```
$RecursionLimit = 10 000

10 000

binomial[z_, k_] := binomial[z, k] = Product[z - j, {j, 0, k - 1}] / k!
zetaHurwitz[n_, s_, y_, 0] := UnitStep[n - 1]
zetaHurwitz[n_, s_, y_, 1] :=
 zetaHurwitz[n, s, y, 1] = HarmonicNumber[Floor[n], s] - HarmonicNumber[y, s]
zetaHurwitz[n_, s_, y_, 2] := zetaHurwitz[n, s, y, 2] =
  Sum[(m^(-2 s)) + 2 (m^-s) (zetaHurwitz[Floor[n / m], s, m, 1]), {m, y + 1, Floor[n^(1 / 2)]}]
zetaHurwitz[n_, s_, y_, k_] := zetaHurwitz[n, s, y, k] =
  Sum[(m^(-s k)) + k (m^(-s (k - 1))) zetaHurwitz[Floor[n / (m^(k - 1))], s, m, 1] +
    Sum[binomial[k, j] (m^-s)^j zetaHurwitz[Floor[n / (m^j)], s, m, k - j], {j, 1, k - 2}],
   {m, y + 1, Floor[n^(1 / k)]}]
zeta[n_, s_, z_] := Expand@Sum[binomial[z, k] zetaHurwitz[n, s, 1, k], {k, 0, Log2@n}]
zetaAlt[n_, s_, x_, z_] :=
 Expand@Sum[(-1)^j binomial[z, j] x^(j (1 - s)) zeta[n / (x^j), s, z], {j, 0, Log[x, n]}]
zetaMinus1Scaled[n_, s_, y_, k_] := y^(k (1 - s)) zetaHurwitz[n y^-k, s, y^-1, k]
zetaScaled[n_, s_, y_, z_] :=
 Expand@Sum[binomial[z, k] zetaMinus1Scaled[n, s, y, k], {k, 0, Log[y + 1, n]}]


Po[n_, s_, x_] := Sum[x^(k (1 - s)) / k, {k, 1, Floor@Log[x, n]}] -
  Sum[x^(k (1 - s)) / k, {k, 1, Floor@Log[x, 1.4513692348833810502839684858]}]

Po[100., -1.5, 1.00001]

9628.14

LogIntegral[100.^2.5]

9629.81

Log[1.00001, 1.4513692348833810502839684858]

37 250.9

Log[1.00001, 1.45136]

37 250.3

Clear[dz, Dz, Ez]
FI[n_] := FactorInteger[n]; FI[1] := {}
dz[n_, z_] := dz[n, z] = Expand@Product[(-1)^p[[2]] binomial[-z, p[[2]]], {p, FI[n]}]
Dz[n_, z_] := Dz[n, z] = Expand@Sum[dz[j, z], {j, 1, n}]

Ez[n_, z_, x_] := Expand@Sum[(-1)^j binomial[z, j] x^j Dz[n / (x^j), z], {j, 0, Log[x, n]}]
E2[n_, k_, x_] := Sum[(-1)^(k - j) binomial[k, j] Ez[n, j, x], {j, 0, k}]
E2z[n_, z_, x_] :=
 Sin[Pi z] / Pi Sum[(-1)^k / (z - k) E2[n, k, x], {k, 0, If[x < 2, Log[x, n], Log[2, n]]}]
z2a[n_, x_] := (D[Ez[n, z, x], z] /. z → 0) +
  Sum[x^k / k, {k, 1, Log[x, 1.4513692348833810502839684858]}]

z2[n_, x_] := (D[zetaAlt[n, 0, x, z], z] /. z → 0) +
  Sum[x^k / k, {k, 1, Log[x, 1.4513692348833810502839684858]}]

z2[100, 1.01]

-1.52211
```

```
Sum[PrimePi[100^(1/k)]/k, {k, 1, Log2@100}] - LogIntegral[100.]
```

-1.59281

```
z2[100, 1.005]
```

-1.61434

```
z2a[100, 1.005]
```

$Aborted

```
Sum[PrimePi[6.^(1/k)]/k, {k, 1, Log2@6.}] - LogIntegral[6.]
```

-0.722222

```
z2[6., 1.005]
```

-0.730088

```
z2[6., 1.002]
```

-0.719911

```
Log[1.002, 6]
```

896.775

```
Table[E2[100, k, 2], {k, 0, 8}]
```

{1, -1, 3, -4, -8, 9, -5, 0, 0}

```
Limit[D[E2z[100, z, 3/2], z], z → 0]
```

$$-\frac{8149753}{2365440}$$

```
D[Ez[100, z, 3/2], z] /. z → 0
```

$$-\frac{8149753}{2365440}$$

```
Clear[ff]
ff[n_, s_, x_, y_, z_] := If[n < x y, 1, Expand@Sum[
    binomial[z, k] (x^(1 - s) y^-s)^k ff[n / (x y)^k, s, x, 1 + y, z - k], {k, 0, Log[x y, n]}]]
f2[n_, s_, x_, z_] := ff[n, s, x, 1 + 1/x, z]
```

```
N@f2[100, -1, 1/3, z]
```

