The Riemann Prime Counting Function $\Pi(n)$ and The Partial Sum Convolution $[\log \zeta(s)]_n$

One main reason for interest in $[\zeta(s)^z]_n$ at all is that the partial sum $[\log \zeta(s)]_n$ is a function that only changes values at prime powers, which we can use to reason about the distribution of primes. In particular, $[\log \zeta(0)]_n = \Pi(n)$, the Riemann Prime Counting function.

More generally, if we have some function $f(n)$ that is multiplicative, then $[\log f]_n$ will be a function that only increases at prime powers as well, which is also potentially interesting depending on the nature of $f(n)$.

3. Riemann's Prime Counting Function $\Pi(n)$

One of the main reasons to generalize $[\zeta(s)^z]_n$ is its tight relationship to $\Pi(n)$, the Riemann Prime counting function. This section lists identities for $\Pi(n)$ based on that relationship.

3.1 $[\log \zeta(s)]_n$ as the log of $[\zeta(s)^z]_n$

Now that we have several ways to express $[\zeta(s)^z]_n$ with *z* a complex continuous value, we can take limits with it.

And that lets us take the following limit, immediately giving us a very important expression for the Riemann Prime Counting function:

$$\boxed{[\log \zeta(s)]_n = \lim_{z \to 0} \frac{[\zeta(s)^z]_n - 1}{z}} \tag{3.1.1}$$

```
ri[]:=RandomInteger[{-10,10}];rr[]:=RandomReal[{-3,3}]+RandomReal[{-3,3}]I
K[n_]:=FullSimplify[MangoldtLambda[n]/Log[n]]
logzeta[n_,s_]:=Sum[j^-s K[j],{j,2,n}]
zeta[n_,s_,z_,k_]:=1+((z+1)/k-1)Sum[j^-s zeta[n/j,s,z,k+1],{j,2,n}]
Table[Chop[logzeta[n=ri[],ss=rr[]]-Limit[(zeta[n,ss,z,1]-1)/z,z->0]],{t,1,100}]
```

The limit of this as *n* approaches infinity, if $\Re(s) > 1$, is

$$\log \zeta(s) = \lim_{z \to 0} \frac{\zeta(s)^z - 1}{z} \tag{3.1.2}$$

```
{Log[Zeta[s]],Limit[ (Zeta[s]^z-1)/z,z->0]}
```

Of particular note, the Riemann Prime Counting function is

$$\Pi(n)=\lim_{z\to 0}\frac{[\zeta(0)^z]_n-1}{z} \tag{3.1.3}$$

```
RiemannPrimeCount[n_]:=Sum[PrimePi[n^(1/j)]/j,{j,1,Log[2,n]}]
zeta[n_,s_,z_,k_]:=1+((z+1)/k-1)Sum[j^-s zeta[n/j,s,z,k+1],{j,2,n}]
Table[RiemannPrimeCount[n]-Limit[(zeta[n,0,z,1]-1)/z,z->0],{n,1,100}]
```

This can be generalized as

$$[\log f]_n=\lim_{z\to 0}\frac{[f^z]_n-1}{z} \tag{3.1.4}$$

3.2 $[\log\zeta(s)]_n$ in Terms of $[(\zeta(s)-1)^k]_n$

This same idea can also be expressed as

$$[\log\zeta(s)]_n=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}[(\zeta(s)-1)^k]_n \tag{3.2.1}$$

```
ri[]:=RandomInteger[{-10,10}];rr[]:=RandomReal[{-3,3}]+RandomReal[{-3,3}]I
K[n_]:=FullSimplify[MangoldtLambda[n]/Log[n]]
logzeta[n_,s_]:=Sum[j^-s K[j],{j,2,n}]
zetam1[n_,s_,0]:=UnitStep[n-1]
zetam1[n_,s_,k_]:=Sum[j^-s zetam1[n/j, s, k-1],{j,2,n}]
altlogzeta[n_, s_] := Sum[ (-1)^(k+1)/k zetam1[n, s, k],{k,1,Log[2,n]}]
Table[Chop[logzeta[a=ri[],b=rr[]]-altlogzeta[a,b]],{n,1,100}]
```

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\log\zeta(s)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}(\zeta(s)-1)^k \tag{3.2.2}$$

```
{Log[Zeta[s]],Sum[(-1)^(k+1)/k (Zeta[s]-1)^k,{k,1,Infinity}]}/.s->0
```

Of particular note, the Riemann Prime Counting function is

$$\Pi(n)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}[(\zeta(0)-1)^k]_n \tag{3.2.3}$$

This is Linnik's identity summed, from pg. 343 of H. Iwaniec and E. Kowalski's "Analytic Number Theory", more or less.

