```
tk[n_, k_, a_] :=
 tk[n, k, a] = Sum[ tk[n / j, k - 1, a], {j, 2, n}] - a Sum[ tk[n / (a j), k - 1, a], {j, 1, n / a}];
tk[n_, 0, a_] := 1
Lina[n_, a_] := Sum[ ((-1) ^ (k + 1) tk[n, k, a] + 1) / k + (a^k - 1) / k, {k, 1, Log[a, n]}]
Linb[n_, a_] :=
 Sum[ (-1) ^ (k + 1) / k tk[n, k, a], {k, 1, Log[2, n]}] + Sum[ a^k / k, {k, 1, Log[a, n]}]
Lin2[n_, a_] := If[a >= 2, Linb[n, a], Lina[n, a]]

FullSimplify[Lin2[100, 2^(1 / 2)]]
```

$$\frac{428}{15}$$

```
FullSimplify[Lin2[100, 2^(1 / 3)]]
```

$$\frac{428}{15}$$

```
FullSimplify[Lin2[100, 2^(1 / 4)]]
```

$$\frac{428}{15}$$

```
Lin2[100, 3.27]

28.5333

f[n_, a_] := Sum[ (a^k - 1) / k, {k, 1, Log[a, n]}]

f[100, 1.00001] + f[1.45, 1.00001]

28.4309

N[LogIntegral[100]]

30.1261

g[n_, a_] := Sum[ (E^(k Log[a]) - 1) / k, {k, 1, Log[n] / Log[a]}]

g[100, 1.00001]

28.0218

Limit[g[100, s], {s → 1}]
```

$$\left\{\text{Limit}\left[-\text{HarmonicNumber}\left[\frac{\text{Log}[100]}{\text{Log}[s]}\right] - 100\, s\, \text{LerchPhi}\left[s, 1, 1 + \frac{\text{Log}[100]}{\text{Log}[s]}\right] - \text{Log}[1 - s], s \to 1\right]\right\}$$

$$\textbf{Limit}\left[-\textbf{HarmonicNumber}\left[\frac{\textbf{Log}[100]}{\textbf{Log}[s]}\right] - \textbf{Log}[1 - s], s \to 1\right]$$

$$-\text{EulerGamma} - i\pi - \text{Log}[\text{Log}[100]]$$

$$\textbf{Limit}\left[-100\, s\, \textbf{LerchPhi}\left[s, 1, 1 + \frac{\textbf{Log}[100]}{\textbf{Log}[s]}\right], s \to 1\right]$$

$$\text{Limit}\left[-100\, s\, \text{LerchPhi}\left[s, 1, 1 + \frac{\text{Log}[100]}{\text{Log}[s]}\right], s \to 1\right]$$

$$\text{pp}[\text{s\_}] := \left(-100\,\text{s LerchPhi}\left[\text{s}, 1, 1+\frac{\text{Log}[100]}{\text{Log}[\text{s}]}\right]\right) - \text{EulerGamma} - i\,\pi - \text{Log}[\text{Log}[100]]$$

$$\text{pp2}[\text{s\_}] := \left(-100\,\text{s LerchPhi}\left[\text{s}, 1, 1+\frac{\text{Log}[100]}{\text{Log}[\text{s}]}\right]\right)$$

$$\text{pp3}[\text{s\_}] := \left(-100\ \text{LerchPhi}\left[\text{s}, 1, 1+\frac{\text{Log}[100]}{\text{Log}[\text{s}]}\right]\right)$$

```
N[pp[1.00001]]
```

$28.0219 + 3.90545 \times 10^{-11}\, i$

```
N[pp2[1.00001]]
```

$30.1263 + 3.14159\, i$

```
N[LogIntegral[100]]
```

30.1261

```
N[pp3[1.00001]]
```

$30.1259 + 3.14156\, i$

```
f3[n_, s_] := -n Sum[ (s^k) / (Log[n] / Log[s] + 1), {k, 0, Infinity}]
```

$$\text{Limit}\left[-\frac{\text{n}}{(1-\text{s})\left(1+\frac{\text{Log}[\text{n}]}{\text{Log}[\text{s}]}\right)}, \{\text{s} \to 1\}\right]$$

$$\left\{\frac{\text{n}}{\text{Log}[\text{n}]}\right\}$$

```
f3[100, 1.00000001]
```

Sum::div : Sum does not converge. ≫

```
lerch[n_, s_] := n Integrate[ E^(- (Log[n] / Log[s]) t) / (1 - (s E^(-t))), {t, 0, Infinity}]
lerch[100, .9]
```

