```
al[s_] := -2 Pi^(s / 2) / (s (1 - s) Gamma[s / 2])
alo[s_] := -1 / ((1 / 2) s (1 - s) Pi^(-s / 2) Gamma[s / 2])
al2[s_] := -((1 / 2) s (1 - s) Pi^(-s / 2) Gamma[s / 2])
```

$$\text{ssosub}[n\_,\ s\_] := \frac{(1-s)\ n\text{^}s}{\frac{1}{1/2\ (1-s)\ \pi^{-\frac{1-s}{2}}\ \text{Gamma}\left[\frac{1-s}{2}\right]}\ n\text{^}(1-s) - \frac{1}{1/2\ s\ \pi^{-s/2}\ \text{Gamma}\left[\frac{s}{2}\right]}\ n\text{^}s}\ \text{HarmonicNumber}[n,\ s]$$

$$\text{ssosub2}[n\_,\ s\_] := \frac{(1-s)\ n\text{^}s}{(1-s)\ \text{al}[s]\ n\text{^}s - s\ \text{al}[1-s]\ n\text{^}(1-s)}\ \text{HarmonicNumber}[n,\ s]$$

$$\text{ssosub3}[n\_,\ s\_] := \frac{1}{\text{al}[s] - \text{al}[1-s]\ s\ /\ (1-s)\ n\text{^}(1-2\,s)}\ \text{HarmonicNumber}[n,\ s]$$

```
sso10c[n_, s_] := ssosub3[n, s] + ssosub3[n, 1 - s]
```

$$\text{ssosub4}[n\_,\ s\_] := \frac{(1-s)\ n\text{^}s}{(1-s)\ n\text{^}s - s\ n\text{^}(1-s)}\ \text{HarmonicNumber}[n,\ s]$$

```
sso10d[n_, s_] := ssosub4[n, s] + ssosub4[n, 1 - s]
zetc[n_, s_] := sso10c[n, s] al[s]
```

```
sso10c[100 000, .5 + I]
```

0.483321 + 0. i

```
Zeta[.3 + I] al2[.3 + I]
```

0.486188 - 0.00450044 i

```
sso10c[1 000 000, .3 + I] al[.3 + I]
```

0.0581739 - 0.591581 i

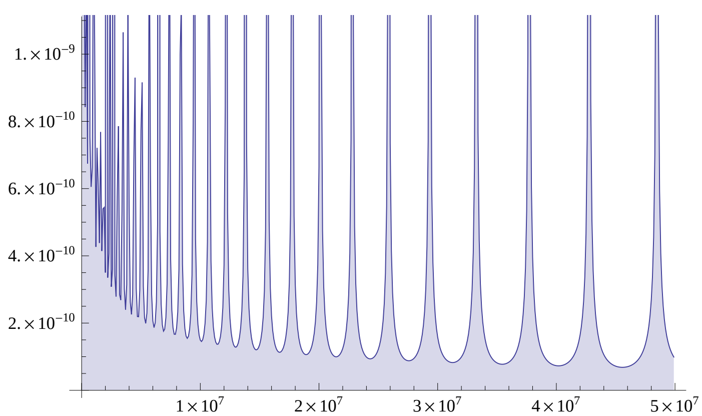
```
Zeta[.3 + I]
```

0.0581511 - 0.591547 i

```
zetc[100 000, .3 + I]
```

0.0579454 - 0.591517 i

```
DiscretePlot[Abs@sso10c[n, N@ZetaZero@3], {n, 1, 50 000 000, 100 000}]
```




```
sso10d[100 000 000 000 000 000 000 000 000, N@ZetaZero@1 + .1 I + .06]
```

0.037632 + 0.0828781 i

```
Zeta[N@ZetaZero@1 + .1 I + .06]
```

0.0377431 + 0.082786 i

```
alo[s]
```

$$-\frac{2\,\pi^{s/2}}{(1-s)\ s\ \text{Gamma}\left[\frac{s}{2}\right]}$$

```
alo[1 - s]
```

$$-\frac{2\,\pi^{\frac{1-s}{2}}}{(1-s)\ s\ \text{Gamma}\left[\frac{1-s}{2}\right]}$$

```
FullSimplify@alo[1 / 2 + s]
```

$$\frac{2\,\pi^{\frac{1}{4}+\frac{s}{2}}}{(-1+2\,s)\ \text{Gamma}\left[\frac{5}{4}+\frac{s}{2}\right]}$$

```
FullSimplify@alo[1 / 2 - s]
```

