

```

Clear[Dd, Ds]
bin[z_, k_] := bin[z, k] = Product[z - j, {j, 0, k - 1}] / k!

Dd[n_, 0, a_] := UnitStep[n - 1]
Dd[n_, 1, a_] := Dd[n, 1, a] = Floor[n] - a
Dd[n_, 2, a_] := Dd[n, 2, a] = Sum[1 + 2 (Dd[Floor[n / m], 1, m]), {m, a + 1, Floor[n^(1 / 2)]]]
Dd[n_, k_, a_] := Dd[n, k, a] = Sum[1 + k Dd[Floor[n / (m^(k - 1))], 1, m] +
    Sum[bin[k, j] Dd[Floor[n / (m^j)], k - j, m], {j, 1, k - 2}], {m, a + 1, Floor[n^(1 / k)]]]
Ddz[n_, z_] := Expand[Sum[bin[z, k] Dd[n, k, 1], {k, 0, Log[2, n]}]]
Ddzeros[n_] := List@@NRoots[Ddz[n, z] == 0, z][[All, 2]]
DdzAlt[n_, z_] := Expand[Product[1 - z / rho, {rho, Ddzeros[n]}]]

Ds[n_, 0, s_, a_] := UnitStep[n - 1]
Ds[n_, 1, s_, a_] := Ds[n, 1, s, a] = HarmonicNumber[Floor[n], s] - HarmonicNumber[a, s]
Ds[n_, 2, s_, a_] := Ds[n, 2, s, a] =
    Sum[(m^(-2 s)) + 2 (m^(-s)) (Ds[Floor[n / m], 1, s, m]), {m, a + 1, Floor[n^(1 / 2)]]]
Ds[n_, k_, s_, a_] := Ds[n, k, s, a] =
    Sum[(m^(-s k)) + k (m^(-s (k - 1))) Ds[Floor[n / (m^(k - 1))], 1, s, m] +
        Sum[bin[k, j] (m^(-s))^j Ds[Floor[n / (m^j)], k - j, s, m], {j, 1, k - 2}],
        {m, a + 1, Floor[n^(1 / k)]]]
Dsz[n_, s_, z_] := Expand[Sum[bin[z, k] Ds[n, k, s, 1], {k, 0, Log[2, n]}]]
Dszeros[n_, s_] := List@@NRoots[Dsz[n, s, z] == 0, z][[All, 2]]
DszAlt[n_, s_, z_] := Expand[Product[1 - z / rho, {rho, Dszeros[n, s]}]]

Ddy[n_, s_, y_, k_] := y^(k (s - 1)) Ds[n y^k, k, s, y]
Dyz[n_, s_, y_, z_] := Expand[Sum[bin[z, k] Ddy[n, s, y, k], {k, 0, Log[(y + 1) / y, n]}]]
Dyzeros[n_, s_, y_] := List@@NRoots[Dyz[n, s, y, z] == 0, z][[All, 2]]
DyzAlt[n_, s_, y_, z_] := Expand[Product[1 - z / rho, {rho, Dyzeros[n, s, y]}]]

Eabz[n_, s_, a_, b_, z_] := Expand[
    Sum[(-1)^j bin[z, j] (a / b)^(j (1 - s)) Dsz[n / ((a / b)^j), s, z], {j, 0, Log[a / b, n]}]]
Eabzeros[n_, s_, a_, b_] := List@@Roots[Eabz[n, s, a, b, z] == 0, z][[All, 2]]

Ez[n_, s_, z_] :=
    Expand[Sum[(-1)^j bin[z, j] 2^(j (1 - s)) Dsz[n / (2^j), s, z], {j, 0, Log[2, n]}]]
Ezeros[n_, s_, y_] := List@@NRoots[Ez[n, s, y, z] == 0, z][[All, 2]]

N[Ddy[100, N[ZetaZero[1]], 10, 2]]
2.34019 - 2.15473 i

tt[n_, s_] := 1 / (s - 1)^2 Gamma[2, 0, (s - 1) Log[n]] / Gamma[2]
N[tt[100, N[ZetaZero[1]]]]
2.39654 - 2.19631 i

Eabz[100, 0, 1, 101, 1, z] // TableForm
0      1 +  $\frac{428 z}{15}$  +  $\frac{16289 z^2}{360}$  +  $\frac{331 z^3}{16}$  +  $\frac{611 z^4}{144}$  +  $\frac{67 z^5}{240}$  +  $\frac{7 z^6}{720}$ 

