```
Table[1 / k ! D[1 / (1 - x), {x, k}] /. x → 0, {k, 0, 10}]
```

{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

```
Table[1 / k ! D[1 / (1 + x), {x, k}] /. x → 0, {k, 0, 10}]
```

{1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1}

```
Table[1 / k ! D[Log[1 / (1 - x)], {x, k}] /. x → 0, {k, 0, 10}]
```

$$\left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}\right\}$$

```
Log[1 / (1 - x)]
```

$$\mathrm{Log}\left[\frac{1}{1-x}\right]$$

```
Clear[sp]
bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k !
sp[n_, k_] := sp[n, k] = Sum[PartitionsP[j] sp[n - j, k - 1], {j, 1, n}]
sp[n_, 0] := UnitStep[n]
sz[n_, z_] := Sum[ bin[z, k] sp[n, k], {k, 0, n}]
```

```
Table[Expand[sz[n, z] - sz[n - 1, z]] /. z → -1, {n, 1, 20}]
```

{-1, -1, 0, 0, 1, 0, 1, 0, 0, 0, 0, -1, 0, 0, -1, 0, 0, 0, 0, 0}

```
Table[PartitionsP[n], {n, 1, 10}]
```

{1, 2, 3, 5, 7, 11, 15, 22, 30, 42}

```
aa[n_] := SeriesCoefficient[Product[1 - x^k, {k, n}], {x, 0, n}]
al[n_] := SeriesCoefficient[Log[Product[1 - x^k, {k, n}]], {x, 0, n}]
```

```
Table[aa[j], {j, 1, 20}]
```

{-1, -1, 0, 0, 1, 0, 1, 0, 0, 0, 0, -1, 0, 0, -1, 0, 0, 0, 0, 0}

```
Table[al[j], {j, 1, 20}]
```

$$\left\{-1, -\frac{3}{2}, -\frac{4}{3}, -\frac{7}{4}, -\frac{6}{5}, -2, -\frac{8}{7}, -\frac{15}{8}, -\frac{13}{9}, -\frac{9}{5}, -\frac{12}{11}, -\frac{7}{3}, -\frac{14}{13}, -\frac{12}{7}, -\frac{8}{5}, -\frac{31}{16}, -\frac{18}{17}, -\frac{13}{6}, -\frac{20}{19}, -\frac{21}{10}\right\}$$

```
Table[-DivisorSigma[1, j] / j, {j, 1, 20}]
```

$$\left\{-1, -\frac{3}{2}, -\frac{4}{3}, -\frac{7}{4}, -\frac{6}{5}, -2, -\frac{8}{7}, -\frac{15}{8}, -\frac{13}{9}, -\frac{9}{5}, -\frac{12}{11}, -\frac{7}{3}, -\frac{14}{13}, -\frac{12}{7}, -\frac{8}{5}, -\frac{31}{16}, -\frac{18}{17}, -\frac{13}{6}, -\frac{20}{19}, -\frac{21}{10}\right\}$$

```
Table[FullSimplify[1 / k ! D[(1 / (1 - x) - 1) ^z, {x, k}]] /. x → 0, {k, 0, 6}]
```

$$\left\{0^{z}, 0^{-1+z}\, z, \frac{1}{2}\, 0^{-2+z} (-1+z)\, z, \frac{1}{6}\, 0^{-3+z}\, z \left(2-3 z+z^{2}\right), \frac{1}{24}\, 0^{-4+z} (-3+z) (-2+z) (-1+z)\, z, \frac{1}{120}\, 0^{-5+z} (-4+z) (-3+z) (-2+z) (-1+z)\, z, \frac{1}{720}\, 0^{-6+z} (-5+z) (-4+z) (-3+z) (-2+z) (-1+z)\, z\right\}$$

