```
ps[d_, s_, t_] := (-s/(s^2 + (1 + t)^2)) - d Sum[Sin[s Log[d j]] (d j)^t, {j, 1, 1/d}]

ps2[d_, s_, t_] := (1/d) (-s/(s^2 + (1 + t)^2)) - Sum[Sin[s Log[d j]] (d j)^t, {j, 1, 1/d}]

ps3[d_, s_, t_] :=
 ((-s/(s^2 + (1 + t)^2)) - d Sum[Sin[s Log[d j]] (d j)^t, {j, 1, 1/d}]) / (d Sin[s Log[d]] d^t)

ps3a[d_, s_, t_] := -Sum[j^t Sin[s Log[d j]] / Sin[s Log[d]], {j, 1, Floor[1/d]}]

Integrate[Sin[s Log[x]] / x^(1/2), {x, 0, 1}]
```

ConditionalExpression$\left[-\frac{4\,s}{1+4\,s^2},\ -\frac{1}{2}<\mathrm{Im}[s]<\frac{1}{2}\right]$

```
Integrate[Sin[s Log[x]] / x^(1/3), {x, 0, 1}]
```

ConditionalExpression$\left[-\frac{9\,s}{4+9\,s^2},\ -\frac{2}{3}<\mathrm{Im}[s]<\frac{2}{3}\right]$

```
Integrate[Sin[s Log[x]] / x^t, {x, 0, 1}]
```

ConditionalExpression$\left[-\frac{s}{s^2+(-1+t)^2},\ \mathrm{Re}[t]<1+\mathrm{Im}[s]\ \&\&\ \mathrm{Im}[s]+\mathrm{Re}[t]<1\right]$

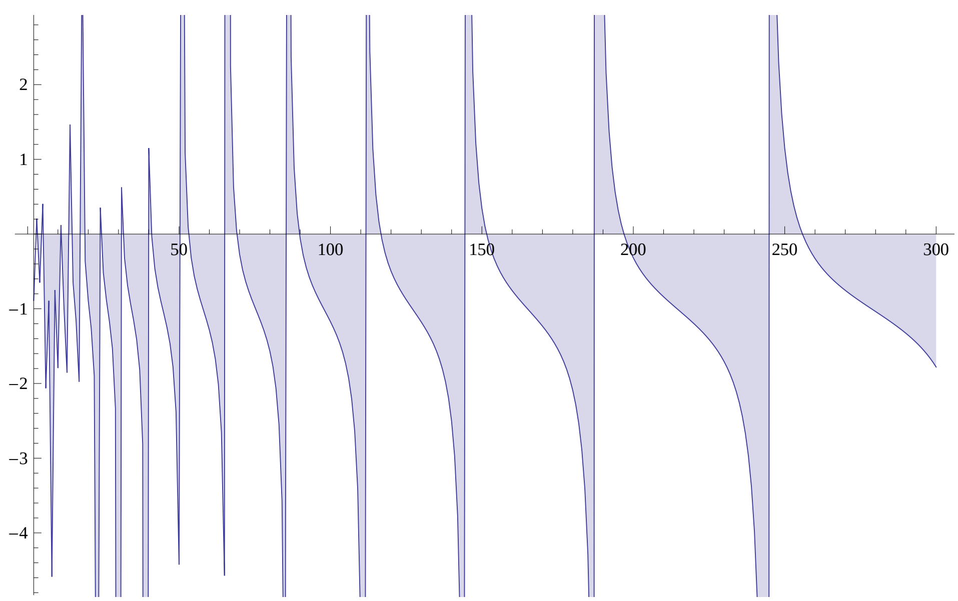
```
Integrate[Sin[s Log[x]] x^t, {x, 0, 1}]
```

ConditionalExpression$\left[-\frac{s}{s^2+(1+t)^2},\ \mathrm{Im}[s]<1+\mathrm{Re}[t]\ \&\&\ 1+\mathrm{Im}[s]+\mathrm{Re}[t]>0\right]$

```
ps2[.00001, N@Im@ZetaZero@1, -1]
```

-15 918.5

```
DiscretePlot[Re@ps3[1/d, 12., -.75], {d, 2, 300}]
```




```
Integrate[Sin[s Log[x]] / x^(1/2), {x, 0, 1}]
```

ConditionalExpression$\left[-\frac{4\,s}{1+4\,s^2},\ -\frac{1}{2}<\mathrm{Im}[s]<\frac{1}{2}\right]$

Integrate[Tan[s Log[x]] / x^(1 / 2), {x, 0, 1}]

$Aborted

Integrate[Sin[s Log[x]] / x^(1 / 2), {x, 0, 1}]

ConditionalExpression$\left[-\frac{4\,s}{1+4\,s^2},\ -\frac{1}{2}<\mathrm{Im}[s]<\frac{1}{2}\right]$

Integrate[Sin[s Log[x] + a] / x^k, {x, 0, 1}]

ConditionalExpression$\left[\frac{-s\,\mathrm{Cos}[a]+\mathrm{Sin}[a]-k\,\mathrm{Sin}[a]}{(-1+k)^2+s^2},\ \mathrm{Re}[k]<1+\mathrm{Im}[s]\ \&\&\ \mathrm{Im}[s]+\mathrm{Re}[k]<1\right]$

FullSimplify[d Sum[Sin[s d t], {t, 1, 1 / d}]]

$\frac{1}{2}\,d\left(-(-1+\mathrm{Cos}[s])\,\mathrm{Cot}\left[\frac{d\,s}{2}\right]+\mathrm{Sin}[s]\right)$

ex[d_, s_] := $\left(\frac{1-\mathrm{Cos}[s]}{s}\right)-\left(\frac{1}{2}\,d\left(-(-1+\mathrm{Cos}[s])\,\mathrm{Cot}\left[\frac{d\,s}{2}\right]+\mathrm{Sin}[s]\right)\right)$

ex2[d_, s_] := (1 / d) $\left(\left(\frac{1-\mathrm{Cos}[s]}{s}\right)-\left(\frac{1}{2}\,d\left(-(-1+\mathrm{Cos}[s])\,\mathrm{Cot}\left[\frac{d\,s}{2}\right]+\mathrm{Sin}[s]\right)\right)\right)$

Limit[ex2[d, s], d → 0]

$-\frac{\mathrm{Sin}[s]}{2}$

Integrate[Sin[s Log[x] + ArcTan[2 s]] / x^(1 / 2), {x, 0, 1}]

ConditionalExpression$\left[0,\ -\frac{1}{2}<\mathrm{Im}[s]<\frac{1}{2}\right]$

Integrate[Sin[s Log[x] + a] x^k, {x, 0, 1}]

ConditionalExpression$\left[\frac{-s\,\mathrm{Cos}[a]+(1+k)\,\mathrm{Sin}[a]}{(1+k)^2+s^2},\ \mathrm{Im}[s]<1+\mathrm{Re}[k]\ \&\&\ 1+\mathrm{Im}[s]+\mathrm{Re}[k]>0\right]$

FullSimplify$\left[\frac{-s\,\mathrm{Cos}[a]+(1+k)\,\mathrm{Sin}[a]}{(1+k)^2+s^2}\ /.\ a\to \mathrm{ArcTan}[2\,s]\ /.\ k\to -1/2\right]$

0

FullSimplify$\left[\frac{-s\,\mathrm{Cos}[a]+(1+k)\,\mathrm{Sin}[a]}{(1+k)^2+s^2}\ /.\ k\to -1/4\ /.\ a\to \mathrm{ArcTan}[4/3\,s]\right]$

0

FullSimplify$\left[\frac{-s\,\mathrm{Cos}[a]+(1+k)\,\mathrm{Sin}[a]}{(1+k)^2+s^2}\ /.\ k\to -1/2\ /.\ a\ \text{->}\ \mathrm{ArcTan}[2\,s]\right]$

0

FullSimplify$\left[\frac{-s\,\mathrm{Cos}[a]+(1+k)\,\mathrm{Sin}[a]}{(1+k)^2+s^2}\ /.\ k\to -1/3\ /.\ a\to \mathrm{ArcTan}[3/2\,s]\right]$

0

```
FullSimplify[(-s Cos[a] + (1 + k) Sin[a])/((1 + k)^2 + s^2) /. k → 0 /. a → ArcTan[s]]
```

0

```
Integrate[Sin[ s Log[x] + ArcTan[s]], {x, 0, 1}]
```

ConditionalExpression[0, -1 < Im[s] < 1]

```
Integrate[Sin[ Log[x^s E^ArcTan[2 s]]] / x^(1 / 2), {x, 0, 1}]
```

ConditionalExpression$\left[0, -\frac{1}{2} < \mathrm{Im}[s] < \frac{1}{2}\right]$

```
TrigToExp[Sin[Log[x]]]
```

$\frac{i\, x^{-i}}{2} - \frac{i\, x^{i}}{2}$

```
2 Integrate[Cosh[-(1 / 2 - s) Log[x]] / x^(1 / 2), {x, 0, 1}]
```

$\frac{2}{2 s - 2 s^2}$

```
FullSimplify[1/(2 s - 2 s^2)]
```

$\frac{1}{2 s - 2 s^2}$

```
Integrate[(2 s Cos[s Log[x]] - Sin[ s Log[x]]) / x^(1 / 2) / (2 s)^(1 / 2), {x, 0, 1}]
```

ConditionalExpression$\left[\frac{4\sqrt{2}\sqrt{s}}{1 + 4 s^2}, s \in \mathrm{Reals}\right]$

```
Integrate[(Cos[s Log[x]] + 2 s Sin[ s Log[x]]) / x^(1 / 2) / (2 s)^(1 / 2), {x, 0, 1}]
```

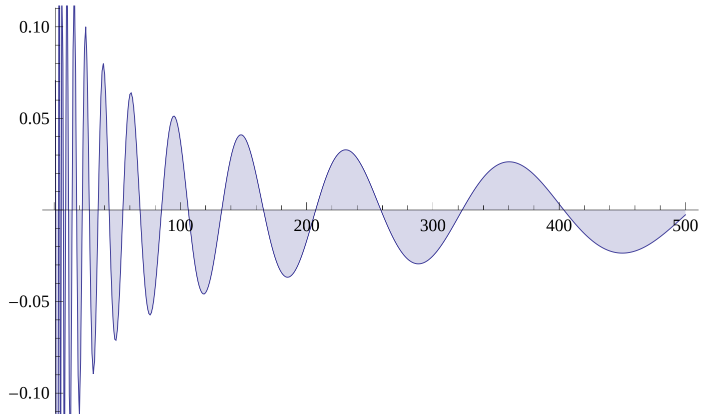
ConditionalExpression$\left[\frac{\sqrt{2}\left(1 - 4 s^2\right)}{\sqrt{s}\left(1 + 4 s^2\right)}, s \in \mathrm{Reals}\right]$

```
Integrate[Sin[s Log[x]] / x^(1 / 2), {x, 0, 113}]
```

