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Chromatic Polynomial in Small Space

Algorithmic Engineering Aspects of Fast Zeta Transform-based Graph Colouring Algorithms

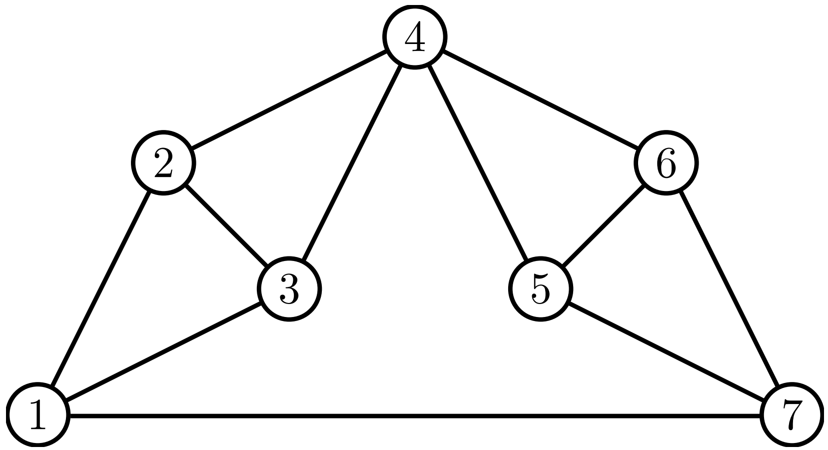
This presentation

- Introduction and theory
  - Graphs & algorithms
  - The BHKK algorithm
  - The Fast Zeta Transform
- My work
- Results
- Questions
  - There are no silly ones.

Introduction, graphs

- What is a graph?
  - Vertices, V
    - Order n
  - Edges, E
    - Size m

The Moser Spindle graph



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  - Edges, *E*
- How can we colour it?
  - Optimally
  - Counting

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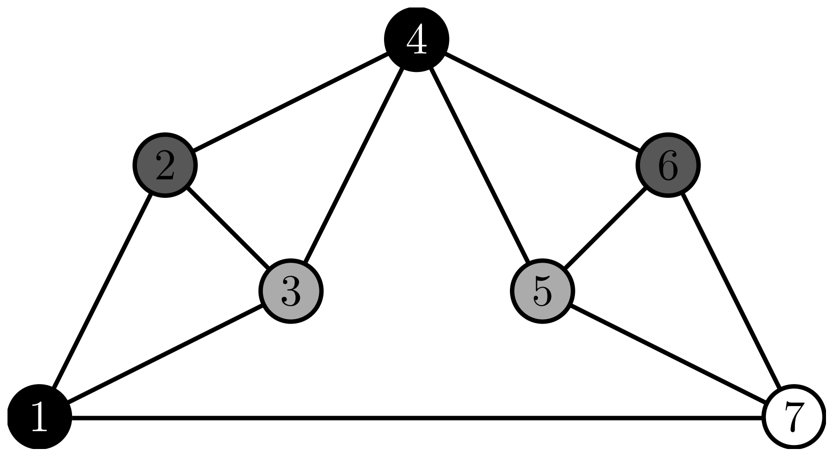

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- What is the Chromatic Polynomial?
  - What does it look like?

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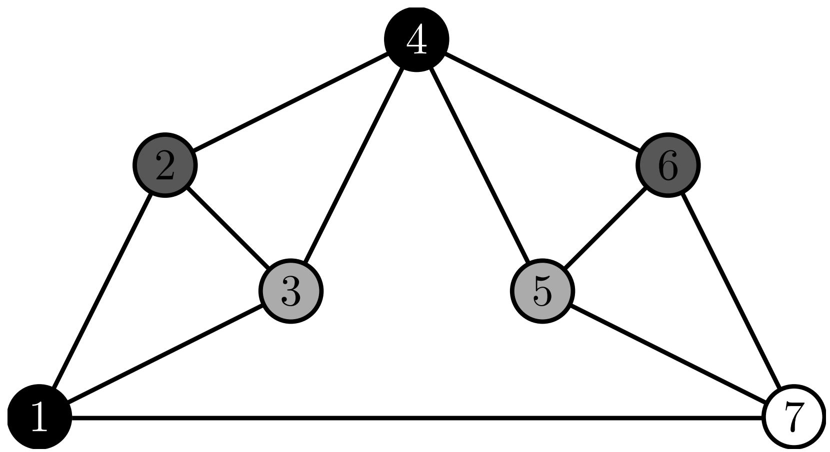