$$-\frac{2\,\pi^{\frac{1}{4}-\frac{s}{2}}}{(1+2\,s)\ \text{Gamma}\left[\frac{5}{4}-\frac{s}{2}\right]}$$

```
alo[s]
```

$$-\frac{2\,\pi^{s/2}}{(1-s)\ s\ \text{Gamma}\left[\frac{s}{2}\right]}$$

```
alo[1 - s]
```

$$-\frac{2\,\pi^{\frac{1-s}{2}}}{(1-s)\ s\ \text{Gamma}\left[\frac{1-s}{2}\right]}$$

```
po[n_, s_, t_] := Zeta[s, n + 1] ((s - 1) n^(s - 1 + t))
po2[n_, s_, t_] :=
 HarmonicNumber[n, s] ((s - 1) n^(s - 1 + t)) - HarmonicNumber[n, 1 - s] ((-s) n^(-s + t))
```

```
po2[1 000 000 000, N@ZetaZero@16, 1]
```

0. + 67.0797 i

```
1.0000000035000012`
```

```
ts[n_, s_, x_] := (1 - s) n^(s - 1) Zeta[s, n + 1] - (1 - s - x) n^(s - 1 + x) Zeta[s + x, n + 1]
tsa[n_, s_, x_] := (1 - s) n^(s - 1) (Zeta[s] - HarmonicNumber[n, s]) -
  (1 - s - x) n^(s - 1 + x) (Zeta[s + x] - HarmonicNumber[n, s + x])
tsb[n_, s_, x_] := (1 - s) n^(s - 1) (-HarmonicNumber[n, s]) -
  (1 - s - x) n^(s - 1 + x) (-HarmonicNumber[n, s + x])
tsc[n_, s_] := (1 - s) n^(s - 1 / 2) (-HarmonicNumber[n, s]) -
  (1 - s - (1 - 2 s)) n^(1 / 2 - s) (-HarmonicNumber[n, s + (1 - 2 s)])
tso[n_, s_, x_, t_] := (1 - s) n^(s - 1 + t) Zeta[s, n + 1] -
  (1 - s - x) n^(s - 1 + x + t) Zeta[s + x, n + 1]
tr[n_, s_] := (1 - s) Zeta[s, n + 1]
```

```
tso[10 000 000 000, N@ZetaZero@1, 1 - 2 N@ZetaZero@1, .5]
```

0. + 0.000141346 i

```
Limit[(1 - s) n^(s - 1) Zeta[s, n + 1] /. n → 30 000, s → 1]
```

$Aborted

```
s - 1 + (1 - 2 s) + 1 / 2
```

$\frac{1}{2} - s$

```
tso[100 000 000 000, N@ZetaZero@1, 1 - N@ZetaZero@1, .5]
```

Infinity::indet : Indeterminate expression (0. + 0. i) ComplexInfinity encountered. ≫

Indeterminate

```
N@ZetaZero@16
```

0.5 + 67.0798 i

```
tr[10 000 000 000, .3 + I]
```

$5.10782 \times 10^6 - 8.5971 \times 10^6\, i$

```
al[s_] := -2 Pi^(s / 2) / (s (1 - s) Gamma[s / 2])
```

$$\text{ssosub3}[n\_, s\_] := \frac{1}{al[s] - al[1 - s]\, s / (1 - s)\, n^{\wedge}(1 - 2 s)}\ \text{HarmonicNumber}[n, s]$$

```
sso10c[n_, s_] := ssosub3[n, s] + ssosub3[n, 1 - s]
```

$$\text{ssosub4}[n\_, s\_] := \frac{(1 - s)\, n^{\wedge}s}{(1 - s)\, n^{\wedge}s - s\, n^{\wedge}(1 - s)}\ \text{HarmonicNumber}[n, s]$$

```
sso10d[n_, s_] := ssosub4[n, s] + ssosub4[n, 1 - s]
ssosub6[n_, s_] :=
```

$$\frac{1}{al[s] / al[1 - s] - al[1 - s] / al[s]\, s / (1 - s)\, n^{\wedge}(1 - 2 s)}\ \text{HarmonicNumber}[n, s]$$

```
sso10f[n_, s_] := ssosub6[n, s] + ssosub6[n, 1 - s]
ssosub7[n_, s_] :=
```

