```
bernb[s_] := If[ s == 1, 1 / 2, BernoulliB[s]]
pp[n_, s_] := Sum[ Binomial[-s, j] bernb[-s - j] n^j, {j, 0, -s}]
harm[n_, s_] := (pp[n, s - 1] - pp[m, s - 1]) / (1 - s) /. m → 0
zet[s_] := -bernb[(1 - s)] / (1 - s)
```

```
Expand[harm[n, -1]]
```

$$\frac{n}{2}+\frac{n^2}{2}$$

```
zet[-1]
```

$$-\frac{1}{12}$$

```
Expand[Sum[ 1 / j^-1, {j, 1, n}]]
```

$$\frac{n}{2}+\frac{n^2}{2}$$

```
zd1[n_, s_] := zet[s] - harm[n, s]
zd2[n_, s_] := (s - 1) (zet[s] - harm[n, s])
zd3[n_, s_] := n^(s - 1) ((s - 1) (zet[s] - harm[n, s]))
```

```
Table[{Expand[zd2[n, -s]], Expand[zd3[n, -s]]}, {s, 0, 6}] // TableForm
```

$\frac{1}{2}+n$	$1+\frac{1}{2\,n}$
$\frac{1}{6}+n+n^2$	$1+\frac{1}{6\,n^2}+\frac{1}{n}$
$\frac{n}{2}+\frac{3\,n^2}{2}+n^3$	$1+\frac{1}{2\,n^2}+\frac{3}{2\,n}$
$-\frac{1}{30}+n^2+2\,n^3+n^4$	$1-\frac{1}{30\,n^4}+\frac{1}{n^2}+\frac{2}{n}$
$-\frac{n}{6}+\frac{5\,n^3}{3}+\frac{5\,n^4}{2}+n^5$	$1-\frac{1}{6\,n^4}+\frac{5}{3\,n^2}+\frac{5}{2\,n}$
$\frac{1}{42}-\frac{n^2}{2}+\frac{5\,n^4}{2}+3\,n^5+n^6$	$1+\frac{1}{42\,n^6}-\frac{1}{2\,n^4}+\frac{5}{2\,n^2}+\frac{3}{n}$
$\frac{n}{6}-\frac{7\,n^3}{6}+\frac{7\,n^5}{2}+\frac{7\,n^6}{2}+n^7$	$1+\frac{1}{6\,n^6}-\frac{7}{6\,n^4}+\frac{7}{2\,n^2}+\frac{7}{2\,n}$

```
Table[{Expand[zd3[n, -s]]}, {s, 0, 10}] // TableForm
```

$$1+\frac{1}{2\,n}$$
$$1+\frac{1}{6\,n^2}+\frac{1}{n}$$
$$1+\frac{1}{2\,n^2}+\frac{3}{2\,n}$$
$$1-\frac{1}{30\,n^4}+\frac{1}{n^2}+\frac{2}{n}$$
$$1-\frac{1}{6\,n^4}+\frac{5}{3\,n^2}+\frac{5}{2\,n}$$
$$1+\frac{1}{42\,n^6}-\frac{1}{2\,n^4}+\frac{5}{2\,n^2}+\frac{3}{n}$$
$$1+\frac{1}{6\,n^6}-\frac{7}{6\,n^4}+\frac{7}{2\,n^2}+\frac{7}{2\,n}$$
$$1-\frac{1}{30\,n^8}+\frac{2}{3\,n^6}-\frac{7}{3\,n^4}+\frac{14}{3\,n^2}+\frac{4}{n}$$
$$1-\frac{3}{10\,n^8}+\frac{2}{n^6}-\frac{21}{5\,n^4}+\frac{6}{n^2}+\frac{9}{2\,n}$$
$$1+\frac{5}{66\,n^{10}}-\frac{3}{2\,n^8}+\frac{5}{n^6}-\frac{7}{n^4}+\frac{15}{2\,n^2}+\frac{5}{n}$$
$$1+\frac{5}{6\,n^{10}}-\frac{11}{2\,n^8}+\frac{11}{n^6}-\frac{11}{n^4}+\frac{55}{6\,n^2}+\frac{11}{2\,n}$$

```
bernz[s_] := If[s == 0, 1, -s Zeta[(1 - s)]]
ppz[n_, s_] := Sum[ Binomial[-s, j] bernz[-s - j] n^j, {j, 0, -s}]
ppzt[n_, s_, t_] := Sum[ Binomial[-s, j] bernz[-s - j] n^j, {j, 0, t}]
harmz[n_, s_] := (ppz[n, s - 1] - ppz[m, s - 1]) / (1 - s) /. m → 0
harmzt[n_, s_, t_] := (ppzt[n, s - 1, t] - ppzt[m, s - 1, t]) / (1 - s) /. m → 0
harmzt2[n_, s_, t_] := harmzt[n, s, t] - n^(1 - s) / (1 - s)
zetz[s_] := -bernz[(1 - s)] / (1 - s)
```

```
Limit[-s Zeta[(1 - s)], s → 0]
```