This can be generalized as

$$[\log f]_n=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}[(f-1)^k]_n \tag{3.2.4}$$

3.3 $[\log\zeta(s)]_n$ as the Derivative of $[\zeta(s)^z]_n$

Other ways to arrive at Riemann's Prime counting function are

$$\boxed{[\log\zeta(s)]_n = \lim_{z\to 0}\frac{\partial}{\partial z}[\zeta(s)^z]_n} \tag{3.3.1}$$

```
ri[]:=RandomInteger[{10,100}];rr[]:=RandomReal[{-3,3}]+RandomReal[{-3,3}]I
K[n_]:=FullSimplify[MangoldtLambda[n]/Log[n]]
logzeta[n_,s_]:=Sum[j^-s K[j],{j,2,n}]
zeta[n_,s_,z_,k_]:=1+((z+1)/k-1)Sum[j^-s zeta[n/j,s,z,k+1],{j,2,n}]
Table[Chop[logzeta[a=ri[],b=rr[]]-(Limit[D[zeta[a,b,z,1],z],z->0])],{n,1,100}]
```

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\log\zeta(s) = \lim_{z\to 0}\frac{\partial}{\partial z}\zeta(s)^z \tag{3.3.2}$$

```
{Log[Zeta[0]],Limit[D[Zeta[0]^z,z],z->0]}
```

Of particular note, the Riemann Prime Counting function is

$$\boxed{\Pi(n) = \lim_{z\to 0}\frac{\partial}{\partial z}[\zeta(0)^z]_n} \tag{3.3.3}$$

This can be generalized as

$$\boxed{[\log f]_n = \lim_{z\to 0}\frac{\partial}{\partial z}[f^z]_n} \tag{3.3.4}$$

3.4 $[\log\zeta(s)]_n$ as a Residue of $[\zeta(s)^z]_n$

and

$$\boxed{[\log\zeta(s)]_n = \operatorname*{Res}_{z=0}\frac{[\zeta(s)^z]_n}{z^2}} \tag{3.4.1}$$

```
ri[]:=RandomInteger[{10,100}];rr[]:=RandomReal[{-3,3}]+RandomReal[{-3,3}]I
K[n_]:=FullSimplify[MangoldtLambda[n]/Log[n]]
logzeta[n_,s_]:=Sum[j^-s K[j],{j,2,n}]
zeta[n_,s_,z_,k_]:=1+((z+1)/k-1)Sum[j^-s zeta[n/j,s,z,k+1],{j,2,n}]
Table[Chop[logzeta[a=ri[],b=rr[]]-Residue[zeta[a,b,z,1]/z^2,{z,0}]],{n,1,100}]
```

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\log\zeta(0) = \operatorname*{Res}_{z=0}\frac{\zeta(0)^z}{z^2} \tag{3.4.2}$$

```
{Log[Zeta[0]],Residue[ Zeta[0]^z/z^2,{z,0}]}
```

Of particular note, the Riemann Prime Counting function is

$$\Pi(n)=\operatorname*{Res}_{z=0}\frac{[\zeta(0)^z]_n}{z^2}$$

(3.4.3)

This can be generalized as

$$[\log f]_n=\operatorname*{Res}_{z=0}\frac{[f^z]_n}{z^2}$$

(3.4.4)

3.5 $\Pi(n)$ as Explicit Sum

Remembering our examples of $[(\zeta(s)-1)^k]_n$ from (1.5), (P3) can be written more explicitly in sum notation as

$$[\log\zeta(s)]_n=\sum_{j=2}^{\lfloor n\rfloor} j^{-s}-\frac{1}{2}\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor \frac{n}{j}\rfloor}(j\cdot k)^{-s}+\frac{1}{3}\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor \frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor \frac{n}{j\cdot k}\rfloor}(j\cdot k\cdot l)^{-s}-\frac{1}{4}\ldots$$

(3.5.1)

