```
binomial[z_, k_] := binomial[z, k] = Product[z - j, {j, 0, k - 1}] / k!
Ds[n_, 0, s_, a_] := UnitStep[n - 1]
Ds[n_, 1, s_, a_] := Ds[n, 1, s, a] = HarmonicNumber[Floor[n], s] - HarmonicNumber[a, s]
Ds[n_, 2, s_, a_] := Ds[n, 2, s, a] =
  Sum[(m^(-2 s)) + 2 (m^-s) (Ds[Floor[n / m], 1, s, m]), {m, a + 1, Floor[n^(1 / 2)]}]
Ds[n_, k_, s_, a_] := Ds[n, k, s, a] =
  Sum[(m^(-s k)) + k (m^(-s (k - 1))) Ds[Floor[n / (m^(k - 1))], 1, s, m] +
    Sum[binomial[k, j] (m^-s)^j Ds[Floor[n / (m^j)], k - j, s, m], {j, 1, k - 2}],
   {m, a + 1, Floor[n^(1 / k)]}]

Ddy[n_, s_, y_, k_] := y^(k (s - 1)) Ds[n y^k, k, s, y]

Dnsyz[n_, s_, y_, z_] := Expand@Sum[binomial[z, k] Ddy[n, s, y, k], {k, 0, Log[(y + 1) / y, n]}]
dd[n_, y_, z_] := Dnsyz[n, 0, y, z]

dss[n_, s_, y_, z_, x_] :=
 If[n < y, 1, Sum[binomial[z, k] (x y^-s)^k dss[n / y^k, s, y + x, z - k, x], {k, 0, Log[y, n]}]]
```

```
dd[100, 1, 2]
```

482

```
dd[100, 2, -2]
```

$$\frac{13529}{1024}$$

```
Expand@dss[100, 0, 1 + 1 / 2, z, 1 / 2]
```

$$1+\frac{202986703\,z}{7096320}+\frac{68602319\,z^2}{1612800}+\frac{622902011\,z^3}{29030400}+\frac{2091660979\,z^4}{371589120}+\frac{52801531\,z^5}{74317824}+\frac{21461041\,z^6}{353894400}+\frac{5689681\,z^7}{2477260800}+\frac{16259\,z^8}{247726080}+\frac{739\,z^9}{743178240}+\frac{37\,z^{10}}{7431782400}+\frac{z^{11}}{81749606400}$$

```
dd[100, 2, z]
```

$$1+\frac{202986703\,z}{7096320}+\frac{68602319\,z^2}{1612800}+\frac{622902011\,z^3}{29030400}+\frac{2091660979\,z^4}{371589120}+\frac{52801531\,z^5}{74317824}+\frac{21461041\,z^6}{353894400}+\frac{5689681\,z^7}{2477260800}+\frac{16259\,z^8}{247726080}+\frac{739\,z^9}{743178240}+\frac{37\,z^{10}}{7431782400}+\frac{z^{11}}{81749606400}$$

```
dss[100, 0, 5 / 2, 3, 1 / 2]
```

1016

```
df[n_, k_, a_, t_] :=  df[n / 2, k - 1, a, t] + Sum[ df[n / j, k - 1, a, t], {j, a, n}]
df[n_, 0, a_, t_] := UnitStep[n - 1]
```

```
df[800, 3, 5, 2] / 8
```

2927

```
dss[20, 0, 3 / 2, 2, 1 / 2]
```

72

```
Ds[20 × 4, 2, 0, 2] / 4
```

33

```
df[20 × 4, 2, 3, 2] / 4
```

$$\frac{209}{4}$$

```
dss[20, 0, 3 / 2, 2, 1 / 2]
```

72

```
Sum[ Binomial[z, k] (x y^-s) ^k (1 + x^ (1 - s) Zeta[ s, y + 1]) ^ (z - k), {k, 0, Infinity}]
```

