Section 10: $\psi(n)$

10. The Difference between n and $\psi(n)$: A Partial Sum Equivalence to the Dirichlet Eta Function for the Chebyshev Function $\psi(n)$

10.1

$$[(\zeta(s)-1)^k]_n = \sum_{j=1} (j+1)^{-s} [(\zeta(s)-1)^{k-1}]_{n(j+1)^{-1}}$$
$$[\zeta(s)^z]_n = \sum_{k=0}^{\infty} \binom{z}{k} [(\zeta(s)-1)^k]_n$$
$$[(\log\zeta(s))^k]_n = \sum_{j=2} \frac{\Lambda(j)}{\log j} j^{-s} [(\log\zeta(s))^{k-1}]_{n\cdot j^{-1}}$$

$$\psi(n) = -\sum_{j=1}^{n} [\nabla\zeta'(0)]_j \cdot [(\zeta(0))^{-1}]_{n j^{-1}} \tag{10.1.1}$$

$$\psi(n) = -\sum_{j=1}^{n} [\nabla\zeta(0)^{-1}]_j \cdot [\zeta'(0)]_{n j^{-1}} \tag{10.1.2}$$

$$\psi(n) = \left[\frac{-\zeta'(0)}{\zeta(0)}\right]_n \tag{10.1.3}$$

```
chebyshev[n_]:=Sum[MangoldtLambda[j],{j,2,n}]
dz[n_,z_, s_]:=(n^-s)Product[(-1)^p[[2]] Binomial[-z,p[[2]]],{p,FI[n]}];FI[n_]:=FactorInteger[n];FI[1]:={}
Dz[n_,z_,s_]:=Sum[dz[j,z,s],{j,1,n}]
Table[Chop[N[chebyshev[n]]-( -N[Sum[ dz[j,-1,0](D[Dz[n/j,1,1,s],s]/.s->0),{j,1,n}]])],{n,10,100,10}]
Table[Chop[N[chebyshev[n]]-( -N[Sum[ (D[Dz[n/j,1,1,s],s]/.s->0)dz[j,-1,0],{j,1,n}]])],{n,10,100,10}]
```

Compare this to $\frac{\zeta'(0)}{\zeta(0)}$

$$\psi(n) = -\lim_{s\to 0} \frac{\partial}{\partial s} \lim_{z\to 0} \frac{\partial}{\partial z} [\zeta(s)^z]_n \tag{10.1.4}$$

```
 chebyshev[n_]:=Sum[MangoldtLambda[j],{j,2,n}]
Dz[n_,z_,k_,s_]:=Dz[n,z,k,s]=1+((z+1)/k-1)Sum[j^-s Dz[Floor[n/j],z,k+1,s],{j,2,n}]
Table[{N[chebyshev[n]], -N[Limit[D[Limit[D[Dz[n,z,1,s],z],z->0],s],s->0]]},{n,10,70,10}]
```

Compare this to $\frac{-\zeta'(0)}{\zeta(0)} = \lim_{s\to 0} \frac{\partial}{\partial s} \lim_{z\to 0} \frac{\partial}{\partial z} \zeta(s)^z$ (10.1.5)

```
{D[Zeta[s],s]/Zeta[s]/.s->0,Limit[D[Limit[D[Zeta[s]^z,z],z->0],s],s->0]}
```

$$\psi(n) = -\lim_{s\to 0}\frac{\partial}{\partial s}[\log\zeta(s)]_n \tag{10.1.6}$$

```
chebyshev[n_]:=Sum[MangoldtLambda[j],{j,2,n}]
logD[n_,0,s_]:=UnitStep[n-1];logD[n_,k_,s_]:=Sum[MangoldtLambda[j]/Log[j]j^-s logD[n/j,k-1,s],{j,2,n}]
Table[{N[chebyshev[n]], -N[Limit[D[logD[n,1,s],s],s->0]]},{n,10,70,10}]
```

