# 1 Constant Coefficient Systems. Fourier Syntheis

## 1.1 The Fourier transform

**Definition 1.1.**

$$\mathscr{S} = \{u \in C^\infty(\mathbb{R}^d) : \forall \alpha, \beta, \sup_{x\in\mathbb{R}^d} |x^\alpha \partial_x^\beta u(x)| \leq C_{\alpha,\beta}\} \tag{1}$$

this space is topology space with topology defined by semi-norm $|u|_n = \sup_{x\in\mathbb{R}^d, \alpha+\beta\leq n} |x^\alpha \partial_x^\beta u(x)|$

**Definition 1.2.** $\forall u \in \mathscr{S}(\mathbb{R}^d), T \in \mathscr{S}'(\mathbb{R}^d), x, \xi \in \mathbb{R}^d$

$$(\mathscr{F}u)(\xi) = \widehat{u}(\xi) = \int e^{-ix\cdot\xi} u(x)\,\mathrm{d}x \tag{2}$$

$$(\mathscr{F}^{-1}u)(x) = (2\pi)^{-d} \int e^{ix\cdot\xi} u(\xi)\,\mathrm{d}\xi \tag{3}$$

$$(\mathscr{F}T, u) = (T, \mathscr{F}u) \tag{4}$$

**Theorem 1.1.** *1. $\mathscr{F}$ is homeomorphism on $\mathscr{S}$ or $\mathscr{S}'$, if $u(x) \in \mathscr{S}$*

$$u(x) = (2\pi)^{-d} \int e^{ix\cdot\xi} \widehat{u}(\xi)\,\mathrm{d}x$$

*2. (Plancherel [?]) $\forall u \in L^2$, then*

$$||u||_{L^2}^2 = (2\pi)^{-d} ||\widehat{u}||_{L^2}^2 \tag{5}$$

*so Fourier transform is isometric(almost) on $L^2$*

*3. $\forall u \in \mathscr{S}$ or $\mathscr{S}'$*

$$\widehat{\partial_x^\alpha u}(\xi) = (i\xi)^\alpha \widehat{u}(\xi) \tag{6}$$

$$\partial_\xi^\alpha \widehat{u}(\xi) = \widehat{(ix)^\alpha u}(\xi) \tag{7}$$

**Definition 1.3.**

$$H^s = \{u \in \mathscr{S}' : ||\lambda^s \widehat{u}||_{L^2} < \infty\} \tag{8}$$

where $\lambda(\xi) = (1 + |\xi|^2)^{1/2}$

## 1.2 The method

### 1.2.1 General method example

this section we consider first order about $t$ and constant coefficient systems:

$$\begin{cases} \partial_t u + A(\partial_x) u = f & on \quad [0,T] \times \mathbb{R}^d \\ u|_{t=0} = h & on \quad \mathbb{R}^d \end{cases} \tag{9}$$

where $A(\partial_x) = \sum_\alpha A_\alpha \partial_x^\alpha u$. $u(t,x)$ is function from $[0,T] \times \mathbb{R}^d$ to $\mathbb{R}^N$ and $A_\alpha$ is $N \times N$ order constant matrix.

**Theorem 1.2.** *$\forall u \in C^0([0,T]; H^s(\mathbb{R}^d)), f \in L^1([0,T]; H^s(\mathbb{R}^d))$ then system on Fourier side is*

$$\begin{cases} \partial_t \widehat{u} + A(i\xi) \widehat{u} = \widehat{f} & on \quad [0,T] \times \mathbb{R}^d \\ \widehat{u}|_{t=0} = \widehat{h} & on \quad \mathbb{R}^d \end{cases} \tag{10}$$

*where $A(i\xi) = \sum_\alpha A_\alpha (i\xi)^\alpha$.*

*this first order ordinary differential equation has solution*

$$\widehat{u}(t,\xi) = e^{-tA(i\xi)} \widehat{h}(\xi) + \int_0^t e^{(t'-t)A(i\xi)} \widehat{f}(t',\xi)\, \mathrm{d}t' \tag{11}$$

**Theorem 1.3.** *if exist function C(t) bounded on any compact set* $[0,T]$*, and* $\forall t \geq 0, \xi \in \mathbb{R}^d, |e^{-tA(i\xi)}| \leq C(t)$*,then:*

