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Clear[Dnz]
binomial[z_, k_] := binomial[z, k] = Product[z - j, {j, 0, k - 1}] / k!
Dnka[n_, 0, a_] := UnitStep[n - 1]
Dnka[n_, 1, a_] := Dnka[n, 1, a] = Floor[n] - a
Dnka[n_, 2, a_] :=
  Dnka[n, 2, a] = Sum[1 + 2 (Dnka[Floor[n / m], 1, m]), {m, a + 1, Floor[n^(1 / 2)]}]
Dnka[n_, k_, a_] := Dnka[n, k, a] =
  Sum[1 + k Dnka[Floor[n / (m^(k - 1))], 1, m] + Sum[binomial[k, j]
    Dnka[Floor[n / (m^j)], k - j, m], {j, 1, k - 2}], {m, a + 1, Floor[n^(1 / k)]}]

Dnz[n_, z_] := Expand@Sum[binomial[z, k] Dnka[n, k, 1], {k, 0, Log2@n}]
DnzRoots[n_] := If[(c = Exponent[f = Dnz[n, z], z]) == 0, {},
  If[c == 1, List@NRoots[f == 0, z][[2]], List@@NRoots[f == 0, z][[All, 2]]]]
DnzRootsm[n_] := If[(c = Exponent[f = (Dnz[n, z] - 1), z]) == 0, {},
  If[c == 1, List@NRoots[f == 0, z][[2]], List@@NRoots[f == 0, z][[All, 2]]]]
DnzRootsm2[n_] := If[(c = Exponent[f = (Dnz[n, z] - 1) / z, z]) == 0, {},
  If[c == 1, List@NRoots[f == 0, z][[2]], List@@NRoots[f == 0, z][[All, 2]]]]

Table[N@Discriminant[Dnz[n, z], z], {n, 1, 100}] // TableForm
DnzRootsm[100]

{-11.1997 - 12.3983 i, -11.1997 + 12.3983 i, -2.65742 - 1.85785 i, -2.65742 + 1.85785 i, -1., 0.}

binomial[z_, k_] := binomial[z, k] = Product[z - j, {j, 0, k - 1}] / k!
Ds[n_, 0, s_, a_] := UnitStep[n - 1]
Ds[n_, 1, s_, a_] := Ds[n, 1, s, a] = HarmonicNumber[Floor[n], s] - HarmonicNumber[a, s]
Ds[n_, 2, s_, a_] := Ds[n, 2, s, a] =
  Sum[(m^(-2 s)) + 2 (m^(-s)) (Ds[Floor[n / m], 1, s, m]), {m, a + 1, Floor[n^(1 / 2)]}]
Ds[n_, k_, s_, a_] := Ds[n, k, s, a] =
  Sum[(m^(-s k)) + k (m^(-s (k - 1))) Ds[Floor[n / (m^(k - 1))], 1, s, m] +
    Sum[binomial[k, j] (m^(-s))^j Ds[Floor[n / (m^j)], k - j, s, m], {j, 1, k - 2}],
    {m, a + 1, Floor[n^(1 / k)]}]
Dnsz[n_, s_, z_] := Expand@Sum[binomial[z, k] Ds[n, k, s, 1], {k, 0, Log2@n}]

Dns112z[n_, s_, z_] :=
  Expand@Sum[(-1)^j binomial[z, j] 2^(j (1 - s)) Dnsz[n / (2^j), s, z], {j, 0, Log2@n}]

Dns112zZeros[n_, s_] := If[(c = Exponent[f = Dns112z[n, s, z], z]) == 0, {},
  If[c == 1, List@NRoots[f == 0, z][[2]], List@@NRoots[f == 0, z][[All, 2]]]]

Dns112zZeros[100 000 000, N@ZetaZero[1]]

{-23.4433 + 115.492 i, -3.00749 + 7.24046 i, -0.00399172 - 0.0024432 i,
  1. + 3.6082 x 10^-6 i, 1.59804 + 38.0367 i, 1.61973 - 1.50287 i, 2.00164 - 0.0052567 i,
  2.70904 + 0.134875 i, 3.0931 - 8.44949 i, 3.1986 - 1.21641 i, 3.56358 + 1.39473 i,
  4.17529 - 3.86039 i, 4.4097 + 16.3853 i, 4.60627 + 3.92814 i, 5.18591 + 0.00864523 i,
  9.63811 - 0.0407831 i, 12.2774 - 154.546 i, 12.4618 - 21.1794 i,
  21.8501 + 3.95519 i, 24.4498 - 2.96747 i, 32.8255 - 0.284314 i, 66.7753 - 26.1226 i,
  68.0094 + 67.8245 i, 87.9623 - 5862.85 i, 133.035 - 68.5173 i, 820.076 - 843.833 i}

