This is an early set of notes trying generalize the smoothing relationship I had already noticed between the log integral and the riemann prime counting function. An early incomplete signpost.

$$\mu(n)=\sum_{k=0} -1^k d_k{}'(n)$$

$$M(n)=\sum_{k=0} -1^k \sum_{j=2}^{n} d_k{}'(j)$$

$$M(n)=1-\sum_{k=2}^{n} 1+\sum_{k=2}^{n}\sum_{m=2}^{\frac{n}{k}} 1-\sum_{k=2}^{n}\sum_{m=2}^{\frac{n}{k}}\sum_{o=2}^{\frac{n}{km}} 1+\ldots$$

$$A(n)=1-\int_{1}^{n} dy+\int_{1}^{n}\int_{1}^{\frac{n}{y}} dz\,dy-\int_{1}^{n}\int_{1}^{\frac{n}{y}}\int_{1}^{\frac{n}{yz}} dw\,dz\,dy+\ldots$$

$$A(n)=1+\sum_{k=1} -1^k\left(-1^k+\sum_{j=0}^{k-1}\frac{-1^{k-j-1} n\log^j n}{j!}\right)$$

$$A(n)=1+\sum_{k=1} 1-n\sum_{j=0}^{k-1}\frac{(-\log n)^j}{j!}$$

$$m=-\log n$$

$$A(n)=1+\sum_{k=1} 1-e^{-m}\sum_{j=0}^{k-1}\frac{m^j}{j!}$$

$$A(n)=1+\sum_{k=1} 1-\frac{1}{(k-1)!}\int_{-\log n}^{\infty} t^{k-1}e^{-t}\,dt$$

$$A(n)=1+\sum_{k=1}\frac{(k-1)!}{(k-1)!}-\frac{1}{(k-1)!}\int_{-m}^{\infty} t^{k-1}e^{-t}\,dt$$

$$A(n)=1+\sum_{k=1}\frac{\Gamma(k)}{(k-1)!}-\frac{1}{(k-1)!}\int_{-\log n}^{\infty} t^{k-1}e^{-t}\,dt$$

$$A(n)=1+\sum_{k=1}\frac{1}{(k-1)!}\int_{0}^{\infty} t^{k-1}e^{-t}\,dt-\frac{1}{(k-1)!}\int_{-\log n}^{\infty} t^{k-1}e^{-t}\,dt$$

$$A(n)=1+\sum_{k=1}\frac{-1}{(k-1)!}\int_{-\log n}^{0} t^{k-1}e^{-t}\,dt$$

$$A(n)=1-\sum_{k=1}\frac{1}{(k-1)!}\int_{-\log n}^{0} t^{k-1}e^{-t}\,dt$$

$$A(n)=1-\int_{-\log n}^{0}\left(\sum_{k=1}\frac{t^{k-1}}{(k-1)!}\right)e^{-t}\,dt$$

$$A(n)=1-\int_{-\log n}^{0} e^{t}e^{-t}\,dt=1-\int_{-\log n}^{0} dt=1-\log n$$

$$1-\log n=1-\int_1^n dx+\int_1^n\int_1^{\frac{n}{x}} dy\,dx-\int_1^n\int_1^{\frac{n}{x}}\int_1^{\frac{n}{xy}} dz\,dy\,dx+\dots$$

$$M(n)=1-\sum_{k=2}^{n}1+\sum_{k=2}^{n}\sum_{m=2}^{\frac{n}{k}}1-\sum_{k=2}^{n}\sum_{m=2}^{\frac{n}{k}}\sum_{o=2}^{\frac{n}{km}}1+\dots$$

$$\begin{aligned}
&M(n)+\log n-1= \\
&\quad -\left(\int_1^n\int_1^{\frac{n}{x}} dy\,dx-\sum_{j=2}^{n}\sum_{k=2}^{\frac{n}{j}}1\right) \\
&\quad +\left(\int_1^n\int_1^{\frac{n}{x}}\int_1^{\frac{n}{xy}} dz\,dy\,dx-\sum_{j=2}^{n}\sum_{k=2}^{\frac{n}{j}}\sum_{m=2}^{\frac{n}{jk}}1\right) \\
&\quad -\left(\int_1^n\int_1^{\frac{n}{x}}\int_1^{\frac{n}{xy}}\int_1^{\frac{n}{xyz}} dw\,dz\,dy\,dx-\sum_{j=2}^{n}\sum_{k=2}^{\frac{n}{j}}\sum_{m=2}^{\frac{n}{jk}}\sum_{s=2}^{\frac{n}{jkm}}1\right) \\
&\quad +\dots
\end{aligned}$$

