```
s1[n_, s_] := Sum[j^s, {j, 1, n}] - Integrate[j^s, {j, 0, n}] - n^s / 2
s2[n_, s_] := Sum[j^(s I) / j^(1 / 2), {j, 1, n}] -
  Integrate[j^(s I) / j^(1 / 2), {j, 0, n}] - n^(s I) / n^(1 / 2) / 2
s3[n_, s_] := Sum[E^(s Log[j] I) / j^(1 / 2), {j, 1, n}] -
  Integrate[E^(s Log[j] I) / j^(1 / 2), {j, 0, n}] - E^((s I - 1 / 2) Log[n]) / 2
s4[n_, s_] := 1 + Sum[Sum[(s Log[j] I)^k / k!, {k, 0, Infinity}] / j^(1 / 2), {j, 2, n}] -
  Integrate[E^(s Log[j] I) / j^(1 / 2), {j, 0, n}] - E^((s I) Log[n]) / n^(1 / 2) / 2
s4a[n_, s_] := (1/2 - ⅈ s)
  (1 - (2 ⅈ/(ⅈ - 2 s)) + Sum[Sum[(s Log[j] I)^k / k!, {k, 0, Infinity}] / j^(1 / 2), {j, 2, n}] -
    Integrate[E^(s Log[j] I) / j^(1 / 2), {j, 1, n}] - E^((s I) Log[n]) / n^(1 / 2) / 2)
s5[n_, s_] := 1 + Sum[Sum[(s Log[j] I)^k / k!, {k, 0, Infinity}] / j^(1 / 2), {j, 2, n}] -
  Integrate[Sum[(s Log[j] I)^k / k!, {k, 0, Infinity}] / j^(1 / 2), {j, 0, n}] -
  Sum[((s I) Log[n])^k / k!, {k, 0, Infinity}] / n^(1 / 2) / 2
s6[n_, s_] := 1 + Sum[ Sum[(I s)^k (Log[j])^k / k!, {k, 0, Infinity}] / j^(1 / 2), {j, 2, n}] -
  Integrate[Sum[(I s)^k ( Log[j] )^k / k!, {k, 0, Infinity}] / j^(1 / 2), {j, 0, n}] -
  Sum[(s I)^k (Log[n])^k / k!, {k, 0, Infinity}] / n^(1 / 2) / 2
s7[n_, s_, l_] := 1 + Sum[ Sum[(I s)^k (Log[j])^k / k!, {k, 0, l}] / j^(1 / 2), {j, 2, n}] -
  Integrate[Sum[(I s)^k ( Log[j] )^k / k!, {k, 0, l}] / j^(1 / 2), {j, 0, n}] -
  Sum[(s I)^k (Log[n])^k / k!, {k, 0, l}] / n^(1 / 2) / 2
s8[n_, s_, l_] := (1/2 - ⅈ s) (1 + Sum[Sum[(I s)^k (Log[j])^k / k! / j^(1 / 2), {j, 2, n}] -
      Integrate[(I s)^k ( Log[j] )^k / k! / j^(1 / 2), {j, 0, n}] -
      (s I)^k (Log[n])^k / k! / n^(1 / 2) / 2, {k, 0, l}])
s9[n_, s_, l_] := (1/2 - ⅈ s) (1 + Sum[(I s)^k / k! Sum[(Log[j])^k / j^(1 / 2), {j, 2, n}] -
      (I s)^k / k! Integrate[( Log[j] )^k / j^(1 / 2), {j, 0, n}] -
      (s I)^k / k! (Log[n])^k / n^(1 / 2) / 2, {k, 0, l}])
s10[n_, s_, l_] := (1/2 - ⅈ s) (1 - (2 ⅈ/(ⅈ - 2 s)) +
    Sum[(I s)^k / k! (Sum[Log[j]^k / j^(1 / 2), {j, 2, n}] -
        Integrate[ Log[j]^k / j^(1 / 2), {j, 1, n}] - Log[n]^k / n^(1 / 2) / 2), {k, 0, l}])
s11[n_, s_, l_] := (1/2 - ⅈ s) (1 - (2 ⅈ/(ⅈ - 2 s)) + Sum[(I s)^k / k!
      (Sum[Log[j]^k / j^(1 / 2), {j, 2, n}] -
        ((-2)^(1+k) (Gamma[1 + k, 0, -Log[n^(1 / 2)]])) - Log[n]^k / n^(1 / 2) / 2), {k, 0, l}])
s12[n_, s_, l_] := ((1/2 - ⅈ s) - (1/2 - ⅈ s) (2 ⅈ/(ⅈ - 2 s)) +
   Sum[(I s)^k / k! (Sum[(1/2 - ⅈ s) Log[j]^k / j^(1 / 2), {j, 2, n}] - ((-2)^(1+k) (1/2 - ⅈ s)
          (Gamma[1 + k, 0, -Log[n^(1 / 2)]])) - (1/2 - ⅈ s) Log[n]^k / n^(1 / 2) / 2), {k, 0, l}])
```

$$\text{s13}[n\_, s\_, l\_] := \left(\frac{1}{2} - i\,s\right)\left(1 - \left(\frac{2\,i}{i - 2\,s}\right) + \text{Sum}\Big[(I\,s)\,^\wedge k / k!\ \big(\text{Sum}[\text{Log}[j]^\wedge k / j^\wedge(1/2), \{j, 2, n\}] - \big((-2)^{1+k}\,(\text{Gamma}[1 + k, 0, -\text{Log}[n^\wedge(1/2)]])\big) - \text{Log}[n]^\wedge k / n^\wedge(1/2)/2\big), \{k, 0, l\}\Big]\right)$$

$$\text{zn}[n\_, k\_] := \text{Sum}[\text{Log}[j]^\wedge k / j^\wedge(1/2), \{j, 2, n\}] - \big((-2)^{1+k}\,(\text{Gamma}[1 + k, 0, -\text{Log}[n^\wedge(1/2)]])\big) - \text{Log}[n]^\wedge k / n^\wedge(1/2)/2$$

$$\text{zn2}[n\_, k\_] := -(-2)^{1+k}\,\text{Gamma}\left[1 + k, 0, -\text{Log}\left[\sqrt{n}\right]\right] - \frac{\text{Log}[n]^k}{2\sqrt{n}} + \sum_{j=2}^{n}\frac{\text{Log}[j]^k}{\sqrt{j}}$$

$$\text{s14}[n\_, s\_, l\_] := \left(\frac{1}{2} - i\,s\right)\left(1 - \left(\frac{2\,i}{i - 2\,s}\right) + \text{Sum}[(I\,s)^\wedge k / k!\ \text{zn2}[n, k], \{k, 0, l\}]\right)$$

$$\text{bo3}[n\_, t\_] := \text{Sum}[\text{Log}[j]^\wedge t / j^\wedge(1/2), \{j, 2, n\}] - \left((-2)^{1+t}\,\text{Gamma}\left[1 + t, 0, -\text{Log}\left[\sqrt{n}\right]\right]\right) - \text{Log}[n]^\wedge t / n^\wedge(1/2) - \text{Sum}[\text{BernoulliB}[k] / k!\ D[\text{Log}[n2]^\wedge t / n^\wedge(1/2), \{n2, k-1\}]\ /.\ n2 \to n, \{k, 1, 10\}]$$

$$\text{s15}[n\_, s\_, l\_] := \left(\frac{1}{2} - i\,s\right)\left(1 - \left(\frac{2\,i}{i - 2\,s}\right) + \text{Sum}[(I\,s)^\wedge k / k!\ \text{bo3}[n, k], \{k, 0, l\}]\right)$$

$$\text{s15o}[n\_, s\_, l\_] := \left(\frac{1}{2} - i\,s\right)\left(1 - \left(\frac{1}{1/2 + s\,I}\right) + \text{Sum}[(I\,s)^\wedge k / k!\ \text{bo3}[n, k], \{k, 0, l\}]\right)$$

$$\text{s15a}[n\_, s\_, l\_] := (1 - s)\left(1 - 1/s + \text{Sum}\left[\left(-\frac{1}{2} + s\right)^\wedge k \Big/ k!\ \text{bo3}[n, k], \{k, 0, l\}\right]\right)$$

```
N@s15a[10, .5 + 8. I, 400]
```

$-2.25578 - 10.1027\,i$

```
N@s15o[10, 8. + .1 I, 400]
```

$-1.96335 - 10.0507\,i$

```
N@s14[100, 5., 100]
```

$-0.805111 - 3.62658\,i$

```
s4a[100, 5.]
```

$-0.804979 - 3.62657\,i$

```
((.6 + 8. I)) Zeta[.6 + 8. I]
```

$-1.96626 + 10.0588\,i$

$$\text{FullSimplify}\left[\left(\frac{1}{1/2 + s\,I}\right)\right]$$

$$\frac{1}{\frac{1}{2} + i\,s}$$

```
Integrate[( Log[j] )^k / j^(1 / 2), {j, 0, n}]
```

$$\text{ConditionalExpression}\left[2^{1+k}\left((-1)^k\,\text{Gamma}[1 + k] + \left(-k\,\text{Gamma}[k] + \text{Gamma}\left[1 + k, -\frac{\text{Log}[n]}{2}\right]\right)(-\text{Log}[n])^{-k}\,\text{Log}[n]^k\right), \text{Re}[k] > -1\right]$$

-Integrate[j^(I s) / j^(1/2), {j, 0, 1}]

ConditionalExpression$\left[-\frac{2 i}{i-2 s}, \mathrm{Im}[s] < \frac{1}{2}\right]$

FullSimplify[Integrate[(Log[j])^k / j^(1/2), {j, 1, n}], Element[k, Integers]]

ConditionalExpression$\left[(-1)^k 2^{1+k} \left(-k! + \mathrm{Gamma}\left[1+k, -\frac{\mathrm{Log}[n]}{2}\right]\right), k \geq 0 \,\&\&\, \mathrm{Log}[n] > 0\right]$

(I s)^k / k (Sum[(Log[j])^k! / j^(1/2), {j, 2, n}] -
$\left((-1)^k 2^{1+k} \left(-k! + \mathrm{Gamma}\left[1+k, -\frac{\mathrm{Log}[n]}{2}\right]\right)\right)$ **- (Log[n])^k / n^(1/2) / 2) /. k → 0**

Power::infy : Infinite expression $\frac{1}{0}$ encountered. ≫

ComplexInfinity

Sum[(Log[j])^k! / j^(1/2), {j, 2, n}] -
$\left((-1)^k 2^{1+k} \left(-k! + \mathrm{Gamma}\left[1+k, -\frac{\mathrm{Log}[n]}{2}\right]\right)\right)$ **- (Log[n])^k / n^(1/2) / 2 /. k → 0**

FullSimplify$\left[-2\left(-1+\sqrt{n}\right) - \frac{1}{2\sqrt{n}} - \frac{1}{4}(2\,\mathrm{EulerGamma} + \pi + \mathrm{Log}[64] + 2\,\mathrm{Log}[\pi])\,\mathrm{Zeta}\left[\frac{1}{2}\right] + \mathrm{Zeta}^{(1,0)}\left[\frac{1}{2}, 1+n\right], \mathrm{Element}[n, \mathrm{Integers}]\right]$

$2 - \frac{1}{2\sqrt{n}} - 2\sqrt{n} - \frac{1}{4}\left(2\,\mathrm{EulerGamma} + \pi + \mathrm{Log}\left[64\,\pi^2\right]\right)\mathrm{Zeta}\left[\frac{1}{2}\right] + \mathrm{Zeta}^{(1,0)}\left[\frac{1}{2}, 1+n\right]$

(1 - s) /. s → 1/2 + s I

$\frac{1}{2} - i s$

FullSimplify$\left[\left(\frac{1}{2} - i s\right)\left(\frac{2 i}{i - 2 s}\right)\right]$

$\frac{i + 2 s}{i - 2 s}$

Integrate[Log[j] / j^(1/2), {j, 1, n}]

ConditionalExpression$\left[4 + 2\sqrt{n}\,(-2 + \mathrm{Log}[n]), \mathrm{Re}[n] \geq 0 \,||\, n \notin \mathrm{Reals}\right]$

FullSimplify[Integrate[Log[j]^k / j^(1/2), {j, 1, n}], Element[k, Integers]]

ConditionalExpression$\left[(-1)^k 2^{1+k} \left(-k! + \mathrm{Gamma}\left[1+k, -\frac{\mathrm{Log}[n]}{2}\right]\right), k \geq 0 \,\&\&\, \mathrm{Log}[n] > 0\right]$

$(-1)^k 2^{1+k} \left(-k! + \mathrm{Gamma}\left[1+k, -\frac{\mathrm{Log}[n]}{2}\right]\right)$ **/. k → 4 /. n → 32.**

$435.435 - 2.66627 \times 10^{-13}\, i$

(-2)$^{1+k}$ (Gamma[1 + k, 0, -Log[n^(1/2)]]) /. k → 4 /. n → 32.

