```
f1[x_, a_] := -(Log[Log[x]] + EulerGamma) + Sum[1 / k, {k, 1, Floor@Log[a, x]}]
f1a[x_, a_] := -(Log[Log[x]] + EulerGamma) + HarmonicNumber[Floor@Log[a, x]]
f2[x_, a_] := Sum[a^k / k, {k, 1, Floor@Log[a, 1.45136923488338105028396848]}]
f3[x_, a_] := Sum[a^k / k, {k, 1 + Floor@Log[a, 1.45136923488338105028396848], Log[a, x]}]
```

```
N@f1a[100, 1.00001]
```

11.5129

```
N@f2[100, 1.00001]
```

11.5129

```
Expand@Sum[a^k / k, {k, 1 + Floor[Log[a, y]], Floor[Log[a, x]]}]
```

$$-a^{1+\text{Floor}\left[\frac{\text{Log}[x]}{\text{Log}[a]}\right]}\,\text{LerchPhi}\left[a,1,1+\text{Floor}\left[\frac{\text{Log}[x]}{\text{Log}[a]}\right]\right]+a^{1+\text{Floor}\left[\frac{\text{Log}[y]}{\text{Log}[a]}\right]}\,\text{LerchPhi}\left[a,1,1+\text{Floor}\left[\frac{\text{Log}[y]}{\text{Log}[a]}\right]\right]$$

```
f3[300, 1.000001]
```

$Aborted

```
LogIntegral[300.]
```

68.3336

```
(Hypergeometric1F1[k, k + 1, Log[x]] Log[x]^k / k!) /. k → 4 /. x → 100.
```

$928.88 - 3.40898 \times 10^{-13}\, i$

```
Integrate[1, {x, 1, 100.}, {y, 1, 100. / x}, {z, 1, 100. / x / y}]
```

698.863

```
D[(Hypergeometric1F1[k, k + 1, Log[x]] Log[x]^k / k!), k] /. k → 0 /. x → 100.
```

$30.1261 + 0.\, i$

```
Hypergeometric1F1[k, k + 1, Log[x]] Log[x]^k / k!
```

$$\frac{(\text{Gamma}[1+k]-k\,\text{Gamma}[k,-\text{Log}[x]])\,(-\text{Log}[x])^{-k}\,\text{Log}[x]^{k}}{k!}$$

$$\mathbf{D}\left[\frac{(\text{Gamma}[1+k]-k\,\text{Gamma}[k,-\text{Log}[x]])\,(-\text{Log}[x])^{-k}\,\text{Log}[x]^{k}}{k!},k\right] /.\ k\to 0\ /.\ x\to 100.$$

$30.1261 + 0.\, i$

```
(x Hypergeometric1F1[1, k + 1, -Log[x]] Log[x]^k / k!) /. k → 4 /. x → 100.
```

$928.88 - 3.40898 \times 10^{-13}\, i$

```
(x Hypergeometric1F1[1, k + 1, -Log[x]] Log[x]^k / k!) /. k → 4 /. x → 100.
FullSimplify[D[x Hypergeometric1F1[1, k + 1, -Log[x]] Log[x]^k / k!, x]]
```

$$\frac{\text{Log}[x]^{-1+k}}{\text{Gamma}[k]}$$

```
FullSimplify[D[Hypergeometric1F1[k, k + 1, Log[x]] Log[x]^k / k!, x]]
```

$$\frac{\text{Log}[x]^{-1+k}}{\text{Gamma}[k]}$$

```
D[x Hypergeometric1F1[1, k + 1, -Log[x]] Log[x]^k / k!, k] /. k → 0 /. x → 100.
```

30.1261 + 0. ⅈ

```
x Hypergeometric1F1[1, k + 1, -Log[x]] /. k → 3 /. x → 7.
```

$4.5858 - 1.6848 \times 10^{-15}$ ⅈ

```
Hypergeometric1F1[k, k + 1, Log[x]] /. k → 3 /. x → 7.
```