$$1. + 1193.42\,z + 2167.06\,z^2 + 1216.08\,z^3 + 373.387\,z^4 + 59.4418\,z^5 + 6.19276\,z^6 + 0.3988\,z^7 + 0.018873\,z^8 + 0.000503346\,z^9 + 0.0000125962\,z^{10} + 1.62017\times10^{-7}\,z^{11} + 1.39947\times10^{-9}\,z^{12} + 1.31732\times10^{-11}\,z^{13} + 2.0211\times10^{-14}\,z^{14} + 6.54708\times10^{-17}\,z^{15} + 1.1078\times10^{-19}\,z^{16}$$

```
N@zetaScaled[10, 0, 1/2, z]
```

$$1. + 5.10521\,z + 3.29167\,z^2 + 0.566406\,z^3 + 0.0364583\,z^4 + 0.000260417\,z^5$$

```
Clear[gg]
gg[n_, s_, x_, j_, z_] := If[n < j, 1, Expand@
   Sum[binomial[z, k] x^-k j^(-s k) gg[n / j^k, s, x, j + 1/x, z - k], {k, 0, Log[j, n]}]]
g2[n_, s_, x_, z_] := gg[n, s, x, 1 + 1/x, z]
```

```
N@g2[10, 0, 2, z]
```

$$1. + 5.10521\,z + 3.29167\,z^2 + 0.566406\,z^3 + 0.0364583\,z^4 + 0.000260417\,z^5$$

```
Clear[hh]
t[n_, x_] :=
 (Floor[n] - Floor[n - 1 / x]) - (x + 1) / x (Floor[n x / (x + 1)] - Floor[(n x - 1) / (x + 1)])
pw[n_, k_] := If[n == 0 && k == 0, 1, n^k]
hh[n_, s_, x_, j_, z_] := If[n < j, 1,
  Expand@Sum[ binomial[z, k] pw[t[j, x], k] j^(-s k) hh[n / j^k, s, x, j + 1 / x, z - k],
    {k, 0, Log[j, n]}]]
h2[n_, s_, x_, z_] := hh[n, s, x, 1 + 1 / x, z]
```

```
h2[10, 0, 3, z]
```

$$1-\frac{589516\,z}{229635}+\frac{1500281\,z^2}{459270}-\frac{1052917\,z^3}{590490}+\frac{341204\,z^4}{295245}-\frac{141968\,z^5}{295245}+\frac{24704\,z^6}{295245}-\frac{17408\,z^7}{2066715}+\frac{512\,z^8}{2066715}$$

```
zetaAlt[10, 0, 4 / 3, z]
```

$$1-\frac{589516\,z}{229635}+\frac{1500281\,z^2}{459270}-\frac{1052917\,z^3}{590490}+\frac{341204\,z^4}{295245}-\frac{141968\,z^5}{295245}+\frac{24704\,z^6}{295245}-\frac{17408\,z^7}{2066715}+\frac{512\,z^8}{2066715}$$

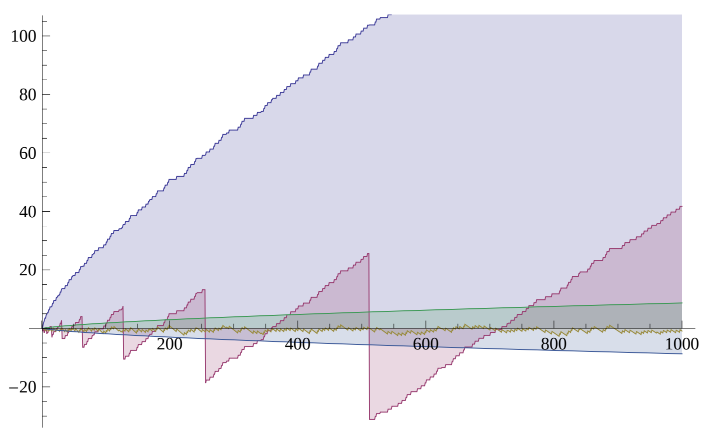
```
Clear[ee, ef]
ef[n_, j_, z_] := ef[n, j, z] = Expand@If[n < j, 1,
    Sum[binomial[z, k] (-1)^((j + 1) k) ef[n / j^k, j + 1, z - k], {k, 0, Log[j, n]}]]
ee[n_, j_, z_] := ee[n, j, z] = Expand@If[n < j, 1,
    Table[binomial[z, k] (-1)^((j + 1) k) ef[n / j^k, j + 1, z - k], {k, 0, Log[j, n]}]]
```

```
D[ee[100, 2, z], z] /. z → 0
```

$$\left\{-\frac{9}{4},\ 2,\ -1,\ \frac{5}{3},\ -\frac{1}{4},\ \frac{4}{5},\ -\frac{1}{6}\right\}$$

```
pl[n_] := Sum[PrimePi[n^(1 / k)] / k, {k, 1, Log2@n}] - Sum[2^k / k, {k, 1, Log2@n}]
pl2[n_] := Sum[PrimePi[n^(1 / k)] / k, {k, 1, Log2@n}]
pl3[n_] := Sum[PrimePi[n^(1 / k)] / k, {k, 1, Log2@n}] - LogIntegral[n]
```

```
DiscretePlot[
 {pl2[n], pl[n], pl3[n], 1 / (8 Pi) Log[n] n^(1 / 2), -1 / (8 Pi) Log[n] n^(1 / 2)}, {n, 1, 1000}]
```




```
N@pl2[100]
```

28.5333