```
ClearAll["Global`*"]

M1[n_, k_, s_] := Sum[ (-1)^(j + 1) j^(-s) M1[n / j, k - 1, s], {j, 1, n}]; M1[n_, 0, s_] := 1
M2[n_, k_, s_] := Sum[ (-1)^(j + 1) j^(-s) M2[n / j, k - 1, s], {j, 2, n}]; M2[n_, 0, s_] := 1

E2a[n_, k_, x_, s_] := E2a[n, k, x, s] = Sum[ j^(-s) E2a[n / j, k - 1, x, s], {j, 2, n}] -
   x Sum[ (j x)^(-s) E2a[n / (x j), k - 1, x, s], {j, 1, n / x}];
E2a[n_, 0, a_, s_] := UnitStep[n - 1]
E2ab[n_, k_, x_, s_] := E2ab[n, k, x, s] =
  Sum[ (j + 1)^(-s) E2ab[n / (j + 1), k - 1, x, s] - x (j x)^(-s) E2ab[n / (x j), k - 1, x, s],
   {j, 1, n - 1}]; E2ab[n_, 0, a_, s_] := UnitStep[n - 1]
E1a[n_, k_, x_, s_] := E1a[n, k, x, s] = Sum[j^(-s) E1a[n / j, k - 1, x, s], {j, 1, n}] -
   x Sum[(j x)^(-s) E1a[n / (x j), k - 1, x, s], {j, 1, n / x}];
E1a[n_, 0, a_, s_] := UnitStep[n - 1]
E1ab[n_, k_, x_, s_] := E1ab[n, k, x, s] =
  Sum[j^(-s) E1ab[n / j, k - 1, x, s] - x (j x)^(-s) E1ab[n / (x j), k - 1, x, s], {j, 1, n}];
E1ab[n_, 0, a_, s_] := UnitStep[n - 1]

Dk[n_, k_, s_] := Dk[n, k, s] = Sum[j^(-s) Dk[Floor[n / j], k - 1, s], {j, 1, n}];
Dk[n_, 0, s_] := UnitStep[n - 1]
D2a[n_, k_, s_] := D2a[n, k, s] = Sum[j^(-s) D2a[Floor[n / j], k - 1, s], {j, 2, n}];
D2a[n_, 0, s_] := UnitStep[n - 1]

bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
D2b[n_, k_, s_] := Sum[(-1)^j Binomial[k, j] Dk[n, k - j, s], {j, 0, k}]
DDb[n_, z_, s_] := Sum[bin[z, k] D2a[n, k, s], {k, 0, Log[2, n]}]
E2b[n_, k_, x_, s_] := Sum[(-1)^j Binomial[k, j] E1a[n, k - j, x, s], {j, 0, k}]
E1b[n_, k_, x_, s_] := Sum[ Binomial[k, j] E2a[n, k - j, x, s], {j, 0, k}]
E1ba[n_, z_, x_, s_] := Sum[ bin[z, k] E2a[n, k, x, s], {k, 0, Log[If[x < 2, x, 2], n]}]
DDc[n_, k_, x_, s_] :=
 Sum[Binomial[k + j - 1, k - 1] x^(j (1 - s)) E1a[n / (x^j), k, x, s], {j, 0, Log[x, n]}]
DzAlt[n_, z_, x_, s_] := Sum[(-1)^j Binomial[-z, j] Binomial[z, k]
   x^(j (1 - s)) E2a[n / x^j, k, x, s], {j, 0, Log[x, n]}, {k, 0, Log[x, n / x^j]}]

E1c[n_, k_, x_, s_] := Sum[(-1)^j Binomial[k, j] x^(j (1 - s)) Dk[n / x^j, k, s], {j, 0, k}]
E2c[n_, k_, x_, s_] := Sum[(-1)^j x^(j (1 - s))
   Binomial[k, j] Binomial[j, m] D2a[n / x^j, k - m, s], {j, 0, k}, {m, 0, j}]
D2E2[n_, k_, x_, s_] := Sum[ (-1)^j x^(j (1 - s)) Binomial[k, j]
   Sum[ Binomial[j, m] If[n / x^j < 1, 0, D2a[n / x^j, k - m, s]], {m, 0, j}], {j, 0, k}]
E2D2[n_, k_, x_, s_] := (-1)^k + Sum[x^(a (1 - s)) / ((k - 1)!) Binomial[k, j]
    Pochhammer[a - k + j + 1, k - 1] E2a[x^-a n, j, x, s], {a, 0, Log[x, n]}, {j, 0, k}]

Lin[n_, s_] := Sum[ (-1)^(k + 1) / k D2a[n, k, s], {k, Log[2, n]}]
LinE[n_, b_, s_] := Sum[ (-1)^(k + 1) / k E2a[n, k, b, s], {k, Log[2, n]}]
DzAlt[100, 2, 2, -1]

26879

E1a[121, 3, 1.6, 0]

-25.904
```

```
E1ab[121, 3, 1.6, 0]

-25.904

bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
Dm1[n_, k_, s_] := Sum[j^(-s) Dm1[n / j, k - 1, s], {j, 2, n}];
Dm1[n_, 0, s_] := UnitStep[n - 1]
Dz[n_, z_, s_] := Sum[bin[z, k] D2a[n, k, s], {k, 0, Log[2, n]}]
Dz[n_, z_, k_, s_] :=
 Dz[n, z, k, s] = 1 + ((z + 1) / k - 1) Sum[j^-s Dz[n / j, z, k + 1, s], {j, 2, n}]
N[Limit[D[Limit[D[DDb[100, z, s], z], z → 0], s], s → 0]]

$Aborted

Dz[100, 1.3, -1]

9083.02

Dz[100, 1.3, 1, -1]

9083.02

-N[Limit[D[Limit[D[Dz[100, z, 1, s], z], z -> 0], s], s -> 0]]

94.0453

N[referenceChebyshev[100]]

94.0453

chebyshev[n_] := Sum[MangoldtLambda[j], {j, 2, n}]
Dz[n_, z_, k_, s_] :=
 Dz[n, z, k, s] = 1 + ((z + 1) / k - 1) Sum[j^-s Dz[Floor[n / j], z, k + 1, s], {j, 2, n}]
Table[{N[chebyshev[n]], -N[Limit[D[Limit[D[Dz[n, z, 1, s], z], z → 0], s], s → 0]]},
 {n, 10, 70, 10}]

{{7.83201, 7.83201}, {19.2657, 19.2657}, {28.4765, 28.4765},
 {36.2146, 36.2146}, {49.4854, 49.4854}, {57.5332, 57.5332}, {66.5419, 66.5419}}

D[Log[Zeta[s]], s]
```

