```
Clear[fa]
bin[z_, k_] := bin[z, k] = Product[z - j, {j, 0, k - 1}] / k!
fa[n_, s_, x_, k_] :=
 fa[n, s, x, k] = Sum[ - (x^(j (1 - s)) / j) fa[ n / (x^j), s, x, k - 1], {j, 1, Log[x, n]}]
fa[n_, s_, x_, 0] := UnitStep[n - 1]
fz[n_, s_, x_, z_] := Sum[ z^k / k! fa[n, s, x, k], {k, 0, Log[x, n]}]
fzz[n_, s_, x_, z_] := Sum[ x^(k (1 - s)) Pochhammer[-z, k] / k! , {k, 0, Log[x, n]}]
roots[n_, s_, x_] := If[(c = Exponent[f = fzz[n, s, x, z], z]) == 0, {},
  If[c == 1, List@NRoots[f == 0, z][[2]], List @@ NRoots[f == 0, z][[All, 2]]]]
rootsa[n_, s_, x_] := If[(c = Exponent[f = fzz[n, s, x, z], z]) == 0, {},
  If[c == 1, List@Roots[f == 0, z][[2]], List @@ Roots[f == 0, z][[All, 2]]]]
proots[n_, s_, x_, z_] := Product[ 1 - z / r, {r, roots[n, s, x]}]
prootsa[n_, s_, x_, z_] := Product[ 1 - z / r, {r, rootsa[n, s, x]}]
srootsa[n_, s_, x_] := Sum[ -1 / r, {r, rootsa[n, s, x]}]
```

```
Table[fz[n, -1, 2, 2], {n, 2, 100}]
```

{-7, -7, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9,
9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9,
9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9}

```
Table[fzz[n, -1, 2, 2], {n, 2, 100}]
```

{-7, -7, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9,
9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9,
9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9}

```
Table[fa[n, 2, 1], {n, 2, 100}]
```

$$\left\{2, 2, 4, 4, 4, 4, \frac{20}{3}, \frac{20}{3}, \frac{20}{3}, \frac{20}{3}, \frac{20}{3}, \frac{20}{3}, \frac{20}{3}, \frac{20}{3}, \frac{32}{3}, \frac{32}{3}, \frac{32}{3}, \frac{32}{3}, \frac{32}{3}, \frac{32}{3}, \frac{32}{3}, \frac{32}{3},\right.$$
$$\frac{32}{3}, \frac{32}{3}, \frac{32}{3}, \frac{32}{3}, \frac{32}{3}, \frac{32}{3}, \frac{32}{3}, \frac{32}{3}, \frac{256}{15}, \frac{256}{15}, \frac{256}{15}, \frac{256}{15}, \frac{256}{15}, \frac{256}{15}, \frac{256}{15}, \frac{256}{15}, \frac{256}{15},$$
$$\frac{256}{15}, \frac{256}{15}, \frac{256}{15}, \frac{256}{15}, \frac{256}{15}, \frac{256}{15}, \frac{256}{15}, \frac{256}{15}, \frac{256}{15}, \frac{256}{15}, \frac{256}{15}, \frac{256}{15}, \frac{256}{15}, \frac{256}{15}, \frac{256}{15},$$
$$\frac{256}{15}, \frac{256}{15}, \frac{256}{15}, \frac{256}{15}, \frac{256}{15}, \frac{256}{15}, \frac{256}{15}, \frac{256}{15}, \frac{416}{15}, \frac{416}{15}, \frac{416}{15}, \frac{416}{15}, \frac{416}{15}, \frac{416}{15}, \frac{416}{15},$$
$$\frac{416}{15}, \frac{416}{15}, \frac{416}{15}, \frac{416}{15}, \frac{416}{15}, \frac{416}{15}, \frac{416}{15}, \frac{416}{15}, \frac{416}{15}, \frac{416}{15}, \frac{416}{15}, \frac{416}{15}, \frac{416}{15}, \frac{416}{15}, \frac{416}{15},$$
$$\left.\frac{416}{15}, \frac{416}{15}, \frac{416}{15}, \frac{416}{15}, \frac{416}{15}, \frac{416}{15}, \frac{416}{15}, \frac{416}{15}, \frac{416}{15}, \frac{416}{15}, \frac{416}{15}, \frac{416}{15}, \frac{416}{15}, \frac{416}{15}, \frac{416}{15}\right\}$$

```
binomial[z_, k_] := binomial[z, k] = Product[z - j, {j, 0, k - 1}] / k!
zetaHurwitz[n_, s_, y_, 0] := UnitStep[n - 1]
zetaHurwitz[n_, s_, y_, 1] :=
 zetaHurwitz[n, s, y, 1] = HarmonicNumber[Floor[n], s] - HarmonicNumber[y, s]
zetaHurwitz[n_, s_, y_, 2] := zetaHurwitz[n, s, y, 2] =
  Sum[(m^(-2 s)) + 2 (m^-s) (zetaHurwitz[Floor[n / m], s, m, 1]), {m, y + 1, Floor[n^(1 / 2)]}]
zetaHurwitz[n_, s_, y_, k_] := zetaHurwitz[n, s, y, k] =
  Sum[(m^(-s k)) + k (m^(-s (k - 1))) zetaHurwitz[Floor[n / (m^(k - 1))], s, m, 1] +
    Sum[binomial[k, j] (m^-s)^j zetaHurwitz[Floor[n / (m^j)], s, m, k - j], {j, 1, k - 2}],
   {m, y + 1, Floor[n^(1 / k)]}]
zeta[n_, s_, z_] := Expand@Sum[binomial[z, k] zetaHurwitz[n, s, 1, k], {k, 0, Log2@n}]

zetaAlt[n_, s_, x_, z_] :=
 Expand@Sum[(-1)^j binomial[z, j] x^(j (1 - s)) zeta[n / (x^j), s, z], {j, 0, Log[x, n]}]
da[n_, x_, z_] := zetaAlt[n, 0, x, z] - zetaAlt[n - 1, 0, x, z]
dzz[n_, z_] := zeta[n, 0, z] - zeta[n - 1, 0, z]

Clear[dv]
dv[n_, x_] := dv[n, x] = - D[da[n, x, z] - dzz[n, z], {z, 1}] /. z → 0


Table[dv[n, 2], {n, 2, 100}]
```

$$\left\{2, 0, 2, 0, 0, 0, \frac{8}{3}, 0, 0, 0, 0, 0, 0, 0, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{32}{5}, 0, 0,\right.$$
$$0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{32}{3}, 0, 0, 0,$$
$$\left. 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right\}$$

```
Clear[fv]
fv[n_, x_, k_] := fv[n, x, k] = Sum[ dv[j, x] fv[ n / j, x, k - 1], {j, 2, n}]
fv[n_, x_, 0] := UnitStep[n - 1]
fvz[n_, x_, z_] := Sum[ z^k / k! fv[n, x, k], {k, 0, Log[2, n]}]
Table[fv[32, 3, k], {k, 1, 10}]
```

$$\left\{\frac{33}{2}, 36, 27, 0, 0, 0, 0, 0, 0, 0\right\}$$

```
Table[fvz[n, 3, 1] - fvz[n - 1, 3, 1], {n, 2, 81}]
```

```
{0, 3, 0, 0, 0, 0, 0, 9, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 27,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 81}


Sum[ x^k, {k, 0, Floor@Log[x, n]}]
```

$$\frac{-1 + x^{1+\text{Floor}\left[\frac{\text{Log}[n]}{\text{Log}[x]}\right]}}{-1 + x}$$

```
x^k Pochhammer[z, k] / k!
```

```
fzz[n_, s_, x_, z_] := 1 + Sum[ x^(k (1 - s)) Pochhammer[z, k] / k! , {k, 1, Log[x, n]}]
fzzb[n_, x_, z_] := Table[ x^k Pochhammer[z, k] / k! , {k, 1, Log[x, n]}]
fzza[n_, x_, z_, j_] :=
 Sum[ D[x^k Pochhammer[z, k] / k!, {z, j}] /. z → 0 , {k, 1, Log[x, n]}]
```

```
Expand@fzz[100, 1, 2, z]
```

$$1+\frac{49\,z}{20}+\frac{203\,z^2}{90}+\frac{49\,z^3}{48}+\frac{35\,z^4}{144}+\frac{7\,z^5}{240}+\frac{z^6}{720}$$

```
fz[100, 1, 2, z]
```

$$1+\frac{49\,z}{20}+\frac{203\,z^2}{90}+\frac{49\,z^3}{48}+\frac{35\,z^4}{144}+\frac{7\,z^5}{240}+\frac{z^6}{720}$$

```
D[z (z + 1) (z + 2) (z + 3), {z, 5}] /. z → 0
```

