Section 1

Here we set up some notation, for Riemann's Prime Counting function, the divisor sum function, an adjusted divisor sum function, and an important prime power sum function.

1.1 The Riemann Prime Counting Function

The notation for Riemann's Prime counting function that we'll use is

$$\Pi(n) = \sum_{k=1}^{\lfloor \frac{\log n}{\log 2} \rfloor} k^{-1} \pi(n^{\frac{1}{k}}) \tag{1.1.1}$$

```
riemannPrimeCount[ n_ ] := Sum[ k^-1 PrimePi[ n^(1/k)], { k, 1, Log[2,n] } ]
Table[ { n, riemannPrimeCount[ n ] }, { n, 1, 100 } ] // TableForm
```

http://mathworld.wolfram.com/RiemannPrimeCountingFunction.html

Here, $\pi(n)$ is the number of primes <= n.

$$[f^0]_n = 1_{[1,\infty)}(|n|)$$

/// !!!! No!

$$[\nabla f^z]_n = [f^z]_n - [f^z]_{n-1}$$

$$[f_n]^{*x} * [g_n]^{*y} = \sum_{j=1} [f_{\Delta j}]^{*x} \cdot [g_{n j^{-1}}]^{*y} = \sum_{j=1} [g_{\Delta j}]^{*y} \cdot [f_{n j^{-1}}]^{*x}$$

1.2 The Partial Sum of the Zeta Function

The divisor sum function is

$$[\zeta(s)^k]_n = \sum_{j=1} j^{-s} \cdot [\zeta(s)^{k-1}]_{n j^{-1}} \tag{1.2.2}$$

```
zeta[ n_, s_, 0 ] := UnitStep[n-1]
zeta[ n_, s_, k_ ]  := Sum[ j^-s Zk[ n / j, s, k - 1 ], {j, 1, n } ]
Table[ zeta[ n, 0, k ], { n, 1, 50 }, { k, 1, 7 } ] // TableForm
```

http://en.wikipedia.org/wiki/Divisor_summatory_function

When s=0, this function is more often written as $D_k(n)$ or $T_k(n)$, but for this paper I want a notation that shows more visual symmetry with the usual exponentiation operation for reasons that will hopefully be clear as the paper progresses.

Writing out (1.2.2) more explicitly, examples of it include

$$[\zeta(s)]_n=\sum_{j=1}^{\lfloor n\rfloor} j^{-s}$$

$$[\zeta(s)^2]_n=\sum_{j=1}^{\lfloor n\rfloor}\sum_{k=1}^{\lfloor \frac{n}{j}\rfloor}(j\cdot k)^{-s}$$

$$[\zeta(s)^3]_n=\sum_{j=1}^{\lfloor n\rfloor}\sum_{k=1}^{\lfloor \frac{n}{j}\rfloor}\sum_{m=1}^{\lfloor \frac{n}{jk}\rfloor}(j\cdot k\cdot m)^{-s}$$

(1.2.3)

```
ri[]:=RandomInteger[{10,200}]; rr[]:=RandomReal[{-3,3}]+RandomReal[{-3,3}]I
Zk1[n_, s_] :=Sum[ j^-s, { j, 1, n } ]
Zk2[n_,s_] :=Sum[ j^-s k^-s, { j, 1, n }, { k, 1, n/j } ]
Zk3[n_,s_] :=Sum[ j^-s k^-s m^-s, { j, 1, n }, { k, 1, n/j }, {m,1, n/(j k) } ]
Zk[ n_, k_,s_ ] := Sum[ j^-s Zk[ n/j, k-1,s ], { j, 1, n } ]; Zk[ n_, 0,s_ ] := UnitStep[n-1]
Table[ Chop[{ Zk1[ n=ri[],s=rr[] ]-Zk[ n, 1,s ], Zk2[ n,s ]-Zk[ n, 2,s ], Zk3[ n,s ]-Zk[ n, 3,s ] }], { t, 1, 50 } ]
```

and so on.

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\boxed{\lim_{n\to\infty}[\zeta(s)^k]_n=\zeta(s)^k}$$

(1.2.6)

```
Zk[n_,k_,s_]:=Zk[n,k,s]=Sum[j^-s Zk[Floor[n/j],k-1,s],{j,1,n}]; Zk[n_,0,s_]:=UnitStep[n-1]
Grid[Table[ Chop[ Zk[30000, k, s]-N[Zeta[s]^k]],{k,1,3},{s,2,4,.5}]]
```

where $\zeta(s)$ is the Riemann Zeta function.