ConditionalExpression$\left[\frac{2\sqrt{113}\,(-2 s \mathrm{Cos}[s \mathrm{Log}[113]] + \mathrm{Sin}[s \mathrm{Log}[113]])}{1 + 4 s^2}, -\frac{1}{2} < \mathrm{Im}[s] < \frac{1}{2}\right]$

```
ap[n_, s_] :=
 Sum[Sin[s Log[x]] / x^(1 / 2), {x, 1, n}] - (2 Sqrt[n] (-2 s Cos[s Log[n]] + Sin[s Log[n]]))/(1 + 4 s^2)
```

```
DiscretePlot[ap[n, N@Im@ZetaZero@1], {n, 1, 500}]
```




```
Integrate[Sin[s Log[x] + ArcTan[2 s]] / x^(1 / 2), {x, 0, 113}]
```

$$\text{ConditionalExpression}\left[\frac{2\sqrt{113}\ \text{Sin}[s\ \text{Log}[113]]}{\sqrt{1+4\,s^2}},\ -\frac{1}{2} < \text{Im}[s] < \frac{1}{2}\right]$$

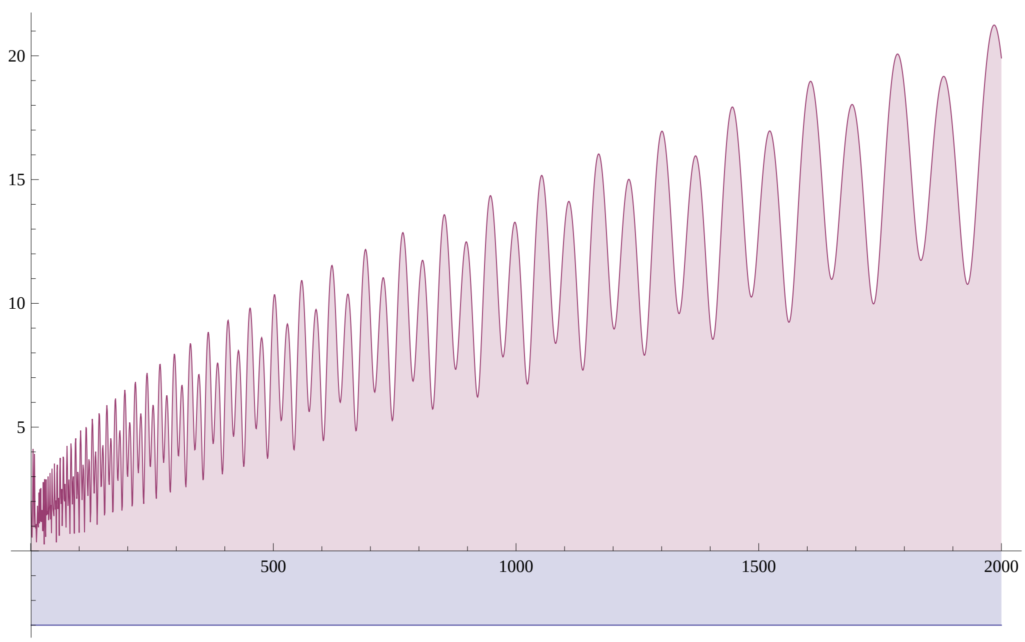
```
ke[d_, s_] := 2 d^(-1 / 2) Sum[t^(-1 / 2) Sin[-s Log[d t] - ArcTan[2 s]], {t, 1, 1 / d}]
```

```
ke[1. / 1000, N@Im@ZetaZero@10]
```

-0.499975

```
ag[s_] := DiscretePlot[{-3, Abs@ke[1 / n, s], Sin[ArcTan[s]]}, {n, 1, 2000}]
```

```
ag[N@Im@ZetaZero@12 + 3 + .1 I]
```




```
ke[.000001, 14.]
```

0.0942849

```
ke[.1, 14.]
```

-0.250178

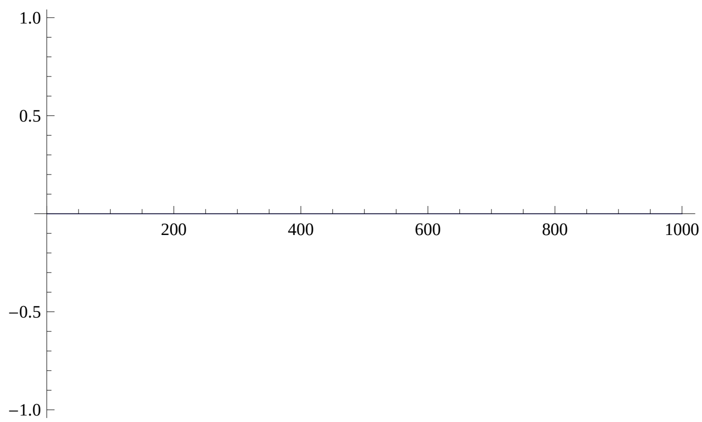
```
Limit[Sum[t^(-1/2) Sin[-s Log[d t] - ArcTan[2 s]], {t, 1, 1/d}] /. s → 10, d → 0]
```

$$\text{Limit}\left[\sum_{t=1}^{\frac{1}{d}} -\frac{\text{Sin}[\text{ArcTan}[20] + 10\,\text{Log}[d\,t]]}{\sqrt{t}},\ d \to 0\right]$$

```
Sin[ArcTan[N@Im@ZetaZero@11]]
```

0.999822

```
ke2a[n_, s_] := (-1/2 - s) n^(1/2 - s) (Zeta[1/2 - s] - HarmonicNumber[n, 1/2 - s]) -
  (-1/2 + s) n^(1/2 + s) (Zeta[1/2 + s] - HarmonicNumber[n, 1/2 + s])
Plot[{Re@ke2a[n, N@Im@ZetaZero@100 I]}, {n, 1, 1000}]
```




```
Im[-Sinh[ArcTanh[2 (N@Im@ZetaZero@1 I + .1)]]]
```

-0.999375

```
Arg[(1 - 2 s I)] /. s → 3 + 2 I
```

$$-\text{ArcTan}\left[\frac{6}{5}\right]$$

```
E^(I ArcTan[2 s]) /. s → 3 + 2 I
```

0.106846 + 0.920588 i

```
Integrate[ (2 s Cos[s Log[x]] + Sin[s Log[x]]) / x^(1/2), {x, 0, 1}]
```

ConditionalExpression[0, s ∈ Reals]

```
Integrate[ (Cos[s Log[x]] - 2 s Sin[s Log[x]]) / x^(1/2), {x, 0, 1}]
```

ConditionalExpression[2, s ∈ Reals]

```
Integrate[ ( Sin[s Log[x] + ArcTan[2 s]]) / x^(1/2), {x, 0, 1}]
```

$$\text{ConditionalExpression}\left[0,\ -\frac{1}{2} < \text{Im}[s] < \frac{1}{2}\right]$$

```
Integrate[ (Sin[s Log[x] - ArcTan[-1 / (2 s)]]) / x^(1 / 2), {x, 0, 1}]
```

$$\text{ConditionalExpression}\left[\frac{2\sqrt{4+\frac{1}{s^2}}\ s\left(1-4 s^2\right)}{\left(1+4 s^2\right)^2},\ -\frac{1}{2}<\text{Im}[s]<\frac{1}{2}\right]$$

```
ec[n_, s_] := 2 Sum[(n / j) ^ (1 / 2) Sin[s Log[n / j] - ArcTan[2 s]], {j, 1, n}] +
  I (E^ (-I ArcTan[2 s]) n^ (1 / 2 + s I) Zeta[1 / 2 + s I] -
     E^ (I ArcTan[2 s]) n^ (1 / 2 - s I) Zeta[1 / 2 - s I]) + Sin[ArcTan[2 s]]
eca[n_, s_] := {2 Sum[(n / j) ^ (1 / 2) Sin[s Log[n / j] - ArcTan[2 s]], {j, 1, n}],
  I E^ (-I ArcTan[2 s]) n^ (1 / 2 + s I) Zeta[1 / 2 + s I],
  -I E^ (I ArcTan[2 s]) n^ (1 / 2 - s I) Zeta[1 / 2 - s I], Sin[ArcTan[2 s]]}
eca2[n_, s_] := {2 Sum[(n / j) ^ (1 / 2) Sin[s Log[n / j] - ArcTan[2 s]], {j, 1, n}],
  {I E^ (-I ArcTan[2 s]), n^ (1 / 2 + s I), Zeta[1 / 2 + s I]},
  {-I E^ (I ArcTan[2 s]), n^ (1 / 2 - s I), Zeta[1 / 2 - s I]}, Sin[ArcTan[2 s]]}
```

```
eca[10 000, N@Im@ZetaZero@3 + 2]
```

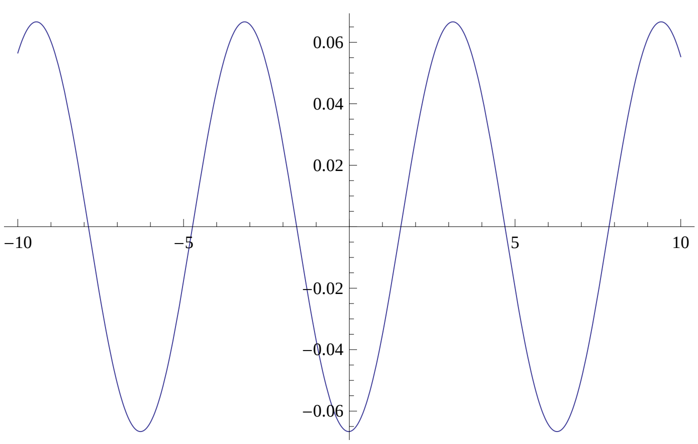
{233.397, -117.199 - 228.355 i, -117.199 + 228.355 i, 0.999829}

```
pe[s_, t_] := (4 Sin[t] - 8 s Cos[t]) / (1 + 4 s^2)
```

```
FullSimplify[pe[s, ArcTan[2 s]]]
```