0

```
Table[ D[zetaAlt[n, 0, 3 / 2, z] - zetaAlt[n - 1 / 2, 0, 3 / 2, z] , z] /. z → 0, {n, 2, 100, 1 / 2}]
```

$$\left\{1,-\frac{9}{8},1,-\frac{9}{8},\frac{1}{2},0,1,-\frac{81}{64},0,0,1,0,-\frac{569}{480},0,\frac{1}{2},0,0,0,1,-\frac{243}{128},0,0,1,0,0,0,\right.$$
$$0,0,\frac{1}{4},0,1,-\frac{2187}{896},0,0,1,0,0,0,0,0,0,0,1,0,0,0,\frac{1}{2},0,-\frac{6561}{2048},0,\frac{1}{3},0,$$
$$0,0,1,0,0,0,1,0,\frac{1}{5},0,0,0,0,0,0,0,0,0,1,0,0,-\frac{2187}{512},0,0,0,0,1,0,0,$$
$$0,1,0,0,0,0,0,0,0,1,0,0,0,\frac{1}{2},0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,$$
$$0,-\frac{59049}{10240},0,1,0,0,0,1,0,0,0,0,0,\frac{1}{6},0,0,0,0,0,1,0,0,0,0,0,0,0,1,$$
$$0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,\frac{1}{4},0,0,0,1,0,0,0,0,0,0,$$
$$\left.-\frac{177147}{22528},0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0\right\}$$

```
Sum[ x^(k (1 - s)) Pochhammer[-z, k] / k!, {k, 0, Infinity}]
```

$$\left(1-x^{1-s}\right)^z$$

```
Sum[ (-x) ^(k (1 - s)) Pochhammer[z, k] / k!, {k, 0, Infinity}]
```

$$\left(1+(-x)^{-s}\,x\right)^{-z}$$

```
FullSimplify@Sum[ -x^(k (1 - s)) / k, {k, 1, Infinity}]
```

$$\mathrm{Log}\left[1-x^{1-s}\right]$$

```
FullSimplify@Sum[ (-x^(j (1 - s)) / j) (-x^(k (1 - s)) / k), {j, 1, Infinity}, {k, 1, Infinity}]
```

$$\mathrm{Log}\left[1-x^{1-s}\right]^2$$

```
Sum[ z^k / k! Log[1 - x^(1-s)]^k, {k, 0, Infinity}]
```

$$\left(1-x^{1-s}\right)^z$$

```
D[ (1 - x^(1-s))^z, {z, 2}] /. z → 0
```

$$\mathrm{Log}\left[1-x^{1-s}\right]^2$$

```
pp[s_, x_, z_] := (1 - x^(1 - s))^z
```

```
Table[{Chop@N@(fzz[100 000 000 000, s, 2, z] - pp[s, 2, z])}, {s, -3, 3}, {z, -3, 3}] // Grid
```

Power::infy : Infinite expression $\frac{1}{0^3}$ encountered. ≫

Power::infy : Infinite expression $\frac{1}{0^2}$ encountered. ≫

Power::infy : Infinite expression $\frac{1}{0}$ encountered. ≫

General::stop : Further output of Power::infy will be suppressed during this calculation. ≫

Power::indet : Indeterminate expression $0^0$ encountered. ≫

| | | | | | | |
|---|---|---|---|---|---|---|
| $\{1.6664\times10^{46}\}$ | $\{8.7855\times10^{44}\}$ | $\{2.37875\times10^{43}\}$ | {0} | {0} | {0} | {0} |
| $\{2.58775\times10^{35}\}$ | $\{1.36695\times10^{34}\}$ | $\{3.70878\times10^{32}\}$ | {0} | {0} | {0} | {0} |
| $\{4.34947\times10^{24}\}$ | $\{2.30871\times10^{23}\}$ | $\{6.29649\times10^{21}\}$ | {0} | {0} | {0} | {0} |
| $\{9.16718\times10^{13}\}$ | $\{4.9478\times10^{12}\}$ | $\{1.37439\times10^{11}\}$ | {0} | {0} | {0} | {0} |
| {ComplexInfinity} | {ComplexInfinity} | {ComplexInfinity} | {Indeterminate} | {0} | {0} | {0} |
| $\{-1.1365\times10^{-8}\}$ | $\{-5.67525\times10^{-10}\}$ | {0} | {0} | {0} | {0} | {0} |
| {0} | {0} | {0} | {0} | {0} | {0} | {0} |

```
Table[{Chop@N@(fzz[100 000 000 000, s, 5, z] - pp[s, 5, z])}, {s, -3, 3}, {z, -3, 3}] // Grid
```

Power::infy : Infinite expression $\frac{1}{0^3}$ encountered. ≫

Power::infy : Infinite expression $\frac{1}{0^2}$ encountered. ≫

Power::infy : Infinite expression $\frac{1}{0}$ encountered. ≫

General::stop : Further output of Power::infy will be suppressed during this calculation. ≫

Power::indet : Indeterminate expression $0^0$ encountered. ≫

| | | | | | | |
|---|---|---|---|---|---|---|
| $\{1.18128\times10^{44}\}$ | $\{1.38986\times10^{43}\}$ | $\{8.68752\times10^{41}\}$ | {0} | {0} | {0} | {0} |
| $\{3.89283\times10^{33}\}$ | $\{4.58184\times10^{32}\}$ | $\{2.86509\times10^{31}\}$ | {0} | {0} | {0} | {0} |
| $\{1.31292\times10^{23}\}$ | $\{1.54816\times10^{22}\}$ | $\{9.70128\times10^{20}\}$ | {0} | {0} | {0} | {0} |
| $\{5.03778\times10^{12}\}$ | $\{6.00815\times10^{11}\}$ | $\{3.8147\times10^{10}\}$ | {0} | {0} | {0} | {0} |
| {ComplexInfinity} | {ComplexInfinity} | {ComplexInfinity} | {Indeterminate} | {0} | {0} | {0} |
| $\{-1.29075\times10^{-9}\}$ | $\{-1.41312\times10^{-10}\}$ | {0} | {0} | {0} | {0} | {0} |
| {0} | {0} | {0} | {0} | {0} | {0} | {0} |

```
Table[{Chop@N@(fzz[10 000 000 000 000, s, 1.5, z] - pp[s, 1.5, z])},
  {s, -3, 3, .5}, {z, -3, 3, .5}] // Grid
```

Power::infy : Infinite expression $\frac{1}{0.^{3.}}$ encountered. ≫

Power::infy : Infinite expression $\frac{1}{0.^{2.5}}$ encountered. ≫

Power::infy : Infinite expression $\frac{1}{0.^{2.}}$ encountered. ≫

General::stop : Further output of Power::infy will be suppressed during this calculation. ≫

Power::indet : Indeterminate expression $0.^{0.}$ encountered. ≫

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\{9.00807 \times 10^{54}\}$ | $\{1.56483 \times 10^{54} + 0.0300619\, i\}$ | $\{2.40993 \times 10^{53}\}$ | $\{3.16102 \times 10^{52} - 0.122127\, i\}$ | $\{3.26753 \times 10^{51}\}$ | $\{2.15764 \times 10^{50} + 0.496139\, i\}$ | $\{0\}$ | $\{-1.49323 \times 10^{48} - 2.01556\, i\}$ | $\{0\}$ | $\{3.14397 \times 10^{46} + 8.18823\, i\}$ | $\{0\}$ | $\{-$ |
| $\{3.55772 \times 10^{48}\}$ | $\{6.1832 \times 10^{47} + 0.0575335\, i\}$ | $\{9.52707 \times 10^{46}\}$ | $\{1.25025 \times 10^{46} - 0.180282\, i\}$ | $\{1.29302 \times 10^{45}\}$ | $\{8.54251 \times 10^{43} + 0.564916\, i\}$ | $\{0\}$ | $\{-5.91817 \times 10^{41} - 1.77017\, i\}$ | $\{0\}$ | $\{1.24741 \times 10^{40} + 5.54686\, i\}$ | $\{0\}$ | $\{-$ |
| $\{1.42884 \times 10^{42}\}$ | $\{2.48491 \times 10^{41} + 0.115038\, i\}$ | $\{3.83133 \times 10^{40}\}$ | $\{5.03134 \times 10^{39} - 0.273215\, i\}$ | $\{5.2071 \times 10^{38}\}$ | $\{3.4426 \times 10^{37} + 0.648886\, i\}$ | $\{0\}$ | $\{-2.3885 \times 10^{35} - 1.5411\, i\}$ | $\{0\}$ | $\{5.04207 \times 10^{33} + 3.66012\, i\}$ | $\{0\}$ | $\{-$ |
| $\{5.87633 \times 10^{35}\}$ | $\{1.02294 \times 10^{35} + 0.244844\, i\}$ | $\{1.57875 \times 10^{34}\}$ | $\{2.07531 \times 10^{33} - 0.429866\, i\}$ | $\{2.15 \times 10^{32}\}$ | $\{1.42292 \times 10^{31} + 0.754706\, i\}$ | $\{0\}$ | $\{-9.89364 \times 10^{28} - 1.32502\, i\}$ | $\{0\}$ | $\{2.09323 \times 10^{27} + 2.3263\, i\}$ | $\{0\}$ | $\{-$ |
| $\{2.50378 \times 10^{29}\}$ | $\{4.36501 \times 10^{28} + 0.572433\, i\}$ | $\{6.74695 \times 10^{27}\}$ | $\{8.88281 \times 10^{26} - 0.715542\, i\}$ | $\{9.21714 \times 10^{25}\}$ | $\{6.11008 \times 10^{24} + 0.894427\, i\}$ | $\{0\}$ | $\{-4.26274 \times 10^{22} - 1.11803\, i\}$ | $\{0\}$ | $\{9.0509 \times 10^{20} + 1.39754\, i\}$ | $\{0\}$ | $\{-$ |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| {1.129⋯22 × $10^{23}$} | {1.9736 × $10^{22}$ + 1.55⋯968 i} | {3.058⋯49 × $10^{21}$} | {4.037⋯45 × $10^{20}$ − 1.30⋯563 i} | {4.200⋯91 × $10^{19}$} | {2.792⋯67 × $10^{18}$ + 1.09⋯297 i} | {0} | {−1.9⋯59⋯86 × $10^{16}$ − 0.9⋯14⋯94⋯1 i} | {0} | {4.187⋯47 × $10^{14}$ + 0.76⋯591⋯3 i} | {0} | {− |
| {5.648⋯12 × $10^{16}$} | {9.921⋯71 × $10^{15}$ + 5.65⋯685 i} | {1.545⋯67 × $10^{15}$} | {2.051⋯56 × $10^{14}$ − 2.82⋯843 i} | {2.146⋯76 × $10^{13}$} | {1.435⋯56 × $10^{12}$ + 1.41⋯421 i} | {0} | {−1.0⋯20⋯16 × $10^{10}$ − 0.7⋯07⋯10⋯7 i} | {0} | {2.209⋯64 × $10^{8}$ + 0.35⋯355⋯3 i} | {0} | {− |
| {3.594⋯17 × $10^{10}$} | {6.407⋯13 × $10^{9}$ + 41.7⋯614 i} | {1.013⋯88 × $10^{9}$} | {1.3684 × $10^{8}$ − 9.38⋯567 i} | {1.457⋯76 × $10^{7}$} | {99378⋯5. + 2.10⋯938 i} | {0} | {−737⋯6.⋯54 − 0.4⋯74⋯07⋯3 i} | {0} | {168.38 + 0.10⋯654⋯5 i} | {0} | {− |
| {ComplexInfinity} | {ComplexInfinity} | {ComplexInfinity} | {ComplexInfinity} | {ComplexInfinity} | {ComplexInfinity} | {Indeterminate} | {0.06⋯592⋯04} | {0} | {−0.00⋯045⋯462⋯4} | {0} | {9. 1 |
| {−0.00⋯533⋯59} | {−0.00⋯089⋯151⋯9} | {−0.00⋯013⋯215⋯1} | {−0.00⋯001⋯669⋯54} | {−1.66⋯333 × $10^{-6}$} | {−1.05⋯926 × $10^{-7}$} | {0} | {6.83⋯017 × $10^{-10}$} | {0} | {0} | {0} | · |
| {−8.40⋯142 × $10^{-10}$} | {−1.42⋯803 × $10^{-10}$} | {0} | {0} | {0} | {0} | {0} | {0} | {0} | {0} | {0} | · |
| {0} | {0} | {0} | {0} | {0} | {0} | {0} | {0} | {0} | {0} | {0} | · |
| {0} | {0} | {0} | {0} | {0} | {0} | {0} | {0} | {0} | {0} | {0} | · |