$$\frac{\text{Zeta}'[s]}{\text{Zeta}[s]}$$

```
{D[Zeta[s], s] / Zeta[s] /. s → 0, Limit[D[Limit[D[Zeta[s]^z, z], z → 0], s], s → 0]}

{Log[2 π], Log[2 π]}

dz[n_, z_, s_] := (n^-s) Product[(-1)^p[[2]] Binomial[-z, p[[2]]], {p, FI[n]}];
FI[n_] := FactorInteger[n]; FI[1] := {}

D[dz[100, 1, s], s] /. s → 0

-Log[100]

N[Sum[ (D[dz[j, 1, s], s] /. s → 0) Dz[100 / j, -1, 1, 0], {j, 1, 100}]]

-94.0453
```

```
N[Sum[ (D[Dz[100 / j, 1, 1, s], s] /. s → 0) dz[j, -1, 0], {j, 1, 100}]]
```

```
-94.0453
```

```
logD[n_, 0, s_] := UnitStep[n - 1];
logD[n_, k_, s_] := Sum[MangoldtLambda[j] / Log[j] j^-s logD[n / j, k - 1, s], {j, 2, n}]
```

```
chebyshev[n_] := Sum[MangoldtLambda[j], {j, 2, n}]
dz[n_, z_, s_] := (n^-s) Product[(-1)^p[[2]] Binomial[-z, p[[2]]], {p, FI[n]}];
FI[n_] := FactorInteger[n]; FI[1] := {}
Dz[n_, z_, s_] := Sum[dz[j, z, s], {j, 1, n}]
Table[
 Chop[N[chebyshev[n]] - ( -N[Sum[ (D[Dz[n / j, 1, 1, s], s] /. s → 0) dz[j, -1, 0], {j, 1, n}]])],
 {n, 10, 100, 10}]
```