Integrate::idiv : Integral of $\frac{e^{44.7087\,t}}{-0.9 + e^{t}}$ does not converge on $\{0, \infty\}$. ≫

$$100 \int_0^\infty \frac{e^{43.7087\,t}}{1 - 0.9\,e^{-t}}\, dt$$

```
lch[z_, s_, a_] :=
 1 / (2 a^s) + Log[1 / z] ^ (s - 1) / z^a Gamma[1 - s, a Log[1 / z]] + 2 / (a^ (s - 1)) Integrate[
    Sin[s ArcTan[t] - t a Log[z]] / ( (1 + t^2 ) ^ (s / 2) (E^ (2 Pi a t) - 1)), {t, 0, Infinity}]
lch[1.00001, 1, Log[100] / Log[1.00001]]
```

$$(-0.30126 - 0.0314159\, i) + 2.\int_0^\infty -\frac{\text{Sin}[4.60517\,t - \text{ArcTan}[t]]}{\left(-1 + e^{2.89353\times 10^{6}\,t}\right)\sqrt{1+t^2}}\, dt$$

$$\mathbf{N}\Big[(-0.30126033010911385\grave{}-0.03141592653589791\grave{}\, i)+2.\grave{}\int_0^\infty -\frac{\mathrm{Sin}[4.605170185988092\grave{}\, t-\mathrm{ArcTan}[t]]}{\left(-1+e^{2.8935282324919426\grave{}*\wedge 6\, t}\right)\sqrt{1+t^2}}\,dt\Big]$$

NIntegrate::ncvb :

NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in t near {t} = $\{8.99367\times10^{-6}\}$.

NIntegrate obtained -7.08303×10^{-13} and 4.4015461258204655`*^-18 for the integral and error estimates. >>

```
-100 (-0.30126033011053044` - 0.03141592653589791` i)
```

30.126 + 3.14159 i

```
lch[r, 1, p]
```

$$\frac{1}{2\,p}+r^{-p}\,\mathrm{Gamma}\left[0,\,p\,\mathrm{Log}\left[\frac{1}{r}\right]\right]+2\int_0^\infty\frac{\mathrm{Sin}[\mathrm{ArcTan}[t]-p\,t\,\mathrm{Log}[r]]}{\left(-1+e^{2\,p\,\pi\,t}\right)\sqrt{1+t^2}}\,dt$$

$$\frac{1}{2\,p}+r^{-p}\,\mathrm{Gamma}\left[0,\,p\,\mathrm{Log}\left[\frac{1}{r}\right]\right]+2\int_0^\infty\frac{\mathrm{Sin}[\mathrm{ArcTan}[t]-p\,t\,\mathrm{Log}[r]]}{\left(-1+e^{2\,p\,\pi\,t}\right)\sqrt{1+t^2}}\,dt\ /.\ \{r\to s,\ p\to(\mathrm{Log}[n]/\mathrm{Log}[s])\}$$

$$\frac{\mathrm{Gamma}\left[0,\,\frac{\mathrm{Log}[n]\,\mathrm{Log}\left[\frac{1}{s}\right]}{\mathrm{Log}[s]}\right]}{n}+2\int_0^\infty\frac{\mathrm{Sin}[\mathrm{ArcTan}[t]-t\,\mathrm{Log}[n]]}{\left(-1+n^{\frac{2\pi t}{\mathrm{Log}[s]}}\right)\sqrt{1+t^2}}\,dt+\frac{\mathrm{Log}[s]}{2\,\mathrm{Log}[n]}$$

```
lchx[z_, s_, a_] :=
 1 / (2 a^s) + Log[1 / z] ^ (s - 1) / z^a Gamma[1 - s, a Log[1 / z]] + 2 / (a^ (s - 1)) Integrate[
   Sin[s ArcTan[t] - t a Log[z]] / ( (1 + t^2 ) ^ (s / 2) (E^ (2 Pi a t) - 1)), {t, 0, Infinity}]
```

```
N[lchx[1.00001, 1, Log[100] / Log[1.00001]]]
```

NIntegrate::ncvb :

NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in t near {t} = $\{8.99367\times10^{-6}\}$.

NIntegrate obtained -7.08303×10^{-13} and 4.4015461258204655`*^-18 for the integral and error estimates. >>

-0.30126 - 0.0314159 i

```
lchxa[z_, s_, a_] := 1 / (2 a^s)
lchxb[z_, s_, a_] := Log[1 / z] ^ (s - 1) / z^a Gamma[1 - s, a Log[1 / z]]
lchxc[z_, s_, a_] := 2 / (a^ (s - 1)) Integrate[
  Sin[s ArcTan[t] - t a Log[z]] / ( (1 + t^2 ) ^ (s / 2) (E^ (2 Pi a t) - 1)), {t, 0, Infinity}]
```

```
N[lchxa[1.00001, 1, Log[100] / Log[1.00001]] +
  lchxb[1.00001, 1, Log[100] / Log[1.00001]] + lchxc[1.00001, 1, Log[100] / Log[1.00001]]]
```

NIntegrate::ncvb :

NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in t near {t} = $\{8.99367\times10^{-6}\}$.