Compare this to $\frac{\zeta'(0)}{\zeta(0)} = -\lim_{s\to 0}\frac{\partial}{\partial s}\log\zeta(s)$ (10.1.7)

```
{D[Zeta[s],s]/Zeta[s]/.s->0,Limit[D[Log[Zeta[s]],s],s->0]}
```

$$\int_s^\infty \left[\frac{-\zeta'(t)}{\zeta(t)}\right]_n dt = [\log\zeta(s)]_n \tag{10.1.8}$$

Compare this to $\int_s^\infty -\frac{\zeta'(t)}{\zeta(t)}\,dt = \log\zeta(s)$ (10.1.9)

```
{Integrate[- Zeta'[t]/Zeta[t],{t,s,Infinity}],Log[Zeta[s]]}
```

10.2

$$[\zeta(s)^z]_n = 1 - \int_s^\infty \frac{\partial}{\partial t}[\zeta(t)^z]_n\,dt$$
$$\zeta(s)^z = 1 - \int_s^\infty \frac{\partial}{\partial t}\zeta(t)^z\,dt$$

$$[\log\zeta(s)]_n = \int_s^\infty \frac{\partial}{\partial t}\lim_{z\to 0}\frac{\partial}{\partial z}[\zeta(t)^z]_n\,dt$$

$$\log\zeta(s) = \int_s^\infty \frac{\partial}{\partial t}\lim_{z\to 0}\frac{\partial}{\partial z}\zeta(t)^z\,dt$$

$$[\log\zeta(s)]_n = \int_s^\infty \left(\left[\frac{-\zeta'(t)}{\zeta(t)}\right]_n\right)dt$$

$$\log\zeta(s) = \int_s^\infty \frac{\zeta'(t)}{\zeta(t)}\,dt$$

10.2

$$\lim_{s\to 0}\frac{\partial}{\partial s}[(\zeta(s)-1)^0]_n=0$$
$$\lim_{s\to 0}\frac{\partial}{\partial s}[\zeta(s)-1]_n=-\sum_{j=2}^{n}\log j$$
$$\lim_{s\to 0}\frac{\partial}{\partial s}[(\zeta(s)-1)^2]_n=-2\sum_{j=2}^{n}\sum_{k=2}^{\lfloor n\cdot j^{-1}\rfloor}\log k$$

(10.2.1)

$$\lim_{s\to 0}\frac{\partial}{\partial s}[(\zeta(s)-1)^k]_n=\frac{k}{k-1}\sum_{j=2}^{\lfloor n\rfloor}\lim_{s\to 0}\frac{\partial}{\partial s}[(\zeta(s)-1)^{k-1}]_{n\,j^{-1}}$$

(10.2.2)

$$\lim_{s\to 0}\frac{\partial}{\partial s}[\zeta(s)^0]_n=0$$
$$\lim_{s\to 0}\frac{\partial}{\partial s}[\zeta(s)]_n=-\sum_{j=1}^{n}\log j$$
$$\lim_{s\to 0}\frac{\partial}{\partial s}[\zeta(s)^2]_n=-2\sum_{j=1}^{n}\sum_{k=1}^{\lfloor n\cdot j^{-1}\rfloor}\log k$$

(10.2.3)

$$\lim_{s\to 0}\frac{\partial}{\partial s}[\zeta(s)^k]_n=\frac{k}{k-1}\sum_{j=1}^{\lfloor n\rfloor}\lim_{s\to 0}\frac{\partial}{\partial s}[\zeta(s)^{k-1}]_{n\,j^{-1}}$$

(10.2.4)

$$\lim_{s\to 0}\frac{\partial}{\partial s}[\zeta(s)^z]_n=\sum_{k=1}^{\infty}\binom{z}{k}\lim_{s\to 0}\frac{\partial}{\partial s}[(\zeta(s)-1)^k]_n$$

(10.2.5)

$$\lim_{s\to 0}\frac{\partial}{\partial s}[\log\zeta(s)]_n=-\sum_{j=2}^{\lfloor n\rfloor}\kappa(j)\log j$$
$$\lim_{s\to 0}\frac{\partial}{\partial s}[(\log\zeta(s))^2]_n=-2\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\kappa(j)\kappa(k)\log(k)$$
$$\lim_{s\to 0}\frac{\partial}{\partial s}[(\log\zeta(s))^3]_n=-3\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\sum_{m=2}^{\lfloor\frac{n}{j\,k}\rfloor}\kappa(j)\kappa(k)\kappa(m)\log(m)$$