0

```
Plot[pe[30, t], {t, -10, 10}]
```




```
N@ArcTan[2 × 30]
```

1.55413

```
FullSimplify[I E^ (-I s) / (-Sin[s])]
```

-1 - i Cot[s]

```
FullSimplify[-I E^ (I s) / (-Sin[s])]
```

-1 + i Cot[s]

```
FullSimplify[(4 Sin[s] - 8 s Cos[s]) / ((1 + 4 s^2) (-Sin[s]))]
```

$$\frac{-4+8\,s\,\mathrm{Cot}[s]}{1+4\,s^2}$$

```
FullSimplify[Sin[a - theta] / Sin[-theta]]
```

Cos[a] - Cot[theta] Sin[a]

```
Cot[ArcTan[2 s]]
```

$$\frac{1}{2\,s}$$

FullSimplify$\left[\frac{-4+8\,s\,\mathrm{Cot}[s]}{1+4\,s^2}\ /.\ s\to\mathrm{ArcTan}[2\,s]\right]$

$$\frac{-4\,s+4\,\mathrm{ArcTan}[2\,s]}{s+4\,s\,\mathrm{ArcTan}[2\,s]^2}$$

```
FullSimplify[FullSimplify[Sin[a - ArcTan[2 s]] / -Sin[ArcTan[2 s]]] /. a → s Log[d t]]
```

$$\mathrm{Cos}[s\,\mathrm{Log}[d\,t]]-\frac{\mathrm{Sin}[s\,\mathrm{Log}[d\,t]]}{2\,s}$$

```
Integrate[x^(-1/2) (1/(2 s) Sin[s Log[x]] + Cos[s Log[x]]), {x, 0, 1}]
```

ConditionalExpression[0, s ∈ Reals]

```
feh[n_, s_] := Sum[(n/j)^(1/2) Sin[s Log[n/j] - ArcTan[2 s]], {j, 1, n}]
feh2[n_, s_] := Sum[(n/j)^(1/2) (Cos[s Log[n/j]] - (1/(2 s)) Sin[s Log[n/j]]), {j, 1, n}]
```

```
feh2[1000, N@Im@ZetaZero@1]
```

0.5

```
ecc[n_, s_] := 2 Sum[(n/j)^(1/2) Sin[s Log[n/j] - ArcTan[2 s]], {j, 1, n}] +
  I (E^(-I ArcTan[2 s]) n^(1/2 + s I) Zeta[1/2 + s I] -
     E^(I ArcTan[2 s]) n^(1/2 - s I) Zeta[1/2 - s I]) + Sin[ArcTan[2 s]]
ecc2[n_, s_] := 2 Sum[(n/j)^(1/2) Sin[s Log[n/j] - ArcTan[2 s]] / Sin[ArcTan[2 s]],
    {j, 1, n}] + I (E^(-I ArcTan[2 s]) / Sin[ArcTan[2 s]] n^(1/2 + s I) Zeta[1/2 + s I] -
     E^(I ArcTan[2 s]) / Sin[ArcTan[2 s]] n^(1/2 - s I) Zeta[1/2 - s I]) +
  Sin[ArcTan[2 s]] / Sin[ArcTan[2 s]]
ecc3[n_, s_] := Sum[(n/j)^(1/2) Sin[s Log[n/j] - ArcTan[2 s]] / Sin[ArcTan[2 s]], {j, 1, n}] +
  I E^(-I ArcTan[2 s]) / Sin[ArcTan[2 s]] / 2 n^(1/2 + s I) Zeta[1/2 + s I] -
  I E^(I ArcTan[2 s]) / Sin[ArcTan[2 s]] / 2 n^(1/2 - s I) Zeta[1/2 - s I] + 1/2
ecc4[n_, s_] := Sum[(n/j)^(1/2) Sin[s Log[n/j] - ArcTan[2 s]] / Sin[ArcTan[2 s]], {j, 1, n}] +
  I E^(-I ArcTan[2 s]) / Sin[ArcTan[2 s]] / 2 n^(1/2 + s I) Zeta[1/2 + s I] -
  I E^(I ArcTan[2 s]) / Sin[ArcTan[2 s]] / 2 n^(1/2 - s I) Zeta[1/2 - s I] + 1/2
```

```
ecc4[1000, 2.2 + 1.7 I]
```

$-4.65661\times10^{-9}-2.44472\times10^{-9}\,i$

```
ecc[1000, 2.2 + 1.7 I]
```

$-3.60283\times10^{-9}-2.34328\times10^{-9}\,i$

```
Integrate[ Sin[s Log[x]] / (x)^(1/2), {x, 0, 1}]
```

ConditionalExpression$\left[-\frac{4\,s}{1+4\,s^2},\ -\frac{1}{2}<\mathrm{Im}[s]<\frac{1}{2}\right]$

```
Integrate[Sin[s Log[x] + ArcTan[2 s]] / x^(1 / 2), {x, 0, 1}]
```

ConditionalExpression$\left[0, -\frac{1}{2} < \mathrm{Im}[s] < \frac{1}{2}\right]$

```
ecx[n_, s_] :=
```

$-\left(-\frac{8\,s}{\left(1+4\,s^2\right)^{3/2}}\right)$ `n + 2 Sum[(n / j)^(1 / 2) Sin[s Log[n / j] - (-ArcTan[2 s])], {j, 1, n}] +`

```
  I (E^(-I (-ArcTan[2 s])) n^(1 / 2 + s I) Zeta[1 / 2 + s I] -
     E^(I (-ArcTan[2 s])) n^(1 / 2 - s I) Zeta[1 / 2 - s I]) + Sin[(-ArcTan[2 s])]
```

```
ecx[1000, 2.2 + 1.7 I]
```

113.793 − 365.831 i

```
ecr[n_, s_, a_] :=
```

$\left(\frac{-8\ s\,\mathrm{Cos}[a] + 4\,\mathrm{Sin}[a]}{1+4\,s^2}\right)$ `n + 2 Sum[(n / j)^(1 / 2) Sin[s Log[n / j] - a], {j, 1, n}] +`

```
  I (E^(-I a) n^(1 / 2 + s I) Zeta[1 / 2 + s I] - E^(I a) n^(1 / 2 - s I) Zeta[1 / 2 - s I]) + Sin[a]
```

`ecr2[n_, s_, a_] :=` $\left(\frac{-8\ s\,\mathrm{Cos}[a] + 4\,\mathrm{Sin}[a]}{1+4\,s^2}\right)$ `n -`

```
  2 Sum[(j / n)^(-1 / 2) Sin[s Log[j / n] + a], {j, 1, n}] +
  I (E^(-I a) n^(1 / 2 + s I) Zeta[1 / 2 + s I] - E^(I a) n^(1 / 2 - s I) Zeta[1 / 2 - s I]) + Sin[a]
```

`ecr3[d_, s_, a_] :=` $\left(\frac{-4\ s\,\mathrm{Cos}[a] + 2\,\mathrm{Sin}[a]}{1+4\,s^2}\right)$ `d^-1 -`

```
  Sum[(d j)^(-1 / 2) Sin[s Log[d j] + a], {j, 1, 1 / d}] + (1 / 2) I
   (E^(-I a) d^(-1 / 2 - s I) Zeta[1 / 2 + s I] - E^(I a) d^(-1 / 2 + s I) Zeta[1 / 2 - s I]) + Sin[a] / 2
```

```
ecr3[1 / 1000., 2.2, ArcTan[2 × 2.2]]
```

$2.77609 \times 10^{-11} + 0.\ i$

```
Integrate[Sin[s Log[x] + a] / x^(1 / 2), {x, 0, 1}]
```

$$\frac{2\,(-2\,s\,\mathrm{Cos}[a] + \mathrm{Sin}[a])}{1+4\,s^2}$$

$\frac{2\,(-2\,s\,\mathrm{Cos}[a] + \mathrm{Sin}[a])}{1+4\,s^2}$ `/. a → Pi / 2`

$$\frac{2}{1+4\,s^2}$$

```
Integrate[x^(s I - 1 / 2), {x, 0, 1}]
```

ConditionalExpression$\left[\frac{2\,i}{i - 2\,s}, \mathrm{Im}[s] < \frac{1}{2}\right]$

```
FullSimplify[Cos[s Log[x]] + I Sin[s Log[x]]]
```

$x^{i\,s}$

```
-2 s / 2 / I
```

$i\ s$

```
Integrate[x^(s), {x, 0, 1}]
```

ConditionalExpression$\left[\frac{1}{1+s}, \mathrm{Re}[s] > -1\right]$

```
Expand[((-1 / n^(-1 / 2 + s) - (1 / 2 + s) / 2 n^(-1 / 2 + s)) -
    (-1 / n^(-1 / 2 - s) - (1 / 2 - s) / 2 n^(-1 / 2 - s))) /
  (((1 / 2 + s) / 2)^(1 / 2) ((1 / 2 - s) / 2)^(1 / 2))]
```

$$\frac{n^{-\frac{1}{2}-s}}{2\sqrt{\frac{1}{2}-s}\,\sqrt{\frac{1}{2}+s}} - \frac{2\,n^{\frac{1}{2}-s}}{\sqrt{\frac{1}{2}-s}\,\sqrt{\frac{1}{2}+s}} - \frac{n^{-\frac{1}{2}+s}}{2\sqrt{\frac{1}{2}-s}\,\sqrt{\frac{1}{2}+s}} + \frac{2\,n^{\frac{1}{2}+s}}{\sqrt{\frac{1}{2}-s}\,\sqrt{\frac{1}{2}+s}} - \frac{n^{-\frac{1}{2}-s}\,s}{\sqrt{\frac{1}{2}-s}\,\sqrt{\frac{1}{2}+s}} - \frac{n^{-\frac{1}{2}+s}\,s}{\sqrt{\frac{1}{2}-s}\,\sqrt{\frac{1}{2}+s}}$$

```
FullSimplify[((1 / 2 + s) (1 / 2 - s))^(1 / 2)]
```

$$\frac{1}{2}\sqrt{1-4\,s^2}$$

```
FullSimplify[((1 / 2 + s) / (1 / 2 - s))^(1 / 2)]
```

$$\sqrt{\frac{1+2\,s}{1-2\,s}}$$

```
FullSimplify[E^(-s Log[n]) - E^(s Log[n])]
```