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\log\zeta(s)= \sum_{j=2}^{\infty} j^{-s}-\frac{1}{2}\sum_{j=2}^{\infty}\sum_{k=2}^{\infty}(j\cdot k)^{-s}+\frac{1}{3}\sum_{j=2}^{\infty}\sum_{k=2}^{\infty}\sum_{l=2}^{\infty}(j\cdot k\cdot l)^{-s}-\frac{1}{4}\ldots= (\zeta(s)-1)-\frac{1}{2}(\zeta(s)-1)^2+\frac{1}{3}(\zeta(s)-1)^3-\frac{1}{4}(\zeta(s)-1)^4+\ldots$$

(3.5.2)

```
{Log[Zeta[s]],Sum[ (-1)^(k+1)/k (Zeta[s]-1)^k,{k,1,Infinity}]}
```

In particular, the Riemann Prime Counting Function is

$$\Pi(n)= \sum_{j=2}^{\lfloor n\rfloor} 1-\frac{1}{2}\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor \frac{n}{j}\rfloor}1+\frac{1}{3}\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor \frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor \frac{n}{j\cdot k}\rfloor}1-\frac{1}{4}\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor \frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor \frac{n}{j\cdot k}\rfloor}\sum_{m=2}^{\lfloor \frac{n}{j\cdot k\cdot l}\rfloor}1+\frac{1}{5}\ldots$$

(3.5.3)

```
RiemannPrimeCount[n_]:=Sum[PrimePi[n^(1/j)]/j,{j,1,Log[2,n]}]
(* logzeta0 is trunctated and stops working after n=2^6-1*)
logzeta0[n_] := Sum[1, { j, 2, n } ] - 1 / 2Sum[1, { j, 2, n }, { k, 2, n / j } ] + 1 / 3Sum[1, { j, 2, n }, { k, 2, n / j }, { l, 2, n / (j k) } ] -
1/4Sum[1,{j,2,n},{k,2,n/j},{l,2,n/(j k)},{m,2,n/(j k l)}]+1/5Sum[1,{j,2,n},{k,2,n/j},{l,2,n/(j k)},{m,2,n/(j k l)},{o,2,n/(j k l m)}]
Table[RiemannPrimeCount[n]-logzeta0[n],{n,1,63}]
```

This can be generalized as

$$[\log f]_n=\sum_{j=2}^{\lfloor n\rfloor} f(j)-\frac{1}{2}\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor \frac{n}{j}\rfloor}f(j)f(k)+\frac{1}{3}\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor \frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor \frac{n}{j\cdot k}\rfloor}f(j)f(k)f(l)-\frac{1}{4}\ldots$$

(3.5.4)

3.6 $\Pi(n)$ as a Recursive Function

The core idea here can be rewritten recursively as

$$F_k(n)=\sum_{j=2}^{\lfloor n\rfloor} j^{-s}\left(\frac{1}{k}-F_{k+1}\left(\frac{n}{j}\right)\right)$$
$$[\log\zeta(s)]_n=F_1(n) \tag{3.6.1}$$

```
rr[]:=RandomReal[{-3,3}]+RandomReal[{-3,3}]I
K[n_]:=FullSimplify[MangoldtLambda[n]/Log[n]]
logzeta[n_,s_]:=Sum[j^-s K[j],{j,2,n}]
F[n_,s_,k_] := Sum[ j^-s(1/k-F[n/j,s,k+1]),{j,2,n}]
Table[Chop[logzeta[n,a=rr[]]-F[n,a,1]],{n,1,100}]
```

Compare this to

$$\log\zeta(s)=F_1 \textit{ where } F_k=(\zeta(s)-1)\left(\frac{1}{k}-F_{k+1}\right) \tag{3.6.2}$$

```
F[k_,s_, t_] := If[t>200,0,(N[Zeta[s]]-1)( 1/k - F[k+1,s,t+1])]
Table[Chop[F[1,s,1]-Log[Zeta[s]]],{s,2,8}]
```

Another way to write this recursively is

$$F_k(n,j)=0 \textit{ if } n<j$$
$$F_k(n,j)=j^{-s}\left(\frac{1}{k}-F_{k+1}\left(\frac{n}{j},2\right)\right)+F_k(n,j+1)$$
$$[\log\zeta(s)]_n=F_1(n,2) \tag{3.6.3}$$

```
rr[]:=RandomReal[{-3,3}]+RandomReal[{-3,3}]I
K[n_]:=FullSimplify[MangoldtLambda[n]/Log[n]]
logzeta[n_,s_]:=Sum[j^-s K[j],{j,2,n}]
F[n_,s_,j_,k_]:=If[n<j,0,j^-s(1/k-F[n/j,s,2,k+1])+F[n,s,j+1,k]]
Table[Chop[logzeta[n,a=rr[]]-F[n,a,2,1]],{n,1,100}]
```

