```
ida[n_] := 1
idb[n_] := n
idc[n_] := n^2
idd[n_, k_] := n^k
id2[n_, k_] := GCD[ n, k]
id3[n_] := If[ n == 1, 1, 0]
id4[n_] := EulerPhi[ n]
id5[n_, k_] := DivisorSigma[ k, n]
id6[n_] := LiouvilleLambda[ n]
id7[n_] := FiniteAbelianGroupCount[ n]
id8[n_] := (-1)^Length[FactorInteger[n]]
id9[n_, k_] := JacobiSymbol[n, k] (* k should be a prime *)
id[n_] := id2[n, 6]
dr[n_, k_] := Sum[ id[j] dr[Floor[n / j], k - 1], {j, 2, n}]; dr[n_, 0] := 1
dx[n_, z_] := Sum[ Binomial[z, k] dr[n, k], {k, 0, Log[2, n]}]
Table[{n, FullSimplify[dx[n, z] - dx[n - 1, z]]}, {n, 2, 40}] // TableForm
Table[{n, D[ FullSimplify[dx[n, z] - dx[n - 1, z]], {z, 1}] /. z → 0}, {n, 2, 40}] // TableForm
```

2	$2 z$
3	$3 z$
4	$2 z^2$
5	z
6	$6 z^2$
7	z
8	$\frac{2}{3} \left(z + 2 z^3\right)$
9	$\frac{3}{2} z (-1 + 3 z)$
10	$2 z^2$
11	z
12	$6 z^3$
13	z
14	$2 z^2$
15	$3 z^2$
16	$\frac{2}{3} z^2 \left(2 + z^2\right)$
17	z
18	$3 z^2 (-1 + 3 z)$
19	z
20	$2 z^3$
21	$3 z^2$
22	$2 z^2$
23	z
24	$2 \left(z^2 + 2 z^4\right)$
25	$\frac{1}{2} z (1 + z)$
26	$2 z^2$
27	$\frac{3}{2} z (2 + 3 (-1 + z) z)$
28	$2 z^3$
29	z
30	$6 z^3$
31	z
32	$\frac{2}{15} z \left(3 + 2 z^2 \left(5 + z^2\right)\right)$
33	$3 z^2$
34	$2 z^2$
35	z^2
36	$3 z^3 (-1 + 3 z)$
37	z
38	$2 z^2$
39	$3 z^2$
40	$\frac{2}{3} \left(z^2 + 2 z^4\right)$

2	2
3	3
4	0
5	1
6	0
7	1
8	$\frac{2}{3}$
9	$-\frac{3}{2}$
10	0
11	1
12	0
13	1
14	0
15	0
16	0
17	1
18	0
19	1
20	0
21	0
22	0
23	1
24	0
25	$\frac{1}{2}$
26	0
27	3
28	0
29	1
30	0
31	1
32	$\frac{2}{5}$
33	0
34	0
35	0
36	0
37	1
38	0
39	0
40	0

```
Table[{n, FullSimplify[Expand[dx[2^n, z] - dx[2^n - 1, z]]]}, {n, 1, 5}] // TableForm
```

1	$z\, f[2]$
2	$\frac{1}{2} (-1+z)\, z\, f[2]^2 + z\, f[4]$
3	$\frac{1}{6} \left((-1+z)\, z\, f[2] \left((-2+z)\, f[2]^2 + 6\, f[4]\right) + 6\, z\, f[8]\right)$
4	$\frac{1}{24} \left((-1+z)\, z \left((-3+z)\, (-2+z)\, f[2]^4 + 12\, (-2+z)\, f[2]^2\, f[4] + 12\, f[4]^2 + 24\, f[2]\, f[8]\right) + 24\, z\, f[16$
5	$\frac{1}{120} \left((-1+z)\, z \left((-4+z)\, (-3+z)\, (-2+z)\, f[2]^5 + 20\, (-3+z)\, (-2+z)\, f[2]^3\, f[4] + 60\, (-2+z)\, f[2]^2$

```
Expand[z ((-1 + z)
    (f[5] f[2] f[3] + f[3] f[2] f[5] + f[2] ((-2 + z) f[3] f[5] + f[3] f[5])) + f[3] f[2] f[5])]
```

$z^3\, f[2]\, f[3]\, f[5]$

```
Expand[(z f[4] + z (z - 1) / 2 f[2]^2) (z f[3])]
```

$$\text{Expand}\left[\left(-\frac{1}{2} z^2 f[2]^2 f[3] + \frac{1}{2} z^3 f[2]^2 f[3] + z^2 f[3] f[4]\right) - \left(\frac{1}{2} \left((-1+z)\, z \left((-2+z)\, f[2]^2 f[3] + 2 f[3] f[4] + 2 f[2] f[2] f[3]\right) + 2 z f[3] f[4]\right)\right)\right]$$