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

```
chebyshev[n_] := Sum[MangoldtLambda[j], {j, 2, n}]
dz[n_, z_, s_] := (n^-s) Product[(-1)^p[[2]] Binomial[-z, p[[2]]], {p, FI[n]}];
FI[n_] := FactorInteger[n]; FI[1] := {}
Dz[n_, z_, s_] := Sum[dz[j, z, s], {j, 1, n}]
Table[
 Chop[N[chebyshev[n]] - ( -N[Sum[ dz[j, -1, 0] (D[Dz[n / j, 1, 1, s], s] /. s → 0), {j, 1, n}]])],
 {n, 10, 100, 10}]
```

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

```
{D[Zeta[s], s] / Zeta[s] /. s → 0, Limit[D[Log[Zeta[s]], s], s → 0]}
```

```
{Log[2 π], Log[2 π]}
```

```
chebyshev[n_] := Sum[MangoldtLambda[j], {j, 2, n}]
logD[n_, 0, s_] := UnitStep[n - 1]; logD[n_, k_, s_] :=
 Sum[FullSimplify[MangoldtLambda[j] / Log[j]] j^-s logD[n / j, k - 1, s], {j, 2, n}]
Table[{N[chebyshev[n]], -N[Limit[D[logD[n, 1, s], s], s → 0]]}, {n, 10, 70, 10}]
```

```
{{7.83201, 7.83201}, {19.2657, 19.2657}, {28.4765, 28.4765},
 {36.2146, 36.2146}, {49.4854, 49.4854}, {57.5332, 57.5332}, {66.5419, 66.5419}}
```

```
N[Limit[D[logD[100, 1, s], s], s → 0]]
```

```
-94.0453
```

```
chebyshev[n_] := Sum[MangoldtLambda[j], {j, 2, n}]
dz[n_, z_, s_] := (n^-s) Product[(-1)^p[[2]] Binomial[-z, p[[2]]], {p, FI[n]}];
FI[n_] := FactorInteger[n]; FI[1] := {}
Dz[n_, z_, s_] := Sum[dz[j, z, s], {j, 1, n}]
Table[
 Chop[N[chebyshev[n]] - (-N[Sum[dz[j, -1, 0] (D[Dz[n / j, 1, 1, s], s] /. s → 0), {j, 1, n}]])],
 {n, 10, 100, 10}]
Table[Chop[N[chebyshev[n]] -
   (-N[Sum[(D[Dz[n / j, 1, 1, s], s] /. s → 0) dz[j, -1, 0], {j, 1, n}]])], {n, 10, 100, 10}]
```

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

```
Dza[ n_, z_, s_] := Sum[ z^k / k ! logD[n, k, s], {k, 0, Log[2, n]}]
```

```
N[D[Dza[100, z, s], s] /. s → 0]
```