**N@Chop@rootsa[10 000, 1, 2]**

{1., 2., 3., 4., 5., 6., 7., 8., 9., 10., 11., 12., 13.}

**pp[1.5, 2, 2]**

0.0857864

**prootsa[100 000 000, 1, 2, 20]**

0

```
fzz[1 000 000 000, .5, 2, z]
```

```
N@srootsa[1 000 000 000, 1, 3 / 2]
```

```
-4.51881
```

```
Log[ 3 / 2, 1 000 000 000.]
```

```
51.1099
```

```
N@HarmonicNumber[51]
```

```
4.51881
```

```
Product[ 1 - 1.5 / k, {k, 1, Log[2, 10 000 000 000 000]}]
```

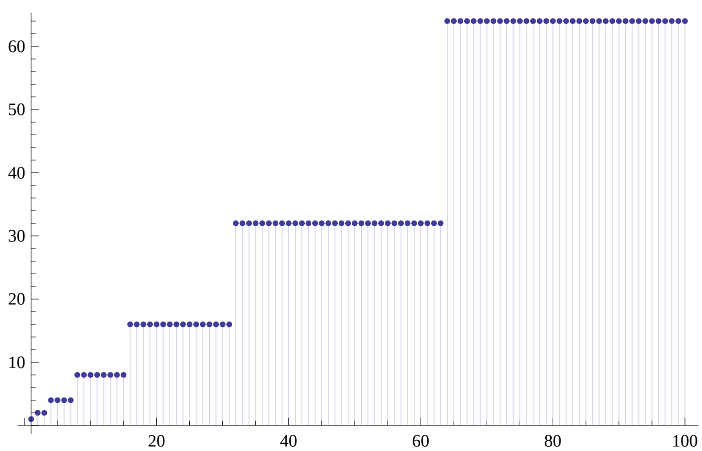
```
-0.00100927
```

```
Product[ 1 - (1.01 + 3000 I) / k, {k, 1, Infinity}]
```

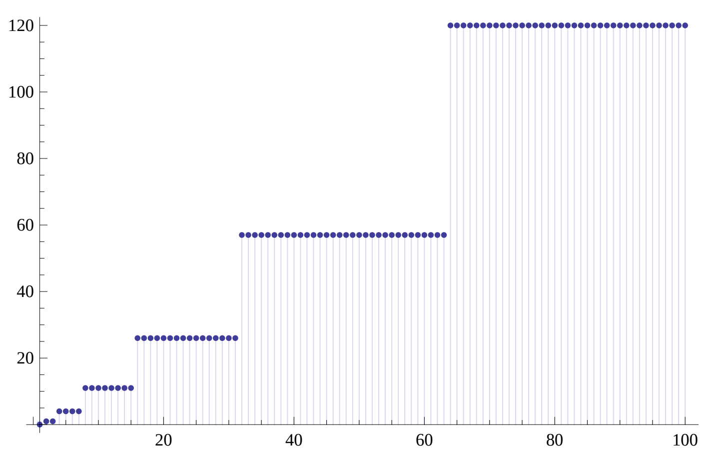
```
0.
```

```
Clear[fob]
fob[n_, s_, x_, k_] :=
 fob[n, s, x, k] = Sum[ - ((x^(j (1 - s)) - 1) / j) fob[ n / (x^j), s, x, k - 1], {j, 1, Log[x, n]}]
fob[n_, s_, x_, 0] := UnitStep[n - 1]
fobz[n_, s_, x_, z_] := Sum[ z^k / k! fob[n, s, x, k], {k, 0, Log[x, n]}]
```

```
DiscretePlot[ fobz[n, 0, 2, -1], {n, 1, 100}]
```




```
DiscretePlot[ fbzz[n, 0, 2, -1], {n, 1, 100}]
```



```
FullSimplify@Sum[ (-1)^(k+1) x^(k (1-s)) / k, {k, 1, Infinity}]
```

$\text{Log}\left[1+x^{1-s}\right]$

```
Sum[ Binomial[z, k] x^(k (1-s)), {k, 0, Infinity}]
```

$\left(1+x^{1-s}\right)^z$

```
FullSimplify@Sum[ x^(k (1-s)) / k, {k, 1, Infinity}]
```

$-\text{Log}\left[1-x^{1-s}\right]$

```
Sum[ z^k / (k!) (-Log[1 - x^(1-s)])^k, {k, 0, Infinity}]
```

$\left(1-x^{1-s}\right)^{-z}$

```
(* alternating sign version *)

Clear[fc]
fc[n_, s_, x_, k_] := fc[n, s, x, k] =
  Sum[ (-1)^(j+1) (x^(j (1-s)) / j) fc[ n / (x^j), s, x, k-1], {j, 1, Log[x, n]}]
fc[n_, s_, x_, 0] := UnitStep[n-1]
fcz[n_, s_, x_, z_] := Sum[ z^k / k! fc[n, s, x, k], {k, 0, Log[x, n]}]
fczz[n_, s_, x_, z_] := Sum[ x^(k (1-s)) bin[z, k] , {k, 0, Log[x, n]}]
croots[n_, s_, x_] := If[(c = Exponent[f = fczz[n, s, x, z], z]) == 0, {},
  If[c == 1, List@NRoots[f == 0, z][[2]], List @@ NRoots[f == 0, z][[All, 2]]]]
crootsa[n_, s_, x_] := If[(c = Exponent[f = fczz[n, s, x, z], z]) == 0, {},
  If[c == 1, List@Roots[f == 0, z][[2]], List @@ Roots[f == 0, z][[All, 2]]]]
cproots[n_, s_, x_, z_] := Product[ 1 - z / r, {r, croots[n, s, x]}]
cprootsa[n_, s_, x_, z_] := Product[ 1 - z / r, {r, crootsa[n, s, x]}]
csrootsa[n_, s_, x_] := Sum[ -1 / r, {r, crootsa[n, s, x]}]
```

```
Table[ fcz[n, -2, 2, -1], {n, 1, 100}]
```

```
{1, -7, -7, 57, 57, 57, 57, -455, -455, -455, -455, -455, -455, -455, -455, 3641,
 3641, 3641, 3641, 3641, 3641, 3641, 3641, 3641, 3641, 3641, 3641, 3641, 3641, 3641,
 3641, -29127, -29127, -29127, -29127, -29127, -29127, -29127, -29127, -29127,
 -29127, -29127, -29127, -29127, -29127, -29127, -29127, -29127, -29127, -29127,
 -29127, -29127, -29127, -29127, -29127, -29127, -29127, -29127, -29127, -29127,
 -29127, -29127, -29127, 233017, 233017, 233017, 233017, 233017, 233017, 233017,
 233017, 233017, 233017, 233017, 233017, 233017, 233017, 233017, 233017, 233017,
 233017, 233017, 233017, 233017, 233017, 233017, 233017, 233017, 233017, 233017,
 233017, 233017, 233017, 233017, 233017, 233017, 233017, 233017, 233017, 233017}
```

```
Table[ fczz[n, 1, 2, -1], {n, 1, 100}]
```

```
{1, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1,
 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
```

```
N@crootsa[10000, 1, 3]
```

```
{-0.821667 - 0.917058 i, -0.821667 + 0.917058 i, 0.476791 - 2.88837 i, 0.476791 + 2.88837 i,
 2.9943 - 4.8656 i, 2.9943 + 4.8656 i, 7.35057 - 6.40637 i, 7.35057 + 6.40637 i}
```

```
fczz[100, 1, 2, 3]
```