$435.435 - 2.66627 \times 10^{-13}\, i$

```
FullSimplify[Sum[Log[j]^k / j^(1 / 2), {j, 2, n}]]
```

$$\sum_{j=2}^{n} \frac{\text{Log}[j]^k}{\sqrt{j}}$$

```
FullSimplify[Sum[Log[j]^k / j^(1 / 2), {j, 2, n}] -
  ((-2)^(1+k) (Gamma[1 + k, 0, -Log[n^(1 / 2)]])) - Log[n]^k / n^(1 / 2) / 2]
```

$$-(-2)^{1+k}\,\text{Gamma}\left[1+k,\,0,\,-\frac{\text{Log}[n]}{2}\right]-\frac{\text{Log}[n]^k}{2\sqrt{n}}+\sum_{j=2}^{n}\frac{\text{Log}[j]^k}{\sqrt{j}}$$

```
Sum[Log[j]^k / j^(1 / 2), {j, 2, n}] -
  ((-2)^(1+k) (Gamma[1 + k, 0, -Log[n^(1 / 2)]])) - Log[n]^k / n^(1 / 2) / 2
```

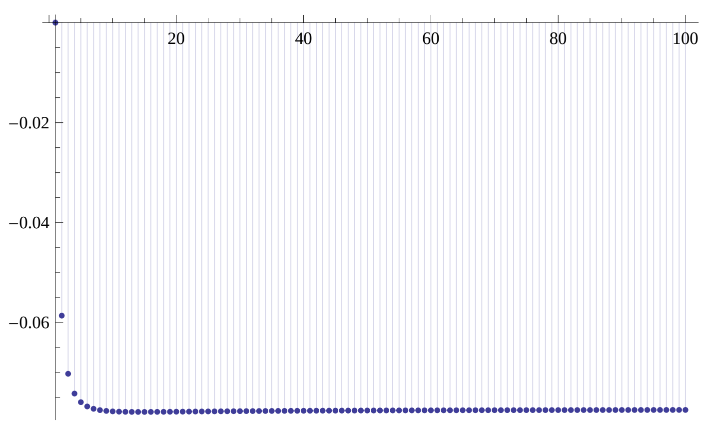
$$-(-2)^{1+k}\,\text{Gamma}\left[1+k,\,0,\,-\text{Log}\left[\sqrt{n}\right]\right]-\frac{\text{Log}[n]^k}{2\sqrt{n}}+\sum_{j=2}^{n}\frac{\text{Log}[j]^k}{\sqrt{j}}$$

$$\texttt{zn2[n\_, k\_] :=} -(-2)^{1+k}\,\text{Gamma}\left[1+k,\,0,\,-\text{Log}\left[\sqrt{n}\right]\right]-\frac{\text{Log}[n]^k}{2\sqrt{n}}+\sum_{j=2}^{n}\frac{\text{Log}[j]^k}{\sqrt{j}}$$

```
Table[zn2[100 000., k], {k, 0, 5}]
```

$\{-0.460355,\ -0.0773539 + 7.36809\times 10^{-13}\,i,\ -0.00835713 + 1.4638\times 10^{-11}\,i,$
$0.00330829 + 2.22842\times 10^{-10}\,i,\ 0.00267317 + 3.06609\times 10^{-9}\,i,\ 0.000564868 + 4.80127\times 10^{-8}\,i\}$

```
DiscretePlot[Re@zn2[n, 1], {n, 1, 100}]
```




```
Zeta[.5]
```

```
zn2[1 000 000 000., 1]
```

$-0.0773539 - 1.60509\times 10^{-18}\,i$

$$-(-2)^{1+k}\,\text{Gamma}\left[1+k,\,0,\,-\text{Log}\left[\sqrt{n}\right]\right]-\frac{\text{Log}[n]^k}{2\sqrt{n}}+\sum_{j=2}^{n}\frac{\text{Log}[j]^k}{\sqrt{j}}\ \texttt{/. k}\to 1$$

$$-4\,\text{Gamma}\left[2,\,0,\,-\text{Log}\left[\sqrt{n}\right]\right]-\frac{\text{Log}[n]}{2\sqrt{n}}-$$
$$\frac{1}{4}\,(2\,\text{EulerGamma}+\pi+\text{Log}[64]+2\,\text{Log}[\pi])\,\text{Zeta}\left[\frac{1}{2}\right]+\text{Zeta}^{(1,0)}\left[\frac{1}{2},\,1+n\right]$$

```
N@Limit[-4 Gamma[2, 0, -Log[√n]] - Log[n]/(2 √n) -
  1/4 (2 EulerGamma + π + Log[64] + 2 Log[π]) Zeta[1/2] + Zeta^(1,0)[1/2, 1 + n], n → Infinity]
```

Attributes::ssle : Symbol, string, or HoldPattern[symbol] expected at position 1 in Attributes[Zeta^(1,0)]. ≫

$$\text{Limit}\left[3.92265 - 4.\ \text{Gamma}\left[2., 0., -1.\ \text{Log}\left[\sqrt{n}\right]\right] - \frac{0.5\ \text{Log}[n]}{\sqrt{n}} + \text{Zeta}^{(1,0)}[0.5, 1. + n], n \to \infty\right]$$

```
zn2[n, 2]
```

$$8\ \text{Gamma}\left[3, 0, -\text{Log}\left[\sqrt{n}\right]\right] - \frac{\text{Log}[n]^2}{2\sqrt{n}} + \sum_{j=2}^{n} \frac{\text{Log}[j]^2}{\sqrt{j}}$$

```
zn2[1000000., 2]
```

$-0.00835703 + 7.03468 \times 10^{-11}\ i$

```
Needs["NumericalCalculus`"]
```

```
NLimit[zn2[n, 1], n → Infinity]
```

$-0.0773539 - 2.34499 \times 10^{-17}\ i$

```
NLimit[zn2[n, 2], n → Infinity]
```

$-0.00835703 + 7.69864 \times 10^{-11}\ i$

```
NLimit[zn2[n, 3], n → Infinity]
```

$0.00330975 + 3.42579 \times 10^{-9}\ i$

```
NLimit[zn2[n, 4], n → Infinity]
```

$0.00287472 + 2.65881 \times 10^{-6}\ i$

```
NLimit[zn2[n, 5], n → Infinity]
```

$0.00081609 + 7.31479 \times 10^{-6}\ i$

```
NLimit[zn2[n, 6], n → Infinity]
```

$-0.000545871 + 5.5106 \times 10^{-6}\ i$

```
NLimit[zn2[n, 7], n → Infinity]
```

$0.000575878 + 0.000119598\ i$

```
NLimit[zn2[n, 8], n → Infinity]
```

$0.0572435 + 0.00187555\ i$

```
NLimit[zn2[n, 9], n → Infinity, WorkingPrecision → 60]
```

NLimit::noise :
Cannot recognize a limiting value. This may be due to noise resulting from roundoff errors in which case higher WorkingPrecision, fewer Terms, or a different Scale might help. ≫

$$\text{NLimit}\left[-1024\ \text{Gamma}\left[10, 0, -\text{Log}\left[\sqrt{n}\right]\right] - \frac{\text{Log}[n]^9}{2\sqrt{n}} + \sum_{j=2}^{n} \frac{\text{Log}[j]^9}{\sqrt{j}},\ n \to \infty, \text{WorkingPrecision} \to 60, \text{Scale} \to .63\right]$$

$0.450829256370052948579220565643530503911331 + 0. \times 10^{-44}\ i$

```
NLimit[zn2[n, 10], n → Infinity]
```

20.8502 + 3.85348 ⅈ

```
NLimit[zn2[n, 11], n → Infinity]
```

758.238 + 15.185 ⅈ

```
NLimit[zn2[n, 12], n → Infinity]
```

NLimit::noise :
Cannot recognize a limiting value. This may be due to noise resulting from roundoff errors in which case higher WorkingPrecision, fewer Terms, or a different Scale might help. ≫

$$\mathrm{NLimit}\left[8192\,\mathrm{Gamma}\left[13,\,0,\,-\mathrm{Log}\left[\sqrt{n}\right]\right]-\frac{\mathrm{Log}[n]^{12}}{2\sqrt{n}}+\sum_{j=2}^{n}\frac{\mathrm{Log}[j]^{12}}{\sqrt{j}},\,n\to\infty\right]$$

```
NLimit[zn2[n, 13], n → Infinity]
```

NLimit::noise :
Cannot recognize a limiting value. This may be due to noise resulting from roundoff errors in which case higher WorkingPrecision, fewer Terms, or a different Scale might help. ≫

$$\mathrm{NLimit}\left[-16\,384\,\mathrm{Gamma}\left[14,\,0,\,-\mathrm{Log}\left[\sqrt{n}\right]\right]-\frac{\mathrm{Log}[n]^{13}}{2\sqrt{n}}+\sum_{j=2}^{n}\frac{\mathrm{Log}[j]^{13}}{\sqrt{j}},\,n\to\infty\right]$$

```
NLimit[zn2[n, 14], n → Infinity]
```

$Aborted

```
NLimit[zn2[n, 15], n → Infinity]
NLimit[zn2[n, 16], n → Infinity]
NLimit[zn2[n, 17], n → Infinity]
NLimit[zn2[n, 18], n → Infinity]
NLimit[zn2[n, 19], n → Infinity]
NLimit[zn2[n, 20], n → Infinity]
NLimit[zn2[n, 21], n → Infinity]
NLimit[zn2[n, 22], n → Infinity]
NLimit[zn2[n, 23], n → Infinity]
NLimit[zn2[n, 24], n → Infinity]
NLimit[zn2[n, 25], n → Infinity]
NLimit[zn2[n, 26], n → Infinity]
NLimit[zn2[n, 27], n → Infinity]
NLimit[zn2[n, 28], n → Infinity]
NLimit[zn2[n, 29], n → Infinity]
NLimit[zn2[n, 30], n → Infinity]
```

```
Sum[j^s, {j, 1, n}] - Integrate[j^s, {j, 0, n}] -
 n^s / 2 + Sum[BernoulliB[k] / k ! D[n^s, {n, k - 1}], {k, 1, 8}]
```

ConditionalExpression$\Big[$

$$-n^{s}+\frac{1}{12}n^{-1+s}s-\frac{1}{720}n^{-3+s}(-2+s)(-1+s)s+\frac{n^{-5+s}(-4+s)(-3+s)(-2+s)(-1+s)s}{30240}-\frac{n^{-7+s}(-6+s)(-5+s)(-4+s)(-3+s)(-2+s)(-1+s)s}{1209600}-\frac{n^{1+s}}{1+s}+\text{HarmonicNumber}[n,-s],\ \text{Re}[s]>-1\Big]$$

$$\texttt{zn2[n\_, k\_] :=} -(-2)^{1+k}\,\text{Gamma}\left[1+k,0,-\text{Log}\left[\sqrt{n}\right]\right]-\frac{\text{Log}[n]^{k}}{2\sqrt{n}}+\sum_{j=2}^{n}\frac{\text{Log}[j]^{k}}{\sqrt{j}}$$

```
bo2[n_, t_] :=
 Sum[Log[j]^t / j^(1 / 2), {j, 2, n}] - Integrate[Log[j]^t / j^(1 / 2), {j, 1, n}] -
  Log[n]^t / n^(1 / 2) - Sum[BernoulliB[k] / k ! D[Log[n]^t / n^(1 / 2), {n, k - 1}], {k, 1, 10}]
```

$$\texttt{bo3[n\_, t\_] := Sum[Log[j]\^{}t / j\^{}(1 / 2), \{j, 2, n\}]} - \left((-2)^{1+t}\,\text{Gamma}\left[1+t,0,-\text{Log}\left[\sqrt{n}\right]\right]\right) -$$

```
  Log[n]^t / n^(1 / 2) - Sum[BernoulliB[k] / k ! D[Log[n]^t / n^(1 / 2), {n, k - 1}], {k, 1, 10}]
```

```
N[bo3[n, 1] /. n → 1 000 000]
```

$-0.0773539 - 3.72219\times10^{-17}\,i$

```
-0.07735386079139062`

-0.07735386079366435`
-0.0773538607863884`

zn2[100 000., 3]
```

$0.00330829 + 2.22842\times10^{-10}\,i$

```
NLimit[zn2[n, 3], n → Infinity]
```

$0.00330975 + 3.42579\times10^{-9}\,i$

```
NLimit[bo[n, 3], n → Infinity]
```

0.00330878

```
NLimit[zn2[n, 11], n → Infinity]
```

$758.238 + 15.185\,i$

```
NLimit[bo[n, 11], n → Infinity]
```

-456.194

```
N[bo2[n, 11] /. n → 10 000]
```

0.00976563

```
N[bo2[n, 8] /. n → 100 000]
```

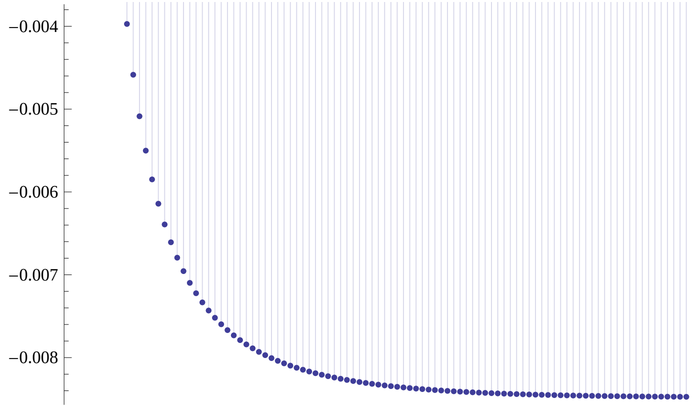
-0.000354767

```
zn2[1 000 000, 8.]
```

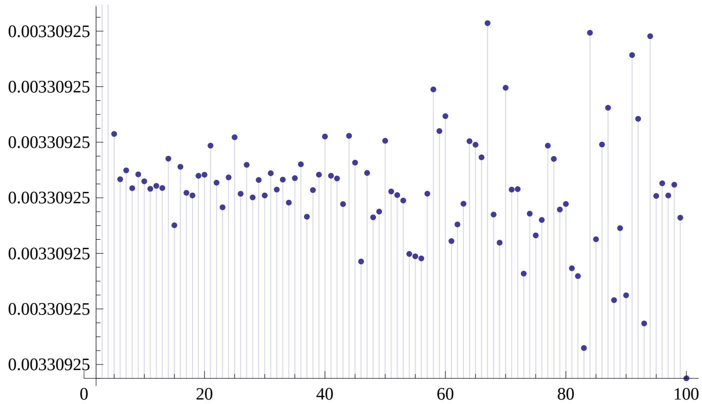
$0.00805664 + 0.00130568\,i$

```
NLimit[zn2[n, 2], n → Infinity]
```

```
DiscretePlot[Re@zn2[n, 2], {n, 1, 100}]
```




```
DiscretePlot[Re@bd[n], {n, 2, 100}]
```




```
FullSimplify@bo[n, 2]
```

$$\text{ConditionalExpression}\Big[\frac{1}{595137134592000\,n^{25/2}}\Big(-16\,\big(22670409511051 + 10920\,n^2\,\big(-607817345 + 352\,n^2\,\big(731043 + 40\,n^2\,\big(-12139 + 336\,n^2\,\big(43 + 120\,n^2\,\big(-1 - 96\,n^{5/2} + 96\,n^3\big)\big)\big)\big)\big)\big) + 105\,\text{Log}[n]\,\big(64\,\big(32146869913 + 260\,n^2\,\big(-38797345 + 4224\,n^2\,\big(4269 - 3254\,n^2 + 4928\,n^4 - 26880\,n^6 - 80640\,n^7 + 645120\,n^9\big)\big)\big) - 45045\,\big(5133439 + 80\,n^2\,\big(-20995 + 8\,n^2\,\big(1287 + 32\,n^2\,\big(-33 + 8\,n^2\,\big(7 + 48\,n^2\,(1 + 4\,n)\,\big(-1 + 8\,n^2\big)\big)\big)\big)\big)\big)\,\text{Log}[n]\big) + 595137134592000\,n^{25/2}\sum_{j=2}^{n}\frac{\text{Log}[j]^2}{\sqrt{j}}\Big),\ \text{Re}[n] \geq 0\ ||\ n \notin \text{Reals}\Big]$$

$$\textbf{bb[n\_] :=} \frac{1}{595137134592000\, n^{25/2}} \Bigg(-16 \Big(22670409511051 + 10920\, n^2 \big(-607817345 + 352\, n^2 \big(731043 + 40\, n^2 \big(-12139 + 336\, n^2 \big(43 + 120\, n^2 \big(-1 - 96\, n^{5/2} + 96\, n^3\big)\big)\big)\big)\big)\Big) + 105\,\mathrm{Log}[n] \Big(64 \big(32146869913 + 260\, n^2 \big(-38797345 + 4224\, n^2 \big(4269 - 3254\, n^2 + 4928\, n^4 - 26880\, n^6 - 80640\, n^7 + 645120\, n^9\big)\big)\big) - 45045 \big(5133439 + 80\, n^2 \big(-20995 + 8\, n^2 \big(1287 + 32\, n^2 \big(-33 + 8\, n^2 \big(7 + 48\, n^2 (1 + 4\, n) \big(-1 + 8\, n^2\big)\big)\big)\big)\big)\big) \mathrm{Log}[n]\Big) + 595137134592000\, n^{25/2} \sum_{j=2}^{n} \frac{\mathrm{Log}[j]^2}{\sqrt{j}}\Bigg)$$