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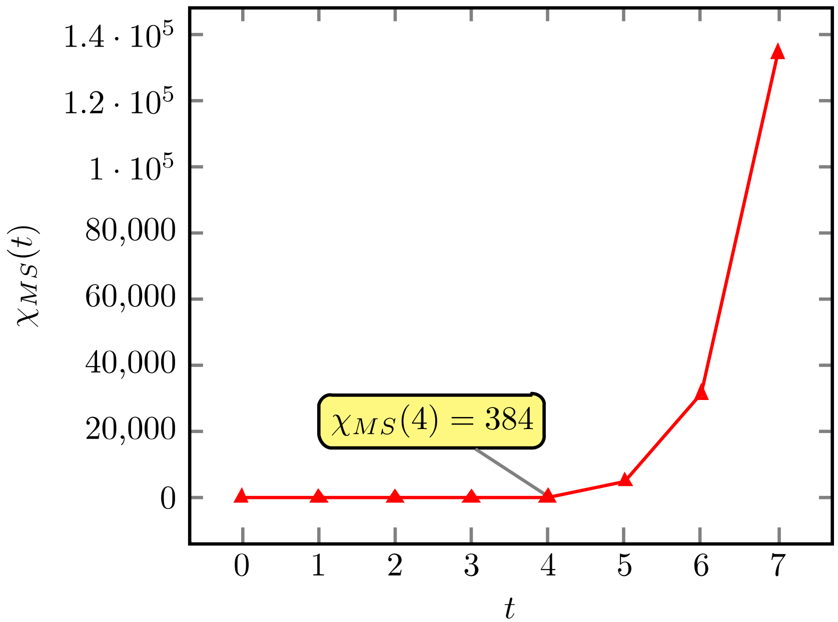
The Moser Spindle graph chromatic polynomial



$$\chi_{MS}(t) = t^7 - 11t^6 + 51t^5 - 129t^4 + 188t^3 - 148t^2 + 48t$$

Introduction, algorithms

- What is an algorithm?

  Stone: ”A set of rules that precisely define a sequence of operations.”

  Wikipedia: ”A step-by-step procedure for calculations.”

Example:

1. *Initialize data structures.*
2. *Magic*
3. *...*
4. *Profit.*

Introduction, algorithms

Step A. For $q = 0, 1, \ldots, n$, do

1. Partition V into V_1 and V_2 of sizes n_1 and n_2.
2. For each $X_1 \subseteq V_1$, do
   a) For each independent $Y_1 \subseteq X_1$, do
   $$h[V_2 \setminus N(Y_1)] \leftarrow h[V_2 \setminus N(Y_1)] + z^{|Y_1|}$$
   b) For each independent $Y_2 \subseteq V_2$, do
   $$l[Y_2] \leftarrow z^{|Y_2|}$$
   c) $h \leftarrow (h\zeta') \cdot l$
   d) $h \leftarrow h\zeta$
   e) For each $X_2 \subseteq V_2$, do
   $$r \leftarrow r + (-1)^{n-|X_1|-|X_2|} \cdot h[X_2]^q$$
3. Return coefficient c_n of z^n in r.

Step B. Construct interpolating polynomial $\chi_G(t)$ on points (q, c_{nq}).

Step C. Return $\chi_G(t)$.