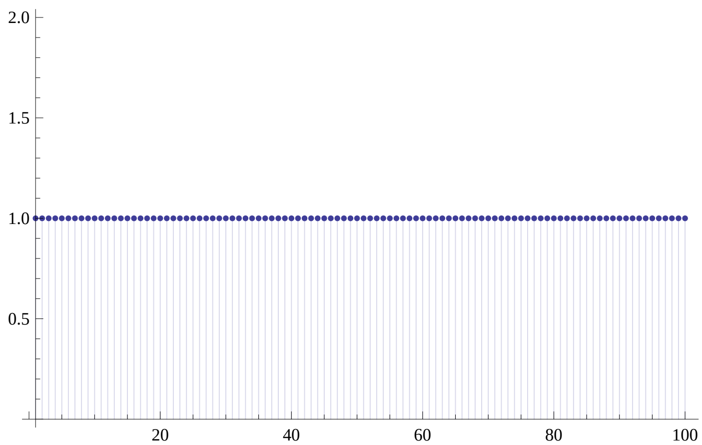
8

```
Chop@N@cprootsa[10000, 1, 2, 3]
```

8.

```
Clear[foc]
foc[n_, s_, x_, k_] := foc[n, s, x, k] =
  Sum[ (-1)^(j+1) ((x^(j (1-s)) - 1) / j) foc[ n / (x^j), s, x, k-1], {j, 1, Log[x, n]}]
foc[n_, s_, x_, 0] := UnitStep[n-1]
focz[n_, s_, x_, z_] := Sum[ z^k / k! foc[n, s, x, k], {k, 0, Log[x, n]}]
```

```
DiscretePlot[ focz[n, 1, 1.2, 2], {n, 1, 100}]
```




```
focz[100, -1, 1.2, z]
```

$1 + 211.465\,z - 528.78\,z^2 + 518.614\,z^3 - 272.456\,z^4 + 87.905\,z^5 - 18.8544\,z^6 + 2.83284\,z^7 - 0.309427\,z^8 + 0.0252538\,z^9 - 0.00157225\,z^{10} + 0.0000758541\,z^{11} - 2.86984\times10^{-6}\,z^{12} + 8.58858\times10^{-8}\,z^{13} - 2.04509\times10^{-9}\,z^{14} + 3.8872\times10^{-11}\,z^{15} - 5.90156\times10^{-13}\,z^{16} + 7.14094\times10^{-15}\,z^{17} - 6.84929\times10^{-17}\,z^{18} + 5.15843\times10^{-19}\,z^{19} - 3.00514\times10^{-21}\,z^{20} + 1.32335\times10^{-23}\,z^{21} - 4.24849\times10^{-26}\,z^{22} + 9.36208\times10^{-29}\,z^{23} - 1.26368\times10^{-31}\,z^{24} + 7.8645\times10^{-35}\,z^{25}$

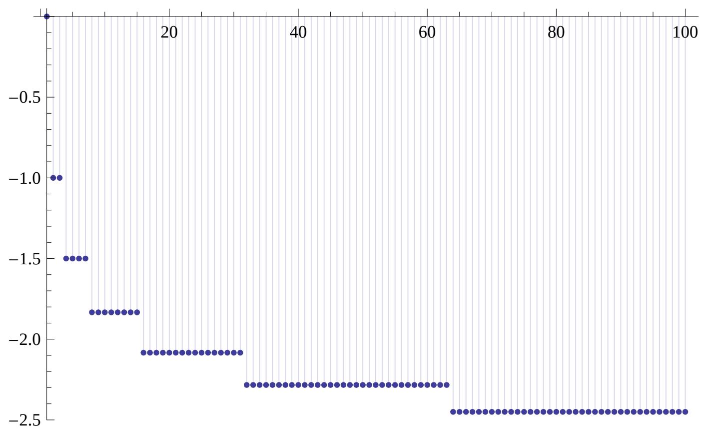
```
fob[100, 1, 1.2, 1]
```

0.

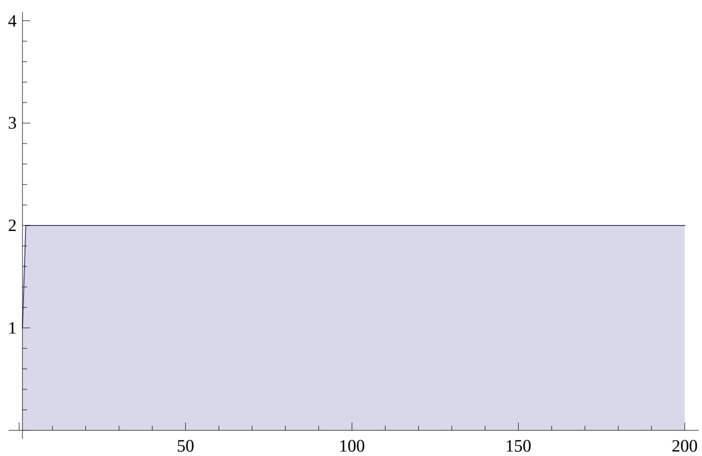
```
Clear[fop]
fop[n_, x_, k_] := fop[n, x, k] = Sum[ - (1/j) fop[ n / (x^j), x, k-1], {j, 1, Log[x, n]}]
fop[n_, x_, 0] := UnitStep[n-1]
fopz[n_, x_, z_] := Sum[ z^k / k! fop[n, x, k], {k, 0, Log[x, n]}]
fopzz[n_, x_, z_] := Sum[(-1)^k bin[z, k], {k, 0, Log[x, n]}]
parootsa[n_, x_] := If[(c = Exponent[f = fopzz[n, x, z], z]) == 0, {},
  If[c == 1, List@Roots[f == 0, z][[2]], List @@ Roots[f == 0, z][[All, 2]]]]
pprootsa[n_, x_, z_] := Product[ 1 - z / r, {r, parootsa[n, x]}]
psrootsa[n_, x_] := Sum[ -1 / r, {r, parootsa[n, x]}]
```

```
Clear[foq]
foq[n_, x_, k_] :=
 foq[n, x, k] = Sum[ (-1)^(j+1) ((1) / j) foq[ n / (x^j), x, k-1], {j, 1, Log[x, n]}]
foq[n_, x_, 0] := UnitStep[n-1]
foqz[n_, x_, z_] := Sum[ z^k / k! foq[n, x, k], {k, 0, Log[x, n]}]
foqzz[n_, x_, z_] := Sum[bin[z, k], {k, 0, Log[x, n]}]
qrootsa[n_, x_] := If[(c = Exponent[f = foqzz[n, x, z], z]) == 0, {},
  If[c == 1, List@Roots[f == 0, z][[2]], List @@ Roots[f == 0, z][[All, 2]]]]
qprootsa[n_, x_, z_] := Product[ 1 - z / r, {r, qrootsa[n, x]}]
qsrootsa[n_, x_] := Sum[ -1 / r, {r, qrootsa[n, x]}]
```

```
DiscretePlot[ fop[n, 2, 1], {n, 1, 100}]
```




```
DiscretePlot[ foqzz[n, 2, 1], {n, 1, 200}]
```




```
Table[ FullSimplify[fopz[2^k, 2, z] - fopz[2^k-1, 2, z]], {k, 1, 5}] // TableForm
```

$z$

$\frac{1}{2} z (1+z)$

$\frac{1}{6} z (1+z) (2+z)$

$\frac{1}{24} z (1+z) (2+z) (3+z)$

$\frac{1}{120} z (1+z) (2+z) (3+z) (4+z)$

```
Table[ FullSimplify[fopzz[2^k, 2, z] - fopzz[2^k - 1, 2, z]], {k, 1, 5}] // TableForm
```

$z$

$\frac{1}{2} z (1+z)$

$\frac{1}{6} z (1+z) (2+z)$

$\frac{1}{24} z (1+z) (2+z) (3+z)$

$\frac{1}{120} z (1+z) (2+z) (3+z) (4+z)$

```
Table[ FullSimplify[foqz[2^k, 2, z] - foqz[2^k - 1, 2, z]], {k, 1, 5}] // TableForm
```

$-z$

$\frac{1}{2} z (1+z)$

$-\frac{1}{6} z (1+z) (2+z)$

$\frac{1}{24} z (1+z) (2+z) (3+z)$

$-\frac{1}{120} z (1+z) (2+z) (3+z) (4+z)$

```
Table[ FullSimplify[foqzz[2^k, 2, z] - foqzz[2^k - 1, 2, z]], {k, 1, 5}] // TableForm
```

$-z$

$\frac{1}{2} z (1+z)$

$-\frac{1}{6} z (1+z) (2+z)$

$\frac{1}{24} z (1+z) (2+z) (3+z)$

$-\frac{1}{120} z (1+z) (2+z) (3+z) (4+z)$

```
Sum[ Pochhammer[-z, k] / k!, {k, 0, Infinity}]
```

```
HypergeometricPFQ[{-z}, {}, 1]
```

```
Sum[(-1)^k Pochhammer[-z, k] / k!, {k, 0, Infinity}]
```