$$\psi(n)=\sum_{j=1}^{n} \log j\, M\left(\frac{n}{j}\right)$$

so

$$\psi(n)=\sum_{j=1}^{n} \log j\left(1-\sum_{k=2}^{\frac{n}{j}} 1+\sum_{k=2}^{\frac{n}{j}} \sum_{m=2}^{\frac{n}{jk}} 1-\sum_{k=2}^{\frac{n}{j}} \sum_{m=2}^{\frac{n}{jk}} \sum_{o=2}^{\frac{n}{jkm}} 1+\ldots\right)$$

$$B(n)=\int_0^n \log x\left(1-\int_1^{\frac{n}{x}} dy+\int_1^{\frac{n}{x}} \int_1^{\frac{n}{xy}} dz\, dy-\int_1^{\frac{n}{x}} \int_1^{\frac{n}{xy}} \int_1^{\frac{n}{xyz}} dw\, dz\, dy+\ldots\right) dx$$

$$B(n)=\int_0^n \log x \cdot\left(1-\log \frac{n}{x}\right) dx$$

$$B(n)=\int_0^n \log x \cdot(1-\log n+\log x) dx$$

$$B(n)=\int_0^n \log x\, dx-\int_0^n \log x \log n\, dx+\int_0^n \log x \log x\, dx$$

$$B(n)=(n \log n-n)-(n \log^2 n-n \log n)+(2\mathrm{n}-2 n \log n+n \log^2 n)$$

$$B(n)=n$$

$$n=\int_0^n \log x\left(1-\int_1^{\frac{n}{x}} dy+\int_1^{\frac{n}{x}} \int_1^{\frac{n}{xy}} dz\, dy-\int_1^{\frac{n}{x}} \int_1^{\frac{n}{xy}} \int_1^{\frac{n}{xyz}} dw\, dz\, dy+\ldots\right) dx$$

$$\psi(n)=\left(\sum_{j=1}^{n} \log j\right)-\left(\sum_{j=1}^{n} \log j \cdot \sum_{k=2}^{\frac{n}{j}} 1\right)+\left(\sum_{j=1}^{n} \log j \cdot \sum_{k=2}^{\frac{n}{j}} \sum_{m=2}^{\frac{n}{jk}} 1\right)-\left(\sum_{j=1}^{n} \log j \cdot \sum_{k=2}^{\frac{n}{j}} \sum_{m=2}^{\frac{n}{jk}} \sum_{o=2}^{\frac{n}{jkm}} 1\right)+\ldots$$

$$n=\left(\int_0^n \log x\, dx\right)-\left(\int_0^n \log x \int_1^{\frac{n}{x}} dy\, dx\right)+\left(\int_0^n \log x \int_1^{\frac{n}{x}} \int_1^{\frac{n}{xy}} dz\, dy\, dx\right)-\left(\int_0^n \log x \int_1^{\frac{n}{x}} \int_1^{\frac{n}{xy}} \int_1^{\frac{n}{xyz}} dw\, dz\, dy\, dx\right)+\ldots$$

$$\begin{aligned}
n-\psi(n)= & \\
& \left(\left(\int_0^n \log x\, dx\right)-\left(\sum_{j=1}^{n} \log j\right)\right) \\
& -\left(\left(\int_0^n \log x \int_1^{\frac{n}{x}} dy\, dx\right)-\left(\sum_{j=1}^{n} \log j \cdot \sum_{k=2}^{\frac{n}{j}} 1\right)\right) \\
& +\left(\left(\int_0^n \log x \int_1^{\frac{n}{x}} \int_1^{\frac{n}{xy}} dz\, dy\, dx\right)-\left(\sum_{j=1}^{n} \log j \cdot \sum_{k=2}^{\frac{n}{j}} \sum_{m=2}^{\frac{n}{jk}} 1\right)\right) \\
& -\left(\left(\int_0^n \log x \int_1^{\frac{n}{x}} \int_1^{\frac{n}{xy}} \int_1^{\frac{n}{xyz}} dw\, dz\, dy\, dx\right)-\left(\sum_{j=1}^{n} \log j \cdot \sum_{k=2}^{\frac{n}{j}} \sum_{m=2}^{\frac{n}{jk}} \sum_{o=2}^{\frac{n}{jkm}} 1\right)\right) \\
& +\ldots
\end{aligned}$$

