left(\bigcup_{i=1}^{n-1} E_i \right).$$

Consider F_n and F_m for $n \neq m$. WLOG we may assume $n < m$. Observe that $F_n \subset E_n$. Since $F_m \cap \cup_{i=1}^{m-1} E_i = \emptyset$, in particular $F_m \cap E_n = \emptyset$. Thus F_m and F_n are disjoint, and $\mathscr{M}$ contains an infinite sequence of disjoint sets.

b) Consider the mapping $\rho : \mathcal{P}(\mathbb{N}) \to \mathscr{M}$ by

$$S \mapsto \bigcup_{i \in S} F_i$$

where $S \subset \mathbb{N}$. Since the F_i are disjoint, ρ is injective.

1.2.4. An algebra $\mathcal{A}$ is a σ-algebra iff $\mathcal{A}$ is closed under countable increasing unions (i.e. if $\{E_j\}_{j=1}^{\infty} \subset \mathcal{A}$ and $E_1 \subset E_2 \subset ...$, then $\bigcup E_j \in \mathcal{A}$).

Solution: The only difference between an algebra and a σ-algebra is that an algebra is closed under finite unions whereas a σ-algebra is closed under countable unions (n.b. in general this is what the prefix σ means, it upgrades a finite closure condition to a countable closure condition). The first direction is therefore trivial; if $\mathcal{A}$ is actually a σ-algebra, it is closed under any countable union, in particular a countable union of increasing sets.

For the other direction, suppose that $\mathcal{A}$ is closed under countable increasing unions. Let $\{E_i\}_{i=1}^{\infty}$ be any collection of sets in $\mathcal{A}$. Define F_j by

$$F_j = \bigcup_{i=1}^{j} E_i,$$

from which we easily see that $F_1 \subset F_2 \subset ....$ Moreover, since $\mathcal{A}$ is an algebra, each $F_j \in \mathcal{A}$, as it is a finite union of sets in $\mathcal{A}$. Hence,

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{j=1}^{\infty} F_j \in \mathcal{A}.$$

1.2.5. If $\mathscr{M}$ is the σ-algebra generated by $\mathcal{E}$, then $\mathscr{M}$ is the union of σ-algebras generated by $\mathcal{F}$ as $\mathcal{F}$ ranges over all countable subsets of $\mathcal{E}$. (Hint: Show that the latter object is a σ-algebra).

Solution: Let $\mathscr{F} = \{\mathcal{F} \subset \mathcal{E} \mid \text{card}(\mathcal{F}) = \aleph_0\}$. We show that

$$\tilde{\mathscr{M}} = \bigcup_{\mathcal{F} \in \mathscr{F}} \mathscr{M}(\mathcal{F})$$

is a σ-algebra. Let $\{E_i\}_{i=1}^{\infty} \subset \tilde{\mathscr{M}}$. Then each $E_i \in \mathscr{M}(\mathcal{F}_i)$ for some $\mathcal{F}_i \in \mathscr{F}$. Define $\mathcal{F} = \bigcup_i \mathcal{F}_i$. We see that $\bigcup_i E_i \in \mathscr{M}(\mathcal{F})$. Since each $\mathcal{F}_i$ was countable, $\mathcal{F}$ is the countable union of countable sets, and is therefore countable. Hence $\mathcal{F} \in \mathscr{F}$ and $\bigcup_i E_i \in \tilde{\mathscr{M}}$.

As for complements, let $E \in \tilde{\mathscr{M}}$. Then $E \in \mathscr{M}(\mathcal{F})$ for some $\mathcal{F} \in \mathscr{F}$, and clearly $E^c \in \mathscr{M}(\mathcal{F})$. Hence $E^c \in \tilde{\mathscr{M}}$.

The σ-algebra $\mathscr{M}$ contains $\mathcal{E}$ since, for any $E \in \mathcal{E}$, $\mathcal{F} := \{E\} \in \mathscr{F}$. Hence $E \in \tilde{\mathscr{M}}$; it follows that $\mathscr{M}(\mathcal{E}) \subset \tilde{\mathscr{M}}$. For the reverse inclusion, if $E \in \tilde{\mathscr{M}}$ then as before there exists an $\mathcal{F} \in \mathscr{F}$ such that $E \in \mathscr{M}(\mathcal{F})$. But, $\mathcal{F} \subset \mathcal{E}$ so that $\mathscr{M}(\mathcal{F}) \subset \mathscr{M}(\mathcal{E})$. Thus, $E \in \mathscr{M}(\mathcal{E})$, and the reverse containment holds.

1.3. **Measures.**

1.3.6. Complete the proof of Theorem 1.9

Solution: Recall that Theorem 1.9 says to suppose $(X, \mathscr{M}, \mu)$ is a measure space and define $\mathscr{N} := \{N \in \mathscr{M} \mid \mu(N) = 0\}$, the set of null-sets. Then $\overline{\mathscr{M}} := \{E \cup F \mid E \in \mathscr{M},\ F \subset N \in \mathscr{N}\}$ is a σ-algebra and there exits a unique complete measure $\overline{\mu}$ on $\overline{\mathscr{M}}$ extending μ. We show that $\overline{\mu}$ is the unique complete extension of μ.

First, we need to show that $\overline{\mu}$ is a measure at all. Since $\overline{\mu}(E \cup F) = \mu(E)$, it follows that $\overline{\mu} : \overline{\mathscr{M}} \to [0, \infty]$. Next, let $\{\overline{E}_j\}_{j=1}^{\infty}$ be a disjoint collection of sets in $\overline{\mathscr{M}}$. Then,

$$\overline{\mu}\left(\bigcup_{j=1}^{\infty} \overline{E}_j\right) = \overline{\mu}\left(\bigcup_{j=1}^{\infty} (E_j \cup F_j)\right) = \overline{\mu}\left(\bigcup_{j=1}^{\infty} E_j \cup \bigcup_{j=1}^{\infty} F_j\right).$$

Since each $F_j \subset N_j \in \mathscr{N}$, and $N = \bigcup_j N_j$ is a null set, we see that $\bigcup_j F_j \subset N$. Hence,

$$\overline{\mu}\left(\bigcup_{j=1}^{\infty} E_j \cup \bigcup_{j=1}^{\infty} F_j\right) = \mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j) = \sum_{j=1}^{\infty} \overline{\mu}(E_j \cup F_j) = \sum_{j=1}^{\infty} \overline{\mu}(\overline{E}_j).$$

Thus $\overline{\mu}$ is countably additive. We easily see that $\overline{\mu}(\emptyset) = \mu(\emptyset) = 0$ so that $\overline{\mu}$ is a measure.

Let $\overline{E}$ be null set in $\overline{\mathscr{M}}$ – that is if we write $\overline{E} = E \cup F$ in the above notation, $\overline{\mu}(\overline{E}) = \mu(E) = 0$. Since E is a null set and F is a subset of a null set, their union is a subset of a null set (namely the union of the two null sets). It follows that $\overline{E}$ is a subset of some null set $N \in \mathscr{N}$. Let $\overline{F} \subset \overline{E}$. Then, immediately, $\overline{F} \subset N$ and $\overline{F} \in \overline{\mathscr{M}}$ by writing $\overline{F} = \emptyset \cup \overline{F}$. So, $\overline{\mu}$ is complete. Clearly $\overline{\mu}$ extends μ since if $E \in \mathscr{M}$ then $E = E \cup \emptyset \in \overline{\mathscr{M}}$. Moreover, $\overline{\mu}(E) = \mu(E)$.

Now suppose ν is another complete measure on $\overline{\mathscr{M}}$ extending μ. Let $\overline{E} \in \mathscr{M}$. Then $\overline{E} = E \cup F \subset E \cup N$ where N is a μ-null set such that $F \subset N$. As in the proof of Theorem 1.9, we may choose N disjoint from E. From this, we have

$$\mu(E) = \nu(E) \leq \nu(\overline{E}) \leq \nu(E) + \nu(N) = \mu(E) + \mu(N) = \mu(E).$$

Hence, $\nu(\overline{E}) = \mu(E)$. But, $\overline{\mu}(\overline{E}) = \mu(E)$, so that $\nu = \overline{\mu}$.

1.3.7. If $\mu_1, ..., \mu_n$ are measures on $(X, \mathscr{M})$ and $a_1, ..., a_n \in [n, \infty)$, then $\sum_j a_j \mu_j$ is a measure on $(X, \mathscr{M})$.

Solution: Let $\mu = \sum_j a_j \mu_j$. First, since each $\mu_j : \mathscr{M} \to [0, \infty]$, and all the $a_j \in [0, \infty)$, it follows that $\mu : \mathscr{M} \to [0, \infty]$. Now,

$$\mu(\emptyset) = \sum_{j=1}^{n} a_j \mu_j(\emptyset) = \sum_{j=1}^{n} a_j \cdot 0 = 0$$

since each μ_j is a measure. Finally let $\{E_i\}_{i=1}^{\infty}$ be a collection of disjoint measurable sets. Then,

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{j=1}^{n} a_j \mu_j \left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{j=1}^{n} a_j \sum_{i=1}^{\infty} \mu_j(E_i) = \sum_{j=1}^{n} \sum_{i=1}^{\infty} a_j \mu_j(E_i).$$

Since all the terms are nonnegative, we can change the order of summation and conclude

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{j=1}^{n} \sum_{i=1}^{\infty} a_j \mu_j(E_i) = \sum_{i=1}^{\infty} \sum_{j=1}^{n} a_j \mu_j(E_i) = \sum_{i=1}^{\infty} \mu(E_i).$$

1.3.8. If $(X, \mathscr{M}, \mu)$ is a measure space and $\{E_j\}_{j=1}^{\infty} \subset \mathscr{M}$ then $\mu(\liminf E_j) \leq \liminf \mu(E_j)$. Also, $\mu(\limsup E_j) \geq \limsup \mu(E_j)$ provided $\mu(\bigcup_j E_j) < \infty$.

Solution: Recall that

$$\liminf E_j = \bigcup_{j=1}^{\infty} \bigcap_{i=j}^{\infty} E_i \quad \text{and} \quad \limsup E_j = \bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} E_i.$$

Now set $F_j = \bigcap_{i=j}^{\infty} E_i$, so that $\liminf E_j = \bigcup_j F_j$. Notice as j increases, we are taking the intersection over fewer sets, so the F_j are increasing. Then by applying monotone convergence of sets, we see that

$$\lim_{j\to\infty} \mu(F_j) = \mu\left(\bigcup_{j=1}^{\infty} F_j\right) = \mu(\liminf E_j).$$

Now it suffices to show that $\lim_{j\to\infty} \mu(F_j) \leq \liminf \mu(E_j)$. By definition of F_j, we see that $F_j \subset E_j$. Hence $\mu(F_j) \leq \mu(E_j)$. Taking liminfs on both sides, and applying the fact that $\lim \mu(F_j)$ exists gives

$$\mu(\liminf E_j) = \lim_{j\to\infty} \mu(F_j) = \liminf_{j\to\infty} \mu(F_j) \leq \liminf_{j\to\infty} \mu(E_j).$$

As for the limsup, we proceed similarly. Let $F_j = \bigcup_{i=j}^{\infty} E_i$. Then as j increases, the F_j are decreasing. The assumption $\mu(\bigcup_j E_j) < \infty$ is precisely that $\mu(F_1) < \infty$. Now apply dominated convergence of sets to conclude that

$$\lim_{j\to\infty} \mu(F_j) = \mu\left(\bigcap_{j=1}^{\infty} F_j\right) = \mu(\limsup E_j).$$

As before, it suffices to show that $\limsup(E_j) \leq \lim \mu(F_j)$. By definition, $E_j \subset F_j$. So, by monotonicity and taking limsups,

$$\limsup \mu(E_j) \leq \limsup \mu(F_j) = \lim \mu(F_j) = \mu(\limsup E_j).$$

1.3.9. If $(X, \mathscr{M}, \mu)$ is a measure space and $E, F \in \mathscr{M}$, then $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$.

Solution: If either E or F has infinite measure, then the above holds trivially since $\mu(E), \mu(F) \leq \mu(E \cup F)$. So, assume WLOG that E and F have finite measure. Then,

$$E = (E \setminus F) \cup (E \cap F) \quad (E \cup F) \setminus F = E \setminus F$$

where the above is a disjoint union. Now, since $F \subset E \cup F$ and $\mu(F) < \infty$ we have

$$\mu(E) = \mu(E \setminus F) + \mu(E \cap F) = \mu((E \cup F) \setminus F) + \mu(E \cap F) = \mu(E \cup F) - \mu(F) + \mu(E \cap F).$$

1.3.10. Given a measure space $(X, \mathscr{M}, \mu)$ and $E \in \mathscr{M}$, define $\mu_E(A) = \mu(A \cap E)$ for $A \in \mathscr{M}$. Then μ_E is a measure.

Solution: It is clear that $\mu_E : \mathscr{M} \to [0, \infty]$. Next let $\{A_j\}_{j=1}^{\infty}$ be a disjoint collection of measurable sets. Then,

$$\mu_E\left(\bigcup_{j=1}^{\infty} A_j\right) = \mu\left(\left(\bigcup_{j=1}^{\infty} A_j\right) \cap E\right) = \mu\left(\bigcup_{j=1}^{\infty} (A_j \cap E)\right).$$

Since the A_j are disjoint, it follows that the $A_j \cap E$ (after throwing away all empty set contributions) are also disjoint. Hence,

$$\mu_E\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j \cap E) = \sum_{j=1}^{\infty} \mu_E(A_j).$$

Finally, we have that $\mu_E(\emptyset) = \mu(\emptyset \cap E) = \mu(\emptyset) = 0$. So, μ_E is a measure.

1.3.11. A finitely additive measure μ is a measure iff monotone convergence for sets holds. If $\mu(X) < \infty$, μ is a measure iff dominated convergence for sets holds.

Solution: The forward direction for each statement holds automatically, as proven in Folland. We need show the reverse directions. First, suppose that monotone convergence for sets holds. We need to show that countable additivity holds. Suppose we have a collection of disjoint measurable sets $\{E_j\}_{j=1}^{\infty}$. Set $F_j = \bigcup_{i=1}^{j} E_i$. Then the F_j are increasing, so monotone convergence applies. Hence,

$$\lim_{j\to\infty} \mu\left(\bigcup_{i=1}^{j} E_i\right) = \mu\left(\bigcup_{i=1}^{\infty} E_i\right).$$

But, μ is finitely additive so that

$$\mu\left(\bigcup_{i=1}^{j} E_i\right) = \sum_{i=1}^{j} \mu(E_i).$$

Putting these two together, we get that

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{j\to\infty} \sum_{i=1}^{j} \mu(E_i) = \sum_{i=1}^{\infty} \mu(E_i).$$

Hence, μ is countably additive.

Next, suppose instead that dominated convergence of sets holds and $\mu(X) < \infty$. Observe that

$$X \setminus \bigcap_{j=1}^{\infty} E_j^c = \bigcup_{j=1}^{\infty} E_j$$

Let $F_j = \bigcap_{i=1}^{j} E_i^c$ it follows that the F_j are decreasing, and since $\mu(X) < \infty$ all measurable sets have finite measure and we can apply dominated convergence of sets. Thus,

$$\lim_{j\to\infty} \mu\left(\bigcap_{i=1}^{j} E_i^c\right) = \mu\left(\bigcap_{i=1}^{\infty} E_i^c\right).$$

Next, for each j we have that

$$\mu\left(X \setminus \bigcap_{i=1}^{j} E_i^c\right) = \mu\left(\bigcup_{i=1}^{j} E_i\right) = \sum_{i=1}^{j} \mu(E_i).$$

Since F_j has finite measure and $F_j \subset X$, we can split the first measure and get

$$\mu(X) - \mu\left(\bigcap_{i=1}^{j} E_i^c\right) = \sum_{i=1}^{j} \mu(E_i).$$

Now, taking limits and recombining the LHS gives

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \mu(X) - \mu\left(\bigcap_{i=1}^{\infty} E_i^c\right) = \mu(X) - \lim_{j\to\infty} \mu\left(\bigcap_{i=1}^{j} E_i^c\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

1.3.12. Let $(X, \mathscr{M}, \mu)$ be a measure space.

a) If $E, F \in \mathscr{M}$ and $\mu(E\Delta F) = 0$ then $\mu(E) = \mu(F)$.
b) Say that $E \sim F$ if $\mu(E\Delta F) = 0$; then $\sim$ is an equivalence relation on $\mathscr{M}$.
c) For $E, F \in \mathscr{M}$, define $\rho(E, F) = \mu(E\Delta F)$. Then $\rho(E, G) \leq \rho(E, F) + \rho(F, G)$, and hence ρ defines a metric on the space $\mathscr{M}/\sim$ of equivalence classes.

Solution:

a) Recall that the symmetric difference is defined by
$$E\Delta F = (E \setminus F) \cup (F \setminus E).$$
Of course, the above is a disjoint union. So,
$$\mu(E\Delta F) = \mu(E \setminus F) + \mu(F \setminus E).$$
If $\mu(E\Delta F) = 0$, then since μ takes on values in $[0, \infty]$ it follows that
$$\mu(E \setminus F) = \mu(F \setminus E) = 0.$$
Now, we can write
$$E = (E \setminus F) \cup (E \cap F) \quad F = (F \setminus E) \cup (E \cap F)$$
where the above are disjoint unions. It follows that
$$\mu(E) = \mu(E \setminus F) + \mu(E \cap F) = \mu(E \cap F) = \mu(F \setminus E) + \mu(E \cap F) = \mu(F).$$

b)
- Reflexivity: Clearly $E\Delta E = \emptyset$ so that $\mu(E\Delta E) = 0$, and $E \sim E$.
- Symmetry: The symmetric difference (as the name suggests) is a symmetric operator. So, if $E \sim F$ then,
$$\mu(F\Delta E) = \mu(E\Delta F) = 0$$
and $F \sim E$.
- Transitivity: Suppose that $E \sim F$ and $F \sim G$. We must show $E \sim G$. It suffices to show that both $E\backslash G$ and $G\backslash E$ are null sets. For the first, observe that $E\backslash G \subset (E\backslash F)\cup(F\backslash G)$. But, both sets on the right are null sets by hypothesis. Similarly, $G\backslash E \subset (G\backslash F)\cup(F\backslash E)$. It follows by monotonicity that both $E \setminus G$ and $G \setminus E$ are null sets.

c) I think this needs the additional assumption $\mu(X) < \infty$. If not, then $\rho(X, \emptyset) = \mu(X\Delta\emptyset) = \mu(X)$, which can be infinite; yet, metrics only take finite nonnegative values. We now prove ρ is a metric.
- Clearly under this assumption $\rho : \mathscr{M}/\sim \to [0, \infty)$. Now assume $\rho(E, F) = 0$; then $\mu(E\Delta F) = 0$ and $E \sim F$. So E, F are in the same equivalence class.
- Since the symmetric difference is symmetric,
$$\rho(E, F) = \mu(E\Delta F) = \mu(F\Delta E) = \rho(F, E).$$
- Recall the subset containments used in the proof of transitivity above. Then, $E\Delta G \subset (E\Delta F) \cup (F\Delta G)$. By monotonicity,
$$\rho(E, G) = \mu(E\Delta G) \le \mu(E\Delta F) + \mu(F\Delta G) = \rho(E, F) + \rho(F, G).$$

1.3.13. Every σ-finite measure is semifinite.

Solution: Recall that a measure μ is semifinite if for every measurable set E with $\mu(E) = 0$, there exists a measurable set $F \subset E$ such that $0 < \mu(F) < \infty$.

Since μ is σ-finite, we can write X as
$$X = \bigcup_{j=1}^{\infty} E_j$$
where all the E_j are such that $\mu(E_j) < \infty$. Now suppose E is such that $\mu(E) = \infty$. Let $F_j = E \cap E_j$. Then,
$$E = \bigcup_{j=1}^{\infty} F_j.$$
Suppose for the sake of contradiction that all the F_j have zero measure. Then,
$$\mu(E) \le \sum_{j=1}^{\infty} \mu(F_j) = 0,$$
a contradiction. So at least one of the F_j has nonzero measure – use this as the desired F.

1.3.14. If μ is a semifinite measure and $\mu(E) = \infty$, for any $C > 0$ there exists $F \subset E$ with $C < \mu(F) < \infty$.

Solution: Since μ is semifinite we may find an $F_1 \subset E$ with $0 < \mu(F_1) < \infty$. If $C < \mu(F_1)$, we are done. So, suppose not. Let $E_1 = E \setminus F_1$. Since $F_1 \subset E$ with finite measure,

$$\mu(E_1) = \mu(E \setminus F_1) = \mu(E) - \mu(F_1) = \infty.$$

We may therefore apply semifiniteness to E_1 and obtain $F_2 \subset E_1 \subset E$ with $0 < \mu(F_2) < \infty$. Note that $F_1 \neq F_2$ since $F_1 \nsubseteq E_1$. Now,

$$0 < \mu(F_1) < \mu(F_1 \cup F_2) < \mu(E).$$

If $C < \mu(F_1 \cup F_2)$ we are done. So, continue inductively in this way. Given F_n define $E_n = E \setminus F_n$. Then extract $F_{n+1} \subset E_n \subset E$ with $0 < \mu(F_{n+1}) < \infty$. By construction, all of the F_i are distinct (in fact, disjoint). Then,

$$0 < \mu(F_1) < \mu(F_1 \cup F_2) < ... < \mu(F_1 \cup ... \cup F_{n+1}) < \mu(E)$$

Eventually this must be greater than C. Define $F = F_1 \cup ... \cup F_{n+1}$ when this occurs.

1.3.15. Given a measure μ on $(X, \mathscr{M})$, define μ_0 on $\mathscr{M}$ by $\mu_0(E) = \sup\{\mu(F) \mid F \subset E \text{ and } \mu(F) < \infty\}$

a) μ_0 is a semifinite measure. It is called the **semifinite part** of μ. (Use Exercise 1.3.14.)
b) If μ is semifinite then $\mu = \mu_0$.
c) There is a measure ν on $\mathscr{M}$ (in general, not unique) which assumes only the values 0 and ∞ such that $\mu = \mu_0 + \nu$.

Solution:

a) We first show that μ_0 is a measure. The only subset of the empty set is the empty set, so that $\mu_0(\emptyset) = 0$. Now let $\{E_j\}_{j=1}^{\infty}$ be a collection of disjoint measurable sets. Let $E = \bigcup_j E_j$. For a measurable set $F \subset E$ and $\mu(F) < \infty$ we set $F_j = F \cap E_j$. Then, this is also a collection of disjoint sets, and each F_j is such that $\mu(F_j) < \infty$. Since $F \subset E$, we see that $\bigcup_j F_j = F$. Hence,

$$\mu(F) = \sum_{j=1}^{\infty} \mu(F_j).$$

It is also easy to see that $\mu(F_j) \leq \mu_0(E_j)$ since $F_j \subset E_j$ and $\mu(F_j) < \infty$. Thus,

$$\mu(F) \leq \sum_{j=1}^{\infty} \mu_0(E_j).$$

Since this bound holds for all $F \subset E$ with $\mu(F) < \infty$, we have that

$$\mu_0(E) \leq \sum_{j=1}^{\infty} \mu_0(E_j).$$

If $\mu_0(E) = \infty$ then

$$\sum_{j=1}^{\infty} \mu_0(E_j) \leq \mu_0(E)$$

trivially. So suppose E is such that $\mu_0(E) < \infty$. Now, every subset of E_j with finite measure is also a subset of E with finite measure. Thus,

$$\mu_0(E_j) \leq \mu_0(E)$$

for all j. Let $\epsilon > 0$ and let $F_j \subset E_j$ be such that

$$\mu_0(E_j) - \epsilon \cdot 2^{-j} \leq \mu(F_j).$$

Such a set exists by properties of the supremum. Set $F^n = \bigcup_{j=1}^n F_j$. Then $\mu(F_n) \le \sum_{j=1}^n \mu(F_j) < \infty$ since we have a finite sum of finite numbers. Each $F^n \subset E_j$ so that the F_j are disjoint and $\mu(F^n) \le \mu_0(E)$. Finally,

$$\mu_0(E) \ge \mu\left(\bigcup_{j=1}^n F_j\right) = \sum_{j=1}^n \mu(F_j) \ge \sum_{j=1}^n \left(\mu_0(E_j) - \epsilon \cdot 2^{-j}\right) \ge \sum_{j=1}^n \mu_0(E_j) - \epsilon.$$

Now taking $n \to \infty$ gives the result.

It is immediate that μ_0 is semifinite. If E is such that $\mu_0(E) = \infty$, then there exists some $F \subset E$ with $0 < \mu(F) < \infty$. If all such F had $\mu(F) = 0$ then $\mu_0(E) = 0$, a contradiction.

b) First, if E is such that $\mu(E) < \infty$ then $\mu_0(E) = \mu(E)$ – all measurable subsets of E have finite measure bounded by $\mu(E)$, including E for which equality holds. Now suppose $\mu(E) = \infty$. Since μ is semifinite, for any $C > 0$ there exists $F_C \subset E$ with $C < \mu(F_C) < \infty$. Then $\mu_0(E) \ge \mu(F_C) > C$ for any $C > 0$. It follows that $\mu_0(E) = \infty$.

c) We saw at the beginning of part b) that μ_0 and μ differ only for sets with infinite measure. If $\mu(E) = \infty$, then clearly $\mu(E) \ge \mu_0(E)$. If we define $\nu(E) = \infty$ for those sets with μ-infinite measure, then clearly $\mu(E) = \mu_0(E) + \nu(E)$. The issue is that ν is not a measure in this way; of course right now it is not even defined on $\mathscr{M}$. A naive remedy is as follows: We define $\nu : \mathscr{M} \to \{0, \infty\}$ by

$$\nu(E) = \begin{cases} 0 & \text{if } \mu(E) < \infty \\ \infty & \text{if } \mu(E) = \infty \end{cases}$$

Once more, it is clear that $\mu(E) = \mu_0(E) + \nu(E)$. We now show that ν is a measure. Since $\mu(\emptyset) < \infty$, $\nu(\emptyset) = 0$. Now let $\{E_j\}_{j=1}^\infty$ be a disjoint collection of measurable sets. However, it is possible for $E = \bigcup E_j$ to have infinite measure while each E_j has finite measure, in which case an equality like

$$\nu(E) = \sum_{j=1}^\infty \nu(E_j)$$

is impossible. So, we must be a little smarter. We need a stronger property that holds under unions – σ-finiteness does exactly this. If all the E_j are σ-finite, then so too is $E = \bigcup_j E_j$. Note that any finite measure set is automatically σ-finite. So we redefine ν as

$$\nu(E) = \begin{cases} 0 & \text{if } E \text{ is } \sigma\text{-finite} \\ \infty & \text{if } E \text{ is not } \sigma\text{-finite} \end{cases}$$

Clearly the empty set is σ-finite since it is finite, so that $\nu(\emptyset) = 0$. Now let $\{E_i\}_{i=1}^\infty$ be a disjoint collection of measurable sets. If all the E_i are σ-finite, then as observed before $E = \bigcup_j E_j$ is σ-finite. Hence,

$$\nu(E) = 0 = \sum_{j=1}^\infty \nu(E_j).$$

Now suppose any one of the E_j is not σ-finite. Then, E cannot be σ-finite. Assume it is, and write $E = \bigcup_j F_j$ where all the F_j have finite measure. Then,

$$E_j = E_j \cap E = E_j \cap \left(\bigcup_{i=1}^\infty F_i\right) = \bigcup_{i=1}^\infty (E_j \cap F_i)$$

All of the $E_j \cap F_i$ have finite measure so that E_j is σ-finite for all j, a contradiction. Thus if at least one of the E_j is not σ-finite, then E is not. It follows that

$$\nu(E) = \infty = \sum_{j=1}^\infty \nu(E_j).$$

We still need to check that $\mu = \mu_0 + \nu_0$. If E has finite measure then equality holds. Now assume that $\mu(E) = \infty$ but is σ-finite. By combining Exercises 1.3.13 and 1.3.14, we see that $\mu_0(E) = \infty$ (similar to what we did in b).), and equality still holds. The remaining case is when $\mu(E) = \infty$ and is not σ-finite. Then, it does not matter what $\mu_0(E)$ is, since $\nu(E) = \infty$.

1.3.16. Let $(X, \mathscr{M}, \mu)$ be a measure space. A set $E \subset X$ is called **locally measurable** if $E \cap A \in \mathscr{M}$ for all $A \in \mathscr{M}$ such that $\mu(A) < \infty$. Let $\tilde{\mathscr{M}}$ be the collection of all locally measurable sets. Clearly $\mathscr{M} \subset \tilde{\mathscr{M}}$; if $\mathscr{M} = \tilde{\mathscr{M}}$, then μ is called **saturated**.

a) If μ is σ-finite, then μ is saturated.
b) $\tilde{\mathscr{M}}$ is a σ-algebra.
c) Define $\tilde{\mu}$ on $\tilde{\mathscr{M}}$ by $\tilde{\mu}(E) = \mu(E)$ if $E \in \mathscr{M}$ and $\tilde{\mu}(E) = \infty$ otherwise. Then $\tilde{\mu}$ is a saturated measure on $\tilde{\mathscr{M}}$, called the saturation of μ.
d) If μ is complete, so is $\tilde{\mu}$.
e) Suppose that μ is semifinite. For $E \in \tilde{\mathscr{M}}$, define $\underline{\mu}(E) = \sup\{\mu(A) \mid A \in \mathscr{M} \text{ and } A \subset E\}$. Then $\underline{\mu}$ is a saturated measure on $\tilde{\mathscr{M}}$ that extends μ.
f) Let X_1, X_2 be disjoint uncountable sets, $X = X_1 \cup X_2$, and $\mathscr{M}$ the σ-algebra of countable or co-countable sets in X. Let μ_0 be counting measure on $\mathcal{P}(X_1)$, and define μ on $\mathscr{M}$ by $\mu(E) = \mu_0(E \cap X_1)$. Then μ is a measure on $\mathscr{M}$, $\tilde{\mathscr{M}} = \mathcal{P}(X)$, and in the notation of parts c) and e), $\tilde{\mu} \neq \underline{\mu}$.

Solution:

a) We must show that each locally measurable set $E \subset X$ is in fact measurable. Since μ is σ-finite, we may write $X = \bigcup_j F_j$ where all the F_j have finite measure. Let E be locally measurable. Then $E \cap F_j \in \mathscr{M}$ for all j, since the F_j are measurable with $\mu(F_j) < \infty$. But,
$$E = E \cap X = E \cap \left(\bigcup_{j=1}^{\infty} F_j \right) = \bigcup_{j=1}^{\infty} (E \cap F_j)$$
so that $E \in \mathscr{M}$. Hence $\tilde{\mathscr{M}} \subset \mathscr{M}$. The reverse inclusion is always true.
b) First let $\{E_j\}_{j=1}^{\infty}$ be a collection of sets in $\tilde{\mathscr{M}}$. Let A be a measurable set with finite measure. Then,
$$\left(\bigcup_{j=1}^{\infty} E_j \right) \cap A = \bigcup_{j=1}^{\infty} (E_j \cap A) \in \mathscr{M}$$
since each $E_j \cap A \in \mathscr{M}$ by local measurability of the E_j. Now let $E \in \tilde{\mathscr{M}}$ and A as before. Then,
$$E^c \cap A = (E \cap A)^c \cap A \in \mathscr{M}$$
since local measurability of E guarantees $E \cap A \in \mathscr{M}$.
c) We first show $\tilde{\mu}$ is a measure. Evidently it is a map $\mathscr{M} \to [0, \infty]$. Since $\emptyset \in \mathscr{M}$, we have that $\tilde{\mu}(\emptyset) = \mu(\emptyset) = 0$. Now let $\{E_j\}_{j=1}^{\infty}$ be a collection of disjoint sets in $\tilde{\mathscr{M}}$. Let $E = \bigcup_j E_j$. A priori we have three cases
   - Case 1: E is measurable with $\mu(E) < \infty$.
   - Case 2: E is measurable with $\mu(E) = \infty$.
   - Case 3: E is locally measurable, but not measurable.