$2^z$

```
Sum[ (-1)^k Binomial[z, k], {k, 0, Infinity}]
```

```
HypergeometricPFQ[{-z}, {}, 1]
```

```
Sum[(-1)^k Pochhammer[z, k] / k!, {k, 0, Infinity}]
```

```
N@qrootsa[100, 1.2]
```

```
{-0.685615 - 1.44102 i, -0.685615 + 1.44102 i, 0.109333 - 3.04474 i, 0.109333 + 3.04474 i,
 1.27778 - 4.75948 i, 1.27778 + 4.75948 i, 2.80018 - 6.53172 i, 2.80018 + 6.53172 i,
 4.6835 - 8.31832 i, 4.6835 + 8.31832 i, 6.9517 - 10.0774 i, 6.9517 + 10.0774 i,
 9.64711 - 11.7615 i, 9.64711 + 11.7615 i, 12.8383 - 13.3094 i, 12.8383 + 13.3094 i,
 16.6384 - 14.6322 i, 16.6384 + 14.6322 i, 21.2502 - 15.5833 i, 21.2502 + 15.5833 i,
 27.0974 - 15.8805 i, 27.0974 + 15.8805 i, 35.3918 - 14.8102 i, 35.3918 + 14.8102 i, -1.}
```

```
FullSimplify@Sum[ Pochhammer[-z, k] / k!, {k, 0, n}]
```

$$\frac{\text{Gamma}[1+n-z]}{\text{Gamma}[1+n]\ \text{Gamma}[1-z]}$$

```
Expand@Sum[ Pochhammer[-z, k] / k!, {k, 0, 8}]
```

$$1-\frac{761 z}{280}+\frac{29531 z^2}{10080}-\frac{267 z^3}{160}+\frac{1069 z^4}{1920}-\frac{9 z^5}{80}+\frac{13 z^6}{960}-\frac{z^7}{1120}+\frac{z^8}{40320}$$

```
Expand@Sum[(-1)^k bin[z, k], {k, 0, 8}]
```

$$1-\frac{761\,z}{280}+\frac{29\,531\,z^2}{10\,080}-\frac{267\,z^3}{160}+\frac{1069\,z^4}{1920}-\frac{9\,z^5}{80}+\frac{13\,z^6}{960}-\frac{z^7}{1120}+\frac{z^8}{40\,320}$$

```
Sum[(-1)^k Binomial[z, k], {k, 0, n}]
```

$(-1)^n$ Binomial[-1 + z, n]

```
Expand[(-1)^n bin[-1 + z, n] /. n → 8]
```

$$1-\frac{761\,z}{280}+\frac{29\,531\,z^2}{10\,080}-\frac{267\,z^3}{160}+\frac{1069\,z^4}{1920}-\frac{9\,z^5}{80}+\frac{13\,z^6}{960}-\frac{z^7}{1120}+\frac{z^8}{40\,320}$$

```
FullSimplify@Sum[ Binomial[z, k], {k, 0, n}]
Sum[ 1 / (3 j + 1) + 1 / (3 j + 2) - 2 / (3 j + 3 ), {j, 0, Infinity}]
```

Log[3]

```
D[Binomial[z, 3], z] /. z → 0
```

$$\frac{1}{3}$$

```
Clear[px, py]
px[n_, k_] := px[n, k] = Sum[(-1)^(j + 1) / j px[n - j, k - 1], {j, 1, n - 1}]
px[n_, 1] := (-1)^(n + 1) / n
px[n_, 0] := 0
py[n_, k_] := py[n, k] = Sum[1 / j py[n - j, k - 1], {j, 1, n - 1}]
py[n_, 1] := 1 / n
py[n_, 0] := 0


Table[ FullSimplify[Sum[ z^k / k ! px[n, k], {k, 0, n}]], {n, 0, 6}] // TableForm
```

$0$
$z$
$\frac{1}{2}(-1+z)\,z$
$\frac{1}{6}(-2+z)(-1+z)\,z$
$\frac{1}{24}(-3+z)(-2+z)(-1+z)\,z$
$\frac{1}{120}(-4+z)(-3+z)(-2+z)(-1+z)\,z$
$\frac{1}{720}(-5+z)(-4+z)(-3+z)(-2+z)(-1+z)\,z$

```
Table[FullSimplify@Binomial[z, n], {n, 0, 6}] // TableForm
```

$1$
$z$
$\frac{1}{2}(-1+z)\,z$
$\frac{1}{6}(-2+z)(-1+z)\,z$
$\frac{1}{24}(-3+z)(-2+z)(-1+z)\,z$
$\frac{1}{120}(-4+z)(-3+z)(-2+z)(-1+z)\,z$
Binomial[z, 6]

```
Table[ FullSimplify[Sum[ z^k / k! py[n, k], {k, 0, n}]], {n, 0, 6}] // TableForm
```

$0$
$z$
$\frac{1}{2} z (1+z)$
$\frac{1}{6} z (1+z) (2+z)$
$\frac{1}{24} z (1+z) (2+z) (3+z)$
$\frac{1}{120} z (1+z) (2+z) (3+z) (4+z)$
$\frac{1}{720} z (1+z) (2+z) (3+z) (4+z) (5+z)$

```
Clear[px]
t[n_, a_, b_] := b (Floor[n / b] - Floor[(n - 1) / b]) - a (Floor[n / a] - Floor[(n - 1) / a])
px[n_, a_, b_, k_] := px[n, k] = Sum[t[j, a, b] / j px[n - j, a, b, k - 1], {j, 1, n - 1}]
px[n_, a_, b_, 1] := t[n, a, b] / n
px[n_, a_, b_, 0] := 0
```

```
Table[ FullSimplify@Sum[ z^k / k! px[n, 5, 1, k], {k, 0, n}], {n, 0, 6}] // TableForm
```

$0$
$z$
$\frac{1}{2} z (1+z)$
$\frac{1}{6} z (1+z) (2+z)$
$\frac{1}{24} z (1+z) (2+z) (3+z)$
$\frac{1}{120} (-1+z) z (6+z) (16+z (5+z))$
$\frac{1}{720} (-1+z) z (-120+z (326+z (101+z (16+z))))$

```
Table[FullSimplify@( (-1)^n Pochhammer[z, n] / n!), {n, 0, 6}] // TableForm
```

$1$
$-z$
$\frac{1}{2} z (1+z)$
$-\frac{1}{6} z (1+z) (2+z)$
$\frac{1}{24} z (1+z) (2+z) (3+z)$
$-\frac{1}{120} z (1+z) (2+z) (3+z) (4+z)$
$\frac{1}{720} z (1+z) (2+z) (3+z) (4+z) (5+z)$

```
Table[FullSimplify@Sum[ z^k / k! px[n, 2, 1, k], {k, 0, n}], {n, 0, 6}] // TableForm
```

$0$
$z$
$\frac{1}{2} (-1+z) z$
$\frac{1}{6} (-2+z) (-1+z) z$
$\frac{1}{24} (-3+z) (-2+z) (-1+z) z$
$\frac{1}{120} (-4+z) (-3+z) (-2+z) (-1+z) z$
$\frac{1}{720} (-5+z) (-4+z) (-3+z) (-2+z) (-1+z) z$

```
Table[FullSimplify[Expand[(Pochhammer[z, n] / n!)]], {n, 0, 8}] // TableForm
```

$1$

$z$

$\frac{1}{2} z (1+z)$

$\frac{1}{6} z (1+z) (2+z)$

$\frac{1}{24} z (1+z) (2+z) (3+z)$

$\frac{1}{120} z (1+z) (2+z) (3+z) (4+z)$

$\frac{1}{720} z (1+z) (2+z) (3+z) (4+z) (5+z)$

$\frac{z (1+z) (2+z) (3+z) (4+z) (5+z) (6+z)}{5040}$

$\frac{z (1+z) (2+z) (3+z) (4+z) (5+z) (6+z) (7+z)}{40320}$

```
sa[n_, x_] := Sum[(-1)^(k) bin[z, k], {k, 0, Log[x, n]}]
si2[n_, x_] :=
 -Sum[ bin[z, k], {k, 0, Log[x, n]}] + 2 Sum[ bin[z, 2 k], {k, 0, Log[n] / (2 Log[x])}]
```

```
Expand@sa[100, 2]
```

$$1-\frac{49 z}{20}+\frac{203 z^2}{90}-\frac{49 z^3}{48}+\frac{35 z^4}{144}-\frac{7 z^5}{240}+\frac{z^6}{720}$$

```
Expand@si2[100, 2]
```

$$1-\frac{49 z}{20}+\frac{203 z^2}{90}-\frac{49 z^3}{48}+\frac{35 z^4}{144}-\frac{7 z^5}{240}+\frac{z^6}{720}$$

```
sx[n_, x_] := Sum[ Pochhammer[-z, k] / k!, {k, 0, Log[x, n]}]
sx2[n_, x_] := -Sum[ (-1)^k Pochhammer[-z, k] / k!, {k, 0, Log[x, n]}] +
  2 Sum[ (-1)^(2 k) Pochhammer[-z, 2 k] / ((2 k)!), {k, 0, Log[n] / (2 Log[x])}]
sx2a[n_, x_] := Sum[ t[k, 2, 1] Pochhammer[-z, k] / k!, {k, 0, Log[x, n]}] -
  2 Sum[ t[2 k, 2, 1] Pochhammer[-z, 2 k] / ((2 k)!), {k, 0, Log[n] / (2 Log[x])}]
sx3[n_, x_] := Sum[ t[k, 3, 1] Pochhammer[-z, k] / k!, {k, 0, Log[x, n]}] -
  3 Sum[ t[3 k, 3, 1] Pochhammer[-z, 3 k] / ((3 k)!), {k, 0, Log[n] / (3 Log[x])}]
```

```
FullSimplify@sx[100, 2]
```

$$\frac{1}{720} (-6+z) (-5+z) (-4+z) (-3+z) (-2+z) (-1+z)$$

```
FullSimplify@sx2[100, 2]
```

$$\frac{1}{720} (-6+z) (-5+z) (-4+z) (-3+z) (-2+z) (-1+z)$$

```
FullSimplify@sx2a[100, 2]
```

$$\frac{1}{720} (-6+z) (-5+z) (-4+z) (-3+z) (-2+z) (-1+z)$$

```
FullSimplify@sx3[100, 2]
```

$$\frac{1}{360} (-3+z) (-480+z (314+z (-483+z (134+z (-27+2 z)))))$$

```
t[0, 3, 1]
```