This will be generalized as

$$\boxed{[f^k]_n=\sum_{j=1} f(j)\cdot[f^{k-1}]_{n j^{-1}}}$$

(1.2.2)

$$[f]_n=\sum_{j=1}^{\lfloor n\rfloor} f(j)$$

$$[f^2]_n=\sum_{j=1}^{\lfloor n\rfloor}\sum_{k=1}^{\lfloor \frac{n}{j}\rfloor} f(j)\cdot f(k)$$

$$[f^3]_n=\sum_{j=1}^{\lfloor n\rfloor}\sum_{k=1}^{\lfloor \frac{n}{j}\rfloor}\sum_{m=1}^{\lfloor \frac{n}{jk}\rfloor} f(j)\cdot f(k)\cdot f(m)$$

(1.2.3)

1.3 Partial sums of the Hurwitz Zeta Function

A closely related function is

$$[\zeta(s,y+1)^k]_n = \sum_{j=1} (j+y)^{-s} \cdot [\zeta(s,y+1)^{k-1}]_{n(j+y)^{-1}}$$

(1.3.2)

```
Zmy[ n_, 0, s_, y_ ] := UnitStep[n-1]
Zmy[ n_, k_ ,s_, y_] := Sum[ (j+y)^-s Zmy[ n(j+y)^-1, k-1,s, y ], { j, 1, n-y } ]
Table[ Zmy[ n, k, s, 2 ], { n, 1, 40 }, { k, 1, 5 }, { s, -2, 2 } ]//TableForm
```

with examples including

$$[\zeta_n(s,y+1)]_n = \sum_{j=1}^{\lfloor n-y \rfloor} (j+y)^{-s}$$

$$[\zeta(s,y+1)^2]_n = \sum_{j=1}^{\lfloor n-y \rfloor} \sum_{k=1}^{\lfloor \frac{n}{j+y} - y \rfloor} ((j+y)\cdot(k+y))^{-s}$$

$$[\zeta(s,y+1)^3]_n = \sum_{j=1}^{\lfloor n-y \rfloor} \sum_{k=1}^{\lfloor \frac{n}{j+y} - y \rfloor} \sum_{m=1}^{\lfloor \frac{n}{(j+y)(k+y)} - y \rfloor} ((j+y)\cdot(k+y)\cdot(m+y))^{-s}$$

If y is a positive integer, this can also be written more simply as

$$[\zeta(s,y)^k]_n = \sum_{j=y} j^{-s} \cdot [\zeta(s,y)^{k-1}]_{n j^{-1}}$$

(1.3.2)

with examples including

$$[\zeta(s,y)]_n = \sum_{j=y}^{\lfloor n \rfloor} j^{-s}$$

$$[\zeta(s,y)^2]_n = \sum_{j=y}^{\lfloor n \rfloor} \sum_{k=y}^{\lfloor \frac{n}{j} \rfloor} (j\cdot k)^{-s}$$

$$[\zeta(s,y)^3]_n = \sum_{j=y}^{\lfloor n \rfloor} \sum_{k=y}^{\lfloor \frac{n}{j} \rfloor} \sum_{m=y}^{\lfloor \frac{n}{j\cdot k} \rfloor} (j\cdot k\cdot m)^{-s}$$

and so on.

The case where y=2 is particularly important. We'll sometimes write it as $[\zeta(s)-1]_n$, which is to say,

$$[\zeta(s,2)^k]_n = [(\zeta(s)-1)^k]_n$$

```
ri[]:=RandomInteger[{10,200}]; rr[]:=RandomReal[{-3,3}]+RandomReal[{-3,3}]I
Zm11[ n_, s_ ] := Sum[ j^-s, { j, 2, n } ]
Zm12[ n_, s_ ] := Sum[ j^-s k^-s, { j, 2, n }, { k, 2, n/j } ]
Zm13[ n_, s_ ] := Sum[ j^-s k^-s m^-s, { j, 2, n }, { k, 2, n/j }, { m, 2, n/(j k) } ]
Zm1[ n_, k_, s_ ] := Sum[ j^-s Zm1[ n/j, k-1, s ], { j, 2,  n } ]; Zm1[ n_, 0, s_ ] := UnitStep[n-1]
Table[ Chop[{ Zm11[ n=ri[],ss=rr[] ]-Zm1[ n, 1, ss ], Zm12[ n, ss ]-Zm1[ n, 2, ss ], Zm13[ n, ss ]-Zm1[ n, 3, ss ] }], { t, 1, 50 } ]
```