1. $\forall h \in H^s, f \in L^1([0,T]; H^s)$*, the solution* $u(t,x)$ *for system is in* $C^0([0,T]; H^s)$*, with*

$$||u(t)||_{H^s} \leq C(t)||h||_{H^s} + \int_0^t C(t-t')||f(t')||_{H^s}\,\mathrm{d}t' \tag{12}$$

2. $\forall h \in L^2, f \in L^1([0,T]; L^2)$*, the solution* $u(t,x)$ *for system is in* $C^0([0,T]; L^2)$*, with*

$$||u(t)||_{L^2} \leq C(t)||h||_{L^2} + \int_0^t C(t-t')||f(t')||_{L^2}\,\mathrm{d}t' \tag{13}$$

### 1.2.2 Specific examples

this section is use general method(that is Fourier transform and analysis solution) just learned to solve particular three eq.

**Theorem 1.4** (heat equation)**.** *if exist function C(t) bounded on any compact set* $[0,T]$*, and* $\forall t \geq 0, \xi \in \mathbb{R}^d, |e^{-tA(i\xi)}| \leq C(t)$*,then heat eq:*

$$\partial_t u - \triangle_x u = f, u|_{t=0} = h \tag{14}$$

*has solution* $u \in C^0([0,T];H^s)$ *for all* $h \in H^s, f \in L^1([0,T];H^s)$

**Theorem 1.5** (Schrödinger equation)**.**

$$\partial_t u - i\triangle_x u = f, u|_{t=0} = h \tag{15}$$

*has solution* $u \in C^0([0,T];H^s)$ *whenever* $h \in H^s, f \in L^1([0,T];H^s)$

**Theorem 1.6** (wave eqution)**.** *if* $h_0 \in H^{s+1}, h_1 \in H^s, f \in L^1([0,T];H^s)$*, then wave eq*

$$\partial_t^2 u - \triangle_x u = f, u|_{t=0} = h_0, \partial_t u|_{t=0} = h_1 \tag{16}$$

*has solution* $u \in C^0([0,T];H^{s+1})$*,and* $\partial_t u, \partial_{x_j} u \in C^0([0,T];H^s)$*,with*

$$||u(t)||_{H^{s+1}} \leq ||h_0||_{H^{s+1}} + 2(1+t)||h_1||_{H^s} + 2(1+t)\int_0^t ||f(t')||_{H^s}\,\mathrm{d}t' \tag{17}$$

$$||\partial_t u(t)||_{H^s}, ||\partial_{x_j} u(t)||_{H^s} \leq ||h_0||_{H^{s+1}} + ||h_1||_{H^s} + \int_0^t ||f(t')||_{H^s}\,\mathrm{d}t' \tag{18}$$

#### 1.2.3 more about wave equation

**Definition 1.4.** $C(t,\xi) = \cos(t|\xi|), S(t,\xi) = \frac{\sin(t\xi)}{\xi|}$

### 1.3 Hyperbolicity for first order system

in this section we consider more simpler system, first order system about $t$ and $x$

$$\begin{cases} \partial_t u + \sum_{j=1}^{d} A_j \partial_{x_j} u = f & on \quad [0,T] \times \mathbb{R}^d \\ u|_{t=0} = h & on \quad \mathbb{R}^d \end{cases} \tag{19}$$

where $A_j$ is $N \times N$ constant matrix, and $u(t,x)$ is $N-$valued function.

this eq is special form of theorem1.2,so use (11), we can get this simper system's solution on Fourier side:

$$\widehat{u}(t,\xi) = e^{-itA(\xi)}\widehat{h}(\xi) + \int_0^t e^{i(t'-t)A(\xi)}\widehat{f}(t',\xi)\,\mathrm{d}t' \tag{20}$$

where $A(\xi) = \sum_{j=1}^{d} \xi_j A_j$

from section 1.2 we see $e^{itA(\xi)}$ play important role in whether solution exist on sobolev space.

**Definition 1.5.** if first order system satisfy $\forall \xi \in \mathbb{R}^d, |e^{iA(\xi)}| \leq C|\xi|^k$, for some $k > 0$,then this system is hyperbolic.