$$\frac{1}{al[1 - s] / al[s] - al[s] / al[1 - s]\, s / (1 - s)\, n^{\wedge}(1 - 2 s)}\ \text{HarmonicNumber}[n, s]$$

```
sso10g[n_, s_] := ssosub7[n, s] + ssosub7[n, 1 - s]
ssosub8[n_, s_] :=
```

$$\frac{(1 - s)\, n^{\wedge}s}{al[s] / al[1 - s]\, (1 - s)\, n^{\wedge}s - al[1 - s] / al[s]\, s\, n^{\wedge}(1 - s)}\ \text{HarmonicNumber}[n, s]$$

```
sso10h[n_, s_] := ssosub8[n, s] + ssosub8[n, 1 - s]
```

```
sso10c[1 000 000, -2.]
```

0.57394

```
Zeta[-2.0000001] al2[-2.0000001]
```

0.57394

```
sso10f[1 000 000, -1.00001]
```

-0.0833317

```
sso10f[10 000, 2.00000001]
```

-0.0833333

```
Zeta[-2.]
```

0.

```
zt[n_, s_] := ((1 - s) n^s HarmonicNumber[n, s] - s n^(1 - s) HarmonicNumber[n, 1 - s]) /
  ((1 - s) n^s - s n^(1 - s) al[1 - s] / al[s])
zt2[n_, s_] := ((1 - s) n^s HarmonicNumber[n, s] - s n^(1 - s) HarmonicNumber[n, 1 - s]) /
  ((1 - s) n^s - s n^(1 - s) al[s] / al[1 - s])
```

```
zt2[100 000 000, -1.]
```

-32.4697

```
Zeta[.7]
```

-2.77839

```
al[.7] / al[.3] zt[100 000 000, .3]
```

```
inv[s_] := al[1 - s] / al[s] Zeta[1 - s]
```

```
inv[-1]
```

$$-\frac{\pi^4}{3}$$

```
pp[n_, s_] := (1 - s) n^s
```

```
Expand[pp[n, 1 / 2 + x] - pp[n, 1 / 2 - x]]
```

$$-\frac{1}{2} n^{\frac{1}{2}-x} + \frac{1}{2} n^{\frac{1}{2}+x} - n^{\frac{1}{2}-x} x - n^{\frac{1}{2}+x} x$$

```
pp[n, 1 / 2 - x]
```

$$n^{\frac{1}{2}-x} \left(\frac{1}{2} + x\right)$$

$$\mathbf{n^\wedge(1/2)} \left(-\frac{1}{2} n^{-x} + \frac{1}{2} n^{+x} - n^{-x} x - n^{+x} x\right)$$

$$\mathbf{n^\wedge(1/2)\ ((1/2)\ (n^{x} - n^{-x}) - x\ (n^{-x} + n^{+x}))}$$

$$-\frac{1}{2} n^{\frac{1}{2}-x} + \frac{1}{2} n^{\frac{1}{2}+x} - n^{\frac{1}{2}-x} x - n^{\frac{1}{2}+x} x \text{ /. } n \to 100 \text{ /. } x \to .3$$

5.95263

$$\mathbf{n^\wedge(1/2)\ ((1/2)\ (n^{x} - n^{-x}) - x\ (n^{-x} + n^{+x}))\ \text{/. } n \to 100 \text{ /. } x \to .3}$$

5.95263

$$\mathbf{n^\wedge(1/2)} \left((1/2) \left(E^{x\, \mathrm{Log@n}} - E^{-x\, \mathrm{Log@n}}\right) - x \left(E^{-x\, \mathrm{Log@n}} + E^{+x\, \mathrm{Log@n}}\right)\right) \text{ /. } n \to 100 \text{ /. } x \to .3$$

5.95263

$$\mathbf{n^\wedge(1/2)} \left(\mathrm{Sinh}[x\ \mathrm{Log@n}] - x \left(E^{-x\, \mathrm{Log@n}} + E^{+x\, \mathrm{Log@n}}\right)\right) \text{ /. } n \to 100 \text{ /. } x \to .3$$