NIntegrate obtained -7.08303×10^{-13} and 4.4015461258204655`*^-18 for the integral and error estimates. >>

-0.30126 - 0.0314159 i

```
lchxa[z_, s_, a_] := 1 / (2 a^1)
lchxb[z_, s_, a_] := Log[1 / z] ^ (1 - 1) / z^a Gamma[1 - 1, a Log[1 / z]]
lchxc[z_, s_, a_] := 2 / (a^ (1 - 1)) Integrate[
  Sin[1 ArcTan[t] - t a Log[z]] / ( (1 + t^2 ) ^ (1 / 2) (E^ (2 Pi a t) - 1)), {t, 0, Infinity}]
```

```
N[lchxa[1.00001, 1, Log[100] / Log[1.00001]] +
  lchxb[1.00001, 1, Log[100] / Log[1.00001]] + lchxc[1.00001, 1, Log[100] / Log[1.00001]]]
```

NIntegrate::ncvb :
NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in t near {t} = $\{8.99367\times10^{-6}\}$.
NIntegrate obtained -7.08303×10^{-13} and 4.4015461258204655`*^-18 for the integral and error estimates. ≫

-0.30126 - 0.0314159 ⅈ

```
lchxa[z_, s_, a_] := 1 / (2 a)
lchxb[z_, s_, a_] := Log[1 / z] ^ (1 - 1) / z ^ a Gamma[1 - 1, a Log[1 / z]]
lchxc[z_, s_, a_] := 2 / (a ^ (1 - 1)) Integrate[
   Sin[1 ArcTan[t] - t a Log[z]] / ((1 + t ^ 2) ^ (1 / 2) (E ^ (2 Pi a t) - 1)), {t, 0, Infinity}]
lchxa[1.00001, 1, Log[100] / Log[1.00001]]
```

1.08573×10^{-6}

```
lchxb[1.00001, 1, Log[100] / Log[1.00001]]
```

-0.301261 - 0.0314159 ⅈ

```
N[lchxc[1.00001, 1, Log[100] / Log[1.00001]]]
```

NIntegrate::ncvb :
NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in t near {t} = $\{8.99367\times10^{-6}\}$.
NIntegrate obtained -7.08303×10^{-13} and 4.4015461258204655`*^-18 for the integral and error estimates. ≫

-1.41661×10^{-12}

```
lchxa[z_, s_, a_] := 0
lchxb[z_, s_, a_] := 1 / z ^ a Gamma[0, a Log[1 / z]]
lchxc[z_, s_, a_] := 2 Integrate[
   Sin[ArcTan[t] - t a Log[z]] / ((1 + t ^ 2) ^ (1 / 2) (E ^ (2 Pi a t) - 1)), {t, 0, Infinity}]
lchxb[1.00001, 1, Log[100] / Log[1.00001]]
```

-0.301261 - 0.0314159 ⅈ

```
N[lchxc[1.00001, 1, Log[100] / Log[1.00001]]]
```

NIntegrate::ncvb :
NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in t near {t} = $\{8.99367\times10^{-6}\}$.
NIntegrate obtained -7.08303×10^{-13} and 4.4015461258204655`*^-18 for the integral and error estimates. ≫

-1.41661×10^{-12}

```
FullSimplify[Sin[ArcTan[t] - t a Log[z]] / ((1 + t ^ 2) ^ (1 / 2) (E ^ (2 Pi a t) - 1))]
```

$$\frac{(-1+\mathrm{Coth}[a\,\pi\,t])\,\mathrm{Sin}[\mathrm{ArcTan}[t]-a\,t\,\mathrm{Log}[z]]}{2\sqrt{1+t^2}}$$

```
2
 Integrate[Sin[ArcTan[t] - t a Log[z]] / ((1 + t ^ 2) ^ (1 / 2) (E ^ (2 Pi a t) - 1)), {t, 0, Infinity}]
```

$$2\int_0^\infty \frac{\mathrm{Sin}[\mathrm{ArcTan}[t]-a\,t\,\mathrm{Log}[z]]}{\left(-1+e^{2\,a\,\pi\,t}\right)\sqrt{1+t^2}}\,dt$$

$$2\,\text{Integrate}\left[\frac{(-1+\text{Coth}[a\,\pi\,t])\,\text{Sin}[\text{ArcTan}[t]-a\,t\,\text{Log}[z]]}{2\sqrt{1+t^2}},\{t,0,\text{Infinity}\}\right]$$

```
$Aborted
```

$$\text{ff}[z\_,a\_,t\_]:=\frac{(-1+\text{Coth}[a\,\pi\,t])\,\text{Sin}[\text{ArcTan}[t]-a\,t\,\text{Log}[z]]}{2\sqrt{1+t^2}}$$

```
N[ff[s, Log[n] / Log[s], t] /. {s → 1.0001, n → 100, t → 80}]

0.

fff[z_, a_, t_] := Sin[ArcTan[t] - t a Log[z]] / ((1 + t^2)^(1/2) (E^(2 Pi a t) - 1))

N[fff[s, Log[n] / Log[s], t]] /. {s → 3, n → 10, t → 10}
```