$4.5858 - 1.6848 \times 10^{-15}$ ⅈ

```
LaguerreL[k, -x] /. k → 3 /. x → 7.
```

152.667

```
Hypergeometric1F1[-k, 1, -x] /. k → 3 /. x → 7.
```

152.667

```
E^-x Hypergeometric1F1[1 + k, 1, x] /. k → 3 /. x → 7.
```

152.667

```
LaguerreL[-k, Log@x] /. k → 3 /. x → 7.
```

47.4957

```
Hypergeometric1F1[k, 1, Log@x] /. k → 3 /. x → 7.
```

47.4957

```
x Hypergeometric1F1[1 - k, 1, -Log@x] /. k → 3 /. x → 7.
```

47.4957

```
Hypergeometric1F1[-z, 1, -x]
x Hypergeometric1F1[1 - z, 1, -Log@x]
x Hypergeometric1F1[1, 1 + k, -Log[x]] Log[x]^k / k!
D[x Hypergeometric1F1[1, k + 1, -Log[x]] Log[x]^k / k!, k] /. k → 0 /. x → 1.01
```

-4.02296 + 0. ⅈ

```
LogIntegral[1.01]
```

-4.02296

```
N@f3[2, 1.000001]
```

1.04516

```
LogIntegral[2.]
```

1.04516

```
Sum[a^k / k, {k, 1, Log[a, 1.451369234883]}]
```

$$-1.45137\, a\, \text{LerchPhi}\left[a, 1, 1 + \frac{0.372507}{\text{Log}[a]}\right] - \text{Log}[1 - a]$$

$$\text{Plot}\left[-1.451369234883\grave{}\ a\, \text{LerchPhi}\left[a, 1, 1 + \frac{0.37250741078110416\grave{}}{\text{Log}[a]}\right] - \text{Log}[1 - a], \{a, 5, 1\}\right]$$

```
Clear[e2]
bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
e2[n_, 0, x_] := UnitStep[n - 1]
e2[n_, k_, x_] :=
 e2[n, k, x] = Sum[e2[n / j, k - 1, x], {j, 2, n}] - x Sum[e2[n / (j x), k - 1, x], {j, 1, n}]
ez[n_, z_, x_] := Sum[bin[z, k] e2[n, k, x], {k, 0, If[x < 2, Log[x, n], Log[2, n]]}]
```

```
e2[100, 2, 2]
```

3

```
D[Expand@ez[100, z, 3 / 2], z] /. z → 0
```

$-\frac{8149753}{2365440}$

```
(D[Expand@ez[100, z, 101], z] /. z → 0) - (Sum[(3 / 2)^k / k, {k, 1, Log[3 / 2, 100]}])
```