$n^{-s} - n^{s}$

```
TrigToExp[2 I Sin[s Log[j]]]
```

$-j^{-i\,s} + j^{i\,s}$

```
Integrate[j^(-1 / 2 + s I), {j, 0, n}]
```

$$\text{ConditionalExpression}\left[\frac{2\,i\,n^{\frac{1}{2}+i\,s}}{i-2\,s},\ \text{Im}[s] < \frac{1}{2}\right]$$

```
TrigToExp[I Sin[s Log[n]]]
```

$$-\frac{1}{2}\,n^{-i\,s} + \frac{n^{i\,s}}{2}$$

```
a1[s_] := 1 / (1 / 2 - s I) - 1 / (1 / 2 + s I)
a2[s_] := ((1 / 2 + s I)^(1 / 2) (1 / 2 - s I)^(1 / 2) / (1 / 2 - s I) - (1 / 2 + s I)^(1 / 2)
      (1 / 2 - s I)^(1 / 2) / (1 / 2 + s I)) / (1 / 2 - s I)^(1 / 2) / (1 / 2 + s I)^(1 / 2)
a3[s_] := ((1 / 2 + s I)^(1 / 2) (1 / 2 - s I)^(1 / 2) / (1 / 2 - s I) - (1 / 2 + s I)^(1 / 2)
      (1 / 2 - s I)^(1 / 2) / (1 / 2 + s I)) / (1 / 2 - s I)^(1 / 2) / (1 / 2 + s I)^(1 / 2)
```

```
a3[.3 + 2 I]
a1[.3 + 2 I]
```

$1.03535 + 0.175524\,i$

$1.03535 + 0.175524\,i$

```
FullSimplify[(1 / 2 + s I) ^ (1 / 2) (1 / 2 - s I) ^ (1 / 2) / (1 / 2 - s I)]
```

$$\frac{\sqrt{1+2\,i\,s}}{\sqrt{1-2\,i\,s}}$$

```
FullSimplify[((1 / 2 + s I) ^ (1 / 2) (1 / 2 - s I) ^ (1 / 2) / (1 / 2 + s I))]
```

$$\frac{\sqrt{1-2\,i\,s}}{\sqrt{1+2\,i\,s}}$$

```
FullSimplify[1 / (1 / 2 - s I) ^ (1 / 2) / (1 / 2 + s I) ^ (1 / 2)]
```

$$\frac{2}{\sqrt{1+4\,s^2}}$$

$$\text{FullSimplify@Log}\left[\frac{\sqrt{1+2\,i\,s}}{\sqrt{1-2\,i\,s}}\right]$$

```
i ArcTan[2 s]
```

$$\text{FullSimplify@Log}\left[\frac{\sqrt{1-2\,i\,s}}{\sqrt{1+2\,i\,s}}\right]$$

```
-i ArcTan[2 s]
```

```
2 Cos[ArcTan[2 s]]
```

$$\frac{2}{\sqrt{1+4\,s^2}}$$

```
b1[n_, s_] := n^(1 / 2 - s I) / (1 / 2 + s I) - n^(1 / 2 + s I) / (1 / 2 - s I)
b2[n_, s_] := n^(1 / 2) (n^(-s I) / (1 / 2 + s I) - n^(s I) / (1 / 2 - s I))
b3[n_, s_] :=
 n^(1 / 2) ((1 / 2 - s I) ^ (1 / 2) (1 / 2 + s I) ^ (1 / 2) n^(-s I) / (1 / 2 + s I) - (1 / 2 - s I) ^ (1 / 2)
       (1 / 2 + s I) ^ (1 / 2) n^(s I) / (1 / 2 - s I)) / (1 / 2 - s I) ^ (1 / 2) / (1 / 2 + s I) ^ (1 / 2)
b4[n_, s_] := 2 Cos[ArcTan[2 s]] n^(1 / 2) ((1 / 2 - s I) ^ (1 / 2) (1 / 2 + s I) ^ (1 / 2)
      n^(-s I) / (1 / 2 + s I) - (1 / 2 - s I) ^ (1 / 2) (1 / 2 + s I) ^ (1 / 2) n^(s I) / (1 / 2 - s I))
b5[n_, s_] := 2 Cos[ArcTan[2 s]] n^(1 / 2) ((1 / 2 - s I) ^ (1 / 2) (1 / 2 + s I) ^ (1 / 2) / (1 / 2 + s I)
      n^(-s I) - (1 / 2 - s I) ^ (1 / 2) (1 / 2 + s I) ^ (1 / 2) / (1 / 2 - s I) n^(s I))
b6[n_, s_] := 2 Cos[ArcTan[2 s]] n^(1 / 2)
  (E^(-i ArcTan[2 s]) n^(-s I) - E^(i ArcTan[2 s]) n^(s I))
b7[n_, s_] := 2 Cos[ArcTan[2 s]] n^(1 / 2)
  (E^(-i ArcTan[2 s]) E^(-s Log[n] I) - E^(i ArcTan[2 s]) E^(s Log[n] I))
b8[n_, s_] := 2 Cos[ArcTan[2 s]] n^(1 / 2)
  (E^(-i ArcTan[2 s] + -s Log[n] I) - E^(s Log[n] I + i ArcTan[2 s]))
b10[n_, s_] := -4 i Cos[ArcTan[2 s]] n^(1 / 2) Sin[s Log[n] + ArcTan[2 s]]
```

```
b10[100, .3 + 1.2 I]
```

```
-1846.35 + 2733.03 i
```

```
b1[100, .3 + 1.2 I]
```

```
-1846.35 + 2733.03 i
```

```
FullSimplify[1 / (1 / 2 - s I) ^ (1 / 2) / (1 / 2 + s I) ^ (1 / 2)]
```

$$\frac{2}{\sqrt{1+4\,s^2}}$$

```
FullSimplify@Log[(1 / 2 - s I) ^ (1 / 2) (1 / 2 + s I) ^ (1 / 2) / (1 / 2 + s I)]
```

-ⅈ ArcTan[2 s]

```
FullSimplify@Log[(1 / 2 - s I) ^ (1 / 2) (1 / 2 + s I) ^ (1 / 2) / (1 / 2 - s I)]
```

ⅈ ArcTan[2 s]

```
ExpToTrig[(E^(-ⅈ (ArcTan[2 s] + s Log[n])) - E^(I ( s Log[n]  + ArcTan[2 s] ))))]
```

-2 ⅈ Sin[ArcTan[2 s] + s Log[n]]

```
c1[n_, s_] := n^(1 / 2 - s I) / (1 / 2 + s I) + n^(1 / 2 + s I) / (1 / 2 - s I)
c8[n_, s_] := 2 Cos[ArcTan[2 s]] n^(1 / 2)
   (E^(-ⅈ ArcTan[2 s] + -s Log[n] I) + E^(s Log[n] I + ⅈ ArcTan[2 s]))
c10[n_, s_] := 4 Cos[ArcTan[2 s]] n^(1 / 2) Cos[s Log[n] + ArcTan[2 s]]
```

```
c10[100, .3 + .2 I]
```

-34.3604 - 43.8865 ⅈ

```
c1[100, .3 + .2 I]
```

-34.3604 - 43.8865 ⅈ

```
Integrate[Cos[13 Log[j]] / j^(1 / 2), {j, 0, n}]
```

$$\frac{2}{677}\sqrt{n}\ (\mathrm{Cos}[13\ \mathrm{Log}[n]]+26\ \mathrm{Sin}[13\ \mathrm{Log}[n]])$$

```
Integrate[Sin[13 Log[j]] / j^(1 / 2), {j, 0, n}]
```

$$-\frac{2}{677}\sqrt{n}\ (26\ \mathrm{Cos}[13\ \mathrm{Log}[n]]-\mathrm{Sin}[13\ \mathrm{Log}[n]])$$

$$\texttt{cal0[n\_, s\_] := Sum[j\^{}(-1 / 2 + s I), \{j, 1, n\}]} - \frac{1}{2}\,n^{-\frac{1}{2}+i\,s} - \frac{n^{\frac{1}{2}+i\,s}}{\frac{1}{2}+i\,s}$$

```
z1[s_] := Zeta[1 / 2 + s I]
```

$$\texttt{cal01[n\_, s\_] := Sum[j\^{}(-1 / 2 - s I), \{j, 1, n\}]} - \frac{1}{2}\,n^{-\frac{1}{2}-i\,s} - \frac{n^{\frac{1}{2}-i\,s}}{\frac{1}{2}-i\,s}$$

```
z1i[s_] := Zeta[1 / 2 - s I]
calx[n_, s_] := (cal0[n, s] - cal01[n, s]) / (2 I)
cal1[n_, s_] := Sum[Sin[s Log[j]] / j^(1 / 2), {j, 1, n}] +
   2 n^(1 / 2) Cos[ArcTan[2 s]] Sin[s Log[n] + ArcTan[2 s]] - Sin[s Log[n]] / (2 n^(1 / 2))
rcal[s_] := 1 / (2 I) (Zeta[1 / 2 - s I] - Zeta[1 / 2 + s I])
```

$$\texttt{calx2[n\_, s\_] :=} \left(\left(\texttt{Sum[j\^{}(-1 / 2 + s I), \{j, 1, n\}]} - \frac{1}{2}\,n^{-\frac{1}{2}+i\,s} - \frac{n^{\frac{1}{2}+i\,s}}{\frac{1}{2}+i\,s}\right) - \right.$$

$$\left(\text{Sum}[j\text{^}(-1/2-s\,I),\{j,1,n\}]-\frac{1}{2}n^{-\frac{1}{2}-i s}-\frac{n^{\frac{1}{2}-i s}}{\frac{1}{2}-i s}\right)\Big/(2 I)$$

$$\text{calx3}[n\_,s\_]:=\left(\left(\text{Sum}[j\text{^}(-1/2+s\,I)-j\text{^}(-1/2-s\,I),\{j,1,n\}]-\left(\frac{1}{2}n^{-\frac{1}{2}+i s}-\frac{1}{2}n^{-\frac{1}{2}-i s}\right)-\left(\frac{n^{\frac{1}{2}+i s}}{\frac{1}{2}+i s}-\frac{n^{\frac{1}{2}-i s}}{\frac{1}{2}-i s}\right)\right)\right)\Big/(2 I)$$

$$\text{calx4}[n\_,s\_]:=\left(\left(\text{Sum}[j\text{^}(-1/2)\,(j\text{^}(s\,I)-j\text{^}(-s\,I)),\{j,1,n\}]-\left(\frac{1}{2}n\text{^}(-1/2)\left(n^{i s}-n^{-i s}\right)\right)-n\text{^}(1/2)\left(\frac{n^{i s}}{\frac{1}{2}+i s}-\frac{n^{-i s}}{\frac{1}{2}-i s}\right)\right)\right)\Big/(2 I)$$

$$\text{calx5}[n\_,s\_]:=\left(\left(2 I\,\text{Sum}[j\text{^}(-1/2)\,\text{Sin}[s\,\text{Log}[j]],\{j,1,n\}]-\left(\frac{1}{2}n\text{^}(-1/2)\left(n^{i s}-n^{-i s}\right)\right)-n\text{^}(1/2)\left(\frac{n^{i s}}{\frac{1}{2}+i s}-\frac{n^{-i s}}{\frac{1}{2}-i s}\right)\right)\right)\Big/(2 I)$$

