```
binomial[z_, k_] := binomial[z, k] = Product[z - j, {j, 0, k - 1}] / k!
zetaHurwitz[n_, s_, y_, 0] := UnitStep[n - 1]
zetaHurwitz[n_, s_, y_, 1] :=
  zetaHurwitz[n, s, y, 1] = HarmonicNumber[Floor[n], s] - HarmonicNumber[y, s]
zetaHurwitz[n_, s_, y_, 2] := zetaHurwitz[n, s, y, 2] =
  Sum[(m^(-2 s)) + 2 (m^-s) (zetaHurwitz[Floor[n / m], s, m, 1]), {m, y + 1, Floor[n^(1 / 2)]}]
zetaHurwitz[n_, s_, y_, k_] := zetaHurwitz[n, s, y, k] =
  Sum[(m^(-s k)) + k (m^(-s (k - 1))) zetaHurwitz[Floor[n / (m^(k - 1))], s, m, 1] +
    Sum[binomial[k, j] (m^-s)^j zetaHurwitz[Floor[n / (m^j)], s, m, k - j], {j, 1, k - 2}],
   {m, y + 1, Floor[n^(1 / k)]}]
zeta[n_, s_, z_] := Expand@Sum[binomial[z, k] zetaHurwitz[n, s, 1, k], {k, 0, Log2@n}]
zetaAlt[n_, s_, x_, z_] :=
  Expand@Sum[(-1)^j binomial[z, j] x^(j (1 - s)) zeta[n / (x^j), s, z], {j, 0, Log[x, n]}]

(leta2[100, 0, 4 / 3, 4, 3] + HarmonicNumber[Floor[Log[100] / Log[4 / 3]]]

 956 240 247 073
 ───────────────
 383 020 157 520

$RecursionLimit = 10 000;
(D[Expand@eta2[100, 4 / 3, z, 4, 3], z] /. z → 0) +
  HarmonicNumber[Floor[Log[100] / Log[4 / 3]]]

   956 240 247 073
 - ───────────────
   383 020 157 520

leta2[100, 4 / 3, 4, 3]

   956 240 247 073
 - ───────────────
   383 020 157 520

leta3[100, 4 / 3, 4, 3] + HarmonicNumber[Floor[Log[100] / Log[4 / 3]]]

   956 240 247 073
 - ───────────────
   383 020 157 520

e2[n_, y_, z_] :=
  If[n < y, 1, Sum[binomial[z, k] (-1)^(k (y + 1)) e2[n / (y^k), y + 1, z - k], {k, 0, Log[y, n]}]]
thet[y_, t_] := 1 - t (Floor[y / t] - Floor[(y - 1) / t])
et[n_, y_, z_, t_] :=
  If[n < y, 1, Sum[binomial[z, k] thet[y, t]^k et[n / (y^k), y + 1, z - k, t], {k, 0, Log[y, n]}]]
that[y_, t_, u_] := u (Floor[y / u] - Floor[(y - 1) / u]) - t (Floor[y / t] - Floor[(y - 1) / t])
eta[n_, y_, z_, t_, u_] :=
  If[n < y, 1, eta[n, y + 1 / u, z, t, u] + If[that[y u, t, u] == 0, 0, Sum[binomial[z, k]
       that[y u, t, u]^k (1 / u)^k eta[n / (y^k), y + 1 / u, z - k, t, u], {k, 1, Log[y, n]}]]]]
that2a[y_, t_, u_] := (u (Floor[u y / u] - Floor[(u y - 1) / u]) -
     t (Floor[u y / t] - Floor[(u y - 1) / t])) / u
that2[y_, t_, u_] := (Floor[y] - Floor[y - 1 / u]) - t / u (Floor[y u / t] - Floor[y u / t - 1 / t])
eta2[n_, y_, z_, t_, u_] :=
  If[n < y, 1, eta2[n, y + 1 / u, z, t, u] + If[that2[y, t, u] == 0, 0, Sum[binomial[z, k]
       that2[y, t, u]^k eta2[n / (y^k), y + 1 / u, z - k, t, u], {k, 1, Log[y, n]}]]]]
leta3[n_, y_, t_, u_] := If[n < y, 0, leta3[n, y + 1 / u, t, u] + If[that2[y, t, u] == 0, 0, Sum[
      (-1)^(k + 1) / k that2[y, t, u]^k eta2[n / (y^k), y + 1 / u, -k, t, u], {k, 1, Log[y, n]}]]]]
leta2[n_, y_, t_, u_] := leta3[n, y + 1 / u, t, u] +
   Sum[(-1)^(k + 1) / k that2[y, t, u]^k eta2[n / (y^k), y + 1 / u, -k, t, u] + 1 / k,
    {k, 1, Log[y, n]}]
eta4[n_, y2_, z_, t_, u_] := 1 + Sum[If[that2[y, t, u] == 0, 0,
     Sum[binomial[z, k] that2[y, t, u]^k eta4[n / (y^k), y + 1 / u, z - k, t, u],
      {k, 1, Log[y, n]}]], {y, y2, n, 1 / u}]
leta4[n_, t_, u_] := Sum[If[that2[y, t, u] == 0, 0, Sum[(-1)^(k + 1) / k that2[y, t, u]^k
     eta4[n / (y^k), y + 1 / u, -k, t, u], {k, 1, Log[y, n]}]], {y, 1 + 1 / u, n, 1 / u}]


Expand@e2[100, 2, z]
```

$$1 + \frac{4 z}{5} - \frac{419 z^2}{72} + \frac{265 z^3}{48} - \frac{241 z^4}{144} + \frac{43 z^5}{240} - \frac{z^6}{144}$$

```
Expand@et[100, 2, z, 5]
```

$$1 + \frac{331 z}{30} - \frac{7711 z^2}{360} + \frac{403 z^3}{48} + \frac{131 z^4}{144} + \frac{17 z^5}{240} + \frac{7 z^6}{720}$$

```
(D[zetaAlt[100, 0, 4 / 3, z], z] /. z → 0) + HarmonicNumber[Floor[Log[100] / Log[4 / 3]]]
```

$$-\frac{956240247073}{383020157520}$$

```
$RecursionLimit = 10000;
(D[Expand@eta2[100, 4 / 3, z, 4, 3], z] /. z → 0) +
 HarmonicNumber[Floor[Log[100] / Log[4 / 3]]]
```

$$-\frac{956240247073}{383020157520}$$

```
leta2[100, 4 / 3, 4, 3]
```

$$-\frac{956240247073}{383020157520}$$

```
leta3[100, 4 / 3, 4, 3] + HarmonicNumber[Floor[Log[100] / Log[4 / 3]]]
```

$$-\frac{956240247073}{383020157520}$$

```
FullSimplify[Sum[MangoldtLambda[j] / Log[j], {j, 2, 100}] -
   Sum[ (4 / 3)^k / k, {k, 1, Floor@Log[4 / 3, 100]}]] +
 HarmonicNumber[Floor[Log[100] / Log[4 / 3]]]
```

$$-\frac{956240247073}{383020157520}$$

```
leta4[100, 4, 3] + HarmonicNumber[Floor[Log[100] / Log[4 / 3]]]
```

$$-\frac{956240247073}{383020157520}$$