   Observe that if all the $E_j \in \mathscr{M}$ then we are done, since
$$\tilde{\mu}(E) = \mu(E) = \sum_{j=1}^{\infty} \mu(E_j) = \sum_{j=1}^{\infty} \tilde{\mu}(E_j).$$
   Let us examine the case when E is measurable and $\mu(E) < \infty$. We show that any E_j is measurable, which is sufficient by the above. Note that $E_j \subset E$ and $\mu(E) < \infty$, so we may

use local measurability of E_j to test against E. Thus,

$$E_j = E_j \cap E \in \mathscr{M}.$$

For the remaining cases, we may suppose that we have some nonzero amount of nonmeasurable E_j. If E is measurable with $\mu(E) = \infty$ or is nonmeasurable then we are done as

$$\tilde{\mu}(E) = \infty = \sum_{j=1}^{\infty} \tilde{\mu}(E_j)$$

and the last equality holds since at least one of the E_j is nonmeasurable.

We now show this measure is saturated. That is, if $E \subset X$ is such that $E \cap A \in \tilde{\mathscr{M}}$ for all $A \in \tilde{\mathscr{M}}$ with $\tilde{\mu}(A) < \infty$ then actually $E \in \tilde{\mathscr{M}}$. Let $A \in \mathscr{M}$ be such that $\mu(A) < \infty$ – we need to show that $E \cap A \in \mathscr{M}$. Then $A \in \tilde{\mathscr{M}}$ and $\tilde{\mu}(A) = \mu(A) < \infty$. By assumption, $E \cap A \in \tilde{\mathscr{M}}$. Now we test this against A. Thus,

$$E \cap A = (E \cap A) \cap A \in \mathscr{M}.$$

d) For $\tilde{\mu}$ to be complete, it must be that if $E \in \tilde{\mathscr{M}}$ is such that $\tilde{\mu}(E) = 0$ then all $F \subset E$ are measurable and $\tilde{\mu}(F) = 0$. But this is true since if $\tilde{\mu}(E) = 0$, in fact $E \in \mathscr{M}$. Then by completeness of μ, all $F \subset E$ are μ-null. Hence,

$$\tilde{\mu}(F) = \mu(F) = 0.$$

e) The definition of $\overline{\mu}$ looks awfully like that of μ_0 in Exercise 1.3.15. We show that in fact

$$\underline{\mu}(E) = \sup\{\mu(A) \mid A \subset E,\ A \in \mathscr{M},\ \mu(A) < \infty\}.$$

Clearly since $\{A \in \mathscr{M} \mid A \subset E,\ \mu(A) < \infty\} \subset \{A \in \mathscr{M} \mid A \subset E\}$ we have

$$\sup\{\mu(A) \mid A \subset E,\ A \in \mathscr{M},\ \mu(A) < \infty\} \leq \sup\{\mu(A) \mid A \subset E,\ A \in \mathscr{M}\} = \underline{\mu}(E).$$

If all such A actually have finite measure, then we have equality instead. So suppose some A have infinite measure. Clearly then $\underline{\mu}(E) = \infty$. On the other hand, choose one such A and call it A'. By semifiniteness we have by Exercise 1.3.14 that for any $C > 0$ there exists $A_C \subset A'$ with $C < \mu(A_C) < \infty$. It follows that

$$\sup\{\mu(B) \mid B \subset A',\ B \in \mathscr{M},\ \mu(B) < \infty\} = \infty = \mu(A').$$

But then,

$$\begin{aligned} \infty = \underline{\mu}(E) &\geq \sup\{\mu(A) \mid A \subset E,\ A \in \mathscr{M},\ \mu(A) < \infty\} \\ &\geq \sup\{\mu(B) \mid B \subset A',\ B \in \mathscr{M},\ \mu(B) < \infty\} = \infty \end{aligned}$$

since, for a fixed measurable subset A of E with $\mu(A) < \infty$, any measurable subset B of A is such that $B \subset E$ with $\mu(B) < \infty$. We simply choose A to be our special set A'. Hence, we actually have equality throughout.

We now show that $\underline{\mu}(E)$ is a measure on $\tilde{\mathscr{M}}$. It is clear that $\underline{\mu} : \tilde{\mathscr{M}} \to [0, \infty]$. Furthermore, any subset of the empty set is empty so that $\underline{\mu}(\emptyset) = 0$. Now let $\{E_j\}_{j=1}^{\infty}$ be a disjoint collection of locally measurable sets. Let $E = \bigcup_j E_j$. Choose $F \subset E$ with $\mu(F) < \infty$. Remember the remark earlier of the similarity of $\underline{\mu}$ with μ_0 in Exercise 1.3.15? Essentially the same proof carries over here: we repeat it for completeness.
By local measurability of the E_j, we have that $F_j = E_j \cap F \in \mathscr{M}$. Furthermore $F_j \subset E$ with $\mu(F_j) < \infty$. It follows that

$$\mu(F_j) \leq \underline{\mu}(E_j)$$

for all j. Summing over both sides gives

$$\mu(F) \leq \sum_{j=1}^{\infty} \underline{\mu}(E_j)$$

This bound holds for all $F \subset E$ with $\mu(F) < \infty$, so that

$$\underline{\mu}(E) = \sup\{\mu(F) \mid F \subset E,\ \mu(F) < \infty\} \leq \sum_{j=1}^{\infty} \underline{\mu}(E_j).$$

If $\underline{\mu}(E) = \infty$ then the reverse inequality holds trivially. So we may assume E is such that $\underline{\mu}(E) < \infty$. Earlier we showed an obvious monotonicity property of $\underline{\mu}$, so we have for any j

$$\underline{\mu}(E_j) \leq \underline{\mu}(E).$$

Now let $\epsilon > 0$ and choose $F_j \subset E_j$ such that $\underline{\mu}(E_j) - \epsilon \cdot 2^{-j} \leq \mu(F_j) < \infty$. Since the F_j have finite measure and are subsets of E_j, they are finite measure subsets of E. In particular, for any n, $F^n = \bigcup_{j=1}^n F_j \subset E$ with finite measure. These are used in the sup. Thus,

$$\sum_{j=1}^{n} \underline{\mu}(E_j) - \epsilon < \sum_{j=1}^{n} \left(\underline{\mu}(E_j) - \epsilon \cdot 2^{-j}\right) \leq \mu(F^n) \leq \underline{\mu}(E).$$

Taking $n \to \infty$ gives the result.

By monotonicty of μ, it follows that if $E \in \mathscr{M}$ then $\mu(A) \leq \mu(E)$ for all $A \subset E$ measurable. On the other hand, E is a subset of E with $\mu(E) = \mu(E)$. Hence $\underline{\mu}(E) = \mu(E)$ when $E \in \mathscr{M}$, so $\underline{\mu}$ extends μ.

Finally we show $\underline{\mu}$ is saturated. Let $E \subset X$ be such that if $A \in \tilde{\mathscr{M}}$ with $\underline{\mu}(A) < \infty$ then $E \cap A \in \tilde{\mathscr{M}}$. We must show that $E \in \tilde{\mathscr{M}}$. The proof is similar to those before. Let $A \in \mathscr{M}$ be such that $\mu(A) < \infty$. Since $\underline{\mu}$ extends μ, we have $\underline{\mu}(A) = \mu(A) < \infty$. Hence by local measurability of E with respect to $\underline{\mu}$, $E \cap A \in \tilde{\mathscr{M}}$. Now we can test local measurability of this set with A itself and get

$$E \cap A = (E \cap A) \cap A \in \mathscr{M}$$

for any measurable A with $\mu(A) < \infty$. It follows that E is locally measurable with respect to μ; that is $E \in \tilde{\mathscr{M}}$.

f) Clearly $\mu : \mathscr{M} \to [0, \infty]$ is well defined. First, $\mu(\emptyset) = \mu_0(\emptyset \cap X) = \mu_0(\emptyset) = 0$. Next let $\{E_j\}_{j=1}^{\infty}$ be a collection of disjoint sets in $\mathscr{M}$. Set $E = \bigcup_j E_j$. Then,

$$\mu(E) = \mu_0\left(\left(\bigcup_{j=1}^{\infty} E_j\right) \cap X_1\right) = \mu_0\left(\bigcup_{j=1}^{\infty} (E_j \cap X_1)\right) = \sum_{j=1}^{\infty} \mu_0(E_j \cap X_1) = \sum_{j=1}^{\infty} \mu(E_j)$$

since μ_0 is a measure. Hence, μ is a measure.

Let E be any subset of X. We show that E is locally measurable. Suppose that $A \in \mathscr{M}$ such that $\mu(A) < \infty$. Then A is either countable or A^c is countable. If A is countable, then $E \cap A$ is countable, and hence $E \cap A \in \mathscr{M}$. Now suppose A is co-countable. By definition,

$$\mu(A) = \mu_0(A \cap X_1) < \infty.$$

Since μ_0 is counting measure, it follows that the portion of A residing in X_1 is finite. But X_1 is uncountable, so that $X_1 \setminus A$ is uncountable (A being finite in X_1). It follows that A cannot be co-countable.

To conclude, consider $E = X_2$. Since E is uncountable and $E^c = X_1$ is uncountable, $E \notin \mathscr{M}$. It follows that $\tilde{\mu}(E) = \infty$. On the other hand, X_1 and X_2 are disjoint so that any μ-measurable subset $A \subset E$ is such that

$$\mu(A) \leq \mu(E) = \mu_0(X_1 \cap X_2) = 0.$$

Hence, $\underline{\mu}(E) = 0$.

1.4. **Outer Measures.**

1.4.17. If μ^* is an outer measure on X and $\{A_j\}_{j=1}^\infty$ is a sequence of disjoint μ^*-measurable sets, then $\mu^*(E \cap (\bigcup_j A_j)) = \sum_j \mu^*(E \cap A_j)$ for any $E \subset X$.

Solution: Since μ^* is an outer measure, it is countably subadditive. Hence,

$$\mu^*\left(E \cap \left(\bigcup_{j=1}^\infty A_j\right)\right) = \mu^*\left(\bigcup_{j=1}^\infty (E \cap A_j)\right) \leq \sum_{j=1}^\infty \mu^*(E \cap A_j).$$

So, we must show the reverse inequality. Set $B_n = \bigcup_{j=1}^n A_j$ and $B_0 = \emptyset$. Now applying Carathéodory's condition to $E \cap B_n$, we have

$$\begin{aligned} \mu^*(E \cap B_n) &= \mu^*((E \cap B_n) \cap A_n) + \mu^*((E \cap B_n) \cap A_n^c) \\ &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}) \end{aligned}$$

since A_n is disjoint from all the other A_j. Inductively we see that

$$\mu^*(E \cap B_n) = \sum_{j=1}^n \mu^*(E \cap A_n).$$

By monotonicity, if $B = \bigcup_j A_j$ we have

$$\mu^*\left(E \cap \left(\bigcup_{j=1}^\infty A_j\right)\right) = \mu^*(E \cap B) \geq \mu^*(E \cap B_n) = \sum_{j=1}^n \mu^*(E \cap A_n)$$

for all n. Hence, taking $n \to \infty$ we obtain the reverse inequality.

1.4.18. Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, $\mathcal{A}_\sigma$ the collection of countable unions of sets in $\mathcal{A}$, and $\mathcal{A}_{\sigma\delta}$ the collection of countable intersections of sets in $\mathcal{A}_\sigma$. Let μ_0 be a premeasure on $\mathcal{A}$ and μ^* the induced outer measure.

a) For any $E \subset X$ and $\epsilon > 0$ there exists $A \in \mathcal{A}_\sigma$ with $E \subset A$ and $\mu^*(A) \leq \mu^*(E) + \epsilon$.
b) If $\mu^*(E) < \infty$, then E is μ^*-measurable iff there exists $B \in \mathcal{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \setminus E) = 0$.
c) If μ_0 is σ-finite, the restriction $\mu^*(E) < \infty$ in b) is superfluous.

Solution:

a) Recall that the outer measure μ^* induced by μ_0 is

$$\mu^*(E) = \inf\left\{\sum_{j=1}^\infty \mu_0(A_j) \;\middle|\; E \subset \bigcup_{j=1}^\infty A_j,\ \{A_j\}_{j=1}^\infty \subset \mathcal{A}\right\}.$$

So, for any $\epsilon > 0$ there exists a covering $E \subset \bigcup_j A_j$ of sets $A_j \in \mathcal{A}$ such that

$$\sum_{j=1}^\infty \mu_0(A_j) \leq \mu^*(E) + \epsilon$$

(this is just by properties of the inf). But, outer measures are subadditive and the outer measure of a set in $\mathcal{A}$ coincides with its premeasure. So,

$$\mu^*(A) \leq \sum_{j=1}^\infty \mu^*(A_j) = \sum_{j=1}^\infty \mu_0(A_j) \leq \mu^*(E) + \epsilon$$

where $A = \bigcup_j A_j$. By definition, $A \in \mathcal{A}_\sigma$.

b) Suppose first that E is μ^*-measurable. Then for each k there exists a set $B_k \in \mathcal{A}_\sigma$ such that $\mu^*(B_k) \leq \mu^*(E) + 1/k$. Set $B = \bigcap B_k$. This set is the desired B, as we will now show. We know that $E \subset B \subset B_k$ for all k, thus

$$\mu^*(E) \leq \mu^*(B) \leq \mu^*(B_k) \leq \mu^*(E) + \frac{1}{k}$$

and we conclude $\mu^*(E) = \mu^*(B)$. Since outer measures are subadditive, we cannot immediately conclude that $\mu^*(B) - \mu^*(E) = \mu^*(B \setminus E)$ whenever $E \subset B$. To do this, we use μ^*-measurability of E. Applying Carathéodory's condition to B gives

$$\mu^*(B) = \mu^*(E) + \mu^*(E^c \cap B) = \mu^*(E) + \mu^*(B \setminus E)$$

Since $\mu^*(E) < \infty$, we can subtract it and conclude $\mu^*(B \setminus E) = 0$.

Now suppose there exits a $B \in \mathcal{A}_{\sigma\delta}$ such that $E \subset B$ and $\mu^*(B \setminus E) = \mu^*(B \cap E^c) = 0$. Similar to above, for each k we can find an $A_k \in \mathcal{A}_\sigma$ such that $(B \setminus E) \subset A_k$ and $\mu^*(A_k) \leq \mu^*(B \setminus E) + 1/k = 1/k$. Set $A = \bigcap_k A_k$, so that $\mu^*(A) = 0$. But A is μ^*-measurable, so by completeness of μ^* we have that $B \setminus E$ is measurable. Since $B \in \mathcal{A}_{\sigma\delta}$, it is also μ^*-measurable. Yet,

$$E = (B^c \cup E) \cap E = (B \cap E^c)^c \cap E = (B \setminus E)^c \cap B$$

since $E \subset B$. It follows that E is μ^*-measurable. Notice we did not need the hypothesis $\mu^*(E) < \infty$ for this direction.

c) We need to show the forward direction of the above without the assumption that $\mu^*(E) < \infty$. Since μ_0 is σ-finite, we can write $X = \bigcup X_j$ where each X_j is such that $\mu_0(X_j) < \infty$. Now set $E_j = E \cap X_j$ which has finite μ^* measure. Also, $\bigcup_j E_j = E$. Fix a natural number k. For each j, using a), we can find a $C_j \in \mathcal{A}_\sigma$ such that $E_j \subset C_j$ and

$$\mu^*(C_j) \leq \mu^*(E_j) + \frac{1}{k \cdot 2^j}.$$

Since E_j is μ^*-measurable,

$$\mu^*(C_j) = \mu^*(C_j \cap E_j) + \mu^*(C_j \cap E_j^c) = \mu^*(E_j) + \mu^*(C_j \setminus E_j)$$

Because $\mu^*(E_j)$ is finite, we can rearrange this and get

$$\mu^*(C_j \setminus E_j) = \mu^*(C_j) - \mu^*(E_j) \leq \mu^*(E_j) + \frac{1}{k \cdot 2^j} - \mu^*(E_j) = \frac{1}{k \cdot 2^j}.$$

Now set $B_k = \bigcup_j C_j \in \mathcal{A}_\sigma$ (it is still a countable union of sets in $\mathcal{A}$, since each C_j is). Now note that $E^c \subset E_j^c$ for any j, so that

$$\mu^*(B_k \setminus E) = \mu^*\left(\bigcup_{j=1}^{\infty}(C_j \cap E^c)\right) \leq \mu^*\left(\bigcup_{j=1}^{\infty}(C_j \cap E_j^c)\right) \leq \sum_{j=1}^{\infty} \mu^*(C_j \setminus E_j) \leq \frac{1}{k}.$$

Thus, for each k we can find a $B_k \in \mathcal{A}_\sigma$ such that

$$\mu^*(B_k \setminus E) \leq \frac{1}{k}.$$

Set $B = \bigcap_k B_k$. Then $B \in \mathcal{A}_{\sigma\delta}$, and by monotonicity

$$\mu^*(B \setminus E) \leq \mu^*(B_k \setminus E) \leq \frac{1}{k}$$

for all k. Hence, $\mu^*(B \setminus E) = 0$ as desired.

1.4.19. Let μ^* be an outer measure on X induced from a finite premeasure μ_0. If $E \subset X$, define the inner measure of E to be $\mu_*(E) = \mu_0(X) - \mu^*(E^c)$. Then E is μ^*-measurable iff $\mu^*(E) = \mu_*(E)$. (Use Exercise 1.4.18).

Solution: First if E is μ^* measurable, since μ^* is induced by μ_0 they agree on μ^*-measurable sets. Thus,

$$\mu_0(X) = \mu_0(E) + \mu_0(E^c) = \mu^*(E) + \mu^*(E^c)$$

Since μ_0 is finite,

$$\mu^*(E) = \mu_0(X) - \mu^*(E^c) = \mu_*(E)$$

as desired.
Conversely, by 18a) for each n there exists an $A_n \in \mathcal{A}_\sigma$ such that $E \subset A_n$ and $\mu^*(A_n) \leq \mu^*(E)+1/n$. Since each A_n is μ^*-measurable,

$$\mu^*(E^c) = \mu^*(E^c \cap A_n) + \mu^*(E^c \cap A_n^c) = \mu^*(A_n \setminus E) + \mu^*(A_n^c).$$

On the other hand, by assumption and since A_n is μ^*-measurable,

$$\mu^*(E) = \mu_*(E) = \mu_0(X) - \mu^*(E^c) = \mu^*(A_n) + \mu^*(A_n^c) - \mu^*(E^c).$$

Therefore,

$$\mu^*(A_n \setminus E) \leq \mu^*(E^c) - \mu^*(A_n^c) = \mu^*(A_n) - \mu^*(E) \leq \frac{1}{n}.$$

Set $A = \bigcap_n A_n$. Then,

$$\mu^*(A \setminus E) \leq \mu^*(A_n \setminus E) \leq \frac{1}{n}$$

for all n. It follows that $\mu^*(A \setminus E) = 0$ where $A \in \mathcal{A}_{\sigma\delta}$. Hence by 18b), E is μ^*-measurable.

1.4.20. Let μ^* be an outer measure on X, $\mathscr{M}^*$ the σ-algebra of μ^*-measurable sets, $\overline{\mu} = \mu^*|_{\mathscr{M}^*}$, and μ^+ the outer measure induced by $\overline{\mu}$ as in (1.12) (with $\overline{\mu}$ and $\mathscr{M}^*$ replacing μ_0 and $\mathcal{A}$).

a) If $E \subset X$, we have $\mu^*(E) \leq \mu^+(E)$, with equality iff there exists $A \in \mathscr{M}^*$ with $E \subset A$ and $\mu^*(A) = \mu^*(E)$.
b) If μ^* is induced from a premeasure, then $\mu^* = \mu^+$. (Use Exercise 18a.)
c) If $X = \{0, 1\}$, there exists an outer measure μ^* on X such that $\mu^* \neq \mu^+$.

Solution:

a) First, let us explicitly write out what μ^+ is:

$$\mu^+(E) = \inf \left\{ \sum_{j=1}^\infty \overline{\mu}(A_j) \;\middle|\; E \subset \bigcup_{j=1}^\infty A_j, A_j \in \mathscr{M}^* \right\}$$

Let $E \subset X$ and $\epsilon > 0$. Then there exists a covering of $E \subset \bigcup_j A_j$ of sets $A_j \in \mathscr{M}^*$ such that

$$\sum_{j=1}^\infty \overline{\mu}(A_j) \leq \mu^+(E) + \epsilon$$

By subadditivity of μ^*,

$$\mu^*(E) \leq \sum_{j=1}^\infty \mu^*(A_j) = \sum_{j=1}^\infty \overline{\mu}(A_j) \leq \mu^+(E) + \epsilon$$

since μ^* and $\overline{\mu}$ coincide on $\mathscr{M}^*$. It follows that $\mu^*(E) \leq \mu^+(E)$.
If there exists $A \in \mathscr{M}^*$ with $E \subset A$ and $\mu^*(A) = \mu^*(E)$, then A is a valid covering of E. Hence,

$$\mu^+(E) \leq \overline{\mu}(A) = \mu^*(A_j) = \mu^*(E)$$

so that equality holds. Next suppose $\mu^*(E) = \mu^+(E)$. For each n there exists a covering $\{A_{j,n}\}_{j=1}^\infty \subset \mathscr{M}^*$ of E such that

$$\sum_{j=1}^\infty \overline{\mu}(A_{j,n}) \leq \mu^+(E) + \frac{1}{n}$$

Now set $A_n = \bigcup_j A_{j,n}$, which is μ^*-measurable. Then,

$$\mu^*(A_n) = \overline{\mu}(A_n) \leq \sum_{j=1}^\infty \overline{\mu}(A_{j,n}) \leq \mu^+(E) + \frac{1}{n}.$$

Finally set $A = \bigcap_n A_n$, which is also μ^*-measurable. Then for each n,

$$\mu^*(A) \leq \mu^*(A_n) \leq \mu^+(E) + \frac{1}{n}.$$

It follows that $\mu^*(A) \leq \mu^+(E) = \mu^*(E)$, where of course the last equality is by assumption. Now, by construction $E \subset A_n$ for all n. Hence $E \subset A$. By subadditivity, $\mu^*(E) \leq \mu^*(A)$. It follows that $\mu^*(A) = \mu^*(E)$, as desired.

b) We need only show the reverse equality. Supposing that μ^* is induced by a premeasure (say μ_0 with some algebra $\mathcal{A}$), for $E \subset X$ and $\epsilon > 0$ we can find an $A \in \mathcal{A}_\sigma$ such that $E \subset A$ and

$$\mu^*(A) \leq \mu^*(E) + \epsilon.$$

Now, the σ-algebra generated by $\mathcal{A}$ is a subset of $\mathscr{M}^*$. So, $A \in \mathscr{M}^*$. Then A is a valid covering of E used in the definition of μ^+. It follows that

$$\mu^+(E) \leq \overline{\mu}(A) = \mu^*(A) \leq \mu^*(E) + \epsilon$$

applying the fact that $\overline{\mu}$ and μ^* coincide on $\mathcal{M}^*$. Since the above holds for all $\epsilon > 0$, we are done.

c) There are four subsets of X, so μ^* and μ^+ only take at most four values. Of course since both are outer measures, $\mu^*(\emptyset) = \mu^+(\emptyset) = 0$. Define μ^* by

$$\begin{aligned} \mu^*(\emptyset) &= 0 \qquad & \mu^*(\{0\}) &= a \\ \mu^*(\{1\}) &= b \qquad & \mu^*(X) &= c \end{aligned}$$

where a, b, c are to be determined. First, let us impose conditions on a, b, c necessary for μ^* to be an outer measure. For monotonicity to hold, we must have that

$$0 \leq a \leq c \text{ and } 0 \leq b \leq c$$

For subadditivity to hold, we must check, in theory, 15 conditions. However, note that if $\bigcup_j A_j = A_i$ for some i, then subadditivity holds no matter what; in particular, we can ignore any case where one of the A_j is X. Also, any case where we choose only one set is trivially true. Furthermore, any case with any nonzero amount of $A_j = \emptyset$ can be reduced to one without any empty sets. So, we really only have one case:

$$c = \mu^*(X) = \mu^*(\{0\} \cup \{1\}) \leq \mu^*(\{0\}) + \mu^*(\{1\}) = a + b$$

We now check what conditions on a, b, c are necessary for $\mu^* \neq \mu^+$. Since $\mathscr{M}^*$ is a σ-algebra, both $\emptyset$ and X are in it – the Carathéodory criterion for each is trivial, and does not matter based on the values prescribed above. If either $\{0\}$ or $\{1\}$ is μ^*-measurable, then $\mathscr{M}* = \mathcal{P}(X)$. Thus $\overline{\mu} = \mu^*$. Since μ^+ extends $\overline{\mu}$, it follows that $\mu^+ = \overline{\mu}$. Thus we must break the Carathéodory condition for $\{0\}$ and $\{1\}$. First writing it out for $\{0\}$ gives

$$\mu^*(A) = \mu^*(A \cap \{0\}) + \mu^*(A \cap \{0\}^c) = \mu^*(A \cap \{0\}) + \mu^*(A \cap \{1\})$$

(of course, we see from this that the condition is exactly equivalent for $\{1\}$ being μ^*-measurable). If $A = \emptyset$ then we trivially have equality; the same occurs if A is $\{0\}$ or $\{1\}$ since they are disjoint. So the only meaningful case is when $A = X$. In this, we have

$$c = \mu^*(X) = \mu^*(X \cap \{0\}) + \mu^*(X \cap \{1\}) = a + b.$$

Thus if $a + b \neq c$, we will have $\{0\}, \{1\} \notin \mathscr{M}^*$.

Now what is μ^+? We already know $\mu^+(\emptyset) = 0$, and since X is the only covering of itself, $\mu^+(X) = \overline{\mu}(X) = \mu^*(X)$. So we must look at $\mu^+(\{0\})$ and $\mu^+(\{1\})$. For each, there is only one covering (since we assume from earlier that only $\emptyset, X \in \mathscr{M}^*$) – namely, X. Thus

$$\mu^+(\{0\}) = \overline{\mu}(X) = \mu^*(X) = c = \overline{\mu}(X) = \mu^+(\{1\})$$

In total, for μ^* to be an outer measure, we must have $0 \leq a \leq c$, $0 \leq b \leq c$ and $c \leq a + b$. Next, for $\mu^+ \neq \mu^*$ we must have $a + b \neq c$ and either $a \neq c$ or $b \neq c$. For convenience, suppose both are unequal to c. We can easily see that if either a or b is zero, then the other must simultaneously be less than c and greater than or equal to c. So, any a, b, c with

$$0 < a < c,\ 0 < b < c,\ c < a + b$$

works.

1.4.21. Let μ^* be an outer measure induced from a premeasure and $\overline{\mu}$ the restriction of μ^* to the μ^*-measurable sets. Then $\overline{\mu}$ is saturated. (Use Exercise 1.4.18.)

Solution: Let the underlying premeasure be μ_0 and the algebra it is defined on $\mathcal{A}$. Let $E \subset X$ be locally $\overline{\mu}$-measurable; that is for any $A \in \mathscr{M}^*$ such that $\mu(A) < \infty$ we have $E \cap A \in \mathscr{M}^*$. We must show that actually $E \in \mathscr{M}^*$. By a remark in Folland, it suffices to show that

$$\mu^*(F) \geq \mu^*(E \cap F) + \mu^*(E^c \cap F)$$

whenever $\mu^*(F) < \infty$. Suppose we have such an F, and let $\epsilon > 0$. Then by Exercise 1.4.18a), there exists an $A \in \mathcal{A}_\sigma$ such that $F \subset A$ and

$$\mu^*(A) \leq \mu^*(F) + \epsilon$$

Since A is μ^*-measurable and has finite outer measure, by local measurability of E it follows that $E \cap A$ is measurable. Hence,

$$\mu^*(A) = \mu^*(A \cap (E \cap A)) + \mu^*(A \cap (E \cap A)^c) = \mu^*(E \cap A) + \mu^*(A \cap E^c)$$

Now, applying the above bound and monotonicity,

$$\mu^*(F) + \epsilon \geq \mu^*(A) = \mu^*(E \cap A) + \mu^*(E^c \cap A) \geq \mu^*(E \cap F) + \mu^*(E^c \cap F).$$

Since this holds for any ϵ, we get

$$\mu^*(F) + \epsilon \geq \mu^*(E \cap F) + \mu^*(E^c \cap F)$$

as desired.

1.4.22. Let $(X, \mathscr{M}, \mu)$ be a measure space, μ^* the outer measure induced by μ according to (1.12), $\mathscr{M}^*$ the σ-algebra of μ^*-measurable sets, and $\overline{\mu} = \mu^*|_{\mathscr{M}^*}$.

a) If μ is σ-finite, then $\overline{\mu}$ is the completion of μ. (Use Exercise 1.4.18a).)
b) In general, $\overline{\mu}$ is the saturation of the completion of μ. (See Exercises 1.3.16 and 1.4.21.)

Solution:

a) Let us first write out what μ^* is:

$$\mu^*(E) = \inf \left\{ \sum_{j=1}^{\infty} \mu(A_j) \;\middle|\; E \subset \bigcup_{j=1}^{\infty} A_j, A_j \in \mathscr{M} \right\}.$$

Recall that the completion of a measure is the unique measure $\hat{\mu}$ defined on $\hat{\mathscr{M}} = \{E \cup F \mid E \in \mathscr{M},\ F \subset N \in \mathscr{N}\}$ where $\mathscr{N} = \{N \in \mathscr{M} \mid \mu(N) = 0\}$ such that $\hat{\mu}$ is complete on $\hat{\mathscr{M}}$ and extends μ.

To show that $\overline{\mu} = \hat{\mu}$, we must first show that $\mathscr{M}^* = \hat{\mathscr{M}}$. Suppose first that $E \in \mathscr{M}^*$. Then there exists a $B^c \in \mathscr{M}$ (as per 1.4.18b) and 1.4.18c)) such that $E^c \subset B^c$ and $\overline{\mu}(B^c \setminus E^c) = \mu^*(B^c \setminus E^c) = \mu^*(E \setminus B) = 0$. Now write $E = B \cup (E \setminus B)$. Since $B^c \in \mathscr{M}^*$, it follows that $B \in \mathscr{M}$. Thus we need only show that $E \setminus B \subset N$ for some $N \in \mathscr{M}$ with $\mu(N) = 0$. By Exercise 1.4.18a), for each n there exists an $A_n \in \mathscr{M}$ such that $E \setminus B \subset A_n$ and

$$\mu^*(A_n) \leq \mu^*(E \setminus B) + \frac{1}{n} = \frac{1}{n}.$$

Now set $N = \bigcap_n A_n$. Then,

$$\mu^*(N) \leq \mu^*(A_n) \leq \frac{1}{n}$$

for all n. It follows that $\mu^*(N) = 0$. But each $A_n \in \mathscr{M}$ so that $N \in \mathscr{M}$ too. Finally, $E \setminus B \subset A_n$ for all n, so that $E \setminus B \subset N$. This is exactly what we want.