$$\text{calx6}[n\_,s\_]:=\left(\left(2 I\,\text{Sum}[j\text{^}(-1/2)\,\text{Sin}[s\,\text{Log}[j]],\{j,1,n\}]-(I\,n\text{^}(-1/2)\,\text{Sin}[s\,\text{Log}[n]])-n\text{^}(1/2)\left(\frac{n^{i s}}{\frac{1}{2}+i s}-\frac{n^{-i s}}{\frac{1}{2}-i s}\right)\right)\right)\Big/(2 I)$$

$$\text{calx7}[n\_,s\_]:=\left(\left(2 I\,\text{Sum}[j\text{^}(-1/2)\,\text{Sin}[s\,\text{Log}[j]],\{j,1,n\}]-(I\,n\text{^}(-1/2)\,\text{Sin}[s\,\text{Log}[n]])-n\text{^}(1/2)/(1/2+I\,s)\text{^}(1/2)/(1/2-I\,s)\text{^}(1/2)\left(\frac{n^{i s}\,(1/2+I\,s)\text{^}(1/2)\,(1/2-I\,s)\text{^}(1/2)}{\frac{1}{2}+i s}-\frac{n^{-i s}\,(1/2+I\,s)\text{^}(1/2)\,(1/2-I\,s)\text{^}(1/2)}{\frac{1}{2}-i s}\right)\right)\right)\Big/(2 I)$$

$$\text{calx8}[n\_,s\_]:=\left(\left(2 I\,\text{Sum}[j\text{^}(-1/2)\,\text{Sin}[s\,\text{Log}[j]],\{j,1,n\}]-(I\,n\text{^}(-1/2)\,\text{Sin}[s\,\text{Log}[n]])-n\text{^}(1/2)/(1/2+I\,s)\text{^}(1/2)/(1/2-I\,s)\text{^}(1/2)\left(\frac{n^{i s}\,(1/2-I\,s)\text{^}(1/2)}{\left(\frac{1}{2}+i s\right)\text{^}(1/2)}-\frac{n^{-i s}\,(1/2+I\,s)\text{^}(1/2)}{\left(\frac{1}{2}-i s\right)\text{^}(1/2)}\right)\right)\right)\Big/(2 I)$$

$$\text{calx9}[n\_,s\_]:=\left(\left(2 I\,\text{Sum}[j\text{^}(-1/2)\,\text{Sin}[s\,\text{Log}[j]],\{j,1,n\}]-(I\,n\text{^}(-1/2)\,\text{Sin}[s\,\text{Log}[n]])-\right.\right.$$

```
    n^(1/2) 2 Cos[ArcTan[2 s]] (n^(i s) (1/2 - I s)^(1/2) / ((1/2 + i s)^(1/2)) - n^(-i s) (1/2 + I s)^(1/2) / ((1/2 - i s)^(1/2))))) / (2 I)
calx10[n_, s_] := ((2 I Sum[j^(-1/2) Sin[s Log[j]], {j, 1, n}] - (I n^(-1/2) Sin[s Log[n]]) -
    n^(1/2) 2 Cos[ArcTan[2 s]] (E^(-i ArcTan[2 s]) n^(i s) - E^(i ArcTan[2 s]) n^(-i s)))) / (2 I)
calx11[n_, s_] := ((2 I Sum[j^(-1/2) Sin[s Log[j]], {j, 1, n}] -
    (I n^(-1/2) Sin[s Log[n]]) - n^(1/2) 2 Cos[ArcTan[2 s]]
     (E^(-i ArcTan[2 s]) E^(i s Log[n]) - E^(i ArcTan[2 s]) E^(-i s Log[n])))) / (2 I)
calx12[n_, s_] := ((2 I Sum[j^(-1/2) Sin[s Log[j]], {j, 1, n}] - (I n^(-1/2) Sin[s Log[n]]) -
    n^(1/2) 2 Cos[ArcTan[2 s]] (E^(i s Log[n] - i ArcTan[2 s]) - E^(-i s Log[n] + i ArcTan[2 s])))) / (2 I)
calx13[n_, s_] := Sum[j^(-1/2) Sin[s Log[j]], {j, 1, n}] -
  n^(1/2) Cos[ArcTan[2 s]] 2 Sin[s Log[n] - ArcTan[2 s]] - n^(-1/2) / 2 Sin[s Log[n]]
rcala[s_] := 1 / (2 I) (Zeta[1/2 - s I] - Zeta[1/2 + s I])

calx13[10 000, 12. + .1 I]

0.746892 + 0.0120629 i

rcala[12. + .1 I]

0.746893 + 0.0120623 i

calx[10 000, 2. + .1 I]

0.310829 - 0.029081 i

FullSimplify[1 / (1/2 + I s)^(1/2) / (1/2 - I s)^(1/2)]
```

$$\frac{2}{\sqrt{1+4\,s^2}}$$

```
FullSimplify[Log[(1/2 - I s)^(1/2) / ((1/2 + i s)^(1/2))]]

-i ArcTan[2 s]

FullSimplify[Log[(1/2 + I s)^(1/2) / ((1/2 - i s)^(1/2))]]

i ArcTan[2 s]

balx13[n_, s_] := Sum[j^(-1/2) Cos[s Log[j]], {j, 1, n}] -
  n^(1/2) Cos[ArcTan[2 s]] 2 Cos[s Log[n] - ArcTan[2 s]] - n^(-1/2) / 2 Cos[s Log[n]]
rcalb[s_] := 1 / (2) (Zeta[1/2 - s I] + Zeta[1/2 + s I])

balx13[10 000, 12. + .1 I]
rcalb[12. + .1 I]

1.01688 - 0.0525293 i

1.01688 - 0.0525302 i

Limit[Cos[(300 + .5 I) Log[n]] / n^(1/2), n → Infinity]
```

$0.5 \times 2.71828^{(0.+2.\,i)\ \mathrm{Interval}[\{-8.9003\times10^{-308},3.14159\}]}$

```
arcal[s_, a_] := E^(a I) Zeta[1/2 - s I] + E^(-a I) Zeta[1/2 + s I]
```

$$\mathtt{acalx2[n\_,\ s\_,\ a\_]} := E^{\wedge}(a\,I)\left(\mathtt{Sum[j^(-1/2+s\,I),\ \{j,1,n\}]} - \frac{1}{2}\,n^{-\frac{1}{2}+i s} - \frac{n^{\frac{1}{2}+i s}}{\frac{1}{2}+i s}\right) +$$

```
E^(-a I) (Sum[j^(-1/2 - s I), {j, 1, n}] - 1/2 n^(-1/2 - I s) - n^(1/2 - I s)/(1/2 - I s))
acalx3[n_, s_, a_] := Sum[E^(a I) j^(-1/2 + s I), {j, 1, n}] - 1/2 E^(a I) n^(-1/2 + I s) -
  E^(a I) n^(1/2 + I s)/(1/2 + I s) + Sum[E^(-a I) j^(-1/2 - s I), {j, 1, n}] - 1/2 E^(-a I) n^(-1/2 - I s) - E^(-a I) n^(1/2 - I s)/(1/2 - I s)
acalx4[n_, s_, a_] := Sum[E^(a I) j^(-1/2 + s I) + E^(-a I) j^(-1/2 - s I), {j, 1, n}] -
  1/2 E^(a I) n^(-1/2 + I s) - 1/2 E^(-a I) n^(-1/2 - I s) - E^(a I) n^(1/2 + I s)/(1/2 + I s) - E^(-a I) n^(1/2 - I s)/(1/2 - I s)
acalx5[n_, s_, a_] := Sum[j^(-1/2) (E^(a I) E^(s Log[j] I) + E^(-a I) E^(-s Log[j] I)),
   {j, 1, n}] - 1/2 n^(-1/2) (E^(a I) n^(I s) + E^(-a I) n^(-I s)) - E^(a I) n^(1/2 + I s)/(1/2 + I s) - E^(-a I) n^(1/2 - I s)/(1/2 - I s)
acalx6[n_, s_, a_] := Sum[j^(-1/2) (E^(s Log[j] I + a I) + E^(-s Log[j] I - a I)), {j, 1, n}] -
  1/2 n^(-1/2) (E^(I s Log[n] + a I) + E^(-I s Log[n] - a I)) - n^(1/2) (E^(I s Log[n] + a I)/(1/2 + I s) + E^(-I s Log[n] - a I)/(1/2 - I s))
acalx7[n_, s_, a_] := 2 Sum[j^(-1/2) Cos[s Log[j] + a], {j, 1, n}] -
  1/2 n^(-1/2) (E^(I s Log[n] + a I) + E^(-I s Log[n] - a I)) - n^(1/2) (E^(I s Log[n] + a I)/(1/2 + I s) + E^(-I s Log[n] - a I)/(1/2 - I s))
acalx8[n_, s_, a_] := 2 Sum[j^(-1/2) Cos[s Log[j] + a], {j, 1, n}] -
  n^(-1/2) Cos[s Log[n] + a] - n^(1/2) (E^(I s Log[n] + a I)/(1/2 + I s) + E^(-I s Log[n] - a I)/(1/2 - I s))
acalx9[n_, s_, a_] := 2 Sum[j^(-1/2) Cos[s Log[j] + a], {j, 1, n}] -
  n^(-1/2) Cos[s Log[n] + a] - n^(1/2) / (1/2 + I s)^(1/2) / (1/2 - I s)^(1/2)
   (((1/2 + I s)^(1/2) (1/2 - I s)^(1/2) E^(I s Log[n] + a I))/(1/2 + I s) +
    ((1/2 + I s)^(1/2) (1/2 - I s)^(1/2) E^(-I s Log[n] - a I))/(1/2 - I s))
acalx10[n_, s_, a_] := 2 Sum[j^(-1/2) Cos[s Log[j] + a], {j, 1, n}] -
  n^(-1/2) Cos[s Log[n] + a] -
  n^(1/2) 2 Cos[ArcTan[2 s]] (E^(-I ArcTan[2 s]) E^(I s Log[n] + a I) + E^(I ArcTan[2 s]) E^(-I s Log[n] - a I))
acalx11[n_, s_, a_] := 2 Sum[j^(-1/2) Cos[s Log[j] + a], {j, 1, n}] -
  n^(-1/2) Cos[s Log[n] + a] -
  n^(1/2) 2 Cos[ArcTan[2 s]] (E^(I s Log[n] + a I - I ArcTan[2 s]) + E^(-I s Log[n] - a I + I ArcTan[2 s]))
acalx12[n_, s_, a_] := 2 Sum[j^(-1/2) Cos[s Log[j] + a], {j, 1, n}] -
  n^(-1/2) Cos[s Log[n] + a] - 4 n^(1/2) Cos[ArcTan[2 s]] Cos[s Log[n] + a - ArcTan[2 s]]
arcal2[s_, a_] := E^(a I) Zeta[1/2 - s I] + E^(-a I) Zeta[1/2 + s I]

acalx12[10 000, 12. + .1 I, .3]
arcal2[12. + .1 I, .3]
```

```
1.50149 - 0.107496 i

1.50149 - 0.107497 i

(1 / 2 + I s) ^ (1 / 2) (1 / 2 - I s) ^ (1 / 2)

ExpToTrig[E^ (a I) Zeta[1 / 2 - s I] + E^ (-a I) Zeta[1 / 2 + s I]]
```