(10.2.6)

$$\lim_{s\to 0}\frac{\partial}{\partial s}[(\log\zeta(s))^k]_n=\frac{k}{k-1}\sum_{j=2}^{\lfloor n\rfloor}\kappa(j)\cdot\lim_{s\to 0}\frac{\partial}{\partial s}[(\log\zeta(s))^{k-1}]_{n\,j^{-1}}$$

(10.2.7)

10.3

$$\boxed{[(y^{s-1}\cdot\zeta(s,1+y))^k]_n=y^{s-1}\cdot\sum_{j=1}(j+y)^{-s}\cdot[(y^{s-1}\cdot\zeta(s,1+y))^{k-1}]_{n\cdot y\,(j+y)^{-1}}}$$

(10.3.1)

$$\lim_{s\to 0}\frac{\partial}{\partial s}[y^{s-1}\cdot\zeta(s,1+y)]_n=-y^{-1}\sum_{j=1}\log(1+\frac{j}{y})$$
$$\lim_{s\to 0}\frac{\partial}{\partial s}[(y^{s-1}\cdot\zeta(s,1+y))^2]_n=-2y^{-2}\sum_{j=1}\sum_{k=1}\log(1+\frac{j}{y})$$
$$\lim_{s\to 0}\frac{\partial}{\partial s}[(y^{s-1}\cdot\zeta(s,1+y))^3]_n=-3y^{-3}\sum_{j=1}\sum_{k=1}\sum_{l=1}\log(1+\frac{j}{y})$$

(10.3.1)

// I haven't checked this or the next few yet.

$$\lim_{s\to 0}\frac{\partial}{\partial s}[y^{s-1}\cdot\zeta(s,1+y)]_n=-y^{-1}\sum_{j=1}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}(\frac{j}{y})^k$$

$$\lim_{s\to 0}\frac{\partial}{\partial s}[y^{s-1}\cdot\zeta(s,1+y)]_n=-y^{-1}\sum_{j=1}\log(1+\frac{j}{y})$$
$$\lim_{s\to 0}\frac{\partial}{\partial s}[(y^{s-1}\cdot\zeta(s,1+y))^2]_n=-2y^{-2}\sum_{j=1}[\nabla(\zeta(0))^2]_j\log(1+\frac{j}{y})$$
$$\lim_{s\to 0}\frac{\partial}{\partial s}[(y^{s-1}\cdot\zeta(s,1+y))^3]_n=-3y^{-3}\sum_{j=1}[\nabla(\zeta(0))^3]_j\log(1+\frac{j}{y})$$

(10.3.1a)

$$\lim_{s\to 0}\frac{\partial}{\partial s}[(y^{s-1}\cdot\zeta(s,1+y))^k]_n=-k\,y^{-k}\sum_{j=1}[\nabla(\zeta(0))^k]_j\log(1+\frac{j}{y})$$

(10.3.1b)

$$[(1+y^{s-1}\cdot\zeta(s,1+y))^z]_n=\sum_{k=0}^{\infty}\binom{z}{k}[(y^{s-1}\cdot\zeta(s,1+y))^k]_n$$

(10.3.1)

Consequently, we can express the difference between them as

$$\boxed{\psi(n)=n-\log n-1-\int_1^{\infty}\frac{\partial}{\partial y}(\lim_{s\to 0}\frac{\partial}{\partial s}[\log(1+y^{s-1}\cdot\zeta(s,1+y))]_n)dy}$$

(10.3.1)