$-\frac{8149753}{2365440}$

```
D[zetaAlt[100, 0, 3 / 2, z], z] /. z → 0
```

$-\frac{8149753}{2365440}$

```
binomial[z_, k_] := binomial[z, k] = Product[z - j, {j, 0, k - 1}] / k!
zetaHurwitz[n_, s_, y_, 0] := UnitStep[n - 1]
zetaHurwitz[n_, s_, y_, 1] :=
 zetaHurwitz[n, s, y, 1] = HarmonicNumber[Floor[n], s] - HarmonicNumber[y, s]
zetaHurwitz[n_, s_, y_, 2] := zetaHurwitz[n, s, y, 2] =
  Sum[(m^(-2 s)) + 2 (m^-s) (zetaHurwitz[Floor[n / m], s, m, 1]), {m, y + 1, Floor[n^(1 / 2)]}]
zetaHurwitz[n_, s_, y_, k_] := zetaHurwitz[n, s, y, k] =
  Sum[(m^(-s k)) + k (m^(-s (k - 1))) zetaHurwitz[Floor[n / (m^(k - 1))], s, m, 1] +
    Sum[binomial[k, j] (m^-s)^j zetaHurwitz[Floor[n / (m^j)], s, m, k - j], {j, 1, k - 2}],
   {m, y + 1, Floor[n^(1 / k)]}]
zeta[n_, s_, z_] := Expand@Sum[binomial[z, k] zetaHurwitz[n, s, 1, k], {k, 0, Log2@n}]

zetaAlt[n_, s_, x_, z_] :=
 Expand@Sum[(-1)^j binomial[z, j] x^(j (1 - s)) zeta[n / (x^j), s, z], {j, 0, Log[x, n]}]

zetaAltZeros[n_, s_, x_] := If[(c = Exponent[f = zetaAlt[n, s, x, z], z]) == 0,
  {}, If[c == 1, List@NRoots[f == 0, z][[2]], List @@ NRoots[f == 0, z][[All, 2]]]]
```

```
zetaAltZeros[100, 0, 1.03]

{-16.5771, -16.4028, -14.379, -14.1703, -13.9603, -13.5314, -13.1481, -12.1554, -10.6526,
 -9.4474, -8.79865, -8.79772, -8.41932, -7.64418, -6.95538, -6.74756, -6.56705, -6.09618,
 -5.4721, -4.79767, -4.65332, -3.40074, -2.10298 - 4.22112 i, -2.10298 + 4.22112 i,
 -1.82758 + 7.10388 i, -1.82758 - 7.10388 i, -1.80097, -1.62803 - 10.5796 i,
 -1.62803 + 10.5796 i, -1.52193, -1.11115 + 23.7191 i, -1.11107 - 23.7191 i,
 -0.933336 - 2.0046 i, -0.933336 + 2.0046 i, -0.901875 + 16.3405 i, -0.901875 - 16.3405 i,
 0. - 32.3428 i, 0. + 32.3452 i, 0. + 55.5238 i, 0. - 55.5456 i, 0.299051 + 0.85901 i,
 0.299051 - 0.85901 i, 0.448547, 0.676548 + 2.25746 i, 0.676548 - 2.25746 i, 0.903063,
 0.983713, 1.23641, 2.04437, 2.31697, 2.49711 + 5.9364 i, 2.49711 - 5.9364 i, 3.17962,
 4.90743 - 12.2382 i, 4.90743 + 12.2382 i, 5.27636 + 5.29972 i, 5.27636 - 5.29972 i, 5.27869,
 6.01875, 6.06018, 6.17294, 6.4873, 6.58938, 7.12996, 7.35205, 7.95852, 8.24719,
 8.44708 - 4.42884 i, 8.44708 + 4.42884 i, 9.53532, 10.3059, 10.5243, 10.6267, 11.258,
 12.7267, 12.9497, 13.4265, 13.4802, 13.744, 13.8824, 14.2778, 15.3369, 15.4706,
 15.5045, 15.7331, 16.4826, 16.5853, 16.8231, 17.2589, 17.5187, 21.2064, 21.6358,
 21.7774, 22.4206, 24.2096, 26.8151, 29.3805, 30.7764, 31.7025, 33.6151, 33.655,
 35.3588, 36.8634, 37.0616, 37.3528, 37.6189, 38.2959, 38.3641, 40.3293, 46.7278,
 54.1758, 55.6732, 58.3167, 60.0754, 69.1297, 70.5178, 72.9714, 76.4211, 85.5868,
 93.7439, 104.411, 113.454, 114.864, 116.604, 122.744, 125.143, 126.986, 140.399,
 140.714, 151.636, 154.075, 156.722, 157.051, 163.568, 163.811, 181.92, 223.581,
 238.563, 246.493, 266.52, 298.584, 332.24, 367.373, 381.977, 400.502, 436.857,
 469.263, 473.262, 524.733, 533.231, 539.666, 581.596, 606.538, 623.608, 642.424}

f1b[a_] := HarmonicNumber[1 / Log[a]] - EulerGamma
f1c[a_] := -Log[a - 1]
f2b[a_] := Sum[a^k / k, {k, 1, Floor@Log[a, 1.451369234883381050283968485 8]}]
f2c[a_] := Sum[a^k / k, {k, 1, Log[a, 1.4513692348833810502839684858]}] - (-Log[a - 1])
f2d[a_] := Sum[a^k / k, {k, 1, Log[a, 1.4513692348833810502839684858]}] -
  Sum[(2 - a) ^k / k, {k, 1, Infinity}]
f2e[a_] := Sum[a^k / k, {k, 1, Log[a, 1.4513692348833810502839684858]}] -
  Sum[(2 - a) ^k / k, {k, 1, Floor@Log[a, 1.4513692348833810502839684858]}] -
  Sum[(2 - a) ^k / k, {k, Floor@Log[a, 1.4513692348833810502839684858], Infinity}]
f2f[a_] := Sum[(a^k - (2 - a) ^k) / k, {k, 1, Log[a, 1.4513692348833810502839684858]}] -
  Sum[(2 - a) ^k / k, {k, Floor@Log[a, 1.4513692348833810502839684858], Infinity}]
f2g[a_] := Sum[((a + 1) ^k - (2 - (a + 1)) ^k) / k,
   {k, 1, Log[(a + 1), 1.4513692348833810502839684858]}] -
  Sum[(2 - (a + 1)) ^k / k, {k, Floor@Log[(a + 1), 1.4513692348833810502839684858], Infinity}]
f2h[a_] := Sum[((a + 1) ^k - (1 - a) ^k) / k,
   {k, 1, Log[(a + 1), 1.4513692348833810502839684858]}] -
  Sum[(1 - a) ^k / k, {k, Floor@Log[(a + 1), 1.4513692348833810502839684858], Infinity}]
f2i[a_] := Sum[((a + 1) ^k - (1 - a) ^k) / k,
   {k, 1, Log[(a + 1), 1.4513692348833810502839684858]}] -
  Sum[(1 - a) ^k / k, {k, Floor@Log[(a + 1), 1.4513692348833810502839684858], Infinity}]
f2j[a_] := Sum[((a + 1) ^k - (1 - a) ^k) / k, {k, 1, Log[(a + 1), n]}] -
  Sum[(1 - a) ^k / k, {k, Log[(a + 1), n], Infinity}]

N@f1c[1.00001]

11.5129

N@f1c[1.00001]

11.5129
```

```
N@f2c[1.000001]
```