$$\mathrm{Cos}[a]\ \mathrm{Zeta}\left[\frac{1}{2}-i\,s\right]+i\,\mathrm{Sin}[a]\ \mathrm{Zeta}\left[\frac{1}{2}-i\,s\right]+\mathrm{Cos}[a]\ \mathrm{Zeta}\left[\frac{1}{2}+i\,s\right]-i\,\mathrm{Sin}[a]\ \mathrm{Zeta}\left[\frac{1}{2}+i\,s\right]$$

```
Cos[ArcTan[2 s]]
```

$$\frac{1}{\sqrt{1+4\,s^2}}$$

```
dcalx12[n_, s_, a_] := 2 Sum[j^ (-1 / 2) Cos[s Log[j] + a], {j, 1, n}] -
  n^ (-1 / 2) Cos[s Log[n] + a] - 4 n^ (1 / 2) Cos[ArcTan[2 s]] Cos[s Log[n] + a - ArcTan[2 s]]
dcalx13[n_, s_] := 2 Sum[j^ (-1 / 2) Cos[s Log[j] + ArcTan[2 s]], {j, 1, n}] -
  n^ (-1 / 2) Cos[s Log[n] + ArcTan[2 s]] -
  4 n^ (1 / 2) Cos[ArcTan[2 s]] Cos[s Log[n] + ArcTan[2 s] - ArcTan[2 s]]
drcal2[s_, a_] := Cos[a] (Zeta[1 / 2 - s I] + Zeta[1 / 2 + s I]) +
  I Sin[a] (Zeta[1 / 2 - s I] - Zeta[1 / 2 + s I])
drcal3[s_] := Cos[ArcTan[2 s]] (Zeta[1 / 2 - s I] + Zeta[1 / 2 + s I]) +
  I Sin[ArcTan[2 s]] (Zeta[1 / 2 - s I] - Zeta[1 / 2 + s I])
```

$$\mathrm{drcal4}[s\_] := \frac{1}{\sqrt{1+4\,s^2}}\ (\mathrm{Zeta}[1/2 - s\,I] + \mathrm{Zeta}[1/2 + s\,I]) + I\left(\frac{2\,s}{\sqrt{1+4\,s^2}}\right)(\mathrm{Zeta}[1/2 - s\,I] - \mathrm{Zeta}[1/2 + s\,I])$$

$$\mathrm{dcalx14}[n\_, s\_] := 2\,\mathrm{Sum}[j^\wedge(-1/2)\ \mathrm{Cos}[s\,\mathrm{Log}[j] + \mathrm{ArcTan}[2\,s]], \{j, 1, n\}] - n^\wedge(-1/2)\ \mathrm{Cos}[s\,\mathrm{Log}[n] + \mathrm{ArcTan}[2\,s]] - 4\,n^\wedge(1/2)\ \frac{1}{\sqrt{1+4\,s^2}}\ \mathrm{Cos}[s\,\mathrm{Log}[n]]$$

```
drcal5[s_] := ((Zeta[1 / 2 - s I] + Zeta[1 / 2 + s I]) + I (2 s) (Zeta[1 / 2 - s I] - Zeta[1 / 2 + s I])) /
  2
```

$$\mathrm{dcalx15}[n\_, s\_] := \sqrt{1+4\,s^2}\ \mathrm{Sum}[j^\wedge(-1/2)\ \mathrm{Cos}[s\,\mathrm{Log}[j] + \mathrm{ArcTan}[2\,s]], \{j, 1, n\}] - n^\wedge(-1/2)\ \sqrt{1+4\,s^2}\Big/ 2\,\mathrm{Cos}[s\,\mathrm{Log}[n] + \mathrm{ArcTan}[2\,s]] - 2\,n^\wedge(1/2)\ \mathrm{Cos}[s\,\mathrm{Log}[n]]$$

```
dcalx15[10 000, 12. + .1 I]
drcal5[12. + .1 I]

-16.9061 - 0.491417 i

-16.9061 - 0.491403 i

Cos[ArcTan[2 s]]
```

$$\frac{1}{\sqrt{1+4\,s^2}}$$

```
FullSimplify[TrigToExp[Cos[s Log[j] + ArcTan[2 s]]]]
```

$$\frac{j^{-i s} \left(1 + j^{2 i s} (1 + 2 i s) - 2 i s\right)}{2 \sqrt{1 + 4 s^2}}$$

```
Cos[ArcTan[2 s] - s Log[n] + Pi / 2]
```

-Sin[ArcTan[2 s] - s Log[n]]

```
ecalx12[n_, s_, a_] := 2 Sum[j^(-1/2) Cos[s Log[j] + a], {j, 1, n}] -
  n^(-1/2) Cos[s Log[n] + a] - 4 n^(1/2) Cos[ArcTan[2 s]] Cos[s Log[n] + a - ArcTan[2 s]]
ercal2[s_, a_] := Cos[a] (Zeta[1/2 - s I] + Zeta[1/2 + s I]) +
  I Sin[a] (Zeta[1/2 - s I] - Zeta[1/2 + s I])
ts1[n_, s_] := ecalx12[n, s, ArcTan[2 s] - Pi/2 - s Log[n]]
ts2[n_, s_] := ercal2[s, ArcTan[2 s] - Pi/2 - s Log[n]]
```

```
ts1[10 000, 13. + .1 I]
ts2[10 000, 13. + .1 I]
```

1.93821 + 0.7392 i

1.93821 + 0.7392 i

```
2 Sum[j^(-1/2) Cos[s Log[j] + a], {j, 1, n}] - n^(-1/2) Cos[s Log[n] + a] -
  4 n^(1/2) Cos[ArcTan[2 s]] Cos[s Log[n] + a - ArcTan[2 s]] /. a → ArcTan[2 s] - Pi/2 - s Log[n]
```

$$-\frac{2 s}{\sqrt{n} \sqrt{1 + 4 s^2}} + 2 \sum_{j=1}^{n} \frac{\text{Sin}[\text{ArcTan}[2 s] + s \text{Log}[j] - s \text{Log}[n]]}{\sqrt{j}}$$

```
FullSimplify[Cos[a] (Zeta[1/2 - s I] + Zeta[1/2 + s I]) +
   I Sin[a] (Zeta[1/2 - s I] - Zeta[1/2 + s I]) /. a → ArcTan[2 s] - Pi/2 - s Log[n]]
```

$$\frac{n^{-i s} \left((-i + 2 s) \text{Zeta}\left[\frac{1}{2} - i s\right] + n^{2 i s} (i + 2 s) \text{Zeta}\left[\frac{1}{2} + i s\right]\right)}{\sqrt{1 + 4 s^2}}$$

```
Chop[N@Sin[2 I + 111.1]^2 + Cos[2 I + 111.1]^2]
```

1.

```
Limit[2 s / (Sqrt[n] Sqrt[1 + 4 s^2]), n → Infinity]
```

0

```
Cos[Pi / 2]
```

0

```
2 Sum[j^(-1/2) Cos[s Log[j] + a], {j, 1, n}] /. a → ArcTan[2 s] - Pi/2 - s Log[n]
```

$$2 \sum_{j=1}^{n} \frac{\text{Sin}[\text{ArcTan}[2 s] + s \text{Log}[j] - s \text{Log}[n]]}{\sqrt{j}}$$

```
- n^(-1/2) Cos[s Log[n] + a] /. a → ArcTan[2 s] - Pi/2 - s Log[n]
```

$$-\frac{2 s}{\sqrt{n} \sqrt{1 + 4 s^2}}$$

```
-4 n^(1/2) Cos[ArcTan[2 s]] Cos[s Log[n] + a - ArcTan[2 s]] /. a → ArcTan[2 s] - Pi/2 - s Log[n]
```

0

FullSimplify[Cos[a] (Zeta[1 / 2 - s I] + Zeta[1 / 2 + s I]) /. a → ArcTan[2 s] - Pi / 2 - s Log[n]]

$$\text{Sin}[\text{ArcTan}[2\,s] - s\,\text{Log}[n]]\left(\text{Zeta}\left[\frac{1}{2} - i\,s\right] + \text{Zeta}\left[\frac{1}{2} + i\,s\right]\right)$$

I Sin[a] (Zeta[1 / 2 - s I] - Zeta[1 / 2 + s I]) /. a → ArcTan[2 s] - Pi / 2 - s Log[n]

$$-i\,\text{Cos}[\text{ArcTan}[2\,s] - s\,\text{Log}[n]]\left(\text{Zeta}\left[\frac{1}{2} - i\,s\right] - \text{Zeta}\left[\frac{1}{2} + i\,s\right]\right)$$

$$\textbf{FullSimplify@Expand}\left[\text{Sin}[\text{ArcTan}[2\,s] - s\,\text{Log}[n]]\left(\text{Zeta}\left[\frac{1}{2} - i\,s\right] + \text{Zeta}\left[\frac{1}{2} + i\,s\right]\right)\right]$$

$$\text{Sin}[\text{ArcTan}[2\,s] - s\,\text{Log}[n]]\left(\text{Zeta}\left[\frac{1}{2} - i\,s\right] + \text{Zeta}\left[\frac{1}{2} + i\,s\right]\right)$$

FullSimplify[Sin[ArcTan[2 s] - s Log[n]]]

Sin[ArcTan[2 s] - s Log[n]]

Cos[a] (Zeta[1 / 2 - s I] + Zeta[1 / 2 + s I]) + I Sin[a] (Zeta[1 / 2 - s I] - Zeta[1 / 2 + s I]) /.
a → ArcTan[2 s] - Pi / 2 - s Log[n] /. n → 100. /. s → 44.