This same general idea can also be expressed as

$$[\log\zeta(s)]_n=$$
$$\sum_{j=2}^{\lfloor n\rfloor}j^{-s}-\frac{1}{2}\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}(j\cdot k)^{-s}+\frac{1}{3}\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor\frac{n}{j\cdot k}\rfloor}(j\cdot k\cdot l)^{-s}-\frac{1}{4}...$$

$$-\Gamma(0,(s-1)\log n)+\Gamma(0,s\log n)+\log(\frac{s}{s-1})=$$
$$\int_1^n x^{-s}dx-\frac{1}{2}\int_1^n\int_1^{\frac{n}{x}}(x\cdot y)^{-s}dy\,dx+\frac{1}{3}\int_1^n\int_1^{\frac{n}{x}}\int_1^{\frac{n}{x\,y}}(x\cdot y\cdot z)^{-s}dz\,dy\,dx-\frac{1}{4}...$$

$$\psi(n)=-\lim_{s\to 0}\frac{\partial}{\partial s}[\log\zeta(s)]_n$$

$$-\lim_{s\to 0}\frac{\partial}{\partial s}\Gamma(0,s\log n)-\Gamma(0,(s-1)\log n)+\log(\frac{s}{s-1})=$$
$$n-\log n-1$$

(10.3.1)

$$\begin{array}{ll}\psi(n) & =\sum_{j=2}^{n}\log j-\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\log j+\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor\frac{n}{j\cdot k}\rfloor}\log j-\ldots \\ n-\log n-1 & =\int_{1}^{n}\log x\,dx-\int_{1}^{n}\int_{1}^{\frac{n}{x}}\log x\,dy\,dx+\int_{1}^{n}\int_{1}^{\frac{n}{x}}\int_{1}^{\frac{n}{xy}}\log x\,dz\,dy\,dx-\ldots\end{array}\tag{10.3.1}$$

Some identities glossed over here are covered in more detail in http://www.icecreambreakfast.com/primecount/ApproximingThePrimeCountingFunctionWithLinniksIdentity_NathanMcKenzie.pdf

10.4

$$\lim_{s\to0}\frac{\partial}{\partial s}[\zeta(s)^z]_n=\sum_{k=1}^{\infty}\binom{z}{k}\lim_{s\to0}\frac{\partial}{\partial s}[(\zeta(s)-1)^k]_n$$

$$\lim_{s\to0}\frac{\partial}{\partial s}[\zeta(s)^\rho]_n-1=0$$

For some fixed n, $[l_n]^{*z}=0$ has $\log_2 n$ solutions in z, denoted ρ. Using those solutions, we have

$$\lim_{s\to0}\frac{\partial}{\partial s}[\zeta(s)^\rho]_n=-1+\prod_{\rho}\left(1-\frac{1}{\rho}\right)$$