5.95263

```
n^(1/2) (Sinh[x Log@n] - 2 x Cosh[x Log@n]) /. n → 100 /. x → .3
```

```
5.95263
```

```
zo[n_, s_] := Sum[
  j^(-1/2) (2 s Cosh[s Log[n/j]] - Sinh[s Log[n/j]]) / (2 s Cosh[s Log[n]] - Sinh[s Log[n]]),
  {j, 1, n}]
```

```
zo[100 000, 1.3]
```

```
1.88223
```

```
Zeta[.8]
```

```
-4.43754
```

```
ssosub4[n_, s_] := ((1 - s) n^s)/((1 - s) n^s - s n^(1 - s)) HarmonicNumber[n, s]
sso10d[n_, s_] := ssosub4[n, s] + ssosub4[n, 1 - s]
sso10d2[n_, s_] := ((1 - s) n^s HarmonicNumber[n, s] - s n^(1 - s) HarmonicNumber[n, 1 - s]) /
  ((1 - s) n^s - s n^(1 - s))
sso10d3[n_, s_] := ((1 - s) n^s HarmonicNumber[n, s] - s n^(1 - s) HarmonicNumber[n, 1 - s]) /
  ((1 - s) n^s - s n^(1 - s))
sso10d4[n_, s_] := (-n^(1/2 - s) (1/2 + s) HarmonicNumber[n, 1/2 - s] +
    n^(1/2 + s) (1/2 - s) HarmonicNumber[n, 1/2 + s]) / (n^(1/2 + s) (1/2 - s) - n^(1/2 - s) (1/2 + s))
sso10d5[n_, s_] := (-n^(1/2 - s) (1/2 + s) HarmonicNumber[n, 1/2 - s] +
    n^(1/2 + s) (1/2 - s) HarmonicNumber[n, 1/2 + s]) / (n^(1/2) (E^(s Log[n]) (1/2 - s) - E^(-s Log[n]) (1/2 + s)))
sso10d6[n_, s_] := (n^(1/2 - s) (1/2 + s) HarmonicNumber[n, 1/2 - s] -
    n^(1/2 + s) (1/2 - s) HarmonicNumber[n, 1/2 + s]) / (n^(1/2) (E^(s Log[n]) (s - 1/2) + E^(-s Log[n]) (s + 1/2)))
sso10d7[n_, s_] := (n^(1/2 - s) (1/2 + s) HarmonicNumber[n, 1/2 - s] -
    n^(1/2 + s) (1/2 - s) HarmonicNumber[n, 1/2 + s]) /
  (n^(1/2) (s (E^(s Log[n]) + E^(-s Log[n])) - (1/2) (E^(s Log[n]) - E^(-s Log[n]))))
sso10d8[n_, s_] := (n^(1/2 - s) (1/2 + s) HarmonicNumber[n, 1/2 - s] -
    n^(1/2 + s) (1/2 - s) HarmonicNumber[n, 1/2 + s]) / (n^(1/2) (2 s Cosh[s Log[n]] - Sinh[s Log[n]]))
sso10d9[n_, s_] := Sum[(n/j)^(1/2) (2 s Cosh[s Log[n/j]] - Sinh[s Log[n/j]]), {j, 1, n}] /
  (n^(1/2) (2 s Cosh[s Log[n]] - Sinh[s Log[n]]))
sso10d10[n_, s_] := Sum[j^(-1/2) (2 s Cosh[s Log[n/j]] - Sinh[s Log[n/j]]) /
    (2 s Cosh[s Log[n]] - Sinh[s Log[n]]), {j, 1, n}]
sso10d11[n_, s_] := Sum[j^(-1/2) (2 s Cos[s Log[n/j]] - Sin[s Log[n/j]]) /
    (2 s Cos[s Log[n]] - Sin[s Log[n]]), {j, 1, n}]
sso10d12[n_, t_] := (2 t Sin[t Log[n]] + Cos[t Log[n]]) / (2 t Cos[t Log[n]] - Sin[t Log[n]])
   Sum[j^(-1/2) Sin[t Log[j]], {j, 1, n}] +
  (2 t Cos[t Log[n]] - Sin[t Log[n]]) / (2 t Cos[t Log[n]] - Sin[t Log[n]])
   Sum[j^(-1/2) Cos[t Log[j]], {j, 1, n}]
sso10d13[n_, t_] := (2 t Sin[t Log[n]] + Cos[t Log[n]]) / (2 t Cos[t Log[n]] - Sin[t Log[n]])
   Sum[j^(-1/2) Sin[t Log[j]], {j, 1, n}] + Sum[j^(-1/2) Cos[t Log[j]], {j, 1, n}]

sso10d11[10 000, .3 + 3 I]

1.12149 + 0.03326 i

sso10d10[10 000, .3 + 3 I]