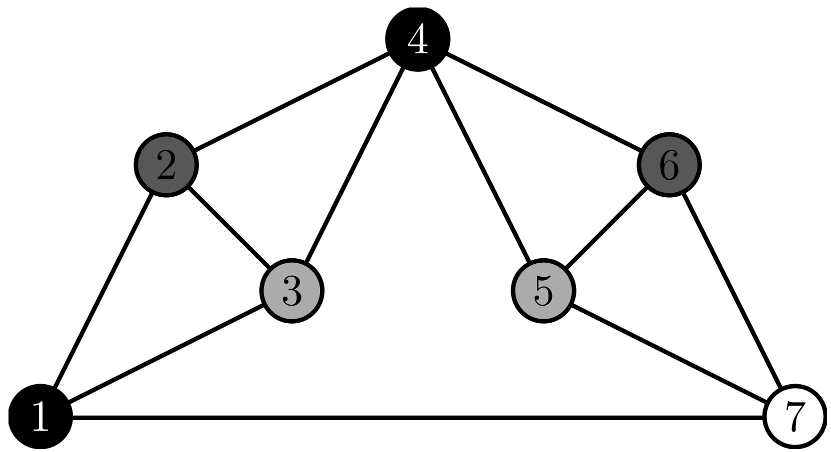
The BHKK Algorithm

- Björklund, Husfeldt, Kaski, Koivisto = BHKK
- Computes the Chromatic Polynomial
  - $O^*(2^n)$ time
  - $O^*(1.2916^n)$ space
    - Main improvement
  - Principle of inclusion-exclusion
  - Iteration over independent subsets
- Fast Zeta Transform

The Fast Zeta Transform

Split V into V1 and V2

	V_1					V_2	
V	1	2	3	4	5	6	7



The Fast Zeta Transform

b) For each independent $Y_2 \subseteq V_2$, do

$$l[Y_2] \leftarrow z^{|Y_2|}$$

$X \subseteq V_2$	$f(X)$
$\{6,7\}$	0
$\{7\}$	z
$\{6\}$	z
$\emptyset$	1

$$\xrightarrow[\textit{BHKK version}]{\textit{Fast Zeta Transform, }\zeta}$$

$X \subseteq V_2$	$f\zeta(X)$
$\{6,7\}$	$2z+1$
$\{7\}$	$z+1$
$\{6\}$	$z+1$
$\emptyset$	1

The Fast Zeta Transform

Sets are encoded as integers (= bitsets). 1 in position *i* means element *i* is in the set (*i = 1,...,n*)

$X \subseteq V_2$	$f(X)$
$1100000 \equiv 96$	0
$1000000 \equiv 64$	z
$0100000 \equiv 32$	z
$0000000 \equiv 0$	1