$-94.0453\,z - 169.15\,z^2 - 81.6195\,z^3 - 17.6846\,z^4 - 1.19616\,z^5 - 0.0438125\,z^6$

```
zeros[n_] := List @@ Roots[N[D[Dza[n, z, s], s] /. s → 0] == 0, z][[All, 2]]
```

```
zeros[1000]
```

```
{0, -147.982, -8.54743 - 14.2051 i, -8.54743 + 14.2051 i, -4.39602 - 3.11948 i,
 -4.39602 + 3.11948 i, -1.97192 - 1.0644 i, -1.97192 + 1.0644 i, -0.923003}
```

```
zeros2[n_] := List @@ Roots[N[Dza[n, z, 0]] == 0, z][[All, 2]]
```

```
zeros2[1000]
```

```
{-145.722, -8.80186 - 14.3448 i, -8.80186 + 14.3448 i, -4.45483 - 3.16845 i,
 -4.45483 + 3.16845 i, -2.04875 - 1.06859 i, -2.04875 + 1.06859 i, -0.961602, -0.00572997}
```

```
D[j^-s k^-s, s] /. s → 0
```

```
-Log[j] - Log[k]
```

```
D[j^-s k^-s l^-s, s] /. s → 0
```

```
-Log[j] - Log[k] - Log[l]
```

```
N[Sum[ (-1)^(k + 1) / k Limit[D[Dm1[100, k, s], s], s → 0], {k, 1, Log[2, 100]}]]
```

```
-94.0453
```

```
Lm1[n_, 1] := Sum[Log[j], {j, 2, n}]; Lm1[n_, 0] := UnitStep[n - 1]
Lm1[n_, k_] := Sum[Lm1[n / j, k - 1], {j, 2, n}]
```

```
bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
Lm1[n_, k_] := Sum[Lm1[n / j, k - 1], {j, 2, n}];
Lm1[n_, 1] := Sum[Log[j], {j, 2, n}]; Lm1[n_, 0] := UnitStep[n - 1]
Lz[n_, z_] := Sum[bin[z, k] Lm1[n, k], {k, 0, Log[2, n]}]
Dm1[n_, k_, s_] := Sum[j^(-s) Dm1[n / j, k - 1, s], {j, 2, n}];
Dm1[n_, 0, s_] := UnitStep[n - 1]
Lzb[n_, z_] := -Sum[1 / k bin[z, k] D[Dm1[n, k, s], s] /. s → 0, {k, 1, Log[2, n]}]
```

```
Limit[N[Lza[100, z] / z], z → 0]
```

```
-94.0453
```

```
N[Lz[100, -1]]
```

```
-93.0453
```

```
Limit[ Expand[N[(D[Dz[100, z, s], s] / z /. s → 0)]], z → 0]
```

```
-94.0453
```

```
N[Lz[100, -1]]
```

```
-93.0453
```

```
N[Lzb[100, -1]]
```

```
-93.0453
```

```
{-1 / 4 N[Limit[D[Dm1[80, 4, s], s], s → 0]], N[Lm1[80, 4]]}
```

```
{110.045, 110.045}
```

```
Dm1[n_, k_, s_] := Sum[j^(-s) Dm1[n / j, k - 1, s], {j, 2, n}]; Dm1[n_, 0, s_] := UnitStep[n - 1]
dsDz[n_, z_] := -Sum[1 / k bin[z, k] D[Dm1[n, k, s], s] /. s → 0, {k, 1, Log[2, n]}]
zeros[n_] := List @@ NRoots[dsDz[n, z] == -1, z][[All, 2]]
Table[{Chop[-1 + Product[1 - 1 / r, {r, zeros[n]}] - N[Sum[Log[j], {j, 2, n}]]],
   Chop[1 - Product[1 + 1 / r, {r, zeros[n]}] - N[Sum[MangoldtLambda[j], {j, 2, n}]]]}, {n,
   4, 10}] // TableForm
```

```
0    0
0    0
0    0
0    0
0    0
0    0
0    0
```

```
zeros[100]
```

```
{-12.9799 - 15.0426 i, -12.9799 + 15.0426 i, -3.66756,
 -3.06482 - 2.95324 i, -3.06482 + 2.95324 i, -0.00522175}
```

```
0    0
0    0
0    0
0    0
0    0
0    0
0    0
```

```
D[E2ab[100, 0, 2, s], s] /. s → 0
```

```
0
```

```
D1xD[n_, k_, x_, s_] := D1xD[n, k, x, s] =
  Sum[(j + 1) ^-s D1xD[n / (j + 1), k - 1, x, s] - x (j x) ^-s D1xD[n / (x j), k - 1, x, s], {j, 1, n}];
D1xD[n_, 0, x_, s_] := UnitStep[n - 1]
L2[n_, 1, b_] := L2[n, 1, b] = Sum[Log[j], {j, 2, n}] - b Sum[Log[j b], {j, 1, n / b}]
L2[n_, k_, b_] := Sum[L2[n / j, k - 1, b], {j, 2, n}] - b Sum[L2[n / (j b), k - 1, b], {j, 1, n}]
{N[D[D1xD[100, 3, 1.5, s], s] /. s → 0], -3 N[L2[100, 3, 1.5]]}
```

```
{-100.193, -100.193}
```

```
chebyshev[n_] := Sum[MangoldtLambda[j], {j, 2, n}]
D1xD[n_, k_, x_, s_] := D1xD[n, k, x, s] =
  Sum[(j + 1) ^-s D1xD[n / (j + 1), k - 1, x, s] - x (j x) ^-s D1xD[n / (x j), k - 1, x, s], {j, 1, n}];
D1xD[n_, 0, x_, s_] := UnitStep[n - 1]
ChebAlt[n_, c_] := Sum[(-1) ^ (k) / k (D[D1xD[n, k, c, s], s] /. s → 0),
   {k, 1, Floor[Log[n] / Log[If[c < 2, c, 2]]]}] +
  Sum[c^k Log[c], {k, 1, Floor[Log[n] / Log[c]]}]
```

```
N[ChebAlt[100, 1.5]]
```

```
94.0453
```

```
N[chebyshev[100]]
```

```
94.0453
```

```
ClearAll["Global`*"]
```

```
r[n_, k_, s_] := Sum[ j^-s ( 1 / k - r[n / j, k + 1, s]), {j, 2, n}]
r2[n_, k_, x_, s_] := r2[n, k, x, s] = Sum[ j^-s ( 1 / k - r2[n / j, k + 1, x, s]), {j, 2, n}] -
   x Sum[ (j x) ^-s ( 1 / k - r2[n / (j x), k + 1, x, s]), {j, 1, n / x}]
r2a[n_, k_, x_, s_] := r2a[n, k, x, s] = If[ n < 1 || k == 0, 0, -1 / k] -
   Sum[ (j + 1) ^-s r2a[n / (j + 1), k + 1, x, s] - x (j x) ^-s r2a[n / (j x), k + 1, x, s], {j, 1, n}]
```

```
r2[100, 1, 3 / 2, 0]
```