Next suppose $E \cup F \in \hat{\mathscr{M}}$. Then there exists an $N \in \mathscr{M}$ such that $F \subset N$ and $\mu(N) = 0$. Since $\mathscr{M} \subset \mathscr{M}^*$ (the set of μ^*-measurable sets always includes the algebra the premeasure is defined on; here we just have a special premeasure and algebra), it follows that both $E, N \in \mathscr{M}^*$. We then have

$$\overline{\mu}(N) = \mu^*(N) = \mu(N) = 0$$

Since $\overline{\mu}$ is complete, $F \in \mathscr{M}^*$ with $\overline{\mu}(F) = 0$. It follows that $E \cup F \in \mathscr{M}^*$. Thus we have shown $\mathscr{M}^* = \hat{\mathscr{M}}$.

Evidently $\overline{\mu}$ extends μ, so that we need only show that $\overline{\mu}$ is complete. But by Carathéodory's theorem, it is. Uniqueness of the completion then gives the result.

b) We want to show that $\overline{\mu}$ is the saturation of the completion of μ, which we will denote by $\tilde{\hat{\mu}}$ ($\sim$ for saturation, $\hat{}$ for completion). We know that $\tilde{\hat{\mu}}$ is defined on the σ-algebra of locally $\hat{\mu}$-measurable sets. On the other hand, $\overline{\mu}$ is defined a priori only on the set of μ^*-measurable sets. So, we first show these two σ-algebras coincide.

Let E be locally $\hat{\mu}$-measurable. Let $A \in \mathscr{M}$ be such that $\mu(A) < \infty$. Then, since $\hat{\mu}$ extends μ, we have $A \in \hat{\mathscr{M}}$ with $\hat{\mu}(A) < \infty$. It follows that $E \cap A \in \hat{\mathscr{M}}$. In a), we showed that $\hat{\mathscr{M}} \subset \mathscr{M}^*$ without the use of σ-finiteness of μ. It follows that $E \cap A \in \mathscr{M}^*$. Applying μ^*-measurability of $E \cap A$ to A gives

$$\mu^*(A) = \mu^*(A \cap (E \cap A)) + \mu^*(A \cap (E \cap A)^c) = \mu^*(E \cap A) + \mu^*(A \cap E^c).$$

Now, by the remark in Folland we need to show that

$$\mu^*(F) \geq \mu^*(E \cap F) + \mu^*(E^c \cap F)$$

whenever $\mu^*(F) < \infty$. In the proof of 1.4.18b), we actually found a set $B \in \mathscr{M}$ such that $E \subset B$ and $\mu^*(E) = \mu^*(B) = \mu(B)$. Applying this to F instead of E gives a $B \in \mathscr{M}^*$ such that

$$\mu(B) = \mu^*(F) < \infty.$$

Then, applying the above with B in place of A gives $E \cap B \in \mathscr{M}^*$ and $E^c \cap B \in \mathscr{M}^*$. Finally,

$$\mu^*(F) = \mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap E^c) \geq \mu^*(F \cap E) + \mu^*(F \cap E^c)$$

owing to monotonicity since $F \subset B$. Hence, E is μ^*-measurable and $\tilde{\hat{\mathscr{M}}} \subset \mathscr{M}^*$.

Suppose now that $E \in \mathscr{M}^*$. Suppose $A \in \hat{\mathscr{M}}$ has $\hat{\mu}(A) < \infty$. As shown at the end of the proof of Exercise 1.4.22a), we showed that $\hat{\mathscr{M}} \subset \mathscr{M}^*$ so that $A \in \mathscr{M}^*$. Writing $A = B \cup C$ where $B \in \mathscr{M}$ and $C \subset N$ with $\mu(N) = 0$, we see that

$$\mu^*(B) \leq \mu^*(A) \leq \mu^*(B) + \mu^*(C) \leq \mu^*(B) + \mu^*(N) = \mu^*(B).$$

By definition, $\hat{\mu}(A) = \mu(B)$ so that $\hat{\mu}(A) = \mu^*(A)$. Using this, we have $\mu^*(A) < \infty$. Since both $E, A \in \mathscr{M}^*$, we have $E \cap A \in \mathscr{M}^*$ with $\mu^*(E \cap A) < \infty$. Then, we can repeat the argument at the beginning of the proof of Exercise 1.4.22a) to conclude that $E \cap A \in \hat{\mathscr{M}}$ (the assumption $\mu^*(E \cap A)$ takes the place of σ-finiteness). Thus, E is locally $\hat{\mu}$-measurable. We have finally shown that $\tilde{\hat{\mathscr{M}}} = \mathscr{M}$.

We now show that $\tilde{\hat{\mu}} = \overline{\mu}$. To this end, first let $E \in \hat{\mathscr{M}}$. By definition of the saturation, $\tilde{\hat{\mu}}(E) = \hat{\mu}(E)$. On the other hand, since $\overline{\mu}$ is a complete measure on $\mathscr{M}^* = \tilde{\hat{\mathscr{M}}}$, and the completion of μ to $\hat{\mathscr{M}}$ is unique, we have that $\hat{\mu}(E) = \overline{\mu}(E)$. If $E \notin \hat{\mathscr{M}}$ is locally $\hat{\mu}$-measurable, then by definition of the saturation $\tilde{\hat{\mu}}(E) = \infty$. Earlier, we saw that if $E \in \mathscr{M}^*$ with $\mu^*(E) < \infty$ then $E \in \hat{\mathscr{M}}$ (we saw this applied directly to $E \cap A$). But, $E \in \mathscr{M}$ and $E \notin \hat{\mathscr{M}}$, so $\mu^*(E) = \infty$. Since μ^* and $\overline{\mu}$ agree on $\mathscr{M}^*$, it follows that $\overline{\mu}(E) = \infty$.

1.4.23. Let $\mathcal{A}$ be the collection of finite unions of sets of the form $(a, b] \cap \mathbb{Q}$ where $-\infty \leq a < b \leq \infty$.

a) $\mathcal{A}$ is an algebra on $\mathbb{Q}$. (Use Proposition 1.7.)
b) The σ-algebra generated by $\mathcal{A}$ is $\mathcal{P}(\mathbb{Q})$.
c) Define μ_0 on $\mathcal{A}$ by $\mu_0(\emptyset) = 0$ and $\mu_0(A) = \infty$ for $A \neq \emptyset$. Then μ_0 is a premeasure on $\mathcal{A}$, and there is more than one measure on $\mathcal{P}(\mathbb{Q})$ whose restriction to $\mathcal{A}$ is μ_0.

Solution:

a) We show first that $\mathcal{E} = \{(a, b] \cap \mathbb{Q} \mid a \leq b \ a, b \in \overline{\mathbb{R}}\}$ is an elementary family (note that this includes more than the sets described in the problem!). Then, by Proposition 1.7 the collection of finite disjoint unions of elements in $\mathcal{E}$, denoted $\mathcal{B}$, is an algebra. So, it suffices to show that $\mathcal{A} = \mathcal{B}$.

First let us show $\mathcal{E}$ is an elementary family. By choosing $a = b$, we see that $\emptyset = (a, a] \in \mathcal{E}$. Now let $E, F \in \mathcal{E}$. Then there exist $e_1 \leq e_2$ and $f_1 \leq f_2$ such that $E = (e_1, e_2] \cap \mathbb{Q}$ and $F = (f_1, f_2] \cap \mathbb{Q}$. Let $g_1 = \max\{e_1, f_1\}$ and $g_2 = \min\{e_2, f_2\}$. Then,

$$E \cap F = (e_1, e_2] \cap (f_1, f_2] \cap \mathbb{Q} = (g_1, g_2] \cap \mathbb{Q} \in \mathcal{E}.$$

Note that $(g_1, g_2]$ makes sense since if not we would have

$$e_2 \leq g_2 < g_2 \leq e_1$$

(similarly for f_1 and f_2). This is a contradiction. Finally, we must check that the complement of any elementary set is a finite disjoint union of elementary sets. Note that since we're working over $\mathbb{Q}$, the complement is the relative complement. So, we take the complement in $\mathbb{Q}$. Now let $E = (e_1, e_2] \cap \mathbb{Q} \in \mathcal{E}$. The complement of E in $\mathbb{Q}$ is

$$E^c = ([-\infty, e_1] \cup (e_2, \infty]) \cap \mathbb{Q} = ((-\infty, e_1] \cap \mathbb{Q}) \cup ((e_2, \infty] \cap \mathbb{Q}).$$

since $-\infty \notin \mathbb{Q}$.

We now show that $\mathcal{A} = \mathcal{B}$. It is clear that $\mathcal{B} \subset \mathcal{A}$, since any element of $\mathcal{E}$ takes the form $(a, b] \cap \mathbb{Q}$ or $\emptyset$. Then, a finite disjoint union of these is certainly included in $\mathcal{A}$. Now let $E \in \mathcal{A}$. Then,

$$E = ((e_1, e_2] \cup (e_3, e_4] \cup ... \cup (e_{n-1}, e_n]) \cap \mathbb{Q}$$

possibly with overlap. To show that $E \in \mathcal{B}$, we must "disjointify" these. WLOG we may assume $e_1 \leq e_3 \leq ... \leq e_{n-1}$, so that the intervals are arranged by increasing left endpoint. For odd i, if $e_i < e_{i-1}$ then we may merge the two intervals $(e_{i-2}, e_{i-1}]$ and $(e_{i-1}, e_i]$ into $(e_{i-2}, e_i]$. If $e_{i-1} \leq e_i$ then the two intervals above are disjoint. Doing this for each overlap gives a new representation of E,

$$E = ((e'_1, e'_2] \cup (e'_3, e'_4] \cup ... \cup (e'_{m-1}, e'_m]) \cap \mathbb{Q}$$

where all the $(e_i, e_{i+1}]$ are disjoint. Thus E is a finite disjoint union of sets in $\mathcal{E}$, and $\mathcal{A} \subset \mathcal{B}$.

b) First note that any subset of $\mathbb{Q}$ is countable. Next, let $q \in \mathbb{Q}$. Then we have

$$\{q\} = \left(\bigcap_{n=1}^{\infty} \left(q - \frac{1}{n}, q \right] \right) \cap \mathbb{Q}.$$

It follows that $\{q\}$ for any $q \in \mathbb{Q}$ is in the σ-algebra generated by $\mathcal{A}$. This implies that any countable subset of rational numbers is in the σ-algebra. But every subset of $\mathbb{Q}$ is such a set.

c) We first show that μ_0 is a premeasure. Evidently $\mu_0 : \mathcal{A} \to [0, \infty]$ is well-defined on an algebra $\mathcal{A}$ (as shown in part a)). Trivially $\mu_0(\emptyset) = 0$ so we only need to check countable additivity. Let $\{A_j\}_{j=1}^{\infty}$ be a collection of disjoint sets in $\mathcal{A}$ such that $A = \bigcup_j A_j \in \mathcal{A}$. If all the A_j are empty, then

$$\mu_0(A) = \mu_0(\emptyset) = 0 = \sum_{j=1}^{\infty} \mu_0(A_j).$$

Now suppose at least one of the A_j is nonempty. Then, so too is A. Hence,

$$\mu_0(A) = \infty = \sum_{j=1}^{\infty} \mu_0(A_j).$$

So μ_0 is a premeasure on $\mathcal{A}$. Let μ be counting measure and let $\overline{\mu}$ be the measure obtained by extending μ_0 to an outer measure and restricting it to the μ^*-measurable sets. Let $\mathscr{M}^*$ denote the μ^*-measurable sets. Since $\mathcal{A} \subset \mathscr{M}^*$, it follows that

$$\mathcal{P}(\mathbb{Q}) = \mathscr{M}(\mathcal{A}) \subset \mathscr{M}^* \subset \mathcal{P}(\mathbb{Q}).$$

Hence, both μ and $\overline{\mu}$ are measures defined on $\mathcal{P}(\mathbb{Q})$. Clearly $\overline{\mu}$ restricts to μ_0 on $\mathcal{A}$. Since μ is a measure, $\mu(\emptyset) = 0 = \mu_0(\emptyset)$. So we need to check for any $A \in \mathcal{A}$ nonempty that $\mu(A) = \infty$. Since the rationals are dense in $\mathbb{Q}$, it follows that any set $(a, b] \cap \mathbb{Q}$ contains infinitely many rationals. Hence, $\mu(A) = \infty = \mu_0(A)$. Thus, both μ and $\overline{\mu}$ restrict to μ_0 on $\mathcal{A}$.

Consider the set $\{0\}$. Evidently $\mu(\{0\}) = 1$. On the other hand, the empty set cannot cover $\{0\}$, so any covering of $\{0\}$ by sets $A_j \in \mathcal{A}$ is such that

$$\sum_{j=1}^{\infty} \mu_0(A_j) = \infty.$$

It follows that $\overline{\mu}(\{0\}) = \infty$.

1.4.24. Let μ be a finite measure on $(X, \mathscr{M})$, and let μ^* be the outer measure induced by μ. Suppose that $E \subset X$ satisfies $\mu^*(E) = \mu^*(X)$ (but not that $E \in \mathscr{M}$).

a) If $A, B \in \mathscr{M}$ and $A \cap E = B \cap E$, then $\mu(A) = \mu(B)$.
b) Let $\mathscr{M}_E = \{A \cap E \mid A \in \mathscr{M}\}$, and define the function ν on $\mathscr{M}_E$ defined by $\nu(A \cap E) = \mu(A)$ (which makes sense by a)). Then $\mathscr{M}_E$ is a σ-algebra on E and ν is a measure on $\mathscr{M}_E$.

Solution:

a) The condition $A \cap E = B \cap E$ is stronger than just saying their outer measures are the same; we should try and gather statements about the sets themselves first using this. That $A \cap E = B \cap E$ implies that, from the point of view of E, both A and B are identical. We therefore should be able to prove something like

$$(B \setminus A) \cap E = \emptyset.$$

Indeed, this follows since

$$(B \setminus A) \cap E = B \cap A^c \cap E = (B \cap E) \cap A^c = (A \cap E) \cap A^c = \emptyset.$$

By a symmetric argument, we see too that $(A \setminus B) \cap E = \emptyset$. We showed in Exercise 1.3.12 that if $\mu(A \Delta B) = 0$ then $\mu(A) = \mu(B)$ (regardless of finiteness). The above shows us that

$$A \Delta B = (A \setminus B) \cup (B \setminus A) \subset E^c.$$

Hence, by monotonicity and the assumption $\mu^*(X) = \mu^*(E)$,

$$\mu^*(X) = \mu^*(E) \leq \mu^*((A \Delta B)^c) \leq \mu^*(X).$$

It follows that $\mu^*(X) = \mu^*((A \cap B)^c)$. Now, $A \Delta B$ is measurable so that

$$\mu^*(X) = \mu^*(X \cap (A \Delta B)) + \mu^*(X \cap (A \Delta B)^c) = \mu^*(A \Delta B) + \mu^*((A \Delta B)^c) = \mu^*(A \Delta B) + \mu^*(X).$$

Since all the sets above are measurable and $\mu^*|_{\mathscr{M}} = \mu$, we see

$$\mu(X) = \mu(A \Delta B) + \mu(X).$$

Finally, owing to finiteness of μ we can subtract by $\mu(X)$ and conclude $\mu(A \Delta B) = 0$.

b) We first show that $\mathscr{M}_E$ is a σ-algebra on E. First, given $A \cap E \in \mathscr{M}_E$, the complement (in E!) is $A^c \cap E$. Since $\mathscr{M}$ is a σ-algebra, $A^c \in \mathscr{M}$ and therefore $A^c \cap E \in \mathscr{M}_E$. Now let $\{A_j \cap E\}_{j=1}^{\infty}$ be a collection of sets in $\mathscr{M}_E$. Then,

$$\bigcup_{j=1}^{\infty}(A_j \cap E) = \left(\bigcup_{j=1}^{\infty} A_j\right) \cap E \in \mathscr{M}_E$$

since $A_j \in \mathscr{M}$ implies $\bigcup_j A_j \in \mathscr{M}$. So $\mathscr{M}_E$ is a σ-algebra.

Since μ is a finite measure, ν is a priori a finite function; hence $\nu : \mathscr{M}_E \to [0, \infty)$. Clearly $\nu(\emptyset) = \nu(\emptyset \cap E) = \mu(\emptyset) = 0$. Now let $\{A_j \cap E\}_{j=1}^{\infty}$ be a collection of disjoint sets in $\mathscr{M}_E$. We need not have that the A_j themselves are disjoint! But, we can disjointify them as seen before. Set $B_n = A_n \setminus \bigcup_{j=1}^{n-1} A_j$. Then all the B_n are disjoint and $\bigcup_{i=1}^{n} B_i = \bigcup_{j=1}^{n} A_j$. Next, notice that

$$B_n \cap E = \left(A_n \setminus \bigcup_{j=1}^{n-1} A_j\right) \cap E = (A_n \cap E) \setminus \bigcup_{j=1}^{n-1}(A_j \cap E) = A_n \cap E$$

Intuitively, the above says that to disjointify the A_n, we only care about disjointifying them outside of E. This makes sense, since the $A_n \cap E$ are disjoint. Now we have

$$\nu\left(\bigcup_{j=1}^{\infty}(A_j \cap E)\right) = \nu\left(\bigcup_{j=1}^{\infty}(B_j \cap E)\right) = \nu\left(\left(\bigcup_{j=1}^{\infty} B_j\right) \cap E\right) = \mu\left(\bigcup_{j=1}^{\infty} B_j\right) = \sum_{j=1}^{\infty} \mu(B_j).$$

On the other hand, we see that

$$\nu(A_j capE) = \nu(B_j \cap E) = \mu(B_j)$$

so that combined with the above we have

$$\nu\left(\bigcup_{j=1}^{\infty}(A_j \cap E)\right) = \sum_{j=1}^{\infty} \nu(A_j \cap E).$$

1.5. **Borel Measures on the Real Line.**

1.5.25. Complete the proof of Theorem 1.19.

Solution: We restate Theorem 1.19 here for convenience.

Theorem 1.19: If $E \subset \mathbb{R}$, the following are equivalent.

a) $E \in \mathscr{M}_\mu$
b) $E = V \setminus N_1$ where V is a G_δ set and $\mu(N_1) = 0$.
c) $E = H \cup N_2$ where H is an F_σ set and $\mu(N_2) = 0$.

(in this context, μ is a complete Lebesgue-Stieltjes measure associated to a right continuous, increasing function $F : \mathbb{R} \to \mathbb{R}$ and $\mathscr{M}_\mu$ is the domain of μ).

Folland already proves this for the bounded case, so we are just left with the implications $a) \Rightarrow b)$ and $a) \Rightarrow c)$ for E with $\mu(E) = \infty$. First observe we can deduce the former implication from the latter. If $E \in \mathscr{M}_\mu$ then so too is E^c. Applying $c)$ gives an F_σ set H and μ-null set N such that $E^c = H \cup N$. Now let $V = H^c$; this is a G_δ set since

$$V = H^c = \left(\bigcup_{j=1}^{\infty} H_j\right)^c = \bigcap_{j=1}^{\infty} H_j^c = \bigcap_{j=1}^{\infty} V_j$$

where $V_j := H_j^c$. Since all the H_j are closed, each V_j is open, and V is G_δ. Now, $E = V \setminus N$ since

$$(V \setminus N)^c = (V \cap N^c)^c = V^c \cup N = H \cup N = E^c.$$

So, $a) \Rightarrow c)$ implies $a) \Rightarrow b)$. Now we need only show the unbounded case in the implication $a) \Rightarrow c)$. To do so, set $E_j = E \cap (j, j+1]$ for $j = 0, \pm 1, \pm 2, ...$ Then each $E_j \in \mathscr{M}_\mu$ with $\mu(E_j) < \infty$. Applying the bounded case yields a corresponding F_σ set H_j and μ-null set N_j such that $E_j = H_j \cup N_j$. Set $H = \bigcup_j H_j$, which is F_σ and $N = \bigcup_j N_j$ which is still μ-null by subadditivity. Then,

$$E = \bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} (H_j \cup N_j) = H \cup N$$

as desired.

1.5.26. Prove Proposition 1.20. (Use Theorem 1.18.)

Solution: We state Proposition 1.20 here for convenience.

Proposition 1.20: If $E \in \mathscr{M}_\mu$ and $\mu(E) < \infty$, then for every $\epsilon > 0$ there is a set A that is a finite union of open intervals such that $\mu(E \Delta A) < \epsilon$.

Let $E \in \mathscr{M}_\mu$. Then by Lemma 1.17

$$\mu(E) = \inf \left\{ \sum_{j=1}^{\infty} (a_j, b_j) \;\middle|\; E \subset \bigcup_{j=1}^{\infty} (a_j, b_j) \right\}.$$

Choose a covering of E such that

$$\sum_{j=1}^{\infty} \mu(a_j, b_j) \leq \mu(E) + \frac{\epsilon}{2}.$$

Set $I_j = (a_j, b_j)$. Since $\mu(E) < \infty$, the sum is convergent. Hence, there exists an N such that

$$\sum_{j=N}^{\infty} \mu(I_j) < \frac{\epsilon}{2}.$$

Now set $A = \bigcup_{j=1}^{N-1} I_j$. It follows that $E \setminus A \subset (\bigcup_j I_j) \setminus A \subset \bigcup_{j=N}^{\infty} I_j$. Hence,

$$\mu(E \setminus A) \leq \mu\left(\bigcup_{j=N}^{\infty} I_j\right) \leq \sum_{j=N}^{\infty} \mu(I_j) < \frac{\epsilon}{2}.$$

On the other hand, $A \setminus E \subset \bigcup_j I_j \setminus E$ so that

$$\mu(A \setminus E) \leq \mu\left(\left(\bigcup_{j=1}^{\infty} I_j\right) \setminus E\right).$$

But $\mu(E) < \infty$ and $E \subset \bigcup_j I_j$ so that we may split the above as follows

$$\mu\left(\left(\bigcup_{j=1}^{\infty} I_j\right) \setminus E\right) = \mu\left(\bigcup_{j=1}^{\infty} I_j\right) - \mu(E) \leq \sum_{j=1}^{\infty} \mu(I_j) - \mu(E) \leq \mu(E) - \mu(E) + \frac{\epsilon}{2} = \frac{\epsilon}{2}.$$

Combining these two estimations yields

$$\mu(E \Delta A) = \mu((E \setminus A) \cup (A \setminus E)) \leq \mu(E \setminus A) + \mu(A \setminus E) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

The above result is commonly known as Littlewood's First Principle of Analysis.

1.5.27. Prove Proposition 1.22a. (Show that if $x, y \in C$ and $x < y$, there exists $z \notin C$ such that $x < z < y$.)

Solution: We restate Proposition 1.22a here for convenience

Proposition 1.22a: Let C be the Cantor set. Then C is compact, nowhere dense, and totally disconnected (i.e., the only connected subsets of C are single points). Moreover, C has no isolated points.

Let $E_0 = [0, 1]$ inductively define E_{n+1} by removing the (open) middle third of each disjoint interval in E_n. So, $E_1 = [0, 1/3] \cup [2/3, 1]$, $E_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$, and so on. It is clear that each E_n is a disjoint union of 2^n many closed intervals of length 3^{-n}. The Cantor set can be written as

$$C = \bigcap_{n=1}^{\infty} E_n$$

so that C is a closed set. Clearly $C \subset [0, 1]$ so that it is bounded; hence, by Heine-Borel C is compact.
A set X is nowhere dense if $(\overline{X})^\circ = \emptyset$. So, we must show that $C^\circ = \emptyset$. If C° was nonempty, then it would contain some interval. So, we show that C contains no interval $I \subset [0, 1]$. Suppose that it did. Then for each n, $I \subset I_n$ where I_n is one of the disjoint intervals in E_n. But, $m(I_n) = 3^{-n}$, which tends to 0. Hence, eventually one of the I_n will have length less than I, and cannot contain it.
To see that C is totally disconnected, we show if $x, y \in C$ and $x < y$ then there exists $z \notin C$ such that $x < z < y$. Let $d = |x - y| < 1$. There exists a largest n such that $x, y \in I_n$, one of the disjoint intervals in E_n. If not, then x and y lie in the same interval I_n for all n. But, the I_n have decreasing length, and eventually decrease below d. Thus, for some n, x and y lie in separate intervals in E_n. Denoting N by the last n before this separation, we choose a z in the middle third of I_N. Then $x < z < y$; since x and y get separated, they must lie in different thirds, hence we have strict inequalities. By choice of z, it is removed upon moving to E_{N+1}. Thus, $z \notin C$.
A point $x \in C$ is isolated if there exists a $B_r(x)$ such that $B_r(x) \cap C = \{x\}$. Suppose that $x \in C$ is isolated. Choose n such that $3^{-n} < r$. By definition, x lies in some I_n composing E_n. Moreover, $I_n \subset B_r(x)$ by choice of n. Now I_n has two endpoints a_n and b_n, both of which are in C. Choose one of them that is not x; then it is an element of C lying in $B_r(x)$, a contradiction.

1.5.28. Let F be increasing and right continuous, and let μ_F be the associated measure. Then $\mu_F(\{a\}) = F(a) - F(a-)$, $\mu_F([a, b)) = F(b-) - F(a-)$, $\mu_F([a, b]) = F(b) - F(a-)$, and $\mu_F((a, b)) = F(b-) - F(a)$.

Solution: First, to clear up some notation that Folland uses:

$$F(t-) = \lim_{x \to t^-} F(x),$$

that is $F(t-)$ is the left-handed limit of F at t. Since F is increasing, this limit always exists. Moreover, if the limit exists it may be calculated by looking at $\{F(x_n)\}_{n=1}^{\infty}$ for any sequence $x_n \to t$ where $x_n < t$. In particular, we may choose $x_n = t - 1/n$. That is, $F(t-) = \lim F(t - 1/n)$.

Note that $\{a\} = \bigcap_n (a - 1/n, a]$. Since μ_F is a measure, and $(a - 1, a]$ has finite measure, dominated convergence holds. Thus,

$$\begin{aligned}
\mu_F(\{a\}) &= \mu_F\left(\bigcap_{n=1}^{\infty}\left(a - \frac{1}{n}, a\right]\right) = \lim_{n\to\infty} \mu_F\left(\left(a - \frac{1}{n}, a\right]\right) = \lim_{n\to\infty}\left[F(a) - F\left(a - \frac{1}{n}\right)\right] \\
&= F(a) - \lim_{n\to\infty} F\left(a - \frac{1}{n}\right) = F(a) - F(-a)
\end{aligned}$$

More generally, $[a,b] = \bigcap_n (a - 1/n, b]$. It follows that

$$\mu_F([a,b]) = F(b) - F(a-)$$

by replacing the right endpoint in the above calculation to b. Now, by countable additivity,

$$F(b) - F(b-) + \mu_F([a,b)) = \mu_F(\{b\}) + \mu_F([a,b)) = \mu_F([a,b]) = F(b) - F(a-).$$

Hence, by rearranging terms,

$$\mu_F([a,b)) = F(b-) - F(a-).$$

Finally, once more by countable additivity,

$$F(b) - F(b-) + \mu_F((a,b)) = \mu_F(\{b\}) + \mu_F((a,b)) = \mu_F((a,b]) = F(b) - F(a).$$

Then,

$$\mu_F((a,b)) = F(b-) - F(a).$$

1.5.29. Let E be a Lebesgue measurable set.

a) If $E \subset N$ where N is the nonmeasurable set described in §1.1, then $m(E) = 0$.
b) If $m(E) > 0$, then E contains a nonmeasurable set. (It suffices to assume $E \subset [0,1]$. In the notation of §1.1, $E = \bigcup_{r \in R} E \cap N_r$.)

Solution:

a) We first recall the construction of N given in §1.1. We define an equivalence relation on $[0,1)$ by $x \sim y$ iff $x - y \in \mathbb{Q}$. Now let $N \subset [0,1)$ be such that N contains precisely one member of each equivalence class. Let $R = [0,1) \cap \mathbb{Q}$ and set

$$E_r = \{x + r \mid x \in E \cap [0, 1-r)\} \cup \{x + r - 1 \mid x \in E \cap [1-r, 1)\}$$

That is, we take the portion of E in $[0, 1-r)$ and translate it by r units to the right, and we take the portion of E in $[1-r, 1)$ and translate this by $1-r$ units to the right. This is easy to visualize when $E = [0, 1-r)$ itself, and imagining $[0, 1-r)$ and $[1-r, 1)$ as solid blocks which you shuffle around. The same phenomenon happens here, but only for the points in E. Since all these translations are by rational numbers, the E_r are disjoint. Moreover, each $E_r \subset [0,1]$. Finally, each E_r is measurable as a union of two measurable sets – e.g. the intersection $E \cap [0, 1-r)$ is measurable, and translating this preserves measurability. Note by translation invariance of the Lebesgue measure, we have $m(E) = m(E_r)$ for all $r \in R$. Then,

$$1 = m([0,1)) \geq m\left(\bigcup_{r \in R} E_r\right) = \sum_{r \in R} m(E_r) = \sum_{r \in R} m(E).$$

Since the sum ranges over a countable set, the only way for this inequality to be true is if $m(E) = 0$.

b) First, it suffices to assume $E \subset [0,1]$ due to translation invariance. That is, E is measurable iff any translate of E is measurable. So, translates of nonmeasurable sets are nonmeasurable. By shifting E to be a subset of $[0,1]$, finding a nonmeasurable subset there, and shifting it back, we are done. So suppose $E \subset [0,1]$ with $m(E) > 0$. By the observation given in the problem, $E = \bigcup_{r \in R} E \cap N_r$. To see this, let $e \in E$. Since $E \subset [0,1]$, it lies in one of the equivalence classes of $\sim$. Let $n \in N$ be the representative chosen for this equivalence class. Let $r = |n - e|$. Either $n \leq e$ or $n > e$. If $n = e$, we are done since $N_0 = N$, and $e \in N$. If $n < e$ then $n \in [0, 1-r)$. This follows since

$$1 - r - n = 1 - (e - n) - n = 1 - e > 0$$

since $e \in [0,1)$. Now, if $n < e$ then it must be $n = e - r$. Hence, $n + r = e \in N_r$, and $e \in E \cap N_r$. So suppose that $n > e$. By reversing roles in the above logic, we see that $e \in [0, 1-r)$ and $n = e + r$. We want to find some $r' \in R$ such that $n + r' = e$, where '$+$" follows the rule described above: if $n \in [0, 1-r')$ then $n \mapsto n + r'$, while if $n \in [1-r', 1)$ then $n \mapsto n + r' - 1$. Let $r' = 1 - r \in R$. Note that

$$n - (1 - r') = e + r - 1 + r' = e + r - 1 + (1 - r) = e \geq 0$$

so that $n \in [1-r', 1)$. Hence, $n \in N$ gets sent to $n + r' - 1 \in N_{r'}$ But $n + r' - 1 = (e + r) + (1 - r) - 1 = e$, which is in $N_{r'}$. Thus $e \in E \cap N_{r'}$. In both cases, there exists an r such that $e \in E \cap N_r$, and therefore $E \subset \bigcup_{r \in R} E \cap N_r$.

Suppose that $E \cap N_r$ is measurable for all $r \in R$. Then,

$$m(E) = m\left(\bigcup_{r\in R} E \cap N_r\right) = \sum_{r\in R} m(E \cap N_r)$$

since the $E \cap N$ are disjoint. Now, by a) we see that $m(E \cap N_r) = 0$ since $E \cap N_r \subset N$ and $E \cap N_r$ is measurable by assumption. It follows that $m(E) = 0$, a contradiction.

1.5.30. If $E \in \mathcal{L}$ and $m(E) > 0$, for any $\alpha < 1$ there is an open interval I such that $m(E \cap I) > \alpha m(I)$.