$$\left(1+x^{1-s}\,\mathrm{Zeta}[s,\,1+y]\right)^{z}\left(\frac{y^{-s}\left(x^{1+s}+x^{s}\,y^{s}+x\,y^{s}\,\mathrm{Zeta}[s,\,1+y]\right)}{x^{s}+x\,\mathrm{Zeta}[s,\,1+y]}\right)^{z}$$

$$\mathbf{Expand}\left[(1+\mathrm{Zeta}[s,\,x+y])^{z}\left(\frac{y^{-s}\,(x+y^{s}+y^{s}\,\mathrm{Zeta}[s,\,x+y])}{1+\mathrm{Zeta}[s,\,x+y]}\right)^{z}\right]$$

$$(1+\mathrm{Zeta}[s,\,x+y])^{z}\left(\frac{y^{-s}\,(x+y^{s}+y^{s}\,\mathrm{Zeta}[s,\,x+y])}{1+\mathrm{Zeta}[s,\,x+y]}\right)^{z}$$

$$\mathbf{FullSimplify}[(y^{-s}\,(x+y^{s}+y^{s}\,\mathrm{Zeta}[s,\,x+y]))\,\hat{}\,z]$$

$$(1+x\,y^{-s}+\mathrm{Zeta}[s,\,x+y])^{z}$$

```
Sum[ Binomial[z, k] x^k (x y) ^ (-s k) (1 + x^ (1 - s) Zeta[ s, y + 1]) ^ (z - k), {k, 0, Infinity}]
```

$$\left(1+x^{1-s}\,\mathrm{Zeta}[s,\,1+y]\right)^{z}\left(\frac{(x\,y)^{-s}\left(x^{1+s}+x^{s}\,(x\,y)^{s}+x\,(x\,y)^{s}\,\mathrm{Zeta}[s,\,1+y]\right)}{x^{s}+x\,\mathrm{Zeta}[s,\,1+y]}\right)^{z}$$

```
Sum[ (1 / 2) j^-s, {j, 3 / 2, Infinity, 1 / 2}]
```

$$2^{-1+s}\,\mathrm{Zeta}[s,\,3]$$

```
(1 + Sum[ (1 / 3) j^-s, {j, 1 + (1 / 3), Infinity, 1 / 3}]) ^z
```

$$\left(1+3^{-1+s}\,\mathrm{Zeta}[s,\,4]\right)^{z}$$

```
px1[x_, y_, s_, z_] :=
 Sum[ Binomial[z, k] (x y^-s) ^k (1 +  x^ (1 - s) Zeta[ s, y + 1 / x]) ^ (z - k), {k, 0, Infinity}]
px2[x_, y_, s_, z_] := (1 +  x^ (1 - s) Zeta[ s, y]) ^z
```

```
px1[1 / 3, 1 + 1 / 3, 0, 3]
```

$$\frac{1}{5832}$$

```
px2[1 / 3, 1, 0, 3]
```

$$\frac{125}{216}$$

```
(1 / y) ^k /. y → 30
```

$$30^{-k}$$

```
FullSimplify@Integrate[ x^s, {x, 1, n}]
```

$$\mathrm{ConditionalExpression}\left[\frac{-1+n^{1+s}}{1+s},\ \mathrm{Re}[n]\geq 0\ ||\ n\notin \mathrm{Reals}\right]$$

$$\text{Expand}\left[\frac{-1+n^{1+s}}{1+s}\right]$$

$$-\frac{1}{1+s}+\frac{n^{1+s}}{1+s}$$

```
1 + Sum[1 / (5 + j / 2) ^s, {j, 0, Infinity}]
```

$1+2^s\,\text{Zeta}[s, 10]$

```
FullSimplify[1 + 2 Sum[1 / (5 + j / 2) ^s, {j, 0, Infinity}] +
  Sum[1 / (5 + j / 2) ^s × 1 / (5 + k / 2) ^s, {j, 0, Infinity}, {k, 0, Infinity}]]
```