FullSimplify@bo2[n, 2]

$$\mathrm{ConditionalExpression}\Bigg[\frac{1}{297568567296000\, n^{23/2}} \Bigg(-8 \Big(1871243360239 + 87360\, n^2 \big(-7473495 + 22\, n^2 \big(177331 + 320\, n^2 \big(-475 + 756 \big(n^2 - 1280\, n^{11/2} + 1280\, n^6\big)\big)\big)\big)\Big) + 105\,\mathrm{Log}[n] \Big(4 \big(21925921301 + 208\, n^2 \big(-39831725 + 264\, n^2 \big(88069 + 160\, n^2 \big(-563 + 56\, n^2 \big(23 + 240\, n^2 \big(-1 + 48\, n^2\big)\big)\big)\big)\big)\big) - 45045 \big(223193 + 16\, n^2 \big(-5525 + 8\, n^2 \big(429 + 160\, n^2 \big(-3 + 8\, n^2 (1 + 4\, n) \big(1 - 4\, n + 192\, n^3\big)\big)\big)\big)\big) \mathrm{Log}[n]\Big) + 297568567296000\, n^{23/2} \sum_{j=2}^{n} \frac{\mathrm{Log}[j]^2}{\sqrt{j}}\Bigg),\ \mathrm{Re}[n] \ge 0 \,||\, n \notin \mathrm{Reals}\Bigg]$$

$$\textbf{bc[n\_] :=} \frac{1}{297568567296000\, n^{23/2}} \Bigg(-8 \Big(1871243360239 + 87360\, n^2 \big(-7473495 + 22\, n^2 \big(177331 + 320\, n^2 \big(-475 + 756 \big(n^2 - 1280\, n^{11/2} + 1280\, n^6\big)\big)\big)\big)\Big) + 105\,\mathrm{Log}[n] \Big(4 \big(21925921301 + 208\, n^2 \big(-39831725 + 264\, n^2 \big(88069 + 160\, n^2 \big(-563 + 56\, n^2 \big(23 + 240\, n^2 \big(-1 + 48\, n^2\big)\big)\big)\big)\big)\big) - 45045 \big(223193 + 16\, n^2 \big(-5525 + 8\, n^2 \big(429 + 160\, n^2 \big(-3 + 8\, n^2 (1 + 4\, n) \big(1 - 4\, n + 192\, n^3\big)\big)\big)\big)\big) \mathrm{Log}[n]\Big) + 297568567296000\, n^{23/2} \sum_{j=2}^{n} \frac{\mathrm{Log}[j]^2}{\sqrt{j}}\Bigg)$$

```
FullSimplify@bo2[n, 3]
```

$$
\begin{aligned}
&\text{ConditionalExpression}\Bigg[\\
&\frac{1}{2\,425\,103\,265\,699\,861\,626\,880\,000\, n^{39/2}}\Bigg(16\,\big(32\,572\,022\,259\,617\,356\,906\,848\,633 + 1408\\
&\quad n^2\,\big(-2\,546\,829\,381\,552\,005\,487\,765 + 133\, n^2\,\big(2\,620\,371\,801\,645\,139\,247 +\\
&\quad\quad 141\,440\, n^2\,\big(-3\,229\,899\,546\,685 + 4\, n^2\,\big(184\,827\,808\,649 + 2240\, n^2\,\big(-25\,713\,055 +\\
&\quad\quad\quad 198\, n^2\,\big(57\,379 + 320\, n^2\,\big(-115 + 84\,\big(n^2 - 11\,520\, n^{11/2} + 11\,520\, n^6\big)\big)\big)\big)\big)\big)\big)\big)\big) +\\
&\quad 5\,\text{Log}[n]\,\big(-8\,\big(11\,884\,150\,070\,640\,380\,526\,438\,893 + 1408\, n^2\,\big(-971\,742\,778\,563\,097\,465\,585 +\\
&\quad\quad 399\, n^2\,\big(351\,548\,638\,800\,930\,241 +\\
&\quad\quad\quad 2176\, n^2\,\big(-30\,074\,857\,160\,075 + 4\, n^2\,\big(1\,871\,243\,360\,239 + 87\,360\, n^2\\
&\quad\quad\quad\quad \big(-7\,473\,495 + 22\, n^2\,\big(177\,331 + 320\, n^2\,\big(-475 + 756\,\big(n^2 + 1280\, n^6\big)\big)\big)\big)\big)\big)\big)\big)\big)\big) +\\
&\quad 3465\,\text{Log}[n]\,\big(2\,\big(3\,505\,722\,679\,891\,501\,697\,411 + 16\, n^2\,\big(-26\,112\,968\,662\,831\,984\,945 +\\
&\quad\quad 1064\, n^2\,\big(3\,689\,649\,641\,824\,737 + 1088\, n^2\\
&\quad\quad\quad \big(-663\,023\,777\,675 + 8\, n^2\,\big(21\,925\,921\,301 + 208\, n^2\,\big(-39\,831\,725 + 264\, n^2\\
&\quad\quad\quad\quad \big(88\,069 + 160\, n^2\,\big(-563 + 56\, n^2\,\big(23 + 240\, n^2\,\big(-1 + 48\, n^2\big)\big)\big)\big)\big)\big)\big)\big)\big)\big) -\\
&\quad\quad 4\,849\,845\,\big(98\,185\,688\,640\,165 + 16\, n^2\,\big(-748\,306\,316\,175 + 56\, n^2\,\big(2\,062\,720\,845 +\\
&\quad\quad\quad 64\, n^2\,\big(-6\,500\,375 + 8\, n^2\,\big(223\,193 + 16\, n^2\,\big(-5525 + 8\, n^2\,\big(429 +\\
&\quad\quad\quad\quad 160\, n^2\,\big(-3 + 8\, n^2\,(1 + 4\, n)\,\big(1 - 4\, n + 192\, n^3\big)\big)\big)\big)\big)\big)\big)\big)\big)\big)\,\text{Log}[n]\big)\big) +\\
&\quad 2\,425\,103\,265\,699\,861\,626\,880\,000\, n^{39/2}\sum_{j=2}^{n}\frac{\text{Log}[j]^3}{\sqrt{j}}\Bigg),\ \text{Re}[n] \geq 0\ ||\ n \notin\\
&\quad \text{Reals}\Bigg]
\end{aligned}
$$

$$\text{bd}[n\_] := \frac{1}{2\,425\,103\,265\,699\,861\,626\,880\,000\, n^{39/2}} \Big(16\,\big(32\,572\,022\,259\,617\,356\,906\,848\,633 + 1408\, n^2\,\big(-2\,546\,829\,381\,552\,005\,487\,765 + 133\, n^2\,\big(2\,620\,371\,801\,645\,139\,247 + 141\,440\, n^2\,\big(-3\,229\,899\,546\,685 + 4\, n^2\,\big(184\,827\,808\,649 + 2240\, n^2\,\big(-25\,713\,055 + 198\, n^2\,\big(57\,379 + 320\, n^2\,\big(-115 + 84\,\big(n^2 - 11\,520\, n^{11/2} + 11\,520\, n^6\big)\big)\big)\big)\big)\big)\big)\big)\big) + 5\,\text{Log}[n]\,\big(-8\,\big(11\,884\,150\,070\,640\,380\,526\,438\,893 + 1408\, n^2\,\big(-971\,742\,778\,563\,097\,465\,585 + 399\, n^2\,\big(351\,548\,638\,800\,930\,241 + 2176\, n^2\,\big(-30\,074\,857\,160\,075 + 4\, n^2\,\big(1\,871\,243\,360\,239 + 87\,360\, n^2\,\big(-7\,473\,495 + 22\, n^2\,\big(177\,331 + 320\, n^2\,\big(-475 + 756\,\big(n^2 + 1280\, n^6\big)\big)\big)\big)\big)\big)\big)\big)\big)\big) + 3465\,\text{Log}[n]\,\big(2\,\big(3\,505\,722\,679\,891\,501\,697\,411 + 16\, n^2\,\big(-26\,112\,968\,662\,831\,984\,945 + 1064\, n^2\,\big(3\,689\,649\,641\,824\,737 + 1088\, n^2\,\big(-663\,023\,777\,675 + 8\, n^2\,\big(21\,925\,921\,301 + 208\, n^2\,\big(-39\,831\,725 + 264\, n^2\,\big(88\,069 + 160\, n^2\,\big(-563 + 56\, n^2\,\big(23 + 240\, n^2\,\big(-1 + 48\, n^2\big)\big)\big)\big)\big)\big)\big)\big)\big)\big)\big) - 4\,849\,845\,\big(98\,185\,688\,640\,165 + 16\, n^2\,\big(-748\,306\,316\,175 + 56\, n^2\,\big(2\,062\,720\,845 + 64\, n^2\,\big(-6\,500\,375 + 8\, n^2\,\big(223\,193 + 16\, n^2\,\big(-5525 + 8\, n^2\,\big(429 + 160\, n^2\,\big(-3 + 8\, n^2\,(1 + 4\, n)\,\big(1 - 4\, n + 192\, n^3\big)\big)\big)\big)\big)\big)\big)\big)\big)\,\text{Log}[n]\big)\big) + 2\,425\,103\,265\,699\,861\,626\,880\,000\, n^{39/2} \sum_{j=2}^{n} \frac{\text{Log}[j]^3}{\sqrt{j}}\Big)$$

E^-0.0773538607863884

0.925562

bo2[n, 1]

$$\text{ConditionalExpression}\Big[-4 - 2\sqrt{n}\,(-2 + \text{Log}[n]) - \frac{\text{Log}[n]}{2\sqrt{n}} + \frac{-\frac{71\,697\,105}{256\, n^{19/2}} + \frac{34\,459\,425\,\text{Log}[n]}{512\, n^{19/2}}}{47\,900\,160} + \frac{\frac{264\,207}{64\, n^{15/2}} - \frac{135\,135\,\text{Log}[n]}{128\, n^{15/2}}}{1\,209\,600} + \frac{-\frac{1689}{16\, n^{11/2}} + \frac{945\,\text{Log}[n]}{32\, n^{11/2}}}{30\,240} + \frac{1}{720}\left(\frac{23}{4\, n^{7/2}} - \frac{15\,\text{Log}[n]}{8\, n^{7/2}}\right) + \frac{1}{12}\left(-\frac{1}{n^{3/2}} + \frac{\text{Log}[n]}{2\, n^{3/2}}\right) - \frac{1}{4}\,(2\,\text{EulerGamma} + \pi + \text{Log}[64] + 2\,\text{Log}[\pi])\,\text{Zeta}\Big[\frac{1}{2}\Big] + \text{Zeta}^{(1,0)}\Big[\frac{1}{2}, 1 + n\Big],\ \text{Re}[n] \geq 0\ ||\ n \notin \text{Reals}\Big]$$

$$\left(\frac{1}{2} - s\right)\left(1 - \left(\frac{2\,i}{i - 2\,s\,/\,I}\right) + \text{Sum}[(s)\text{^}k\,/\,k!\ \text{bbo3}[n, k], \{k, 0, l\}]\right)\ /.\ s \to s - 1/2$$

$$(1 - s)\left(1 - \frac{1}{s} + \sum_{k=0}^{l} \frac{\left(-\frac{1}{2} + s\right)^k \text{bbo3}[n, k]}{k!}\right)$$

```
bo3[n_, t_] := Sum[Log[j]^t / j^(1 / 2), {j, 2, n}] -
  ((-2)^(1+t) Gamma[1 + t, 0, -Log[Sqrt[n]]]) - Log[n]^t / n^(1 / 2) -
  Sum[BernoulliB[k] / k! D[Log[n2]^t / n^(1 / 2), {n2, k - 1}] /. n2 → n, {k, 1, 10}]
s15a[n_, s_, l_] := 1 - 1 / (1 - s) + Sum[(1/2 - s)^k / k! bo3[n, k], {k, 0, l}]
```

```
s15a[10, .500001, 200]
```

$-1.46168 - 1.44844 \times 10^{-21}\, i$

```
Zeta[s] /. s → .500001
```

-1.46036

```
FullSimplify[Integrate[(Log[j])^k / j^(1 / 2), {j, 0, n}], Element[k, Integers]]
```

ConditionalExpression$\left[(-1)^k 2^{1+k} \text{Gamma}\left[1+k, -\frac{\text{Log}[n]}{2}\right], k > -1\right]$

```
FullSimplify[Integrate[(Log[j])^k / j^(1 / 2), {j, 1, n}], Element[k, Integers]]
```

ConditionalExpression$\left[(-1)^k 2^{1+k} \left(-k! + \text{Gamma}\left[1+k, -\frac{\text{Log}[n]}{2}\right]\right), k \geq 0 \,\&\&\, \text{Log}[n] > 0\right]$

$(-1)^k 2^{1+k} \left(-k! + \text{Gamma}\left[1+k, -\frac{\text{Log}[n]}{2}\right]\right)$ /. n → 100. /. k → 3

$754.862 - 3.69776 \times 10^{-13}\, i$

$(-1)^{k+1} 2^{1+k} \left(\text{Gamma}\left[1+k, 0, -\frac{\text{Log}[n]}{2}\right]\right)$ /. n → 100. /. k → 2

$199.738 - 7.33826 \times 10^{-14}\, i$

$(-2)^{1+k} \left(\text{Gamma}\left[1+k, 0, -\frac{\text{Log}[n]}{2}\right]\right)$ /. n → 100. /. k → 3

$754.862 - 3.69776 \times 10^{-13}\, i$

$(-2)^{1+k}$ (Gamma[1 + k, 0, Log[1 / n^(1 / 2)]]) /. n → 100. /. k → 3

$754.862 - 3.69776 \times 10^{-13}\, i$

```
br[k_] := D[Zeta[1 / 2 - s] - 1 + 1 / (1 / 2 + s), {s, k}] /. s → 0
```

Table[{k, N[FullSimplify[br[k]]]}, {k, 0, 20}] // TableForm

0	-0.460355
1	-0.0773539
2	-0.00835701
3	0.00330925
4	0.00268028
5	0.000600662
6	-0.000546537
7	-0.000702627
8	-0.000337601
9	0.000105381
10	0.000360489
11	0.000335693
12	0.
13	0.
14	0.
15	0.
16	0.
17	0.
18	0.
19	0.
20	0.

```
bo4[k_] := D[Zeta[1/2 - s] - 1 + 1/(1/2 + s), {s, k}] /. s → 0
s16[s_, l_] := 1 - 1/(1/2 + s) + Sum[s^k/k! bo4[k], {k, 0, l}]
s17[s_] := 1 - (1/(1/2 - s)) + Sum[s^k/k! bo4[k], {k, 0, Infinity}]
```

$MaxExtraPrecision = 500

500

N[s16[1/10 + 16 I, 70], 100]

0.9218819970008730243451677330022114094459566304567690837478634134011003572018268165217 6 9431551351908 - 1.3236511927061844668711008581115520429140761428998284219539460529970159086798963033718 31077097092885 ⅈ