$$-\frac{8149753}{2365440}$$

D[E1ba[100, z, 3 / 2, 0], z] /. z → 0

$-\frac{8149753}{2365440}$

r2a[100, 0, 3 / 2, 0]

$-\frac{8149753}{2365440}$

FullSimplify[r2[100, 1, 2, s]]

$$-21\, 2^{-1-6 s} - 3 \times 2^{-1-2 s} - \frac{31 \times 2^{-5 s}}{5} - 2^{-s} + 3^{-1-3 s} + \frac{3^{-4 s}}{4} + 3^{-s} - 15 \times 4^{-1-2 s} + \frac{5^{-2 s}}{2} + 5^{-s} + \frac{7^{-2 s}}{2} + 7^{-s} - \frac{7 \times 8^{-s}}{3} + \frac{9^{-s}}{2} + 11^{-s} + 13^{-s} + 17^{-s} + 19^{-s} + 23^{-s} + 29^{-s} + 31^{-s} + 37^{-s} + 41^{-s} + 43^{-s} + 47^{-s} + 53^{-s} + 59^{-s} + 61^{-s} + 67^{-s} + 71^{-s} + 73^{-s} + 79^{-s} + 83^{-s} + 89^{-s} + 97^{-s}$$

$$\mathbf{N\Big[D\Big[-21\, 2^{-1-6 s} - 3 \times 2^{-1-2 s} - \frac{31 \times 2^{-5 s}}{5} - 2^{-s} + 3^{-1-3 s} + \frac{3^{-4 s}}{4} + 3^{-s} - 15 \times 4^{-1-2 s} + \frac{5^{-2 s}}{2} + 5^{-s} + \frac{7^{-2 s}}{2} + 7^{-s} - \frac{7 \times 8^{-s}}{3} + \frac{9^{-s}}{2} + 11^{-s} + 13^{-s} + 17^{-s} + 19^{-s} + 23^{-s} + 29^{-s} + 31^{-s} + 37^{-s} + 41^{-s} + 43^{-s} + 47^{-s} + 53^{-s} + 59^{-s} + 61^{-s} + 67^{-s} + 71^{-s} + 73^{-s} + 79^{-s} + 83^{-s} + 89^{-s} + 97^{-s}, s\Big] \mathbin{/.} s \to 0\Big]}$$

-6.70877

```
$RecursionLimit = 1 000 000
r3[n_, k_, x_, s_] := r3[n, k, x, s] = Sum[ (j + 1) ^-s ( 1 / k - r3[n / (j + 1), k + 1, x, s]) -
    x (j x) ^-s ( 1 / k - r3[n / (j x), k + 1, x, s]), {j, 1, Floor[n - 1]}]
```

1 000 000

r3[4, 1, 1.0002, 0]