Of particular note, the Riemann Prime Counting function is

$$F_k(n)=\sum_{j=2}^{\lfloor n\rfloor}\frac{1}{k}-F_{k+1}\left(\frac{n}{j}\right)$$
$$\Pi(n)=F_1(n) \tag{3.6.4}$$

and

$$F_k(n,j)=0 \textit{ if } n<j$$
$$F_k(n,j)=\frac{1}{k}-F_{k+1}\left(\frac{n}{j},2\right)+F_k(n,j+1)$$
$$\Pi(n)=F_1(n,2) \tag{3.6.5}$$

It can be generalized to

$$F_k(n)=\sum_{j=2}^{\lfloor n\rfloor} f(j)\left(\frac{1}{k}-F_{k+1}\left(\frac{n}{j}\right)\right)$$
$$[\log f]_n=F_1(n)$$
(3.6.6)

$$F_k(n,j)=0 \; if \; n<j$$
$$F_k(n,j)=f(j)\left(\frac{1}{k}-F_{k+1}\left(\frac{n}{j},2\right)\right)+F_k(n,j+1)$$
$$[\log f]_n=F_1(n,2)$$
(3.6.7)

3.7 Miscellaneous

A slight variant of (P5) is

$$z\cdot\Pi(n)=\sum_{j=2}^{\lfloor n\rfloor} d_z(j)-\frac{1}{2}\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor \frac{n}{j}\rfloor} d_z(j)d_z(k)+\frac{1}{3}\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor \frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor \frac{n}{j\cdot k}\rfloor} d_z(j)\,d_z(k)\,d_z(l)-\frac{1}{4}...$$
(3.7.1)

Compare to $z\cdot\log\zeta(s)=\frac{(\zeta(s)^z-1)}{1}-\frac{(\zeta(s)^z-1)^2}{2}+\frac{(\zeta(s)^z-1)^3}{3}-\frac{(\zeta(s)^z-1)^4}{4}+\frac{(\zeta(s)^z-1)^5}{5}...$
(3.7.2)

```
{Log[Zeta[s]],Sum[ (-1)^(k-1)/k (Zeta[s]^z-1)^{k},{k,1,Infinity}]/z}
```

$$t\cdot\Pi(n)=\lim_{z\to 0}\frac{\partial}{\partial z}([\zeta(0)^z]_n)^t$$

$$\Pi(n)+\Pi(m)=\lim_{z\to 0}\frac{\partial}{\partial z}([\zeta(0)^z]_n\cdot[\zeta(0)^z]_m)$$

$$\Pi(n)-\Pi(m)=\lim_{z\to 0}\frac{\partial}{\partial z}\left(\frac{[\zeta(0)^z]_n}{[\zeta(0)^z]_m}\right)$$

$$t\cdot[\log\zeta(s)]_n=\lim_{z\to 0}\frac{\partial}{\partial z}([\zeta(s)^z]_n)^t$$

$$[\log\zeta(s)]_n+[\log\zeta(s)]_m=\lim_{z\to 0}\frac{\partial}{\partial z}([\zeta(s)^z]_n\cdot[\zeta(s)^z]_m)$$

$$[\log\zeta(s)]_n-[\log\zeta(s)]_m=\lim_{z\to 0}\frac{\partial}{\partial z}\left(\frac{[\zeta(s)^z]_n}{[\zeta(s)^z]_m}\right)$$

3.8 Identities for $[(\log\zeta(s))^k]_n$

Generalizing, $[(\log\zeta(s))^k]_n=\sum_{j=2} j^{-s}\frac{\Lambda(j)}{\log j}\cdot[(\log\zeta(s))^{k-1}]_{n\,j^{-1}}$ from (1.6) also has a few useful identities.