Zeta[.5 - (.1 + 16 I)]

0.921882 - 1.32365 ⅈ

N[s16[Im@ZetaZero@1 I, 100], 120]

-8.5939786072054024919539719592717108795005717843121713438372869531696345259933533383418 4684899911438944929780417220955 11 $\times 10^{-27}$ + 4.1291589952135603156685701835464121906432484726952455835715537018655206097809657951050 9414645337584626725585827502962179 $\times 10^{-26}$ ⅈ

Zeta[.5 - (30 I)]

-0.120642 + 0.583691 ⅈ

```
Integrate[x^z / x^(1 / 2), {x, 0, 1}]
```

$\text{ConditionalExpression}\left[\frac{2}{1+2z}, \text{Re}[z] > -\frac{1}{2}\right]$

```
sa1[n_, s_] := Sum[j^s / j^(1 / 2), {j, 1, n}] - Integrate[j^s / j^(1 / 2), {j, 0, n}]
```

$\text{sa2}[n\_, s\_] := 1 - \frac{1}{1/2+s} + \text{Sum}[j\text{^}s / j\text{^}(1/2), \{j, 2, n\}] - \text{Integrate}[j\text{^}s / j\text{^}(1/2), \{j, 1, n\}]$

```
sa2[100000., .2]
```

-0.888745

```
Zeta[.5 - .2]
```

-0.904559

```
sa1[100000., .2]
```

-0.886481

```
Integrate[j^s / j^(1 / 2), {j, 0, 1}]
```

$\text{ConditionalExpression}\left[\frac{2}{1+2s}, \text{Re}[s] > -\frac{1}{2}\right]$

$N\left[1 - \frac{1}{1/2+s} \text{ /. } s \to \text{Im@ZetaZero@1 I}\right]$

0.997501 + 0.0706593 i

```
Integrate[Cos[s I Log[x]] / x^(1 / 2), {x, 0, 1}]
```

$\text{ConditionalExpression}\left[\frac{2}{1-4s^2}, -\frac{1}{2} < s < \frac{1}{2}\right]$

```
Integrate[Sin[(s) Log[x]] / x^(1 / 2), {x, 0, 1}]
```

$\text{ConditionalExpression}\left[-\frac{4s}{1+4s^2}, -\frac{1}{2} < \text{Im}[s] < \frac{1}{2}\right]$

```
Integrate[Sin[s Log[x] + ArcTan[2 s]] / x^(1 / 2), {x, 0, 1}]
```

$\text{ConditionalExpression}\left[0, -\frac{1}{2} < \text{Im}[s] < \frac{1}{2}\right]$

$$\texttt{bo4[k\_] := D}\left[\texttt{Zeta[1/2 - s] - 1 + }\frac{1}{1/2+s}\texttt{, \{s, k\}}\right]\texttt{ /. s} \to \texttt{0}$$

$$\texttt{ex[s\_, l\_] := 1 - }\frac{1}{1/2+s}\texttt{ + Sum[s\^{}k / k! bo4[k], \{k, 0, l\}]}$$

$$\texttt{cs[s\_, l\_] := 1 - }\frac{2}{1+4s^2}\texttt{ + Sum[(-1)\^{}k s\^{}(2 k) / ((2 k)!) bo4[2 k], \{k, 0, Floor[l / 2]\}]}$$

$$\texttt{sn[s\_, l\_] := }\frac{4s}{1+4s^2}\texttt{ + Sum[(-1)\^{}k s\^{}(2 k + 1) / ((2 k + 1)!) bo4[2 k + 1], \{k, 0, Floor[l / 2]\}]}$$

```
exr[s_] := Zeta[1 / 2 - s]
csr[s_] := (Zeta[1 / 2 - s I ] + Zeta[1 / 2 + s I ]) / 2
snr[s_] := (Zeta[1 / 2 - s I] - Zeta[1 / 2 + s I ]) / (2 I)
```

$$\texttt{exd[s\_] := Zeta[1/2 - s] - }\left(1-\frac{1}{1/2+s}\right)$$

$$\texttt{csd[s\_] := (Zeta[1/2 - s I] + Zeta[1/2 + s I]) / 2 - }\left(1-\frac{2}{1+4s^2}\right)$$

$$\texttt{snd[s\_] := (Zeta[1/2 - s I] - Zeta[1/2 + s I]) / (2 I) - }\left(\frac{4s}{1+4s^2}\right)$$

```
N[sn[14, 100], 100]
```

0.1032581232664500579023630952954409749087096055462790296425727172916244189547968076883762335440600151

```
snr[14.]
```

0.103258 + 0. i

$$\texttt{FullSimplify}\left[\frac{4 i s}{-1+4s^2}\right]$$

$$\frac{4 i s}{-1+4s^2}$$

```
N[sn[Im@ZetaZero@1, 100], 100]
```

$-6.94623715230812265951780970543963259784066947126405781358140341560183628642421537789113646050584903 \times 10^{-28}$

```
N[cs[Im@ZetaZero@1, 100], 100]
```

$-8.59397860720540249195397195927171087950057178431217134383728695316963452599335333834184684899911438 9 \times 10^{-27}$

```
N[ex[5 / 2, 50], 50]
```

$7.1503225982775753754183080074996625489028301530513 \times 10^{-44}$

```
exa[s_, l_] := Sum[s^k / k! bo4[k], {k, 0, l}]
exa2[s_, l_] := Table[s^k / k! bo4[k], {k, 0, l}]
sna[s_, l_] := Sum[(-1)^k s^(2 k + 1) / ((2 k + 1)!) bo4[2 k + 1], {k, 0, Floor[l / 2]}]
csa[s_, l_] := Sum[(-1)^k s^(2 k) / ((2 k)!) bo4[2 k], {k, 0, Floor[l / 2]}]
tana[s_, l_] := Sum[(D[Tan[x], {x, k}] /. x → 0) s^k / (k!) bo4[k], {k, 0, l}]
```

```
tana[2.5, 30]
```

-0.111743

```
1 - 1 / (1 / 2 + (3.5))
```

0.75

```
-7. / 12
```

−0.583333

```
Zeta[-3] - 3 / 4
```

$$-\frac{89}{120}$$

```
Expand@FullSimplify[snd[1]]
```

$$-\frac{4}{5}-\frac{1}{2}\,i\,\text{Zeta}\left[\frac{1}{2}-i\right]+\frac{1}{2}\,i\,\text{Zeta}\left[\frac{1}{2}+i\right]$$

```
Table[D[Tan[x], {x, k}] /. x → 0, {k, 0, 10}]
```

{0, 1, 0, 2, 0, 16, 0, 272, 0, 7936, 0}

```
snt[s_, l_] := Flatten[{N[4 s / (1 + 4 s^2)], Table[
    N[(-1) ^k s^(2 k + 1) / ((2 k + 1) !) bo4[2 k + 1], 70], {k, 0, Floor[l / 2]}]}] // TableForm
sant[s_, l_] := Table[N[s^(2 k + 1), 70], {k, 0, Floor[l / 2]}] // TableForm
sant2[s_, l_] :=
 Table[N[(-1) ^k bo4[2 k + 1] / ((2 k + 1) !), 70], {k, 0, Floor[l / 2]}] // TableForm
```

```
snt[Im@ZetaZero@1, 100]
```

```
snt[Im@ZetaZero@1 + (1 / 20) I, 100]
```

```
sant2[Im@ZetaZero@1 + (1 / 20) I, 100]
```

```
sot[s_, k_] := N[(-1) ^k s^(2 k + 1) / ((2 k + 1) !) bo4[2 k + 1], 70]
```

```
Series[Sin[x + a], {x, 0, 10}]
```

$$\text{Sin}[a]+\text{Cos}[a]\,x-\frac{1}{2}\,\text{Sin}[a]\,x^2-\frac{1}{6}\,\text{Cos}[a]\,x^3+\frac{1}{24}\,\text{Sin}[a]\,x^4+\frac{1}{120}\,\text{Cos}[a]\,x^5-\frac{1}{720}\,\text{Sin}[a]\,x^6-\frac{\text{Cos}[a]\,x^7}{5040}+\frac{\text{Sin}[a]\,x^8}{40\,320}+\frac{\text{Cos}[a]\,x^9}{362\,880}-\frac{\text{Sin}[a]\,x^{10}}{3\,628\,800}+O[x]^{11}$$

```
Table[Sin[a - k Pi / 2], {k, 0, 10}] /. a → 0
```

{0, −1, 0, 1, 0, −1, 0, 1, 0, −1, 0}

```
Sin[ArcTan[2 s]]
```

$$\frac{2\,s}{\sqrt{1+4\,s^2}}$$

```
Cos[ArcTan[2 s]]
```

$$\frac{1}{\sqrt{1+4\,s^2}}$$

```
Integrate[Cos[s Log[x] + a] / x^(1 / 2), {x, 0, 1}]
```

$$\frac{2\,(\text{Cos}[a]+2\,s\,\text{Sin}[a])}{1+4\,s^2}$$

$$\texttt{bo4[k\_] := D}\left[\texttt{Zeta[1 / 2 - s] - 1 +} \frac{1}{1/2+s}\texttt{, \{s, k\}}\right] \texttt{ /. s} \to \texttt{0}$$

$$\texttt{ex[s\_, l\_] := 1 -} \frac{1}{1/2+s} \texttt{ + Sum[s\^{}k / k! bo4[k], \{k, 0, l\}]}$$

$$\texttt{cs[s\_, l\_] := 1 -} \frac{2}{1+4\,s^2} \texttt{ + Sum[(-1)\^{}k s\^{}(2 k) / ((2 k)!) bo4[2 k], \{k, 0, Floor[l / 2]\}]}$$

$$\texttt{sn[s\_, l\_] :=} \frac{4\,s}{1+4\,s^2} \texttt{ + Sum[(-1)\^{}k s\^{}(2 k + 1) / ((2 k + 1)!) bo4[2 k + 1], \{k, 0, Floor[l / 2]\}]}$$

$$\texttt{asn[s\_, l\_, a\_] :=}$$
$$\texttt{Cos[a] -} \frac{2\,(\texttt{Cos[a]} + 2\,s\,\texttt{Sin[a]})}{1+4\,s^2} \texttt{ + Sum[Cos[a + k Pi / 2] s\^{}k / (k!) bo4[k], \{k, 0, l\}]}$$

$$\texttt{arcn[s\_, l\_] :=} \frac{1}{\sqrt{1+4\,s^2}} \texttt{ + Sum[Sin[ArcTan[2 s] + k Pi / 2] s\^{}k / (k!) bo4[k], \{k, 0, l\}]}$$

```
exr[s_] := Zeta[1 / 2 - s]
csr[s_] := (Zeta[1 / 2 - s I ] + Zeta[1 / 2 + s I ]) / 2
snr[s_] := (Zeta[1 / 2 - s I] - Zeta[1 / 2 + s I ]) / (2 I)
```

```
Cos[s Log[1] + a] / 1^(1 / 2)
```

Cos[a]

```
asn[3., 30, .3 I]
```

0.55689 - 0.0240256 i

```
Cos[.3 I] cs[3., 30] - Sin[.3 I] sn[3., 30]
```

0.55689 - 0.0240256 i

$$\texttt{Sin[a] -} \frac{-s\,\texttt{Cos[a]} + \texttt{Sin[a]}}{1+s^2} \texttt{ /. a} \to \texttt{0}$$

$$\frac{s}{1+s^2}$$

$$\texttt{FullSimplify}\left[\texttt{Sin[a] -} \frac{2\,(-2\,s\,\texttt{Cos[a]} + \texttt{Sin[a]})}{1+4\,s^2} \texttt{ /. a} \to \texttt{-ArcTan}\left[\frac{1}{4\,s} - s\right] \texttt{- Pi / 2}\right]$$