0.589196 - 0.0983814 i
```

```
Zeta[.8 + 3 I]

0.590541 - 0.0980708 i

ssosub8[n_, s_] :=
                    (1 - s) n^s
  ------------------------------------------------------ HarmonicNumber[n, s]
   al[s] / al[1 - s] (1 - s) n^s - al[1 - s] / al[s] s n^(1 - s)
sso10h[n_, s_] := ssosub8[n, s] + ssosub8[n, 1 - s]
sso10ha[n_, s_] := ((1 - s) n^s HarmonicNumber[n, s] - s n^(1 - s) HarmonicNumber[n, 1 - s]) /
  (al[s] / al[1 - s] (1 - s) n^s - al[1 - s] / al[s] s n^(1 - s))
sso10hb[n_, s_] := ((1 / 2 - s) n^(1 / 2 + s) HarmonicNumber[n, 1 / 2 + s] -
    (1 / 2 + s) n^(1 / 2 - s) HarmonicNumber[n, 1 / 2 - s]) /
  (al[1 / 2 + s] / al[1 / 2 - s] (1 / 2 - s) n^(1 / 2 + s) -
    al[1 / 2 - s] / al[1 / 2 + s] (1 / 2 + s) n^(1 / 2 - s))
as[s_] := al[1 / 2 + s] / al[1 / 2 - s]
sso10hc[n_, s_] := ((1 / 2 - s) n^(1 / 2 + s) HarmonicNumber[n, 1 / 2 + s] -
    (1 / 2 + s) n^(1 / 2 - s) HarmonicNumber[n, 1 / 2 - s]) /
  (as[s] (1 / 2 - s) n^(1 / 2 + s) - 1 / as[s] (1 / 2 + s) n^(1 / 2 - s))
sso10hd[n_, s_] := ((1 / 2 - s) n^(1 / 2 + s) HarmonicNumber[n, 1 / 2 + s] -
    (1 / 2 + s) n^(1 / 2 - s) HarmonicNumber[n, 1 / 2 - s]) /
  (n^(1 / 2) (as[s] (1 / 2 - s) n^s - 1 / as[s] (1 / 2 + s) n^-s))
sso10he[n_, s_] := ((1 / 2 - s) n^(1 / 2 + s) HarmonicNumber[n, 1 / 2 + s] -
    (1 / 2 + s) n^(1 / 2 - s) HarmonicNumber[n, 1 / 2 - s]) /
  (n^(1 / 2) ((1 / 2 - s) as[s] n^s - (1 / 2 + s) (as[s] n^s)^-1))
sso10hf[n_, s_] := ((1 / 2 - s) n^(1 / 2 + s) HarmonicNumber[n, 1 / 2 + s] -
    (1 / 2 + s) n^(1 / 2 - s) HarmonicNumber[n, 1 / 2 - s]) /
  (n^(1 / 2) ((1 / 2 - s) E^(Log@as[s] + s Log[n]) - (1 / 2 + s) E^(-(Log@as[s] + s Log[n]))))
sso10hg[n_, s_] := (-(1 / 2 - s) n^(1 / 2 + s) HarmonicNumber[n, 1 / 2 + s] +
    (1 / 2 + s) n^(1 / 2 - s) HarmonicNumber[n, 1 / 2 - s]) /
  (n^(1 / 2) ((s - 1 / 2) E^(Log@as[s] + s Log[n]) + (s + 1 / 2) E^(-(Log@as[s] + s Log[n]))))
sso10hi[n_, s_] := (-(1 / 2 - s) n^(1 / 2 + s) HarmonicNumber[n, 1 / 2 + s] +
    (1 / 2 + s) n^(1 / 2 - s) HarmonicNumber[n, 1 / 2 - s]) /
  (n^(1 / 2) (s (E^(Log@as[s] + s Log[n]) + E^-(Log@as[s] + s Log[n])) -
      (1 / 2) (E^(Log@as[s] + s Log[n]) - E^-(Log@as[s] + s Log[n]))))
sso10hj[n_, s_] := (-(1 / 2 - s) n^(1 / 2 + s) HarmonicNumber[n, 1 / 2 + s] +
    (1 / 2 + s) n^(1 / 2 - s) HarmonicNumber[n, 1 / 2 - s]) /
  (n^(1 / 2) (2 s Cosh[Log@as[s] + s Log[n]] - Sinh[Log@as[s] + s Log[n]]))
sso10hk[n_, s_] := Sum[j^(-1 / 2) (2 s Cosh[s Log[n / j]] - Sinh[s Log[n / j]]) /
    (2 s Cosh[s Log[n] + Log@as[s]] - Sinh[s Log[n] + Log@as[s]]), {j, 1, n}]
```

