Observer Design, Kalman Filter

by Sergei Savin

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- From the previous lecture we remember that we wanted the estimate of the state to have the same dynamics as the actual state.
- We also remember that we want to take into account the error between the estimated and measured output $\mathbf{y}$

Assume the measurement is perfect: $\hat{\mathbf{y}} = \mathbf{y}$. Then we can propose observer as the following dynamical system:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{y} - \hat{\mathbf{y}})$$

Remember that state estimation error is $\epsilon = \hat{\mathbf{x}} - \mathbf{x}$ and the actual dynamics of the system is:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases}$$

Then, we can write the state estimation error dynamics:

$$\dot{\hat{\mathbf{x}}} - \dot{\mathbf{x}} = \mathbf{A}\hat{\mathbf{x}} - \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} - \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{y} - \hat{\mathbf{y}})$$

or:

$$\dot{\epsilon} = \mathbf{A}\epsilon + \mathbf{L}(\mathbf{C}\mathbf{x} - \mathbf{C}\hat{\mathbf{x}})$$

or:

$$\dot{\epsilon} = \mathbf{A}\epsilon - \mathbf{L}\mathbf{C}\epsilon$$

Thus, with the proposed observer, state estimation error is:

$$\dot{\epsilon} = (\mathbf{A} - \mathbf{LC})\epsilon$$

From which immediately follows that the observer is *stable* (i.e., the state estimation error tends to zero), as long as the following matrix is negative-definite:

$$\mathbf{A} - \mathbf{LC} < 0$$

We need to find $\mathbf{L}$.

Let us observe the key difference between observer design and controller design.

Controller design: find such $\mathbf{K}$ that:

$$\mathbf{A} - \mathbf{BK} < 0$$

Observer design: find such $\mathbf{L}$ that:

$$\mathbf{A} - \mathbf{LC} < 0$$

We have instruments for finding $\mathbf{K}$, what about $\mathbf{L}$?

Assume that $\mathbf{C}$ is identity $\mathbf{I}$. Then the observer design problem becomes:

- find such $\mathbf{L}$ that $\mathbf{A} - \mathbf{LI} < 0$.

Which is equivalent to the problem:

- find such $\mathbf{L}$ that $\mathbf{A} - \mathbf{BL} < 0$, where $\mathbf{B} = \mathbf{I}$.

And in that formulation, it is equivalent to the controller design problem, which we know how to solve.

In general, we can observe that if $\mathbf{A} - \mathbf{LC}$ is negative-definite, then $(\mathbf{A} - \mathbf{LC})^\top$ is negative-definite too (by definition of the negative-definiteness).

Therefore, we can solve the following *dual problem*:

- find such $\mathbf{L}$ that $\mathbf{A}^\top - \mathbf{C}^\top \mathbf{L}^\top < 0$.

The dual problem is *equivalent* to the control design problem. We can solve it by producing and solving algebraic Riccati equation, as in the LQR formulation. In pseudo-code it can be represented the following way:

$\mathbf{L}^\top$ = lqr($\mathbf{A}^\top$, $\mathbf{C}^\top$, $\mathbf{Q}$, $\mathbf{R}$).
where $\mathbf{Q}$ and $\mathbf{R}$ are weight matrices, determining the "sensitivity" or "aggressiveness" of the observer.

Assume we have a discrete linear system with disturbance and noise:

$$\begin{cases} \mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i + \mathbf{w} \\ \mathbf{y}_i = \mathbf{C}\mathbf{x}_i + \mathbf{v} \end{cases}$$

where $\mathbf{w}$ and $\mathbf{v}$ are disturbance and noise, both are random processes and both assumed to have 0 expected value. Covariance matrix of $\mathbf{w}$ is $\mathbf{W}$, covariance matrix of $\mathbf{v}$ is $\mathbf{V}$.

Our estimation $\hat{\mathbf{x}}$ is now done is two steps: after dynamics update, and after sensor update. We will use the following notation for it:

- $\hat{\mathbf{x}}_{i|i-1}$ is the i-th step estimate after dynamics update; also called an *a priori* estimate.
- $\hat{\mathbf{x}}_{i|i}$ is the estimate after sensor update; also called an *a posteriori* estimate.

Both estimates $\hat{\mathbf{x}}_{i|i-1}$ and $\hat{\mathbf{x}}_{i|i}$ have associated state estimation errors:

- $\epsilon_{i|i-1} = \mathbf{x}_i - \hat{\mathbf{x}}_{i|i-1}$
- $\epsilon_{i|i} = \mathbf{x}_i - \hat{\mathbf{x}}_{i|i}$

Those state estimation errors have their covariance matrices:

- $\mathbf{P}_{i|i-1} = \mathrm{cov}(\epsilon_{i|i-1})$,
- $\mathbf{P}_{i|i} = \mathrm{cov}(\epsilon_{i|i})$

We can consider the output estimation error $\varepsilon_i = \mathbf{y}_i - \mathbf{C}\hat{\mathbf{x}}_{i|i-1}$ and its covariance matrix $\mathbf{Y}_i$:

$$\mathbf{Y}_i = \mathrm{cov}(\mathbf{y}_i - \mathbf{C}\hat{\mathbf{x}}_{i|i-1})$$

Kalman Filter

Estimate updates

We know that $\hat{\mathbf{x}}_{i|i-1}$ can be found as follows:

$$\hat{\mathbf{x}}_{i|i-1} = \mathbf{A}\hat{\mathbf{x}}_{i-1|i-1} + \mathbf{B}\mathbf{u}_i$$

where $\hat{\mathbf{x}}_{i-1|i-1}$ is the estimate on the previous time step.

We will search for the update law for the a posteriori estimate $\hat{\mathbf{x}}_{i|i}$ in the following form:

$$\hat{\mathbf{x}}_{i|i} = \hat{\mathbf{x}}_{i|i-1} + \mathbf{K}_i(\mathbf{y}_i - \mathbf{C}\hat{\mathbf{x}}_{i|i-1})$$

We need to find $\mathbf{K}_i$, which is the Kalman filter gain.

We know or measure:

- $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$
- $\mathbf{W}$ and $\mathbf{V}$
- $\mathbf{y}_i$

We need to find:

- $\hat{\mathbf{x}}_{i|i-1}$ and $\hat{\mathbf{x}}_{i|i}$
- $\mathbf{K}_i$
- $\mathbf{Y}_i$, $\mathbf{P}_{i|i-1}$ and $\mathbf{P}_{i|i}$

There is a formula for the a priori estimation covariance update:

$$\mathbf{P}_{i|i-1} = \mathbf{A}\mathbf{P}_{i-1|i-1}\mathbf{A}^\top + \mathbf{W}$$

And a formula for the measurement covariance:

$$\mathbf{Y}_i = \mathbf{C}\mathbf{P}_{i|i-1}\mathbf{C}^\top + \mathbf{V}$$

With those two, we can find the Kalman filter gain:

$$\mathbf{K}_i = \mathbf{P}_{i|i-1}\mathbf{C}^\top\mathbf{Y}_i^{-1}$$

Now we can find the a posteriori covariance:

$$\mathbf{P}_{i|i} = (\mathbf{I} - \mathbf{K}_i\mathbf{C})\mathbf{P}_{i|i-1}$$

Lecture slides are available via Moodle.

You can help improve these slides at:
`https://github.com/SergeiSa/Linear-Control-Slides-Spring-2020`

Check Moodle for additional links, videos, textbook suggestions.