$-2$

**Sum[ Pochhammer[z, k] / k!, {k, 0, Infinity}]**

HypergeometricPFQ[{z}, {}, 1]

**Sum[ Pochhammer[z, k] / k!, {k, 0, n}]**

$$\frac{(1+n)\ \text{Gamma}[1+n+z]}{z\ \text{Gamma}[2+n]\ \text{Gamma}[z]}$$

**Expand[Pochhammer[-z, k] / k! /. k → 5]**

$$-\frac{z}{5}+\frac{5 z^2}{12}-\frac{7 z^3}{24}+\frac{z^4}{12}-\frac{z^5}{120}$$

**Expand[(-1)^k Binomial[z, k] /. k → 5]**

$$-\frac{z}{5}+\frac{5 z^2}{12}-\frac{7 z^3}{24}+\frac{z^4}{12}-\frac{z^5}{120}$$

**Table[t[n, 1, 2], {n, 0, 10}]**

{1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1}

**Table[ px[n, 2, 1, k], {n, 0, 9}, {k, 0, n}] // TableForm**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | | |
| 0 | 1 | | | | | | | | |
| 0 | $-\frac{1}{2}$ | 1 | | | | | | | |
| 0 | $\frac{1}{3}$ | -1 | 1 | | | | | | |
| 0 | $-\frac{1}{4}$ | $\frac{11}{12}$ | $-\frac{3}{2}$ | 1 | | | | | |
| 0 | $\frac{1}{5}$ | $-\frac{5}{6}$ | $\frac{7}{4}$ | -2 | 1 | | | | |
| 0 | $-\frac{1}{6}$ | $\frac{137}{180}$ | $-\frac{15}{8}$ | $\frac{17}{6}$ | $-\frac{5}{2}$ | 1 | | | |
| 0 | $\frac{1}{7}$ | $-\frac{7}{10}$ | $\frac{29}{15}$ | $-\frac{7}{2}$ | $\frac{25}{6}$ | -3 | 1 | | |
| 0 | $-\frac{1}{8}$ | $\frac{363}{560}$ | $-\frac{469}{240}$ | $\frac{967}{240}$ | $-\frac{35}{6}$ | $\frac{23}{4}$ | $-\frac{7}{2}$ | 1 | |
| 0 | $\frac{1}{9}$ | $-\frac{761}{1260}$ | $\frac{29531}{15120}$ | $-\frac{89}{20}$ | $\frac{1069}{144}$ | -9 | $\frac{91}{12}$ | -4 | 1 |

**Table[ px[n, 6, 1, k], {n, 0, 9}, {k, 0, n}] // TableForm**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | | |
| 0 | 1 | | | | | | | | |
| 0 | $\frac{1}{2}$ | 1 | | | | | | | |
| 0 | $\frac{1}{3}$ | 1 | 1 | | | | | | |
| 0 | $\frac{1}{4}$ | $\frac{11}{12}$ | $\frac{3}{2}$ | 1 | | | | | |
| 0 | $\frac{1}{5}$ | $\frac{5}{6}$ | $\frac{7}{4}$ | 2 | 1 | | | | |
| 0 | $-\frac{5}{6}$ | $\frac{137}{180}$ | $\frac{15}{8}$ | $\frac{17}{6}$ | $\frac{5}{2}$ | 1 | | | |
| 0 | $\frac{1}{7}$ | $-\frac{13}{10}$ | $\frac{29}{15}$ | $\frac{7}{2}$ | $\frac{25}{6}$ | 3 | 1 | | |
| 0 | $\frac{1}{8}$ | $-\frac{197}{560}$ | $-\frac{251}{240}$ | $\frac{967}{240}$ | $\frac{35}{6}$ | $\frac{23}{4}$ | $\frac{7}{2}$ | 1 | |
| 0 | $\frac{1}{9}$ | $-\frac{79}{1260}$ | $-\frac{15829}{15120}$ | $\frac{9}{20}$ | $\frac{1069}{144}$ | 9 | $\frac{91}{12}$ | 4 | 1 |

```
Clear[px, pxa]
t[n_, a_, b_] := b (Floor[n / b] - Floor[(n - 1) / b]) - a (Floor[n / a] - Floor[(n - 1) / a])
px[n_, a_, b_, k_] := px[n, a, b, k] = Sum[t[j, a, b] / j px[n - j, a, b, k - 1], {j, 1, n - 1}]
px[n_, a_, b_, 1] := t[n, a, b] / n
px[n_, a_, b_, 0] := 0
pxa[n_, a_, b_, k_] :=
 pxa[n, a, b, k] = Sum[t[j, a, b] / j pxa[n - j, a, b, k - 1], {j, 1, n - 1}]
pxa[n_, a_, b_, 1] := t[n, a, b] / n
pxa[n_, a_, b_, 0] := 0
```

```
Clear[pm]
pm[n_, a_, b_, k_] :=
 pm[n, a, b, k] = Sum[(-1)^(a j + b) / j pm[n - j, a, b, k - 1], {j, 1, n - 1}]
pm[n_, a_, b_, 1] := (-1)^(a n + b) / n
pm[n_, a_, b_, 0] := 0
```

```
Table[FullSimplify@Sum[ z^k / k! pm[n, 0, I / 2, k], {k, 0, n}], {n, 0, 6}] // TableForm
```

$0$

$(-1)^{\frac{i}{2}} z$

$\frac{1}{2} (-1)^{\frac{i}{2}} z \left(1 + (-1)^{\frac{i}{2}} z\right)$

$\frac{1}{6} (-1)^{\frac{i}{2}} z \left(2 + 3 (-1)^{\frac{i}{2}} z + (-1)^{i} z^2\right)$

$\frac{1}{24} (-1)^{\frac{i}{2}} z \left(1 + (-1)^{\frac{i}{2}} z\right) \left(2 + (-1)^{\frac{i}{2}} z\right) \left(3 + (-1)^{\frac{i}{2}} z\right)$

$\frac{1}{120} (-1)^{\frac{i}{2}} z \left(1 + (-1)^{\frac{i}{2}} z\right) \left(2 + (-1)^{\frac{i}{2}} z\right) \left(3 + (-1)^{\frac{i}{2}} z\right) \left(4 + (-1)^{\frac{i}{2}} z\right)$

$\frac{1}{720} (-1)^{\frac{i}{2}} z \left(1 + (-1)^{\frac{i}{2}} z\right) \left(2 + (-1)^{\frac{i}{2}} z\right) \left(3 + (-1)^{\frac{i}{2}} z\right) \left(4 + (-1)^{\frac{i}{2}} z\right) \left(5 + (-1)^{\frac{i}{2}} z\right)$

```
FullSimplify[Sum[ (-1)^(k (a + b I)) Binomial[z, k], {k, 0, Infinity}]]
```

$\left(1 + (-1)^{a+i b}\right)^z$

```
FullSimplify[Sum[ (-1)^(k (a + b I)) Pochhammer[-z, k] / k!, {k, 0, Infinity}]]
```

$\left(1 - (-1)^{a+i b}\right)^z$

```
Clear[px, py]
px[n_, k_] := px[n, k] = Sum[(-1)^(j + 1) / j px[n - j, k - 1], {j, 1, n - 1}]
px[n_, 1] := (-1)^(n + 1) / n
px[n_, 0] := 0
py[n_, k_] := py[n, k] = If[n < 1, 0, Sum[1 / j py[n - j, k - 1], {j, 1, n - 1}]]
py[n_, 1] := If[n < 1, 0, 1 / n]
py[n_, 0] := 0
```

```
px[100, 2]
```

$$\frac{360\,968\,703\,235\,711\,654\,233\,892\,612\,988\,250\,163\,157\,207}{3\,486\,018\,761\,485\,623\,858\,226\,690\,446\,765\,615\,177\,840\,000}$$

```
Clear[px, py, pxx]
t[n_, a_, b_] := b (Floor[n / b] - Floor[(n - 1) / b]) - a (Floor[n / a] - Floor[(n - 1) / a])
px[n_, k_] := px[n, k] = Sum[(-1) ^ (j + 1) / j px[n - j, k - 1], {j, 1, n}]
px[n_, 0] := UnitStep[n]
pxx[n_, a_, b_, k_] := pxx[n, a, b, k] = Sum[t[j, a, b] / j pxx[n - j, a, b, k - 1], {j, 1, n}]
pxx[n_, a_, b_, 0] := UnitStep[n]
py[n_, k_] := py[n, k] = If[n < 1, 0, Sum[1 / j py[n - j, k - 1], {j, 1, n}]]
py[n_, 0] := UnitStep[n]
dpy[n_, k_] := py[n, k] - py[n - 1, k]
pym[n_, m_, a_, b_] := If[b == 0, py[n, a],
  If[a == 0, py[n / m, b], Sum[ dpy[j, a] dpy[k, b], {j, 1, n}, {k, 1, (n - j) / m}]]]
pyx[n_, k_] := Sum[ (-1) ^j Binomial[k, j] pym[ n, 2, k - j, j], {j, 0, k}]
pyxx[n_, m_, k_] := Sum[ (-1) ^j Binomial[k, j] pym[ n, m, k - j, j], {j, 0, k}]
```

```
{px[120, 5], pyx[120, 5]}
```