$$\texttt{sso10hl[n\_, s\_] := Sum}\Big[\texttt{j\^{}(-1/2) (2 s Cosh[s Log[n/j]] - Sinh[s Log[n/j]])}\Big/ \left(2\,s\,\text{Cosh}\left[s\,\text{Log}[n] + \text{Log@}\left(\frac{\pi^s\,\text{Gamma}\left[\frac{1}{4}-\frac{s}{2}\right]}{\text{Gamma}\left[\frac{1}{4}+\frac{s}{2}\right]}\right)\right] - \text{Sinh}\left[s\,\text{Log}[n] + \text{Log@}\left(\frac{\pi^s\,\text{Gamma}\left[\frac{1}{4}-\frac{s}{2}\right]}{\text{Gamma}\left[\frac{1}{4}+\frac{s}{2}\right]}\right)\right]\right),\ \{j, 1, n\}\Big]$$

$$\texttt{sso10hm[n\_, s\_] := Sum}\Big[\texttt{j\^{}(-1/2) (2 s Cosh[s Log[n/j]] - Sinh[s Log[n/j]])}\Big/ \left(2\,s\,\text{Cosh}\left[s\,\text{Log}[n] + s\,\text{Log@Pi} + \text{Log@Gamma}\left[\frac{1}{4}-\frac{s}{2}\right] - \text{Log@Gamma}\left[\frac{1}{4}+\frac{s}{2}\right]\right] -\right.$$

```
      Sinh[s Log[n] + s Log@Pi + Log@Gamma[1/4 - s/2] - Log@Gamma[1/4 + s/2]]), {j, 1, n}]
sso10hn[n_, s_] := ((2 s Sinh[s Log[n]] + Cosh[s Log[n]])
     Sum[j^(-1/2) Sinh[s Log[j]], {j, 1, n}] +
    (2 s Cosh[s Log[n]] - Sinh[s Log[n]]) Sum[j^(-1/2) Cosh[s Log[j]], {j, 1, n}]) /
  (2 s Cosh[s Log[n] + s Log@Pi + Log@Gamma[1/4 - s/2] - Log@Gamma[1/4 + s/2]] -
    Sinh[s Log[n] + s Log@Pi + Log@Gamma[1/4 - s/2] - Log@Gamma[1/4 + s/2]])
```

```
sso10h[100 000 000 000 000 000 000 000 000 000, .45 + I]
```

0.113281 - 0.690186 i

```
sso10ha[10 000, .49 + I]
```

-0.382372 - 0.109796 i

```
Chop@sso10hk[10 000, -2.]
```

-0.0254852

```
Zeta[.45 + I]
```

0.11771 - 0.689418 i

```
FullSimplify[as[s]]
```

$$\frac{\pi^s\, \text{Gamma}\left[\frac{1}{4} - \frac{s}{2}\right]}{\text{Gamma}\left[\frac{1}{4} + \frac{s}{2}\right]}$$

```
Abs[-3 - 2 I]
```

$\sqrt{13}$

```
su50[n_, s_] := {2 s Cosh[s Log[n]] - Sinh[s Log[n]],
  2 s Cosh[s Log[n] + s Log@Pi + Log@Gamma[1/4 - s/2] - Log@Gamma[1/4 + s/2]] -
   Sinh[s Log[n] + s Log@Pi + Log@Gamma[1/4 - s/2] - Log@Gamma[1/4 + s/2]]}
su50[1 000 000, N@ZetaZero@1]
```

{-6772.82 + 12 406.4 i, 6917.81 - 6399.6 i}

```
al[.3]
```

-1.81792

```
FullSimplify[1 / al[s]]
```

$\pi^{-s/2} (-1 + s) \text{Gamma}\left[1 + \frac{s}{2}\right]$

```
al2o[s_] := -((1 / 2) s (1 - s) Pi^(-s / 2) Gamma[s / 2])
```