```

```
Floor[Log[4 / 3, 100.]]
```

```
16
```

```
Floor[Log[3 / 2, 100.]]
```

```
11
```

```
Floor[Log[4 / 3, 100.]]
```

```
16
```

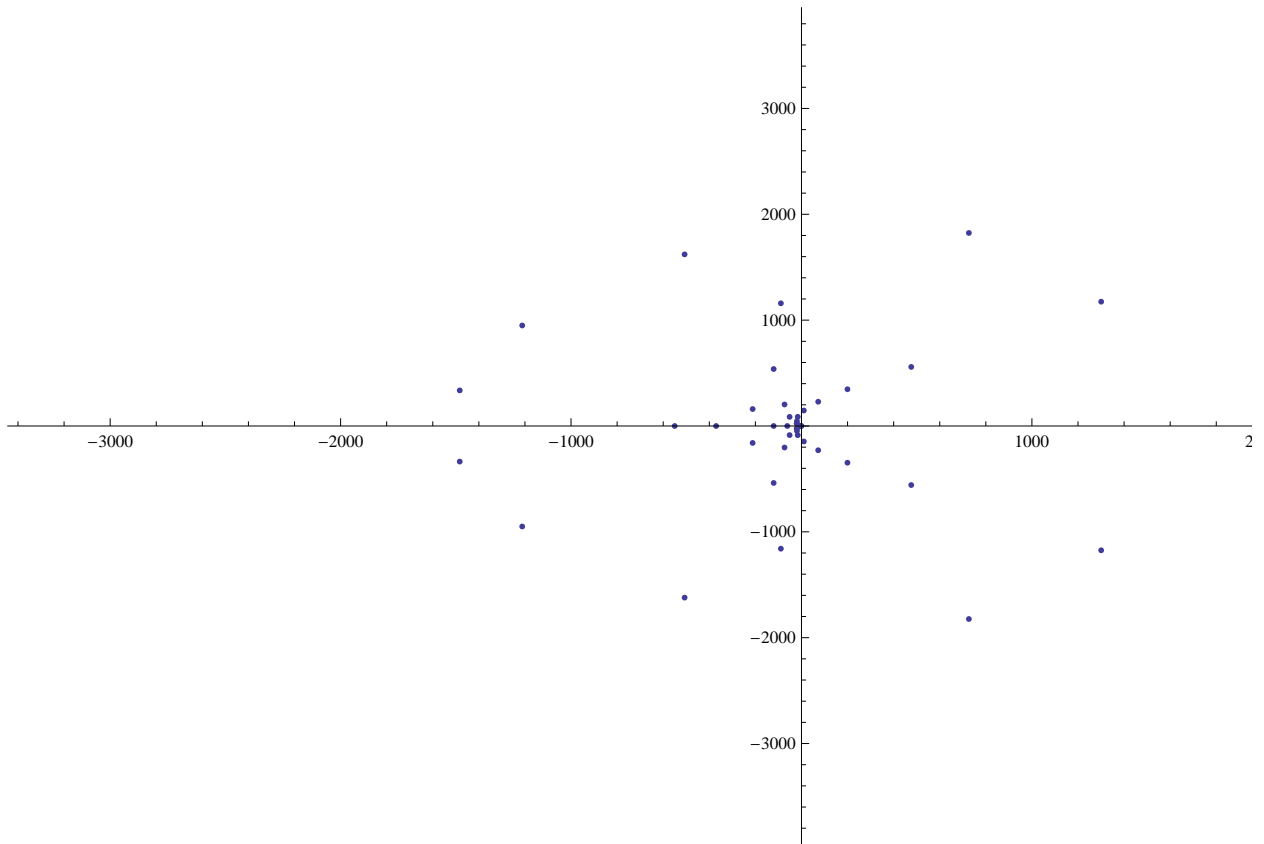
```
Floor[Log[5 / 4, 100.]]
```

```
20
```

```
Dyzeros[20, 0, 18]
```

```
{-16617.9 - 5237.57 i, -16617.9 + 5237.57 i, -9290.99 - 11911. i, -9290.99 + 11911. i,
-3934.78 - 3646.03 i, -3934.78 + 3646.03 i, -1482.99 - 336.133 i, -1482.99 + 336.133 i,
-1211.85 - 949.506 i, -1211.85 + 949.506 i, -550.272, -527.403 - 11813.1 i,
-527.403 + 11813.1 i, -507.145 - 1621.42 i, -507.145 + 1621.42 i, -370.357,
-211.949 - 160.191 i, -211.949 + 160.191 i, -120.769, -120.697 - 538.177 i,
-120.697 + 538.177 i, -89.3935 - 1159.2 i, -89.3935 + 1159.2 i, -73.4803 - 203.301 i,
-73.4803 + 203.301 i, -61.6191, -51.4836 - 86.044 i, -51.4836 + 86.044 i, -20.9516 - 23.3877 i,
-20.9516 + 23.3877 i, -19.6897 - 46.7512 i, -19.6897 + 46.7512 i, -16.2911 - 86.5774 i,
-16.2911 + 86.5774 i, -16.035 - 12.1799 i, -16.035 + 12.1799 i, -15.5, -7.09099 - 1.61247 i,
-7.09099 + 1.61247 i, -2.16918, -0.136789, 10.3212 - 146.17 i, 10.3212 + 146.17 i,
72.5801 - 229.225 i, 72.5801 + 229.225 i, 199.522 - 347.037 i, 199.522 + 347.037 i,
476.359 - 557.924 i, 476.359 + 557.924 i, 726.459 - 1823.6 i, 726.459 + 1823.6 i,
1300.57 - 1174.34 i, 1300.57 + 1174.34 i, 5492.2 - 5648.27 i, 5492.2 + 5648.27 i}
```

```
ListPlot[Table[{Re[n], Im[n]}, {n, Dyzeros[20, 0, 18]}]]
```



```
AbsoluteTiming[Dz[10 000 000, z]]
```