$$\frac{\left(\frac{x}{1-x}\right)^{-1+z} z}{(-1+x)^2} \text{ /. } x \to 0$$

$0^{-1+z}\, z$

```
Clear[lb]
bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
lb[n_, k_, f_] := lb[n, k, f] = Sum[f[j] lb[n - j, k - 1, f], {j, 1, n}]
lb[n_, 0, f_] := UnitStep[n]
llb[n_, f_] := Sum[(-1) ^ (k + 1) / k lb[n, k, f], {k, 1, n}]
lbz[n_, z_, f_] := Sum[bin[z, k] lb[n, k, f], {k, 0, n}]
llz[n_, z_, f_] := Sin[Pi z] / Pi Sum[ (-1) ^k / (z - k) lb[n, k, f], {k, 0, n}]
lroots[n_, ff_] := If[(c = Exponent[f = (lbz[n, z, ff]), z]) == 0, {},
  If[c == 1, List@Roots[f == 0, z][[2]], List @@ Roots[f == 0, z][[All, 2]]]]
dlroots[n_, ff_] := If[(c = Exponent[f = (lbz[n, z, ff] - lbz[n - 1, z, ff]), z]) == 0,
  {}, If[c == 1, List@Roots[f == 0, z][[2]], List @@ Roots[f == 0, z][[All, 2]]]]
bpoly[n_, z_, ff_] := Expand@FullSimplify@
   Expand[llz[n, z, ff] / (Sin[Pi z] / Pi) FactorialPower[z, n + 1] ]
dbpoly[n_, z_, ff_] := Expand@FullSimplify@
   Expand[(llz[n, z, ff] - llz[n - 1, z, ff]) / (Sin[Pi z] / Pi) FactorialPower[z, n + 1] ]
l2roots[n_, ff_] := If[(c = Exponent[f = (bpoly[n, z, ff]), z]) == 0, {},
  If[c == 1, List@Roots[f == 0, z][[2]], List @@ Roots[f == 0, z][[All, 2]]]]
dl2roots[n_, ff_] := If[(c = Exponent[f = (dbpoly[n, z, ff]), z]) == 0, {},
  If[c == 1, List@Roots[f == 0, z][[2]], List @@ Roots[f == 0, z][[All, 2]]]]


id[n_] := 1
idd[n_] := PartitionsP[n]
lb[10, 3, idd]
```

2544

```
lbz[10, 3, idd]
```

5773

```
llb[10, idd]
```

$$\frac{7583}{504}$$

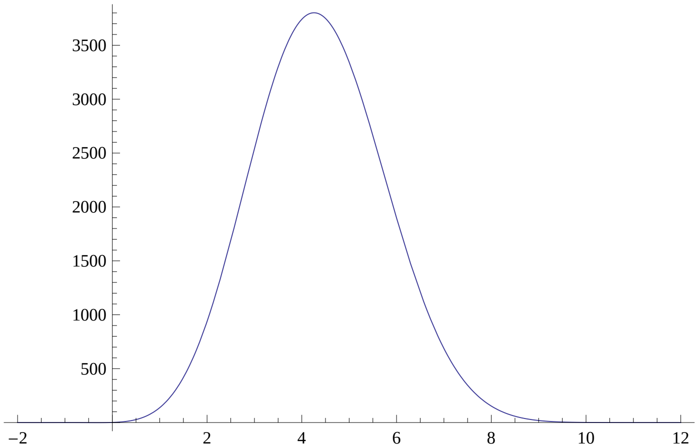
```
Expand[llz[10, z, idd]]
```

$$\frac{\mathrm{Sin}[\pi z]}{\pi(-10+z)} - \frac{19\,\mathrm{Sin}[\pi z]}{\pi(-9+z)} + \frac{153\,\mathrm{Sin}[\pi z]}{\pi(-8+z)} - \frac{687\,\mathrm{Sin}[\pi z]}{\pi(-7+z)} + \frac{1898\,\mathrm{Sin}[\pi z]}{\pi(-6+z)} - \frac{3343\,\mathrm{Sin}[\pi z]}{\pi(-5+z)} + \frac{3741\,\mathrm{Sin}[\pi z]}{\pi(-4+z)} - \frac{2544\,\mathrm{Sin}[\pi z]}{\pi(-3+z)} + \frac{938\,\mathrm{Sin}[\pi z]}{\pi(-2+z)} - \frac{138\,\mathrm{Sin}[\pi z]}{\pi(-1+z)} + \frac{\mathrm{Sin}[\pi z]}{\pi z}$$

```
Sum[PartitionsP[j], {j, 1, 10}]
```