1

```
N@Expand[n^(-2.5 - 1) (-2.5 - 1) (zetz[-2.5] - harmzt[n, -2.5, 25])] /. n → 20
```

-5.04136×10^{23}

```
(s - 1) n^(s - 1) Zeta[s, n + 1] /. n → 100 000 000 000 /. s → -20.0
```

1.

```
(s - 1) n^(s - 1) (Zeta[s] - HarmonicNumber[n, s]) /. n → 1 000 000 000 000 000 000 000 /. s → -1.1
```

1.

```
FullSimplify[Sum[ Binomial[-s, j] (x - 1)^(-s - j), {j, 0, Infinity}]]
```

$(-1 + x)^{-s} \left(\frac{x}{-1 + x}\right)^{-s}$

```
bin[z_, k_] := bin[z, k] = Product[z - j, {j, 0, k - 1}] / k!
pa[n_, s_, t_] :=  Expand[Sum[bin[-s, j] n^(-s - j), {j, 0, t}]]
pas[n_, s_, t_] := Sum[ pa[j, s, t], {j, 1, n}]
px[n_, r_, t_] := Table[bin[-r, j] n^(-r - j), {j, 0, t}]
```

```
Expand[n^s pa[n, s, 3] /. s → -3 / 2]
```

$1 - \frac{1}{16 n^3} + \frac{3}{8 n^2} + \frac{3}{2 n}$

```
Sum[ j^-.5, {j, 1, 100}]
```

18.5896

```
101^.5
```

10.0499

```
pas[n, -2, 12]
```

3 n

```
rr[n_, s_, t_] := 1 + Sum[ Binomial[-s, k] HarmonicNumber[n - 1, s + k], {k, 0, t}]
rr2[n_, s_, t_] := 1 - n^-s + Sum[ Binomial[-s, k] HarmonicNumber[n - 1, s + k], {k, 1, t}]
rri[s_, t_] := 1 + Sum[ Binomial[-s, k] Zeta[s + k], {k, 0, t}]
rri2[s_, t_] := Sum[ Binomial[-s, k] Zeta[s + k], {k, 1, t}]
```

```
rr[100, -.5, 4000]
```

671.463

```
Sum[1 / j ^ (-.5), {j, 1, 100}]
```

671.463

```
Sum[ Binomial[-s, k] Zeta[s + k], {k, 0, Infinity}]
```

$$\sum_{k=0}^{\infty} \text{Binomial}[-s, k]\ \text{Zeta}[k+s]$$

```
rri[2, 8000]
```

$Aborted

```
Zeta[2]
```

0.403783 − 0.0172953 i

```
Binomial[-s, 0]
```

1

```
rri2[-.5 + 2 I, 8000]
```

−0.999999 + 0.0000132855 i

```
rr2[100, -.5, 4000]
```

-5.57486×10^{-7}

```
rr3[n_, s_, t_] :=
  (1 + HarmonicNumber[n - 1, s] + Sum[ Binomial[-s, k] HarmonicNumber[n - 1, s + k], {k, 1, t}]) -
   Sum[j ^ -s, {j, 1, n}]
rr4[n_, s_, t_] := (1 + Sum[ Binomial[-s, k] HarmonicNumber[n - 1, s + k], {k, 1, t}]) +
   HarmonicNumber[n - 1, s] - HarmonicNumber[n, s]
rr5[n_, s_, t_] := (1 + Sum[ Binomial[-s, k] HarmonicNumber[n - 1, s + k], {k, 1, t}]) - n^-s
```

```
rr5[100, -.5, 4000]
```