$$\left\{ 3.4481973, 1 + \frac{3\,559\,637\,526\,370\,229\,z}{5\,354\,228\,880} + \frac{1\,989\,544\,871\,269\,240\,547\,z^2}{921\,858\,537\,600} + \frac{2\,021\,824\,016\,451\,264\,335\,171\,z^3}{677\,566\,025\,136\,000} + \frac{7\,019\,677\,821\,920\,298\,561\,119\,z^4}{2\,956\,651\,746\,048\,000} + \frac{6\,419\,737\,164\,240\,558\,941\,381\,z^5}{5\,217\,620\,728\,320\,000} + \frac{355\,971\,199\,127\,948\,600\,783\,z^6}{800\,296\,713\,216\,000} + \frac{87\,671\,088\,330\,394\,680\,791\,z^7}{750\,278\,168\,640\,000} + \frac{647\,852\,562\,694\,427\,393\,z^8}{28\,245\,766\,348\,800} + \frac{1\,156\,246\,192\,125\,011\,873\,z^9}{6\,208\,422\,327\,896\,021\,939\,z^{10}} + \frac{250\,225\,399\,924\,000\,051\,z^{11}}{718\,931\,434\,240\,000} + \frac{6\,106\,970\,322\,634\,813\,z^{12}}{63\,437\,608\,022\,863\,169\,z^{13}} + \frac{47\,189\,432\,328\,823\,z^{14}}{9\,038\,645\,231\,616\,000} + \frac{2\,549\,361\,475\,584\,000\,z^{15}}{120\,162\,307\,939\,z^{16}} + \frac{179\,878\,582\,253\,z^{17}}{31\,635\,258\,310\,656\,000} + \frac{105\,450\,861\,035\,520\,000\,z^{18}}{53\,393\,233\,z^{19}} + \frac{94\,223\,z^{20}}{2\,919\,482\,409\,811\,968\,000} + \frac{2\,845\,499\,424\,768\,000\,z^{21}}{53\,z^{22}} + \frac{25\,852\,016\,738\,884\,976\,640\,000\,z^{23}}{224\,800\,145\,555\,521\,536\,000} \right\}$$