Sum[-1 / r, {r, Dns112zZeros[100 000 000, N@ZetaZero[1]]}]

178.811 - 111.793 i

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**Dnz[100, z] - 1**

$$\frac{428 z}{15} + \frac{16289 z^2}{360} + \frac{331 z^3}{16} + \frac{611 z^4}{144} + \frac{67 z^5}{240} + \frac{7 z^6}{720}$$

**(1 - 1 / Dns112zZeros[100 000 000, N@ZetaZero[1]]) // TableForm**

1.00169 + 0.00831598 i  
 1.04893 + 0.11779 i  
 183.245 - 111.546 i  
 $9.6754 \times 10^{-7} + 3.60819 \times 10^{-6} i$   
 0.998897 + 0.0262441 i  
 0.668234 - 0.30783 i  
 0.500413 - 0.00131201 i  
 0.631778 + 0.0183327 i  
 0.961795 - 0.104365 i  
 0.726865 - 0.103871 i  
 0.756659 + 0.0952398 i  
 0.870876 - 0.119385 i  
 0.984685 + 0.0569084 i  
 0.87431 + 0.107186 i  
 0.80717 + 0.000321459 i  
 0.896247 - 0.000439024 i  
 0.999489 - 0.00642998 i  
 0.979363 - 0.0350731 i  
 0.955686 + 0.00802156 i  
 0.959694 - 0.00489197 i  
 0.969538 - 0.000263842 i  
 0.987012 - 0.0050809 i  
 0.992628 + 0.0073519 i  
 0.999997 - 0.000170527 i  
 0.994059 - 0.00305976 i  
 0.999408 - 0.000609451 i

**et[s2\_] := Limit[(1 - 2^(1 - s)) Zeta[s], s → s2]**

**etl[s2\_] := Limit[Log[(1 - 2^(1 - s)) Zeta[s]], s → s2]**

**etl[ZetaZero[1]]**

-∞

**(1 - 1 / Dns112zZeros[100 000 000, .3 I + N@ZetaZero[1]]) // TableForm**

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1. - 0.000171978 i
1.00179 + 0.00810083 i
1.05191 + 0.11297 i
214.287 + 52.6102 i
0.00504334 + 0.0382296 i
0.998976 + 0.0259758 i
0.660484 - 0.321229 i
0.530403 + 0.00901208 i
0.631505 + 0.00716893 i
0.961785 - 0.104965 i
0.72649 - 0.11007 i
0.757225 + 0.0949964 i
0.870745 - 0.119918 i
0.983983 + 0.0569748 i
0.874383 + 0.106685 i
0.807182 + 0.000314091 i
0.999611 - 0.00671136 i
0.896247 - 0.000448431 i
0.9797 - 0.035095 i
0.955788 + 0.00796822 i
0.959899 - 0.00459833 i
0.96915 - 0.000489731 i
0.987058 - 0.00520052 i
0.992642 + 0.00744833 i
0.99412 - 0.00308359 i
0.999435 - 0.00063322 i

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(1 - 1 / Dns112zZeros[100 000 000, .3 + N@ZetaZero[1]]) // TableForm
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1.00148 + 0.00820281 i
1.04431 + 0.114926 i
3.24155 - 0.599365 i
0.626285 + 0.627762 i
0.765937 - 0.51369 i
0.998638 + 0.0261585 i
0.755644 - 0.213942 i
0.744934 + 0.140448 i
0.9612 - 0.104398 i
0.760763 - 0.0796203 i
0.760931 + 0.0599455 i
0.87052 - 0.119119 i
0.984806 + 0.0575523 i
0.87381 + 0.107098 i
0.808209 - 0.000505609 i
0.896238 - 0.000439371 i
0.979354 - 0.0354349 i
0.999198 - 0.00654648 i
0.955627 + 0.00793327 i
0.959957 - 0.00511056 i
0.969347 + 0.000116328 i
0.986885 - 0.00513023 i
0.992724 + 0.00733781 i
0.994034 - 0.00312255 i
0.999996 - 0.00017854 i
0.999383 - 0.000635052 i

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