$(1+2^s\,\text{Zeta}[s, 10])^2$

```
(*
In the following case, x is 1/2 and y is 11/2.  And therefore we end up with

 (1+x^-s Zeta[s,y/x])^2
 *)
FullSimplify[1 + 2 Sum[1 / (11 / 2 + j / 2) ^s, {j, 0, Infinity}] +
  Sum[1 / (11 / 2 + j / 2) ^s × 1 / (11 / 2 + k / 2) ^s, {j, 0, Infinity}, {k, 0, Infinity}]]
```

$(1+2^s\,\text{Zeta}[s, 11])^2$

$(1+2^s\,\text{Zeta}[s, 11])^2$

$(1+2^s\,\text{Zeta}[s, 11])^2$

```
FullSimplify[1 + Sum[1 / (11 / 3 + j / 2) ^s, {j, 0, Infinity}]]
```

$$1+2^s\,\text{Zeta}\left[s, \frac{22}{3}\right]$$

```
FullSimplify[1 + 2 Sum[1 / (12 / 2 + j / 2) ^s, {j, 0, Infinity}] +
  Sum[1 / (12 / 2 + j / 2) ^s × 1 / (12 / 2 + k / 2) ^s, {j, 0, Infinity}, {k, 0, Infinity}]]
```

$(1+2^s\,\text{Zeta}[s, 12])^2$

```
(3 / 2) / (1 / 2)
```

3

```
Expand[(1 + (x)) / (x)]
```

$$1+\frac{1}{x}$$

$$\text{FullSimplify}\left[\left(1+x^{1-s}\,\text{Zeta}[s, 1+y]\right)^z \left(\frac{y^{-s}\left(x^{1+s}+x^s y^s+x\,y^s\,\text{Zeta}[s, 1+y]\right)}{x^s+x\,\text{Zeta}[s, 1+y]}\right)^z\right]$$

$$\text{FullSimplify}\left[\text{Expand}\left[\left(1+x^{1-s}\,\text{Zeta}[s, 1+y]\right)^z \left(1+\frac{x^{1+s}\,y^{-s}}{x^s+x\,\text{Zeta}[s, 1+y]}\right)^z \;/.\; z\to 2\right]\right] \;/.$$
$$\{x\to 1/2,\ s\to 2,\ y\to 3/2\}$$

$$\frac{256}{81}\left(\frac{11}{16}+\frac{9}{8}\left(-\frac{40}{9}+\frac{\pi^2}{2}\right)\right)^2$$

```
Sum[ Binomial[z, k] x^k ((x y)^(-s k)) (1 + x^(1 - s) Zeta[ s, y + 1])^(z - k),
  {k, 0, Infinity}] /. s → -1
```

$$\left(1+x^2\,\text{Zeta}[-1,1+y]\right)^z\left(\frac{x\,y\left(1+\frac{1}{x^2\,y}+\frac{\text{Zeta}[-1,1+y]}{y}\right)}{\frac{1}{x}+x\,\text{Zeta}[-1,1+y]}\right)^z$$

```
FullSimplify[x^((1 - s) k) (y^(-s k)) (1 + x^(1 - s) Zeta[ s, y + 1])^(z - k)]
```