AbsoluteTiming[Dsz[10 000 000, 0, z]]

$$\left\{ 5.2973030, 1 + \frac{3\,559\,637\,526\,370\,229\,z}{5\,354\,228\,880} + \frac{1\,989\,544\,871\,269\,240\,547\,z^2}{921\,858\,537\,600} + \right. \\ \frac{2\,021\,824\,016\,451\,264\,335\,171\,z^3}{677\,566\,025\,136\,000} + \frac{7\,019\,677\,821\,920\,298\,561\,119\,z^4}{2\,956\,651\,746\,048\,000} + \frac{6\,419\,737\,164\,240\,558\,941\,381\,z^5}{5\,217\,620\,728\,320\,000} + \\ \frac{355\,971\,199\,127\,948\,600\,783\,z^6}{800\,296\,713\,216\,000} + \frac{87\,671\,088\,330\,394\,680\,791\,z^7}{750\,278\,168\,640\,000} + \frac{647\,852\,562\,694\,427\,393\,z^8}{28\,245\,766\,348\,800} + \\ \frac{1\,156\,246\,192\,125\,011\,873\,z^9}{337\,983\,528\,960\,000} + \frac{6\,208\,422\,327\,896\,021\,939\,z^{10}}{15\,817\,629\,155\,328\,000} + \frac{250\,225\,399\,924\,000\,051\,z^{11}}{7\,189\,831\,434\,240\,000} + \\ \frac{6\,106\,970\,322\,634\,813\,z^{12}}{2\,549\,361\,475\,584\,000} + \frac{63\,437\,608\,022\,863\,169\,z^{13}}{497\,125\,487\,738\,880\,000} + \frac{47\,189\,432\,328\,823\,z^{14}}{9\,038\,645\,231\,616\,000} + \\ \frac{17\,111\,105\,280\,953\,z^{15}}{120\,162\,307\,939\,z^{16}} + \frac{179\,878\,582\,253\,z^{17}}{31\,635\,258\,310\,656\,000} + \frac{2\,688\,996\,956\,405\,760\,000}{2\,349\,779\,z^{18}} + \frac{53\,393\,233\,z^{19}}{2\,845\,499\,424\,768\,000} + \frac{94\,223\,z^{20}}{7\,298\,706\,024\,529\,920\,000} + \frac{z^{21}}{2\,919\,482\,409\,811\,968\,000} + \\ \frac{13\,z^{21}}{53\,z^{22}} + \frac{z^{23}}{25\,852\,016\,738\,884\,976\,640\,000} \left. \right\}$$

Chop[Expand[DdAlt[100, z]]]

$$1 + 28.5333\,z + 45.2472\,z^2 + 20.6875\,z^3 + 4.24306\,z^4 + 0.279167\,z^5 + 0.00972222\,z^6$$

Chop[Expand[Product[1 - z / j, {j, Dzeros[100, 0]}]]]

$$1 + 28.5333\,z + 45.2472\,z^2 + 20.6875\,z^3 + 4.24306\,z^4 + 0.279167\,z^5 + 0.00972222\,z^6$$

Chop[Expand[Product[1 - z / j, {j, Dyzeros[100, 0, 10]}]]]

$$1 + 28.1899\,z + 40.6568\,z^2 + 22.3167\,z^3 + 6.51591\,z^4 + 1.16695\,z^5 + 0.140813\,z^6 + 0.0121152\,z^7 + \\ 0.000776657\,z^8 + 0.0000383027\,z^9 + 1.48643 \times 10^{-6}\,z^{10} + 4.64176 \times 10^{-8}\,z^{11} + 1.18316 \times 10^{-9}\,z^{12}$$

N[Eabz[100, -1, 20, 19, z]]