$$\xrightarrow[\textit{BHKK version}]{\textit{Fast Zeta Transform, } \zeta}$$

$X \subseteq V_2$	$f\zeta(X)$
$1100000 \equiv 96$	$2z+1$
$1000000 \equiv 64$	$z+1$
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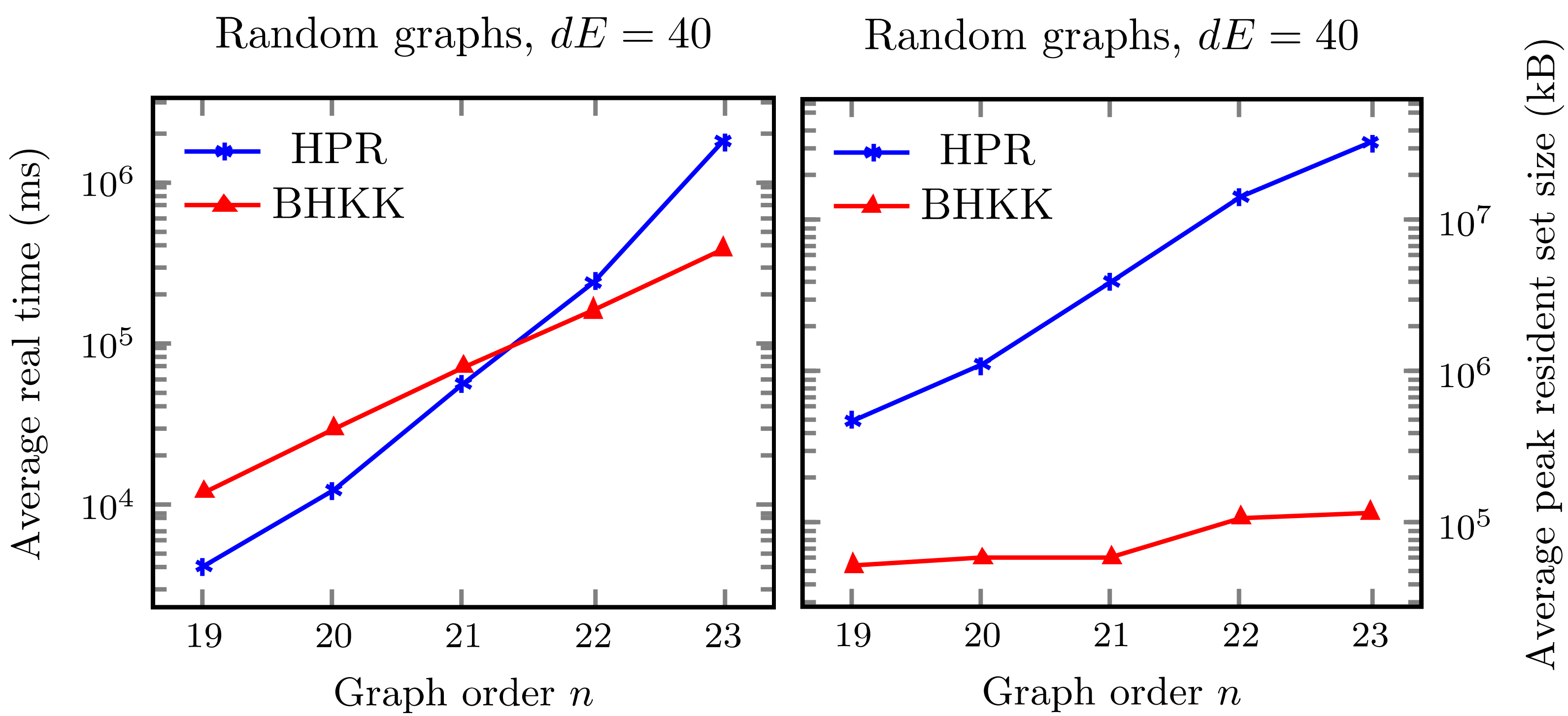
$0000000 \equiv 0$	1