$\text{al2}[s\_] := \pi^{-s/2} (-1 + s) \text{Gamma}\left[1 + \frac{s}{2}\right]$

```
xi[s_] := Zeta[s] al2[s]
ssosub8[n_, s_] :=
```

$$\frac{(1 - s)\ n\hat{}s}{\text{al}[s] / \text{al}[1 - s]\ (1 - s)\ n\hat{}s - \text{al}[1 - s] / \text{al}[s]\ s\ n\hat{}(1 - s)} \text{HarmonicNumber}[n, s]$$

```
sso10h[n_, s_] := ssosub8[n, s] + ssosub8[n, 1 - s]
xi2[n_, s_] := sso10h[n, s] al2[If[Re[s] < 1 / 2, s, 1 - s]]
```

$$\text{ssosub4}[n\_, s\_] := \frac{(1 - s)\ n\hat{}s}{(1 - s)\ n\hat{}s - s\ n\hat{}(1 - s)} \text{HarmonicNumber}[n, s]$$

```
sso10d[n_, s_] := ssosub4[n, s] + ssosub4[n, 1 - s]
xi3[n_, s_] := sso10d[n, s] al2[If[Re[s] > 1 / 2, s, 1 - s]]
```

```
xi[.6 + 1 I]
```

0.485865 + 0.00224879 ⅈ

```
xi2[100 000 000 000 000, .6 + 1 I]
```

0.48708 + 0.00225269 ⅈ

```
FullSimplify[-((1 / 2) s (1 - s) Pi^(-s / 2) Gamma[s / 2])]
```

$\pi^{-s/2} (-1 + s) \text{Gamma}\left[1 + \frac{s}{2}\right]$

fao[n_, s_] := (1 - s) n^s

$$\text{fa}[n\_, s\_] := 1 \Big/ \left(\frac{n^{-s}}{2} + \frac{n^{1-s}}{1-s} - \frac{1}{12} n^{-1-s} s + \frac{1}{720} n^{-3-s} s (1+s)(2+s) - \frac{n^{-5-s} s (1+s)(2+s)(3+s)(4+s)}{30240} + \frac{n^{-7-s} s (1+s)(2+s)(3+s)(4+s)(5+s)(6+s)}{1209600} - \frac{n^{-9-s} s (1+s)(2+s)(3+s)(4+s)(5+s)(6+s)(7+s)(8+s)}{47900160} + \frac{1}{1307674368000} 691 n^{-11-s} s (1+s)(2+s)(3+s)(4+s)(5+s)(6+s)(7+s)(8+s)(9+s)(10+s) - \frac{1}{74724249600} n^{-13-s} s (1+s)(2+s)(3+s)(4+s)(5+s)(6+s)(7+s)(8+s)(9+s)(10+s)(11+s)(12+s) + \left(3617 n^{-15-s} s (1+s)(2+s)(3+s)(4+s)(5+s)(6+s)(7+s)(8+s)(9+s)(10+s)(11+s)(12+s)(13+s)(14+s)\right) \Big/ 10670622842880000 - \left(43867 n^{-17-s} s (1+s)(2+s)(3+s)(4+s)(5+s)(6+s)(7+s)(8+s)(9+s)(10+s)(11+s)(12+s)(13+s)(14+s)(15+s)(16+s)\right) \Big/ 5109094217170944000 + \left(174611 n^{-19-s} s (1+s)(2+s)(3+s)(4+s)(5+s)(6+s)(7+s)(8+s)(9+s)(10+s)(11+s)(12+s)(13+s)(14+s)(15+s)(16+s)(17+s)(18+s)\right) \Big/ 802857662698291200000\right)$$

$$\text{sss}[n\_, s\_] := \frac{\text{fa}[n, s]}{\text{fa}[n, s] - \text{fa}[n, 1-s]} \text{HarmonicNumber}[n, s]$$

zets[n_, s_] := sss[n, s] + sss[n, 1 - s]

$$\text{sss2}[n\_, s\_] := \frac{\text{fao}[n, s]}{\text{fao}[n, s] - \text{fao}[n, 1-s]} \text{HarmonicNumber}[n, s]$$

zets2[n_, s_] := sss2[n, s] + sss2[n, 1 - s]

zets[1 000 000 000, .6 + 3 I]