Solution: Suppose not. Let $\alpha < 1$ be such that every open interval I has $m(E \cap I) \leq \alpha m(I)$. Assume that E is bounded. By construction of the Lebesgue measure, for any $\epsilon > 0$ we may find a covering $\{I_k\}_{k=1}^{\infty}$ of E such that

$$\sum_{k=1}^{\infty} m(I_k) < (1+\epsilon) m(E).$$

But then we have

$$\sum_{k=1}^{\infty} m(E \cap I_k) \leq \alpha \sum_{k=1}^{\infty} m(I_k) < \alpha(1+\epsilon) m(E).$$

Now, by monotonicity and the fact that $\{I_k\}_{k=1}^{\infty}$ covers E,

$$m(E) = m\left(E \cap \left(\bigcup_{k=1}^{\infty} I_k\right)\right) = m\left(\bigcup_{k=1}^{\infty} (E \cap I_k)\right) \leq \sum_{k=1}^{\infty} m(E \cap I_k).$$

Combining the two, we get

$$m(E) < \alpha(1+\epsilon) m(E)$$

which is a contradiction so long as $\epsilon < 1/\alpha - 1$. Such an ϵ can be chosen since $\alpha < 1$ implies $1/\alpha - 1 > 0$.

In the unbounded case, we use σ-finiteness to write $E = \bigcup_k E_k$ where each E_k has finite measure. Since $m(E) = \infty$, at least one E_k has positive finite measure. For this E_k, we may apply the above and conclude that for any $\alpha < 1$ there is an open interval I such that $m(E_k \cap I) > \alpha m(I)$. But by monotonicity, $m(E \cap I) \geq m(E_k \cap I) > \alpha m(I)$.

Note that the bounded case implies that $m(I) < \infty$. Since we use the same interval from the bounded case as in the unbounded, it follows we can always assume $m(I) < \infty$.

1.5.31. If $E \in \mathcal{L}$ and $m(E) > 0$, the set $E - E = \{x - y \mid x, y \in E\}$ contains an interval centered at 0. (If I is as in Exercise 1.5.30 with $\alpha > 3/4$, then $E - E$ contains $(-1/2m(I), 1/2m(I))$.)

Solution: Let us first look at translations of intervals. We may assume here that I has finite measure – we apply the work done here to the interval extracted from Exercise 1.5.30, which by the note at the end of the proof can be assumed to have finite measure. Let $\lambda > 0$ and for concreteness set $I = (a, b)$. Then,

$$m(I \cup (I+\lambda)) = \begin{cases} b + \lambda - a & 0 < \lambda < m(I) \\ b - \lambda - a & 0 < \lambda < m(I) \\ 2m(I) & \text{else} \end{cases} = \begin{cases} m(I) + |\lambda| & |\lambda| < m(I) \\ 2m(I) & \text{else} \end{cases}$$

In particular, if $|\lambda| < m(I)$ then we may write $\lambda = \eta m(I)$ where $\eta \in (-1, 1)$. With this, we have that

$$m(I \cup (I+\lambda)) = m(I) + |\eta| m(I) = (1 + |\eta|) m(I).$$

We restrict our focus to $\eta \in (-1/2, 1/2)$, so the above is bounded by $3/2m(I)$. Note that in this case, $\lambda \in (-1/2m(I), 1/2m(I))$. So if we can show that $\lambda \in E - E$ then we are done. For λ to be in $E - E$, it must be that $\lambda = x - y$ for $x, y \in E$. That is, $E \cap (E + \lambda)$ has nonempty intersection. We will show this by using I from 1.5.30, which of course has special properties.

For measurable E, we then get by monotonicity

$$m((E \cap I) \cup (E \cap I + \lambda)) \leq (1 + |\eta|)m(I) < \frac{3}{2}m(I).$$

Let $\alpha < 1$ and choose I so that $m(E \cap I) > \alpha m(I)$. By translation invariance we then also have $m(E \cap I + \lambda) > \alpha m(I)$. Now suppose that $E + \lambda$ and E are disjoint. Then so too are $(E \cap I) + \lambda$ and $E \cap I$. Hence,

$$m((E \cap I) \cup ((E \cap I) + \lambda)) = m(E \cap I) + m((E \cap I) + \lambda) > 2\alpha m(I).$$

But we computed the measure on the LHS explicitly earlier. In the case $\lambda \in (-1/2m(I), 1/2m(I))$ we have

$$\frac{3}{2}m(I) \geq m((E \cap I) \cup ((E \cap I) + \lambda)) > 2\alpha m(I).$$

Choosing $\alpha > 3/4$ yields a contradiction. That is, E and $E + \lambda$ are not disjoint. Hence, $\lambda \in E - E$.

1.5.32. Suppose $\{\alpha_j\}_{j=1}^{\infty} \subset (0, 1)$

a) $\prod_j (1 - \alpha_j) > 0$ iff $\sum_j \alpha_j < \infty$. (Compare $\sum_j \log(1 - \alpha_j)$ to $\sum_j \alpha_j$.)

b) Given $\beta \in (0, 1)$, exhibit a sequence $\{\alpha_j\}$ such that $\prod_j (1 - \alpha_j) = \beta$.

Solution:

a) First note that the product is always bounded by 1, since all the terms $1 - \alpha_j$ are between 0 and 1. Suppose first that $\prod_j (1 - \alpha_j) > 0$. Then, the product is finite and we may take the logarithm of both sides. Doing so yields

$$\log\left(\prod_{j=1}^{\infty}(1 - \alpha_j)\right) = \sum_{j=1}^{\infty}\log(1 - \alpha_j).$$

This series is finite and negative owing to the fact that the product is bounded above (strictly) by 1. Recall that the Taylor series expansion of $\log(1 + x)$ is

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ...$$

which is valid for $x \in (-1, 1]$. Hence, the Taylor series expansion of $-\log(1 - x)$ is

$$-\log(1 - x) = x + \frac{x^2}{2} + \frac{x^3}{3} + ...$$

which is valid for $x \in [-1, 1)$, in particular for $x \in (0, 1)$. It follows that for $x \in (0, 1)$, since all the terms in the expansion are positive, $-\log(1 - x) \geq x$. Hence,

$$\infty > -\sum_{j=1}^{\infty}\log(1 - \alpha_j) \geq \sum_{j=1}^{\infty}\alpha_j > 0$$

so that $\sum_j \alpha_j$ converges.

Now suppose $\sum_j \alpha_j$ converges. Then for any $\epsilon > 0$ there exists an N such that

$$0 < \sum_{j=N}^{\infty}\alpha_j < \epsilon.$$

We may suppose that $\epsilon < 1$. Define the n-th partial product as

$$P_n = \prod_{j=1}^{n}(1 - \alpha_j).$$

Since none of the α_j are 1, P_n is never zero. We now prove the following:

$$P_n = \prod_{j=1}^{n}(1-\alpha_j) \geq 1 - \sum_{j=1}^{n}\alpha_j.$$

Clearly the base case $n = 1$ holds. Now assume it holds for $j = n$, then

$$\begin{aligned}\prod_{j=1}^{n+1}(1-\alpha_j) &= (1-\alpha_{n+1})\prod_{j=1}^{n}(1-\alpha_j) = \prod_{j=1}^{n}(1-\alpha_j) - \alpha_{n+1}\prod_{j=1}^{n}(1-\alpha_j) \\ &\geq 1 - \sum_{j=1}^{n}\alpha_j - \alpha_{n+1} + \alpha_{n+1}\sum_{j=1}^{n}\alpha_j. \\ &\geq 1 - \sum_{j=1}^{n+1}\alpha_j + \alpha_{n+1}\sum_{j=1}^{n}\alpha_j \geq 1 - \sum_{j=1}^{n+1}\alpha_j.\end{aligned}$$

where the last inequality follows since all the α_j are positive. Note that if we choose a different starting point, the same result still holds. Therefore,

$$\frac{P_n}{P_{N-1}} = \prod_{j=N}^{n}(1-\alpha_j) \geq 1 - \sum_{j=N}^{n}\alpha_j \geq 1 - \sum_{j=N}^{\infty}\alpha_j > 1-\epsilon$$

for $n \geq N$. Since $P_{N-1} > 0$ and $1 - \epsilon > 0$,

$$\prod_{j=1}^{\infty}(1-\alpha_j) = \lim_{n\to\infty} P_n \geq (1-\epsilon)P_{N-1} > 0.$$

b) Let $\beta \in (0,1)$. Recall that $\sum_j 1/2^j = 1$. Hence,

$$b = b^{\sum_j 1/2^j} = \prod_{j=1}^{\infty} b^{1/2^j}.$$

Now, simply set $\alpha_j = 1 - (b^{1/2^j})$. Since $f_j(x) = x^{1/2^j}$ is monotonically increasing and $f(0) = 0$, $f(1) = 1$, it follows that $0 < f(b) < 1$. Therefore $0 < \alpha_j < 1$, and we have exhibited such a sequence.

1.5.33. There exists a Borel set $A \subset [0,1]$ such that $0 < m(A \cap I) < m(I)$ for every subinterval I of $[0,1]$. (Hint: Every subinterval of $[0,1]$ contains Cantor-type sets of positive measure.)

Solution: Enumerate the subintervals of I with rational endpoints. We saw in Folland that we can adapt the construction of the Cantor set to make a so called fat Cantor set with positive measure. One might think to try and cover a dense subfamily of subintervals (such as those with rational intervals) with fat Cantor sets and take the union of all of them. If K denotes this union and I is any nondegenerate subinterval of $[0,1]$, then there exists an n such that $I_n \subset I$. Hence, the fat Cantor set constructed for this I_n, denoted K_n, is such that $K_n \subset I_n$. It follows that

$$0 < m(K_n) = m(K_n \cap I_n) \leq m(K \cap I_n) \leq m(K \cap I) \leq m(I).$$

This almost proves the result, except that the final inequality is not strict. There is no good way to make this strict, however, with the setup we have. Indeed, we must be a little smarter. To remedy, this, instead of constructing one such K, we construct another disjoint K' having the same properties.

To start, we construct disjoint fat Cantor sets $K_1, K_1' \subset I_1$. To do this, simply split I_1 into to intervals with positive measure, and construct K_1 inside one, and K_1' inside the other. Next, assuming we have $K_1, ..., K_n$ and $K_1', ..., K_n'$, we construct K_{n+1} and K_{n+1}'. Let $L_n = (K_1 \cup ... \cup K_n) \cup (K_1' \cup ... \cup K_n')$. One can show that the finite union of Cantor sets (compact, perfect, totally disconnected) is also a Cantor set. In particular, L_{n+1} is totally disconnected, and hence $I_{n+1} \setminus L_{n+1}$ contains an interval J_{n+1} of positive measure. Construct K_{n+1} and K_{n+1}' similarly as K_1 and K_1' from splitting J_{n+1}. From our construction, K_{n+1}' is disjoint from K_{n+1}. Even more, since $K_{n+1}' \subset J_n \subset I_n \setminus L_{n+1}$, it is disjoint from K_i for all $1 \leq i \leq n+1$. If $K = \bigcup_n K_n$ and $K' = \bigcup_n K_n'$, we see that K_n' is disjoint from K for any n. Finally, each K_n is a Cantor set, and hence is Borel (as an intersection of unions of intervals). Thus K is Borel, as a union of Borel sets K_n.

We now show the desired inequality. As before, let I be any nondegenerate subinterval of $[0,1]$. There exists an n such that $I_n \subset I$; hence $K_n, K_n' \subset I_n \subset I$. It follows that

$$0 < m(K_n) = m(K_n \cap I_n) \leq m(K \cap I)$$

since K_n is a fat Cantor set (positive measure) and $K_n \cap I_n \subset K \cap I$. Now, recall that we have K_n' also a fat Cantor set living in I disjoint from K. Clearly $(K \cup K_n') \cap I \subset I$. Since we have a disjoint union,

$$m(K \cap I) + m(K_n') = m(K \cap I) + m(K_n' \cap I) = m((K \cup K_n') \cap I) \leq m(I).$$

But K_n' is also a fat Cantor set and therefore has positive measure. Hence,

$$0 < m(K \cap I) < m(K \cap I) + m(K_n') \leq m(I).$$

2. Integration

2.1. **Measurable Functions.**

In Exercises 1-7, $(X, \mathscr{M})$ is a measurable space. Notationally, I use $\{f \geq t\}$ to denote $\{x \in X \mid f(x) \geq t\} = f^{-1}[t, \infty)$.

2.1.1. Let $f : X \to \overline{\mathbb{R}}$ and $Y = f^{-1}(\mathbb{R})$. Then f is measurable iff $f^{-1}(\{-\infty\}) \in \mathscr{M}$, $f^{-1}(\{\infty\}) \in \mathscr{M}$, and f is measurable on Y.

Solution: Recall that f is measurable on $E \subset X$ if $f^{-1}(B) \cap E \in \mathscr{M}$ for all Borel sets B. Also recall that $\mathscr{B}_{\overline{\mathbb{R}}}$ is defined as $\mathscr{B}_{\overline{\mathbb{R}}} := \{E \subset \overline{\mathbb{R}} \mid E \cap \mathbb{R} \in \mathscr{B}_{\mathbb{R}}\}$.

Suppose first that $f : X \to \overline{\mathbb{R}}$ is measurable. Then $f^{-1}(E) \in \mathscr{M}$ whenever $E \cap \mathbb{R} \in \mathscr{B}_{\mathbb{R}}$. Now, f is measurable on $Y = f^{-1}(\mathbb{R})$ if $f^{-1}(B) \cap f^{-1}(\mathbb{R}) = f^{-1}(B) \in \mathscr{M}$ for all Borel sets $B \in \mathscr{B}_{\mathbb{R}}$. But, $B \cap \mathbb{R} = B \in \mathscr{B}_{\mathbb{R}}$ for any Borel B. Hence, since f is measurable, $f^{-1}(B) \in \mathscr{B}$. It follows that f is measurable on Y. To show that $f^{-1}(\{\infty\}) \in \mathscr{M}$, note that for any real a we have $f^{-1}(a, \infty] \in \mathscr{M}$ since $(a, \infty] \cap \mathbb{R} = (a, \infty) \in \mathscr{B}_{\mathbb{R}}$. Thus,

$$f^{-1}(\{\infty\}) = \bigcap_{n \in \mathbb{N}} f^{-1}(n, \infty] \in \mathscr{M}.$$

Similar reasoning can be used to show $f^{-1}(\{-\infty\}) \in \mathscr{M}$.

Now suppose $f^{-1}(\{\pm\infty\}) \in \mathscr{M}$ and f is measurable on Y. We need to show that $f^{-1}(E) \in \mathscr{M}$ whenever $E \cap \mathbb{R} \in \mathscr{B}_{\mathbb{R}}$. Since f is measurable on Y, $f^{-1}(B) \in \mathscr{M}$ for any Borel B. But if $E \in \mathscr{B}_{\overline{\mathbb{R}}}$ we can write it as $E = B \cup \{\pm\infty\}$ (possibly without one of $\pm\infty$ or both) where B is Borel. Thus

$$f^{-1}(E) = f^{-1}(B) \cup f^{-1}(\{\pm\infty\}) \in \mathscr{M}$$

since each individual set in the union is in $\mathscr{M}$.

2.1.2. Suppose $f, g : X \to \overline{\mathbb{R}}$ are measurable.

a) fg is measurable (where $0 \cdot (\pm\infty) = 0$).
b) Fix $a \in \overline{\mathbb{R}}$ and define $h(x) = a$ if $f(x) = -g(x) = \pm\infty$ and $h(x) = f(x) + g(x)$ otherwise. Then h is measurable.

Solution:

a) By Exercise 2.1.1 it suffices to show that $(fg)^{-1}(\{\pm\infty\}) \in \mathscr{M}$ and fg is measurable on $Y = (fg)^{-1}(\mathbb{R})$. First note that

$$(fg)^{-1}(\mathbb{R}) = (f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R})) \cup (f^{-1}\{\pm\infty\} \cap g^{-1}(0)) \cup (f^{-1}(0) \cap g^{-1}\{\pm\infty\}).$$

Since $f, g : X \to \overline{\mathbb{R}}$ are measurable, by applying the forward direction of 2.1.1 we see that $(fg)^{-1}(\mathbb{R}) \in \mathscr{M}$.

Next, we have that

$$\begin{aligned}
(fg)^{-1}\{\infty\} &= (f^{-1}\{\infty\} \cap g^{-1}(0, \infty]) \cup (f^{-1}(0, \infty] \cap g^{-1}\{\infty\}) \\
&\quad \cup (f^{-1}\{-\infty\} \cap g^{-1}[-\infty, 0)) \cup (f^{-1}[-\infty, 0) \cap g^{-1}\{-\infty\}), \\
(fg)^{-1}(\{-\infty\}) &= (f^{-1}\{\infty\} \cap g^{-1}[-\infty, 0)) \cup (f^{-1}[-\infty, 0) \cap g^{-1}\{\infty\}) \\
&\quad \cup (f^{-1}\{-\infty\} \cap g^{-1}(0, \infty]) \cup (f^{-1}(0, \infty] \cap g^{-1}\{-\infty\}).
\end{aligned}$$

All I have done here is split up the various cases when $fg = \infty$ based on the values of f and g independently. Once more by 2.1.1 we see that each of these is in $\mathscr{M}$. So, fg is measurable.

b) We must break this into cases. Depending if a is real or $\pm\infty$, the sets $f^{-1}\{\pm\infty\} \cap g^{-1}\{\mp\infty\}$ will fall into a different preimage of h.

- Case 1: $a \in \mathbb{R}$. In this case, we have that $h(x) = \pm\infty$ iff $f(x)$ or $g(x)$ is $\pm\infty$ and $f(x) \neq -g(x)$. That is,
$$h^{-1}\{\infty\} = (f^{-1}\{\infty\} \cap g^{-1}(-\infty, \infty]) \cup (f^{-1}(-\infty, \infty] \cap g^{-1}\{\infty\})$$
and similarly for $h^{-1}\{-\infty\}$. Furthermore, $h(x)$ is real iff $f(x)$ and $g(x)$ are real or $f(x) = -g(x) = \pm\infty$. That is,
$$h^{-1}(\mathbb{R}) = (f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R})) \cup (f^{-1}\{\pm\infty\} \cap g^{-1}\{\mp\infty\}).$$
So, $h^{-1}(\mathbb{R})$ and $h^{-1}\{\pm\infty\}$ are measurable when f and g are.
- Case 2: $a = \infty$. The only change here is that when $f(x) = -g(x) = \pm\infty$, h is infinite instead of real. Hence, $h^{-1}\{\infty\}$ and $h^{-1}(\mathbb{R})$ are altered as
$$\begin{aligned} h^{-1}\{\infty\} &= (f^{-1}\{\infty\} \cap g^{-1}(-\infty, \infty]) \cup (f^{-1}(-\infty, \infty] \cap g^{-1}\{\infty\}) \\ &\quad \cup (f^{-1}\{\pm\infty\} \cap g^{-1}\{\mp\infty\}) \\ &= (f^{-1}\{\infty\} \cap g^{-1}(\overline{\mathbb{R}})) \cup (f^{-1}(\overline{\mathbb{R}}) \cap g^{-1}\{\infty\}), \\ h^{-1}(\mathbb{R}) &= f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}). \end{aligned}$$
still, all of the preimages $h^{-1}(\mathbb{R})$ and $h^{-1}\{\pm\infty\}$ are measurable when f and g are.

There is probably a cleaner way of doing this problem, but I think it is clarifying to break down each set; it is a common analysis technique. Another approach is to maybe write fg and h in terms of characteristic functions. XXX

2.1.3. If $\{f_n\}$ is a sequence of measurable functions on X, then $\{x \mid \lim f_n(x) \text{ exists}\}$ is a measurable set.

Solution: I present two proofs of this, each with their own merit. First, remember that $\lim f_n(x)$ exists iff $\liminf f_n(x) = \limsup f_n(x)$. In the notation of Proposition 2.7, $g_3(x) = \limsup f_n(x)$ and $g_4(x) = \liminf f_n(x)$ are measurable whenever $\{f_n\}$ is a sequence of measurable functions. So, $\{x \mid \lim f_n(x) \text{ exists}\} = \{g_3 = g_4\}$. Thus, it suffices to show if f, g are two measurable functions then $\{f = g\}$ is a measurable set. To do this, we show that $\{f \geq g\}$ is measurable. Ideally, we would look at $\{f - g \geq 0\}$ and conclude measurability since $f - g$ is a difference of measurable functions. However, $f - g$ is well defined. Indeed, it is possible that $f(x) = g(x) = \infty$. But we have
$$\{f \geq g\} = (\{f - g \geq 0\} \cap g^{-1}[-\infty, \infty)) \cup (\{f = \infty\} \cap \{g = \infty\})$$
By the above discussion, the first set is measurable. We need only investigate measurability of $\{f = \infty\} \cap \{g = \infty\}$. But, by Exercise 2.1.1 each of these is measurable. Now since $\{f \geq g\}$ is measurable whenever f and g are, by reversing roles of f and g we see that $\{g \geq f\}$ is also measurable. Hence, their intersection $\{f = g\}$ is measurable. Applying this with $f = g_3$ and $g = g_4$ gives the result.

Alternatively, remember that $\lim f_n(x)$ exists iff the sequence is Cauchy (due to completeness of $\mathbb{R}$). Thus, $\lim f_n(x)$ exists iff for every $\epsilon > 0$ there exists an N such that for $n, m \geq N$ we have $|f_n(x) - f_m(x)| < \epsilon$. A standard trick for these types of problems is to write out the set we're interested in by using unions and intersections; in particular "for all" turns into intersection, whereas "there exists" turns into union. Hence, we have
$$\left\{x \;\middle|\; \lim_{n\to\infty} f_n(x) \text{ exists}\right\} = \bigcap_{\epsilon>0} \bigcup_{N=1}^{\infty} \bigcap_{n,m\geq N} \{x \mid |f_n(x) - f_m(x)| < \epsilon\}.$$
There are three things to notice here. First, as before, it doesn't quite make sense to talk about $|f_n(x) - f_m(x)|$ if $f_n(x) = f_m(x) = \infty$. But, we still say that the limit exists even when $\lim f_n(x) = \infty$. So, we should really split our set into two sets, one where the limit exists and is finite, and one where the limit exists and is infinite. The second is to notice that, when it makes sense, $\{|f_n(x) - f_m(x)| < \epsilon\} = g^{-1}(-\epsilon, \epsilon)$ where $g(x) = f_n(x) - f_m(x)$. As the difference of measurable functions, g is measurable. Hence $\{|f_n(x) - f_m(x)| < \epsilon\}$ is a measurable set. Finally, notice that each union and intersection is countable except for the first, over ϵ. But we can simply take any

sequence going to zero, in particular we can take $\epsilon = 1/k$. Thus, we express our set in terms of countable unions/intersections of measurable sets, and is therefore measurable.

In total, we have that

$$\left\{x \;\middle|\; \lim_{n\to\infty} f_n(x) \text{ exists}\right\} = \left\{x \;\middle|\; \lim_{n\to\infty} f_n(x) \text{ exists and is finite}\right\} \cup \left\{x \;\middle|\; \lim_{n\to\infty} f_n(x) = \pm\infty\right\}.$$

The first set can be written as

$$\left\{x \;\middle|\; \lim_{n\to\infty} f_n(x) \text{ exists and is finite}\right\} = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n,m\geq N} \{x \mid |f_n(x) - f_m(x)| < 1/k\}$$

which makes sense since eventually the functions themselves become finite. Let us now look at the set $\{x \mid \lim f_n(x) = \infty\}$. If $\lim f_n(x) = \infty$, this means for every $M > 0$ there exists an N such that for $n \geq N$, $|f_n(x)| \geq M$. That is,

$$\left\{x \;\middle|\; \lim_{n\to\infty} f_n(x) = \infty\right\} = \bigcap_{M=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n\geq N} \{x \mid |f_n(x)| \geq M\}.$$

The sets $\{|f_n| \geq M\}$ are measurable since the f_n are, so that $\{x \mid \lim f_n(x) = \infty\}$ is measurable. Similar reasoning works for the $-\infty$ case.

It is a little ambiguous in this problem whether the functions are real or complex valued. If they are complex valued, we simply have that the limit exists iff the limits of the real and imaginary parts exist. So, we can focus on those, which are each real valued functions, and apply the above.

2.1.4. If $f : X \to \overline{\mathbb{R}}$ and $f^{-1}((r,\infty]) \in \mathscr{M}$ for each $r \in \mathbb{Q}$, then f is measurable.

Solution: The sets $(a,\infty]$ generate $\mathscr{B}_{\overline{\mathbb{R}}}$, so that we simply need to show $f^{-1}(a,\infty] \in \mathscr{M}$ for all real a. Fix a real a and let q_n be a sequence of rational numbers decreasing to a. Then,

$$f^{-1}(a,\infty] = \bigcup_{n=1}^{\infty} f^{-1}(q_n,\infty]$$

which is therefore in $\mathscr{M}$.

2.1.5. If $X = A \cup B$ where $A, B \in \mathscr{M}$, a function f on X is measurable iff f is measurable on A and on B.

Solution: Recall that f is measurable on $E \subset X$ if $f^{-1}(K) \cap E \in \mathscr{M}$ for all Borel sets K. If f is measurable on A and B separately, then $f^{-1}(K) \cap A, f^{-1}(K) \cap B \in \mathscr{M}$ for any Borel K. But then

$$f^{-1}(K) = f^{-1}(K) \cap X = f^{-1}(K) \cap (A \cup B) = (f^{-1}(K) \cap A) \cup (f^{-1}(K) \cap B) \in \mathscr{M}.$$

Now if f is measurable, $f^{-1}(K) \in \mathscr{M}$ for any Borel set K. Since $A, B \in \mathscr{M}$ we have that $f^{-1}(K) \cap A$ and $f^{-1}(K) \cap B$ are both in $\mathscr{M}$. So f is measurable on A and on B.

2.1.6. The supremum of an uncountable family of measurable $\overline{\mathbb{R}}$-valued functions on X can fail to be measurable (unless the σ-algebra $\mathscr{M}$ is very special).

Solution: Consider any uncountable nonmeasurable set Y. For each $y \in Y$ let $f_y(x) = \chi_{\{y\}}$. Then the supremum over all $y \in Y$ is simply $f(x) = \chi_Y$. But Y is not measurable so that f is not.

2.1.7. Suppose that for each $\alpha \in \mathbb{R}$ we are given a set $E_\alpha \in \mathscr{M}$ such that $E_\alpha \subset E_\beta$ whenever $\alpha < \beta$, $\bigcup_{\alpha\in\mathbb{R}} E_\alpha = X$, and $\bigcap_{\alpha\in\mathbb{R}} E_\alpha = \emptyset$. Then there is a measurable function $f : X \to \mathbb{R}$ such that $f(x) \le \alpha$ on E_α and $f(x) \ge \alpha$ on E_α^c for every α (Use Exercise 2.1.4)

Solution: As an example, let us consider $X = \mathbb{R}$ and $E_\alpha = (-\infty, \alpha)$. Then clearly $E_\alpha \subset E_\beta$ whenever $\alpha < \beta$, $\bigcup_{\alpha\in\mathbb{R}} = \mathbb{R}$, and $\bigcap_{\alpha\in\mathbb{R}} = (-\infty, -\infty) = \emptyset$. We now want to construct a measurable function f with the desired properties. To do this, let us look at the conditions on the E_α more carefully. The first one gives us a monotonicity type property. The second tells us that any real number x is in an E_α for some α. The last, in combination with the containment, tells us that there is a smallest α such that $x \in E_\beta$ for all $\beta > \alpha$. So, given x we can place it in some E_β. Then by the containment we can keep decreasing β further and further. That the intersection is empty tells us this must halt at some point. In other words, $\inf\{\alpha \mid x \in E_\alpha\}$ is a finite number.

With this heuristic in mind, a candidate f for the above example E_α is simply $f(x) = x$. However, note that $x = \inf\{\alpha \in \mathbb{R} \mid x \in E_\alpha\}$! For any $\alpha > x$ we have that $(-\infty, x) \subset (-\infty, \alpha)$, and therefore $x \in (-\infty, \alpha)$. It follows that x is the greatest lower bound.

In general, given the collection E_α, we consider the quantity $\inf\{\alpha \in \mathbb{R} \mid x \in E_\alpha\}$ for $x \in X$. Ideally we would define $f(x)$ to be this quantity and look at preimages of intervals to conclude measurability. Indeed, it satisfies the requirement that $f(x) \le \alpha$ on E_α while $f(x) \ge \alpha$ on E_α^c. The issue is that the inf is over too many numbers. To see this we investigate what $f^{-1}(\gamma)$ is. Remember that α is the smallest real such that $x \in E_\beta$ for all $\beta > \gamma$ (possibly equal to). In other words, $x \in E_\beta$ for all $\beta > \gamma$ but $x \notin E_\alpha$ for all $\alpha < \gamma$ (one of these will be an equality, but not both). Symbolically,

$$f^{-1}(\gamma) = \left(\bigcap_{\beta>\gamma} E_\beta\right) \cap \left(\bigcap_{\alpha<\gamma} E_\alpha^c\right) = \left(\bigcap_{\beta>\gamma} E_\beta\right) \cap \left(\bigcup_{\alpha<\gamma} E_\alpha\right)^c.$$

Note that we *cannot* simplify these. One may want to, for example, simplify the first to just be E_γ since $E_\alpha \subset E_\beta$ when $\alpha < \beta$. But, from the example given, we see that

$$\bigcap_{\beta>\gamma} E_\beta = \bigcap_{\beta>\gamma} (-\infty, \beta) = (-\infty, \gamma] \ne E_\gamma$$

However, we do have for this choice of E_α that

$$\bigcup_{\alpha<\gamma} E_\alpha = \bigcup_{\alpha<\gamma} (-\infty, \alpha) = (-\infty, \gamma) = E_\gamma.$$

But, if consider a different example with E_α as $(-\infty, \alpha]$ instead, then

$$\bigcup_{\alpha<\gamma} E_\alpha = \bigcup_{\alpha<\gamma} (-\infty, \alpha] = (-\infty, \gamma) \ne E_\gamma.$$

As an aside, notice when $E_\alpha = (-\infty, \alpha)$ we have that

$$f^{-1}(\gamma) = (-\infty, \gamma] \cap (-\infty, \gamma)^c = \{\gamma\}.$$

We know that $f(x) = x$, so this is consistent.
Next, let us look at $f^{-1}[\gamma, \delta]$. Since $\gamma < \delta$, we see that

$$\bigcap_{\beta>\gamma} E_\beta = \left(\bigcap_{\beta>\delta} E_\beta\right) \cap \left(\bigcap_{\delta\ge\beta>\gamma} E_\beta\right) \subset \bigcap_{\beta>\delta} E_\beta.$$

Similarly, we have that

$$\bigcup_{\alpha<\gamma} E_\alpha \subset \bigcup_{\alpha<\delta} E_\alpha$$

so that we get a reversed containment upon taking the complement. It follows that

$$f^{-1}[\gamma,\delta] = \left(\bigcap_{\beta>\delta} E_\beta\right) \cap \left(\bigcup_{\alpha<\gamma} E_\alpha\right)^c$$

The first set on the right hand side says that the infimum can be no larger than δ, while the second set says that it can be no smaller than γ. Now, taking δ to infinity gives

$$f^{-1}[\gamma,\infty) = \left(\bigcup_{\alpha<\gamma} E_\alpha\right)^c.$$

Note that, since the E_α are measurable, we are done if the union is over a countable set. But, it is not. So, let us shrink the domain over which we inf (remembering that this told us how to construct the union/intersections above). An obvious choice is to work with $\mathbb{Q}$ instead, so let us prove that $\inf\{\alpha \in \mathbb{R} \mid x \in E_\alpha\} = \inf\{q \in \mathbb{Q} \mid x \in E_q\}$. The inequality $\inf\{\alpha \in \mathbb{R} \mid x \in E_\alpha\} \leq \inf\{q \in \mathbb{Q} \mid x \in E_q\}$ is trivial since the inf on the left is over a larger set, and can therefore only decrease. To prove equality, suppose that $\inf\{\alpha \in \mathbb{R} \mid x \in E_\alpha\} < \inf\{q \in \mathbb{Q} \mid x \in E_q\}$. Now consider the interval $(\inf\{\alpha \in \mathbb{R} \mid x \in E_\alpha\}, \inf\{q \in \mathbb{Q} \mid x \in E_q\})$. Since the two endpoints are unequal, this is nonempty. Hence, there exists a rational r in it. By definition of $\inf\{\alpha \in \mathbb{R} \mid x \in E_\alpha\}$, we have that $x \in E_r$. On the other hand, if $x \in E_r$ then $r \geq \inf\{q \in \mathbb{Q} \mid x \in E_q\}$, a contradiction. So, let us set $f(x) = \inf\{q \in \mathbb{Q} \mid x \in E_q\}$.