-0.00364376

```
F[k_, s_, t_] := If[t > 200, 0, (N[Zeta[s]] - 1) ( 1 / k - F[k + 1, s, t + 1]))]
Table[Chop[F[1, s, 1] - Log[Zeta[s]]], {s, 2, 8}]
```

{0, 0, 0, 0, 0, 0, 0}

0.4977

0.4977

```
F[k_, z_, s_, t_] := If[t > 200, 0, 1 + ((z + 1) / k - 1) (N[Zeta[s]] - 1) F[k + 1, z, s, t + 1]]
Table[Chop[F[1, z, s, 1] - Zeta[s]^z], {s, 2, 8, .7}, {z, -3, 4, .4}]
```

```
{{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

```
{Zeta[s], Product[ Sum[Prime[j]^(-s a), {a, 0, Infinity}], {j, 1, Infinity}]}
```

```
{Zeta[s], Zeta[s]}
```

```
{ N[Zeta[2]]^z,
 N[Product[ (Sum[Prime[j]^(-s a ), {a, 0, Infinity}])^z, {j, 1, 400}] /. s → 2]}
```

$\{1.64493^z, 1.64487^z\}$

```
Table[
 Chop[N[Zeta[s]]^z - N[Product[ (Sum[Prime[j]^(-s a ), {a, 0, Infinity}])^z, {j, 1, 400}]]],
 {s, 3, 8}]
```

$\{-1.20206^z + 1.20206^z, -1.08232^z + 1.08232^z, -1.03693^z + 1.03693^z, 0, 0, 0\}$

```
FullSimplify[{-1.2020568938437024`^z + 1.2020569031595942`^z,
  -1.0823232337090738`^z + 1.0823232337111381`^z,
  -1.0369277551433693`^z + 1.03692775514337`^z, 0, 0, 0}]
```

$\{-1.20206^z + 1.20206^z, -1.08232^z + 1.08232^z, -1.03693^z + 1.03693^z, 0, 0, 0\}$

```
Sum[Prime[j]^(-s a ), {a, 0, Infinity}]
```

$$\frac{\text{Prime}[j]^s}{-1 + \text{Prime}[j]^s}$$

```
1 / (1 - 1 / Prime[j]^s)^z
```

```
{Zeta[s]^z, Product[ (1 - Prime[j]^-s)^-z, {j, 1, Infinity}]}
```

$$\left\{\text{Zeta}[s]^z, \prod_{j=1}^{\infty} (1 - \text{Prime}[j]^{-s})^{-z}\right\}$$

```
logD[n_, k_] := Sum[FullSimplify[MangoldtLambda[j] / Log[j]] logD[n / j, k - 1], {j, 2, n}];
logD[n_, 0] := UnitStep[n - 1]
Dz[n_, z_] := Sum[z^k / k ! logD[n, k], {k, 0, Log[2, n]}]
```

```
bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
Dm1[n_, k_, s_] := Sum[j^(-s) Dm1[n / j, k - 1, s], {j, 2, n}];
Dm1[n_, 0, s_] := UnitStep[n - 1]
dsDz[n_, z_] := -Sum[1 / k bin[z, k] D[Dm1[n, k, s], s] /. s -> 0, {k, 1, Log[2, n]}]
zeros[n_] := List @@ NRoots[dsDz[n, z] == -1, z][[All, 2]]
Table[{Chop[-1 + Product[1 - 1 / r, {r, zeros[n]}] - N[Sum[Log[j], {j, 2, n}]]],
   Chop[1 - Product[1 + 1 / r, {r, zeros[n]}] - N[Sum[MangoldtLambda[j], {j, 2, n}]]]}, {n,
   4, 10}] // TableForm
```

0	0
0	0
0	0
0	0
0	0
0	0
0	0

```
1 / Log[x] + Integrate[ 1 / (t Log[t]^2), {t, 2, x}]
```

$$\text{ConditionalExpression}\left[\frac{1}{\text{Log}[2]}, \text{Re}[x] \geq 1 \,||\, x \notin \text{Reals}\right]$$

```
FullSimplify[1 / Log[x] + Integrate[ 1 / (t Log[t]^2), {t, 4, x}]]
```

$$\text{ConditionalExpression}\left[\frac{1}{\text{Log}[4]}, \text{Re}[x] \geq 1 \,||\, x \notin \text{Reals}\right]$$