$$\psi(n)=n-\sum_{\rho}\frac{n^{\rho}}{\rho}-\frac{\zeta'(0)}{\zeta(0)}-\frac{1}{2}\log(1-n^{-2})$$

$$n-\psi(n)=\sum_{\rho}\frac{n^{\rho}}{\rho}+\frac{\zeta'(0)}{\zeta(0)}+\frac{1}{2}\log(1-n^{-2})$$

$$E(n)=\log 2\pi+\frac{1}{2}\log(1-n^{-2})$$

E(n) is of O(epsilon)

$$\sum_{\rho}\frac{n^{\rho}}{\rho}+E(n)=$$

$$((\int_0^n \log x\,dx)-(\sum_{j=1}^{n}\log j))$$

$$-((\int_0^n \log x\int_1^{\frac{n}{x}} dy\,dx)-(\sum_{j=1}^{n}\log j\cdot\sum_{k=2}^{\frac{n}{j}}1))$$

$$+((\int_0^n \log x\int_1^{\frac{n}{x}}\int_1^{\frac{n}{xy}} dz\,dy\,dx)-(\sum_{j=1}^{n}\log j\cdot\sum_{k=2}^{\frac{n}{j}}\sum_{m=2}^{\frac{n}{jk}}1))$$

$$-((\int_0^n \log x\int_1^{\frac{n}{x}}\int_1^{\frac{n}{xy}}\int_1^{\frac{n}{xyz}} dw\,dz\,dy\,dx)-(\sum_{j=1}^{n}\log j\cdot\sum_{k=2}^{\frac{n}{j}}\sum_{m=2}^{\frac{n}{jk}}\sum_{o=2}^{\frac{n}{jkm}}1))$$

$$+\ldots$$

$$D_k'(n) \approx \frac{-1^k}{(k-1)!} \int_{-\log n}^{0} t^{k-1} e^{-t} dt$$

$$D_k'(n) \approx -1^k \frac{\Gamma(k) - \Gamma(k, -\log n)}{\Gamma(k)}$$

$$D_k'(n) \approx -1^k (1 - n \sum_{j=0}^{k-1} \frac{(-\log n)^j}{j!})$$

$$\sum_{j=2}^{n} \psi(\frac{n}{j}) = \sum_{j=2}^{n} \log j$$

$$\psi(n) = \sum_{j=2}^{n} p_1'(j) \log j$$

$$\psi(n) = \sum_{j=2}^{n} M(\frac{n}{j}) \log j$$

$$\psi(n) = \sum_{j=1}^{n} \log j (1 - \sum_{k=2}^{\frac{n}{j}} 1 + \sum_{k=2}^{\frac{n}{j}} \sum_{m=2}^{\frac{n}{jk}} 1 - \sum_{k=2}^{\frac{n}{j}} \sum_{m=2}^{\frac{n}{jk}} \sum_{o=2}^{\frac{n}{jkm}} 1 + \ldots)$$

$$\psi(n) = (\sum_{j=1}^{n} \log j) - (\sum_{j=1}^{n} \log j \cdot \sum_{k=2}^{\frac{n}{j}} 1) + (\sum_{j=1}^{n} \log j \cdot \sum_{k=2}^{\frac{n}{j}} \sum_{m=2}^{\frac{n}{jk}} 1) - (\sum_{j=1}^{n} \log j \cdot \sum_{k=2}^{\frac{n}{j}} \sum_{m=2}^{\frac{n}{jk}} \sum_{o=2}^{\frac{n}{jkm}} 1) + \ldots$$

$$\psi(n) = \sum_{x=2}^{n} \log x - \sum_{x=2}^{n} \sum_{y=2}^{\frac{n}{x}} \log y + \sum_{x=2}^{n} \sum_{y=2}^{\frac{n}{x}} \sum_{z=2}^{\frac{n}{xy}} \log z - \sum_{x=2}^{n} \sum_{y=2}^{\frac{n}{x}} \sum_{z=2}^{\frac{n}{xy}} \sum_{w=2}^{\frac{n}{xyz}} \log w + \ldots$$