$$\log(n!)=-1+\prod_{\rho}\left(1-\frac{1}{\rho}\right)\tag{4.10}$$

$$\psi(n)=-\sum_{\rho}\rho^{-1}\tag{4.11}$$

$$n!=e^{\prod_{\rho}\left(1-\frac{1}{\rho}\right)}\cdot e^{-1}$$

```
bin[ z_, k_ ] := Product[ z-j, { j, 0, k-1 } ]/k!
Lm1[n_,k_] := Sum[ Lm1[n/j,k-1],{j,2,n}];Lm1[n_, 1] := Sum[ Log[j],{j,2,n}];Lm1[n_, 0] := UnitStep[n-1]
 Lz[ n_, z_] := Sum[ bin[z,k] Lm1[n,k],{k,0,Log[2,n]}]
zeros[n_]:=List@@NRoots[Lz[n,z]==0,z][[All,2]]
Table[{Chop[-1+Product[1-1/r,{r,zeros[n]}]-N[Sum[Log[j],{j,2,n}]]], Chop[1-Product[1+1/r,{r,zeros[n]}]-
N[Sum[MangoldtLambda[j],{j,2,n}]]]},{n,4,100}]//TableForm
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Continuing with the approach from section 8, a similar technique can be applied to the Chebyshev function, $\psi(n)=\sum_{j=2}^{n}\Lambda(j)$.

First, let's define the following function, analogous to $[((1-x^{1-s})\zeta(s)-1)^k]_n$ from (8.16), with x some rational constant fraction of the form $x=\frac{a}{b}, a>b$,

$$\lim_{s\to 0}\frac{\partial}{\partial s}[((1-x^{1-s})\zeta(s)-1)^0]_n=0$$
$$\lim_{s\to 0}\frac{\partial}{\partial s}[(1-x^{1-s})\zeta(s)-1]_n=b^{-1}\sum_{j=b+1}^{\lfloor n\cdot b\rfloor}\alpha(j,x)\log\frac{j}{b}$$
$$\lim_{s\to 0}\frac{\partial}{\partial s}[((1-x^{1-s})\zeta(s)-1)^k]_n=b^{-1}\sum_{j=b+1}^{\lfloor n\cdot b\rfloor}\alpha(j,x)\lim_{s\to 0}\frac{\partial}{\partial s}[((1-x^{1-s})\zeta(s)-1)^{k-1}]_{n\cdot b\cdot j^{-1}}$$

(9.1)

This value can also be defined in terms of $[((1-x^{1-s})\zeta(s)-1)^k]_n$ from (8.16),

$$\lim_{s\to 0}\frac{\partial}{\partial s}[((1-x^{1-s})\zeta(s)-1)^0]_n=0$$
$$\lim_{s\to 0}\frac{\partial}{\partial s}[((1-x^{1-s})\zeta(s)-1)^k]_n=b^{-1}\sum_{j=b+1}^{\lfloor n\cdot b\rfloor}\alpha(j,x)\log\frac{j}{b}\lim_{s\to 0}[((1-x^{1-s})\zeta(s)-1)^{k-1}]_{n\cdot b\cdot j^{-1}}$$

(9.2)

It can also be defined for a real number parameter instead, with x a real number > 1,

$$\lim_{s\to 0}\frac{\partial}{\partial s}[((1-x^{1-s})\zeta(s)-1)^0]_n=0$$
$$\lim_{s\to 0}\frac{\partial}{\partial s}[(1-x^{1-s})\zeta(s)-1]_n=\sum_{j=2}^{n}\log j-x\sum_{j=1}^{\lfloor\frac{n}{x}\rfloor}\log j\,x$$
$$\lim_{s\to 0}\frac{\partial}{\partial s}[((1-x^{1-s})\zeta(s)-1)^k]_n=\sum_{j=2}^{n}\lim_{s\to 0}\frac{\partial}{\partial s}[((1-x^{1-s})\zeta(s)-1)^{k-1}]_{n\,j^{-1}}-x\sum_{j=1}^{\lfloor\frac{n}{x}\rfloor}\lim_{s\to 0}\frac{\partial}{\partial s}[((1-x^{1-s})\zeta(s)-1)^{k-1}]_{n(j\cdot x)^{-1}}$$

(

```
L2[n_, 1,b_] := L2[n,1,b]=Sum[ Log[j],{j,2,n}]-b Sum[ Log[j b],{j,1,n/b}]
L2[n_, k_,b_ ]:= Sum[ L2[ n/j,k-1,b],{j,2,n}]-b Sum[ L2[ n/(j b),k-1,b],{j,1,n}]
[Mathematica]
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)

(9.3)

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num[c_]:=Numerator[c];den[c_]:=Denominator[c]
alpha[n_,c_]:=den[c] (Floor[n/den[c]]-Floor[(n-1)/den[c]])-num[c] (Floor[n/num[c]]-Floor[(n-1)/num[c]])
L2[n_,1,c_]:=L2[n,1,c]=(1/den[c])Sum[alpha[ j, c]Log[j/den[c]],{j,den[c]+1,den[c] n}]
L2[n_,k_,c_]:=L2[n,k,c]=(1/den[c])Sum[If[ alpha[j,c] == 0, 0, alpha[j,c]L2[den[c] n/j,k-1,c]],{j,den[c]+1,den[c] n}]
 E2[n_,k_,c_]:=E2[n,k,c]=(1/den[c])Sum[If[alpha[j,c]==0,0,alpha[j,c]E2[(den[c] n)/j,k-1,c]],{j,den[c]+1,den[c] n}]
E2[n_,0,c_]:=UnitStep[n-1]
L2Alt[n_,k_,c_]:=(1/den[c])Sum[If[alpha[j,c]==0,0,alpha[j,c]  Log[j/den[c]]E2[den[c] n/j,k-1,c]],{j,den[c]+1,den[c] n}]
L2Alt[n_,0,c_]:=UnitStep[n-1]
 [Mathematica]
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$$(L)^z(n,x)=\sum_{k=0}^{\lfloor\frac{\log n}{\log x}\rfloor}\binom{z}{k}(L-1)^k(n,x)$$

