Various integrators

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1 Numerical solutions of ordinary differential equations

Of the below integrators, Euler's method, the improved Euler's method,and the Runge-Kunta 4 method are general integrators. The symplectic integrators solve a Hamiltonian system (such as Newton's second law without nonconservative forces) better, since they conserve energy, momentum, angular momenum, etc., better. This is because symplectic integrators solve a nearby Hamiltonian exactly. All energy losses are due to round-off error. All methods require a step-size, h.

1.1 Euler's Method

This is a first order nonsymplectic method.

```
1. Find a(x[i],v[i],t)
2. Set x[i+1]=x[i]+v[i]*h
3. Set v[i+1]=v[i]+a[i]*h.
```

1.2 Improved Euler's Method

With little extra work, we can increase the accuracy (to order 2) by using

```
1. Find a(x[i],v[i],t)
2. Set x[i+1]=x[i]+v[i]*h+0.5*a[i]*h*h
3. Set v[i+1]=v[i]+a[i]*h.
```

1.3 Runge-Kunta 4

This is a fourth order nonsymplectic method.

```
1. Find a(x[i],v[i],t)
2. Set k1=h*a, j1=h*v
3. Find a(x[i]+0.5*k1,v[i]+0.5*j1,t+0.5*h)
4. Set k2=h*a, j2=h*v
```

```
5. Find a(x[i]+0.5*k2,v[i]+0.5*j2,t+0.5*h)
6. Set k3=h*a, j3=h*v
7. Find a(x[i]+k3,v[i]+j3,t+h)
8. Set k4=h*a, j4=h*v
9. Set x[i+1]=x[i]+1/6*(j1+2*j2+2*j3+j4)
10. Set v[i+1]=v[i]+1/6*(k1+2*k2+2*k3+k4).
```

1.4 Yoshida Methods (and Verlet)

The Yoshida methods are a general set of symplectic integrators of even order. The general outline is

```
1. Set x=x[i], v=v[i]
2. For i=1 to n-1:
    2a. x=x+0.5*h*p[i]*v
    2b. Find a(x,v)
    2c. v=v+0.5*h*p[i]*a
    2d. x=x+0.5*h*p[i]*v
3. x=x+0.5*h*p[n]*v
4. Find a(x,v)
5. v=v+0.5*h*p[n]*a
6. x=x+0.5*h*p[n]*v
7. For i=n-1 to 1:
    7a. x=x+0.5*h*p[i]*v
    7b. Find a(x,v)
    7c. v=v+0.5*h*p[i]*a
    7d. x=x+0.5*h*p[i]*v
8. x[i+1]=x
9. v[i+1]=v.
```

For the Yoshida order 2 method (otherwise known as the Verlet method), $n = 1$ and $p_1 = 1$.

For the Yoshida order 4 method, $n = 2$ and $p_1 = 1.351207191959657$, $p_2 = -1.702414383919315$.

For the Yoshida order 6 method, $n = 4$ and

$$
\begin{aligned}
p_1 &= 0.784513610477560 \\
p_2 &= 0.235573213359357 \\
p_3 &= -1.17767998417887 \\
p_4 &= 1.31518632068391.
\end{aligned}
$$

For the Yoshida order 8 method, $n = 8$ and

$$\begin{aligned}
p_1 &= 1.04242620869991 \\
p_2 &= 1.82020630970714 \\
p_3 &= 0.157739928123617 \\
p_4 &= 2.44002732616735 \\
p_5 &= -0.00716989419708120 \\
p_6 &= -2.44699182370524 \\
p_7 &= -1.61582374150097 \\
p_8 &= -1.7808286265894516.
\end{aligned}$$

1.5 Ruth Method

This is a third order symplectic integrator.

Let `p=[2/3,-2/3,1]` and `q=[7/24,3/4,-1/24]`

```
1. Set x=x[i], v=v[i]
2. For i=1 to 3:
    2a. Find a(x,v)
    2b. Set v=v+p[i]*a*h
    2c. Set x=x+q[i]*v*h
3. Set x[i+1]=x
4. Set v[i+1]=v.
```

2 Forest-Ruth

This is a fourth order symplectic integrator.

Let $f_1 = 1.35120719195966$, $f_2 = 1 - f_1$, $f_3 = 1 - 2 * f_1$.

```
1. Set x=x[i], v=v[i]
2. Set x=x+f1*v*h/2
3. Find a(x,v)
4. Set v=v+f1*a*h
5. Set x=x+f2*v*h/2
6. Find a(x,v)
7. Set v=v+f3*a*h
8. Set x=x+f2*v*h/2
9. Find a(x,v)
10. Set v[i]=v+f1*a*h
11. Set x[i]=x+f1*v*h/2.
```

2.1 Position Extended Forest Ruth Method

This is another fourth order Forest-Ruth algorithm.

Let $f_1 = 0.1786178958448091$, $f_2 = -0.2123418310626054$, $f_3 = -0.0662645266981849$.

```
1. Set x=x[i], v=v[i]
2. Set x=x+f1*v*h
3. Find a(x,v)
4. Set v=v+(1-2*f2)*a*h/2
5. Set x=x+f3*v*h
6. Find a(x,v)
7. Set v=v+f2*a*h
8. Set x=x+(1-2*f3-2*f1)*v*h
9. Find a(x,v)
10. Set v=v+f2*a*h
11. Set x=x+f3*v*h
12. Find a(x,v)
13. Set v[i+1]=v+(1-2*f2)*a*h/2
14. Set x[i+1]=x+f1*v*h.
```

2.2 Candy Rozmus

This is a fourth-order symplectic method.

Let $p_1 = p_4 = \frac{1}{2(2-2^{1/3})}$, $p_2 = p_3 = \frac{1-2^{1/3}}{2(2-2^{1/3})}$, $q_1 = 0$, $q_2 = q_4 = \frac{1}{2-2^{1/3}}$, and $q_3 = -\frac{2^{1/3}}{2-2^{1/3}}$.

```
1. Set x=x[i], v=v[i]
2. For i=1 to 4:
    2a. Find a(x,v)
    2b. v=v+q[i]*a*h
    2c. x=x+p[i]*v*h
3. x[i+1]=x
4. v[i+1]=v.
```

We could also optimize this algorithm by setting

$$
\begin{aligned}
p_1 &= 0.5153528374311229364 \\
p_2 &= -0.085782019412973646 \\
p_3 &= 0.4415830236164665242 \\
p_4 &= 0.1288461583653841854 \\
q_1 &= 0.1344961992774310892 \\
q_2 &= -0.2248198030794208058 \\
q_3 &= 0.7563200005156682911 \\
q_4 &= 0.3340036032863214255.
\end{aligned}
$$

3 Major factors on integration accuracy

References

[1] Rodney Dunning, *VPNBody—Solar-System Dynamics for VPython*. http://www.longwood.edu/staff/dunningrb/vpnbody/