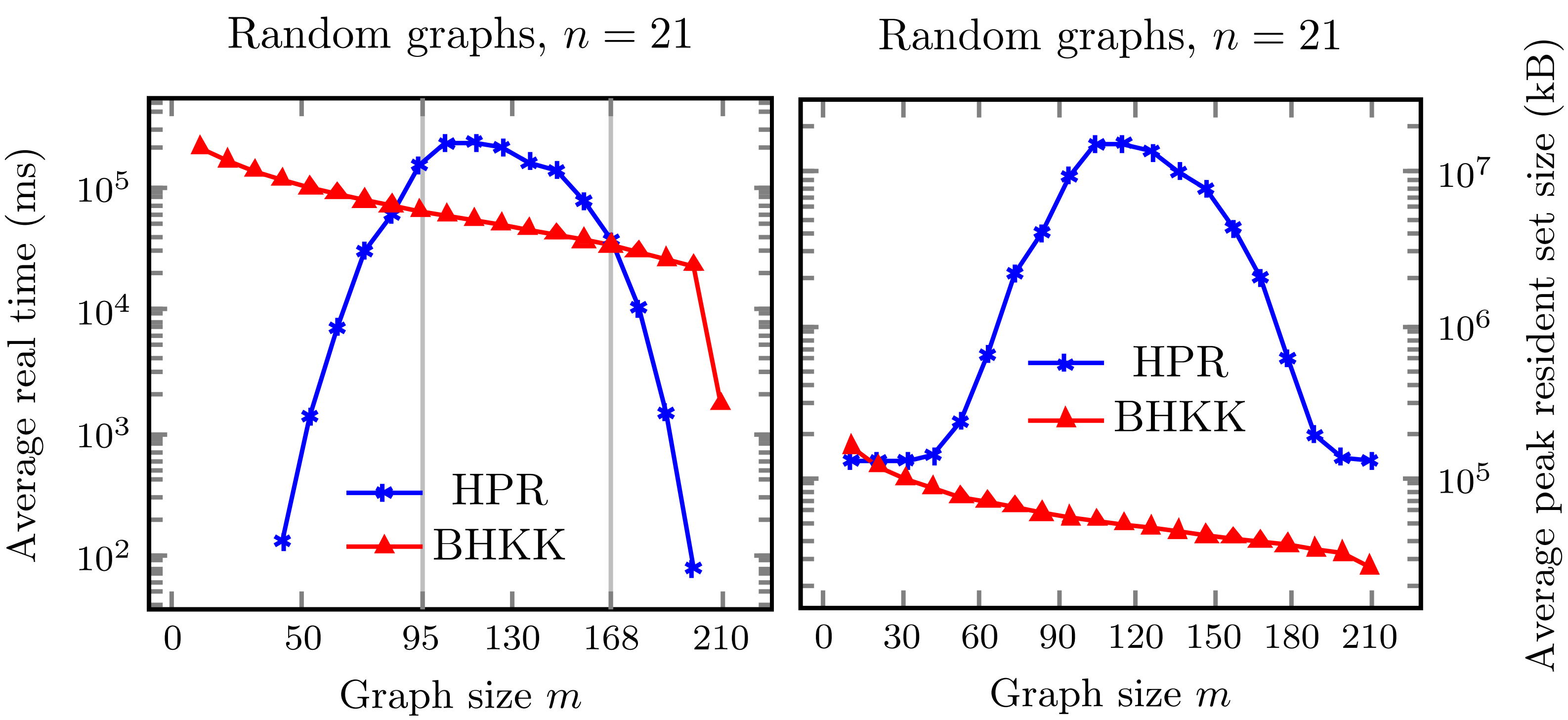
My work

- Implement the BHKK algorithm
  - Polynomial arithmetic externalized
  - Parallelization
- Run simulations, compare to competition
  - Haggard, Pearce, Royle = HPR algorithm
    - Deletion-contraction, dynamic programming
  - Random graphs
  - Optimizations