**Definition 1.6.** if first order system satisfy $\forall \xi \in \mathbb{R}^d, |e^{iA(\xi)}| \leq C$,then this system is strongly hyperbolic.

**Lemma 1.7.** *if* $\forall |x_0| = 1, f(\mu x_0) = O(\mu^k),$ *as* $k \to \infty$,*then* $f(x) = O(|x|^k),$ *as* $|x| \to \infty$

**Lemma 1.8.** $\lambda(\xi)$ *is eigenvalue for matrix* $A(\xi) = \sum_j \xi_j A_j$. *then* $\lambda(\xi) = O(|\xi|),$ *as* $|\xi| \to \infty$

**Lemma 1.9.** *$J$ is $m-$order Jordan block with parameter $\lambda$, then when $k \geq m$*

- $$J^k = \begin{pmatrix} \lambda^k & \binom{k}{1}\lambda^{k-1} & \cdots & \binom{k}{m-1}\lambda^{k-m+1} \\ & \lambda^k & \ddots & \binom{k}{m-2}\lambda^{k-m+2} \\ & & \ddots & \vdots \\ & & & \lambda^k \end{pmatrix} \tag{21}$$

- $$e^J = \begin{pmatrix} e^\lambda & e^\lambda & \cdots & \frac{1}{(m-1)!}e^\lambda \\ & \ddots & \ddots & \vdots \\ & & e^\lambda & e^\lambda \\ & & & e^\lambda \end{pmatrix} \tag{22}$$

**Theorem 1.10.** *system is hyperbolic, then for all $\xi \in \mathbb{R}^d$, $A(\xi)$ only have real eigenvalue.*

**Theorem 1.11** (not proved)**.** *if all eigenvalue of $A(\xi)$ are real and semi-simple, then*

$$A(\xi) = \sum_{j=1}^{N} \lambda_j(\xi)\Pi_j(\xi) \tag{23}$$

*where $\lambda_j(\xi)$ are eigenvalues, and $\Pi_j(\xi)$ are eigenprojectors which can defined by*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-is\lambda} e^{isA(\xi)}\,\mathrm{d}s \tag{24}$$

**Theorem 1.12.** *system is strongly hyperbolic is equals to*

1. *$\forall \xi \in \mathbb{R}^d$, then eigenvalue of $A(\xi)$ is real and simisimple.*
2. *there exist constant $C$, such that, forall $\xi \in \mathbb{R}^d$, the eigenprojectors of $A(\xi)$ is bounded by $C$ uniformly.*

**Definition 1.7.**

1. system is hyperbolic symmtric if $\forall j$, $A_j$ is self adjoint(Hermitian).
2. system is hyperbolic symmtriable if exist positive define matrix $S$, such that $\forall j$, $SA_j$ is self adjoint.

**Theorem 1.13.** *hyperbolic symmtriable system is strongly hyperbolic*

**Definition 1.8.** system is smooth diagonalizable if $A(\xi)$ can diagonalizable and exist real valued eigenvalues $\lambda_j(\xi)$ and eigenprojectors $\Pi_j(\xi)$ which are analytic on unit sphere, such that $A(\xi) = \sum_j \lambda_j(\xi)\Pi_j(\xi)$

**Theorem 1.14.** *smooth diagonalizable system is strongly hyperbolic.*

**Definition 1.9.** system is strictly hyperbolic if $\forall \xi \neq 0, A(\xi)$ has $N$ distinct real eigenvalues.

**Theorem 1.15** (perturbation theory,not proved)**.** *if matrix's eigenvalues have local constant multiplicity, then the eigenvalues and eigenprojectors are both smooth (real analytic)*

**Theorem 1.16.** *strictly hyperbolic system is smooth diagonalizable*

*Remark* 1.1. although strongly hyperbolic is like strong condition, but preview definition all belong to strongly hyperbolic.

**Theorem 1.17.** *system is strongly hyperbolic then for all $h \in H^s, f \in L^1([0,T]; H^s)$, the solution $u \in C^0([0,T]; H^s)$ with*

$$||u(t)||_{H^s} \leq C||h||_{H^s} + C\int_0^t ||f(t')||_{H^s}\,\mathrm{d}t' \tag{25}$$