$$\begin{aligned}
 &1. - 60.5515 z + 39.6222 z^2 + 274.169 z^3 - 767.594 z^4 + 1030.97 z^5 - 981.742 z^6 + 774.004 z^7 - \\
 &484.117 z^8 + 249.592 z^9 - 107.668 z^{10} + 39.2515 z^{11} - 12.2222 z^{12} + 3.28829 z^{13} - 0.77276 z^{14} + \\
 &0.160122 z^{15} - 0.0294805 z^{16} + 0.00485235 z^{17} - 0.000717251 z^{18} + 0.0000954802 z^{19} - \\
 &0.0000114556 z^{20} + 1.23646 \times 10^{-6} z^{21} - 1.19357 \times 10^{-7} z^{22} + 1.01712 \times 10^{-8} z^{23} - 7.43486 \times 10^{-10} z^{24} + \\
 &4.32745 \times 10^{-11} z^{25} - 1.47541 \times 10^{-12} z^{26} - 6.67 \times 10^{-14} z^{27} + 1.92003 \times 10^{-14} z^{28} - 2.36181 \times 10^{-15} z^{29} + \\
 &2.2285 \times 10^{-16} z^{30} - 1.79429 \times 10^{-17} z^{31} + 1.28291 \times 10^{-18} z^{32} - 8.30777 \times 10^{-20} z^{33} + 4.92861 \times 10^{-21} z^{34} - \\
 &2.69861 \times 10^{-22} z^{35} + 1.37086 \times 10^{-23} z^{36} - 6.48586 \times 10^{-25} z^{37} + 2.86666 \times 10^{-26} z^{38} - \\
 &1.18652 \times 10^{-27} z^{39} + 4.6083 \times 10^{-29} z^{40} - 1.68233 \times 10^{-30} z^{41} + 5.78112 \times 10^{-32} z^{42} - 1.87234 \times 10^{-33} z^{43} + \\
 &5.72132 \times 10^{-35} z^{44} - 1.651 \times 10^{-36} z^{45} + 4.50283 \times 10^{-38} z^{46} - 1.16146 \times 10^{-39} z^{47} + 2.83496 \times 10^{-41} z^{48} - \\
 &6.55122 \times 10^{-43} z^{49} + 1.4338 \times 10^{-44} z^{50} - 2.97282 \times 10^{-46} z^{51} + 5.84041 \times 10^{-48} z^{52} - \\
 &1.08732 \times 10^{-49} z^{53} + 1.91829 \times 10^{-51} z^{54} - 3.20687 \times 10^{-53} z^{55} + 5.07912 \times 10^{-55} z^{56} - \\
 &7.61942 \times 10^{-57} z^{57} + 1.08226 \times 10^{-58} z^{58} - 1.45487 \times 10^{-60} z^{59} + 1.84997 \times 10^{-62} z^{60} - \\
 &2.22368 \times 10^{-64} z^{61} + 2.52475 \times 10^{-66} z^{62} - 2.70538 \times 10^{-68} z^{63} + 2.7332 \times 10^{-70} z^{64} - \\
 &2.60053 \times 10^{-72} z^{65} + 2.32732 \times 10^{-74} z^{66} - 1.95631 \times 10^{-76} z^{67} + 1.54214 \times 10^{-78} z^{68} - \\
 &1.138 \times 10^{-80} z^{69} + 7.8457 \times 10^{-83} z^{70} - 5.04228 \times 10^{-85} z^{71} + 3.01326 \times 10^{-87} z^{72} - 1.66968 \times 10^{-89} z^{73} + \\
 &8.55099 \times 10^{-92} z^{74} - 4.03269 \times 10^{-94} z^{75} + 1.74396 \times 10^{-96} z^{76} - 6.88211 \times 10^{-99} z^{77} + \\
 &2.46415 \times 10^{-101} z^{78} - 7.95143 \times 10^{-104} z^{79} + 2.29373 \times 10^{-106} z^{80} - 5.85714 \times 10^{-109} z^{81} + \\
 &1.30788 \times 10^{-111} z^{82} - 2.51441 \times 10^{-114} z^{83} + 4.07773 \times 10^{-117} z^{84} - 5.42456 \times 10^{-120} z^{85} + \\
 &5.68363 \times 10^{-123} z^{86} - 4.39796 \times 10^{-126} z^{87} + 2.23442 \times 10^{-129} z^{88} - 5.5912 \times 10^{-133} z^{89}
 \end{aligned}$$

ListPlot[Table[{Re[n], Im[n]}, {n, Eabzeros[100, 0, 35, 34]}]]

\$Aborted

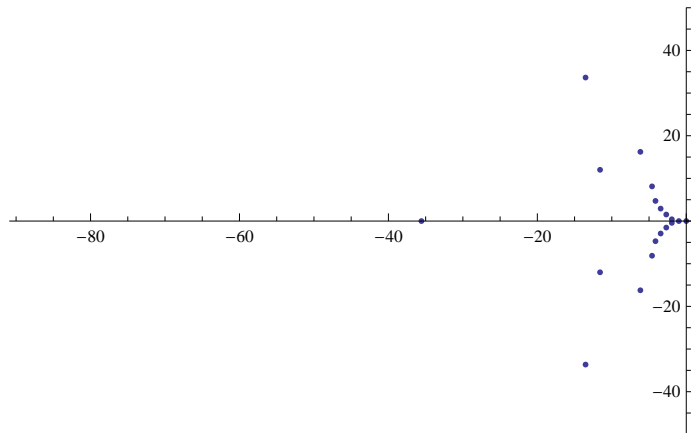
Ddzeros[100 000 000]