138

```
Plot[llz[10, z, idd], {z, -2, 12}]
```




```
Expand[lbz[10, z, idd]]
```

$$1+\frac{7583\,z}{504}+\frac{750\,731\,z^{2}}{16\,800}+\frac{8\,720\,689\,z^{3}}{181\,440}+\frac{1\,708\,153\,z^{4}}{72\,576}+\frac{201\,641\,z^{5}}{34\,560}+\frac{133\,993\,z^{6}}{172\,800}+\frac{6779\,z^{7}}{120\,960}+\frac{13\,z^{8}}{6048}+\frac{29\,z^{9}}{725\,760}+\frac{z^{10}}{3\,628\,800}$$

```
FullSimplify@Product[1 - 1 / r, {r, lroots[10, id]}]
```

11

```
lbz[10, 1, id]
```

11

```
bpoly[n_, z_, ff_] :=
 Expand@FullSimplify@Expand[llz[n, z, ff] / (Sin[Pi z] / Pi) FactorialPower[z, n + 1] ]
Limit[bpoly[10, z, id] / FactorialPower[z, 11] Sin[Pi z] / Pi, z → 1]
```

10

```
N@l2roots[10, idd]
```

{−0.920496, −0.0856497, 2.08088 − 6.96201 ⅈ, 2.08088 + 6.96201 ⅈ,
7.16714 − 5.02105 ⅈ, 7.16714 + 5.02105 ⅈ, 9.56332 − 2.74592 ⅈ,
9.56332 + 2.74592 ⅈ, 10.6917 − 0.821476 ⅈ, 10.6917 + 0.821476 ⅈ}

```
Clear[rb]
bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
rb[n_, k_, f_] := rb[n, k, f] = Sum[f[j] rb[Floor[n / j], k - 1, f], {j, 2, n}]
rb[n_, 0, f_] := UnitStep[n - 1]
lrb[n_, f_] := Sum[(-1) ^ (k + 1) / k rb[n, k, f], {k, 1, Log2@n}]
rbz[n_, z_, f_] := Sum[bin[z, k] rb[n, k, f], {k, 0, Log2@n}]
lrz[n_, z_, f_] := Sin[Pi z] / Pi Sum[ (-1) ^k / (z - k) rb[n, k, f], {k, 0, Log2@n}]
rlroots[n_, ff_] := If[(c = Exponent[f = (rbz[n, z, ff]), z]) == 0, {},
  If[c == 1, List@Roots[f == 0, z][[2]], List @@ Roots[f == 0, z][[All, 2]]]]
rdlroots[n_, ff_] := If[(c = Exponent[f = (rbz[n, z, ff] - rbz[n - 1, z, ff]), z]) == 0,
  {}, If[c == 1, List@Roots[f == 0, z][[2]], List @@ Roots[f == 0, z][[All, 2]]]]
rbpoly[n_, z_, ff_] := Expand@FullSimplify@
   Expand[lrz[n, z, ff] / (Sin[Pi z] / Pi) FactorialPower[z, Floor[Log2@n] + 1] ]
rbpolya[n_, z_, ff_] := FullSimplify[lrz[n, z, ff]
   Floor[Log2@n]! / Binomial[Floor[Log2@n], z]]
rbpolyb[n_, z_, ff_] := lrz[n, z, ff] / FactorialPower[z, Floor[Log2@n]]
rdbpoly[n_, z_, ff_] := Expand@FullSimplify@Expand[
    (lrz[n, z, ff] - lrz[n - 1, z, ff]) / (Sin[Pi z] / Pi) FactorialPower[z, Floor[Log2@n] + 1] ]
rl2roots[n_, ff_] := If[(c = Exponent[f = (rbpoly[n, z, ff]), z]) == 0, {},
  If[c == 1, List@Roots[f == 0, z][[2]], List @@ Roots[f == 0, z][[All, 2]]]]
rl2rootsa[n_, ff_] := If[(c = Exponent[f = (rbpolya[n, z, ff]), z]) == 0, {},
  If[c == 1, List@Roots[f == 0, z][[2]], List @@ Roots[f == 0, z][[All, 2]]]]
rdl2roots[n_, ff_] := If[(c = Exponent[f = (rdbpoly[n, z, ff]), z]) == 0, {},
  If[c == 1, List@Roots[f == 0, z][[2]], List @@ Roots[f == 0, z][[All, 2]]]]
```

```
Expand@lrz[100, z, id]
```