{-6 766 304 837 416 110 487 180 817 591 213 595 916 178 240 456 075 784 958 965 213 649 424 740 665 222 943 301 612 996 720 006 043 424 833 044 359 / 551 740 808 903 438 229 989 722 739 812 058 181 212 700 729 612 774 147 964 353 004 731 191 621 078 401 554 848 864 018 189 373 849 600 000 000,
-6 766 304 837 416 110 487 180 817 591 213 595 916 178 240 456 075 784 958 965 213 649 424 740 665 222 943 301 612 996 720 006 043 424 833 044 359 / 551 740 808 903 438 229 989 722 739 812 058 181 212 700 729 612 774 147 964 353 004 731 191 621 078 401 554 848 864 018 189 373 849 600 000 000}

```
{pxx[50, 4, 1, 4], pyxx[50, 4, 4]}
```

$$\left\{\frac{1\,149\,514\,901\,403\,120\,187\,801\,632\,636\,324\,162\,183\,013}{657\,739\,628\,513\,519\,800\,564\,069\,482\,676\,147\,200\,000}, \frac{1\,149\,514\,901\,403\,120\,187\,801\,632\,636\,324\,162\,183\,013}{657\,739\,628\,513\,519\,800\,564\,069\,482\,676\,147\,200\,000}\right\}$$

```
px[16, 1]
```

$$\frac{95\,549}{144\,144}$$

```
py[16, 1] - py[8, 1]
```

$$\frac{95\,549}{144\,144}$$

```
px[12, 2]
```

$$\frac{150\,781}{207\,900}$$

```
Sum[ (-1) ^ (j + 1) / j px[12 - j, 1], {j, 1, 12}]
```

$$\frac{150\,781}{207\,900}$$

```
Sum[ 1 / j px[12 - j, 1], {j, 1, 12}] - 2 Sum[ 1 / (j) px[12 - j, 1], {j, 2, 12, 2}]
```

$$\frac{150\,781}{207\,900}$$

```
Sum[ 1 / j px[12 - j, 1], {j, 1, 12}] - 2 Sum[ 1 / (2 j) px[12 - 2 j, 1], {j, 1, 6}]
```

$$\frac{150\,781}{207\,900}$$

```
Sum[1/j px[12-j, 1], {j, 1, 12}] - Sum[1/ j px[12-2 j, 1], {j, 1, 6}]
```

$$\frac{150781}{207900}$$

```
Sum[1/j (py[12-j, 1] - py[(12-j)/2, 1]), {j, 1, 12}] - Sum[1/ j px[12-2 j, 1], {j, 1, 6}]
```

$$\frac{150781}{207900}$$

```
Sum[1/j (py[12-j, 1] - py[(12-j)/2, 1]), {j, 1, 12}] -
 Sum[1/ j (py[12-2 j, 1] - py[(6- j), 1]), {j, 1, 6}]
```

$$\frac{150781}{207900}$$

```
Sum[1/j py[12-j, 1], {j, 1, 12}] - Sum[1/j py[(12-j)/2, 1], {j, 1, 12}] -
 Sum[1/ j py[12-2 j, 1], {j, 1, 6}] + Sum[1/ j py[(6- j), 1], {j, 1, 6}]
```

$$\frac{150781}{207900}$$

```
py[12, 2] - Sum[1/j py[(12-j)/2, 1], {j, 1, 12}] -
 Sum[1/ j py[12-2 j, 1], {j, 1, 6}] + py[6, 2]
```

$$\frac{150781}{207900}$$

```
py[12, 2] - 2 Sum[1/j py[(12-j)/2, 1], {j, 1, 12}] + py[6, 2]
```

$$\frac{150781}{207900}$$

```
{Sum[1/j py[(12-j)/2, 1], {j, 1, 12}],
 Sum[1/ j py[12-2 j, 1], {j, 1, 6}], pym[12, 2, 1, 1]}
```

$$\left\{\frac{3733}{630}, \frac{3733}{630}, \frac{3733}{630}\right\}$$

```
py[12, 2] - 2 pym[12, 2, 1, 1] + py[6, 2]
```

$$\frac{150781}{207900}$$

```
{px[50, 3], py[50, 3] - 3 pym[50, 2, 2, 1] + 3 pym[50, 2, 1, 2] - py[50/2, 3]}
```

$$\left\{-\frac{69289605682595051818652751455399}{318853045822253797241687139840000}, -\frac{69289605682595051818652751455399}{318853045822253797241687139840000}\right\}$$

```
Grid@Table[Binomial[k, j], {k, 0, 8}, {j, 0, k}]
```

```
1
1 1
1 2  1
1 3  3  1
1 4  6  4  1
1 5 10 10  5  1
1 6 15 20 15  6  1
1 7 21 35 35 21  7 1
1 8 28 56 70 56 28 8 1
```

```
Grid@Table[Pochhammer[k, j] / j !, {k, 0, 8}, {j, 0, k}]
```

```
1
1 1
1 2  3
1 3  6   10
1 4  10  20   35
1 5  15  35   70  126
1 6  21  56  126  252  462
1 7  28  84  210  462  924  1716
1 8  36  120 330  792  1716 3432  6435
```

```
Clear[px]
t[n_, a_, b_] := b (Floor[n / b] - Floor[(n - 1) / b]) - a (Floor[n / a] - Floor[(n - 1) / a])
px[n_, a_, b_, k_] := px[n, a, b, k] = Sum[t[j, a, b] / j px[n - j, a, b, k - 1], {j, 1, n - 1}]
px[n_, a_, b_, 1] := t[n, a, b] / n
px[n_, a_, b_, 0] := 0
pz[n_, a_, b_, z_] := Sum[ z^k / k ! px[n, a, b, k], {k, 0, n}]
pz[n_, a_, b_, 0] := 1
pz[0, a_, b_, z_] := 1
Table[ FullSimplify@pz[n, 2, 1, z], {n, 0, 6}] // TableForm
```

$1$

$z$

$\frac{1}{2}(-1+z)z$

$\frac{1}{6}(-2+z)(-1+z)z$

$\frac{1}{24}(-3+z)(-2+z)(-1+z)z$

$\frac{1}{120}(-4+z)(-3+z)(-2+z)(-1+z)z$

$\frac{1}{720}(-5+z)(-4+z)(-3+z)(-2+z)(-1+z)z$

```
Grid@Table[pz[j, 16, 1, k] - pz[j, 2, 1, k], {k, 0, 11}, {j, 0, k}]
```

```
0
0 0
0 0  2
0 0  3   9
0 0  4   16   34
0 0  5   25   65   125
0 0  6   36   111  246   461
0 0  7   49   175  441   917   1715
0 0  8   64   260  736   1688  3424   6434
0 0  9   81   369  1161  2919  6399   12 861  24 309
0 0  10  100  505  1750  4795  11 320  24 265  48 610  92 377
0 0  11  121  671  2541  7546  19 118  43 593  92 323  184 745  352 715
```

```
FullSimplify[Pochhammer[z, k] / k ! - Binomial[z, k]]
```

```
(-1) ^4 Pochhammer[-13, 4] / 4 !
```

715

```
FullSimplify[(-1) ^k bin[-z, k] - bin[z, k] /. k → 5]
```

$\frac{1}{6} z^2 \left(5+z^2\right)$

```
Table[FullSimplify@Sum[pz[a, k, 1, z], {k, 1, 10}], {a, 0, 10}] // TableForm
```

$$10$$
$$9\,z$$
$$\frac{1}{2}\,z\,(7+9\,z)$$
$$\frac{1}{2}\,z\,(1+z)\,(4+3\,z)$$
$$\frac{1}{8}\,z\,(6+z\,(25+z\,(14+3\,z)))$$
$$\frac{1}{120}\,z\,(96+z\,(170+z\,(255+z\,(70+9\,z))))$$
$$\frac{1}{240}\,z\,(-80+z\,(502+z\,(285+z\,(215+z\,(35+3\,z)))))$$
$$\frac{z\,(480+z\,(-224+z\,(3262+z\,(1015+z\,(455+z\,(49+3\,z))))))}{1680}$$
$$\frac{z\,(-25\,200+z\,(35\,628+z\,(16\,324+z\,(42\,441+z\,(8680+z\,(2562+z\,(196+9\,z)))))))}{40\,320}$$
$$\frac{z\,(-13\,440+z\,(-20\,272+z\,(37\,260+z\,(17\,892+z\,(16\,009+z\,(2352+z\,(490+z\,(28+z))))))))}{40\,320}$$
$$\frac{(-1+z)\,z\,(322\,560+z\,(425\,424+z\,(404\,284+z\,(146\,524+z\,(51\,849+z\,(5936+z\,(826+z\,(36+z))))))))}{403\,200}$$

```
Sum[n^3, {n, 1, 10}]
```