-3.78188×10^{-7}

```
FullSimplify@Sum[(2 - a)^k / k, {k, 1, Infinity}]
```

```
-Log[-1 + a]
```

```
N@f2i[.000001]
```

-2.2278×10^{-6}

```
N@f2f[1.000001]
```

-2.22783×10^{-6}

```
FullSimplify@Sum[a^k / k, {k, 1, n}]
```

$-a^{1+n}$ LerchPhi[a, 1, 1 + n] - Log[1 - a]

```
Table[(a + 1)^k - (1 - a)^k, {k, 1, 5}] // TableForm
```

$2\,a$
$-(1-a)^2+(1+a)^2$
$-(1-a)^3+(1+a)^3$
$-(1-a)^4+(1+a)^4$
$-(1-a)^5+(1+a)^5$

```
FullSimplify[(a + 1)^k - (1 - a)^k]
```

$-(1-a)^k+(1+a)^k$

```
FullSimplify@f2i[a] /. a → .000001
```

$-1.00589\times10^{-6}+1.35909\times10^{-11}\,i$

```
f2j[a] /. a → .00001
```

$$(0.-3.14159\,i)-\frac{\text{HurwitzLerchPhi}[0.99999,1,100\,000.\,\text{Log}[n]]}{n^{1.00001}}+\frac{0.99999\,\text{LerchPhi}[0.99999,1,1+100\,000.\,\text{Log}[n]]}{n^{1.00001}}-1.00001\,n\,\text{LerchPhi}[1.00001,1,1+100\,000.\,\text{Log}[n]]$$

FullSimplify[

$$\left(0.\grave{}-3.141592653589793\grave{}\,i\right)-\frac{\text{HurwitzLerchPhi}[0.99999\grave{},1,100000.49999851156\grave{}\,\text{Log}[n]]}{n^{1.000010000038898\grave{}}}+\frac{1}{n^{1.000010000038898\grave{}}}0.99999\grave{}\,\text{HurwitzLerchPhi}[0.99999\grave{},1,1+100000.49999851156\grave{}\,\text{Log}[n]]-1.00001\grave{}\,n\,\text{HurwitzLerchPhi}[1.00001\grave{},1,1+100000.49999851156\grave{}\,\text{Log}[n]]$$