With this, we see that for any rational r,

$$f^{-1}[r,\infty) = \left(\bigcup_{q<r, q\in\mathbb{Q}} E_q\right)^c$$

which is a countable combination of measurable sets, hence measurable. By Exercise 2.1.4, we conclude measurability of f.

2.1.8. If $f : \mathbb{R} \to \mathbb{R}$ is monotone, then f is Borel measurable.

Solution: We may assume WLOG that f is monotone nondecreasing. For, if f is monotone nonincreasing then $-f$ is monotone nondecreasing. Fix a real t and let $s = \inf\{x \in \mathbb{R} \mid f(x) \geq t\}$. If $\{f \geq t\}$ is empty of all of $\mathbb{R}$, then we are done (since these are Borel sets). Otherwise, we have that s is a finite number. For all $x > s$ we have $f(x) \geq t$ (by definition of the infimum). Then, if $f(s) \geq t$ we have $\{f \geq t\} = [s,\infty)$. Otherwise, $f(s) < t$ and $\{f \geq t\} = (s,\infty)$. In either case, we have a Borel set. Hence f is Borel measurable, since preimages of generators of Borel sets are Borel.

2.1.9. Let $f : [0,1] \to [0,1]$ be the Cantor function (§1.5), and let $g(x) = f(x) + x$.

a) g is a bijection from $[0,1]$ to $[0,2]$, and $h = g^{-1}$ is continuous from $[0,2]$ to $[0,1]$.
b) If C is the Cantor set, $m(g(C)) = 1$.
c) By Exercise 1.5.29, $g(C)$ contains a Lebesgue nonmeasurable set A. Let $B = g^{-1}(A)$. Then B is Lebesgue measurable but not Borel.
d) There exists a Lebesgue measurable function F and a continuous function G on $\mathbb{R}$ such that $F \circ G$ is not Lebesgue measurable.

Solution:

a) Note that $f(x)$ and $y = x$ are both increasing functions so that $g(x)$ is too. Furthermore, $g(x)$ is continuous. Via the contrapositive, one can show that an increasing function is injective. By IVT, we see that g takes on all values between $g(0) = 0 + 0 = 0$ and $g(1) = 1 + 1 = 2$. Hence g is surjective. Finally, we show that $h = g^{-1}$ is continuous by showing that g is an open map. Let $I = (a,b)$ be an open interval contained in $[0,1]$. I claim that $g(a,b) = (g(a), g(b))$ since g is increasing. If $a < x < b$ then clearly $g(a) < g(x) < g(b)$ by monotonicity. On the other hand, let $y \in (g(a), g(b))$. Then by IVT there exists an

$x \in (a,b)$ such that $f(x) = y$. Hence $y \in g(a,b)$. Since any open subset of $[0,1]$ is the union of intervals, it follows that g is an open map.

b) Let $K = [0,1] \setminus C$. By the construction of C, we see that K is the union of countably many disjoint intervals. Write $K = \bigcup_n I_n$. We first examine $g(I_n)$ for each I_n. By construction of f, it is constant on each of the I_n. So $g(I_n) = c_n + I_n$ for some constant c_n. It follows that $m(g(I_n)) = m(I_n)$ by translation invariance. Hence,

$$2 = m(g[0,1]) = m(g(K) \cup g(C)) = m(g(K)) + m(g(C)) = \sum_{n=1}^{\infty} m(I_n) + m(g(C)).$$

Now $C \cup (\bigcup_n I_n) = [0,1]$ so that

$$1 = m[0,1] = m(C) + m\left(\bigcup_{n=1}^{\infty} I_n\right) = m(C) + \sum_{n=1}^{\infty} m(I_n).$$

Since $m(C) = 0$, we see that $\sum_n m(I_n) = 1$. Combining this with the above gives $m(g(C)) = 1$.

c) As seen in Exercise 1.5.29, any measurable set with positive measure contains a Lebesgue nonmeasurable set. Since $m(g(C)) = 1 > 0$, we can find such a set A. We proved in a) that g is a homeomorphism of $[0,1]$ onto $[0,2]$. So,

$$B = g^{-1}(A) \subset g^{-1}(g(C)) = C.$$

But, C has measure zero and m is complete. Therefore B is Lebesgue measurable. Suppose that B is Borel. Since g^{-1} is continuous, it is measurable. Thus,

$$(g^{-1})^{-1}(B) = A$$

is measurable, a contradiction.

d) Let $F(x) = \chi_B(x)$ and $G(x) = g^{-1}(x)$ continuously extended to all of $\mathbb{R}$. Then F is measurable since B is, and G is continuous since g^{-1} is, and we extend it continuously. Now consider the Borel set $E = (1/2, 3/2)$. Since F only takes values in $\{0,1\}$ we see that $F^{-1}(E) = B$. But, $G^{-1}(B) = g(B) = g(g^{-1}(A)) = A$, which is not measurable. Hence $F \circ G$ cannot be Lebesgue measurable.

2.1.10. Prove Proposition 2.11.

Solution: We repeat Proposition 2.11 here for convenience.

Proposition 2.11: The following implications are valid iff the measure μ is complete:

a) If f is measurable and $f = g$ μ-a.e., then g is measurable.

b) If f_n is measurable for all $n \in \mathbb{N}$ and $f_n \to f$ μ-a.e., then f is measurable.

Suppose first that μ is complete.

a) The set $\{f = g\}^c$ has μ measure zero (since $f = g$ μ-a.e.) and therefore any subset is a measurable set. Now observe that

$$\{g \geq t\} = (\{g \geq t\} \cap \{f = g\}) \cup (\{g \geq t\} \cap \{f = g\}^c) = \{f \geq t\} \cup (\{g \geq t\} \cap \{f = g\}^c)$$

simply by definition of $\{f = g\}$. Since f is measurable, $\{f \geq t\}$ is. Moreover, $\{g \geq t\} \cap \{f = g\}^c \subset \{f = g\}^c$, and so it too is measurable. Hence $\{g \geq t\}$ is measurable for any t.

b) Let E be the set where f_n does not converge to f. Then E has μ measure zero since $f_n \to f$ μ-a.e. By Proposition 2.7, f is measurable on E^c (since the limit f_n exists everywhere on E^c). Since f is measurable on E, $\{f \geq t\} \cap E^c$ is measurable for any t. Then,

$$\{f \geq t\} = (\{f \geq t\} \cap E^c) \cup (\{f \geq t\} \cap E).$$

But $\{f \geq t\} \cap E \subset E$, which has measure zero. Since μ is complete, $\{f \geq t\} \cap E$ is measurable. We conclude that $\{f \geq t\}$ is measurable for any t, and hence f is measurable.

Suppose now that a) holds. Let N be a μ-null set and let $F \subset N$. We wish to show that F is μ measurable. To do this, let $f = \chi_N$ (a measurable function since N is measurable) and $g = \chi_F$. Certainly f and g agree on N^c (they are both 1 here). Since N has measure zero, N^c has full measure, and therefore $f = g$ μ-a.e. It follows that g is measurable.

Suppose now that b) holds. Let N be a μ-null set and $F \subset N$. Let f_n be identically zero for all n and let $f = \chi_F$. Evidently $f_n \to f$ on F^c (since f is zero here) but not on F. Since $F \subset N$ (and thus $N^c \subset F^c$) this implies that $f_n \to f$ on N^c. Since N has measure zero, it follows that $f_n \to f$ μ-a.e. Thus f is measurable, and F is a measurable set.

2.1.11. Suppose that f is a function on $\mathbb{R} \times \mathbb{R}^k$ such that $f(x, \cdot)$ is Borel measurable for each $x \in \mathbb{R}$ and $f(\cdot, y)$ is continuous for each $y \in \mathbb{R}^k$. For $n \in \mathbb{N}$, define f_n as follows. For $i \in \mathbb{Z}$, let $a_i = i/n$, and for $a_i \leq x \leq a_{i+1}$ let

$$f_n(x, y) = \frac{f(a_{i+1}, y)(x - a_i) - f(a_i, y)(x - a_{i+1})}{a_{i+1} - a_i}.$$

Then f_n is Borel measurable on $\mathbb{R} \times \mathbb{R}^k$ and $f_n \to f$ pointwise; hence f is Borel measurable on $\mathbb{R} \times \mathbb{R}^k$. Conclude by induction that every function on $\mathbb{R}^n$ that is continuous in each variable separately is Borel measurable.

Solution: Throughout, "measurable" means Borel measurable. First note that we may write $f_n(x, y)$ as

$$f_n(x, y) = \sum_{i \in \mathbb{Z}} \left(\frac{f(a_{i+1}, y)(x - a_i) - f(a_i, y)(x - a_{i+1})}{a_{i+1} - a_i} \right) \chi_{[a_i, a_{i+1}]}$$

Since each $[a_i, a_{i+1}]$ is measurable, it suffices to show that

$$f_{n,i}(x, y) = \frac{f(a_{i+1}, y)(x - a_i) - f(a_i, y)(x - a_{i+1})}{a_{i+1} - a_i}$$

is measurable. Then $f_n(x, y)$ is the sum of products of measurable functions, and is therefore measurable. But continuous functions are Borel measurable, and so $f(a_{i+1}, y), f(a_i, y), x - a_i, x - a_{i+1}$ are Borel measurable. Therefore $f_{n,i}(x, y)$ is measurable as the sum/product of measurable functions.

We now show that $f_n \to f$ pointwise. Fix $(x, y) \in \mathbb{R} \times \mathbb{R}^k$. Let $\epsilon > 0$; by continuity of $f(\cdot, y)$ there exists a $\delta > 0$ such that if $z \in B_\delta(x)$, $|f(z, y) - f(x, y)| < \epsilon$. There exists an N such that $N\delta > 1$. For any $n \geq N$, choose i such that $x \in [a_i, a_{i+1}]$; there exists such an i since the intervals $[a_i, a_{i+1}]$ cover all of $\mathbb{R}$. The size of this interval is $|(i + 1)/n - i/n| = 1/n \leq 1/N < \delta$. Hence, both $a_i, a_{i+1} \in B_\delta(x)$. Then

$$\begin{aligned}
|f_n(x, y) - f(x, y)| &= \left| \frac{f(a_{i+1}, y)(x - a_i) - f(a_i, y)(x - a_{i+1})}{a_{i+1} - a_i} - f(x, y) \right| \\
&= \left| \frac{f(a_{i+1}, y)(x - a_i) - f(a_i, y)(x - a_{i+1})}{a_{i+1} - a_i} - f(x, y) \left(\frac{x - a_i - (x - a_{i+1})}{a_{i+1} - a_i} \right) \right| \\
&= \left| \frac{(f(a_{i+1}, y) - f(x, y))(x - a_i) - (f(a_i, y) - f(x, y))(x - a_{i+1})}{a_{i+1} - a_i} \right| \\
&\leq \frac{|f(a_{i+1}, y) - f(x, y)||x - a_i|}{|a_{i+1} - a_i|} + \frac{|f(a_i, y) - f(x, y)||x - a_{i+1}|}{|a_{i+1} - a_i|} \\
&< \frac{\epsilon(x - a_i)}{a_{i+1} - a_i} + \frac{\epsilon(a_{i+1} - x)}{a_{i+1} - a_i} = \frac{\epsilon(a_{i+1} - a_i)}{a_{i+1} - a_i} = \epsilon
\end{aligned}$$

owing to the fact that $x \in [a_i, a_{i+1}]$ so that $a_i \leq x \leq a_{i+1}$ (hence, for example, $|x - a_i| = x - a_i$). So, we have shown for fixed (x, y) and every $\epsilon > 0$ there exists an N such that for $n \geq N$, $|f_n(x, y) - f(x, y)| < \epsilon$. Thus $f_n \to f$ pointwise.

Before continuing, let us provide a word of caution. Note that if $f : \mathbb{R}^n \to \mathbb{R}$ is continuous in each variable, it need not be continuous. To see this, consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} xy/(x^2+y^2) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Then, for fixed $y \neq 0$ we have $f(x,y) = xy/(x^2+y^2)$, which is some rational function with nonzero denominator. Hence $f(x,y)$ is continuous. Now let $y = 0$. Then, f is identically zero and is therefore continuous. So, f is continuous in x. Since f is symmetric in x and y, the same reasoning shows that f is continuous in y with fixed x. However, if we take the limit of f to $(0,0)$ along the line $y = x$, we get

$$\lim_{(x,y)\to(0,0), y=x} f(x,y) = \lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{x^2}{x^2+x^2} = \frac{1}{2} \neq 0 = f(0,0)$$

so f is not continuous at $(0,0)$.

What's the point of this? We know that every continuous function is Borel. If continuous in each variable implied continuous, we would be done. This reveals the condition of $f(x,\cdot)$ being Borel. In our above analysis, we only used this fact once, and of course we can assume the stronger condition that $f(x,\cdot)$ is continuous. If we do this, however, we would not be able to do the induction properly. From the result, we get a Borel function, and we will feed this back into the hypotheses – if our hypothesis had continuous instead of Borel, we would not be able to continue.

Clearly a function $f : \mathbb{R} \to \mathbb{R}$ continuous in "each" variable is Borel. Now suppose that every $f : \mathbb{R}^n \to \mathbb{R}$ continuous in each variable is Borel. We show that any $g : \mathbb{R}^{n+1} \to \mathbb{R}$ which is continuous in each variable separately is Borel. To do this, suppose we have such a g. We can identify $\mathbb{R}^{n+1}$ with $\mathbb{R} \times \mathbb{R}^n$ via $(x_1, ..., x_{n+1}) \mapsto (x, y)$ where $x = x_1$ and $y = (x_2, ..., x_{n+1})$. Define $x \wedge y$ where $x \in \mathbb{R}$ and $y \in \mathbb{R}^n$ as the inverse of this, that is if $y = (y_1, ..., y_n)$ we have $x \wedge y = (x, y_1, ..., y_n)$. Let $\tilde{g} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be defined by $\tilde{g}(x,y) = g(x \wedge y)$. Then $\tilde{g}(x,\cdot) = g(x,\cdot,\cdot,...,\cdot)$ is a function on $\mathbb{R}^n$ which is continuous in each variable separately, and therefore by the inductive hypothesis is Borel. Moreover, $\tilde{g}(\cdot,y) = g(\cdot, y_1, ..., y_n)$ is continuous since g is continuous in the first variable. Thus, $\tilde{g}$ is Borel (n.b., this is the step we would not be able to do if we replace Borel to continuous in the hypothesis; we only have that $\tilde{g}(x,\cdot)$ is Borel not continuous). But $\tilde{g}$ is Borel iff g is Borel, so we are done.

2.2. **Integration of Nonnegative Functions.**

2.2.12. Prove Proposition 2.20. (See Proposition 0.20, where a special case is proved).

Solution: We state Proposition 2.20 here for convenience.

Proposition 2.20: If $f \in L^+$ and $\int f < \infty$ then $\{x \mid f(x) = \infty\}$ is a null set and $\{x \mid f(x) > 0\}$ is σ-finite.

Let $E = \{f = \infty\}$. Then $\phi_n = n\chi_E$ is a measurable simple function such that $0 \leq \phi_n \leq f$. Hence,

$$n\mu(E) = \int \phi_n \leq \int f < \infty.$$

If $\mu(E) \neq 0$, we can increase n enough to make this inequality false.

Let F_n be defined by

$$F_n = \left\{ \frac{1}{n+1} \leq f < \frac{1}{n} \right\}$$

so that

$$F := \bigcup_{n=1}^{\infty} F_n = \{0 < f < 1\}.$$

Next let E_n be defined by

$$E_n = \{n \le f < n+1\}$$

so that

$$E := \bigcup_{n=1}^{\infty} E_n = \{f \ge 1\}$$

Hence $F \cup E = \{f > 0\}$. Our goal is to show that each E_n and F_n has finite measure. To do this, let ϕ and ψ be given by

$$\phi(x) = \sum_{n=1}^{\infty} n\chi_{E_n} \quad \psi(x) = \sum_{n=1}^{\infty} \frac{1}{n+1}\chi_{F_n}.$$

By definition of E_n and F_n, we see that $0 \le \phi(x) + \psi(x) \le f(x)$. And $\phi(x) + \psi(x)$ is a simple function (we do not need to pass to a common refinement since all the F_n and E_n are disjoint). We then have

$$\sum_{n=1}^{\infty} n\mu(E_n) + \sum_{n=1}^{\infty} \frac{\mu(F_n)}{n+1} \le \int \phi_n + \psi_n \le \int f < \infty.$$

Since all the terms in the series are nonnegative, it follows that $\mu(E_n)$ and $\mu(F_n)$ are finite. This likely isn't the cleanest setup, but it has the advantage of all the E_n and F_n being disjoint; also by modifying F_n slightly (say to have bounds with $1/(n+1)^2$ and $1/n^2$), you can obtain a bound on the measures of the F_n.

2.2.13. Suppose $\{f_n\} \subset L^+$, $f_n \to f$ pointwise, and $\int f = \lim \int f_n < \infty$. Then $\int_E f = \lim \int_E f_n$ for all $E \in \mathscr{M}$. However, this need not be true if $\int f = \lim \int f_n = \infty$.

Solution: Let $E \in \mathscr{M}$. Then $f_n\chi_E \to f\chi_E$. Fatou gives us

$$\int_E f = \int f\chi_E = \int \lim_{n\to\infty} f_n\chi_E = \int \liminf_{n\to\infty} f_n\chi_E \le \liminf_{n\to\infty} \int f_n\chi_E \le \lim_{n\to\infty} \int f_n\chi_E = \lim_{n\to\infty} \int_E f_n.$$

So, we just need to show the reverse inequality. Note that $\int f = \int_E f + \int_{E^c} f$. Since $\int f < \infty$, both integrals are finite and we can write $\int_{E^c} f = \int f - \int_E f$. Now apply Fatou exactly as above but replacing E with E^c. Thus,

$$\int f - \int_E f = \int_{E^c} f \le \lim_{n\to\infty} \int_{E^c} f_n = \lim_{n\to\infty} \int f_n - \int_E f_n = \lim_{n\to\infty} \int f_n - \lim_{n\to\infty} \int_E f_n$$

Since $\int f = \lim \int f_n < \infty$, we can cancel them from both sides. Hence, $\lim \int_E f_n \le \int_E f$ as desired.

To show a case when this fails, we provide an "escape to infinity". Folland provides the example $\chi_{[n,n+1]}$ as an example of a sequence of measurable functions for which Fatou fails to be an equality. Importantly, $\chi_{[n,n+1]} \to 0$ but $\int \chi_{[n,n+1]} = 1$ for all n. Now consider $f_n = \chi_{(-\infty,0)} + \chi_{[n,n+1]}$. For $E = [0,\infty)$ we have that $f_n|_E$ is the above example, and so $\int_E f \ne \lim \int_E f_n$.

2.2.14. If $f \in L^+$, let $\lambda(E) = \int_E f\, d\mu$ for $E \in \mathscr{M}$. Then λ is a measure on $\mathscr{M}$, and for any $g \in L^+$, $\int g\, d\lambda = \int fg\, d\mu$. (First suppose that g is simple.)

Solution: Clearly

$$\lambda(\emptyset) = \int_{\emptyset} f\, d\mu = \int f\chi_{\emptyset}\, d\mu = \int 0 = 0.$$

Now let $\{E_n\}_{n=1}^{\infty} \subset \mathscr{M}$ be a collection of disjoint sets. Since $f \in L^+$, $f_n = f\chi_{E_n}$ is a sequence of measurable functions. Let $E = \bigcup_n E_n$, a union of disjoint sets. It follows that $\chi_E = \sum_{n=1}^{\infty} \chi_{E_n}$. So, $f\chi_E = \sum_{n=1}^{\infty} f\chi_{E_n}$. By Proposition 2.15,

$$\lambda(E) = \int_E f\, d\mu = \sum_{n=1}^{\infty} \int f\chi_{E_n}\, d\mu = \sum_{n=1}^{\infty} \int_{E_n} f\, d\mu = \sum_{n=1}^{\infty} \lambda(E_n),$$

hence λ is a measure.

Now let $\phi \in L^+$ be a simple function. Write $\phi = \sum_{j=1}^n a_j \chi_{E_j}$ where the E_j are disjoint measurable sets. Then by definition

$$\int \phi \, d\lambda = \sum_{j=1}^n a_j \lambda(E_j) = \sum_{j=1}^n a_j \int_{E_j} f \, d\mu = \int f \sum_{j=1}^\infty a_j \chi_{E_j} \, d\mu = \int f\phi \, d\mu$$

owing to the fact that $f \in L^+$ implies that $a_j f \chi_{E_j} \in L^+$ for fixed j. Let $g \in L^+$ and let ϕ_n be an increasing sequence of simple functions with $0 \leq \phi_n \leq g$ such that $\phi_n \to g$. By monotone convergence it follows that $\int g \, d\lambda = \lim \int \phi_n \, d\lambda$. Similarly, $f\phi_n$ increases to fg, so that monotone convergence also gives $\int fg \, d\mu = \lim \int f\phi_n \, d\mu$. Combining these two with the above result for simple functions yields

$$\int g \, d\lambda = \lim_{n\to\infty} \int \phi_n \, d\lambda = \lim_{n\to\infty} \int f\phi_n \, d\mu = \int fg \, d\mu.$$

Note that we can apply monotone convergence regardless of the measure we integrate against, even though λ is defined in terms of an integral itself.

2.2.15. If $\{f_n\} \subset L^+$, f_n decreases pointwise to f, and $\int f_1 < \infty$, then $\int f = \lim \int f_n$.

Solution: Note the resemblance to dominated convergence for sets (indeed, this is a generalization; If μ is counting measure we recover DCT for sets). So, hopefully a similar proof for DCT for sets will work here. By Proposition 2.16, we may modify f_1 on a set of measure zero without changing the integral. In particular, let $E = \{f_1 = \infty\}$. Since $\int f_1 < \infty$, it follows that $\mu(E) = 0$. Let $g_n = (f_1 - f_n)\chi_{E^c}$ (so, if we ever have $\infty - \infty$, we simply change it to zero). Since the f_n are decreasing, g_n is nonnegative, and as a sequence increases. It is clear that $g_n \to f_1 - f$ a.e. By monotone convergence/Corollary 2.17,

$$\int g = \lim_{n\to\infty} \int g_n = \lim_{n\to\infty} \left(\int f_1 - \int f_n \right) = \int f_1 - \lim_{n\to\infty} \int f_n.$$

We use $\int f_1 < \infty$ to break up the integral. Next,

$$\int g = \int_{E^c} f_1 - f = \int_{E^c} f_1 - \int_{E^c} f = \int f_1 - \int f.$$

Combining the two gives the result.

2.2.16. If $f \in L^+$ and $\int f < \infty$, for every $\epsilon > 0$ there exists $E \in \mathscr{M}$ such that $\mu(E) < \infty$ and $\int_E f > (\int f) - \epsilon$.

Solution: Let $E_n = \{f > 1/n\}$. Clearly the E_n are increasing sets so that $f_n = f\chi_{E_n}$ increases to f. By monotone convergence,

$$\lim_{n\to\infty} \int_{E_n} f = \lim_{n\to\infty} \int f_n = \int f.$$

Let $\epsilon > 0$. Then there exists an N such that

$$\int f - \int_{E_n} f = \left| \int f - \int_{E_n} f \right| < \epsilon$$

for all $n \geq N$. In particular, choose E_N. Then,

$$\left(\int f \right) - \epsilon < \int_{E_N} f.$$

We just need to show that $\mu(E_N) < \infty$. Observe that $\phi = 1/N\chi E_N$ is a simple function with $0 \leq \phi \leq f$. Then,

$$\int \phi = \frac{\mu(E_N)}{N} \leq \int f < \infty$$

from which $\mu(E_N) < \infty$ follows.

2.2.17. Assume Fatou's lemma and deduce the monotone convergence theorem from it.

Solution: Let $\{f_j\}_{j=1}^{\infty} \subset L^+$ such that $f_j \leq f_{j+1}$ for all j and $f = \lim_j f_j$. Since the limit exists, $f = \liminf_j f_j$. Hence,

$$\int f = \int \liminf_{j\to\infty} f_j \leq \liminf_{j\to\infty} \int f_j \leq \lim_{j\to\infty} \int f_j$$

where the middle inequality is due to Fatou. Since the f_j are increasing we have that $f_j \leq f$ for all j. Hence, $\int f_j \leq \int f$ for all j. Taking limits gives the reverse inequality.

2.3. **Integration of Complex Functions.**

2.3.18. Fatou's lemma remains valid if the hypothesis that $f_n \in L^+$ is replaced by the hypothesis that f_n is measurable and $f_n \geq -g$ where $g \in L^+ \cap L^1$. What is the analogue of Fatou's lemma for nonpositive functions?

Solution: Suppose that f_n is a sequence of measurable functions such that $f_n \geq -g$ for $g \in L^+ \cap L^1$. Let $h_n = f_n + g \geq 0$. Then by Fatou,

$$\int \liminf_{n\to\infty} f_n + \int g = \int \liminf_{n\to\infty}(f_n + g) \leq \liminf_{n\to\infty} \int (f_n + g) = \liminf_{n\to\infty} \int f_n + \int g$$

Since g is integrable, we can subtract it from both sides and obtain the conclusion of Fatou. Note that this is essentially just one of the inequalities used in DCT.

For nonnegative functions we have the following. Recall that

$$\liminf_{n\to\infty}(-a_n) = -\limsup_{n\to\infty} a_n.$$

So, suppose that f_n is a sequence of nonpositive functions. Then $g_n = -f_n$ is a sequence of nonnegative functions for which we can apply Fatou. Then,

$$\int \liminf_{n\to\infty}(-f_n) = \int \liminf_{n\to\infty} g_n \leq \liminf_{n\to\infty} \int g_n = \liminf_{n\to\infty} \left(-\int f_n\right).$$

Applying the above gives

$$\int -\limsup_{n\to\infty} f_n = \int \liminf_{n\to\infty}(-f_n) \leq \liminf_{n\to\infty} \left(-\int f_n\right) = -\limsup_{n\to\infty} \int f_n.$$

Rearranging yields

$$\limsup_{n\to\infty} \int f_n \leq \int \limsup_{n\to\infty} f_n.$$

2.3.19. Suppose $\{f_n\} \subset L^1(\mu)$ and $f_n \to f$ uniformly.

a) If $\mu(X) < \infty$, then $f \in L^1(\mu)$ and $\int f_n \to \int f$.
b) If $\mu(X) = \infty$, the conclusions of a) can fail. (Find examples on $\mathbb{R}$ with Lebesgue measure.)

Solution:

a) Let $\epsilon > 0$. Then there exists an N such that for all $n \geq N$ and $x \in X$, $|f_n(x) - f(x)| < \epsilon$. Now, for such n
$$\left|\int f_n - \int f\right| \leq \int |f_n - f| \leq \epsilon\mu(X).$$
Since $\mu(X)$ is finite, we are done.
b) Define f_n by
$$f_n(x) = \begin{cases} x/n^2 & 0 \leq x \leq n \\ 2/n - x/n^2 & n < x \leq 2n \\ 0 & x > 2n \end{cases}$$

These look like "moving hats" which decrease in height but increase in width. The graph of f_n looks like a triangle with vertices at $(0,0)$, $(2n,0)$, and $(0,1/n)$. Also each f_n is nonnegative. Hence,

$$\int_0^\infty f_n(x)\,dx = \frac{1}{2}(2n-0)\left(\frac{1}{n}\right) = 1$$

On the other hand, f_n is bounded by $1/n$. It follows that for every $\epsilon > 0$ we can find an N such that if $n \geq N$, $|f_n(x)| \leq 1/n \leq 1/N < \epsilon$. Hence, f_n uniformly converges to 0. Thus $1 = \lim \int f_n \neq \int f = 0$.

2.3.20. (A generalized Dominated Convergence Theorem) If $f_n, g_n, f, g \in L^1$, $f_n \to f$, and $g_n \to g$ a.e., $|f_n| \leq g_n$, and $\int g_n \to \int g$, then $\int f_n \to \int f$. (Rework the proof of the dominated convergence theorem).

Solution: Similar to the beginning of the proof of DCT in Folland, we may modify f and g so that they are measurable. We also pass to real and imaginary parts to reduce to the real case. Owing to the assumption $|f_n| \leq g_n$ we have $-f_n \leq g_n$ and $f_n \leq g_n$. Hence, $g_n + f_n \geq 0$ and $g_n - f_n \geq 0$. Before continuing, we prove a small lemma. Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be sequences of real numbers. In general it is not true that

$$\liminf_{n\to\infty}(a_n + b_n) = \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n.$$

For example, take the sequence $a_n = n$ and $b_n = -n$. We can show one inequality rather easily. Let $A_k = \inf_{n\geq k}\{a_n\}$, $B_k = \inf_{n\geq k}\{b_n\}$ and $C_k = \inf_{n\geq k}\{a_n + b_n\}$. Then $A_k \leq a_n, B_k \leq b_n, C_k \leq a_n + b_n$ for all $n \geq k$. It follows that

$$a_n + b_n \geq A_k + b_n \geq A_k + B_k$$

for all $n \geq k$. Hence $A_k + B_k$ is a lower bound for $\{a_n + b_n \mid n \geq k\}$ and $A_k + B_k \leq C_k$. Taking limits shows

$$\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n \leq \liminf_{n\to\infty}(a_n + b_n).$$

Recall now that $\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty} a_n$ so that $-\limsup_{n\to\infty} a_n = \liminf_{n\to\infty}(-a_n)$. Combining this with the above yields

$$\liminf_{n\to\infty} a_n = \liminf_{n\to\infty}(a_n + b_n - b_n) \geq \liminf_{n\to\infty}(a_n + b_n) + \liminf_{n\to\infty}(-b_n) = \liminf_{n\to\infty}(a_n + b_n) - \limsup_{n\to\infty} b_n.$$

Thus,

$$\liminf_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \geq \liminf_{n\to\infty}(a_n + b_n).$$

In particular, if $b_n \to b$ then

$$\liminf_{n\to\infty}(a_n + b_n) \leq \liminf_{n\to\infty} a_n + \lim_{n\to\infty} b_n = \liminf_{n\to\infty} a_n + b.$$

We apply this to the sequences $\int f_n$ and $\int g_n$, the latter of which converges to $\int g$. Thus

$$\liminf_{n\to\infty}\left(\int f_n + g_n\right) \leq \liminf_{n\to\infty}\int f_n + \int g.$$

Now recall that $g_n + f_n \geq 0$. Hence by Fatou,

$$\int f + \int g = \int \liminf_{n\to\infty}(f_n + g_n) \leq \liminf_{n\to\infty}\left(\int f_n + g_n\right) \leq \liminf_{n\to\infty}\int f_n + \int g$$

We also have $g_n - f_n \geq 0$, so by another application of Fatou,

$$\begin{aligned} -\int f + \int g &= \int \liminf_{n\to\infty}(-f_n + g_n) \leq \liminf_{n\to\infty}\left(\int -f_n + g_n\right) \\ &\leq \liminf_{n\to\infty}\left(-\int f_n\right) + \int g = -\limsup_{n\to\infty}\int f_n + \int g. \end{aligned}$$

Since g is integrable, we can subtract it from both sides of the above and get

$$\limsup_{n\to\infty}\int f_n \leq \int f \leq \liminf_{n\to\infty}\int f_n$$

as desired.