-5.57486×10^{-7}

```
FullSimplify[HarmonicNumber[n - 1, s] - HarmonicNumber[n, s]]
```

$-n^{-s}$

```
BernoulliB[3, n]
```

$$\frac{n}{2} - \frac{3 n^2}{2} + n^3$$

```
bernz[s_] := If[s == 0, 1, -s Zeta[(1 - s)]]
ppz[n_, s_] := Sum[ Binomial[-s, j] bernz[-s - j] n^j, {j, 0, -s}]
ppzx[n_, s_] := Sum[ Binomial[-s, -s - j] bernz[j] n^ (-s - j), {j, 0, 20}]
ppzx2[n_, s_] := Sum[ Binomial[-s, j] bernz[j] n^ (-s - j), {j, 0, 20}]
ppzxt[n_, s_] := Table[ Binomial[-s, -s - j] bernz[j] n^ (-s - j), {j, 0, 20}]
ppzxt2[n_, s_] := Table[ Binomial[-s, j] bernz[-s - j] n^ (-s - j), {j, 0, 20}]
ppzxt2x[n_, s_] := Table[ bernz[j], {j, 0, 20}]
```

```
ppzxt2x[n, -3]
```

$$\left\{1, \frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, -\frac{1}{30}, 0, \frac{5}{66}, 0, -\frac{691}{2730}, 0, \frac{7}{6}, 0, -\frac{3617}{510}, 0, \frac{43867}{798}, 0, -\frac{174611}{330}\right\}$$

```
N[ppzx2[n, -5 / 2] /. n → 10]
```

356.744

```
N@Sum[ j^(-5/2), {j, 1, 10}]
```

```
1.32192
```

```
zd3f[n_, s_] := ((((s-1) n^(s-1) (Zeta[s] - Sum[j^-s, {j, 1, n}]) - 1) n -
        Binomial[1-s, 1] (-BernoulliB[1])) n -
     Binomial[1-s, 2] BernoulliB[2]) n^2 - Binomial[1-s, 4] BernoulliB[4]
```

```
Table[{s, N@Expand[zd3f[1000, -s]]}, {s, -3, 6, 1/2}] // TableForm
```

Infinity::indet : Indeterminate expression 0 ComplexInfinity encountered. ≫

-3	0.
$-\frac{5}{2}$	-4.75
-2	0.
$-\frac{3}{2}$	0.0090332
-1	Indeterminate
$-\frac{1}{2}$	0.00012207
0	0.
$\frac{1}{2}$	0.
1	0.
$\frac{3}{2}$	-0.000183105
2	0.
$\frac{5}{2}$	-0.00012207
3	0.
$\frac{7}{2}$	-0.00012207
4	0.
$\frac{9}{2}$	-0.000244141
5	2.38095×10^{-8}
$\frac{11}{2}$	0.00012207
6	1.66667×10^{-7}

```
Table[N@ Binomial[s+1, 4] BernoulliB[4], {s, 0, 6, 1/2}] // TableForm
```

```
0.
-0.00078125
0.
0.00130208
0.
-0.00911458
-0.0333333
-0.0820313
-0.166667
-0.300781
-0.5
-0.782031
-1.16667
```

```
BernoulliB[1]
```

$-\frac{1}{2}$

```
Binomial[s-1, 1]
```

$-1 + s$

```
Binomial[1 - s, 1]

1 - s

tt[n_, s_, t_] := Zeta[s] + 1 / (1 - s) Sum[ Binomial[1 - s, k] bernb[k] n^(-s - k + 1), {k, 0, t}]

Expand@tt[10 000 000, 1.0001, 8]

16.6823

Sum[j^(-1.0001), {j, 1, 10 000 000.0}]

16.6823

zd3f2[n_, s_] :=
 ((((-s - 1) n^(-s - 1) (Zeta[-s] - Sum[j^s, {j, 1, n}]) - 1) n - Binomial[1 + s, 1]
          (-BernoulliB[1])) n -
     Binomial[1 + s, 2] BernoulliB[2]) n^2 - Binomial[1 + s, 4] BernoulliB[4]

Table[{s, N@Expand[zd3f2[1000, s]]}, {s, 0, 6, 1 / 2}] // TableForm
```