$-2.79703 + 0.\,i$

$$-i\,\text{Cos}[4.60517\,s - \text{ArcTan}[2\,s]]\left(\text{Zeta}\left[\frac{1}{2} - i\,s\right] - \text{Zeta}\left[\frac{1}{2} + i\,s\right]\right) -$$
$$\text{Sin}[4.60517\,s - \text{ArcTan}[2\,s]]\left(\text{Zeta}\left[\frac{1}{2} - i\,s\right] + \text{Zeta}\left[\frac{1}{2} + i\,s\right]\right)$$

TrigToExp[Cos[a] (Zeta[1 / 2 - s I] + Zeta[1 / 2 + s I]) +
I Sin[a] (Zeta[1 / 2 - s I] - Zeta[1 / 2 + s I]) /. a → ArcTan[2 s] - Pi / 2 - s Log[n]]

$$-i\,e^{\frac{1}{2}(-\text{Log}[1-2\,i\,s]+\text{Log}[1+2\,i\,s])}\,n^{-i\,s}\,\text{Zeta}\left[\frac{1}{2} - i\,s\right] + i\,e^{\frac{1}{2}(\text{Log}[1-2\,i\,s]-\text{Log}[1+2\,i\,s])}\,n^{i\,s}\,\text{Zeta}\left[\frac{1}{2} + i\,s\right]$$

$$\textbf{FullSimplify}\left[e^{\frac{1}{2}(\text{Log}[1-2\,i\,s]-\text{Log}[1+2\,i\,s])}\right]$$

$$\frac{\sqrt{1-2\,i\,s}}{\sqrt{1+2\,i\,s}}$$

$$-i\,e^{\frac{1}{2}(-\text{Log}[1-2\,i\,s]+\text{Log}[1+2\,i\,s])}\,n^{-i\,s}\,\text{Zeta}\left[\frac{1}{2} - i\,s\right] +$$
$$i\,e^{\frac{1}{2}(\text{Log}[1-2\,i\,s]-\text{Log}[1+2\,i\,s])}\,n^{i\,s}\,\text{Zeta}\left[\frac{1}{2} + i\,s\right] \text{ /. s → 44. /. n → 100.}$$

$-2.79703 + 0.\,i$

$$\textbf{FullSimplify}\left[e^{\frac{1}{2}(-\text{Log}[1-2\,i\,s]+\text{Log}[1+2\,i\,s])}\right]$$

$$\frac{\sqrt{1+2\,i\,s}}{\sqrt{1-2\,i\,s}}$$

$$-i\,\frac{\sqrt{1+2\,i\,s}}{\sqrt{1-2\,i\,s}}\,n^{-i\,s}\,\text{Zeta}\left[\frac{1}{2} - i\,s\right] + i\,\frac{\sqrt{1-2\,i\,s}}{\sqrt{1+2\,i\,s}}\,n^{i\,s}\,\text{Zeta}\left[\frac{1}{2} + i\,s\right] \text{ /. s → 44. /. n → 100.}$$

$-2.79703 + 8.67362 \times 10^{-19}\,i$

$$-i \frac{\sqrt{1/2+i s}}{\sqrt{1/2-i s}} n^{-i s}\,\text{Zeta}\left[\frac{1}{2}-i s\right]+i \frac{\sqrt{1/2-i s}}{\sqrt{1/2+i s}} n^{i s}\,\text{Zeta}\left[\frac{1}{2}+i s\right] \text{ /. } s\to 44.\text{ /. } n\to 100.$$

$$-2.79703+2.21177\times 10^{-16}\, i$$

```
FullSimplify[Cos[a] (Zeta[1/2 - s I] + Zeta[1/2 + s I]) /. a → ArcTan[2 s]]
```

$$\frac{\text{Zeta}\left[\frac{1}{2}-i s\right]+\text{Zeta}\left[\frac{1}{2}+i s\right]}{\sqrt{1+4 s^2}}$$

$$\frac{\text{Zeta}\left[\frac{1}{2}-i s\right]+\text{Zeta}\left[\frac{1}{2}+i s\right]}{\sqrt{1+4 s^2}} \text{ /. } s\to 3/2\, I$$

$$-\frac{i\left(-\frac{1}{12}+\frac{\pi^2}{6}\right)}{2\sqrt{2}}$$

```
Cos[ArcTan[2 s]]
```

$$\frac{1}{\sqrt{1+4 s^2}}$$

```
2 Sum[j^(-1/2) Cos[s Log[j] + a], {j, 1, n}] - n^(-1/2) Cos[s Log[n] + a] -
  4 n^(1/2) Cos[ArcTan[2 s]] Cos[s Log[n] + a - ArcTan[2 s]] /. a → -s Log[n] + ArcTan[2 s] + Pi/2
```

$$\frac{2 s}{\sqrt{n}\sqrt{1+4 s^2}}+2\sum_{j=1}^{n}-\frac{\text{Sin}[\text{ArcTan}[2 s]+s\,\text{Log}[j]-s\,\text{Log}[n]]}{\sqrt{j}}$$

```
scal0[n_, s_] := (1/2 + I s) Sum[j^(-1/2 + s I), {j, 1, n}] - 1/2 (1/2 + I s) n^(-1/2 + i s) - n^(1/2 + i s)
sz1[s_] := (1/2 + I s) Zeta[1/2 - s I]
scal01[n_, s_] := (1/2 - I s) Sum[j^(-1/2 - s I), {j, 1, n}] - 1/2 (1/2 - I s) n^(-1/2 - i s) - n^(1/2 - i s)
sz1i[s_] := (1/2 - I s) Zeta[1/2 + s I]
sba[n_, s_, a_] :=
 E^(I a) ((1/2 + I s) Sum[j^(-1/2 + s I), {j, 1, n}] - 1/2 (1/2 + I s) n^(-1/2 + i s) - n^(1/2 + i s)) +
  E^(-I a) ((1/2 - I s) Sum[j^(-1/2 - s I), {j, 1, n}] - 1/2 (1/2 - I s) n^(-1/2 - i s) - n^(1/2 - i s))
sra[s_, a_] := E^(I a) ((1/2 + I s) Zeta[1/2 - s I]) + E^(-I a) ((1/2 - I s) Zeta[1/2 + s I])
sba2[n_, s_, a_] :=
 Sum[E^(I a) (1/2 + I s) j^(-1/2 + s I) + (1/2 - I s) E^(-I a) j^(-1/2 - s I), {j, 1, n}] -
  1/2 (1/2 + I s) E^(I a) n^(-1/2 + i s) - E^(I a) n^(1/2 + i s) - 1/2 (1/2 - I s) E^(-I a) n^(-1/2 - i s) - E^(-I a) n^(1/2 - i s)
sba3[n_, s_, a_] := Sum[E^(I a) (1/2 + I s) j^(-1/2 + s I) + (1/2 - I s) E^(-I a) j^(-1/2 - s I),
    {j, 1, n}] - 1/2 (1/2 + I s) E^(I a) n^(-1/2 + i s) -
  1/2 (1/2 - I s) E^(-I a) n^(-1/2 - i s) - E^(I a) n^(1/2 + i s) - E^(-I a) n^(1/2 - i s)
sba4[n_, s_, a_] := Sum[j^(-1/2) (E^(I a) (1/2 + I s) j^(s I) + (1/2 - I s) E^(-I a) j^(-s I)),
    {j, 1, n}] - 1/2 n^(-1/2) ((1/2 + I s) E^(I a) n^(i s) + (1/2 - I s) E^(-I a) n^(-i s)) -
```

$$n^\wedge(1/2)\,\left(E^\wedge(I\,a)\,n^{i\,s}+E^\wedge(-I\,a)\,n^{-i\,s}\right)$$

$$\text{sba5}[n\_,s\_,a\_]:=\text{Sum}[j^\wedge(-1/2)\,(E^\wedge(I\,a)\,(1/2+I\,s)\,j^\wedge(s\,I)+(1/2-I\,s)\,E^\wedge(-I\,a)\,j^\wedge(-s\,I)),\{j,1,n\}]-\frac{1}{2}\,n^\wedge(-1/2)\,\left((1/2+I\,s)\,E^\wedge(I\,a)\,n^{i\,s}+(1/2-I\,s)\,E^\wedge(-I\,a)\,n^{-i\,s}\right)-n^\wedge(1/2)\,\left(E^\wedge(I\,a)\,E^{i\,s\,\text{Log}[n]}+E^\wedge(-I\,a)\,E^{-i\,s\,\text{Log}[n]}\right)$$

$$\text{sba6}[n\_,s\_,a\_]:=\text{Sum}[j^\wedge(-1/2)\,(E^\wedge(I\,a)\,(1/2+I\,s)\,j^\wedge(s\,I)+(1/2-I\,s)\,E^\wedge(-I\,a)\,j^\wedge(-s\,I)),\{j,1,n\}]-\frac{1}{2}\,n^\wedge(-1/2)\,\left((1/2+I\,s)\,E^\wedge(I\,a)\,n^{i\,s}+(1/2-I\,s)\,E^\wedge(-I\,a)\,n^{-i\,s}\right)-n^\wedge(1/2)\,2\,\text{Cos}[s\,\text{Log}[n]+a]$$

$$\text{sba7}[n\_,s\_,a\_]:=\text{Sum}[j^\wedge(-1/2)\,((1/2+I\,s)\,E^\wedge(s\,I\,\text{Log}[j]+I\,a)+(1/2-I\,s)\,E^\wedge(-s\,I\,\text{Log}[j]-I\,a)),\{j,1,n\}]-\frac{1}{2}\,n^\wedge(-1/2)\,\left((1/2+I\,s)\,E^{i\,s\,\text{Log}[n]+a\,I}+(1/2-I\,s)\,E^{-i\,s\,\text{Log}[n]-I\,a}\right)-n^\wedge(1/2)\,2\,\text{Cos}[s\,\text{Log}[n]+a]$$

$$\text{sba8}[n\_,s\_,a\_]:=(1/2+I\,s)^\wedge(1/2)\,(1/2-I\,s)^\wedge(1/2)\,\text{Sum}[j^\wedge(-1/2)\,((1/2+I\,s)/(1/2+I\,s)^\wedge(1/2)/(1/2-I\,s)^\wedge(1/2)\,E^\wedge(s\,I\,\text{Log}[j]+I\,a)+(1/2-I\,s)/(1/2+I\,s)^\wedge(1/2)/(1/2-I\,s)^\wedge(1/2)\,E^\wedge(-s\,I\,\text{Log}[j]-I\,a)),\{j,1,n\}]-\frac{1}{2}\,n^\wedge(-1/2)\,\left((1/2+I\,s)\,E^{i\,s\,\text{Log}[n]+a\,I}+(1/2-I\,s)\,E^{-i\,s\,\text{Log}[n]-I\,a}\right)-n^\wedge(1/2)\,2\,\text{Cos}[s\,\text{Log}[n]+a]$$