2.3.21. Suppose $f_n, f \in L^1$ and $f_n \to f$ a.e. Then $\int |f_n - f| \to 0$ iff $\int |f_n| \to \int |f|$. (Use Exercise 2.3.20.)

Solution: First note that $|f_n - f| \leq |f_n| + |f|$. Let $g_n = |f_n| + |f|$, which is integrable. We have that $g_n \to g := 2|f|$ and

$$\int |g_n| = \int |f_n| + \int |f| \to 2\int |f| = \int g$$

whenever $\int |f_n| \to \int |f|$. Thus we can apply Exercise 2.3.20 and conclude $\int |f_n - f| \to 0$ since $|f_n - f| \to 0$. Next, by the triangle inequality

$$\int |f_n| \leq \int |f| + \int |f_n - f| \to \int |f|$$

when $\int |f_n - f| \to 0$.

2.3.22. Let μ be counting measure on $\mathbb{N}$. Interpret Fatou's lemma and the monotone and dominated convergence theorems as statements about infinite series.

Solution: A function $f : \mathbb{N} \to \mathbb{C}$ is just a sequence $\{c_n\}_{n=1}^{\infty}$ of complex numbers; that is, $f(n) = a_n + ib_n$. Note that each such sequence is really a sequence of two real numbers, so we may take $f : \mathbb{N} \to \mathbb{R}$ instead. Integration of f against counting measure is simply an infinite series,

$$\int_{\mathbb{N}} f \; d\mu = \sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} a_n.$$

This can be seen by taking a sequence of simple functions converging to f. Namely, take the sequence

$$\phi_k(n) = f\chi_{\{1,\ldots,k\}}.$$

Since each singleton is disjoint,

$$\int \phi_k \; d\mu = \sum_{j=1}^{k} \int_{\{j\}} f \; d\mu = \sum_{j=1}^{k} f(j)\mu(\{j\}) = \sum_{j=1}^{k} f(j)$$

which are precisely the partial sums. We have that $f \in L^1(\mu)$ iff

$$\int_{\mathbb{N}} |f| \; d\mu = \sum_{n=1}^{\infty} |a_n| < \infty$$

That is, iff $\{a_n\}_{n=1}^{\infty}$ is absolutely convergent.

Suppose that $\{f_k\}_{k=1}^{\infty}$ is an increasing sequence of nonnegative $L^1(\mu)$ functions – that is $f_k(n) \leq f_{k+1}(n)$ for all $n \in \mathbb{N}$. Then for each k we have a sequence of real numbers $\{a_{k,n}\}_{n=1}^{\infty}$ given by $f_k(n) = a_{k,n}$. Let

$$f(n) = a_n = \lim_{k\to\infty} a_{k,n} = \lim_{k\to\infty} f_k(n)$$

This limit always exists (possibly infinite) since the sequence in k is monotone. Monotone convergence says that $\lim_k \int f_k = \int f$, that is

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} a_{k,n} = \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \lim_{k\to\infty} a_{k,n}$$

whenever $a_{n,k}$ increases to a_n (all nonnegative) and $\{a_{n,k}\}_{n=1}^{\infty}$ is convergent for all k.

Suppose that $\{f_k\}_{k=1}^{\infty}$ is a sequence of $L^1(\mu)$ functions. Similarly, we obtain a collection of sequences $\{a_{k,n}\}_{n=1}^{\infty}$ each of which is absolutely convergent. Further assume that $a_{k,n} \to a_n$ for almost every

n. Suppose that $|f_k| \leq g$ for some $g \in L^1(\mu)$, meaning there is an absolutely convergent sequence $\{b_n\}_{n=1}^{\infty}$ such that $|a_{k,n}| \leq b_n$ for all k. Then,

$$\sum_{n=1}^{\infty} \lim_{k\to\infty} a_{k,n} = \sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} \sum_{n=1}^{\infty} a_{k,n},$$

where the series in the middle is absolutely convergent.

Finally, suppose that $\{f_k\}_{k=1}^{\infty}$ is a sequence of $L^1(\mu)$ functions. Then,

$$\sum_{n=1}^{\infty} \liminf_{k\in\mathbb{N}} a_{k,n} \leq \liminf_{k\in\mathbb{N}} \sum_{n=1}^{\infty} a_{k,n}.$$

where the $a_{k,n}$ are all nonnegative.

2.3.23. Given a bounded function $f : [a, b] \to \mathbb{R}$, let

$$H(x) = \lim_{\delta\to 0} \sup_{|y-x|\leq\delta} f(y), \quad h(x) = \lim_{\delta\to 0} \inf_{|y-x|\leq\delta} f(y).$$

Prove Theorem 2.28b by establishing the following lemmas:

a) $H(x) = h(x)$ iff f is continuous at x.
b) In the notation of the proof of Theorem 2.28a, $H = G$ a.e. and $h = g$ a.e. Hence H and h are Lebesgue measurable, and $\int_{[a,b]} H\ dm = \overline{I}_a^b(f)$ and $\int_{[a,b]} h\ dm = \underline{I}_a^b(f)$.

Solution: XXX

2.3.24. Let $(X, \mathscr{M}, \mu)$ be a measure space with $\mu(X) < \infty$, and let $(X, \overline{\mathscr{M}}, \overline{\mu})$ be its completion. Suppose $f : X \to \mathbb{R}$ is bounded. Then f is $\overline{\mathscr{M}}$-measurable (and hence in $L^1(\overline{\mu})$) iff there exists sequences $\{\phi_n\}$ and $\{\psi_n\}$ of $\mathscr{M}$-measurable simple functions such that $\phi_n \leq \psi_n$ and $\int(\psi_n - \phi_n)\ d\mu < 1/n$. In this case, $\lim \int \phi_n\ d\mu = \lim \int \psi_n\ d\mu = \int f\ d\overline{\mu}$.

Solution: XXX

2.3.25. Let $f(x) = x^{-1/2}$ if $0 < x < 1$, $f(x) = 0$ otherwise. Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rationals, and set $g(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n)$.

a) $g \in L^1(m)$, and in particular $g < \infty$ a.e.
b) g is discontinuous on every point and unbounded on every interval, and it remains so after modification on a Lebesgue null set.
c) $g^2 < \infty$ a.e., but g^2 is not integrable on any interval.

Solution:

a) We have that

$$\int_{-\infty}^{\infty} |g(x)|\ dx \leq \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{f(x-r_n)}{2^n}\ dx$$

Set $f_n(x) = f(x - r_n)/2^n$ for $n \geq 1$. Since $f_n(x)$ is just a shift of f multiplied by a constant, it is $1/(2^n\sqrt{x})$ on $r_n < x < r_n + 1$ and zero else. Moreover, f_n is nonnegative so that

$$\int_{-\infty}^{\infty} |f_n(x)|\ dx = \lim_{a\to r_n} \int_a^{r_n+1} \frac{1}{2^n\sqrt{x - r_n}}\ dx = \lim_{a\to r_n} \frac{\sqrt{x - r_n}}{2^{n-1}}\bigg|_a^{r_n+1} = \frac{1}{2^{n-1}} < \infty.$$

Hence, each f_n is integrable. Thus, by Theorem 2.15 we can interchange the summation and integration. Doing this yields

$$\int_{-\infty}^{\infty} |g(x)|\ dx \leq \sum_{n=1}^{\infty} \int_{\infty}^{\infty} f_n(x)\ dx = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 2.$$

b) Fix an $x_0 \in \mathbb{R}$ where $g(x_0)$ is finite. Suppose g is continuous. Then for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $x \in B_\delta(x_0)$ then $|g(x) - g(x_0)|$. WLOG we may assume $\delta < 1$. By density of $\mathbb{Q}$ there exists an $r_n \in B_\delta(x_0)$. Note that, since all the f_n are nonnegative, $g(x) \geq f_n(x)$ for any x. In particular, if $r_n \in [x_0, x_0 + \delta)$ then $f_n(x)$ is nonnegative on $[r_n, x_0 + \delta)$. Now, choose x such that

$$r_n < x < \frac{1}{4^n(\epsilon + g(x_0))^2} + r_n,$$

which can certainly be done for some $x \in [r_n, x_0 + \delta)$ by just taking x close enough to r_n. Then, we have

$$g(x) \geq f_n(x) = \frac{1}{2^n\sqrt{x - r_n}} > \epsilon + g(x_0).$$

It follows that g is discontinuous. Now assume that $g(x_0)$ is infinite. Then clearly there can be no $\epsilon > 0$ for which $|g(x_0) - g(x)| < \epsilon$, for any x. The only possibility is if $g(x)$ is also infinite (even though $g(x_0) - g(x)$ is strictly undefined). However, any interval around x_0 contains points x where $g(x)$ is finite (since $g < \infty$ a.e.). It thus follows that we can always find an x for which $|g(x_0) - g(x)| < \epsilon$ fails.

Finally, let I be any interval. Then there exists a rational $r_n \in I$. Since $x^{-1/2} \to \infty$ as $x \to 0$, it follows that $f_n(x) = f(x - r_n)/2^n$ goes to ∞ as $x \to r_n$. Hence f_n is unbounded on I. Combining this with the fact that g is the sum of nonnegative functions tells us that g is unbounded on every interval.

c) Since $g < \infty$ a.e. by part a), we immediately get that $g^2 < \infty$ a.e. Now for nonnegative numbers $a_1, ..., a_n$ we have that

$$\sum_{k=1}^{n} a_k^2 \leq \left(\sum_{k=1}^{n} a_k\right)^2$$

which can be proven by simple induction. Hence, we have

$$\sum_{k=1}^{\infty} a_k^2 \leq \left(\sum_{k=1}^{\infty} a_k\right)^2$$

if all the a_k are nonnegative. Applying this with $f_n(x)$ gives

$$\sum_{n=1}^{\infty} f_n(x)^2 \leq g(x)^2$$

Now let $I = [a, b]$ be an interval and let $r_k \in I$. Trivially we have

$$f_k(x) \leq \sum_{n=1}^{\infty} f_n(x)^2 \leq g(x)^2.$$

On the other hand, $f_k(x)^2$ is $1/4^k|x|$ on $r_k < x < r_k + 1$ and 0 else. Let $c = \min\{b, r_k + 1\}$. Then,

$$\begin{aligned}\int_I g(x)^2\, dx &\geq \int_I f_k(x)^2\, dx = \lim_{a \to r_k} \int_a^c \frac{1}{4^k|x - r_k|}\, dx \\ &= \frac{1}{4^k} \lim_{a \to r_k} \log(x - r_k)\Big|_a^c = \frac{1}{4^k}\log(c - r_k) + \frac{1}{4^k}\lim_{a \to 0}\log(1/a) = \infty.\end{aligned}$$

Thus g^2 is not integrable on any interval.

2.3.26. If $f \in L^1(m)$ and $F(x) = \int_{-\infty}^{x} f(t)\, dt$, then F is continuous on $\mathbb{R}$.

Solution: Let $\epsilon > 0$, and by Theorem 2.26 there exists a continuous g such that $\int |f - g| < \epsilon$

(every integrable function is nearly continuous). Let $x \le y$. Since g is continuous on $[x, y]$, it is bounded (say by $M > 0$). Thus,

$$|F(y) - F(x)| \le \left| \int_x^y f(t)\ dt \right| \le \int_x^y |f(t)|\ dt \le \int_x^y |g(t)|\ dt + \int_x^y |f(t) - g(t)|\ dt \le |y - x|(M + \epsilon).$$

Taking $\epsilon \to 0$ shows that F is Lipschitz continuous.

2.3.27. Let $f_n(x) = ae^{-nax} - be^{-nbx}$ where $0 < a < b$.

a) $\sum_{n=1}^{\infty} \int_0^{\infty} |f_n(x)|\ dx = \infty$.

b) $\sum_{n=1}^{\infty} \int_0^{\infty} f_n(x)\ dx = 0$.

c) $\sum_{n=1}^{\infty} f_n \in L^1([0, \infty), m)$, and $\int_0^{\infty} \sum_{n=1}^{\infty} f_n(x)\ dx = \log(b/a)$.

Solution: This is meant to show why the finiteness condition in Theorem 2.25 is necessary. I will actually do these out of order.

b) Fix n and let $E_n(x, a) = ae^{-nax}$ on $\mathbb{R} \times (0, \infty)$. Then,

$$\int_0^{\infty} E_n(x, a)\ dx = \int_0^{\infty} ae^{-nax}\ dx = -\frac{a}{na} e^{-nax} \Big|_0^{\infty} = \frac{1}{n},$$

which is independent of a. Hence,

$$\int_0^{\infty} f_n(x)\ dx = \int_0^{\infty} E_n(x, a) - E_n(x, b)\ dx = \frac{1}{n} - \frac{1}{n} = 0.$$

Thus taking the sum over n gives the result.

a) One can show (graphically, or by other means) that $f_n(x)$ is negative for $0 \le x < c$ and positive for $x > c$. We find c now. If $f_n(x) = 0$, then

$$e^{nbx - nax} = \frac{b}{a} \Rightarrow nx(b - a) = \log(b) - \log(a) \Rightarrow x = \frac{\log(b) - \log(a)}{n(b - a)}.$$

So, c is this value. Then we can split the integral as

$$\int_0^{\infty} |f_n(x)|\ dx = \int_0^c -f_n(x)\ dx + \int_c^{\infty} f_n(x)\ dx.$$

The indefinite integral of $f_n(x)$ is

$$\int f_n(x)\ dx = \int E_n(x, a) - E_n(x, b)\ dx = \frac{e^{-nbx}}{n} - \frac{e^{-nax}}{n} + C$$

so that

$$\int_0^{\infty} |f_n(x)|\ dx = \left(\frac{e^{-nax}}{n} - \frac{e^{-nbx}}{n} \right) \Big|_0^c + \left(\frac{e^{-nbx}}{n} - \frac{e^{-nax}}{n} \right) \Big|_c^{\infty}.$$

Note that at the limits 0 and ∞, both of these vanish. Hence, evaluating this gives

$$\begin{aligned} \int_0^{\infty} |f_n(x)|\ dx &= \frac{e^{-nac} - e^{-nbc}}{n} - \frac{e^{-nbc} - e^{-nac}}{n} = \frac{2(e^{-nac} - e^{-nbc})}{n} \\ &= \frac{2}{n} \left(e^{a(\log(a) - \log(b))/(b-a)} - e^{b(\log(a) - \log(b))/(b-a)} \right) \end{aligned}$$

Since this is just some constant multiple of $1/n$, we see that the series diverges.

c) Still using $E_n(x, a)$ as before, note that for fixed x and a,

$$E_n(x, a) = ae^{-nax} = a \left(\frac{1}{e^{ax}} \right)^n.$$

Thus, $\sum_{n=1}^{\infty} E_n(x, a)$ is a geometric series with ratio $1/e^{ax}$, which converges pointwise to

$$\sum_{n=1}^{\infty} E_n(x, a) = \sum_{n=0}^{\infty} E_n(x, a) - a = \frac{a}{1 - 1/e^{ax}} - a = \frac{ae^{ax}}{e^{ax} - 1} - a = \frac{a}{e^{ax} - 1}.$$

It follows that

$$f(x) = \sum_{n=1}^{\infty} f_n(x) = \frac{a}{e^{ax}-1} - \frac{b}{e^{bx}-1}.$$

It is easily verified that

$$\int \frac{a}{e^{ax}-1}\ dx = \log\left(1 - \frac{1}{e^{ax}}\right)$$

for $x \in (0,\infty]$ via a substitution $u = e^{ax}$ and partial fraction decomposition. Notice that we can extend the above to a continuous function on $(0,\infty]$ by defining it to be 0 in at ∞ (indeed, taking a formal limit shows that it is be zero in the limit). Thus,

$$\begin{aligned}
\int_0^\infty f(x)\ dx &= \left(\log\left(1-\frac{1}{e^{ax}}\right) - \log\left(1-\frac{1}{e^{bx}}\right)\right)\Bigg|_0^\infty \\
&= -\lim_{x\to 0^+} \log\left(\frac{e^{bx}}{e^{ax}}\left(\frac{e^{ax}-1}{e^{bx}-1}\right)\right) \\
&= -\log\left(\lim_{x\to 0^+}\frac{e^{bx}}{e^{ax}}\left(\frac{e^{ax}-1}{e^{bx}-1}\right)\right)
\end{aligned}$$

since the logarithm is continuous on $(0,\infty)$. Let us evaluate this limit independently,

$$\begin{aligned}
\lim_{x\to 0}\frac{e^{bx}}{e^{ax}}\left(\frac{e^{ax}-1}{e^{bx}-1}\right) &= \left(\lim_{x\to 0^+}\frac{e^{bx}}{e^{ax}}\right)\left(\lim_{x\to 0^+}\frac{e^{ax}-1}{e^{bx}-1}\right) = \lim_{x\to 0^+}\left(\frac{e^{ax}-1}{e^{bx}-1}\right) = \frac{0}{0} \\
&= \lim_{x\to 0^+}\frac{ae^{ax}}{be^{bx}} = \frac{a}{b}
\end{aligned}$$

where we have formally used L'Hopitals between the first and second line. Thus,

$$\int_0^\infty \sum_{n=1}^\infty f_n(x)\ dx = \int_0^\infty f(x)\ dx = -\log\left(\frac{a}{b}\right) = \log\left(\frac{b}{a}\right).$$

2.3.28. Compute the following limits and justify the calculations:

a) $\lim_{n\to\infty}\int_0^\infty (1+(x/n))^{-n}\sin(x/n)\ dx$.

b) $\lim_{n\to\infty}\int_0^1 (1+nx^2)(1+x^2)^{-n}\ dx$.

c) $\lim_{n\to\infty}\int_0^\infty n\sin(x/n)[x(1+x^2)]^{-1}\ dx$.

d) $\lim_{n\to\infty}\int_a^\infty n(1+n^2x^2)^{-1}\ dx$. (The answer depends on whether $a>0$, $a=0$, or $a<0$. How does this accord with the various convergence theorems?)

Solution: We will apply Theorem 2.27a several times.

a) Let $f : [0,\infty)\times[1,\infty)\to\mathbb{R}$ be defined by $f(x,n) = (1+(x/n))^{-n}\sin(x/n)$. Observe that as $n\to\infty$, $x/n\to 0$ so that $\lim_{n\to\infty}(1+x/n)^{-n}\sin(x/n)\to 0$, so we can continuously extend this to a function $f:[0,\infty)\times[1,\infty]\to\mathbb{R}$. For fixed n we see that

$$\left|\int_0^\infty f(x,n)\ dx\right| \le \int_0^\infty |f(x,n)|\ dx \le \int_0^\infty \frac{|\sin(x/n)|}{(1+x/n)^n}\ dx \le \int_0^\infty \frac{1}{(1+x/n)^n}\ dx$$

owing to the fact that $|\sin(x)|\le 1$. Now if $n>1$ we have

$$\left|\int_0^\infty f(x,n)\ dx\right| \le \int_0^\infty \frac{1}{(1+x/n)^n}\ dx = \frac{-n}{(n-1)(1+x/n)^{n-1}}\Bigg|_0^\infty = \frac{n}{n-1} < \infty$$

So, $f(\cdot,n)$ is integrable. By the above construction, $\lim_{n\to\infty} f(x,n) = f(x,\infty) = 0$. Finally, note that for fixed x the sequence $(1+x/n)^{-n}$ monotonically decreases to e^{-x}. It follows that

$$\left|\frac{\sin(x/n)}{(1+x/n)^n}\right| \le \frac{1}{(1+x/n)^n} \le \frac{1}{(1+x/2)^2}$$

where we know the bounding function is integrable by a previous calculation. This estimate holds for all x and $n \geq 2$. Since we're interested in the limit as $n \to \infty$, we only care that this estimate holds eventually for all n. Hence we can interchange limits and get

$$\lim_{n\to\infty} \int_0^\infty \frac{\sin(x/n)}{(1+x/n)^n}\ dx = \int_0^\infty \lim_{n\to\infty} \frac{\sin(x/n)}{(1+x/n)^n}\ dx = 0.$$

b) Let $f : [0,1]\times[1,\infty)$ be defined by $f(x,n) = (1+nx^2)(1+x^2)^{-n}$. We extend f to $[0,1]\times[1,\infty]$ by setting $f(x,\infty) = \chi_{\{0\}}$. Indeed, $\lim_{n\to\infty} f(0,n) = 1/1 = 1$, whereas for any $x \in (0,1]$,

$$\lim_{n\to\infty} \frac{1+nx^2}{(1+x^2)^n} \leq \lim_{n\to\infty} \frac{1+n}{x^{2n}} = 0$$

since $1/(1+x^2) \leq 1/x^2$ and $1+nx^2$ is maximized when $x = 1$. So $\lim_{n\to\infty} f(x,n) = \chi_{\{0\}}$. Now fix an $n \geq 1$. Since $1/(1+x^2) \leq 1$, we see that $1/(1+x^2)^n \leq 1/(1+x^2)$. Next, observe that $1+nx^2$ is bounded by $1+n$ on $[0,1]$, as seen before. Thus,

$$\int_0^\infty f(x,n)\ dx \leq \int_0^\infty \frac{1+n}{1+x^2}\ dx \leq \frac{(1+n)\pi}{2} < \infty$$

so that $f(\cdot,n)$ is integrable. Finally, we note that for any n, $f(x,n)$ is bounded by 1, since it is monotonically decreasing in x and $f(0,n) = 1$. Alternatively one can use the binomial theorem

$$(1+x^2)^n = \sum_{k=0}^n \binom{n}{k} x^{2k} = 1 + nx^2 + ... + nx^{2n-2} + x^{2n} \geq 1 + nx^2$$

for $n \geq 1$. Hence, $|f(x,n)| \leq 1$, which is integrable on $[0,1]$. Thus we can interchange limits and get

$$\lim_{n\to\infty} \int_0^1 f(x,n)\ dx = \int_0^\infty \chi_{\{0\}}\ dx = 0.$$

c) Let $f : (0,\infty)\times[1,\infty)$ be defined by $f(x,n) = n\sin(x/n)/(x(1+x^2))$. For fixed x, observe that

$$\lim_{n\to\infty} n \sin\left(\frac{x}{n}\right) = \lim_{n\to 0} \frac{1}{n} \sin(nx) = x.$$

Hence, for fixed x, $\lim_{n\to\infty} f(x,n) = x/(x(1+x^2)) = 1/(1+x^2)$. We extend f to $(0,\infty)\times[1,\infty]$ by this rule. Note that $f(x,n)$ monotonically increases (in n) to $1/(1+x^2)$. So,

$$\int_0^\infty f(x,n)\ dx \leq \int_0^\infty \frac{1}{1+x^2}\ dx = \frac{\pi}{2}.$$

Hence, $f(x,n)$ is integrable and there exists an integrable function $g(x) = 1/(1+x^2)$ such that $f(x,n) \leq g(x)$. It follows that we can interchange limits, and get

$$\lim_{n\to\infty} \int_0^\infty f(x,n)\ dx = \int_0^\infty \lim_{n\to\infty} f(x,n)\ dx = \int_0^\infty \frac{1}{1+x^2}\ dx = \frac{\pi}{2}.$$

d) We can actually integrate this function in the Riemann sense. Let $f_n(x) = n/(1+n^2x^2)$. Then,

$$\int_a^\infty f_n(x)\ dx = \lim_{b\to\infty} \int_a^b \frac{n}{1+(nx)^2}\ dx = \lim_{b\to\infty} \arctan(nx)\bigg|_a^b = \lim_{b\to\infty} \arctan(nb) - \arctan(na) = \frac{\pi}{2} - \arctan(na).$$

Now, if $a = 0$ then clearly $\arctan(na) = 0$. If $a > 0$ then $\arctan(na) \to \pi/2$ as $n \to \infty$. If $a < 0$, then $\arctan(na) \to -\pi/2$ as $n \to \infty$. Thus,

$$\lim_{n\to\infty} \int_a^\infty f_n(x)\ dx = \begin{cases} 0 & a > 0 \\ \pi/2 & a = 0 \\ \pi & a < 0 \end{cases}.$$

One can compute the pointwise limit of $n/(1+n^2x^2)$ as $n \to \inf$, and it approaches something like the Dirac delta. The integral of this is clearly 0 if $a > 0$, but for $a = 0$ or $a < 0$, the value depends on the convergence to the distribution.

2.3.29. Show that $\int_0^\infty x^n e^{-x}\ dx = n!$ by differentiating the equation $\int_0^\infty e^{-tx}\ dx = 1/t$. Similarly, show that $\int_{-\infty}^\infty x^{2n} e^{-x^2}\ dx = (2n)!\sqrt{\pi}/4^n n!$ by differentiating the equation $\int_{-\infty}^\infty e^{-tx^2}\ dx = \sqrt{\pi/t}$ (see Proposition 2.53).

Solution: Define $f_n : [0,\infty) \times [1/2, 2] \to \mathbb{R}$ by $f_n(x) = x^n e^{-tx}$ where $n \geq 0$. The strategy here is to differentiate the integral equation given and set $t = 1$. To do this, we don't need to consider values of t between $(0,\infty)$; indeed this does not work since $|\partial(e^{-tx})/\partial t| = xe^{-tx}$ tends to x as $t \to 0$, which is not integrable. But, note that xe^{-tx} monotonically increases as t decreases. So if we take just some interval around 1, we can choose the smallest value from this interval and bound xe^{-tx}. Here, I have chosen $t = 1/2$, so we wish to bound $xe^{-x/2}$. Let us find a $\lambda > 0$ such that

$$xe^{-x/2} \leq e^{-\lambda x}$$

since we know that $e^{-\lambda x}$ is integrable by the condition

$$\int_0^\infty e^{-\lambda x}\ dx = 1/\lambda < \infty.$$

which is integrable. There is some hope in doing this since we know that $xe^{-x/2}$ is bounded by $2/e$, and as $\lambda \to 0$ $e^{-\lambda x}$ approaches $1 > 2/e$ (so it becomes flatter and flatter, eventually rising above the graph of $xe^{-x/2}$). Explicitly, suppose that there exists an x where

$$xe^{-x/2} = e^{-\lambda x} \;\Rightarrow\; x = e^{(1/2-\lambda)x}$$

Then, we get

$$\log(x) = \left(\frac{1}{2} - \lambda\right) x$$

Note that $\log(x)$ and cx are tangent precisely when $c = 1/e$. Hence, if $1/2 - \lambda = 1/e$, then $xe^{-x/2}$ and $e^{-\lambda x}$ are tangent, and $e^{-\lambda x} \geq xe^{-x/2}$ (since in general the straight line $1/ex$ sits above $\log(x)$). It follows that

$$\left|\frac{\partial f_0}{\partial t}\right| = \left|\frac{\partial(e^{-tx})}{\partial t}\right| = |xe^{-tx}| \leq xe^{-x/2} \leq e^{-\lambda x}$$

where $\lambda = 1/2 - 1/e > 0$. As stated, $\int_0^\infty f_0(x,t)\ dx = \int_0^\infty e^{-tx}\ dx = 1/t < \infty$ so that $f_0(\cdot, t)$ is integrable for fixed t. Hence by Theorem 2.27b, we can interchange integration and differentiation and get

$$-\frac{1}{t^2} = \frac{\partial}{\partial t}\int_0^\infty e^{-tx}\ dx = \int_0^\infty \frac{\partial(e^{-tx})}{\partial t}\ dx = \int_0^\infty -xe^{-tx}\ dx = -\int_0^\infty f_1(x,t)\ dx.$$

This establishes the base case. We now assume inductively that

$$\int_0^\infty f_n(x,t)\ dx = \frac{n!}{t^{n+1}}.$$

By the exact same logic, we have that

$$\left|\frac{\partial f_n(x,t)}{\partial t}\right| = |x^{n+1}e^{-tx}| \leq x^{n+1}e^{-x/2} \leq x^n(xe^{-x/2}) \leq x^n e^{-\lambda x} = f_n(x,\lambda)$$

where again $\lambda = 1/2 - 1/e$. The inductive hypothesis precisely says that $f_n(\cdot, t)$ is integrable for fixed t. Hence, we can interchange differentiation and integration and get

$$-\frac{(n+1)!}{t^{n+2}} = \frac{\partial}{\partial t}\int_0^\infty f_n(x,t)\ dx = \int_0^\infty \frac{\partial f_n(x,t)}{\partial t}\ dx = \int_0^\infty -x^{n+1}e^{-tx}\ dx = -\int_0^\infty f_{n+1}(x,t)\ dx$$

establishing the inductive result. Applying this at $t = 1$ gives the desired integral.

Now we turn to work with $\int_{-\infty}^\infty x^{2n}e^{-x^2}\ dx$. Similarly, we will define $f_n(x,t) = x^{2n}e^{-tx^2}$ on $(-\infty,\infty) \times [1/2, 2]$ for $n \geq 0$. We are given that

$$\int_{-\infty}^\infty f_0(x,t)\ dx = \sqrt{\frac{\pi}{t}}.$$

Observe that in general

$$\left|\frac{\partial f_n(x,t)}{\partial t}\right| = |-x^2x^{2n}e^{-tx^2}| = x^{2(n+1)}e^{-tx^2} = f_{n+1}(x,t).$$

Note that $f_n(x,t)$ monotonically decreases in t, so that to bound $|\partial_t f_n(x,t)|$ we just need to bound $f_{n+1}(x,1/2)$. I will work with the $n=0$ case first. As before, we can find a constant $\lambda > 0$ such that $x^2e^{-x^2/2}$ and $e^{-\lambda x^2}$ are just tangent, and the latter bounds the former. Since $e^{-\lambda x^2} = f_0(x,\lambda)$, which we know is integrable, such a bound suffices. Suppose that there is a single solution to $e^{-\lambda x^2} = x^2e^{-x^2/2}$. Then,

$$e^{(1/2-\lambda)x^2} = x^2 \;\Rightarrow\; \left(\frac{1}{2}-\lambda\right)x^2 = 2\log(x)$$

Once again, this occurs when $\lambda = 1/2 - 1/e$ (one can manually verify this). So, we have found an integrable bound on $|\partial_t f_0(x,t)|$, and we can interchange differentiation and integration. Doing this yields

$$-\frac{1}{2t}\sqrt{\frac{\pi}{t}} = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} f_0(x,t)\;dx = \int_{-\infty}^{\infty}\frac{\partial f_0(x,t)}{\partial t}\;dx = \int_{-\infty}^{\infty} -x^2e^{-tx}\;dx = -\int_{-\infty}^{\infty} f_1(x,t)\;dx.$$

Now suppose that

$$\int_{-\infty}^{\infty} f_n(x,t)\;dx = \frac{(2n)!}{n!(4t)^n}\sqrt{\frac{\pi}{t}}.$$

Note that this is consistent with the $n=0$ and $n=1$ cases. The $n=0$ case is obvious, while the $n=1$ case gives

$$\frac{2!}{1!(4t)}\sqrt{\frac{\pi}{t}} = \frac{1}{2t}\sqrt{\frac{\pi}{t}}.$$

The need to have something like $(2n)!/(2^n n!)$ is because the powers on t are something like $(2k+1)/2$. So when we differentiate, the numerator of the constant is a product of odd integers only; we can convert it to a factorial using something like this. The inductive hypothesis tells us that $f_n(\cdot,t)$ is integrable for fixed t. We saw before that

$$\left|\frac{\partial f_n(x,t)}{\partial t}\right| = x^{2n}(x^2e^{-tx^2}) \le x^{2n}(x^2e^{-x^2/2}) \le x^{2n}e^{-\lambda x^2} = f_n(x,\lambda)$$

where $\lambda = 1/2 - 1/e > 0$ as before. Hence we bound the partial derivative by an integrable function. Thus,

$$\begin{aligned}
\int_{-\infty}^{\infty} -f_{n+1}(x,t)\;dx &= \int_{-\infty}^{\infty} -\frac{\partial f_n(x,t)}{\partial t}\;dx = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} f_n(x,t)\;dx = \frac{\partial}{\partial t}\left(\frac{(2n)!}{n!(4t)^n}\sqrt{\frac{\pi}{t}}\right)\\
&= \frac{\partial}{\partial t}\left(\frac{(2n)!\sqrt{\pi}}{n!4^nt^{(2n+1)/2}}\right) = \frac{-(2n+1)(2n)!\sqrt{\pi}}{2\cdot n!4^nt^{(2(n+1)+1)/2}} = -\frac{2(2n+1)(2n)!}{n!4^{n+1}t^{n+1}}\sqrt{\frac{\pi}{t}}\\
&= -\frac{2(n+1)(2n+1)(2n)!}{(n+1)!4^{n+1}t^{n+1}}\sqrt{\frac{\pi}{t}} = -\frac{(2(n+1))!}{(n+1)!4^{n+1}t^{n+1}}\sqrt{\frac{\pi}{t}}
\end{aligned}$$

which proves the inductive step. Setting $t=1$ gives the desired integral.