0

```
ff[ p_, a_, z_, k_] := (z - k + 1) / k Sum[ f[ p^j] ff[ p, a - j, z, k + 1], {j, 1, a}];
ff[p_, 0, z_, k_] := 1
```

```
FullSimplify[Expand[ff[ 2, 3, z, 1]]]
```

$$\frac{1}{6} \left((-1+z)\, z\, f[2] \left((-2+z)\, f[2]^2 + 6 f[4]\right) + 6 z f[8]\right)$$

```
Table[{n, Expand[( FullSimplify[Expand[dx[2^n, z] - dx[2^n - 1, z]]]) - ff[2, n, z, 1]]},
  {n, 1, 5}] // TableForm
```

1	0
2	0
3	0
4	0
5	0

```
FullSimplify[ff[ 2, 3, z, 1] ff[ 5, 1, z, 1]]
```

$$\frac{1}{6} z^2 f[5] \left((-1+z)\, f[2] \left((-2+z)\, f[2]^2 + 6 f[4]\right) + 6 f[8]\right)$$

$$\text{FullSimplify}\left[\frac{1}{6} \left((-1+z)\, z \left((-3+z)\, (-2+z)\, f[2]^3 f[5] + 3\, (-2+z)\, f[2]^2 f[2] f[5] + 6\, (f[5] f[8] + f[4] f[2] f[5]) + 6 f[2]\, ((-2+z)\, f[4] f[5] + f[4] f[5])\right) + 6 z f[8] f[5]\right)\right] - \text{FullSimplify}[\text{ff}[2, 3, z, 1]\ \text{ff}[5, 1, z, 1]]$$

0

```
ff[ p_, a_, z_, k_] := (z - k + 1) / k Sum[ f[ p^j] ff[ p, a - j, z, k + 1], {j, 1, a}];
ff[p_, 0, z_, k_] := 1
```

```
ff[ p, 3, z, 1]
```

$$z \left(\frac{1}{2} (-1+z)\, f[p]\, f\left[p^2\right] + \frac{1}{2} (-1+z)\, f[p] \left(\frac{1}{3} (-2+z)\, f[p]^2 + f\left[p^2\right]\right) + f\left[p^3\right]\right)$$

```
EulerPhi[ 32]
```

16

```
32 * (1 - 1 / 2)
```

16

```
Binomial[3, 3]
```

1

```
FullSimplify[
 Binomial[3, 0] Binomial[ z, 1] p^3 (p - 1) + Binomial[3, 1] Binomial[ z, 2] p^2 (p - 1)^2 +
  Binomial[3, 2] Binomial[ z, 3] p (p - 1)^3 + Binomial[3, 3] Binomial[ z, 4] (p - 1)^4]
```

```
FullSimplify[
 Expand[1/24 z (6 (-1 + p^4) + (-1 + p)^2 (11 + p (14 + 11 p)) z + 6 (-1 + p)^3 (1 + p) z^2 + (-1 + p)^4 z^3)]]
```

$$\frac{1}{24} z \left(6 \left(-1+p^4\right)+(-1+p)^2 (11+p (14+11 p)) z+6 (-1+p)^3 (1+p) z^2+(-1+p)^4 z^3\right)$$

```
Expand[(a + b)^3]
```

$a^3+3 a^2 b+3 a b^2+b^3$

```
FI[n_] := FactorInteger[n]; FI[1] := {}
phi1[p_, a_, z_] :=
 Sum[Binomial[a - 1, j] Binomial[z, j + 1] (p - 1)^(j + 1) p^(a - 1 - j), {j, 0, a - 1}]
phiz[n_, z_] := Product[Sum[Binomial[p[[2]] - 1, j] Binomial[z, j + 1]
    (p[[1]] - 1)^(j + 1) p[[1]]^(p[[2]] - 1 - j), {j, 0, p[[2]] - 1}], {p, FI[n]}]
phi2[p_, a_, z_] := (-1 + p) p^(-1+a) z Hypergeometric2F1[1 - a, 1 - z, 2, (-1 + p)/p]
phiz2[n_, z_] := Product[(-1 + p[[1]]) p[[1]]^(-1+p[[2]])
   z Hypergeometric2F1[1 - p[[2]], 1 - z, 2, (-1 + p[[1]])/p[[1]]], {p, FI[n]}]
```

```
Expand[phiz2[30, z]]
```

$8 z^3$

```
Expand[dx[aa = 30, z] - dx[aa - 1, z]]
```

$8 z^3$

```
Expand[ (p - 1)^3 p]
```

$-p+3 p^2-3 p^3+p^4$

```
Expand[ (p - 1)^2 p^2]
```

$p^2-2 p^3+p^4$

```
Expand[((p - 1) + p)^3]
```

$-1+6 p-12 p^2+8 p^3$

```
Sum[Binomial[a - 1, j] Binomial[z, j + 1] (p - 1)^(j + 1) p^(a - 1 - j), {j, 0, a - 1}]
```

$(-1+p)\, p^{-1+a}\, z\, \text{Hypergeometric2F1}\left[1-a, 1-z, 2, \frac{-1+p}{p}\right]$

```
Expand[(-1 + p) p^(-1+a)]
```

$-p^{-1+a}+p^a$

```
Expand[ (a + b) ^2]
```

$a^2 + 2 a b + b^2$

```
FullSimplify[Expand[Binomial[ z, 1] p (p - 1) + Binomial[ z, 2] (p - 1) ^2]]
```