$$\text{sba9}[n\_,s\_,a\_]:=(1/2+I\,s)^\wedge(1/2)\,(1/2-I\,s)^\wedge(1/2)\,\text{Sum}[j^\wedge(-1/2)\,((1/2+I\,s)/(1/2+I\,s)^\wedge(1/2)/(1/2-I\,s)^\wedge(1/2)\,E^\wedge(s\,I\,\text{Log}[j]+I\,a)+(1/2-I\,s)/(1/2+I\,s)^\wedge(1/2)/(1/2-I\,s)^\wedge(1/2)\,E^\wedge(-s\,I\,\text{Log}[j]-I\,a)),\{j,1,n\}]-\frac{1}{2}\,(1/2+I\,s)^\wedge(1/2)\,(1/2-I\,s)^\wedge(1/2)\,n^\wedge(-1/2)\,\left((1/2+I\,s)/(1/2+I\,s)^\wedge(1/2)/(1/2-I\,s)^\wedge(1/2)\,E^{i\,s\,\text{Log}[n]+a\,I}+(1/2-I\,s)/(1/2+I\,s)^\wedge(1/2)/(1/2-I\,s)^\wedge(1/2)\,E^{-i\,s\,\text{Log}[n]-I\,a}\right)-n^\wedge(1/2)\,2\,\text{Cos}[s\,\text{Log}[n]+a]$$

$$\text{sba10}[n\_,s\_,a\_]:=(1/2+I\,s)^\wedge(1/2)\,(1/2-I\,s)^\wedge(1/2)\,\text{Sum}\left[j^\wedge(-1/2)\left(\frac{\sqrt{\frac{1}{2}+i\,s}}{\sqrt{\frac{1}{2}-i\,s}}\,E^\wedge(s\,I\,\text{Log}[j]+I\,a)+\frac{\sqrt{\frac{1}{2}-i\,s}}{\sqrt{\frac{1}{2}+i\,s}}\,E^\wedge(-s\,I\,\text{Log}[j]-I\,a)\right),\{j,1,n\}\right]-\frac{1}{2}\,(1/2+I\,s)^\wedge(1/2)\,(1/2-I\,s)^\wedge(1/2)\,n^\wedge(-1/2)\left(\frac{\sqrt{\frac{1}{2}+i\,s}}{\sqrt{\frac{1}{2}-i\,s}}\,E^{i\,s\,\text{Log}[n]+a\,I}+\frac{\sqrt{\frac{1}{2}-i\,s}}{\sqrt{\frac{1}{2}+i\,s}}\,E^{-i\,s\,\text{Log}[n]-I\,a}\right)-n^\wedge(1/2)\,2\,\text{Cos}[s\,\text{Log}[n]+a]$$

$$\text{sba11}[n\_,s\_,a\_]:=\frac{1}{2}\sqrt{1+4\,s^2}\,\text{Sum}[j^\wedge(-1/2)\,(E^\wedge(s\,I\,\text{Log}[j]+I\,a+i\,\text{ArcTan}[2\,s])+E^\wedge(-s\,I\,\text{Log}[j]-I\,a-i\,\text{ArcTan}[2\,s])),\{j,1,n\}]-\frac{1}{2}\,(1/2+I\,s)^\wedge(1/2)\,(1/2-I\,s)^\wedge(1/2)\,n^\wedge(-1/2)\,\left(E^{i\,s\,\text{Log}[n]+a\,I+i\,\text{ArcTan}[2\,s]}+E^{-i\,s\,\text{Log}[n]-I\,a-i\,\text{ArcTan}[2\,s]}\right)-n^\wedge(1/2)\,2\,\text{Cos}[s\,\text{Log}[n]+a]$$

$$\text{sba12}[n\_,s\_,a\_]:=\sqrt{1+4\,s^2}\,\text{Sum}[j^\wedge(-1/2)\,(\text{Cos}[a+\text{ArcTan}[2\,s]+s\,\text{Log}[j]]),\{j,1,n\}]-$$

$$\frac{1}{2}\sqrt{1+4s^2}\ \texttt{Cos[a + ArcTan[2 s] + s Log[n]] n^(-1/2) - 2 Cos[s Log[n] + a] n^(1/2)}$$

```
sra[s_, a_] := E^(I a) ((1/2 + I s) Zeta[1/2 - s I]) + E^(-I a) ((1/2 - I s) Zeta[1/2 + s I])
```

$$\texttt{srb[s\_, a\_] :=}\ \frac{1}{2}\sqrt{1+4s^2}\ \texttt{(E^(I a) ((1/2 + I s)^(1/2) / (1/2 - I s)^(1/2) Zeta[1/2 - s I]) + E^(-I a) ((1/2 - I s)^(1/2) / (1/2 + I s)^(1/2) Zeta[1/2 + s I]))}$$

$$\texttt{src[s\_, a\_] :=}\ \frac{1}{2}\sqrt{1+4s^2}\ \texttt{(E^(I a + i ArcTan[2 s]) Zeta[1/2 - s I] + E^(-I a - i ArcTan[2 s]) Zeta[1/2 + s I])}$$

$$\texttt{sba13[n\_, s\_, a\_] :=}\ \sqrt{1+4s^2}\ \texttt{Sum[j^(-1/2) (Cos[a + ArcTan[2 s] + s Log[j]]), \{j, 1, n\}] -}\ \frac{1}{2}\sqrt{1+4s^2}\ \texttt{Cos[a + ArcTan[2 s] + s Log[n]] n^(-1/2) - 2 Cos[s Log[n] + a] n^(1/2)}$$

```
srd[s_, a_] := E^(I a + i ArcTan[2 s]) Zeta[1/2 - s I] + E^(-I a - i ArcTan[2 s]) Zeta[1/2 + s I]
sre[s_, a_] := E^(I a + i ArcTan[2 s]) Zeta[1/2 - s I] + E^(-I a - i ArcTan[2 s]) Zeta[1/2 + s I]
```

$$\texttt{sba14[n\_, s\_, a\_] := 2 Sum[j^(-1/2) (Cos[a + ArcTan[2 s] + s Log[j]]), \{j, 1, n\}] - Cos[a + ArcTan[2 s] + s Log[n]] n^(-1/2) - 4} \Big/ \sqrt{1+4s^2}\ \texttt{Cos[s Log[n] + a] n^(1/2)}$$

```
sba15[n_, s_, a_] := 2 Sum[j^(-1/2) (Cos[a + ArcTan[2 s] + s Log[j]]), {j, 1, n}] -
  4 Cos[ArcTan[2 s]] Cos[s Log[n] + a] n^(1/2) -  Cos[a + ArcTan[2 s] + s Log[n]] n^(-1/2)

sba15[1000., 23 + .2 I, 2.2]
```

$-2.61179 + 0.150694\, i$

```
sre[23 + .2 I, 2.2]
```

$-2.61196 + 0.150863\, i$

```
FullSimplify[TrigToExp[E^(I a + i ArcTan[2 s])]]
```

$$\frac{\sqrt{1+2 i s}}{\sqrt{1-2 i s}}$$

```
FullSimplify[e^(i a)]
```

$e^{i a}$

```
Cos[ArcTan[2 s]]
```

$$\frac{1}{\sqrt{1+4s^2}}$$

```
ExpToTrig[E^(I a + i ArcTan[2 s])]
```

Cos[a + ArcTan[2 s]] + i Sin[a + ArcTan[2 s]]

```
ExpToTrig[E^(-I a - i ArcTan[2 s])]
```

Cos[a + ArcTan[2 s]] - i Sin[a + ArcTan[2 s]]

```
Expand[(2 Sum[j^(-1/2) (Cos[a + ArcTan[2 s] + s Log[j]]), {j, 1, n}] -
     4 Cos[ArcTan[2 s]] Cos[s Log[n] + a] n^(1/2) -  Cos[a + ArcTan[2 s] + s Log[n]] n^(-1/2))
   ((1 + 4 s^2)^(1/2)) / (2) /. a → -s Log[n] - Pi/2]
```

$$-\frac{s}{\sqrt{n}} + \sqrt{1+4s^2}\ \sum_{j=1}^{n} \frac{\text{Sin}[\text{ArcTan}[2 s] + s\,\text{Log}[j] - s\,\text{Log}[n]]}{\sqrt{j}}$$

```
FullSimplify[E^(I a + i ArcTan[2 s]) / Cos[ArcTan[2 s]]] / 2
```

$$\frac{1}{2} e^{i a} (1 + 2 i s)$$

```
FullSimplify[E^(I a + i ArcTan[2 s]) Zeta[1 / 2 - s I] +
   E^(-I a - i ArcTan[2 s]) Zeta[1 / 2 + s I] /. a → -s Log[n]]
```

$$e^{i \text{ArcTan}[2 s]} n^{-i s} \text{Zeta}\left[\frac{1}{2} - i s\right] + e^{-i \text{ArcTan}[2 s]} n^{i s} \text{Zeta}\left[\frac{1}{2} + i s\right]$$

```
FullSimplify@TrigToExp[e^(i ArcTan[2 s])]
```

$$\frac{\sqrt{1 + 2 i s}}{\sqrt{1 - 2 i s}}$$

```
E^(I a + i ArcTan[2 s]) Zeta[1 / 2 - s I] + E^(-I a - i ArcTan[2 s]) Zeta[1 / 2 + s I] /.
 a → -s Log[n] - Pi / 2
```

$$e^{i \text{ArcTan}[2 s] + i \left(-\frac{\pi}{2} - s \text{Log}[n]\right)} \text{Zeta}\left[\frac{1}{2} - i s\right] + e^{-i \text{ArcTan}[2 s] - i \left(-\frac{\pi}{2} - s \text{Log}[n]\right)} \text{Zeta}\left[\frac{1}{2} + i s\right]$$

$$\text{FullSimplify}\left[\text{TrigToExp}\left[i\, e^{-i \text{ArcTan}[2 s]}\right]\right]$$

$$\frac{i \sqrt{1 - 2 i s}}{\sqrt{1 + 2 i s}}$$

$$\text{FullSimplify}\left[\text{TrigToExp}\left[e^{-i \text{ArcTan}[2 s] - i \left(-\frac{\pi}{2} - s \text{Log}[n]\right)} \text{Zeta}\left[\frac{1}{2} + i s\right]\right]\right]$$

$$\frac{i\, n^{i s} \sqrt{1 - 2 i s}\, \text{Zeta}\left[\frac{1}{2} + i s\right]}{\sqrt{1 + 2 i s}}$$