$$\frac{7\,\mathrm{Sin}[\pi z]}{\pi(-6+z)} - \frac{51\,\mathrm{Sin}[\pi z]}{\pi(-5+z)} + \frac{184\,\mathrm{Sin}[\pi z]}{\pi(-4+z)} - \frac{324\,\mathrm{Sin}[\pi z]}{\pi(-3+z)} + \frac{283\,\mathrm{Sin}[\pi z]}{\pi(-2+z)} - \frac{99\,\mathrm{Sin}[\pi z]}{\pi(-1+z)} + \frac{\mathrm{Sin}[\pi z]}{\pi z}$$

```
Limit[FullSimplify[
  6! Product[1 - z / r, {r, rl2roots[100, id]}] Sin[Pi z] / Pi / FactorialPower[z, 7]], z → 1 / 2]
```

$$\frac{113\,678}{1155\,\pi}$$

```
Limit[(FullSimplify[6! Product[1 - z / r, {r, rl2roots[100, id]}]
      Sin[Pi z] / Pi / FactorialPower[z, 7]] - 1) / z, z → 0]
```

$$\frac{428}{15}$$

```
FullSimplify@Sum[-1 / r, {r, rl2roots[100, id]}]
```

$$\frac{313}{12}$$

```
Limit[(FullSimplify[6! Sin[Pi z] / Pi / FactorialPower[z, 7]] - 1) / z, z → 0]
```

$$\frac{49}{20}$$

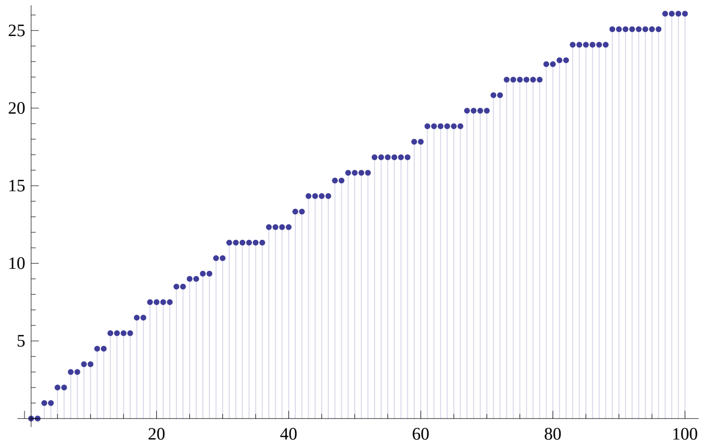
```
HarmonicNumber[Floor[Log2@100]]
```

$$\frac{49}{20}$$

```
(D[Expand[rbpolya[100, z, id]], z] /. z → 0) / 6 !
```

$$\frac{313}{12}$$

```
DiscretePlot[Re@Chop@N@Sum[-1 / r, {r, rl2rootsa[n, id]}], {n, 1, 100}]
```




```
Expand@FullSimplify[lrz[100, z, id] / Sin[Pi z] * Pi * FactorialPower[z, 7]]
```