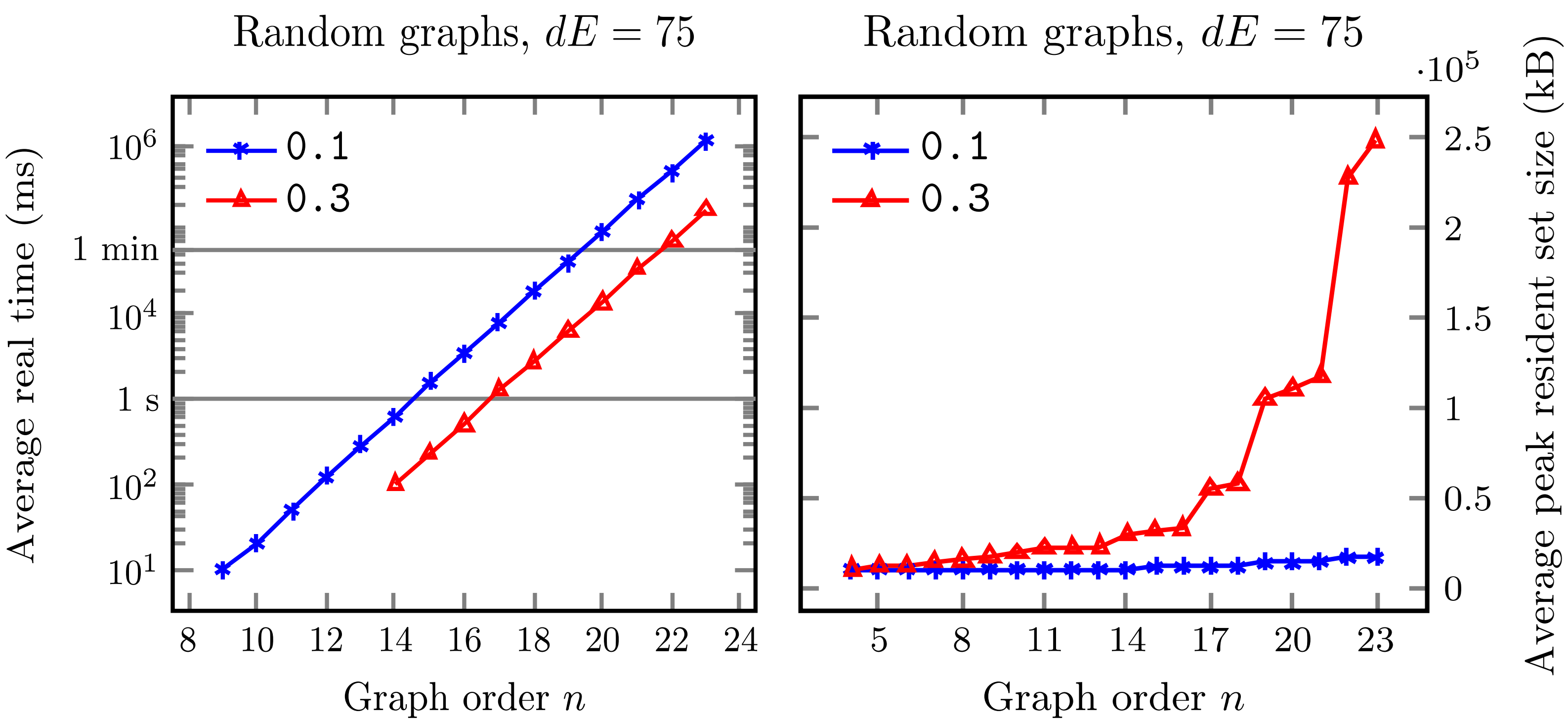
Results, complexity



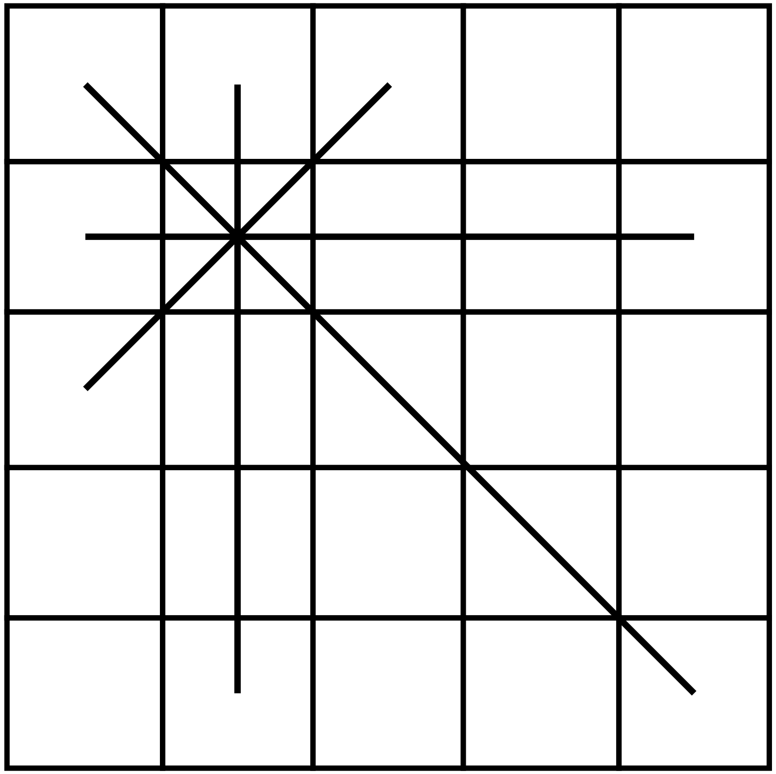
Results, size dependency



Results, parallelization



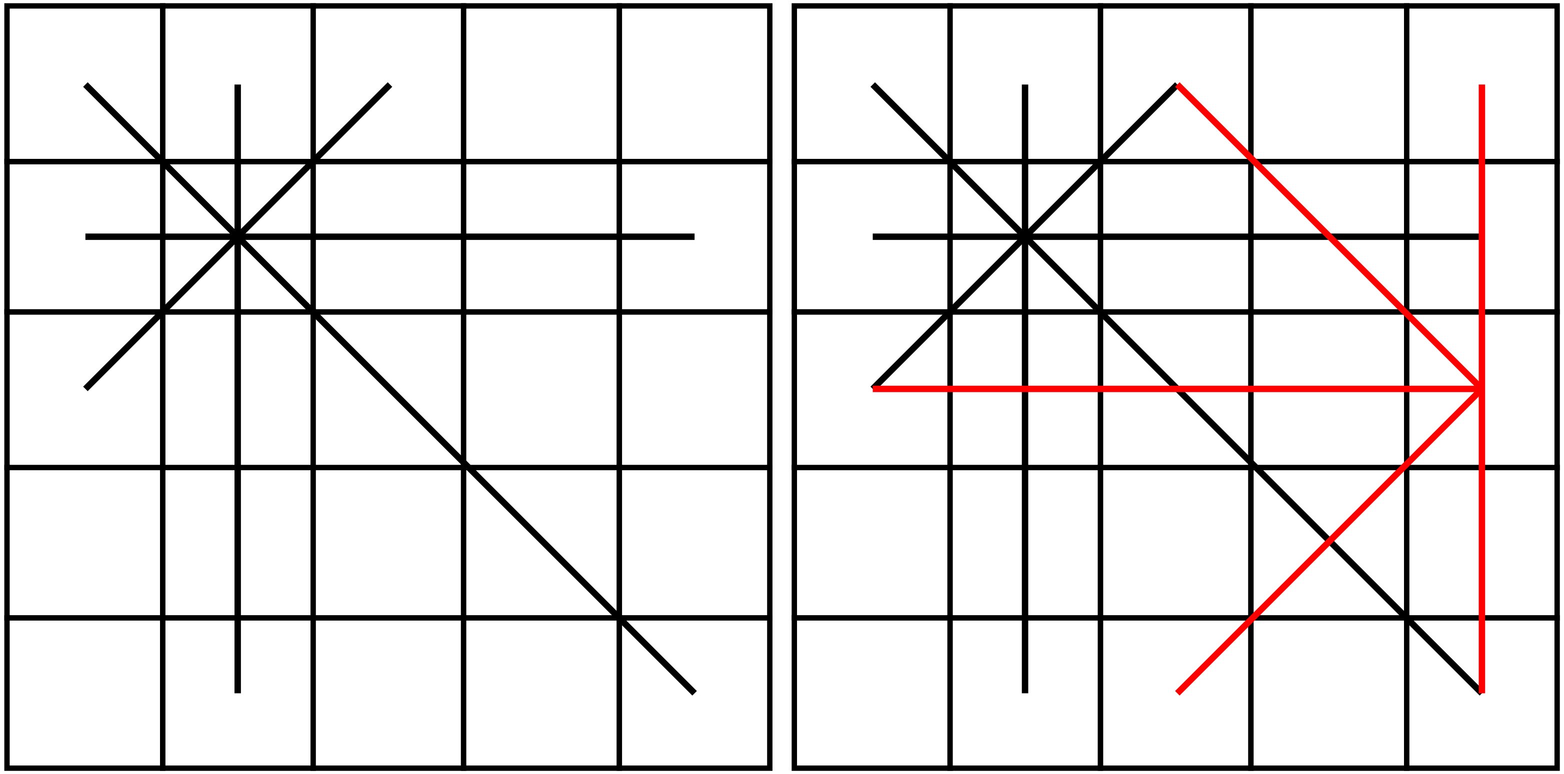
Results, Queen graph



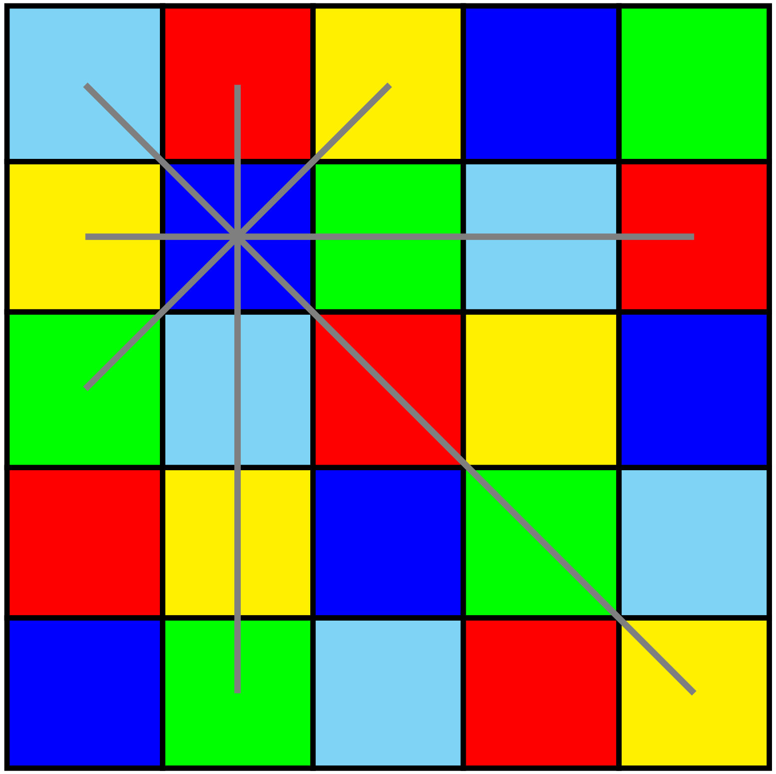
5x5 Queen graph

- Order 25
- Size 160
- Chromatic nbr 5

Results, Queen graph



Results, Queen graph



5x5 Queen graph

- Order 25
- Size 160
- Chromatic nbr 5

Results, Queen graph

$$
\begin{aligned}
\chi_{Q_5}(t) = t^{25} &- 160t^{24} + 12400t^{23} - 619000t^{22} + 22326412t^{21} - 618664244t^{20} \\
&+ 13671395276t^{19} - 246865059671t^{18} + 3702615662191t^{17} \\
&- 46639724773840t^{16} + 496954920474842t^{15} - 4497756322484864t^{14} \\
&+ 34633593670260330t^{13} - 226742890673713726t^{12} \\