$\frac{1}{2} (-1 + p) z (1 + p + (-1 + p) z)$

```
Expand[Binomial[ z, 1] (p - 1)]
```

$-z + p z$

```
Binomial[a - 1, 2]
```

$\frac{1}{2} (-2 + a) (-1 + a)$

```
jord[n_, k_] := n^k Product[(1 - 1 / p[[1]] ^k), {p, FI[n]}]
j2[n_, k_] := Product[(p[[1]] ^ (p[[2]] k) - (p[[1]] ^ ((p[[2]] - 1) k))), {p, FI[n]}]
j3[n_, k_] := Product[p[[1]] ^ ((p[[2]] - 1) k) ( p[[1]] ^k - 1), {p, FI[n]}]
```

```
jord[100, 2]
```

7200

```
j3[100, 2]
```

7200

```
FullSimplify[p^ (a k ) / p^ ((a - 1) k)]
```

p^k

```
Sum[Binomial[a - 1, j] Binomial[z, j + 1] (p^k - 1) ^ (j + 1) p^ ((a - 1 - j) k), {j, 0, a - 1}]
```

$p^{(-1+a) k} \left(-1 + p^k\right) z \text{ Hypergeometric2F1}\left[1 - a, 1 - z, 2, 1 - p^{-k}\right]$

```
lio[n_] := Product[(-1) ^p[[2]], {p, FI[n]}]
```

```
Table[ { LiouvilleLambda[n] - lio[n]}, {n, 2, 20}]
```

{{0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}}

```
FullSimplify[ Binomial[ z, 1] + 3 Binomial[z, 2] + 3 Binomial[z, 3] + Binomial[z, 4]]
```

$\frac{1}{24} z (1 + z) (2 + z) (3 + z)$

```
Table[ GCD[ 8, k], {k, 2, 40}]
```

{2, 1, 4, 1, 2, 1, 8, 1, 2, 1, 4, 1, 2, 1, 8, 1, 2,
1, 4, 1, 2, 1, 8, 1, 2, 1, 4, 1, 2, 1, 8, 1, 2, 1, 4, 1, 2, 1, 8}

```
Table[ GCD[2, k] - ( 1 + (Floor[ k / 2] - Floor[(k - 1) / 2]) ), {k, 2, 40}]
```

{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

```
Table[ GCD[4, k] -
  ( 1 + (Floor[ k / 2] - Floor[(k - 1) / 2]) + 2 (Floor[ k / 4] - Floor[(k - 1) / 4]) ), {k, 2, 40}]
```

{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

```
Table[ GCD[8, k] - ( 1 + (Floor[ k / 2] - Floor[(k - 1) / 2]) +
    2 (Floor[ k / 4] - Floor[(k - 1) / 4]) + 4 (Floor[ k / 8] - Floor[(k - 1) / 8]) ), {k, 2, 40}]
```

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

```
gc[ k_, p_, a_] := 1 + Sum[ (p^j - p^(j - 1)) (Floor[ k / p^j] - Floor[(k - 1) / p^j]), {j, 1, a}]
```

```
Table[ gc[k, 2, 3] gc[k, 3, 1] - GCD[2^3 × 3, k], {k, 2, 80}]
```

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

```
Table[ gc[k, 2, 3] gc[k, 3, 1], {k, 2, 80}]
```

```
{2, 3, 4, 1, 6, 1, 8, 3, 2, 1, 12, 1, 2, 3, 8, 1, 6, 1, 4, 3, 2, 1, 24, 1, 2,
 3, 4, 1, 6, 1, 8, 3, 2, 1, 12, 1, 2, 3, 8, 1, 6, 1, 4, 3, 2, 1, 24, 1, 2, 3, 4, 1,
 6, 1, 8, 3, 2, 1, 12, 1, 2, 3, 8, 1, 6, 1, 4, 3, 2, 1, 24, 1, 2, 3, 4, 1, 6, 1, 8}
```

```
FullSimplify[gc[k, p, 4]]
```

$$1 + (-1 + p)\, p^3 \left(\text{Ceiling}\left[\frac{1-k}{p^4}\right] + \text{Floor}\left[\frac{k}{p^4}\right]\right) + (-1 + p)\, p^2 \left(\text{Ceiling}\left[\frac{1-k}{p^3}\right] + \text{Floor}\left[\frac{k}{p^3}\right]\right) + (-1 + p)\, p \left(\text{Ceiling}\left[\frac{1-k}{p^2}\right] + \text{Floor}\left[\frac{k}{p^2}\right]\right) + (-1 + p) \left(\text{Ceiling}\left[\frac{1-k}{p}\right] + \text{Floor}\left[\frac{k}{p}\right]\right)$$

```
Table[ D[ FullSimplify[dx[n, z] - dx[n - 1, z]], {z, 1}] /. z → 0, {n, 2, 40}]
```

$$\left\{2, 3, 0, 1, 0, 1, \frac{2}{3}, -\frac{3}{2}, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, \frac{1}{2}, 0, 3, 0, 1, 0, 1, \frac{2}{5}, 0, 0, 0, 0, 1, 0, 0, 0\right\}$$