3025

```
bt[fn_, n_] := Sum[ (-1) ^k Binomial[n, k] fn[k], {k, 0, n}]
bt2[fn_, n_] := Sum[ t[k - 1, 2, 1] pz[k, 2, 1, n] fn[k], {k, 0, n}]
bt3[fn_, n_] := Sum[ t[k, 3, 1] pz[k, 3, 1, n] fn[k], {k, 0, 2 n}]
id1[n_] := If[n == 0, 0, (-1) ^ (n + 1) 1 / n]
id2[n_] := HarmonicNumber[n]
```

```
Table[bt[id2, n], {n, 1, 10}]
```

$$\left\{-1,\,-\frac{1}{2},\,-\frac{1}{3},\,-\frac{1}{4},\,-\frac{1}{5},\,-\frac{1}{6},\,-\frac{1}{7},\,-\frac{1}{8},\,-\frac{1}{9},\,-\frac{1}{10}\right\}$$

```
Table[ bt[id1, n], {n, 1, 10}]
```

$$\left\{-1,\,-\frac{5}{2},\,-\frac{29}{6},\,-\frac{103}{12},\,-\frac{887}{60},\,-\frac{1517}{60},\,-\frac{18\,239}{420},\,-\frac{63\,253}{840},\,-\frac{332\,839}{2520},\,-\frac{118\,127}{504}\right\}$$

```
Table[ pz[k, 4, 1, 6], {k, 0, 18}]
```

{1, 6, 21, 56, 120, 216, 336, 456, 546, 580, 546, 456, 336, 216, 120, 56, 21, 6, 1}

```
Table[ pz[k, 6, 3, 6], {k, 0, 30}]
```

{1, 0, 0, 6, 0, 0, 15, 0, 0, 20, 0, 0, 15, 0, 0, 6, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

```
Table[ pz[k, 3, 1, n], {n, 0, 7}, {k, 0, 2 n}] // TableForm
```

```
Table[ CoefficientList[(1 + x + x^2) ^k, x], {k, 0, 7}] // TableForm
```

```
Table[ pz[k, 4, 1, n], {n, 0, 7}, {k, 0, 3 n}] // TableForm
```

```
Table[ CoefficientList[(1 + x + x^2 + x^3) ^k, x], {k, 0, 7}] // TableForm
```

```
Table[bt3[id2, n], {n, 1, 10}]
```

$$\left\{-\frac{5}{2},\,-\frac{5}{4},\,-\frac{47}{60},\,-\frac{151}{280},\,-\frac{11}{28},\,-\frac{551}{1848},\,-\frac{1873}{8008},\,-\frac{2159}{11\,440},\,-\frac{68\,303}{437\,580},\,-\frac{48\,761}{369\,512}\right\}$$

```
Table[ t[k, 2, 1], {k, 0, 5}]
```

{-1, 1, -1, 1, -1, 1}

```
Table[ t[k, 3, 1], {k, 0, 5}]
{-2, 1, 1, -2, 1, 1}
Table[ t[k, 4, 1], {k, 0, 5}]
{-3, 1, 1, 1, -3, 1}

Sum[ pz[k, 2, 1, .5] .5^k, {k, 0, 100}]^2
1.5
Sum[ pz[k, 3, 1, .5] .5^k, {k, 0, 100}]
1.32288
3^.5
1.73205
Sum[ pz[k, 3, 1, .5] .5^k, {k, 0, 100}]^2
1.75
Sum[ pz[k, 4, 1, .5], {k, 0, 130}]
2.00004
Sum[ pz[k, 4, 1, .5] .5^k, {k, 0, 130}]^2
1.875
Sum[ pz[k, 5, 1, .5], {k, 0, 130}]
2.23568
Sum[ pz[k, 5, 1, .5] .5^k, {k, 0, 130}]^2
1.9375
Sum[ pz[k, 5, 1, 1] .5^k, {k, 0, 130}]
1.9375
Sum[ pz[k, 6, 1, 1] .5^k, {k, 0, 130}]
1.96875
Sum[ pz[k, 3, 1, 1] .5^k, {k, 0, 100}]
1.75
Sum[ pz[k, 3, 1, 1] 1^k, {k, 0, 100}]
3
Sum[ pz[k, 3, 1, 1] 1.5^k, {k, 0, 100}]
4.75
Sum[ pz[k, 3, 1, 1] 2^k, {k, 0, 100}]
7
Sum[ pz[k, 5, 1, 1] (x)^k, {k, 0, 100}]
```

$1 + x + x^2 + x^3 + x^4$

```
Table[ pz[k, 5, 1, 1] (x) ^k, {k, 0, 12}]
```

$\{1, x, x^2, x^3, x^4, 0, 0, 0, 0, 0, 0, 0, 0\}$

```
Table[ pz[k, 5, 1, z] (x) ^k, {k, 0, 6}]
```

$$\left\{1, x z, x^2 \left(\frac{z}{2} + \frac{z^2}{2}\right), x^3 \left(\frac{z}{3} + \frac{z^2}{2} + \frac{z^3}{6}\right), x^4 \left(\frac{z}{4} + \frac{11 z^2}{24} + \frac{z^3}{4} + \frac{z^4}{24}\right),\right.$$
$$\left. x^5 \left(-\frac{4 z}{5} + \frac{5 z^2}{12} + \frac{7 z^3}{24} + \frac{z^4}{12} + \frac{z^5}{120}\right), x^6 \left(\frac{z}{6} - \frac{223 z^2}{360} + \frac{5 z^3}{16} + \frac{17 z^4}{144} + \frac{z^5}{48} + \frac{z^6}{720}\right)\right\}$$

```
FullSimplify@Sum[ pza[j, 3, 1, z] pza[5 - j, 3, 1, w], {j, 0, 5}]
```

$$\frac{1}{120} (-2 + w + z) (-1 + w + z) (w + z) (1 + w + z) (12 + w + z)$$

```
FullSimplify@pza[5, 3, 1, z + w]
```

$$\frac{1}{120} (-2 + w + z) (-1 + w + z) (w + z) (1 + w + z) (12 + w + z)$$

```
Clear[pxa]
t[n_, a_, b_] := b (Floor[n / b] - Floor[(n - 1) / b]) - a (Floor[n / a] - Floor[(n - 1) / a])
pxa[n_, a_, b_, k_] := pxa[n, a, b, k] = Sum[t[j, a, b] / j pxa[n - j, a, b, k - 1], {j, 1, n - 1}]
pxa[n_, a_, b_, 1] := t[n, a, b] / n
pxa[n_, a_, b_, 0] := 0
pza[n_, a_, b_, z_] := Sum[ z^k / k ! pxa[n, a, b, k], {k, 0, n}]
pza[n_, a_, b_, 0] := 0
pza[0, a_, b_, z_] := 0
pzm[n_, a_, b_, k_] := Sum[ (-1) ^ (k - j) bin[k, j] pza[n, a, b, j], {j, 0, k}]
pzmz[n_, a_, b_, z_] := Sum[ bin[z, j] pzm[n, a, b, j], {j, 0, n}]
```

```
Table[Expand@Sum[ Sum[pza[j, 2, 1, z] pza[r - j, 2, 1, z], {j, 0, r}], {r, 0, s}], {s, 0, 24}] /.
  z → 3
```

{1, 7, 22, 42, 57, 63, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64}

```
Table[ D[pza[j, 2, 1, z], z] /. z → 0, {j, 1, 8}]
```

$$\left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \frac{1}{7}, -\frac{1}{8}\right\}$$

```
Table[D[ pza[j, 6, 1, z], z] /. z → 0, {j, 1, 8}]
```

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, -\frac{5}{6}, \frac{1}{7}, \frac{1}{8}\right\}$$

```
Table[(D[ pza[j, 2, 1, z], z] /. z → 0) + (D[ pza[j, 3, 1, z], z] /. z → 0), {j, 1, 8}]
```

$$\left\{2, 0, -\frac{1}{3}, 0, \frac{2}{5}, -\frac{1}{2}, \frac{2}{7}, 0\right\}$$

```
Table[ Expand@Sum[ pza[j, 4, 1, z], {j, 0, s}], {s, 0, 24}] /. z → 3
```

{1, 4, 10, 20, 32, 44, 54, 60, 63, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64, 64}

```
Table[ pzm[9, 3, 1, k], {k, 0, 10}]
```

{0, 0, 0, 0, 0, 5, 20, 21, 8, 1, 0}

```
pza[5, 2, 1, z]
```

$$\frac{z}{5} - \frac{5 z^2}{12} + \frac{7 z^3}{24} - \frac{z^4}{12} + \frac{z^5}{120}$$

```
Expand@pzmz[5, 2, 1, z]
```

$$\frac{z}{5} - \frac{5 z^2}{12} + \frac{7 z^3}{24} - \frac{z^4}{12} + \frac{z^5}{120}$$

```
Limit[ HarmonicNumber[n] - HarmonicNumber[n / 2.71], n → Infinity]
```

0.996949