2.3.30. Show that $\lim_{k\to\infty}\int_0^k x^n(1-k^{-1}x)^k\;dx = n!$.

Solution: Fix an $n \ge 0$ and define $f : [0,\infty)\times[0,\infty) \to \mathbb{R}$ by $f(x,k) = x^n(1-x/k)^k\chi_{[0,k]}$. Observe that, for fixed k, $g_k(x) = f(x,k) = x^n(1-x/k)^k\chi_{[0,k]}$ is an integrable function. Since $g_k(0) = g_k(k) = 0$, it is a continuous function on a bounded set. Therefore it is bounded, say by M, and

$$\left|\int_0^k g_k(x)\;dx\right| \le \int_0^k |g_k(x)|\;dx \le \int_0^k M\;dx = Mk < \infty.$$

Next, note that for fixed x, $h_x(k) = x^n(1-x/k)^k\chi_{[0,k]}$ is an increasing function. Moreover, $\lim_{k\to\infty} h_x(k)$ is easily seen to be $x^n e^{-x}$ (via the standard limit definition for e). It follows that for any x, k,

$$|f(x,k)| \leq x^n e^{-x}.$$

The bounding function above is precisely the integrand of $\Gamma(n+1)$. Hence, $|f(x,k)|$ is bounded by a function in L^1. Finally, let us return to $h_x(k)$. Interpreting $\chi_{[0,k]}(x)$ can be a little tricky when x is fixed but k varies. However, one sees that if $k < x$, then certainly $\chi_{[0,k]}(x) = 0$, but if $k \geq x$ then it is 1. Hence, $\chi_{[0,k]}(x) = \chi_{[x,\infty)}(k)$. Now, notice that $h_x(k)$ is zero when $k = x$. Thus, h_x is continuous for fixed x. We may thus apply Proposition 2.27a and conclude

$$\lim_{k\to\infty}\int_0^k x^n\left(1-\frac{x}{k}\right)^k\,dx = \lim_{k\to\infty}\int_0^\infty f(x,k)\,dx = \int_0^\infty \lim_{k\to\infty} f(x,k)\,dx.$$

Evaluating the limit (noticing that x is currently fixed since f is being integrated along x) yields

$$\int_0^\infty \lim_{k\to\infty} f(x,k)\,dx = \int_0^\infty \lim_{k\to\infty} h_x(k)\,dx = \int_0^\infty x^n e^{-x}\,dx = \Gamma(n+1) = n!$$

2.3.31. Derive the following formulas by expanding part of the integrand into an infinite series and justifying the term-by-term integration. Exercise 2.3.29 may be useful. (Note: in d) and e), term-by-term integration works, and the resulting series converges, only for $a > 1$, but the formulas as stated are actually valid for all $a > 0$.)

a) For $a > 0$, $\int_{-\infty}^\infty e^{-x^2}\cos(ax)\,dx = \sqrt{\pi}e^{-a^2/4}$.
b) For $a > -1$, $\int_0^1 x^a(1-x)^{-1}\log(x)\,dx = \sum_{k=1}^\infty (a+k)^{-2}$.
c) For $a > 1$, $\int_0^\infty x^{a-1}(e^x-1)^{-1}\,dx = \Gamma(a)\zeta(a)$, where $\zeta(a) = \sum_{n=1}^\infty n^{-a}$.
d) For $a > 1$, $\int_0^\infty e^{-ax}x^{-1}\sin(x)\,dx = \arctan(a^{-1})$.
e) For $a > 1$, $\int_0^\infty e^{-ax}J_0(x)\,dx = (a^2+1)^{-1/2}$, where $J_0(x) = \sum_{n=1}^\infty (-1)^n x^{2n}/4^n(n!)^2$ is the Bessel function of order zero.

Solution:

a) Using the Taylor expansion of $\cos(ax)$, we have

$$\int_{-\infty}^\infty e^{-x^2}\cos(ax)\,dx = \int_{-\infty}^\infty e^{-x^2}\sum_{n=0}^\infty \frac{(-1)^n(ax)^{2n}}{(2n)!}\,dx = \int_{-\infty}^\infty \sum_{n=0}^\infty \frac{(-1)^n(ax)^{2n}e^{-x^2}}{(2n)!}\,dx.$$

Let $f_n(x) = (-1)^n(ax)^{2n}e^{-x^2}/(2n)!$. We wish to interchange the limit and summation. Theorem 2.25 gives a sufficient condition when we can do this, namely if all the f_n are in L^1 and if $\sum_{n=0}^\infty \int |f_n| < \infty$.

First let us compute $\int |f_n|$. Note that all the individual terms in f_n are nonnegative except possibly for $(-1)^n$. We have that

$$\int_{-\infty}^\infty |f_n(x)|\,dx = \int_{-\infty}^\infty \frac{(ax)^{2n}e^{-x^2}}{(2n)!}\,dx = \frac{a^{2n}}{(2n)!}\int_{-\infty}^\infty x^{2n}e^{-x^2}\,dx = \frac{a^{2n}}{(2n)!}\left(\frac{(2n)!\sqrt{\pi}}{4^n n!}\right) = \frac{\sqrt{\pi}}{n!}\left(\frac{a}{2}\right)^{2n}.$$

Hence each f_n is integrable. By recalling the Taylor series for e^x, we see that

$$\sum_{n=0}^\infty\int_{-\infty}^\infty |f_n(x)|\,dx = \sqrt{\pi}\sum_{n=0}^\infty \frac{(a^2/4)^n}{n!} = \sqrt{\pi}e^{a^2/4}.$$

Thus we can interchange the summation and integration. Doing this yields

$$\begin{aligned}\int_{-\infty}^\infty e^{-x^2}\cos(ax)\,dx &= \int_{-\infty}^\infty \sum_{n=0}^\infty \frac{(-1)^n(ax)^{2n}e^{-x^2}}{(2n)!}\,dx = \sum_{n=0}^\infty \frac{(-1)^n a^{2n}}{(2n)!}\int_{-\infty}^\infty x^{2n}e^{-x^2}\,dx \\ &= \sum_{n=0}^\infty \frac{(-1)^n a^{2n}}{(2n)!}\left(\frac{(2n)!\sqrt{\pi}}{4^n n!}\right) = \sqrt{\pi}\sum_{n=0}^\infty \frac{(-a^2/4)^n}{n!} = \sqrt{\pi}e^{-a^2/4}\end{aligned}$$

Note that this actually holds for all a.

b) Series expand $1/(1-x)$ part. Observe that, for $-1 < x < 1$,

$$\frac{x^a \log(x)}{1-x} = x^a \log(x) \sum_{k=0}^{\infty} x^k = \log(x) \sum_{k=0}^{\infty} x^{a+k}.$$

Now let $f_k(x) = x^{a+k} \log(x)$ for $k \geq 0$. We have that

$$|f_k(x)| = -f_k(x) = -x^{a+k} \log(x)$$

for $0 < x \leq 1$. By a formal use of IBP,

$$\begin{aligned}
\lim_{c\to 0^+} \int_c^1 -x^{a+k} \log(x)\ dx &= \lim_{c\to 0^+} \left(\frac{x^{a+k+1} \log(x)}{a+k+1} \right)\Big|_c^1 + \frac{1}{a+k+1} \int_0^1 x^{a+k}\ dx \\
&= -\lim_{c\to 0^+} \left(\frac{x^{a+k+1} \log(x)}{a+k+1} \right) + \frac{x^{a+k+1}}{(a+k+1)^2} \Big|_0^1 \\
&= \frac{1}{(a+k+1)^2} - \frac{1}{a+k+1} \lim_{c\to 0^+} \left(\frac{\log(x)}{1/x^{a+k+1}} \right) \\
&= \frac{1}{(a+k+1)^2} - \frac{1}{a+k+1} \lim_{c\to 0^+} \left(\frac{1/x}{-(a+k+1)/x^{a+k+2}} \right) \\
&= \frac{1}{(a+k+1)^2} + \frac{1}{(a+k+1)^2} \lim_{c\to 0^+} x^{a+k+1} = \frac{1}{(a+k+1)^2}
\end{aligned}$$

where we have used L'Hopital's rule. Note that since $a > -1$ and $k \geq 0$, $a+k+1 > 0$, and therefore $x^{a+k+1} \to 0$ as $x \to 0$. It follows that each $f_k(x)$ is integrable. Moreover,

$$\sum_{k=0}^{\infty} \int_0^1 |f_k(x)|\ dx = \sum_{k=0}^{\infty} \int_0^1 -x^{a+k} \log(x)\ dx = \sum_{k=1}^{\infty} \frac{1}{(a+k+1)^2} < \infty.$$

But, the integral in question is just the negative of the one above (since $|f_k(x)| = -f_k(x)$), thus the integral is $-\sum_{k=0}^{\infty} 1/(a+k)^2$ after a change in index.

c) Observe that, on $(0, \infty)$,

$$\frac{1}{e^x - 1} = \frac{e^{-x}}{1 - e^{-x}} = \sum_{n=1}^{\infty} e^{-nx}.$$

Thus,

$$\int_0^{\infty} \frac{x^{a-1}}{e^x - 1}\ dx = \int_0^{\infty} \sum_{n=1}^{\infty} x^{a-1} e^{-nx}\ dx$$

Now in Exercise 2.3.29, we proved the formula

$$\int_0^{\infty} x^n e^{-tx}\ dx = \frac{n!}{t^{n+1}}$$

so long as $t \in [1/2, 2]$. The argument we used can be adapted to extend to $t \in [1, \infty)$. Hence, applying this with $t = n$ and $n = a - 1$ (note: valid only if $a > 1$!) gives

$$\int_0^{\infty} x^{a-1} e^{-nx}\ dx = \frac{\Gamma(a)}{n^a}$$

It follows that if $f_n(x) = x^{a-1} e^{-nx}$, then $f_n(x)$ is integrable. Moreover,

$$\sum_{n=1}^{\infty} \int_0^{\infty} x^{a-1} e^{-nx}\ dx = \sum_{n=1}^{\infty} \frac{\Gamma(a)}{n^a} = \Gamma(a)\zeta(a) < \infty.$$

Thus, we can exchange the order of summation. But since our functions were nonnegative, we've already calculated the integral above.

d) We expand $\sin(x)$ using its Taylor series to get

$$\int_0^\infty \frac{e^{-ax}\sin(x)}{x}\,dx = \int_0^\infty \frac{e^{-ax}}{x}\sum_{n=0}^\infty \frac{(-1)^n x^{2n+1}}{(2n+1)!}\,dx = \int_0^\infty \sum_{n=0}^\infty \frac{(-1)^n x^{2n} e^{-ax}}{(2n+1)!}\,dx.$$

Let $f_n(x) = (-1)^n x^{2n} e^{-ax}/(2n+1)!$ for $n \geq 0$. Then,

$$\int_0^\infty |f_n(x)|\,dx = \int_0^\infty \frac{1}{2n+1}\frac{x^{2n}e^{-ax}}{(2n)!}\,dx$$

Applying the fact

$$\int_0^\infty x^n e^{-tx}\,dx = \frac{n!}{t^{n+1}}$$

with $n = 2n$ and $t = a$ gives

$$\int_0^\infty |f_n(x)|\,dx = \frac{1}{(2n+1)(2n)!}\int_0^\infty x^{2n}e^{-ax}\,dx = \frac{(2n)!}{(2n+1)(2n)!a^{2n+1}} = \frac{1}{(2n+1)a^{2n+1}}.$$

Hence, the f_n are integrable. Moreover,

$$\sum_{n=0}^\infty \int_0^\infty |f_n(x)|\,dx = \sum_{n=0}^\infty \frac{1}{(2n+1)a^{2n+1}} = \sum_{n=0}^\infty \frac{(1/a)^{2n+1}}{2n+1}.$$

Recall that the Taylor series $\operatorname{arctanh}(x)$ and $\arctan(x)$ are

$$\operatorname{arctanh}(x) = \sum_{n=0}^\infty \frac{x^{2n+1}}{2n+1} \qquad \arctan(x) = \sum_{n=0}^\infty \frac{(-1)^n x^{2n+1}}{2n+1}$$

so that the above series is just $\operatorname{arctanh}(1/a) < \infty$. Now, we can exchange summation and integration to get

$$\int_0^\infty \frac{e^{-ax}\sin(x)}{x}\,dx = \int_0^\infty \sum_{n=0}^\infty f_n(x)\,dx = \sum_{n=0}^\infty \frac{(-1)^n(1/a)^{2n+1}}{2n+1} = \arctan(1/a).$$

e) Using the series expansion of $J_0(x)$,

$$\int_0^\infty e^{-ax}J_0(x)\,dx = \int_0^\infty e^{-ax}\sum_{n=1}^\infty \frac{(-1)^n x^{2n}}{4^n(n!)^2}\,dx = \int_0^\infty \sum_{n=1}^\infty \frac{(-1)^n x^{2n}e^{-ax}}{4^n(n!)^2}\,dx.$$

Let $f_n(x) = (-1)^n x^{2n}e^{-ax}/(4^n(n!)^2)$. Then,

$$\int_0^\infty |f_n(x)|\,dx = \int_0^\infty \frac{x^{2n}e^{-ax}}{4^n(n!)^2}\,dx = \frac{1}{4^n(n!)^2}\frac{(2n)!}{a^{2n+1}}$$

using the formula derived in Exercise 2.3.29. So, each f_n is integrable. Now,

$$\sum_{n=0}^\infty \int_0^\infty |f_n(x)|\,dx = \sum_{n=0}^\infty \frac{(2n)!(1/a)^{2n+1}}{4^n(n!)^2}.$$

There is a combinatorial bound which states that

$$\frac{(2n)!}{(n!)^2} = \binom{2n}{n} \leq 4^n.$$

Hence,

$$\sum_{n=0}^\infty \int_0^\infty |f_n(x)|\,dx \leq \frac{1}{a}\sum_{n=0}^\infty \left(\frac{1}{a^2}\right)^n = \frac{1}{a}\left(\frac{1}{1-1/a^2}\right) = \frac{a}{a^2-1}$$

which converges as a geometric series since $a > 1$ implies $0 < 1/a < 1$.

Recall that $\operatorname{arcsinh}(x)$ Taylor series and derivative

$$\operatorname{arcsinh}(x) = \sum_{n=0}^\infty \frac{(-1)^n(2n)!x^{2n+1}}{4^n(n!)^2(2n+1)} \qquad \frac{d}{dx}\operatorname{arcsinh}(x) = \frac{1}{\sqrt{x^2+1}},$$

thus
$$\frac{1}{\sqrt{x^2+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)! x^{2n}}{4^n (n!)^2}.$$
Next, note that for $x > 0$ we have
$$\frac{1}{\sqrt{x^2+1}} = \frac{1}{x\sqrt{(1/x)^2+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)! (1/x)^{2n}}{4^n (n!)^2 x} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (n!)^2 x^{2n+1}}.$$
Exchanging summation and integration gives
$$\int_0^\infty e^{-ax} J_0(x)\, dx = \int_0^\infty \sum_{n=0}^{\infty} f_n(x)\, dx = \sum_{n=0}^{\infty} (-1)^n \int_0^\infty \frac{x^{2n} e^{-ax}}{4^n (n!)^2}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (n!)^2 a^{2n+1}},$$
which yields the result by combining it with the above.

XXX when can you differentiate a series.

2.4. **Modes of Convergence.**

2.4.32. Suppose $\mu(X) < \infty$. If f and g are complex-valued measurable functions on X, define
$$\rho(f,g) = \int \frac{|f-g|}{1+|f-g|}\, d\mu.$$
Then ρ is a metric on the space of measurable functions if we identify functions that are equal a.e., and $f_n \to f$ with respect to this metric iff $f_n \to f$ in measure.

Solution: Let $\epsilon > 0$ First, if $f_n \to f$ in measure then for every $\eta > 0$ there exists an $n \geq N$ such that
$$\mu(\{|f_n - f| \geq \epsilon\}) < \eta.$$
In particular, we may choose $\eta = \epsilon$. Observe that for any f, g we have $|f-g|/(1+|f-g|) \leq 1$. Hence,
$$\begin{aligned}\rho(f_n, f) &= \int_{\{|f_n - f| < \epsilon\}} \frac{|f_n - f|}{1+|f_n - f|}\, d\mu + \int_{\{|f_n - f| \geq \epsilon\}} \frac{|f_n - f|}{1+|f_n - f|}\, d\mu \\ &\leq \int_{\{|f_n - f| < \epsilon\}} \epsilon\, d\mu + \mu(\{|f_n - f| \geq \epsilon\}) < \epsilon(1 + \mu(\{|f_n - f| < \epsilon\})\end{aligned}$$
for any $n \geq N$. It follows that $\rho(f_n, f) \to 0$.

Now suppose that $\rho(f_n, f) \to 0$. Then for every $\epsilon > 0$ there exists an N such that if $n \geq N$, $\rho(f_n, f) < \epsilon$. Consequently,
$$\int_{\{|f_n - f| < \epsilon\}} \frac{|f_n - f|}{1+|f_n - f|}\, d\mu + \int_{\{|f_n - f| \geq \epsilon\}} \frac{|f_n - f|}{1+|f_n - f|}\, d\mu < \epsilon$$
whenever $n \geq N$. XXX

2.4.33. If $f_n \geq 0$ and $f_n \to f$ in measure, then $\int f \leq \liminf \int f_n$.

Solution: Suppose that for all $N \in \mathbb{N}$ there exists an $n \geq N$ such that f_n is not integrable. Then $\liminf \int f_n = \infty$, and the result holds trivially. Now suppose against this; that is there exists an N such that if $n \geq N$ then f_n is integrable. We can redefine our sequence to start from N; hence WLOG assume that all the $f_n \in L^+$.

Recall that given a sequence of real numbers $\{a_n\}_{n=1}^\infty$, for any L such that $\liminf a_n \leq L \leq \limsup a_n$, there exists a subsequence $\{a_{n_k}\}_{k=1}^\infty$ such that $a_{n_k} \to L$; in particular this holds for $L = \liminf a_n$. We apply this to the sequence $\int f_n$. Let $\{g_k\}_{k=1}^\infty$ be a subsequence such that
$$\lim_{k\to\infty} \int g_k = \liminf_{n\to\infty} \int f_n.$$

(here, $g_k = f_{n_k}$ for some n_k). Next we must show that $g_k \to f$ in measure. Let $\epsilon, \eta > 0$. Then there exists an N such that for for $n \geq N$,

$$\mu(\{|f_n - f| \geq \epsilon\}) \leq \eta.$$

Since this holds for all $n \geq N$, there exists a smallest n_K such that $n_K \geq N$. Hence for all $k \geq K$,

$$\mu(\{|g_k - f| \geq \epsilon\}) \leq \eta,$$

showing that $g_k \to f$ in measure. By Theorem 2.30, since $g_k \to f$ in measure, there exists a subsequence $g_{k_j} \to f$ a.e. Finally, by Fatou

$$\int f = \int \liminf_{j\to\infty} g_{k_j} \leq \liminf_{j\to\infty} \int g_{k_j} = \lim_{j\to\infty} \int g_{k_j} = \lim_{k\to\infty} \int g_k = \liminf_{n\to\infty} \int f_n$$

since subsequences of a convergent sequence converge to the same limit.

2.4.34. Suppose $|f_n| \leq g \in L^1$ and $f_n \to f$ in measure.

a) $\int f = \lim \int f_n$.
b) $f_n \to f$ in L^1.

Solution:

a) Note that this is essentially the same theorem as DCT except we do not have $f_n \to f$ a.e. Instead, this is replaced by convergence in measure. However, we saw in 2.4.33 that Fatou still holds when we have convergence in measure. The proof of DCT relied solely on Fatou, so we are done.
b) By part a) we can adapt the hypotheses 2.3.20 to $f_n \to f$ and $g_n \to g$ in measure (n.b., we technically need that $g_n \pm f_n \to g \pm f$ in measure, but this is justified by 2.4.38). However, the additional assumption that $\int g_n \to \int g$ is that $g_n \to g$ in L^1, and therefore $g_n \to g$ in measure. Hence by reworking the proof of 2.3.20, one can show if $f_n, g_n, f, g \in L^1$, $f_n \to f$ in measure, $g_n \to g$ in L^1, then $\int f_n \to \int f$. The proof of 2.3.21 then carries over here.

   Alternatively,

   $$|f_n - f| \leq |f_n| + |f| \leq 2|g|.$$

   We now show that $f_n - f \to 0$ in measure. This is almost trivial once we write it out: let $\epsilon > 0$. Then,

   $$\mu(\{|(f_n - f) - 0| > \epsilon\}) = \mu(\{|f_n - f| > \epsilon\}) \to 0$$

   since $f_n \to f$ in measure. Thus, we can apply part a) with $f_n - f$ and $2g$, yielding

   $$\lim_{n\to\infty} \int |f_n - f| = \int 0 = 0,$$

   that is $f_n \to f$ in L^1.

2.4.35. $f_n \to f$ in measure iff for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\mu(\{x \mid |f_n(x) - f(x)| \geq \epsilon\}) < \epsilon$ for $n \geq N$.

Solution: One direction is obvious. If $f_n \to f$ in measure, then for any $\epsilon, \eta > 0$ there exists an N such that for $n \geq N$,

$$\mu(\{|f_n - f| \geq \epsilon\}) < \eta.$$

Now choose $\eta = \epsilon$. On the other hand, suppose that for every $\epsilon > 0$ there exists an N_ϵ such that $\mu(\{|f_n - f| \geq \epsilon\}) < \epsilon$ whenever $n \geq N_\epsilon$. If $\eta \geq \epsilon$ then we are done. So suppose $\eta < \epsilon$. Next, find N_η such that $\mu(\{|f_n - f| \geq \eta\}) < \eta$ for $n \geq N_\eta$. Since $\eta < \epsilon$, if $|f_n - f| \geq \epsilon$ then $|f_n - f| \geq \eta$. Let $N = \max\{N_\epsilon, N_\eta\}$. Note that if $|f_n - f| \geq \epsilon$ then $|f_n - f| \geq \eta$. So for fixed $n \geq N$, by monotonicity

$$\mu(\{|f_n - f| \geq \epsilon\}) \leq \mu(\{|f_n - f| \geq \eta\}) < \eta.$$

2.4.36. If $\mu(E_n) < \infty$ for $n \in \mathbb{N}$ and $\chi_{E_n} \to f$ in L^1, then f is (a.e. equal to) the characteristic function of a measurable set.

Solution: By Corollary 2.32 there exists a subsequence $\{\chi_{E_{n_k}}\}_{k=1}^{\infty}$ such that $\chi_{E_{n_k}} \to f$ a.e. Since all the $\chi_{E_{n_k}}$ take values 0 or 1, it follows that f does too a.e. Moreover, by Propositions 2.11b and 2.12 there exists a measurable function g that is a.e. equivalent to f. This is our desired g since it is measurable and takes values 0 or 1, thus is a characteristic function of a measurable set (namely $g^{-1}\{1\}$).

2.4.37. Suppose that f_n and f are measurable complex-valued functions and $\phi : \mathbb{C} \to \mathbb{C}$.

a) If ϕ is continuous and $f_n \to f$ a.e., then $\phi \circ f_n \to \phi \circ f$ a.e.
b) If ϕ is uniformly continuous and $f_n \to f$ uniformly, almost uniformly, or in measure, then $\phi \circ f_n \to \phi \circ f$ uniformly, almost uniformly, or in measure, respectively.
c) There are counterexamples when the continuity assumptions on ϕ are not satisfied.

Solution:

a) Let $\epsilon > 0$ and $x \in X$ where $f_n \to f$. By continuity of ϕ at $f(x)$ there exists a $\delta > 0$ where any point $z \in B_\delta(f(x))$ is such that $|\phi(z) - \phi(f(x))| < \epsilon$. Then there exists an N such that if $n \geq N$, $|f_n(x) - f(x)| < \delta$. This means that $f_n(x) \in B_\delta(f(x))$ for all $n \geq N$. Hence,
$$|(\phi \circ f_n)(x) - (\phi \circ f)(x)| = |\phi(f_n(x)) - \phi(f(x))| < \epsilon,$$
that is $\phi \circ f_n \to \phi \circ f$ wherever $f_n \to f$.
b) We prove each separately.
   - (Uniformly) Let $\epsilon > 0$. By uniform continuity of ϕ there exists a $\delta > 0$ such that if $|z - w| < \delta$, then $|\phi(z) - \phi(w)| < \epsilon$. Moreover by uniform convergence of f_n, there exists an N such that if $n \geq N$ then $|f_n(x) - f(x)| < \delta$ for all $x \in X$. Combining these yields
   $$|(\phi \circ f_n)(x) - (\phi \circ f)(x)| = |\phi(f_n(x)) - \phi(f(x))| < \epsilon$$
   by choosing $z = f_n(x)$ and $w = f(x)$. Thus $\phi \circ f_n \to \phi \circ f$ uniformly.
   - (Almost uniformly) Recall that $f_n \to f$ almost uniformly if for every $\epsilon > 0$ there exists a measurable set E such that $\mu(E^c) < \epsilon$ and $f_n \to f$ uniformly on E. By the first part of b), we see that if $f_n \to f$ uniformly on E then $\phi \circ f_n \to \phi \circ f$ on E (we still need uniform continuity of ϕ!). But then $\phi \circ f_n \to \phi \circ f$ almost uniformly by using the same E extracted from almost uniform convergence of f_n.
   - (In measure) Let $\epsilon > 0$. By uniform continuity of ϕ there exists a $\delta > 0$ so that whenever $|z - w| < \delta$ we have $|\phi(z) - \phi(w)| < \epsilon$. Since $f_n \to f$ in measure there exists an N such that if $n \geq N$, $\mu(\{|f_n - f| \geq \delta\}) < \epsilon$. Now let $x \in \{|(\phi \circ f_n) - (\phi \circ f)| \geq \epsilon\}$. By our continuity assumption, this certainly cannot happen when $|f_n(x) - f(x)| < \delta$. Hence, it only possibly happens when $|f_n(x) - f(x)| \geq \delta$. That is,
   $$\{|(\phi \circ f_n) - (\phi \circ f)| \geq \epsilon\} \subset \{|f_n - f| \geq \delta\}.$$
   So, by monotonicity,
   $$\mu(\{|(\phi \circ f_n) - (\phi \circ f)| \geq \epsilon\}) \subset \mu(\{|f_n - f| \geq \delta\}) < \epsilon.$$
   Appealing to Exercise 2.4.35 shows that $\phi \circ f_n \to \phi \circ f$ in measure.
c) We go through each separately.
   - (Pointwise a.e.)
   - (Uniformly) $\phi(x) = 1/x$?
   - (Almost uniformly)
   - (In measure)

2.4.38. Suppose $f_n \to f$ in measure and $g_n \to g$ in measure.

a) $f_n + g_n \to f + g$ in measure.
b) $f_n g_n \to fg$ in measure if $\mu(X) < \infty$, but not necessarily if $\mu(X) = \infty$.

Solution:

a) Let $\epsilon > 0$. Then there exist N_f and N_g such that if $n \geq N_f$, $\mu(\{|f_n - f| \geq \epsilon/2\}) < \epsilon/2$ and similarly for g. Let $N = \max\{N_f, N_g\}$. By the triangle inequality,
$$|(f_n + g_n) - (f + g)| \leq |f_n - f| + |g_n - g|,$$
so that
$$\{|(f_n + g_n) - (f + g)| \geq \epsilon\} \subset \{|f_n - f| \geq \epsilon/2\} \cup \{|g_n - g| \geq \epsilon/2\}.$$
Hence by monotonicity,
$$\mu(\{|(f_n + g_n) - (f + g)| \geq \epsilon\}) \leq \mu(\{|f_n - f| \geq \epsilon/2\}) + \mu(\{|g_n - g| \geq \epsilon/2\}) < \epsilon/2 + \epsilon/2 = \epsilon$$
for all $n \geq N$.

b) Let $\epsilon > 0$; we similarly extract N_f and N_g and define $N = \max\{N_f, N_g\}$. Next, by the triangle inequality
$$|f_n g_n - fg| \leq |f_n - f||g_n - g| + |f||g_n - g| + |g||f_n - f|.$$
We now show that for any $\eta > 0$ there exists an $M \geq 0$ such that $\mu(\{|f| \geq M\}) < \eta$. Let $E_n = \{|f| > n\}$. Then E_n is a decreasing sequence of sets, and since $\mu(X) < \infty$ we may apply DCT of sets. Hence $\mu(E_n) \to 0$, since $\bigcap_{n=1}^{\infty} E_n = \emptyset$. It follows that for any η we can find an N such that if $n \geq N$, $\mu(E_n) < \eta$. So, choose $M = N$. Thus when $\mu(X) < \infty$, any measurable function is bounded except on a set with small measure. We now have that
$$\{|fg - f_n g_n| \geq \epsilon\} \subset \{|f - f_n||g - g_n| \geq \epsilon/3\} \cup \{|f||g - g_n| \geq \epsilon/3\} \cup \{|g||f - f_n| \geq \epsilon/3\}.$$
Let us analyze each set on the RHS individually. First we have
$$\{|f - f_n||g - g_n| \geq \epsilon/3\} \subset \{|f - f_n| \geq \sqrt{\epsilon/3}\} \cup \{|g - g_n| \geq \sqrt{\epsilon/3}\}.$$
Each set on the RHS tends to zero in measure so that the set on the left does too. Next we have
$$\{|g||f - f_n| \geq \epsilon/3\} \subset \{|f - f_n| \geq \epsilon/M_g\} \cup \{|g| \geq M_g\}$$
for some $M_g > 0$. Once more, both of these sets go to zero in measure as $M_g \to \infty$. The same reasoning applies with the roles of g and f reversed. Hence $\{|f_n g_n - fg| \geq \epsilon\} \to 0$.

XXX

2.4.39. If $f_n \to f$ almost uniformly, then $f_n \to f$ a.e. and in measure.

Solution: First for each $k \geq 1$ there exists a set E_k such that $\mu(E_k^c) \leq 1/k$ but $f_n \to f$ uniformly on E_k. Let $E = \bigcup_{k=1}^{\infty} E_k$. Then $\mu(E^c) = 0$ and if $x \in E$ then $f_n \to f$ uniformly on E_k for some k. Since pointwise convergence implies uniform convergence, $f_n \to f$ pointwise on E_k, hence on E.

