

```

chi[j_] := ((-1)^(j-1) (2 (j-1))!) / ((1-2 (j-1)) ((j-1)!)^2 (4^(j-1)))
chi2[j_] := Binomial[1/2, j-1]
Table[{chi[j], chi2[j]}, {j, 1, 10}]

{ {1, 1}, {1/2, 1/2}, {-1/8, -1/8}, {1/16, 1/16}, {-5/128, -5/128}, {7/256, 7/256},
  {-21/1024, -21/1024}, {33/2048, 33/2048}, {-429/32768, -429/32768}, {715/65536, 715/65536} }
Sum[Binomial[1/2, j-1], {j, 1, Infinity}]

 $\sqrt{2}$ 

Clear[pp, palt]
pp[n_, j_, k_, z_] := pp[n, j, k] =
  If[n < j, 0, Binomial[1/2, j-1] ((z+1)/k-1) (1+pp[n/j, 2, k+1, z]) + pp[n, j+1, k, z]]
S[n_, z_] := 1 + pp[n, 2, 1, z]
palt[n_, z_, y_] := palt[n, z, y] = If[n < y, 1,
  Sum[Binomial[z, k] Binomial[1/2, y-1]^k palt[n/y^k, z-k, y+1], {k, 0, Log[y, n]}]]
Sv2[n_, z_] := palt[n, z, 2]
SRoots[n_] := If[(c = Exponent[f = S[n, z], z]) == 0, {},
  If[c == 1, List@NRoots[f == 0, z][[2]], List@@NRoots[f == 0, z][[All, 2]]]]
Expand@(1 + pp[100, 2, 1, z])

1 + (130 856 389 339 316 386 294 377 686 430 089 439 999 946 485 663 219 916 659 871 z) /
  376 626 104 123 200 845 830 147 365 392 459 984 966 141 326 667 842 070 773 760 +
  205 986 324 639 431 802 057 210 443 617 z^2 + 15 706 379 949 301 z^3 - 28 877 z^4 + 113 z^5 - z^6
  3 565 267 313 141 895 191 709 477 765 120 + 1 688 849 860 263 936 - 75 497 472 + 491 520 - 92 160
N@(1 + pp[1000, 2, 1, 2])

2.

N@Log[2^(1/2)]

0.346574

$RecursionLimit = 10 000

10 000

Table[D[Expand@dp[n, z], z] /. z -> 0, {n, 1, 30}]

{0, 1/2, -1/8, -1/16, -5/128, 23/256, -21/1024, 163/6144, -685/32768, 1995/65536, -2431/262144,
  15257/15257, -29393/29393, 138019/138019, -349565/349565, 337439/337439, -9694845/9694845,
  -524288/524288, 4194304/4194304, -8388608/8388608, 33554432/33554432, -67108864/67108864, 2147483648/2147483648,
  94028275/94028275, -64822395/64822395, 319681525/319681525, -1588274667/1588274667, 4190118361/4190118361,
  4294967296/4294967296, -17179869184/17179869184, 34359738368/34359738368, -274877906944/274877906944, 549755813888/549755813888,
  6116566755/6116566755, -59590124679/59590124679, 225216898025/225216898025, -815608746719/815608746719,
  -219902325552/219902325552, 4398046511104/4398046511104, -70368744177664/70368744177664, 140737488355328/140737488355328,
  7509782883841/7509782883841, -5523270154941/5523270154941, 17383387729001/17383387729001, -214593265675675/214593265675675,
  -1688849860263936/1688849860263936, 1125899906842624/1125899906842624, -9007199254740992/9007199254740992, 18014398509481984/18014398509481984}

proots[16]

{-2.0198 - 2.20486 i, -2.0198 + 2.20486 i, 2.5198 - 6.0497 i, 2.5198 + 6.0497 i}

```

Expand@pp[50, 2, 1, z] + 1

$$1 + \frac{68\,494\,658\,516\,445\,813\,984\,128\,836\,249\,z}{198\,070\,406\,285\,660\,843\,983\,859\,875\,840} + \frac{8\,951\,771\,050\,407\,z^2}{140\,737\,488\,355\,328} + \frac{44\,819\,z^3}{12\,582\,912} + \frac{7\,z^4}{4096} - \frac{z^5}{15\,360}$$

Expand@palt[50, z, 2]

$$1 + \frac{68\,494\,658\,516\,445\,813\,984\,128\,836\,249\,z}{198\,070\,406\,285\,660\,843\,983\,859\,875\,840} + \frac{8\,951\,771\,050\,407\,z^2}{140\,737\,488\,355\,328} + \frac{44\,819\,z^3}{12\,582\,912} + \frac{7\,z^4}{4096} - \frac{z^5}{15\,360}$$