```
Sum[ (-1)^(k - 1) / k
  (1 / (s - 1)^k - Integrate[ D[ (((Zeta[s, y + 1] - 1) y^(s - 1))^k), y], {y, 1, Infinity}]), {k,
  1, Infinity}]
```

$$\sum_{k=1}^{\infty} \frac{1}{k} (-1)^{-1+k} \left((-1+s)^{-k} - \int_1^{\infty} k \left(y^{-1+s} (-1 + \text{Zeta}[s, 1+y])\right)^{-1+k} \left((-1+s)\, y^{-2+s} (-1+\text{Zeta}[s, 1+y]) - s\, y^{-1+s}\, \text{Zeta}[1+s, 1+y]\right) dy\right)$$

```
Integrate[ D[ (((Zeta[s] - 1) y)^k), y], {y, 1, Infinity}]
```

$$\text{ConditionalExpression}\left[-(-1+\text{Zeta}[s])^k, \text{Re}[k] < 0\right]$$

```
Sum[ (-1)^(k - 1) / k (1 / (s - 1)^k - (-(-1 + Zeta[s])^k)), {k, 1, Infinity}]
Sum[ (-1)^(k - 1) / k (1 / (s - 1)^k ), {k, 1, Infinity}]
```

$$\text{Log}\left[\frac{s}{-1+s}\right]$$

```
Log[ s / (s - 1)] - Integrate[ D[ Log[ (Zeta[s, y + 1] - 1) y^(s - 1)], y], {y, 1, Infinity}]
```

$$-\int_1^{\infty} \frac{y^{1-s}\left((-1+s)\, y^{-2+s} (-1+\text{Zeta}[s, 1+y]) - s\, y^{-1+s}\, \text{Zeta}[1+s, 1+y]\right)}{-1+\text{Zeta}[s, 1+y]}\, dy + \text{Log}\left[\frac{s}{-1+s}\right]$$

```
Log[s / (s - 1)] - Integrate[D[Log[(Zeta[s] - 1) y^(s - 1)], y], {y, 1, Infinity}]
```

Integrate::idiv : Integral of $\frac{1}{y}$ does not converge on $\{1, \infty\}$. ≫

$$-\int_1^\infty \frac{-1+s}{y}\, dy + \text{Log}\left[\frac{s}{-1+s}\right]$$

```
(s / (s - 1))^k - Integrate[D[(((Zeta[s, y + 1]) y^(s - 1) + 1)^z), y], {y, 1, Infinity}]
```

$$\left(\frac{s}{-1+s}\right)^k - \int_1^\infty z\left(1 + y^{-1+s}\,\text{Zeta}[s, 1+y]\right)^{-1+z}\left((-1+s)\, y^{-2+s}\,\text{Zeta}[s, 1+y] - s\, y^{-1+s}\,\text{Zeta}[1+s, 1+y]\right) dy$$

```
Sum[(-1)^(k + 1) / k (1 / (s - 1)^k - Integrate[
    D[(((HurwitzZeta[s, y + 1]) y^(s - 1))^k), y], {y, 1, Infinity}]), {k, 1, Infinity}]
```

$$\sum_{k=1}^{\infty} \frac{1}{k}(-1)^{1+k}\left((-1+s)^{-k} - \int_1^\infty k\left(y^{-1+s}\,\text{HurwitzZeta}[s, 1+y]\right)^{-1+k} \left((-1+s)\, y^{-2+s}\,\text{HurwitzZeta}[s, 1+y] - s\, y^{-1+s}\,\text{HurwitzZeta}[1+s, 1+y]\right) dy\right)$$

```
FullSimplify[D[(((Zeta[s, y + 1]) y^(s - 1))^k), y]]
```

$$k\, y^{-2+s}\left(y^{-1+s}\,\text{Zeta}[s, 1+y]\right)^{-1+k}\left((-1+s)\,\text{Zeta}[s, 1+y] - s\, y\,\text{Zeta}[1+s, 1+y]\right)$$