$720 + 18\,780\,z - 9400\,z^2 + 1947\,z^3 - 165\,z^4 - 3\,z^5 + z^6$

```
rbpoly[100, z, id]
```

$720 + 18\,780\,z - 9400\,z^2 + 1947\,z^3 - 165\,z^4 - 3\,z^5 + z^6$

```
N@Sum[-1 / r, {r, rl2roots[100, id]}]
```

$26.0833 + 0.\,i$

```
{N@Product[1 - z / r, {r, rl2roots[100, id]}] Binomial[6, z] /. z → -1 / 2,
 N@lrz[100, -1 / 2, id]}
```

$\{-3.39902 + 0.\,i,\ -3.39902\}$

```
Table[6 ! Limit[Sin[Pi z] / Pi / FactorialPower[z, 7], z → n], {n, 0, 6}]
```

{1, 6, 15, 20, 15, 6, 1}

```
Table[Binomial[6, k], {k, 0, 6}]
```

{1, 6, 15, 20, 15, 6, 1}

```
rbpoly[100, z, id]
```

$720 + 18\,780\,z - 9400\,z^2 + 1947\,z^3 - 165\,z^4 - 3\,z^5 + z^6$

```
Limit[D[rbpolya[100, z, id], z], z → 0]
```

18 780

```
FullSimplify[n ! / Binomial[n, z]]
```

$$\frac{n!}{\text{Binomial}[n, z]}$$

```
Expand@FullSimplify@rbpolya[10, z, id]
```

$6 + 21\,z - 10\,z^2 + z^3$

```
lrz[n_, z_, f_] := Sin[Pi z] / Pi Sum[ (-1) ^k / (z - k) rb[n, k, f], {k, 0, Log2@n}]
rbpoly[n_, z_, ff_] := Expand@
  FullSimplify@Expand[lrz[n, z, ff] / (Sin[Pi z] / Pi) FactorialPower[z, Floor[Log2@n] + 1] ]
rbpolyx[n_, z_, ff_] := Expand@FullSimplify@
   Expand[Sin[Pi z] / Pi Sum[ (-1) ^k / (z - k) rb[n, k, ff], {k, 0, Log2@n}] / (Sin[Pi z] / Pi)
     FactorialPower[z, Floor[Log2@n] + 1] ]
rbpolyy[n_, z_, ff_] := Sum[ (-1) ^k FactorialPower[z, Floor[Log2@n] + 1] / (z - k)
   rb[n, k, ff], {k, 0, Log2@n}]
rbpolyy2[n_, z_, ff_] := Sum[ (-1) ^k / (z - k) rb[n, k, ff], {k, 0, Log2@n}]
lrza[n_, z_, f_] := rbpolyy[n, z, f] Binomial[Floor[Log2@n], z] / Floor[Log2@n] !
```

```
Expand@FullSimplify@Expand@rbpolyy[100, z, id]
```

$720 + 18780 z - 9400 z^2 + 1947 z^3 - 165 z^4 - 3 z^5 + z^6$

```
rbpoly[10, z, id]
```

$-6 - 21 z + 10 z^2 - z^3$

```
(-1) ^k Floor[Log2@n + 1] ! / Binomial[z, Floor[Log2@n] + 1] / (z - k)
```

$$\frac{(-1)^k \left(1 + \text{Floor}\left[\frac{\text{Log}[n]}{\text{Log}[2]}\right]\right)!}{(-k + z)\ \text{Binomial}\left[z, 1 + \text{Floor}\left[\frac{\text{Log}[n]}{\text{Log}[2]}\right]\right]}$$

```
FullSimplify@lrza[100, z, id] /. z → 2
```

283

```
Table[{n, Chop@N[Sum[-1 / r, {r, rl2rootsa[n, id]}] - Sum[-1 / r, {r, rl2rootsa[n - 1, id]}]]},
  {n, 1, 100}] // TableForm
```