The limit of this as *n* approaches infinity, if $\Re(s)>1$, is

$$\lim_{n\to\infty} [\zeta(s,y)^k]_n = \zeta(s,y)^k$$

(1.3.4)

```
Dmy[n_,0,s_,y_]:=UnitStep[n-1]
```

```
Dmy[n_,k_,s_,y_]:=Sum[ (j+y)^-s Dmy[n (j+y)^-1,k-1,s,y],{j,1,n-y}]
Grid[Table[Chop[Dmy[30000,k,s,y]-N[Zeta[s,y+1]^k]],{k,1,3},{s,2,4},{y,1,4}]]
```

where $\zeta(s,y)$ is the Hurwitz Zeta function, defined for $\Re(s)>1$ and $\Re(y)>0$ as

$$\zeta(s,y)=\sum_{j=0}^{\infty}(j+y)^{-s}$$

This can be generalized as

$$\boxed{[f(y+1)^k]_n=\sum_{j=1} f(j+y)\cdot[f(y+1)^{k-1}]_{n(j+y)^{-1}}} \tag{1.3.2}$$

$$[f(y+1)]_n=\sum_{j=1}^{\lfloor n-y\rfloor} f(j+y)$$

$$[f(y+1)^2]_n=\sum_{j=1}^{\lfloor n-y\rfloor}\sum_{k=1}^{\lfloor \frac{n}{j+y}-y\rfloor} f((j+y)\cdot(k+y))$$

$$[f(y+1)^3]_n=$$

$$\sum_{j=1}^{\lfloor n-y\rfloor}\sum_{k=1}^{\lfloor \frac{n}{j+y}-y\rfloor}\sum_{m=1}^{\lfloor \frac{n}{(j+y)(k+y)}-y\rfloor} f((j+y)\cdot(k+y)\cdot(m+y))$$

If y is a positive integer, this can also be written as

$$\boxed{[f(y)^k]_n=\sum_{j=y} f(j)[f(y)^{k-1}]_{n\cdot j^{-1}}} \tag{1.3.2}$$

with examples

$$[f(y)]_n=\sum_{j=y}^{\lfloor n\rfloor} f(j)$$

$$[f(y)^2]_n=\sum_{j=y}^{\lfloor n\rfloor}\sum_{k=y}^{\lfloor \frac{n}{j}\rfloor} f(j)\cdot f(k)$$

$$[f(y)^3]_n=\sum_{j=y}^{\lfloor n\rfloor}\sum_{k=y}^{\lfloor \frac{n}{j}\rfloor}\sum_{m=y}^{\lfloor \frac{n}{j\cdot k}\rfloor} f(j)\cdot f(k)\cdot f(m)$$

1.4 The Partial Sum of the Log of the Zeta Function

A final function we'll use is

$$\boxed{\kappa(n)=a^{-1} if\ n=p^a, 0\ otherwise=\frac{\Lambda(n)}{\log n}} \tag{1.4.1}$$

```
K[ n_] := FullSimplify[ MangoldtLambda[ n ] / Log[ n ] ]
Table[ {n,K[n]}, {n,2,100}]
```

$$\boxed{[(\log\zeta(s))^k]_n=\sum_{j=2}\kappa(j)\cdot j^{-s}[(\log\zeta(s))^{k-1}]_{n\,j^{-1}}} \tag{1.4.3}$$

```
logZ[ n_, k_, s_ ] := Sum[ j^-s MangoldtLambda[ j ]/Log[ j ] logZ[ n/j, k-1 ], { j, 2, n } ]
logZ[ n_, 0, s_ ] := UnitStep[n-1]
Table[ FullSimplify[ logZ[ n, k, s ] ], { n, 1, 50 }, { k, 1, 5 }, {s,-1,1} ]//TableForm
```