$$(L)^0(n,x)=0$$
$$(L)^k(n,x)=b^{-1}\sum_{j=1}^{\lfloor n\cdot b\rfloor}\alpha(j,x)(\log j+(L)^{k-1}(\frac{n\cdot b}{j},x))$$

$$(L)^z(n,x,k)=b^{-1}\frac{z-k+1}{k}\sum_{j=b+1}^{\lfloor b\cdot n\rfloor}\alpha(j,x)(\log\frac{j}{b}+(L)^z(\frac{n\cdot b}{j},x,k+1))$$

(9.3)

```
num[c_]:=Numerator[c];den[c_]:=Denominator[c]
alpha[n_,c_]:=den[c] (Floor[n/den[c]]-Floor[(n-1)/den[c]])-num[c] (Floor[n/num[c]]-Floor[(n-1)/num[c]])
L2[n_,1,c_]:=L2[n,1,c]=(1/den[c])Sum[alpha[ j, c]Log[j/den[c]],{j,den[c]+1,den[c] n}]
L2[n_,k_,c_]:=L2[n,k,c]=(1/den[c])Sum[If[ alpha[j,c] == 0, 0, alpha[j,c]L2[den[c] n/j,k-1,c]],{j,den[c]+1,den[c] n}]
 E2[n_,k_,c_]:=E2[n,k,c]=(1/den[c])Sum[If[alpha[j,c]==0,0,alpha[j,c]E2[(den[c] n)/j,k-1,c]],{j,den[c]+1,den[c] n}]
E2[n_,0,c_]:=UnitStep[n-1]
L2[n_,k_,c_]:=(1/den[c])Sum[If[alpha[j,c]==0,0,alpha[j,c]  Log[j/den[c]]E2[den[c] n/j,k-1,c]],{j,den[c]+1,den[c] n}]
L2[n_,0,c_]:=UnitStep[n-1]
L1[n_,z_,c_]:=Sum[Binomial[z,k] L2[n,k,c],{k,0,Floor[Log[n]/Log[c]]}]
[Mathematica]
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and we will find that $\psi(n)$ can be expressed as

$$\psi(n)=-\sum_{k=0}^{\lfloor\frac{\log n}{\log x}\rfloor}(-1)^k(L-1)^k(n,x)+\sum_{k=1}^{\lfloor\frac{\log n}{\log x}\rfloor}x^k\cdot\log x$$

(9.4)

or

$$\psi(n)=-(L)^{-1}(n,x)+\sum_{k=1}^{\lfloor\frac{\log n}{\log x}\rfloor}x^k\cdot\log x$$

(9.5)

```
referenceChebyshev[n_]:=Sum[MangoldtLambda[j],{j,2,n}]
num[c_]:=Numerator[c];den[c_]:=Denominator[c]
alpha[n_,c_]:=den[c] (Floor[n/den[c]]-Floor[(n-1)/den[c]])-num[c] (Floor[n/num[c]]-Floor[(n-1)/num[c]])
 E2[n_,k_,c_]:=E2[n,k,c]=(1/den[c])Sum[If[alpha[j,c]==0,0,alpha[j,c]E2[(den[c] n)/j,k-1,c]],{j,den[c]+1,den[c] n}]
E2[n_,0,c_]:=UnitStep[n-1]
L2[n_,k_,c_]:=(1/den[c])Sum[If[alpha[j,c]==0,0,alpha[j,c]  Log[j/den[c]]E2[den[c] n/j,k-1,c]],{j,den[c]+1,den[c] n}]
L2[n_,0,c_]:=UnitStep[n-1]
ChebAlt[n_,c_]:=1+Sum[(-1)^(k-1) L2[n,k,c],{k,0,Floor[ Log[n]/Log[If[c<2,c,2]]]}]+Sum[c^k Log[c],
{k,1,Floor[Log[n]/Log[c]]}]
Grid[Table[ {N[ referenceChebyshev[n]], N[ChebAlt[n,(b+1)/b]] },{n,5,100,5},{b,1,4}]]
[Mathematica]
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Now, given the following limit,