]

$$(0.-3.14159\,i)+\frac{1}{n^{1.00001}}(-1.\,\text{HurwitzLerchPhi}[0.99999,1,100\,000.\,\text{Log}[n]]+0.99999\,\text{HurwitzLerchPhi}[0.99999,1,1+100\,000.\,\text{Log}[n]])-1.00001\,n\,\text{HurwitzLerchPhi}[1.00001,1,1+100\,000.\,\text{Log}[n]]$$

```
FullSimplify[
 (0.` - 3.141592653589793` ⅈ) - LerchPhi[0.99999`, 1, 100000.49999851156` Log[n]] / n^1.000010000038898` +
  0.99999` LerchPhi[0.99999`, 1, 1 + 100000.49999851156` Log[n]] / n^1.000010000038898` -
  1.00001` n LerchPhi[1.00001`, 1, 1 + 100000.49999851156` Log[n]]]
```

$(0. - 3.14159\,i) + \frac{1}{n^{1.00001}} (-1.\,\text{LerchPhi}[0.99999, 1, 100000.\,\text{Log}[n]] + 0.99999\,\text{LerchPhi}[0.99999, 1, 1 + 100000.\,\text{Log}[n]]) - 1.00001\,n\,\text{LerchPhi}[1.00001, 1, 1 + 100000.\,\text{Log}[n]]$

```
f2k[a_] :=
 Sum[(a + 1)^k / k, {k, 1, Log[(a + 1), 1.45136923488338105028396848 58]}] - (-Log[(a + 1) - 1])
f2l[a_] := Sum[(1 + a)^k / k, {k, 1, Log[(a + 1), 1.45136923488338105028396848 58]}] + Log[a]
f2m[a_] := Sum[(1 + a)^k / k, {k, 1, Log[(a + 1), 1.4513692348833810502839684858]}] -
  Sum[(1 - a)^k / k, {k, 1, Infinity}]
f2n[a_] := -(1 + a) n LerchPhi[1 + a, 1, 1 + Log[n] / Log[1 + a]] - Log[-a] + Log[a]
f2o[a_] := -I Pi - n Sum[(1 + a)^(k + 1) / (k + Log[1 + a, n]), {k, 0, Infinity}]
f2l[.00001]
```

-0.0000166497

```
Sum[(a + 1)^k / k, {k, 1, Infinity}]
```

$-\text{Log}[-a]$

```
-Sum[(1 - a)^k / k, {k, 1, Infinity}]
```

$\text{Log}[a]$

```
FullSimplify@Sum[(1 + a)^k / k, {k, 1, Log[(a + 1), n]}]
```

$-(1 + a)\, n\, \text{LerchPhi}\left[1 + a, 1, 1 + \frac{\text{Log}[n]}{\text{Log}[1 + a]}\right] - \text{Log}[-a]$

```
-(1 + a) n LerchPhi[1 + a, 1, 1 + Log[n] / Log[1 + a]] - Log[-a] + Log[a] /. a → 1 / 10
```

$-i\pi - \frac{11}{10}\, n\, \text{LerchPhi}\left[\frac{11}{10}, 1, 1 + \frac{\text{Log}[n]}{\text{Log}\left[\frac{11}{10}\right]}\right]$

```
f2n[.001] /. n → 1.45
```

$-0.0017307 + 1.59872 \times 10^{-14}\, i$

```
FullSimplify[D[Sum[(1 + a)^k / k, {k, 1, Log[(a + 1), n]}] + Log[a], n]]
```

$(1 + a) \left(-\text{LerchPhi}\left[1 + a, 1, 1 + \frac{\text{Log}[n]}{\text{Log}[1 + a]}\right] + \frac{\text{LerchPhi}\left[1 + a, 2, 1 + \frac{\text{Log}[n]}{\text{Log}[1+a]}\right]}{\text{Log}[1 + a]} \right)$

```
f2m2[a_] := Sum[(1 + a)^k / k, {k, 1, Log[1.4513692348833810502839684858] / a}] -
  Sum[(1 - a)^k / k, {k, 1, Infinity}]
f2m2a[a_] := Sum[(1 + a)^k / k, {k, 1, n / a}] - Sum[(1 - a)^k / k, {k, 1, Infinity}]
f2m2a[.000001] /. n -> Log[1.4513692348833810502839684858]
```

$1.22222 \times 10^{-6} - 1.23289 \times 10^{-9}\, i$

```
FullSimplify[Sum[(1 + a) ^k / k, {k, 1, n / a}]]
```

$-(1+a)^{1+\frac{n}{a}}\,\mathrm{LerchPhi}\left[1+a, 1, \frac{a+n}{a}\right] - \mathrm{Log}[-a]$