$$x^{k-k\,s}\,y^{-k\,s}\left(1+x^{1-s}\,\text{Zeta}[s,1+y]\right)^{-k+z}$$

```
binomial[z_, k_] := binomial[z, k] = Product[z - j, {j, 0, k - 1}] / k!
Ds[n_, 0, s_, a_] := UnitStep[n - 1]
Ds[n_, 1, s_, a_] := Ds[n, 1, s, a] = HarmonicNumber[Floor[n], s] - HarmonicNumber[a, s]
Ds[n_, 2, s_, a_] := Ds[n, 2, s, a] =
  Sum[(m^(-2 s)) + 2 (m^-s) (Ds[Floor[n / m], 1, s, m]), {m, a + 1, Floor[n^(1 / 2)]}]
Ds[n_, k_, s_, a_] := Ds[n, k, s, a] =
  Sum[(m^(-s k)) + k (m^(-s (k - 1))) Ds[Floor[n / (m^(k - 1))], 1, s, m] +
    Sum[binomial[k, j] (m^-s)^j Ds[Floor[n / (m^j)], k - j, s, m], {j, 1, k - 2}],
   {m, a + 1, Floor[n^(1 / k)]}]

Ddy[n_, s_, y_, k_] := y^(k (s - 1)) Ds[n y^k, k, s, y]

Dnsyz[n_, s_, y_, z_] := Expand@Sum[binomial[z, k] Ddy[n, s, y, k], {k, 0, Log[(y + 1) / y, n]}]

dss[n_, s_, y_, z_, x_] :=
 If[n < y, 1, Sum[binomial[z, k] (x y^-s)^k dss[n / y^k, s, y + x, z - k, x], {k, 0, Log[y, n]}]]
dsr[n_, s_, y_, z_, x_] := If[n < x y, 1,
  Sum[binomial[z, k] (x (x y)^-s)^k dsr[n / (x y)^k, s, y + 1, z - k, x], {k, 0, Log[(x y), n]}]]
Expand@dss[100, -1, 1 + 1 / 2, z, 1 / 2]
```

$$1+\frac{5\,872\,221\,148\,009\,z}{4\,844\,421\,120}+\frac{115\,599\,501\,233\,317\,z^2}{52\,848\,230\,400}+\frac{28\,467\,067\,739\,779\,z^3}{23\,488\,102\,400}+\frac{6\,102\,361\,373\,993\,z^4}{16\,911\,433\,728}+\frac{887\,595\,700\,367\,z^5}{18\,790\,481\,920}+\frac{1\,080\,797\,829\,851\,z^6}{241\,591\,910\,400}+\frac{93\,618\,628\,703\,z^7}{563\,714\,457\,600}+\frac{19\,613\,385\,z^8}{3\,758\,096\,384}+\frac{762\,507\,z^9}{9\,395\,240\,960}+\frac{79\,461\,z^{10}}{187\,904\,819\,200}+\frac{2187\,z^{11}}{2\,066\,953\,011\,200}$$

```
Expand@dsr[100, -1, 3, z, 1 / 2]
```

$$1+\frac{5\,872\,221\,148\,009\,z}{4\,844\,421\,120}+\frac{115\,599\,501\,233\,317\,z^2}{52\,848\,230\,400}+\frac{28\,467\,067\,739\,779\,z^3}{23\,488\,102\,400}+\frac{6\,102\,361\,373\,993\,z^4}{16\,911\,433\,728}+\frac{887\,595\,700\,367\,z^5}{18\,790\,481\,920}+\frac{1\,080\,797\,829\,851\,z^6}{241\,591\,910\,400}+\frac{93\,618\,628\,703\,z^7}{563\,714\,457\,600}+\frac{19\,613\,385\,z^8}{3\,758\,096\,384}+\frac{762\,507\,z^9}{9\,395\,240\,960}+\frac{79\,461\,z^{10}}{187\,904\,819\,200}+\frac{2187\,z^{11}}{2\,066\,953\,011\,200}$$

```
Expand@Dnsyz[100, -1, 2, z]
```

$$1+\frac{5872221148009\,z}{4844421120}+\frac{115599501233317\,z^2}{52848230400}+\frac{28467067739779\,z^3}{23488102400}+\frac{6102361373993\,z^4}{16911433728}+\frac{887595700367\,z^5}{18790481920}+\frac{1080797829851\,z^6}{241591910400}+\frac{93618628703\,z^7}{563714457600}+\frac{19613385\,z^8}{3758096384}+\frac{762507\,z^9}{9395240960}+\frac{79461\,z^{10}}{187904819200}+\frac{2187\,z^{11}}{2066953011200}$$

```
esr[n_, s_, y_, z_, x_] := If[n < x y, 1,
  Sum[binomial[z, k] (x (x y)^-s)^k esr[n / (x y)^k, s, y + 1, z - k, x], {k, 0, Log[(x y), n]}]]
```