with examples

$$[\log\zeta(s)]_n=\sum_{j=2}^{\lfloor n\rfloor}\kappa(j)\,j^{-s}$$
$$[(\log\zeta(s))^2]_n=\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor \frac{n}{j}\rfloor}\kappa(j)\kappa(k)(j\cdot k)^{-s}$$
$$[(\log\zeta(s))^3]_n=\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor \frac{n}{j}\rfloor}\sum_{m=2}^{\lfloor \frac{n}{jk}\rfloor}\kappa(j)\kappa(k)\kappa(m)(j\cdot k\cdot m)^{-s} \tag{1.4.2}$$

```
K[ n_ ] := FullSimplify[ MangoldtLambda[ n ]/Log[ n ] ]
logD1[ n_ ] := Sum[ K[ j ], { j, 2, n } ]
logD2[ n_ ] := Sum[ K[ j ] K[ k ], { j, 2, n }, { k, 2, n/j } ]
logD3[ n_ ] := Sum[ K[ j ] K[ k ] K[ m ], { j, 2, n }, { k, 2, n/j }, { m, 2, n/(j k) } ]
logD[ n_, k_ ] := Sum[ K[ j ] logD[ n/j, k-1 ], { j, 2, n } ]
logD[ n_, 0 ] := UnitStep[n-1]
 Table[{logD1[n]-logD[n,1],logD2[n]-logD[n,2],logD3[n]-logD[n,3]},{n,1,50}]//TableForm
```

$$[\log\zeta(0)]_n=\sum_{j=2}^{\lfloor n\rfloor}\kappa(j)=\Pi(n) \tag{1.4.2}$$

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\boxed{\lim_{n\to\infty}[(\log\zeta(s))^k]_n=(\log\zeta(s))^k} \tag{1.3.4}$$

```
 Dk[n_,k_,s_]:=Dk[n,k,s]=Sum[j^-s Dk[Floor[n/j],k-1,s],{j,1,n}];Dk[n_,0,s_]:=UnitStep[n-1]
Grid[Table[ Chop[ Dk[30000, k, s]-N[Zeta[s]^k]],{k,1,3},{s,2,4,.5}]]
```

If $f(n)$ is a multiplicative function, this can be generalized as

$$\boxed{[(\log f)^k]_n=\sum_{j=2}\kappa(j)\cdot f(j)[(\log f)^{k-1}]_{n j^{-1}}}$$

with examples

$$[\log f]_n=\sum_{j=2}^{\lfloor n\rfloor}\kappa(j)\cdot f(j)$$
$$[(\log f)^2]_n=\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor \frac{n}{j}\rfloor}\kappa(j)\kappa(k)\cdot f(j)\cdot f(k)$$
$$[(\log f)^3]_n=\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor \frac{n}{j}\rfloor}\sum_{m=2}^{\lfloor \frac{n}{jk}\rfloor}\kappa(j)\kappa(k)\kappa(m)\,f(j)\cdot f(k)\cdot f(m)$$

1.5 Notes on Convergence

A quick glance at the examples above should make clear that $[(\log\zeta(s))^k]_n = 0$ when $n < 2^k$, and $[\zeta(s, y+1)^k]_n = 0$ when $n < (y+1)^k$, but that $[\zeta(s)^k]_n$ is non-zero for any k.

```
Dk[ n_, k_ ]  := Sum[ Dk[ n / j, k - 1 ], { j, 1, n } ];Dk[ n_, 0 ] := UnitStep[n-1]
Dm1[ n_, k_ ] := Sum[ Dm1[ n/j, k-1 ], { j, 2, n } ];Dm1[ n_, 0 ] := UnitStep[n-1]
logD[ n_, k_ ] := Sum[ FullSimplify[MangoldtLambda[ j ]/Log[ j ]] logD[ n/j, k-1 ], { j, 2, n } ];logD[ n_, 0 ] := UnitStep[n-1]
Table[ {n, Dm1[ n, 4 ],Dm1[ n, 5 ], Dm1[ n, 6 ], logD[ n, 4 ],logD[ n, 5 ], logD[ n, 6 ], Dk[ n, 4 ],Dk[ n, 5 ], Dk[ n, 6 ] }, { n, 1,
64 } ] // TableForm
```

When we use these first two functions in analogs to power series or polynomials, this property guarantees convergence after $\log_2 n$ terms, which is crucial.