0	0.
$\frac{1}{2}$	0.
1	0.
$\frac{3}{2}$	-0.000183105
2	0.
$\frac{5}{2}$	-0.00012207
3	0.
$\frac{7}{2}$	-0.00012207
4	0.
$\frac{9}{2}$	-0.000244141
5	2.38095×10^{-8}
$\frac{11}{2}$	0.00012207
6	1.66667×10^{-7}

```
tta[n_, s_, t_] :=
 Zeta[s] + Sum[(1 / (1 - s)) Binomial[1 - s, k] bernb[k] n^(-s - k + 1), {k, 0, t}]
tta2[n_, s_, t_] := Table[(1 / (1 - s)) bin[1 - s, k] bernb[k] n^(-s - k + 1), {k, 0, t}]
ttb[n_, s_, t_] :=
 Sum[ j^-s, {j, 1, n}] - Sum[(1 / (1 - s)) bin[1 - s, k] bernb[k] n^(-s - k + 1), {k, 0, t}]

tta2[10^9, -1.5, 3]
```

$\{1.26491 \times 10^{22}, 1.58114 \times 10^{13}, 3952.85, 0.\}$

```
ttb[100 000, -.5, 3]

-0.207886

Zeta[-.5]

-0.207886

tta2[n, s, 8]
```

$$\left\{\frac{n^{1-s}}{1-s}, \frac{n^{-s}}{2}, -\frac{1}{12} n^{-1-s} s, 0, \frac{1}{720} n^{-3-s} (-2-s)(-1-s) s, 0, -\frac{n^{-5-s}(-4-s)(-3-s)(-2-s)(-1-s)s}{30240}, 0, \frac{n^{-7-s}(-6-s)(-5-s)(-4-s)(-3-s)(-2-s)(-1-s)s}{1209600}\right\}$$

Floor[1 - s] /. s → .5

0

Table[FullSimplify[bin[1 - s, k] / (1 - s)], {k, 1, 6}] // TableForm

1
$-\frac{s}{2}$
$\frac{1}{6} s (1+s)$
$-\frac{1}{24} s (1+s)(2+s)$
$\frac{1}{120} s (1+s)(2+s)(3+s)$
$-\frac{1}{720} s (1+s)(2+s)(3+s)(4+s)$

bin[-s, 2]

$$-\frac{1}{2}(-1-s) s$$

ee[n_, s_] := Sum[j^-s, {j, 1, n}] - n^(1 - s) / (1 - s) - (n^-s) / 2 + (s n^(-s - 1)) / 12
eet[n_, s_] := {Sum[j^-s, {j, 1, n}], -n^(1 - s) / (1 - s), -(n^-s) / 2, (s n^(-s - 1)) / 12}

ee[1000, -2.5 + 3 I]

0.0678485 + 0.132484 i

Zeta[-2.5 + 3 I]

0.0687637 + 0.13398 i

tta[n_, s_, t_] :=
Zeta[s] + Sum[(1 / (1 - s)) Binomial[1 - s, k] bernb[k] n^(-s - k + 1), {k, 0, t}]
tta2[n_, s_, t_] := Table[(1 / (1 - s)) bin[1 - s, k] bernb[k] n^(-s - k + 1), {k, 0, t}]

tta2[n, -3 / 2, 5]

$$\left\{\frac{2 n^{5/2}}{5}, \frac{n^{3/2}}{2}, \frac{\sqrt{n}}{8}, 0, \frac{1}{1920 n^{3/2}}, 0\right\}$$

tta2[40, -3 / 2, 5]

$$\left\{1280 \sqrt{10}, 40 \sqrt{10}, \frac{\sqrt{\frac{5}{2}}}{2}, 0, \frac{1}{153600 \sqrt{10}}, 0\right\}$$

N@Sum[j^(3 / 2), {j, 1, 40}]

4174.97

N@tta[1000, -3 / 2, 4]

1.26649×10^{7}

Zeta[-1.5]

-0.0254852

4174.971595635413`

4174.971597694072`

ee[n_, s_] := Sum[j^-s, {j, 1, n}] - n^(1 - s) / (1 - s) -
(n^-s) / 2 + (s n^(-s - 1)) / 12 - $\frac{1}{720} n^{-3-s}$ **(-2 - s) (-1 - s) s**

eet[n_, s_] := {Sum[j^-s, {j, 1, n}], -n^(1 - s) / (1 - s),
-(n^-s) / 2, (s n^(-s - 1)) / 12, -$\frac{1}{720} n^{-3-s}$ **(-2 - s) (-1 - s) s}**

eet2[n_, s_] := {(1 - s) Sum[j^-s, {j, 1, n}], -n^(1 - s), -(n^-s) (1 - s) / 2,
(s (1 - s) n^(-s - 1)) / 12, -$\frac{1}{720} n^{-3-s}$ **(-2 - s) (-1 - s) s}**