0

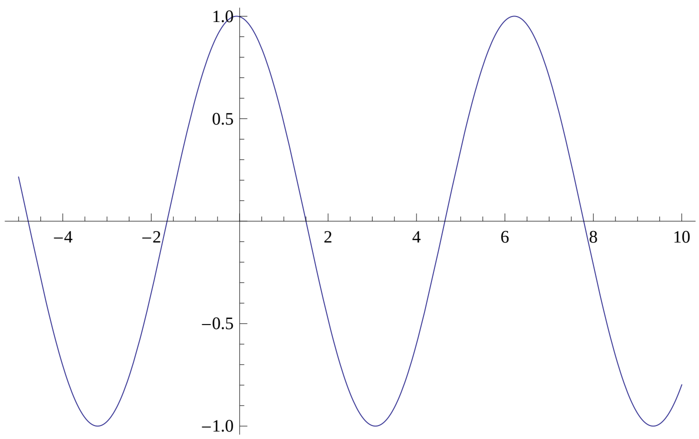
```
arcnt[s_, l_] := Flatten[
   {Table[Sin[ArcTan[(4 s^2 - 1) / (4 s)] + k Pi / 2] s^k / (k!) bo4[k], {k, 0, l}]}] // TableForm
N[arcnt[Im@ZetaZero@1, 70], 100]
```

```
-0.4592038552850402002687492390332820849046744214246302173817348967944218203219523173774388
-0.0772571880192635663500017501996270307244258562487576419466119533537818560384883943116939
0.8327391644804705999532009861380395336657508853464738364519636861893134809336217691689677
-0.1100548916748698815728444063269818679201009731517266718726880389320884749522869485325748
4.4466346425513413433621004796450441644787472847930124941365460639064844109355615682534098
0.1995516547110514459020373419972482472254381774485456799783714516442298729428602023710149
6.0384401764925590570183258280870321045042764811615519948036112707337979773187750137181694
1.1103890334552129816591647083022925033521322929685484155295299814966188401634243219814224
-13.307496154422527185850321975028304017249585248143921349434859246547116387700172782786601
0.4621909069730663416190113550145962101522817998627143177514724680530108351821173163901148
-31.513310770527057003176247095026619961836791723055360150980388130404795106899078595977857
-2.7497697605526084526293094425532334721975543298825118468574226512523871001879952364242071
```

```
16.022598408263413795038809643072916604002022524389021084649745163472994231170104318793318
-1.749422839682394592603480027008434148249090210193464432410958814503461489982299116014633
54.435578555718600972224542205733464252648531832361337364000163215982598752774326594224383
3.61202429478724956968211557072997466986518658722376135899773189128343289361210848167598981
-16.510574651996578900217285491276019615815146510290566929098069945505289104496315457298161
1.80152787416700054005638408446591958281134590569360146634458978061324505997928753521639787
-48.824909349232730668512219928384830244114170813508826511309638604187590454728560397224128
-3.06018380876169479560617270557576320225020298911163635645079049125023469248906348015157
17.47245170709799448859694914468007122261715959251077579369756660248111439412812359522524181
-0.708153328637055430172782334554315675286169332633543681321611545426939466265972075902740
24.825777372696508466384773979272952838374960366419452847101788834353249621015134319566824
1.68001364900740357691946174429929285087524926492501701823532122611911889005180561733798373
-12.697498071449673352904343766120710441812415888513107995233588084494745518628423031078798
-0.0412045493004202768481153071812857480402746868209102125115516624841004515490779886657760
-6.7078595067597362802106385229221189707899328934488345541870341300662833823955700268067689
-0.566739747327456930004749076254744139682881153850712531152619112129786334692525452846074
5.43755265893706025017964053431480560769194162445151198696099222482462734480380220988961580
0.13758836549318202264476447325025636044592520109859662825389455543387032367413527984584347
0.54442187022594870534744751299708758110599705956864448782736717910403789817491490116048409
0.10616409452932128454316188639506067516141955710264557919329816391096632848871806160592271
-1.339100520848578889521258020571081843362705292686967187195193885867882372738418814873589
-0.05260596016783529476570620553311486896179671921422802711711134227620579818506408535418
0.20046173859858181758775740121687185105320653663851285837332596634181777902359573988709509
-0.006932029728944249565556982149449564600338962418999014612631649360921789619463823284286
0.17610594633894654625905469471936989683029161419521901845099244276507023358927629941574032
0.00966450462425537208623277616552158465634642434887409253925391401022483548989445004824505
-0.068184098887164685603273104261449446990515682784977080471330765187916693842395181436259
-0.001212138356965194490473938925410072569841655612413968858702902082370563872182582004493
-0.007160963881604301543799226348010833581384214678770421038275355916881193789201351902444
-0.000876000068738644927813373867772966513771317977350110359154081057022386855382759047876
0.00909940577967396287538361077088724280851872701963845437950570772364232060715696353755637
0.00031348729928249688689052163174707778036748199841021703646643800413504597105762081331151
-0.001222587352877254285647648756512473887600447475759786912721323593131394024978311259979
0.0000153523448418725269588223626813587424208666213131820289918944040787095050111644432562
-0.000540331980342557922455305496748641941633297605824997827311012817027473442953090932577
-0.00002895110259897659040219354102399922892015204235570597257318855065127008301805104822
0.0002043447771771558490044381290786370216823735006467306901989070497856059857639440663967
4.6463463642584519101139499962797345884768528495934969914741500762781045369372724608959426
-2.547476397628985329891202446142113147716752101875459228075953646782736887357139106082701
1.01306323801731880175430247188234810711101239172558778544967000440875307047743755870589190
-0.000012360097058149483254086025222320748697663164883246820481988958830193280187731014101
-4.699521140991757464327104091778013644702233223527282659507216508987008364202396902063955
2.46159373582410015884713507386365137184401366881841629445498401780402612821714679440110091
3.21785230601012600505859281316302541008231494204025596900724079530585923214150421941082499
1.87049751278648398090633418460418124779836447610218473268610740516916667320432910869885525
1.72769736124474629698020736576526424349178934776887112971975986431693764858573869068912419
-1.486389091806213221263840951638552737791953829068139148825091428561228770737513119823380
-4.475935944177638663364139897998271212350604752106326683551110601426736004816236181837695
1.77944614399943229707065364876451626423056395691213497811732379514306393834007191737633327
6.06804891557486985310322878650592373218727317932191615595760746294968026744997682485010917
2.80150595590245265719488929459235911691371760910692115082657566266716454276995510070616965
1.57358350134853628532846465695052531148674555972799799782236186779592353380786816088904327
-1.099444831189142337816962360226105608662935040609794603788345357172062477059025079462429
```

```
-2.76834161423900382287945901886250781399654103209548230370952552369454493245560406636318O
8.42786722571182544978430562658322765516685525267728940733207025050646329596606666646382992
-5.85187228386375604371716504502956998410757065548210579895529315348586864691703427109846
2.01786145431487565859852964581268906530617626636876009616974573642212981037794866550557952
8.97757081092993887527340534466996778112695544886607004049155228475214941104159042973745071
-5.44291245586780741680250946652193561171231585789723888617476119588739957437731776967186
```

Plot[$\frac{(-1+4\,s\char`^2)\ \mathrm{Cos}[a] - 4\,s\ \mathrm{Sin}[a]}{1+4\,s^2}$ /. s → Im@ZetaZero@1, {a, -5, 10}]



FullSimplify[$\frac{(-1+4\,s\char`^2)\ \mathrm{Cos}[a] - 4\,s\ \mathrm{Sin}[a]}{1+4\,s^2}$ /. a → ArcTan[(4 s^2 - 1) / (4 s)]]

0

FullSimplify[ArcTan[(4 s^2 - 1) / (4 s)]]

-ArcTan[$\frac{1}{4\,s}$ - s]

N[-ArcTan[$\frac{1}{4\,s}$ - s] /. s → Im@ZetaZero@4]

1.53793

FullSimplify[Cos[ArcTan[(4 s^2 - 1) / (4 s)]]]

$$\frac{4}{\sqrt{8+\frac{1}{s^2}+16\,s^2}}$$

ArcTan[(4 s^2 - 1) / (4 s)] /. s -> Im@ZetaZero@1

ArcTan[$\frac{-1+4\,\mathrm{Im}[\mathrm{ZetaZero}[1]]^2}{4\,\mathrm{Im}[\mathrm{ZetaZero}[1]]}$]

N[asn[Im@ZetaZero@1, 100, ArcTan[$\frac{-1+4\,\mathrm{Im}[\mathrm{ZetaZero}[1]]^2}{4\,\mathrm{Im}[\mathrm{ZetaZero}[1]]}$]], 100]

-4.1795626651088380891324091390296429030526007102129066489291459013466074946652565322403\
34183868042705 × 10^{-26}

FullSimplify@Expand[(-1 / 2 + z) / (1 / 2 + z)]

$1-\frac{2}{1+2z}$

FullSimplify@Expand[-(1 - 4 z^2) / (1 + 4 z^2)]

$1-\frac{2}{1+4z^2}$

(1 - 2 z) (1 + 2 z) / ((1 - 2 z I) (1 + 2 z I))

$\frac{(1-2z)(1+2z)}{(1-2iz)(1+2iz)}$

-(1 - 4 z^2) / (1 + 4 z^2)

$\frac{-1+4z^2}{1+4z^2}$

bo4[k_] := D[Zeta[1 / 2 - s] - 1 + 1/(1 / 2 + s), {s, k}] /. s → 0

N[(bo4[1] + 2 bo4[0] - 2) / (1 + bo4[0])]

-5.55562

Sum[1 / N[Im@ZetaZero@k] I + 1 / N[Im@ZetaZero@-k I], {k, 1, 20}]

0. + 0.982304 i

N[Sum[1 / Im@ZetaZero@k + 1 / Im@ZetaZero@k, {k, 1, 2000}]]

5.68099

N[1 / Im@ZetaZero@2001] I

0. + 0.000397366 i

Sum[1 / (I N[Im@ZetaZero@k]) + 1 / (N[Im@ZetaZero@-k] I), {k, 1, 20}]

0. + 0. i

```
bo4[k_] := D[Zeta[1 / 2 - s] - 1 + 1/(1 / 2 + s), {s, k}] /. s → 0
ex[s_, l_] := 1 - 1/(1 / 2 + s) + Sum[s^k / k ! bo4[k], {k, 0, l}]
ex2[s_, l_] := (1 / (1 + 2 s)) ((-1 + 2 s) + Sum[(1 + 2 s) s^k / k ! bo4[k], {k, 0, l}])
ex3[s_, l_] := (1 / (1 + 2 s)) (-1 + 2 s + Sum[s^k / k ! bo4[k] + 2 s^(k + 1) / k ! bo4[k], {k, 0, l}])
ex4[s_, l_] := (1 / (1 + 2 s))
  (-1 + 2 s + Sum[s^k / k ! bo4[k], {k, 0, l}] + 2 Sum[s^k / (k - 1) ! bo4[(k - 1)], {k, 1, l + 1}])
ex5[s_, l_] := (1 / (1 + 2 s)) (-1 + 2 s + bo4[0] + Sum[s^k / k ! bo4[k], {k, 1, l}] +
    2 Sum[s^k / (k - 1) ! bo4[(k - 1)], {k, 1, l + 1}])
ex6[s_, l_] := (1 / (1 + 2 s)) (-1 + 2 s + bo4[0] + Sum[s^k / k ! bo4[k], {k, 1, l}] +
    2 Sum[s^k / (k - 1) ! bo4[(k - 1)], {k, 1, l}] + 2 s^(l + 1) / l ! bo4[l])
ex7[s_, l_] := (1 / (1 + 2 s)) (-1 + 2 s + bo4[0] + Sum[(1 / k ! bo4[k]) s^k, {k, 1, l}] +
    Sum[(2 / (k - 1) ! bo4[(k - 1)]) s^k, {k, 1, l}] + 2 s^(l + 1) / l ! bo4[l])
ex8[s_, l_] := (1 / (1 + 2 s)) (-1 + 2 s + bo4[0] +
    Sum[((1 / k ! bo4[k]) + (2 / (k - 1) ! bo4[(k - 1)])) s^k, {k, 1, l}] + 2 s^(l + 1) / l ! bo4[l])
ex9[s_, l_] := (1 / (1 + 2 s)) (-1 + bo4[0])
  (1 + (2 s + Sum[((1 / k ! bo4[k]) + (2 / (k - 1) ! bo4[(k - 1)])) s^k, {k, 1, l}] +
       2 s^(l + 1) / l ! bo4[l]) / (-1 + bo4[0]))
ex9a[s_, l_] := (1 + (2 s + Sum[((1 / k ! bo4[k]) + (2 / (k - 1) ! bo4[(k - 1)])) s^k, {k, 1, l}] +
      2 s^(l + 1) / l ! bo4[l]) / (-1 + bo4[0]))
ex9b[s_, l_] := (1 + ((2 + bo4[1] + 2 bo4[0]) s + Sum[((1 / k ! bo4[k]) + (2 / (k - 1) ! bo4[(k - 1)]))
        s^k, {k, 2, l}] + 2 s^(l + 1) / l ! bo4[l]) / (-1 + bo4[0]))
ex9c[s_, l_] := 1 + (2 + bo4[1] + 2 bo4[0]) / (-1 + bo4[0]) s +
  (Sum[((1 / k ! bo4[k]) + (2 / (k - 1) ! bo4[(k - 1)])) s^k, {k, 2, l}] + 2 s^(l + 1) / l ! bo4[l]) /
   (-1 + bo4[0])
ex9d[s_, l_] := 1 + (2 - Zeta'[1/2]/Zeta[1/2]) s +
  (Sum[((1 / k ! bo4[k]) + (2 / (k - 1) ! bo4[(k - 1)])) s^k, {k, 2, l}] + 2 s^(l + 1) / l ! bo4[l]) /
   (-1 + bo4[0])
ex9e[s_, l_] := 1 + (2 - Zeta'[1/2]/Zeta[1/2]) s +
  (Sum[((1 / k ! bo4[k]) + (2 / (k - 1) ! bo4[(k - 1)])) / (-1 + bo4[0]) s^k, {k, 2, l}] +
    2 s^(l + 1) / l ! bo4[l] / (-1 + bo4[0]))
ex9f[s_, l_] := 1 + (2 - Zeta'[1/2]/Zeta[1/2]) s +
  (Sum[((-1)^k (-2 k (D[Zeta[r], {r, k - 1}] /. r → 1 / 2) + (D[Zeta[r], {r, k}] /. r → 1 / 2)) /
          (k!) / Zeta[1 / 2]) s^k, {k, 2, l}] + 2 s^(l + 1) / l ! bo4[l] / (-1 + bo4[0]))
ex10[s_, l_] := Zeta[1/2]/(1 + 2 s) ex9f[s, l]


spow1[s_] := (2 + bo4[1] + 2 bo4[0]) / (-1 + bo4[0])
```

N@ex9f[z, 50]

$$1. - 0.686092\,z + 0.1088\,z^2 + 0.00534492\,z^3 - 0.000831825\,z^4 - 0.000156374\,z^5 - 6.33542\times10^{-6}\,z^6 + 1.13505\times10^{-6}\,z^7 + 1.9666\times10^{-7}\,z^8 + 1.12683\times10^{-8}\,z^9 - 4.65739\times10^{-10}\,z^{10} - 1.41875\times10^{-10}\,z^{11} - 1.11685\times10^{-11}\,z^{12} + 4.24249\times10^{-8}\,z^{28} + 4.12363\times10^{-7}\,z^{31} + 3.01076\times10^{-6}\,z^{34} - 0.000467447\,z^{41} + 0.00308412\,z^{44} + 0.0122054\,z^{46}$$

ex10[.3, 70]

-0.733921

Zeta[.5 - .3]

-0.733921

$$\mathbf{Expand}\left[1 - \frac{1}{1/2 + s}\right] \text{ /. } s \to .2 + .3\,I$$

$-0.206897 + 0.517241\,i$

$$1 - \frac{1}{1/2 + s}$$

Expand[(-1 + 2 s) / (1 + 2 s)] /. s → .2 + .3 I

$-0.206897 + 0.517241\,i$

FullSimplify@spow1[z]

$$2 - \frac{\mathrm{Zeta}'\left[\frac{1}{2}\right]}{\mathrm{Zeta}\left[\frac{1}{2}\right]}$$

N[ex9[Im@ZetaZero@1 I, 70], 70]

$$-3.591519090866761403755167584429295467207246744758991175645416976250 77\times10^{-13} + 1.717168032339678214985062948237644543463127022775810879118152015848094\times10^{-12}\,i$$

N[ex9b[Im@ZetaZero@1 I, 70], 70]

$$3.348676498252417461613789760803280989135997098774009358580974791546566\times10^{-11} + 5.776582983727527146225151740358877863980237006067824998578458591522 83\times10^{-12}\,i$$

N[ex9b[-1 / 2, 70], 70]

$1.369530472179873046191086702641627602118204017431031048432863457270020$

FullSimplify[(2 + bo4[1] + 2 bo4[0]) / (-1 + bo4[0])]

$$2 - \frac{\mathrm{Zeta}'\left[\frac{1}{2}\right]}{\mathrm{Zeta}\left[\frac{1}{2}\right]}$$

FullSimplify[((1 / 2 ! bo4[2]) + (2 / (2 - 1) ! bo4[(2 - 1)])) / (-1 + bo4[0])]

$$\frac{-4\,\mathrm{Zeta}'\left[\frac{1}{2}\right] + \mathrm{Zeta}''\left[\frac{1}{2}\right]}{2\,\mathrm{Zeta}\left[\frac{1}{2}\right]}$$