-2.70307×10^{-59}

```
ffff[z_, a_, t_] := Sin[ArcTan[t] - t (Log[n] / Log[s]) Log[s]] /
  ((1 + t^2)^(1/2) (E^(2 Pi (Log[n] / Log[s]) t) - 1))

N[ffff[s, Log[n] / Log[s], t]] /. {s → 3, n → 10, t → 10}
```

-2.70307×10^{-59}

```
FullSimplify[Sin[ArcTan[t] - t (Log[n] / Log[s]) Log[s]] /
  ((1 + t^2)^(1/2) (E^(2 Pi (Log[n] / Log[s]) t) - 1))]
```

$$\frac{\text{Sin}[\text{ArcTan}[t]-t\,\text{Log}[n]]}{\left(-1+n^{\frac{2\pi t}{\text{Log}[s]}}\right)\sqrt{1+t^2}}$$

```
fffff[z_, a_, t_] :=
 Sin[ArcTan[t] - t (Log[n])] / ((1 + t^2)^(1/2) (E^(2 Pi (Log[n] / Log[s]) t) - 1))

N[fffff[s, Log[n] / Log[s], t]] /. {s → 3, n → 10, t → 10}
```

-2.70307×10^{-59}

```
fp[z_, a_, t_] :=
 Sin[ArcTan[t] - t (Log[n])] / ((1 + t^2)^(1/2) (E^(2 Pi (Log[n] / Log[s]) t) - 1))


lchxb[z_, s_, a_] := 1 / z^a Gamma[0, a Log[1 / z]]

lchxb[1.00001, 1, Log[100] / Log[1.00001]]

-0.301261 - 0.0314159 i

lchxb2[z_, a_] := 1 / z^a Gamma[0, a Log[1 / z]]

lchxb2[1.00001, Log[100] / Log[1.00001]]

-0.301261 - 0.0314159 i

lchxb3[n_, s_, a_] := 1 / s^a Gamma[0, a Log[1 / s]]

lchxb3[100, 1.00001, Log[100] / Log[1.00001]]

-0.301261 - 0.0314159 i
```

```
lchxb4[n_, s_] := 1 / s^(Log[n] / Log[s]) Gamma[0, (Log[n] / Log[s]) Log[1 / s]]

lchxb4[100, 1.00001]

-0.301261 - 0.0314159 i

lchxb5[n_, s_] := 1 / s^(Log[n] / Log[s]) Gamma[0, (Log[n] / Log[s]) Log[1 / s]]

lchxb5[100, 1.00001]

-0.301261 - 0.0314159 i

FullSimplify[1 / s^(Log[n] / Log[s]) Gamma[0, (Log[n] / Log[s]) Log[1 / s]]]
```

$$\text{chx1}[n\_, s\_] := \frac{\text{Gamma}\left[0, \frac{\text{Log}[n]\,\text{Log}\left[\frac{1}{s}\right]}{\text{Log}[s]}\right]}{n}$$

```
chx1[100, 1.00001]

-0.301261 - 0.0314159 i
```

$$\text{chx2}[n\_, s\_] := \frac{\text{Gamma}\left[0, \frac{\text{Log}[n]\,(-\text{Log}[s])}{\text{Log}[s]}\right]}{n}$$

```
chx2[100, 1.00001]

-0.301261 - 0.0314159 i
```

$$\text{chx3}[n\_, s\_] := (1 / n)\,\text{Gamma}\left[0, \frac{\text{Log}[n]\,(-\text{Log}[s])}{\text{Log}[s]}\right]$$

```
chx3[100, 1.00001]

-0.301261 - 0.0314159 i
```

$$\text{chx4}[n\_, s\_] := (1 / n)\,\text{Gamma}\left[0, -\frac{\text{Log}[n]\,\text{Log}[s]}{\text{Log}[s]}\right]$$

```
chx4[100, 1.00001]

-0.301261 - 0.0314159 i

chx5[n_, s_] := (1 / n) Gamma[0, -Log[n]]
chx5[100, 1.00001]
```

$$\frac{1}{100}\,\text{Gamma}[0, -\text{Log}[100]]$$