It can be expressed most naturally as the derivative of $[\zeta_n(s)]^{*z}$ as

$$\boxed{[(\log\zeta(s))^k]_n=\lim_{z\to 0}\frac{\partial^k}{\partial z^k}[\zeta(s)^k]_n}$$
(3.8.1)

```
rr[]:=RandomReal[{-3,3}]+RandomReal[{-3,3}]I
logzeta[n_,s_,k_]:=Sum[j^-s FullSimplify[MangoldtLambda[j]/Log[j]] logzeta[n/j,s,k-1],{j,2,n}]
logzeta[n_,s_,0]:=UnitStep[n-1]
zeta[n_,s_,z_,k_]:=1+((z+1)/k-1)Sum[j^-s zeta[n/j,s,z,k+1],{j,2,n}]
Table[Chop[logzeta[n,a=rr[],k]-(Limit[D[zeta[n,a,z,1],{z,k}],z->0])],{n,1,50},{k,1,5}]
```

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$(\log\zeta(s))^k=\lim_{z\to 0}\frac{\partial^k}{\partial z^k}\zeta(s)^z \tag{3.8.2}$$

```
{Log[Zeta[s]]^j,Limit[ D[Zeta[s]^z,z]^j, z->0]}
```

It can be generalized to

$$\boxed{[(\log f)^k]_n=\lim_{z\to 0}\frac{\partial^k}{\partial z^k}[f^z]_n} \tag{3.8.3}$$

It can also be expressed in terms of $[(\zeta(s)-1)^k]_n$ as

$$\boxed{[(\log\zeta(s))^j]_n=\sum_{k=0}^{\infty}\frac{1}{k!}(\lim_{y\to 0}\frac{\partial^k}{\partial y^k}(\log(1+y))^j)\cdot[(\zeta(s)-1)^k]_n} \tag{3.8.4}$$

```
rr[]:=RandomReal[{-3,3}]+RandomReal[{-3,3}]I
zeta[n_,s_,z_,k_]:=1+((z+1)/k-1)Sum[j^-s zeta[n/j,s,z,k+1],{j,2,n}]
logzeta[n_,s_, k_]:=Limit[D[zeta[n,s,z,1],{z,k}],z->0]
zetam1[n_,s_,0]:=UnitStep[n-1]
zetam1[n_,s_,k_]:=Sum[j^-s zetam1[n/j,s,k-1],{j,2,n}]
logzetaalt[n_, s_,j_] := Sum[ 1/k!(Limit[D[Log[1+y]^j,{y,k}],y->0]) zetam1[n,s,k],{k,0,Log[2,n]}]
Table[Chop[logzeta[n,a=rr[],k]- logzetaalt[n,a,k]],{n,1,50},{k,1,5}]
```

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$(\log\zeta(s))^j=\sum_{k=0}^{\infty}\frac{1}{k!}(\lim_{y\to 0}\frac{\partial^k}{\partial y^k}(\log(1+y))^j)\cdot(\zeta(s)-1)^k \tag{3.8.5}$$

```
fn[n_, j_] := N[Sum[ (k!)^-1 coef[j,k](n-1)^k, {k,0,150}]]
coef[j_,k_] := coef[j,k]=Limit[ D[ Log[1+y]^j,{y,k}],y->0]
Grid[ Table[ Chop[ Log[n]^k - fn[n,k]],{n,.12, 1.8, .1},{k,1,5}]]
```

This can be generalized as

$$\boxed{[(\log f)^j]_n=\sum_{k=0}^{\infty}\frac{1}{k!}(\lim_{y\to 0}\frac{\partial^k}{\partial y^k}(\log(1+y))^j)\cdot[(f-1)^k]_n} \tag{3.8.6}$$

And it can be expressed as a residue as

$$\boxed{[(\log \zeta(s))^k]_n = k! \operatorname*{Res}_{z=0} \frac{[\zeta(s)^z]_n}{z^{k+1}}}$$

(3.8.7)

```
rr[]:=RandomReal[{-3,3}]+RandomReal[{-3,3}]I
zeta[n_,s_,z_,k_]:=1+((z+1)/k-1)Sum[zeta[n/j,s,z,k+1],{j,2,n}]
logzeta[n_,s_,k_]:=Limit[D[zeta[n,s,z,1],{z,k}],z->0]
Table[logzeta[n,a=rr[],k]-k! Residue[zeta[n,a,z,1]/z^(k+1),{z,0}],{n,1,50},{k,1,5}]
```

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$(\log \zeta(s))^k = k! \operatorname*{Res}_{z=0} \frac{\zeta(s)^z}{z^{k+1}}$$

(3.8.8)

```
Table[{Log[Zeta[s]]^k,k! Residue[Zeta[s]^z/z^(k+1),{z,0}]},{k,1,10}]//TableForm
```

This can be generalized as

$$\boxed{[(\log f)^k]_n = k! \operatorname*{Res}_{z=0} \frac{[f^z]_n}{z^{k+1}}}$$

(3.8.9)