((1 / k ! bo4[k]) + (2 / (k - 1) ! bo4[(k - 1)])) / (-1 + bo4[0])

bk[k_] := ((1 / k ! bo4[k]) + (2 / (k - 1) ! bo4[(k - 1)])) / (-1 + bo4[0])

```
Table[FullSimplify[bk[k]], {k, 2, 10}]
```

$$\left\{\frac{-4\,\text{Zeta}'\left[\frac{1}{2}\right]+\text{Zeta}''\left[\frac{1}{2}\right]}{2\,\text{Zeta}\left[\frac{1}{2}\right]},\ -\frac{-6\,\text{Zeta}''\left[\frac{1}{2}\right]+\text{Zeta}^{(3)}\left[\frac{1}{2}\right]}{6\,\text{Zeta}\left[\frac{1}{2}\right]},\ \frac{-8\,\text{Zeta}^{(3)}\left[\frac{1}{2}\right]+\text{Zeta}^{(4)}\left[\frac{1}{2}\right]}{24\,\text{Zeta}\left[\frac{1}{2}\right]},\ -\frac{-10\,\text{Zeta}^{(4)}\left[\frac{1}{2}\right]+\text{Zeta}^{(5)}\left[\frac{1}{2}\right]}{120\,\text{Zeta}\left[\frac{1}{2}\right]},\ \frac{-12\,\text{Zeta}^{(5)}\left[\frac{1}{2}\right]+\text{Zeta}^{(6)}\left[\frac{1}{2}\right]}{720\,\text{Zeta}\left[\frac{1}{2}\right]},\ -\frac{-14\,\text{Zeta}^{(6)}\left[\frac{1}{2}\right]+\text{Zeta}^{(7)}\left[\frac{1}{2}\right]}{5040\,\text{Zeta}\left[\frac{1}{2}\right]},\ \frac{-16\,\text{Zeta}^{(7)}\left[\frac{1}{2}\right]+\text{Zeta}^{(8)}\left[\frac{1}{2}\right]}{40\,320\,\text{Zeta}\left[\frac{1}{2}\right]},\ -\frac{-18\,\text{Zeta}^{(8)}\left[\frac{1}{2}\right]+\text{Zeta}^{(9)}\left[\frac{1}{2}\right]}{362\,880\,\text{Zeta}\left[\frac{1}{2}\right]},\ \frac{-20\,\text{Zeta}^{(9)}\left[\frac{1}{2}\right]+\text{Zeta}^{(10)}\left[\frac{1}{2}\right]}{3\,628\,800\,\text{Zeta}\left[\frac{1}{2}\right]}\right\}$$

```
FullSimplify[(1 / (1 + 2 s)) (-1 + bo4[0])]
```

$$\frac{\text{Zeta}\left[\frac{1}{2}\right]}{1+2\,s}$$

```
Table[
 (-1)^k (-2 k (D[Zeta[s], {s, k - 1}] /. s → 1 / 2) + (D[Zeta[s], {s, k}] /. s → 1 / 2)) / (k!) /
   Zeta[1 / 2], {k, 2, 10}]
```

$$\left\{\frac{-4\,\text{Zeta}'\left[\frac{1}{2}\right]+\text{Zeta}''\left[\frac{1}{2}\right]}{2\,\text{Zeta}\left[\frac{1}{2}\right]},\ -\frac{-6\,\text{Zeta}''\left[\frac{1}{2}\right]+\text{Zeta}^{(3)}\left[\frac{1}{2}\right]}{6\,\text{Zeta}\left[\frac{1}{2}\right]},\ \frac{-8\,\text{Zeta}^{(3)}\left[\frac{1}{2}\right]+\text{Zeta}^{(4)}\left[\frac{1}{2}\right]}{24\,\text{Zeta}\left[\frac{1}{2}\right]},\ -\frac{-10\,\text{Zeta}^{(4)}\left[\frac{1}{2}\right]+\text{Zeta}^{(5)}\left[\frac{1}{2}\right]}{120\,\text{Zeta}\left[\frac{1}{2}\right]},\ \frac{-12\,\text{Zeta}^{(5)}\left[\frac{1}{2}\right]+\text{Zeta}^{(6)}\left[\frac{1}{2}\right]}{720\,\text{Zeta}\left[\frac{1}{2}\right]},\ -\frac{-14\,\text{Zeta}^{(6)}\left[\frac{1}{2}\right]+\text{Zeta}^{(7)}\left[\frac{1}{2}\right]}{5040\,\text{Zeta}\left[\frac{1}{2}\right]},\ \frac{-16\,\text{Zeta}^{(7)}\left[\frac{1}{2}\right]+\text{Zeta}^{(8)}\left[\frac{1}{2}\right]}{40\,320\,\text{Zeta}\left[\frac{1}{2}\right]},\ -\frac{-18\,\text{Zeta}^{(8)}\left[\frac{1}{2}\right]+\text{Zeta}^{(9)}\left[\frac{1}{2}\right]}{362\,880\,\text{Zeta}\left[\frac{1}{2}\right]},\ \frac{-20\,\text{Zeta}^{(9)}\left[\frac{1}{2}\right]+\text{Zeta}^{(10)}\left[\frac{1}{2}\right]}{3\,628\,800\,\text{Zeta}\left[\frac{1}{2}\right]}\right\}$$

$$\text{ex9f}[s\_,\ l\_] := 1+\left(2-\frac{\text{Zeta}'\left[\frac{1}{2}\right]}{\text{Zeta}\left[\frac{1}{2}\right]}\right)s+$$

```
  (Sum[((-1)^k (-2 k (D[Zeta[r], {r, k - 1}] /. r → 1 / 2) + (D[Zeta[r], {r, k}] /. r → 1 / 2)) /
          (k!) / Zeta[1 / 2]) s^k, {k, 2, l}] + 2 s^(l + 1) / l! bo4[l] / (-1 + bo4[0]))
roots[n_] := If[(c = Exponent[f = N[ex9f[z, 100], 100], z]) == 0, {},
  If[c == 1, List@NRoots[f == 0, z][[2]], List @@ NRoots[f == 0, z][[All, 2]]]]
```

```
roots[10]
```

```
N[ex9f[z, 100], 100]
```

```
FullSimplify[(1 / (1 + 2 s)) (-1 + bo4[0])]
```

$$\frac{\text{Zeta}\left[\frac{1}{2}\right]}{1+2\,s}$$

```
ex9f[s_, l_] := 1 + (2 - Zeta′[1/2]/Zeta[1/2]) s +
  Sum[((-1)^k (-2 k (D[Zeta[r], {r, k - 1}] /. r → 1 / 2) + (D[Zeta[r], {r, k}] /. r → 1 / 2)) /
       (k!) / Zeta[1 / 2]) s^k, {k, 2, l}]

ex9g[s_, l_] :=
 1 + Sum[((-1)^k (-2 k (D[Zeta[r], {r, k - 1}] /. r → 1 / 2) + (D[Zeta[r], {r, k}] /. r → 1 / 2)) /
       (k!) / Zeta[1 / 2]) s^k, {k, 1, l}]

ex10[s_, l_] := Zeta[1/2]/(1 + 2 s) ex9g[s, l]

ex10[.4 + .1 I, 50]
```

$-0.589453 + 0.11391\, i$

```
Zeta[.5 - (.4 + .1 I)]
```

$-0.589453 + 0.11391\, i$

```
Table[FullSimplify[
   ((-1)^k (-2 k (D[Zeta[r], {r, k - 1}] /. r → 1 / 2) + (D[Zeta[r], {r, k}] /. r → 1 / 2)) / (k!) /
     Zeta[1 / 2])] s^k, {k, 1, 10}]
```

$$\left\{s\left(2-\frac{\text{Zeta}'\left[\frac{1}{2}\right]}{\text{Zeta}\left[\frac{1}{2}\right]}\right), \frac{s^2\left(-4\,\text{Zeta}'\left[\frac{1}{2}\right]+\text{Zeta}''\left[\frac{1}{2}\right]\right)}{2\,\text{Zeta}\left[\frac{1}{2}\right]},\right.$$
$$-\frac{s^3\left(-6\,\text{Zeta}''\left[\frac{1}{2}\right]+\text{Zeta}^{(3)}\left[\frac{1}{2}\right]\right)}{6\,\text{Zeta}\left[\frac{1}{2}\right]}, \frac{s^4\left(-8\,\text{Zeta}^{(3)}\left[\frac{1}{2}\right]+\text{Zeta}^{(4)}\left[\frac{1}{2}\right]\right)}{24\,\text{Zeta}\left[\frac{1}{2}\right]},$$
$$-\frac{s^5\left(-10\,\text{Zeta}^{(4)}\left[\frac{1}{2}\right]+\text{Zeta}^{(5)}\left[\frac{1}{2}\right]\right)}{120\,\text{Zeta}\left[\frac{1}{2}\right]}, \frac{s^6\left(-12\,\text{Zeta}^{(5)}\left[\frac{1}{2}\right]+\text{Zeta}^{(6)}\left[\frac{1}{2}\right]\right)}{720\,\text{Zeta}\left[\frac{1}{2}\right]},$$
$$-\frac{s^7\left(-14\,\text{Zeta}^{(6)}\left[\frac{1}{2}\right]+\text{Zeta}^{(7)}\left[\frac{1}{2}\right]\right)}{5040\,\text{Zeta}\left[\frac{1}{2}\right]}, \frac{s^8\left(-16\,\text{Zeta}^{(7)}\left[\frac{1}{2}\right]+\text{Zeta}^{(8)}\left[\frac{1}{2}\right]\right)}{40\,320\,\text{Zeta}\left[\frac{1}{2}\right]},$$
$$\left.-\frac{s^9\left(-18\,\text{Zeta}^{(8)}\left[\frac{1}{2}\right]+\text{Zeta}^{(9)}\left[\frac{1}{2}\right]\right)}{362\,880\,\text{Zeta}\left[\frac{1}{2}\right]}, \frac{s^{10}\left(-20\,\text{Zeta}^{(9)}\left[\frac{1}{2}\right]+\text{Zeta}^{(10)}\left[\frac{1}{2}\right]\right)}{3\,628\,800\,\text{Zeta}\left[\frac{1}{2}\right]}\right\}$$

```
FullSimplify[
 ((-1)^k (-2 k (D[Zeta[r], {r, k - 1}] /. r → 1 / 2) + (D[Zeta[r], {r, k}] /. r → 1 / 2)) / (k!) /
   Zeta[1 / 2])]
```

$$\frac{(-1)^k\left(-2\,k\,\text{Zeta}^{(-1+k)}\left[\frac{1}{2}\right]+\text{Zeta}^{(k)}\left[\frac{1}{2}\right]\right)}{k!\,\text{Zeta}\left[\frac{1}{2}\right]}$$

```
big[N@Im@ZetaZero@1 I]
```

$-2.76898\times10^{-13} - 3.66901\times10^{-13}\, i$

```
N[Im@ZetaZero@1, 100]
```

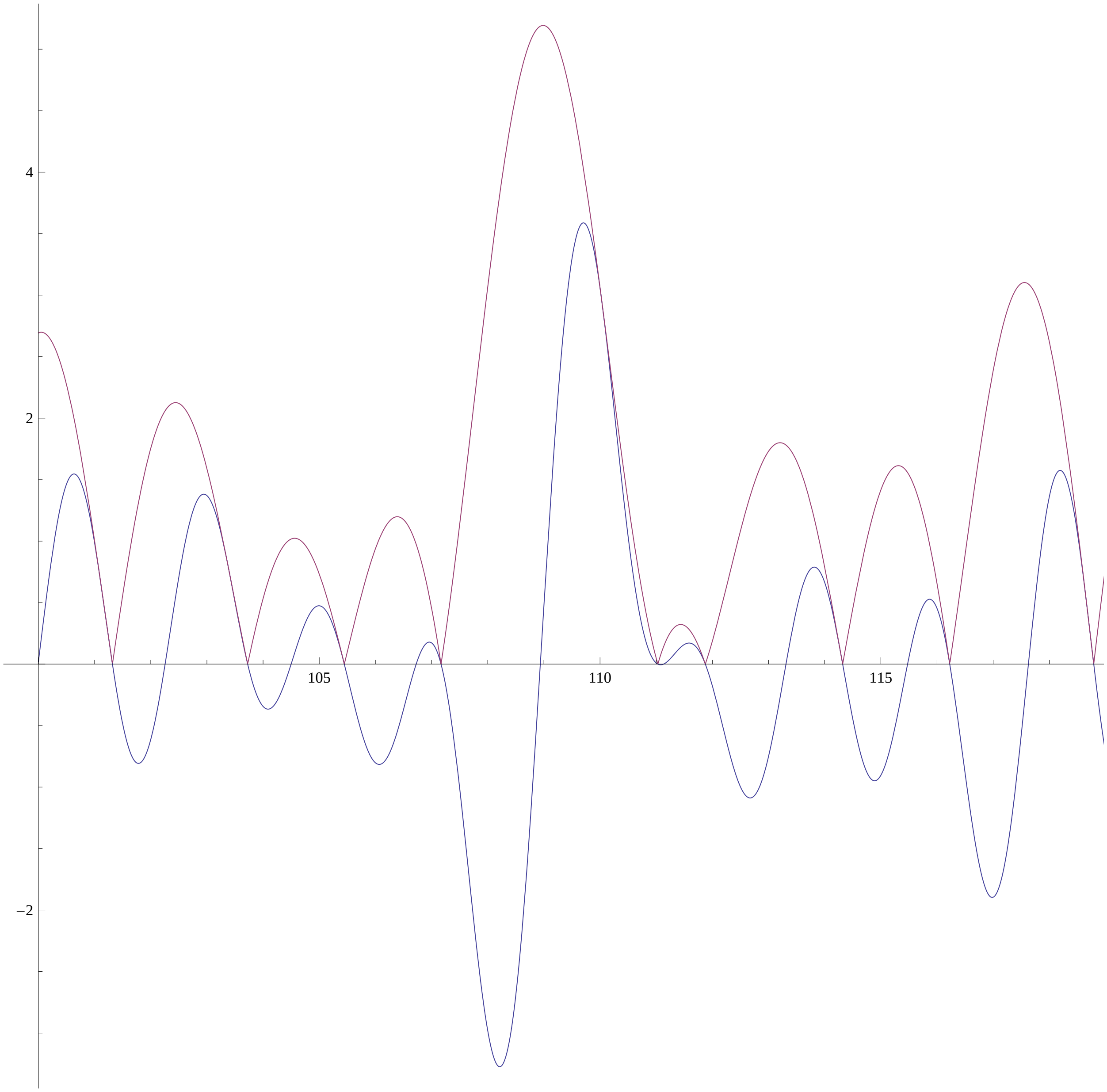
14.134725141734693790457251983562470270784257115699243175685567460149963429809256764949010393171561 01

```
14.134725141734695`

14.1347
```

