All Results Generalized

Quick

$$\kappa(n)=\frac{\Lambda(n)}{\log n}$$

$$\boxed{x^{[0,f]}=1}$$

$$\boxed{x^{[k,f]}=\sum_{j=1}^{\lfloor x\rfloor} f(j)\left(\frac{x}{j}\right)^{[k-1,f]}}$$

$$\boxed{x^{[k,-1,f]}=\sum_{j=2}^{\lfloor x\rfloor} f(j)\left(\frac{x}{j}\right)^{[k-1,-1,f]}}$$

$$\boxed{x^{[(\log)^k,f]}=\sum_{j=2}^{\lfloor x\rfloor} \kappa(j)\, f(j)\left(\frac{x}{j}\right)^{[(\log)^{k-1},f]}}$$

$$\boxed{1^{[k,-1,f]}=1^{[(\log)^k,f]}=0}$$

$$\boxed{x^{\Delta[k,f]}=x^{[k,f]}-(x-1)^{[k,f]}}$$

$$\boxed{x^{[k,f]}=\sum_{j=1}^{n} j^{\Delta[k,f]}}$$

$$x^{[1,f]}=\sum_{j=1}^{\lfloor x\rfloor} f(j) \qquad x^{[2,f]}=\sum_{j=1}^{\lfloor x\rfloor}\sum_{k=1}^{\lfloor \frac{x}{j}\rfloor} f(j)\cdot f(k) \qquad x^{[3,f]}=\sum_{j=1}^{\lfloor x\rfloor}\sum_{k=1}^{\lfloor \frac{x}{j}\rfloor}\sum_{m=1}^{\lfloor \frac{x}{jk}\rfloor} f(j)\cdot f(k)\cdot f(m)$$

$$x^{[1,-1,f]}=\sum_{j=2}^{\lfloor x\rfloor} f(j) \qquad x^{[2,-1,f]}=\sum_{j=2}^{\lfloor x\rfloor}\sum_{k=2}^{\lfloor \frac{x}{j}\rfloor} f(j)\cdot f(k) \qquad x^{[3,-1,f]}=\sum_{j=2}^{\lfloor x\rfloor}\sum_{k=2}^{\lfloor \frac{x}{j}\rfloor}\sum_{m=2}^{\lfloor \frac{x}{jk}\rfloor} f(j)\cdot f(k)\cdot f(m)$$

$$x^{[\log,f]}=\sum_{j=2}^{\lfloor x\rfloor} \kappa(j)\cdot f(j) \qquad x^{[(\log)^2,f]}=\sum_{j=2}^{\lfloor x\rfloor}\sum_{k=2}^{\lfloor \frac{x}{j}\rfloor} \kappa(j)\cdot f(j)\cdot\kappa(k)\cdot f(k)$$

$$\binom{z}{k}=\frac{z(z-1)\dots(z-k+1)}{k!}$$

$$x^{\Delta[z,f]}=f(x)\prod_{p^\alpha | x}(-1)^\alpha\binom{-z}{\alpha}$$

$$x^{[z,f]} = \sum_{j=1}^{\lfloor x \rfloor} j^{\Delta[z,f]}$$

$$\boxed{x^{[z,f]} = \sum_{k=0}^{\lfloor \log_2 x \rfloor} \binom{z}{k} x^{[z,-1,f]}}$$

Compare this to $x^z = \sum_{k=0}^{\infty} \binom{z}{k} (x-1)^k$

$$\boxed{x^{[z,f]} = \sum_{k=0}^{\lfloor \log_2 x \rfloor} \frac{z^k}{k!} x^{[(\log)^k, f]}}$$

Compare this to $x^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} (\log x)^k$

$$\boxed{\frac{\partial}{\partial z} x^{[z,f]} = \sum_{k=0}^{\lfloor \log_2 x \rfloor - 1} \frac{z^k}{k!} x^{[(\log)^{k+1}, f]}}$$

Compare this to $\frac{\partial}{\partial z} x^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} (\log x)^{k+1}$

$$\boxed{\frac{\partial^\alpha}{\partial z^\alpha} x^{[z,f]} = \sum_{k=0}^{\lfloor \log_2 x \rfloor - \alpha} \frac{z^k}{k!} x^{[(\log)^{k+\alpha}, f]}}$$

Compare this to $\frac{\partial^\alpha}{\partial z^\alpha} x^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} (\log x)^{k+\alpha}$

$$\boxed{x^{[z,f]} = \sum_{k=0}^{\lfloor \log_2 x \rfloor} \frac{z^k}{k!} \left(\frac{\partial^k}{\partial y^k} x^{[y,f]} \text{at } y=0 \right)}$$

Compare this to $x^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \left(\frac{\partial^k}{\partial y^k} x^y \text{ at } y=0 \right)$

$$\boxed{x^{[z,f]} = \sum_{k=0}^{\lfloor \log_2 x \rfloor} z^k \operatorname*{Res}_{m=0} \frac{x^{[m,f]}}{m^{k+1}}}$$

Compare this to $x^z = \sum_{k=0}^{\infty} z^k \operatorname*{Res}_{m=0} \frac{x^m}{m^{k+1}}$

$$\boxed{x^{[\log, f]} = \lim_{z \to 0} \frac{x^{[z,f]} - 1}{z}}$$

$\log x = \lim_{z \to 0} \frac{x^z - 1}{z}$

$$\boxed{x^{[\log, f]} = \sum_{k=1}^{\lfloor \log_2 x \rfloor} \frac{(-1)^{k+1}}{k} x^{[k,-1,f]}}$$