$\mathbf{f2na[a\_] := -(1+a)^{1+\frac{n}{a}}\,LerchPhi\left[1+a, 1, \frac{a+n}{a}\right] - Log[-a] + Log[a]}$

```
f2na[.000001] /. n -> Log[1.451369234883381050283968485 8]
```

$1.2223 \times 10^{-6} + 1.35905 \times 10^{-11}\, i$

```
FullSimplify[-Log[-a] + Log[a] /. a → 1 / 20]
```

$-i\pi$

```
f2m2a[.0000001] /. n -> Log[1.4513692348833810502839684858]
```

$1.22495 \times 10^{-7} - 6.52108 \times 10^{-9}\, i$

```
N@Log[1.4513692348833810502839684858]
```

0.372507

```
Sum[a^k / k, {k, 1, Log[a, 100]}] -
  Sum[a^k / k, {k, 1, Log[a, 1.4513692348833810502839684858]}] /. a → 1.001
```

$30.135 + 1.84741 \times 10^{-13}\, i$

```
Sum[((1 + a) ^k - 1) / k, {k, 1, 2 / a}] /. a → .00001
```

$3.68385 + 1.46136 \times 10^{-11}\, i$

```
ExpIntegralEi[2.] - Log[2] - EulerGamma
```

3.68387

```
Sum[(1 + a) ^k / k, {k, 1, 2 / a}] -
  Sum[(1 + a) ^k / k, {k, 1, Log[1.4513692348833810502839684858] / a}] /. a → .000001
```

$4.95423 - 9.72253 \times 10^{-11}\, i$

```
ExpIntegralEi[2.]
```

4.95423

```
D[LaguerreL[z, -x], z] /. z → 0 /. x → 4.
```

1.96729

```
Gamma[0, 4.] + Log[4.] + EulerGamma
```

1.96729

```
-(ExpIntegralEi[-4.] - Log[4.] - EulerGamma)
```

1.96729

```
FullSimplify@Sum[(-1) ^ (4 - j) Binomial[4, j] LaguerreL[j, -x], {j, 0, 4}]
```

$\frac{x^4}{24}$

```
po[x_, k_] := Sum[(-1) ^ (k - j) Binomial[k, j] (x + j) ! / x ! / j !, {j, 0, k}]
```

```
po[15, 5]
```

3003

```
Binomial[15, 5]
```

3003

```
ex[n_, a_] := Product[(1 + a)^k, {k, 1, n / a}]
```

```
ex[30, -.01]
```

1

```
D[LogIntegral[x], x]
```

$$\frac{1}{\mathrm{Log}[x]}$$

```
D[ExpIntegralEi[x], x]
```

$$\frac{e^x}{x}$$

```
Table[{Sum[(1 + (1. / 10^b))^k / k, {k, 1, Log[1.451369234] / (1 / 10^b)}], Log[(1. / 10^b)]},
  {b, 0, 7}] // TableForm
```

0	0.
2.14867	-2.30259
4.60765	-4.60517
6.907	-6.90776
9.21043	-9.21034
11.5129	-11.5129
13.8155	-13.8155
$16.1181 + 2.5157 \times 10^{-10}\, i$	-16.1181

```
f2m2b[a_] := Sum[(1 + a)^k / k, {k, 1, n / a}] + Log[a]
f2m2c[a_] := Sum[(1 + 1 / a)^k / k, {k, 1, n a }] - Log[a]
f2m2d[a_] := Sum[1 / k, {k, 1, n a }] - Log[a]
```

```
f2m2b[.000001] /. n → Log[1.451369234]
```

$1.21993 \times 10^{-6} + 6.79255 \times 10^{-10}\, i$

```
f2m2c[1000.] /. n → Log[1.451369234]
```

$0.00122188 + 2.82253 \times 10^{-13}\, i$

```
Limit[ (1 + 1 / x)^x, x → Infinity]
```

e

```
Limit[ (1 + z / x)^x, x → Infinity]
```

e^z

```
Limit[(x^z - 1) / z, z → 0]
```

Log[x]

```
Limit[1 / k (1 + 1 / x) ^k, x → Infinity]
```

$\frac{1}{k}$