N@ee[1 000 000, 1 / 2] - N@Zeta[1 / 2]

3.3995×10^{-13}

N@Zeta[1 / 2]

-1.46035

-1.460354508809246`

-1.460354508809586`

N@eet[1 000 000, 1 / 2]

{1998.54, -2000., -0.0005, 4.16667×10^{-11}}

N@ee[1 000 000, -1 / 2] - N@Zeta[-1 / 2]

1.84767×10^{-7}

N@ee[100 000, -3 / 2]

-0.0247803

N@Zeta[-3 / 2]

-0.0254852

N@eet[1 000 000, -3 / 2]

{4.00001×10^{14}, $-4.\times10^{14}$, $-5.\times10^{8}$, -125.}

Integrate[j^-s, {j, 0, n}]

ConditionalExpression[$-\frac{n^{1-s}}{-1+s}$, Re[s] < 1]

Limit[eet2[n, s], s → 1]

{0, -1, 0, 0}

N@ee[1 000 000, ZetaZero[1]]

```
7.49623×10^-13 - 3.90799×10^-13 i
```

```
N@eet[1 000 000, ZetaZero[1]]
```

```
{36.0516 + 60.8217 i, -36.0512 - 60.8219 i, -0.000438863 + 0.000239581 i,
 6.00974×10^-10 + 1.0139×10^-9 i, 2.919×10^-21 + 2.74646×10^-21 i}
```

```
ew[n_, s_] := s / (1 - s) n Zeta[1 + s, n + 1]
ew2[n_, s_] := n^(1 - s) / (1 - s)
```

```
ew[n, .5 + I] - ew2[n, .5 + I] /. n → 100 000 000 000
```

```
6.84435×10^-7 - 1.42457×10^-6 i
```

```
tr[n_, s_, x_] := Sum[ j^-s, {j, 1, Floor[n / x]}]
tr2[n_, s_, x_] := Sum[ (j^-s - (j + Floor[n / x]) ^-s), {j, 1, Infinity}]
tr3[n_, s_, x_] := -HurwitzZeta[s, 1 + Floor[n/x]] + Zeta[s]
```

```
N[tr[4697, .3, 3] - tr3[4697, .3, 3]]
```

```
-4.54747×10^-13
```

```
N@tr2[97, 1, 3]
```

```
4.0687
```

```
D[x^(1 - s) (j^-s - (j + Floor[n / x]) ^-s), x] /. x → 1
```

```
(1 - s) (j^-s - (j + Floor[n])^-s) - n s (j + Floor[n])^(-1-s) Floor′[n]
```

```
Sum[ (j^-s - (j + Floor[n / x]) ^-s), {j, 1, Infinity}]
```

```
-HurwitzZeta[s, 1 + Floor[n/x]] + Zeta[s]
```

```
D[-HurwitzZeta[s, 1 + Floor[n/x]] + Zeta[s], x] /. x → 1
```

```
-n s HurwitzZeta[1 + s, 1 + Floor[n]] Floor′[n]
```

```
N[-n s HurwitzZeta[1 + s, 1 + Floor[n]] Floor′[n] /. s → 3 /. n → 10]
```

```
-0.108307
```

```
N[-n s HurwitzZeta[1 + s, 1 + n] /. s → 3 /. n → 10]
```

```
-0.00859951
```

```
Floor′[19]
```

```
Floor′[19]
```

```
D[Sum[j^-s, {j, 1, n}] - x^(1 - s) Sum[ j^-s, {j, 1, Floor[n / x]}], x] /. x → 1
```

```
-(1 - s) HarmonicNumber[Floor[n], s] +
 n s (-HarmonicNumber[Floor[n], 1 + s] + Zeta[1 + s]) Floor′[n]
```

```
N@HarmonicNumber[10 / 3, 2]
```

```
1.38551
```

```
Zeta[2] - Zeta[2, 10.0 / 3 + 1]
```

```
1.38551
```