Next by almost uniform convergence for any $\epsilon > 0$ there exists a measurable E such that $\mu(E^c) < \epsilon$ but $f_n \to f$ uniformly on E. Hence there exists an N such that if $n \geq N$ then $|f_n(x) - f(x)| < \epsilon$ for all $x \in E$. It follows that $\{|f_n - f| \geq \epsilon\} \subset E^c$ so that $\mu(\{|f_n - f| \geq \epsilon\}) < \epsilon$. Thus $f_n \to f$ in measure.

2.4.40. In Egoroff's theorem, the hypothesis "$\mu(X) < \infty$" can be replaced by "$|f_n| \leq g$ for all n, where $g \in L^1(\mu)$".

Solution: XXX

2.4.41. If μ is σ-finite and $f_n \to f$ a.e., there exist measurable $E_1, E_2, ... \subset X$ such that $\mu((\bigcup_{j=1}^{\infty} E_j)^c) = 0$ and $f_n \to f$ uniformly on each E_j.

Solution: Let $X = \bigcup_j F_j$ where each F_j is disjoint with finite positive measure, which can be done by σ-finiteness. Then applying Egoroff's theorem to $f_n \to f$ on each F_j with $\epsilon = 1/j$ gives a measurable set E_j such that $\mu(F_j \setminus E_j) < 1/j$ and $f_n \to f$ uniformly on E_j. Now by monotonicity,

$$\mu\left(\left(\bigcup_{j=1}^{\infty} E_j\right)^c\right) = \mu\left(\bigcup_{j=1}^{\infty}(F_j \setminus E_j)\right) \leq \mu(F_j \setminus E_j) < \frac{1}{j}$$

for all j. Hence, $\mu((\bigcup_{j=1}^{\infty} E_j)^c) = 0$. XXX

2.4.42. Let μ be counting measure on $\mathbb{N}$. Then $f_n \to f$ in measure iff $f_n \to f$ uniformly.

Solution: XXX

2.4.43. Suppose that $\mu(X) < \infty$ and $f : X \times [0,1] \to \mathbb{C}$ is a function such that $f(\cdot, y)$ is measurable for each $y \in [0,1]$ and $f(x, \cdot)$ is continuous for each $x \in X$.

a) If $0 < \epsilon, \delta < 1$ then $E_{\epsilon,\delta} = \{x \mid |f(x,y) - f(x,0)| \leq \epsilon \text{ for all } y < \delta\}$ is measurable.
b) For any $\epsilon > 0$ there is a set $E \subset X$ such that $\mu(E) < \epsilon$ and $f(\cdot, y) \to f(\cdot, 0)$ uniformly on E^c as $y \to 0$.

Solution: XXX

2.4.44. (Lusin's Theorem) If $f : [a,b] \to \mathbb{C}$ is Lebesgue measurable and $\epsilon > 0$, there is a compact set $E \subset [a,b]$ such that $\mu(E^c) < \epsilon$ and $f|_E$ is continuous. (Use Egoroff's theorem and Theorem 2.26).

Solution: XXX

2.5. **Product Measures.**

2.5.45. If $(X_j, \mathscr{M}_j)$ is a measurable space for $j = 1, 2, 3$, then $\bigotimes_{j=1}^{3} \mathscr{M}_j = (\mathscr{M}_1 \otimes \mathscr{M}_2) \otimes \mathscr{M}_3$. Moreover, if μ_j is a σ-finite measure on $(X_j, \mathscr{M}_j)$, then $\mu_1 \times \mu_2 \times \mu_3 = (\mu_1 \times \mu_2) \times \mu_3$.

Solution:

2.5.46. Let $X = Y = [0,1]$, $\mathscr{M} = \mathscr{N} = \mathscr{B}_{[0,1]}$, $\mu =$ Lebesgue measure, and $\nu =$ counting measure. If $D = \{(x,x) \mid x \in [0,1]\}$ is the diagonal in $X \times Y$, then $\iint \chi_D \, d\mu d\nu$, $\iint \chi_D \, d\nu d\mu$, and $\int \chi_D \, d(\mu \times \nu)$ are all unequal. (To compute $\int \chi_D \, d(\mu \times \nu) = (\mu \times \nu)(D)$, go back to the definition of $\mu \times \nu$.)

Solution: We should technically show that $D \in \mathscr{M} \otimes \mathscr{N}$ first. Fix an $n \in \mathbb{N}$ and let E_n be defined by

$$E_n = \bigcup_{i=0}^{2^{n-1}-1} \left[\frac{i}{2^{n-1}}, \frac{i+1}{2^{n-1}}\right] \times \left[\frac{i}{2^{n-1}}, \frac{i+1}{2^{n-1}}\right].$$

The picture is to cover D by a bunch of squares of side length $1/2^{n-1}$ evenly spaced along the diagonal. Then,

$$D = \bigcap_{n=1}^{\infty} E_n$$

It is clear that $D \in \mathscr{M} \otimes \mathscr{N}$ since each E_n is a union of squares. Alternatively, we can define $h : [0,1]^2 \to \mathbb{R}$ by $h(x,y) = x - y$. Then h is continuous and hence Borel measurable. Notice that $D = h^{-1}\{0\}$.

We now compute D_x and D^y for $x, y \in [0,1]$. It is clear that $D_x = \{(x,x)\}$ and similarly for D^y. In light of Lebesgue measure, $\mu(D^y) = 0$ (since it is a singleton), but $\nu(D_x) = 1$ (since ν is counting measure). Note that since these are both constant in y and x respectively, we can pull them out of the iterated integral. Next,

$$\begin{aligned}\iint \chi_D \, d\mu d\nu &= \int_{[0,1]} \left(\int_{[0,1]} \chi_{D^y} \, d\mu \right) d\nu = \int_{[0,1]} \mu(D^y) \, d\nu = \mu(D^y)\nu([0,1]) = 0 \\ \iint \chi_D \, d\nu d\mu &= \int_{[0,1]} \left(\int_{[0,1]} \chi_{D_x} \, d\nu \right) d\mu = \int_{[0,1]} \nu(D_x) \, d\mu = \nu(D_x)\mu([0,1]) = 1\end{aligned}$$

Let $\epsilon > 0$, then there exists a covering of D by rectangles $A_j \times B_j$, where each $A_j, B_j \in \mathscr{B}_{[0,1]}$, such that

$$\sum_{j=1}^{\infty} \mu(A_j)\nu(B_j) \leq (\mu \times \nu)(D) + \epsilon$$

We can assume this covering is minimal in the sense that if there exists an $x \in A_j$ such that $\{x\} \times B_j$ has empty intersection with D, then we remove x from A_j. Similarly, if $y \in B_j$ is such that $A_j \times \{y\}$ does not intersect D, then remove y from B_j. Since removing points decreases the size of these sets, by monotonicty we are only decreasing the above series.

Suppose that $(\mu \times \nu)(D)$ is finite. Then the series above converges. In particular, if $\nu(B_j) = \infty$ then $\mu(A_j) = 0$. For our collection to cover D, it must be that both $\bigcup_j A_j$ and $\bigcup_j B_j$ cover $[0,1]$ (since otherwise we can find an $x \in [0,1]$ that is not covered by one of the A_j or B_j, and hence (x, x) is not in the covering).

As subsets of $[0,1]$, all of the A_j have finite μ-measure. Since the A_j cover $[0,1]$, at least one of them has positive finite measure. There is a corresponding B_j for which $A_j \times B_j$ is in the cover. We assumed that $A_j \times B_j$ is minimal. That is, for each $x \in A_j$, $(\{x\} \times B_j) \cap D \neq \emptyset$. Since D is comprised of points of the form (z, z), we see that $x \in B_j$. Hence if $x \in A_j$ then $x \in B_j$. Similarly, by minimality in B_j if $y \in B_j$ then $y \in A_j$. Thus $A_j = B_j$. If A_j were countable, then (due to properties of Lebesgue measure) $\mu(A_j) = 0$. Since $\mu(A_j) > 0$, it follows that A_j is countable, and hence $\nu(B_j) = \nu(A_j) = \infty$. Thus the sum is infinite, a contradiction. Hence $(\mu \times \nu)(D) = \infty$.

2.5.47. Let $X = Y$ be an uncountable linearly ordered set such that for each $x \in X$, $\{y \in X \mid y < x\}$ is countable. (Example: The set of countable ordinals.) Let $\mathscr{M} = \mathscr{N}$ be the σ-algebra of countable or co-countable sets, and let $\mu = \nu$ be defined on $\mathscr{M}$ by $\mu(A) = 0$ if A is countable and $\mu(A) = 1$ if A is co-countable. Let $E = \{(x, y) \in X \times X \mid y < x\}$. Then E_x and E^y are measurable for all x, y, and $\iint \chi_E \, d\mu d\nu$ and $\iint \chi_E \, d\nu d\mu$ exist but are not equal. (If one believes in the continuum hypothesis, one can take $X = [0,1]$ [with a nonstandard ordering] and thus obtain a set $E \subset [0,1]$ such that E_x is countable and E^y is co-countable [in particular, Borel] for all x, y, but E is not Lebesgue measurable.)

Solution:

2.5.48. Let $X = Y = \mathbb{N}$, $\mathscr{M} = \mathscr{N} = \mathcal{P}(\mathbb{N})$, $\mu = \nu =$ counting measure. Define $f(m, n) = 1$ if $m = n$, $f(m, n) = -1$ if $m = n + 1$, and $f(m, n) = 0$ otherwise. Then $\int |f| \, d(\mu \times \nu) = \infty$, and $\iint f \, d\mu d\nu$ and $\iint f \, d\nu d\mu$ exist and are unequal.

Solution: Recall that for $f : \mathbb{N} \to \mathbb{R}$, integration against counting measure is the same as an infinite series

$$\int_{\mathbb{N}} f = \sum_{n=1}^{\infty} f(n).$$

We first compute the slice integrals. For each fixed $n \in \mathbb{N}$, there is exactly one point where $f^n = 1$, one where $f^n = -1$, and all others are zero. Hence,

$$\int_{\mathbb{N}} f^n \, d\mu = \sum_{m=1}^{\infty} f^n(m) = 0 + 0 + ... + 0 + f(n) + f(n+1) + 0 + ... = -1 + 1 = 0.$$

Then,

$$\iint f \, d\mu d\nu = \int_{\mathbb{N}} \left(\int_{\mathbb{N}} f^n \, d\mu \right) d\nu = \int_{\mathbb{N}} 0 \, d\nu = 0.$$

Now consider fixed $m \in \mathbb{N}$. For $m \geq 2$ there similarly exists a single point where $f_m = 1$ and a single point where $f_m = -1$, and all others are zero (the two points are at $n = m$ and $n = m - 1$, respectively). Thus for $m \geq 2$, $\int f_m \, d\nu = 0$. However, when $m = 1$ there is no point where $f_m = -1$, since it would occur at $m - 1 = 0 \notin \mathbb{N}$. Thus,

$$\int_{\mathbb{N}} f_1 \, d\nu = \sum_{n=1}^{\infty} f_1(n) = f(1,1) + f(1,2) + ... = 1 + 0 + 0 + ... = 1.$$

Then,

$$\iint f \, d\nu d\mu = \int_{\mathbb{N}} \left(\int_{\mathbb{N}} f_m \, d\nu \right) d\mu = \int_{\mathbb{N}} f_m \, d\mu = \int f_1 + \int f_2 + ... = 1 + 0 + 0 = 1$$

since $h(m) = \int f_m \, d\nu$ is 1 when $m = 1$ and 0 else.

Finally, let $D_1 = \{(n,n) \mid n \in \mathbb{N}\}$ and $D_2 = \{(n+1,n) \mid n \in \mathbb{N}\}$. Note that both D_1 and D_2 are uncountable. Moreover, $|f|$ is identically 1 on these sets and 0 off of them. Hence,

$$\begin{aligned} \iint_{\mathbb{N}\times\mathbb{N}} |f| \, d(\mu \times \nu) &= \iint_{D_1 \cup D_2} |f| \, d(\mu \times \nu) = \iint_{D_1} |f| \, d(\mu \times \nu) + \iint_{D_2} |f| \, d(\mu \times \nu) \\ &= (\mu \times \nu)(D_1) + (\mu \times \nu)(D_2) \end{aligned}$$

since D_1 and D_2 are disjoint. Note that we can define a counting measure τ on the product σ-algebra, and by properties of Cartesian products we have $\tau(A \times B) = \mu(A)\nu(B)$. Since μ and ν are σ-finite and τ agrees with $\mu \times \nu$ on rectangles, by uniqueness of the extension we have that $\tau = \mu \times \nu$. Clearly both D_1 and D_2 are countably infinite, so that their measure is ∞.

2.5.49. Prove Theorem 2.39 by using Theorem 2.37 and Proposition 2.12 together with the following lemmas

a) If $E \in \mathscr{M} \times \mathscr{N}$ and $(\mu \times \nu)(E) = 0$ then $\nu(E_x) = \mu(E^y) = 0$ for a.e. x and y.
b) If f is $\mathscr{L}$-measurable and $f = 0$ λ-a.e., then f_x and f^y are integrable for a.e. x and y, and $\int f_x \, d\nu = \int f^y \, d\mu = 0$ for a.e. x and y. (Here the completeness of μ and ν is needed).

Solution:

2.5.50. Suppose $(X, \mathscr{M}, \mu)$ is a σ-finite measure space and $f \in L^+(X)$. Let

$$G_f = \{(x,y) \in X \times [0,\infty] \mid y \leq f(x)\}.$$

Then G_f is $\mathscr{M} \otimes \mathscr{B}_{\mathbb{R}}$-measurable and $(\mu \times m)(G_f) = \int f \, d\mu$; the same is also true if the inequality $y \leq f(x)$ in the definition of G_f is replaced by $y < f(x)$. (To show measurability of G_f, note that the map $(x,y) \mapsto f(x) - y$ is the composition of $(x,y) \mapsto (f(x), y)$ and $(z,y) \mapsto z - y$.) This is the definitive statement of the familiar theorem from calculus, "the integral of a function is the area under its graph."

Solution: The map $g(x,y) = (f(x), y)$ is $(\mathscr{M} \otimes \mathscr{B}_{\overline{\mathbb{R}}}, \mathscr{B}_{\overline{\mathbb{R}}} \otimes \mathscr{B}_{\overline{\mathbb{R}}})$-measurable since if $B_1 \times B_2 \in \mathscr{B}_{\overline{\mathbb{R}}} \otimes \mathscr{B}_{\overline{\mathbb{R}}}$ then $g^{-1}(E \times B) = f^{-1}(E) \times B \in \mathscr{M} \otimes \mathscr{B}_{\overline{\mathbb{R}}}$, owing to measurability of f. Let $\pi_1(z,y) = z$ and $\pi_2(z,y) = y$. Then, as projections, π_1 and π_2 are $(\mathscr{B}_{\overline{\mathbb{R}}}, \mathscr{B}_{\overline{\mathbb{R}}})$-measurable. Now define $h(z,y) = z - y$ unless $z = y = \pm\infty$ in which case set $h(z,y) = 0$. By Exercise 2.1.2, h is $(\mathscr{B}_{\overline{\mathbb{R}}} \otimes \mathscr{B}_{\overline{\mathbb{R}}}, \mathscr{B}_{\overline{\mathbb{R}}})$-measurable. Hence the composition $h(g(x,y)) = f(x) - y$ is $(\mathscr{M} \otimes \mathscr{B}_{\overline{\mathbb{R}}}, \mathscr{B}_{\overline{\mathbb{R}}})$-measurable. It follows that

$$G_f = (h \circ g)^{-1}[0, \infty]$$

is $\mathscr{M} \otimes \mathscr{B}_{\overline{\mathbb{R}}}$ measurable (note that the case $y = f(x) = \infty$ is included in $(h \circ g)^{-1}\{0\}$) since we set $h = 0$ when $z = y = \infty$). If G_f is redefined with $y < f(x)$ instead, it in fact $\mathscr{M} \otimes \mathscr{B}_{\mathbb{R}}$ measurable, and we do not need to go through the fuss of working with extended reals. Measurability follows since $G_f = (h \circ g)^{-1}(0, \infty]$ instead.

Before continuing, let us look at the x-slices of G_f. For fixed x, we have that $(G_f)_x = \{y \in$

$[0,\infty] \mid f(x) \geq y\} = [0, f(x)]$. Now write $X = \bigcup_j E_j$ where each E_j has finite measure and all are disjoint. Let $I_j = [j-1, j]$. Then

$$X \times [0,\infty] = \bigcup_{j,k=1}^{\infty} E_j \times I_k.$$

Let $G_{f,j,k} = G_f \cap (E_j \times I_k)$. Since $E_j \times I_k$ has finite $(\mu \times m)$ measure, it follows that

$$\iint \chi_{G_{f,j,k}} \; d(\mu \times m) = (\mu \times m)(G_{f,j,k}) < \infty.$$

Hence, we can apply Tonelli. Doing so yields

$$(\mu \times m)(G_{f,j,k}) = \iint_{E_j \times I_k} \chi_{G_f} \; d(\mu \times m) = \int_{E_j} \left(\int_{I_k} \chi_{(G_f)_x} \; dm \right) d\mu = \int_{E_j} m([0, f(x)] \cap I_k) d\mu$$

Now by additivity

$$(\mu \times m)(G_f) = \sum_{j=1} \sum_{k=1}^{\infty} (\mu \times m)(G_{f,j,k}) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \int_{E_j} m([0, f(x)] \cap I_k) d\mu.$$

By Theorem 2.15 we can interchange the integral and sum to get

$$\begin{aligned} (\mu \times m)(G_f) &= \sum_{j=1}^{\infty} \int_{E_j} \sum_{k=1}^{\infty} m([0, f(x)] \cap I_k) d\mu = \sum_{j=1}^{\infty} \int_{E_j} m\left(\bigcup_{k=1}^{\infty} ([0, f(x)] \cap I_k) \right) d\mu \\ &= \int_{\bigcup_j E_j} m\left([0, f(x)] \cap \left(\bigcup_{k=1}^{\infty} I_k \right) \right) d\mu = \int_X m[0, f(x)] \; d\mu = \int_X f(x) \; d\mu \end{aligned}$$

owing to the disjointness of the E_j and the I_k. When we define G_f with $y < f(x)$ instead we simply have an open interval $[0, f(x))$ wherever we have $[0, f(x)]$.

2.5.51. Let $(X, \mathscr{M}, \mu)$ and $(Y, \mathscr{N}, \nu)$ be arbitrary measure spaces (not necessarily σ-finite).

a) If $f : X \to \mathbb{C}$ is $\mathscr{M}$-measurable, $g : Y \to \mathbb{C}$ is $\mathscr{N}$-measurable, and $h(x,y) = f(x)g(y)$, then h is $\mathscr{M} \otimes \mathscr{N}$-measurable.
b) If $f \in L^1(\mu)$ and $g \in L^1(\nu)$, then $h \in L^1(\mu \times \nu)$ and $\int h \; d(\mu \times \nu) = [\int f d\mu][\int g d\nu]$.

Solution:

a) Let $\pi : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be given by $\pi(x,y) = xy$. We show that π is measurable, and note that $h(x,y) = \pi(f(x), g(y))$, a composition of measurable functions. Observe that $x = u + iv$ and $y = z + iw$ so that $xy = (uz - vw) + i(vz + uw)$. The real and imaginary parts of this are continuous functions in four variables, hence are measurable functions. By Corollary 2.5 it follows that π is measurable.

   Another way to see this is as follows. We know that the product of measurable functions is measurable. The issue is that $f(x)$ and $g(y)$ are measurable, but not measurable in the same sense; the former is $(\mathscr{M}, \mathscr{B}_{\mathbb{C}})$-measurable while the latter is $(\mathscr{N}, \mathscr{B}_{\mathbb{C}})$-measurable. To remedy this, let $F(x,y) = f(x)$ and $G(x,y) = g(y)$. Then for any Borel set $B \in \mathscr{B}_{\mathbb{C}}$, we have that $F(x,y) = f^{-1}(B) \times Y \in \mathscr{M} \otimes \mathscr{N}$ owing to measurability of f. A similar analysis shows that $G(x,y)$ is $(\mathscr{M} \otimes \mathscr{N}, \mathscr{B}_{\mathbb{C}})$-measurable. Now $h(x,y) = F(x,y)G(x,y) = f(x)g(x)$ is the product of $(\mathscr{M} \otimes \mathscr{N}, \mathscr{B}_{\mathbb{C}})$-measurable functions, and is therefore $(\mathscr{M} \otimes \mathscr{N}, \mathscr{B}_{\mathbb{C}})$-measurable.
b) Let $\{\phi_n\}_{n=1}^{\infty}$ and $\{\psi_n\}_{n=1}^{\infty}$ be an increasing sequences of measurable simple functions converging to nonnegative f,g as usual. We first show that $\phi_n \psi_n$ is a simple function. Let $E \in \mathscr{M}$ and $F \in \mathscr{N}$. Then $\chi_E(x)\chi_F(y)$ is defined by

$$\chi_E(x)\chi_F(y) = \begin{cases} 1 & x \in E, y \in F \\ 0 & \text{else} \end{cases} = \begin{cases} 1 & (x,y) \in E \times F \\ 0 & \text{else} \end{cases} = \chi_{E \times F}(x,y).$$

Writing ϕ_n and ψ_n in standard form gives

$$\eta_n(x,y) = \phi_n(x)\psi_n(y) = \left(\sum_{j=1}^{J} c_j \chi_{E_j}\right)\left(\sum_{k=1}^{K} d_k \chi_{F_k}\right) = \sum_{(j,k)=(1,1)}^{(J,K)} c_j d_k \chi_{E_j \times F_k}(x,y)$$

Since $\phi_n \to f$ and $\psi_n \to g$, it follows that $\eta_n \to h$. Finally, η_n is increasing to h since

$$\eta_n(x,y) = \phi_n(x)\psi_n(y) \leq \phi_{n+1}(x)\psi_n(y) \leq \phi_{n+1}(x)\psi_{n+1}(y) = \eta_{n+1}(x,y).$$

So, by monotone convergence it will suffice to show the desired equality for characteristic functions.

Let $f = \chi_E$ and $g = \chi_F$ where $E \in \mathscr{M}$ and $F \in \mathscr{N}$. Then, as observed, $h = f(x)g(y) = \chi_{E\times F}(x,y)$. Now the equality is obvious by definition of the product measure,

$$\int h \; d(\mu \times \nu) = (\mu \times \nu)(E \times F) = \mu(E)\nu(F) = \left(\int f \; d\mu\right)\left(\int g \; d\nu\right).$$

Now let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$. The positive and negative parts of $h = fg$ are

$$h^+ = f^+g^+ + f^-g^- \qquad h^- = f^-g^+ + f^+g^-$$

By linearity and the case when f, g are nonnegative,

$$\begin{aligned}
\int h \; d(\mu\times\nu) &= \int h^+ \; d(\mu\times\nu) - \int h^- \; d(\mu\times\nu) \\
&= \int f^+g^+ \; d(\mu\times\nu) + \int f^-g^- \; d(\mu\times\nu) - \int f^-g^+ \; d(\mu\times\nu) - \int f^+g^- \; d(\mu\times\nu) \\
&= \left(\int f^+ \; d\mu\right)\left(\int g^+ \; d\nu\right) + \left(\int f^- \; d\mu\right)\left(\int g^- \; d\nu\right) \\
&\quad - \left(\int f^- \; d\mu\right)\left(\int g^+ \; d\nu\right) - \left(\int f^+ \; d\mu\right)\left(\int g^- \; d\nu\right) \\
&= \left(\int f^+ \; d\mu\right)\left(\int g^+ \; d\nu - \int g^- \; d\nu\right) - \left(\int f^-\right)\left(\int g^+ \; d\nu - \int g^- d\nu\right) \\
&= \left(\int f^+ \; d\mu - \int f^- \; d\mu\right)\left(\int g^+ \; d\nu - \int g^- \; d\nu\right) = \left(\int f \; d\mu\right)\left(\int g \; d\nu\right)
\end{aligned}$$

as desired. Now we move to the complex case. Let $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ denote the real and imaginary parts of f, and similarly for g. Then,

$$\begin{aligned}
h &= fg = (\mathrm{Re}(f) + i\,\mathrm{Im}(f))(\mathrm{Re}(g) + i\,\mathrm{Im}(g)) \\
&= (\mathrm{Re}(f)\,\mathrm{Re}(g) - \mathrm{Im}(f)\,\mathrm{Im}(g)) + i(\mathrm{Re}(f)\,\mathrm{Im}(g) + \mathrm{Im}(f)\,\mathrm{Re}(g))
\end{aligned}$$

So, the real and imaginary parts of h are

$$\mathrm{Re}(h) = \mathrm{Re}(f)\,\mathrm{Re}(g) - \mathrm{Im}(f)\,\mathrm{Im}(g) \qquad \mathrm{Im}(h) = \mathrm{Re}(f)\,\mathrm{Im}(g) + \mathrm{Im}(f)\,\mathrm{Re}(g).$$

Then by linearity and the case when f, g are real,

$$\begin{aligned}
\int h\ d(\mu\times\nu) &= \int \mathrm{Re}(h)\ d(\mu\times\nu) + i\int \mathrm{Im}(h)\ d(\mu\times\nu) \\
&= \int \mathrm{Re}(f)\,\mathrm{Re}(g)\ d(\mu\times\nu) - \int \mathrm{Im}(f)\,\mathrm{Im}(g)\ d(\mu\times\nu) \\
&\quad + i\int \mathrm{Re}(f)\,\mathrm{Im}(g)\ d(\mu\times\nu) + i\int \mathrm{Im}(f)\,\mathrm{Re}(g)\ d(\mu\times\nu) \\
&= \left(\int \mathrm{Re}(f)\ d\mu\right)\left(\int \mathrm{Re}(g)\ d\nu\right) - \left(\int \mathrm{Im}(f)\ d\mu\right)\left(\int \mathrm{Im}(g)\ d\nu\right) \\
&\quad + i\left(\int \mathrm{Re}(f)\ d\mu\right)\left(\int \mathrm{Im}(g)\ d\nu\right) + i\left(\int \mathrm{Im}(f)\ d\mu\right)\left(\int \mathrm{Re}(g)\ d\nu\right) \\
&= \left(\int \mathrm{Re}(f)\ d\mu\right)\left(\int \mathrm{Re}(g)\ d\nu + i\int \mathrm{Im}(g)\ d\nu\right) \\
&\quad + \left(\int \mathrm{Im}(f)\ d\mu\right)\left(i\int \mathrm{Re}(g)\ d\nu - \int \mathrm{Im}(g)\ d\nu\right) \\
&= \left(\int \mathrm{Re}(f)\ d\mu\right)\left(\int \mathrm{Re}(g)\ d\nu + i\int \mathrm{Im}(g)\ d\nu\right) \\
&\quad + i\left(\int \mathrm{Im}(f)\ d\mu\right)\left(\int \mathrm{Re}(g)\ d\nu + i\int \mathrm{Im}(g)\ d\nu\right) \\
&= \left(\int \mathrm{Re}(f)\ d\mu + i\int \mathrm{Im}(f)\ d\mu\right)\left(\int \mathrm{Re}(g)\ d\nu + i\int \mathrm{Im}(g)\ d\nu\right) = \left(\int f\ d\mu\right)\left(\int g\ d\nu\right)
\end{aligned}$$

as desired.

All of the above is a bit overkill, but I felt it was appropriate to at least show a case of splitting integrals into real/imaginary and positive/negative parts at least once. It is a standard argument that, once comfortable with, saves a lot of time!

2.5.52. The Fubini-Tonelli theorem is valid when $(X, \mathscr{M}, \mu)$ is an arbitrary measure space and Y is a countable set, $\mathscr{N} = \mathcal{P}(Y)$, and ν is counting measure on Y. (Cf. Theorems 2.15 and 2.25.)

Solution: First let us get an understanding of the problem. We can naturally identify Y with $\mathbb{N}$, so throughout I will choose an enumeration of the members of Y and work with $Y = \mathbb{N}$. If $f : X \times \mathbb{N} \to \mathbb{R}$ is a function then the slices $f^n(x)$ are really just elements of a sequence of functions. For fixed x, the slices $f_x(n)$ is just some sequence of real numbers. As seen, integration against counting measure is the same as summation, so that

$$\begin{aligned}
\int_X \left(\int_{\mathbb{N}} f_x(n)\ d\nu\right) d\mu &= \int_X \left(\sum_{n=1}^{\infty} f_x(n)\right) d\mu \\
\int_{\mathbb{N}} \left(\int_X f^n(x)\ d\mu\right) d\nu &= \sum_{n=1}^{\infty} \int_X f^n(x)\ d\mu
\end{aligned}$$

Let $g(x) = \sum_{n=1}^{\infty} f_x(n)$. Then also $g(x) = \sum_{n=1}^{\infty} f^n(x)$. Both definitions are equivalent and are just a matter of how you view g; to evaluate $g(x)$ you must also fix an x, and observe that $f^n(x) = f_x(n) = f(x, n)$. So, Theorems 2.15 and 2.25 are just a Fubini-Tonelli type theorem.

One can also observe that the proof of Fubini-Tonelli relies on σ-finiteness only in applying Theorem 2.36. So, it suffices to modify Theorem 2.36 with the above hypothesis and prove the same result. XXX

6. L^p SPACES

6.1. **Basic Theory of L^p Spaces.**

6.1.1. When does equality hold in Minkowski's inequality (the answer is different for $p = 1$ and for $1 < p < \infty$. What about $p = \infty$?)

Solution: Recall that the proof of Minkowski's inequality used Hölder's inequality twice; equality in Hölder occurs iff there exist nonzero constants α, β such that $\alpha|f|^p = \beta|g|^q$. We can restate this as equality holds iff there exists a positive constant c such that $c|f|^p = |g|^q$ by setting $c = \alpha/\beta$, which is valid since $\beta \neq 0$.

First, in the case $p = 1$, Minkowski follows easily from the triangle inequality and $|f + g| = |f| + |g|$ iff f and g have the same sign.

For the $1 < p < \infty$ case, there is a step which uses the triangle inequality:

$$|f + g|^p = |f + g||f + g|^{p-1} \leq |f||f + g|^{p-1} + |g||f + g|^{p-1}.$$

So, we have that f and g have the same sign. The two instances of Hölder are

$$\begin{aligned} \|f(f + g)^{p-1}\|_1 &\leq \|f\|_p\|(f + g)^{p-1}\|_q \\ \|g(f + g)^{p-1}\|_1 &\leq \|g\|_p\|(f + g)^{p-1}\|_q \end{aligned}$$

with $q = p/(p - 1)$. So, applying the equality case of Hölder we see that equality in both of the above occurs iff there exist nonzero constants c_1, c_2 such that

$$c_1|f|^p = |f + g|^{p-1} \qquad c_2|g|^p = |f + g|^{p-1}$$

Hence, equality in both holds iff there exist constants $c_1, c_2 > 0$ such that $c_1|f|^p = c_2|g|^p$. By taking p-th roots, we see that $c_1^{1/p}|f| = c_2^{1/p}|g|$. This says that $|f|$ and $|g|$ are multiples of each other. Now remember that equality in the triangle inequality holds iff f and g have the same sign. Combining these tells us that equality in Minkowski holds iff f and g are multiples of each other.

For the $p = \infty$, note that the essential sup is attained. That is, $\|f\|_\infty$ is the smallest number such that $\mu\{|f| > \|f\|_\infty\} = 0$. Let $F = \{|f| > \|f\|_\infty\}$ and similarly for G. Suppose that $F \cap G = \emptyset$.

Assuming $\|f\|_\infty < \infty$.