$\log x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (x-1)^k$

$$\boxed{x^{[\log, f]} = \frac{\partial}{\partial z} x^{[z, f]} \, at \, z=0}$$

$$\log x = \frac{\partial}{\partial z} x^z \, at \, z=0$$

$$\boxed{x^{[\log, f]} = \operatorname*{Res}_{z=0} \frac{x^{[z,f]}}{z^2}}$$

Compare to $\log x = \operatorname*{Res}_{z=0} \frac{x^z}{z^2}$

$$\boxed{x^{[\log, f]} = \sum_{j=2}^{\lfloor x \rfloor} f(j) - \frac{1}{2} \sum_{j=2}^{\lfloor x \rfloor} \sum_{k=2}^{\lfloor \frac{x}{j} \rfloor} f(j) \cdot f(k) + \frac{1}{3} \sum_{j=2}^{\lfloor x \rfloor} \sum_{k=2}^{\lfloor \frac{x}{j} \rfloor} \sum_{l=2}^{\lfloor \frac{x}{j \cdot k} \rfloor} f(j) \cdot f(k) \cdot f(l) - \frac{1}{4} \ldots}$$

$$F_k(x, f) = \sum_{j=2}^{\lfloor x \rfloor} f(j) \left(\frac{1}{k} - F_{k+1}\left(\lfloor \frac{x}{j} \rfloor, f\right)\right)$$
$$x^{[\log, f]} = F_1(x, f)$$

$$x^{[\log, f]} = z^{-1} \left(\sum_{j=2}^{\lfloor x \rfloor} j^{[z,f]} - \frac{1}{2} \sum_{j=2}^{\lfloor x \rfloor} \sum_{k=2}^{\lfloor \frac{x}{j} \rfloor} j^{[z,f]} k^{[z,f]} + \frac{1}{3} \sum_{j=2}^{\lfloor x \rfloor} \sum_{k=2}^{\lfloor \frac{x}{j} \rfloor} \sum_{l=2}^{\lfloor \frac{x}{j \cdot k} \rfloor} j^{[z,f]} k^{[z,f]} l^{[z,f]} - \frac{1}{4} \ldots \right)$$

Compare to $\log x = z^{-1} \left(\frac{(x^z-1)}{1} - \frac{(x^z-1)^2}{2} + \frac{(x^z-1)^3}{3} - \frac{(x^z-1)^4}{4} + \frac{(x^z-1)^5}{5} \ldots \right)$

$$\boxed{x^{[(\log)^j, f]} = \sum_{k=0}^{\lfloor \log_2 x \rfloor} \frac{1}{k!} \left(\frac{\partial^k}{\partial y^k} (\log(1+y))^j \text{ at } y=0 \right) \cdot x^{[k,-1,f]}}$$

Compare to $(\log x)^j = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\partial^k}{\partial y^k} (\log(1+y))^j \text{ at } y=0 \right) \cdot (x-1)^k$

$$\boxed{x^{[(\log)^k, f]} = \frac{\partial^k}{\partial z^k} x^{[z,f]} \, at \, z=0}$$

Compare to $(\log x)^j = \frac{\partial^k}{\partial z^k} x^z \, at \, z=0$

$$\boxed{x^{[(\log)^k, f]} = k! \operatorname*{Res}_{z=0} \frac{x^{[z,f]}}{z^{k+1}}}$$

Compare to $(\log x)^k = k! \operatorname*{Res}_{z=0} \frac{x^z}{z^{k+1}}$

$$x^{[(\log)^k, f]} = \sum_{j=2}^{\lfloor x \rfloor} \kappa(j) f(j) \left(\frac{x}{j} \right)^{[(\log)^{k-1}, f]} \text{ and } x^{[z, f]} = \sum_{k=0}^{\lfloor \log_2 x \rfloor} \frac{z^k}{k!} x^{[(\log)^k, f]}$$

There will be $\log_2 x$ solutions for z where $x^{[z, f]} = 0$. Call those solutions ρ.

$$\boxed{x^{[z,f]}=\prod_{\rho}\left(1-\frac{z}{\rho}\right)}$$

$$\boxed{x^{[\log,f]}=-\sum_{\rho}\frac{1}{\rho}}$$

$$x^{[k,-a,f]}=\sum_{j=0}^{k}\binom{k}{j}f(a)^{j}\left(\frac{x}{a^{j}}\right)^{[k-j,-a-1,f]}$$

$$x^{[k,-a,f]}=\sum_{j=0}^{k}(-1)^{j}\binom{k}{j}f(a-1)^{j}\left(\frac{x}{(a-1)^{j}}\right)^{[k-j,-a+1,f]}$$

$$x^{[k,-a,f]}=0 \text{ when } x<a^{k}$$

$$\boxed{x^{[k,a,f]}=\sum_{j=1}^{k}\binom{k}{j}\sum_{m=a}^{\lfloor x^{\frac{1}{k}}\rfloor}f(m)^{j}\left(\frac{x}{m^{j}}\right)^{[k-j,-a-1,f]}}$$