$$\begin{aligned}
 &\{-7494.07, -1440.44, -281.526, -37.2362 - 148.713 i, -37.2362 + 148.713 i, -35.5639, \\
 &-13.9034 - 54.5839 i, -13.9034 + 54.5839 i, -13.5241 - 33.6412 i, -13.5241 + 33.6412 i, \\
 &-11.5772 - 12.0093 i, -11.5772 + 12.0093 i, -6.16326 - 16.2264 i, -6.16326 + 16.2264 i, \\
 &-4.59605 - 8.12566 i, -4.59605 + 8.12566 i, -4.13439 - 4.72012 i, -4.13439 + 4.72012 i, \\
 &-3.44066 - 2.91838 i, -3.44066 + 2.91838 i, -2.68701 - 1.52872 i, -2.68701 + 1.52872 i, \\
 &-1.94007 - 0.419915 i, -1.94007 + 0.419915 i, -0.99477, -1.73548 \times 10^{-7}\}
 \end{aligned}$$

Ddz[100 000 000, z]

$$\begin{aligned}
 & 1 + \frac{6\,427\,431\,691\,337\,929\,z}{1\,115\,464\,350} + \frac{2\,516\,314\,672\,020\,796\,036\,867\,z^2}{128\,629\,994\,613\,120} + \frac{3\,575\,746\,713\,648\,621\,345\,062\,531\,z^3}{124\,672\,148\,625\,024\,000} + \\
 & \frac{20\,380\,394\,053\,499\,739\,865\,496\,567\,z^4}{831\,147\,657\,500\,160\,000} + \frac{1\,351\,629\,191\,016\,384\,695\,492\,785\,649\,z^5}{97\,569\,507\,619\,584\,000\,000} + \\
 & \frac{240\,272\,276\,318\,313\,590\,611\,783\,219\,z^6}{43\,364\,225\,608\,704\,000\,000} + \frac{41\,833\,655\,627\,451\,907\,360\,857\,929\,z^7}{25\,545\,471\,085\,854\,720\,000} + \\
 & \frac{3\,761\,102\,376\,291\,956\,378\,646\,751\,z^8}{5\,175\,675\,474\,335\,907\,053\,437\,241\,z^9} + \frac{10\,218\,188\,434\,341\,888\,000}{80\,669\,908\,692\,172\,800\,000} + \\
 & \frac{1\,418\,826\,547\,912\,276\,016\,443\,177\,z^{10}}{279\,266\,312\,199\,608\,569\,103\,z^{11}} + \frac{161\,339\,817\,384\,345\,600\,000}{292\,017\,769\,021\,440\,000} + \\
 & \frac{1\,386\,818\,791\,325\,659\,005\,283\,z^{12}}{4\,814\,640\,871\,442\,135\,348\,159\,z^{13}} + \frac{3\,482\,068\,013\,538\,497\,942\,357\,z^{14}}{10\,857\,220\,652\,217\,139\,200\,000} + \\
 & \frac{16\,703\,416\,388\,026\,368\,000}{4\,192\,807\,585\,604\,237\,z^{16}} + \frac{17\,963\,977\,627\,832\,867\,z^{17}}{83\,517\,081\,940\,131\,840\,000} + \frac{1\,290\,718\,539\,074\,764\,800\,000}{390\,742\,194\,213\,977\,z^{18}} + \\
 & \frac{1\,290\,718\,539\,074\,764\,800\,000}{1\,555\,110\,247\,813\,z^{19}} + \frac{31\,437\,955\,243\,z^{20}}{306\,545\,653\,030\,256\,640\,000} + \frac{490\,473\,044\,848\,410\,624\,000}{14\,797\,988\,921\,z^{21}} + \\
 & \frac{24\,523\,652\,242\,420\,531\,200\,000}{976\,022\,221\,z^{22}} + \frac{47\,726\,800\,133\,326\,110\,720\,000}{786\,869\,z^{23}} + \\
 & \frac{1493\,z^{24}}{269\,760\,174\,666\,625\,843\,200\,000} + \frac{727\,z^{25}}{403\,291\,461\,126\,605\,635\,584\,000\,000} + \\
 & \frac{38\,778\,025\,108\,327\,464\,960\,000}{31\,022\,420\,086\,661\,971\,968\,000\,000} + \frac{403\,291\,461\,126\,605\,635\,584\,000\,000}{z^{26}}
 \end{aligned}$$

ListPlot[Table[{Re[n], Im[n]}, {n, Ddzeros[100 000 000]}]]



⋮