$$\lim_{x\to 1^+}\sum_{k=1}^{\lfloor\frac{\log n}{\log x}\rfloor}x^k\cdot\log x=n-1$$

(9.6)

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{Limit[Sum[c^k Log[c],{k,1,Log[n]/Log[c]}],c->1],n-1}
```

this means that the relationship between $\psi(n)$ and n can be expressed, with $x=\frac{b+1}{b}$, as

$$\boxed{\psi(n)=n-1-\lim_{x\to1+}\sum_{k=0}^{\lfloor\frac{\log n}{\log x}\rfloor}(-1)^k(L-1)^k(n,x)} \tag{9.7}$$

or more tersely as

$$\boxed{\psi(n)=n-1-\lim_{x\to1+}(L)^{-1}(n,x)} \tag{9.8}$$

Since, by (9.8), the error term in the prime number theory ($\psi(n)-n$) is $-1-\lim_{b\to1^+}(L)^{-1}(n,y)$, different representations of $(L)^{-1}(n,y)$, where y is a real number > 1 or, in some cases, a rational fraction > 1, are potentially interesting to study. So let's collect some such representations.

$$(L)^{-1}(n,c)=b^{-1}\sum_{j=b+1}^{\lfloor n\cdot b\rfloor}\alpha(j,c)\left(-\log\frac{j}{b}-(L)^{-1}\left(\frac{n\cdot b}{j},c\right)\right)$$

$$(L-x\,L)^{-1}(n)=-\sum_{j=2}^{n}\log j+(L-x\,L)^{-1}\left(\frac{n}{j}\right)+x\sum_{j=1}^{\lfloor\frac{n}{x}\rfloor}\log(j\,x)+(L-x\,L)^{-1}\left(\frac{n}{j\,x}\right)$$

$$(E)^{-1}(n,x)=1-b^{-1}\sum_{j=b+1}^{\lfloor n\cdot b\rfloor}\alpha(j,x)(E)^{-1}\left(\frac{n\cdot b}{j},x\right)$$

$$(D-x\,D)^{-1}(n)=1-\sum_{j=2}^{n}(D-x\,D)^{-1}\left(\frac{n}{j}\right)+x\sum_{j=1}^{\lfloor\frac{n}{x}\rfloor}(D-x\,D)^{-1}\left(\frac{n}{j\,x}\right)$$

```
num[c_]:=Numerator[c];den[c_]:=Denominator[c]
alpha[n_,c_]:=alpha[n,c]=den[c] (Floor[n/den[c]]-Floor[(n-1)/den[c]])-num[c] (Floor[n/num[c]]-Floor[(n-1)/num[c]])
Lm1[ n_, c_ ] := (1/den[c])Sum[If[alpha[j,c]==0,0,alpha[j,c]( -Log[j/den[c]]-Lm1[den[c] n/j,c])],{j,den[c]+1,den[c] n}]
Em1[ n_, c_ ] := 1-(1/den[c])Sum[If[alpha[j,c]==0,0,alpha[j,c]( Em1[den[c] n/j,c])],{j,den[c]+1,den[c] n}]
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$$(L)^{-1}(n,x)=b^{-1}\sum_{j=1}^{\lfloor n\cdot b\rfloor}\alpha(j,x)\log\frac{j}{b}(E)^{-1}\left(\frac{n\cdot b}{j},x\right)$$

```
num[c_]:=Numerator[c];den[c_]:=Denominator[c]
alpha[n_,c_]:=alpha[n,c]=den[c] (Floor[n/den[c]]-Floor[(n-1)/den[c]])-num[c] (Floor[n/num[c]]-Floor[(n-1)/num[c]])
Lm1[ n_, c_ ] := (1/den[c])Sum[If[alpha[j,c]==0,0,alpha[j,c]( Log[j/den[c]]-Lm1[den[c] n/j,c])],{j,den[c]+1,den[c] n}]
Em1[ n_, c_ ] := 1-(1/den[c])Sum[If[alpha[j,c]==0,0,alpha[j,c]( Em1[den[c] n/j,c])],{j,den[c]+1,den[c] n}]
L1mAlt[ n_, c_] := den[c]^-1 Sum[ Em1[ n den[c]/j, c ]N[alpha[ j, c]Log[j/den[c]]],{j,1,n den[c]}]
```