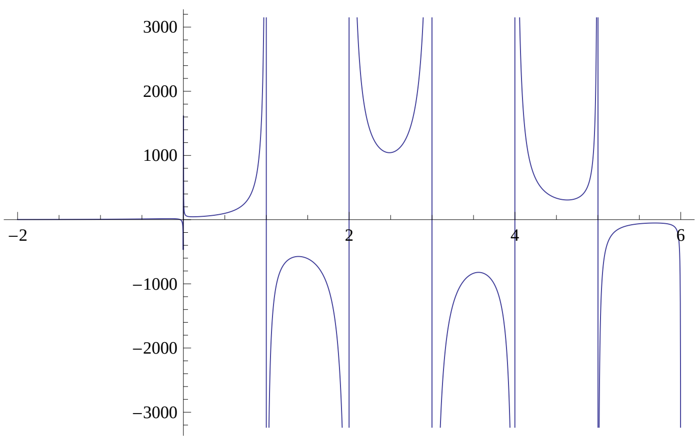
1	0
2	0
3	1.
4	0
5	1.
6	0
7	1.
8	0
9	0.5
10	0
11	1.
12	0
13	1.
14	0
15	0
16	0
17	1.
18	0
19	1.
20	0
21	0
22	0
23	1.
24	0
25	0.5
26	0
27	0.333333

28	0
29	1.
30	0
31	1.
32	0
33	0
34	0
35	0
36	0
37	1.
38	0
39	0
40	0
41	1.
42	0
43	1.
44	0
45	0
46	0
47	1.
48	0
49	0.5
50	0
51	0
52	0
53	1.
54	0
55	0
56	0
57	0
58	0
59	1.
60	0
61	1.
62	0
63	0
64	0
65	0
66	0
67	1.
68	0
69	0
70	0
71	1.
72	0
73	1.
74	0
75	0
76	0
77	0
78	0
79	1.
80	0
81	0.25
82	0
83	1.

84	0
85	0
86	0
87	0
88	0
89	1.
90	0
91	0
92	0
93	0
94	0
95	0
96	0
97	1.
98	0
99	0
100	0

mm[z_] := $\frac{7}{-6+z} - \frac{51}{-5+z} + \frac{184}{-4+z} - \frac{324}{-3+z} + \frac{283}{-2+z} - \frac{99}{-1+z} + \frac{1}{z}$

Plot[mm[z], {z, -2, 6}]



Residue[mm[z], {z, 0}]

1

Expand@lrz[100, z, id]

$$\frac{7\,\mathrm{Sin}[\pi z]}{\pi\,(-6+z)} - \frac{51\,\mathrm{Sin}[\pi z]}{\pi\,(-5+z)} + \frac{184\,\mathrm{Sin}[\pi z]}{\pi\,(-4+z)} - \frac{324\,\mathrm{Sin}[\pi z]}{\pi\,(-3+z)} + \frac{283\,\mathrm{Sin}[\pi z]}{\pi\,(-2+z)} - \frac{99\,\mathrm{Sin}[\pi z]}{\pi\,(-1+z)} + \frac{\mathrm{Sin}[\pi z]}{\pi z}$$

FullSimplify[1 / Gamma[z] / Gamma[1 - z]]

$$\frac{\mathrm{Sin}[\pi z]}{\pi}$$

1 / Gamma[5 - z] / Gamma[z] /. z → 2.3

0.554876

1 / Gamma[z] / Gamma[4 - z] / (4 - z) /. z → 2.3

0.554876

```
1 / Gamma[z] / Gamma[1 - z] / (4 - z) / (3 - z) / (2 - z) / (1 - z) /. z → 2.3
```

```
0.554876
```

```
Sin[Pi z] / Pi / (4 - z) / (3 - z) / (2 - z) / (1 - z) /. z → 2.3
```

```
0.554876
```

```
Sin[Pi z] / Pi / FactorialPower[z - 1, 4] /. z → 2.3
```

```
0.554876
```