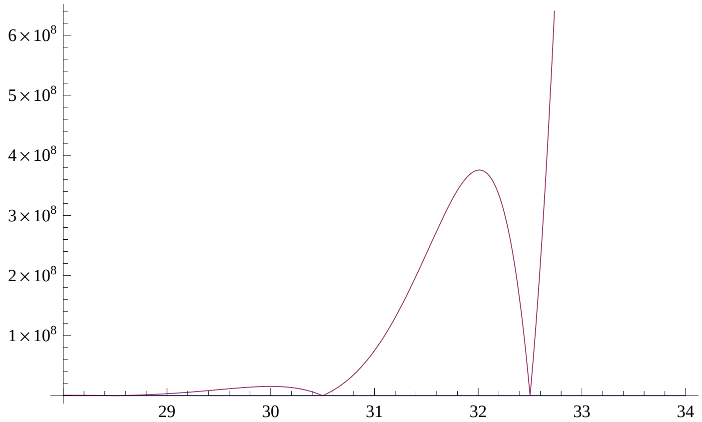
$$\texttt{ex9f[s\_, l\_] := 1 +} \left(2 - \frac{\texttt{Zeta}'\left[\frac{1}{2}\right]}{\texttt{Zeta}\left[\frac{1}{2}\right]}\right) \texttt{ s +}$$

```
  (Sum[((-1)^k (-2 k (D[Zeta[r], {r, k - 1}] /. r → 1 / 2) + (D[Zeta[r], {r, k}] /. r → 1 / 2)) /
        (k!) / Zeta[1 / 2]) s^k, {k, 2, l}] + 2 s^(l + 1) / l! bo4[l] / (-1 + bo4[0]))
roots[n_] := If[(c = Exponent[f = ex9f[z, 100], z]) == 0, {},
  If[c == 1, List@Roots[f == 0, z][[2]], List @@ Roots[f == 0, z][[All, 2]]]]
rootsa[n_] := If[(c = Exponent[f = ex9f[z, 100], z]) == 0, {},
  If[c == 1, List@Roots[f == 0, z][[2]], N[List @@ Roots[f == 0, z][[All, 2]], 100]]]

rootsa[1]

$Aborted

ba[s_] := (Zeta[1 / 2 - s I] + Zeta[1 / 2 + s I]) / 2
ba2[s_] := (Zeta[1 / 2 - s I] - Zeta[1 / 2 + s I]) / (2 I)
```

```
Plot[{ba2[t], Abs[Zeta[1 / 2 + t I]]}, {t, 100, 130}]
```

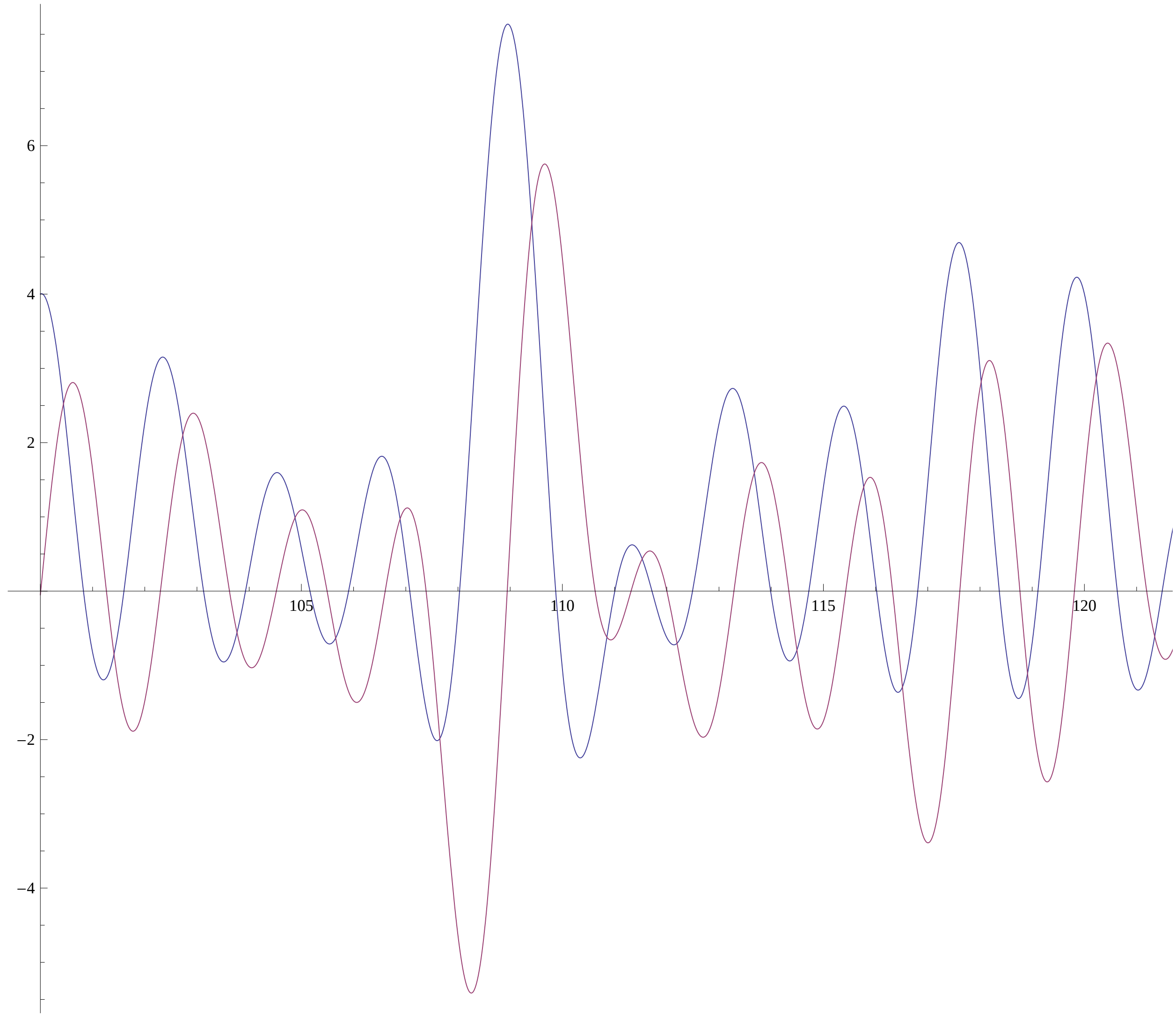


```
Plot[{0, Abs[ba[I t]]}, {t, 28, 34}]
```




```
baa[s_] := (Zeta[1 / 4 - s I] + Zeta[1 / 4 + s I]) / 2
ba2a[s_] := (Zeta[1 / 4 - s I] - Zeta[1 / 4 + s I]) / (2 I)
```

```
Plot[{baa[t], ba2a[t]}, {t, 100, 130}]
```




```
pa[t_] := Cos[t] - 2 (Cos[t] + 2 z Sin[t]) / (1 + 4 z^2)
pa2[t_] := (Cos[t] (1 + 4 z^2) - 2 Cos[t] - 4 z Sin[t]) / (1 + 4 z^2)
pa3[t_] := ((4 z^2 - 1) Cos[t] - 4 z Sin[t]) / (1 + 4 z^2)
```

```
pa3[ArcTan[(4 z^2 - 1) / (4 z)]]
```

0

FullSimplify[pa3[ArcTan[$z - \frac{1}{4z}$]]]

```
FullSimplify[
 Cos[t] - Integrate[Cos[z Log[x] + t] / x^(1 / 2), {x, 0, 1}] /. t → ArcTan[z - 1 / (4 z)]]
```

0

```
$MaxPrecision = 1000
$MaxExtraPrecision = 1000
```

```
1000

1000
```

```
bo4[k_] := D[Zeta[1 / 2 - s] - 1 + 1/(1 / 2 + s), {s, k}] /. s → 0
Table[N[bo4[k], 200], {k, 0, 50}] // TableForm
```

```
-0.460354508809586812889499152515298012467229331012581490542886087825530529474500625276419
-0.077353860790848272528468553285400486269676028493494790431701514745279196849661715119349
-0.008357013928661422691306505944962785185593619636354535309295753667809246014498013380680
0.00330924531907009738976722069545930251401884655572805429990806567091944187631603406556932
0.00268027955257018572669537881543557270389101782335102580920922381262001829685645516708573
0.00060066206666342563365592409801332863415701793005835291161639522087328085461588030919781
-0.000546537458946057963715903974343103029364511201846071563059321366785206713767403236124
-0.000702626606664312438388776803254796515986970775849368416362295081309268884470888829544
-0.000337601840801398243612455726278852133264275766172186272935859486535437780793907520849
0.00010539708153449920136429387481484248467690018135246829077504332830432207035156482328062
0.00036013908995572346665746324295093641983569040458572071275753762788909903044254247743924
0.00034524028748512509901258475796457009057546571949600843229663412894338866701363934384492
0.00012097854967658662849904339600957356452057428865042743565962960009423137656617313226401
-0.000171502238457934262000437218962002378402759933715451043480715631975349427858485813850
-0.000374416450333153638680165127262560497693165907718188583847756269166965639010101787132
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-0.000136417715955298860243198153953640465720586221401954469837995050132888205713897134590
0.00025272878787139626047383609789374705748244609077643528714323763876893713172414216823975
0.00061786944823174286302684761403761458856533908080096613154586949853972151551451303115309
0.00073487354235035253333422746277112302564470742015264876281285550947212516811771344373685
0.00042055033706637091770480853262386310331865472475587388780628591601920288451774726472684
-0.000357492516037905560300601087321776951067441415408650836628204651393920905638822910092
-0.001381765143958376242718382597987901516571048528669398301336604881021289201865004017153
-0.002147970165792253883365651913925855724285799146636055392655447119364803582408334166908
-0.001952602649529718638101739526335807206576872119904187936474157953844623789752383879499
-0.000158211161096251562702296838331263435274148080944343957023013155105409920406320902826
0.00335597176057297473759206583273593628924358779523962968285375151816981452106049865924903
0.00764606522416076959695467254241749620419953403739471837336079824373286127559895500938819
0.01029401686785582365205906827360012889278768450445416155171593381188019650888750865719968
0.00754427188304734933624813979723244955023931945258578166477346062778172607343232789153418
-0.004488089200365022562660063365381485571924945014009199722265873224017010003323510793807
-0.027097017565142774686081677008815792592782618964161024249137453900579797333466889078181
-0.054812048369935778008901938415227465381805172614872413769549039598944612867539353218555
-0.070968882582817233880190220998766795431291905597935461241837722855877420895964676270174
-0.046079999674452814208082518903089470469622340819192261352585895250823541690971744512306
0.05570134189589911763108606836779381613112971180260449336415021822343796316236060551126325
0.25529999405621157188712999668057640775464692228976798261438721212936295379769230039505082
0.51774311072975240243173854316440661441300233431422655900982708982227249285570106954155569
0.69561746057856236860976305055472340240164942580366201270886211446592269312013710177432019
0.48168197275081710751296248395060760188157470541279019411468644383207748912362023124757261
-0.570438350445663529332680310814633439414912629074401028724764854442201303631956934354400
-2.857468498458942255422374836966982094338273846842634507597422447489905492719408730880083
-6.247534687244425170228635521797940012837985944466271714842002185716972302871664605595850
-9.243581438933812032637602242510663020221237294511264673686229638110490767730859780780922
-7.949172821941956874451068577874197553950256524140494324304217351295732398196530412631303
4.486270884068259603972988816756643095773262481406918295878801418998577612343780226984282
36.399743447610999558115408376567828941943187736496809633136880525377578223407244600603877
91.549681684236337202933124916326309148202121933557996211218552531785765478946937213561385
155.440989371233012799153075873903653937563182982316041703992124489739377063815529612644832
172.968051621068670188275798980128819610891395724837119829254514539828067331866220297105703
23.763123630745759320145161145671042869236489469707080668105373255906904887752752269090205
```

bo4[50]

$$68\,486\,449\,405\,023\,952\,496\,492\,865\,790\,416\,371\,950\,237\,350\,107\,438\,397\,655\,831\,308\,926\,976\,000\,000\,000\,000 + \mathrm{Zeta}^{(50)}\left[\frac{1}{2}\right]$$

Table[N[bo4[k], 200], {k, 51, 100}] // TableForm

-481.344831462131222631166239214403366770610988697316781533491240450750630946184353343978
-1530.431950425695204429207153191495258712281812302370493647608021663053390130081651344588
-3080.192380120985144452997131411158657173120132581377517115077672614133996791533133273774
-4366.197881606460660465638339955818519142662340516202670314612583219845165398611837909812
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5114.699847116144021356549566369405598715559862710162408960498007523734042770976912057194
26 894.394145363252078094837820347422413965763928594057634138632931529946190672553238266644
67 254.854985530303633065451495785421752638341862525661849155349146243715768607195890850243
119 339.228974188222250095062491868358364451377356327355030827778082999965926557435176590
142 660.750997953369294220873434024535973656620316575942938233018981658686157032310820791
29 638.402243605631792944951561123653390007783822297097043552983915899345009095087211286
-425 165.480494446834275367482427754871524902042625512567503871780360557055054814307814669
-1.502633308942747166474677196069032273422201215400592381637700695460375939305447812023
-3.367329974191590708976862123384894766049235937863456060109435687146256216352799360564
-5.504292373196187192994997630689910833106857025243995395710790516821155595304785958728
-5.542586453476717349787955158022724759630766261883613223443431454442012085011344339075
2.575257466716813711515165017341756882060147510774617169984558975218143681402768104522
3.026182498031853805884296569661785376072350732980112273580386662613684036816271224682
9.278289582999680088966367516810006868758199898779102361071864958618593144729880163937
1.973365426707593583102913021732912815648120820034404203073423404106836483719489566661
3.074367772346947323109343934977448654575573253255219960838153132994505039599164562762
2.663923531515148441887099955264593497329236367281965004599562610521888374718932938059
-3.262987260362711385202689533826823265012664642887189285824691894702072790634375424461
-2.235759348371671087908802662547189425624642330884441860457443489298237794855109956810
-6.504499146318527731785604789565042281314575022797810157962235104884598869726651386092
-1.362845939204457991255747799930337592645073346729550714744901988876497279771444808415
-2.095938864923195190915661821391639751185043077277501110632123454557346246601200048367
-1.684154540672194121482731803436443438852640328350404410119643146016290691557402717973
2.945849610256966799027793920064583181110277251406353835151009704776093066312805480928
1.789358745358634371713397375996350822720509975178358082016970653835485373903933687751
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1.104523084163848108507966385474486483273075607878981339396651043114236161068894447488
1.738226318587496725468405032895980768250734250273678939388707964925211511864282816517
1.458413852746652965480254698262914352580543745635781911061076109853704848831437127820
-2.427128850842059709999225330321095071585361232038855373329093791205844488919253379820
-1.558691219412294855450858035513644810584158747315056664424585328767385748734446532555
-4.692558569891784334740586979793805129685161616681122456682012786879581511709171745295
-1.037886339014140104389101668445421723380238889269080821997949365627083734860700282163
-1.725411553501819584275237378059803195563119310580858463749228544291214440606165728014
-1.670707662236092996141880538254415149938993382003847172163805711991902738967756090328
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1.458026250476367739442659809493451486928596347271639140251197406494716976708880213946
4.722335445246532595045400426073963986888080881409469633240851939012812817010675435106
1.110961685261269497313723896176104595497011594989938679528733049729054912068892852207

```
1.99778009568883908196364725597921065568895834780213411773831144715415440427700880118923331
2.32822148722152392968034301130449579281418456574254716665368497418152545835330656743224445
-7.18153687458281435367569305487051400200286072182703627721520850113521021142000804299546
-1.42067063622058821316802576913572859304783595892550432333178566751136990494649509293908
-5.18808846510828063089913009706205756462859301798831551229097786633341875563892196025342
-1.32620620448471212106317892125189200269902603323207497750634932745944653465736580820910
```

**-1 326 206 204 484 712 121.06317892125189200269902603323207497750634932745944653465736580820\
910589397578559349375280672222217294601282232496430410424207290232358646166777453342 49\
9673890339056058195215885761 31**

```
Table[N[bo4[k], 30], {k, 0, 10}] // TableForm
```