```
Sum[Binomial[z, k] (1 / (s - 1)^k - Integrate[k y^(-2+s) (y^(-1+s) Zeta[s, 1 + y])^(-1+k)
    ((-1 + s) Zeta[s, 1 + y] - s y Zeta[1 + s, 1 + y]), {y, 1, Infinity}]), {k, 0, Infinity}]
```

```
$Aborted
```

```
Integrate[
 k y^(-2+s) (y^(-1+s) Zeta[s, 1 + y])^(-1+k) ((-1 + s) Zeta[s, 1 + y] - s y Zeta[1 + s, 1 + y]), {y, 1, Infinity}]
```

$$\int_1^\infty k\, y^{-2+s}\left(y^{-1+s}\,\text{Zeta}[s, 1+y]\right)^{-1+k}\left((-1+s)\,\text{Zeta}[s, 1+y] - s\, y\,\text{Zeta}[1+s, 1+y]\right) dy$$

```
FullSimplify[D[Log[Zeta[s, y + 1] y^(s - 1) + 1], y]]
```

$$\frac{(-1+s)\,\text{Zeta}[s, 1+y] - s\, y\,\text{Zeta}[1+s, 1+y]}{y^{2-s} + y\,\text{Zeta}[s, 1+y]}$$

```
Integrate[((-1 + s) Zeta[s, 1 + y] - s y Zeta[1 + s, 1 + y]) / (y^(2-s) + y Zeta[s, 1 + y]), {y, 1, Infinity}] /. s → 2
```

$$\int_1^\infty \frac{\text{Zeta}[2, 1+y] - 2\, y\,\text{Zeta}[3, 1+y]}{1 + y\,\text{Zeta}[2, 1+y]}\, dy$$

```
FullSimplify[-(1 / x - 1) ((x - 1)^k + (x - 1)^(k - 1))]
```

$$(-1+x)^k$$

```
FullSimplify[- (x - 1) ((1 / x - 1) ^k + (1 / x - 1) ^ (k - 1))]
```

$$\left(-1+\frac{1}{x}\right)^k$$

```
FullSimplify[
 Sum[ (x - 1) ^a / a !, {a, 1, Infinity}] Sum[ BernoulliB[b] / b ! (x - 1) ^b, {b, 0, Infinity}]]
```

$$-1+x$$

```
FullSimplify[Sum[ (x - 1) ^ (a + c) / a !, {a, 1, Infinity}]
  Sum[ BernoulliB[b] / b ! (x - 1) ^ (b + k - c - 1), {b, 0, Infinity}]]
```

$$(-1+x)^k$$

```
FullSimplify[Sum[ (1 / x - 1) ^ (a + c) / a !, {a, 1, Infinity}]
  Sum[ BernoulliB[b] / b ! (1 / x - 1) ^ (b + k - c - 1), {b, 0, Infinity}]]
```

$$\left(-1+\frac{1}{x}\right)^k$$

```
Sum[ (-1) ^ (j + 1) / j (x - 1) ^ (j + k), {j, 1, Infinity}]
```

$$(-1+x)^k \,\mathrm{Log}[x]$$

```
Sum[ (-1) ^ (j + 1) / j (x^n - 1) ^ (j + k), {j, 1, Infinity}]
```

$$(-1+x^n)^k \,\mathrm{Log}[x^n]$$

```
Integrate[ ff[t], {t, a, b}] +
 Sum[BernoulliB[k] / k ! D[ ff[a], {a, k}] - D[ ff[b], {b, k}], {k, 0, Infinity}]
```

$$\int_a^b \mathrm{ff}[t]\, dt + \sum_{k=0}^{\infty}\left(\frac{\mathrm{BernoulliB}[k]\, \mathrm{ff}^{(k)}[a]}{k!} - \mathrm{ff}^{(k)}[b]\right)$$

```
(x - 1) Sum[BernoulliB[ k ] / k ! Log[x] ^ (k + a - 1), {k, 0, Infinity}]
```

$$\mathrm{Log}[x]^a$$

```
Limit[D[x^n, {n, 3}], n → 0]
```

$$\mathrm{Log}[x]^3$$

```
(Zeta[s] - 1) Sum[BernoulliB[ k ] / k ! Log[Zeta[s]] ^ (k + a - 1), {k, 0, Infinity}]
```

$$\mathrm{Log}[\mathrm{Zeta}[s]]^a$$