&+ 1258486280066672806t^{11} - 5890734492089539317t^{10} \\
&+ 23071456910844580538t^{9} - 74774310771536397886t^{8} \\
&+ 197510077615138465516t^{7} - 416375608854898733286t^{6} \\
&+ 680208675481930270860t^{5} - 824635131668099993614t^{4} \\
&+ 692768396747228503860t^{3} - 356298290543726707632t^{2} \\
&+ 83353136564448062208t
\end{aligned}
$$

Results, Queen graph

```
$ bins/chr_pol_pari input/adjm/dimacs/dim_36_290 12
     Evaluating x(0)...  = 0
     Evaluating x(1)...  = 0
     Evaluating x(2)...  = 0
     Evaluating x(3)...  = 0
     Evaluating x(4)...  = 0
     Evaluating x(5)...  = 0
     Evaluating x(6)...  = 0
     Evaluating x(7)...  = 100800
     Evaluating x(8)...  = 539993341440
     Evaluating x(9)...  = 28523818425553920
     Evaluating x(10)...  = 168848393250572582400
     Evaluating x(11)...  = 242888062409233489987200
     Evaluating x(12)...  = 125418745435123389172492800
     Evaluating x(13)...  = 29561851852939196036141575680
     Evaluating x(14)...  = 3746380075607039378376563735040
     Evaluating x(15)...  = 287237024180965294490230843867200
     ...
```

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Open questions

- How fast can we run on exponential amount of threads?
  - One subset per thread.
  - ”Worst” subset is the one with most independent ”secondary” subsets.
  - Requires ~33 million CPUs for Queen 5x5.
- How fast can we run if actively separating ”bad” subsets?
  - We group subsets into 12 groups (or some other constant).
  - If we are unlucky, we might get all ”bad” subsets in one group.