-0.460354508809586812889499152515
-0.0773538607908482725284685532854
-0.00835701392866142269130650594496
0.00330924531907009738976722069546
0.00268027955257018572669537881544
0.000600662066663425633655924098013
-0.000546537458946057963715903974343
-0.000702626606643124383887768032548
-0.000337601840801398243612455726279
0.000105397081534499201364293874815
0.000360139089955723466657463242951

```
TrigToExp@Cos[Log[x] + t]
```

$$\frac{1}{2} e^{-i t} x^{-i} + \frac{1}{2} e^{i t} x^{i}$$

$$\frac{1}{2} e^{-i t - i x} + \frac{1}{2} e^{i t + i x} \text{ /. } t \to -\text{Pi}/2$$

$$\frac{1}{2} e^{\frac{i \pi}{2} - i x} + \frac{1}{2} e^{-\frac{i \pi}{2} + i x}$$

```
FullSimplify[E^(-I t)]
```

$$e^{-i t}$$

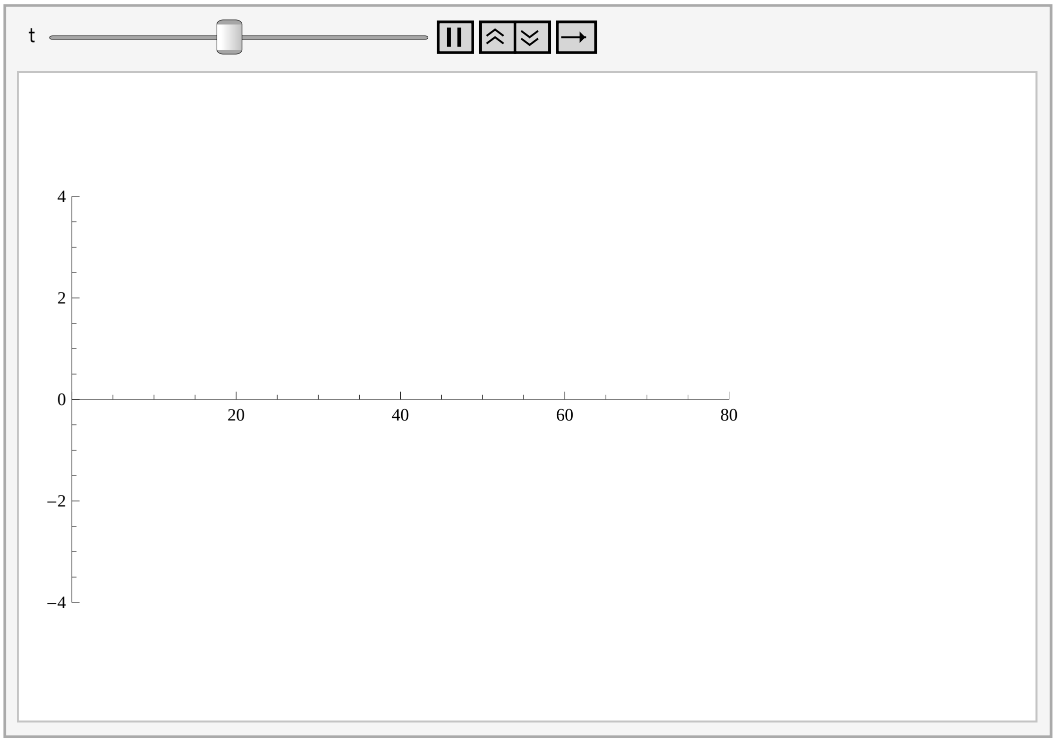
```
pr[s_, t_] := (1/2) (E^(t I) Zeta[1/2 - s I] + E^(-t I) Zeta[1/2 + s I])
```

```
N@pr[10, 4 + 2 I]
```

$-3.47072 + 4.51388 i$

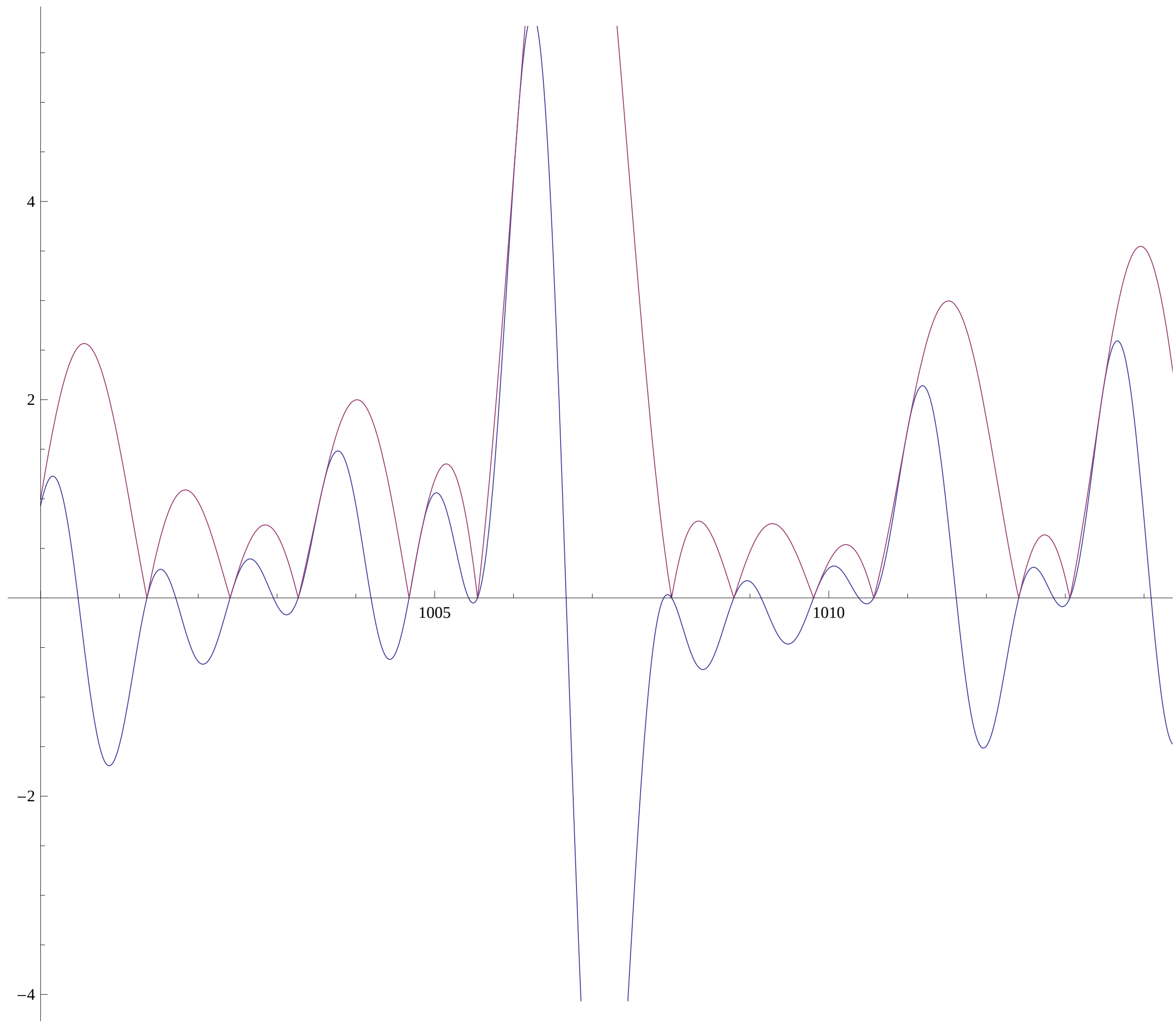
(* ArcTan$\left[n-\frac{1}{4 n}\right]$ *)

```
Animate[Plot[{Re@pr[n, -t]}, {n, 0, 80}, PlotRange → {{0, 80}, {-4, 4}}], {t, 0, 6.28}]
```

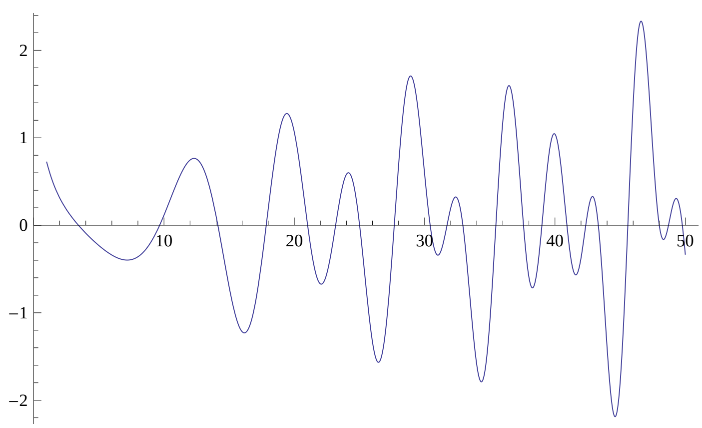



```
Graphics[{Pink, Disk[]}, PlotRange → {{-.5, .5}, {0, 1.5}},
 PlotRangeClipping → True, Frame → True]
```

```
Plot[{pr[z, ArcTan[z - 1/(4 z)]], Abs[Zeta[1/2 - z I]]}, {z, 1000, 1020}]
```



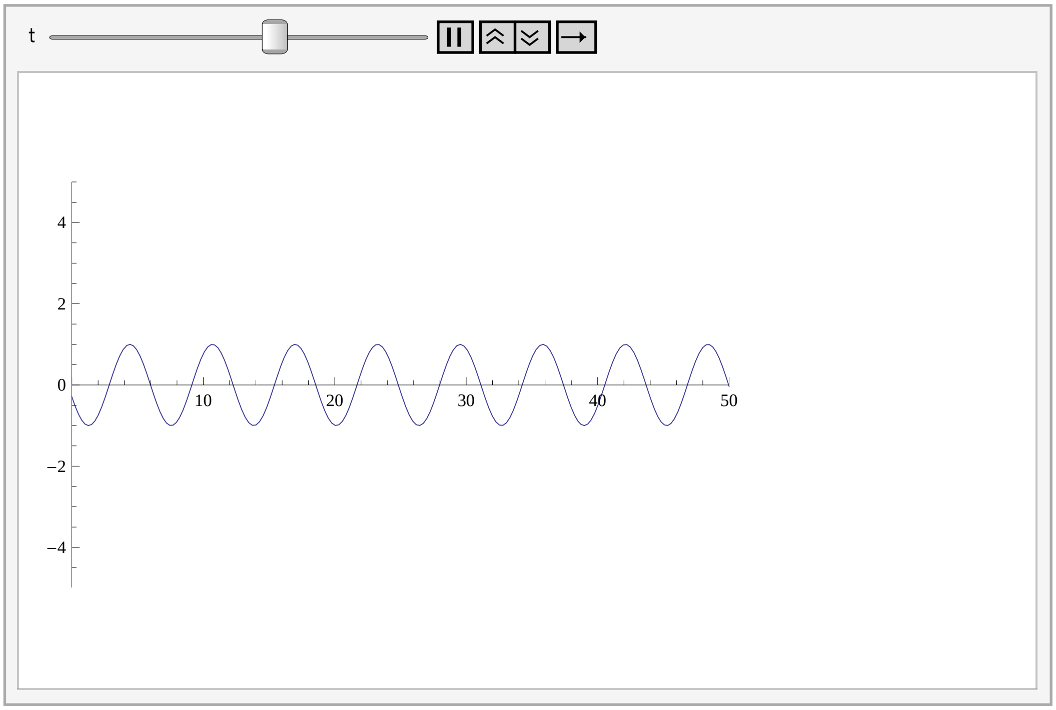

```
Plot[pr[z, -Pi/2], {z, 1, 50}]
```



```
Table[Im[Zeta[1 / 2 - z I]], {z, 0, 1000, .1}]
Table[N[Im@ZetaZero@k], {k, 1, 300}]
```

$Aborted

```
Animate[Plot[Cos[n - t], {n, 0, 50}, PlotRange → {{0, 50}, {-5, 5}}], {t, 0, 6.28}]
```



$$\mathtt{pr}\left[z,\ \mathtt{ArcTan}\left[z-\frac{1}{4\,z}\right]\right]$$

$$\frac{1}{2}\left(e^{-i\,\mathrm{ArcTan}\left[\frac{1}{4z}-z\right]}\,\mathrm{Zeta}\left[\frac{1}{2}-i\,z\right]+e^{i\,\mathrm{ArcTan}\left[\frac{1}{4z}-z\right]}\,\mathrm{Zeta}\left[\frac{1}{2}+i\,z\right]\right)$$

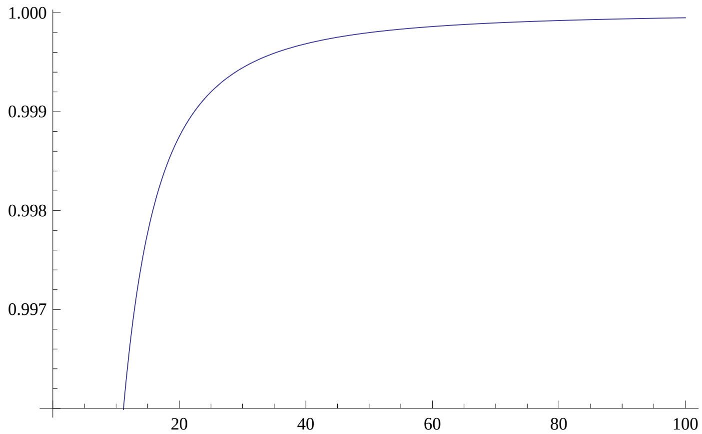
```
pr[s_, t_] := (1 / 2) ( E^ (t I ) Zeta[1 / 2 - s I] + E^ (-t I) Zeta[1 / 2 + s I])
pra[s_, t_] := Cos[t] Re[Zeta[1 / 2 - s I]] - Sin[t] Im[Zeta[1 / 2 - s I]]
pr[7, .3]
```

1.0929 + 0. i

```
pra[7, .3]
```

1.0929

```
Plot[{1 - 2 / (1 + 4 z^2)}, {z, 0, 100}]
```




```
D[Sin[z Log[t]], {z, 1}] /. z → 0
```

```
Log[t]
```

```
Integrate[ Log[x] / x^(1 / 2), {x, 0, 1}]
```

```
-4
```

```
Integrate[ Log[x]^3 / x^(1 / 2), {x, 0, 1}]
```

```
-96
```

```
Sum[(-1)^k z^(2 k + 1) / ((2 k + 1) !), {k, 0, Infinity}]
```

```
Sin[z]
```

```
sl[z2_] := Limit[Zeta[1 / 2 - z] - 1 + 1 / (1 / 2 + z), z → z2]
```

```
sl[-1 / 2]
```

```
-1 + EulerGamma
```

```
Zeta[1 / 2 - z] - 1 + 1 / (1 / 2 + z) /. z → -s + 1 / 2
```

$$-1 + \frac{1}{1 - s} + \text{Zeta}[s]$$

```
1000. / Pi
```

```
318.31
```