$$(L)^{-1}(n,y)=-\sum_{j=2}^{n}\log j\,(E)^{-1}\left(\frac{n}{j},y\right)+y\sum_{j=1}^{\lfloor\frac{n}{y}\rfloor}\log(j\,y)(E)^{-1}\left(\frac{n}{j\,y}\right)$$

$$(L)^{-1}(n,y)=\log y\,(E)^{-1}(\frac{n}{y},y)+\sum_{j=2}^{\lfloor\frac{n}{y}\rfloor}(y\log(j\,y)(E)^{-1}(\frac{n}{j\,y},y)-\log j\,(E)^{-1}(\frac{n}{j},y))-\sum_{j=\lfloor\frac{n}{y}\rfloor+1}^{n}\log j\,(E)^{-1}(\frac{n}{j},y)$$

$$(L)^{z}(n,y)=\sum_{k=0}^{\lfloor\frac{\log n}{\log y}\rfloor}\frac{z^k}{k!}(\frac{\partial^k}{\partial r^k}(L)^{r}(n,y)\text{ at } r=0)$$

$$(L)^{-1}(n,y)=\sum_{k=0}^{\lfloor\frac{\log n}{\log y}\rfloor}\frac{(-1)^k}{k!}(\frac{\partial^k}{\partial r^k}(L)^{r}(n,y)\text{ at } r=0)$$

$$(L)^{-1}(n,y)=-\sum_{k=0}^{\lfloor\frac{\log n}{\log y}\rfloor}(-1)^k(L-1)^k(n,y)$$

If $1<y\leq 2$, for some fixed value of *n,* the were will be $\lfloor\frac{\log n}{\log y}\rfloor$ values for *z* such that $(L)^{z}(n,y)-1=0$. Let's call those zeros ρ .

$$(L)^{z}(n,y)=1-\prod_{\rho}1-\frac{z}{\rho}$$

$$(L)^{-1}(n,y)=1-\prod_{\rho}1+\frac{1}{\rho}$$

```
num[c_]:=Numerator[c];den[c_]:=Denominator[c]
alpha[n_,c_]:=den[c] (Floor[n/den[c]]-Floor[(n-1)/den[c]])-num[c] (Floor[n/num[c]]-Floor[(n-1)/num[c]])
L2[n_,1,c_]:=L2[n,1,c]=(1/den[c])Sum[alpha[j,c]Log[j/den[c]],{j,den[c]+1,den[c] n}];L2[n_,0,c_]:=0
L2[n_,k_,c_]:=L2[n,k,c]=(1/den[c])Sum[If[alpha[j,c] □0,0,alpha[j,c]L2[den[c] n/j,k-1,c]],{j,den[c]+1,den[c] n}]
bin[z_,k_]:=Product[z-j,{j,0,k-1}]/k!
L1[n_,z_,c_]:=Sum[bin[z,k] L2[n,k,c],{k,0,Floor[Log[n]/Log[c]]}]
zeros[n_, c_]:=List@@Roots[L1[n,z,c]-1□0,z][[All,2]]
L1Alt[ n_, z_, c_] := 1-Product[ 1-z/r,{r,zeros[n,c]}]
L1m[ n_, c_] := 1-Product[ 1+r^-1,{r,zeros[n,c]}]
[Mathematica]
```