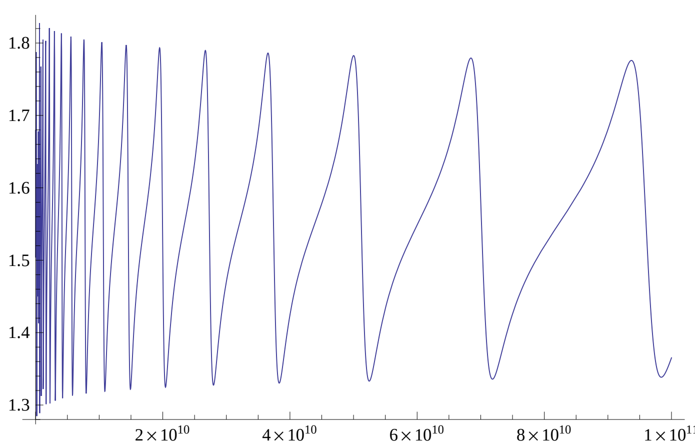
0.54955 - 0.0850346 i

Abs@Zeta[.51 + 10 I]

1.54561

0.557271 - 0.882999 i

Plot[Abs@zets[n, .51 + 10 I], {n, 1, 100 000 000 000}]



```
Expand[1 / (1 - s / (1 - s) n^(1 - 2 s)) j^-s - 1 / ((1 - s) / s n^(2 s - 1) - 1) j^(s - 1)]
```

$$-\frac{j^{-1+s}}{-1+\frac{n^{-1+2s}(1-s)}{s}}+\frac{j^{-s}}{1-\frac{n^{1-2s}s}{1-s}}$$

```
bb[n_, t_, t2_] := {(2 t Cos[t Log[n]] - Sin[t Log[n]]) Zeta[t2],
  (2 t Sin[t Log[n]] + Cos[t Log[n]]) Sum[ j^(-1 / 2) Sin[t Log[j]], {j, 1, n}] +
   (2 t Cos[t Log[n]] - Sin[t Log[n]]) Sum[ j^(-1 / 2) Cos[t Log[j]], {j, 1, n}]}

bb[100 000, .3 I + 2, .8 + 2 I]

{5.37521 + 39.8772 i, -34.0654 + 21.4869 i}

bc[n_, t_, t2_] :=
 {Zeta[t2], (2 t Sin[t Log[n]] + Cos[t Log[n]]) / (2 t Cos[t Log[n]] - Sin[t Log[n]])
    Sum[j^(-1 / 2) Sin[t Log[j]], {j, 1, n}] + Sum[j^(-1 / 2) Cos[t Log[j]], {j, 1, n}]}
bc2[n_, t_, t2_] := {Zeta[t2], (2 t Sinh[t Log[n]] + Cosh[t Log[n]]) /
     (2 t Cosh[t Log[n]] - Sinh[t Log[n]]) Sum[j^(-1 / 2) Sinh[t Log[j]], {j, 1, n}] +
   Sum[j^(-1 / 2) Cosh[t Log[j]], {j, 1, n}]}
```

$$\text{ssosub4[n\_, s\_] := } \frac{(1-s)\ n\text{^}s}{(1-s)\ n\text{^}s - s\ n\text{^}(1-s)} \text{ HarmonicNumber[n, s]}$$

```
sso10d[n_, s_] := ssosub4[n, s] + ssosub4[n, 1 - s]
sso10d10[n_, s_] := Sum[j^(-1 / 2)
   (2 s Cosh[s Log[n / j]] - Sinh[s Log[n / j]]) / (2 s Cosh[s Log[n]] - Sinh[s Log[n]]), {j, 1, n}]

(* Imaginary part is conjugate here.  What am I doing wrong? *)
bc2[100 000, .3 + 2 I, .8 + 2 I]

{0.533102 - 0.342462 i, -4297.91 + 1834.17 i}

sso10d10[100 000, .3 + 2 I]

0.533592 - 0.342803 i

sso10d11[n_, s_] := Sum[j^(-1 / 2)
   (2 s Cos[s Log[n / j]] - Sin[s Log[n / j]]) / (2 s Cos[s Log[n]] - Sin[s Log[n]]), {j, 1, n}]

sso10d11[100 000, .3 I + 2]

0.533592 + 0.342803 i

Zeta[.8 + 2 I]

0.533102 - 0.342462 i

(.3 I + 2 ) I

-0.3 + 2. i

pr[n_, x_] := ((2 x Sin[x Log[n]] + Cos[x Log[n]]) / (2 x Cos[x Log[n]] - Sin[x Log[n]]))
   Sum[ j^(-1 / 2) Sin[ x Log[j]], {j, 1, n}] + Sum[ j^(-1 / 2) Cos[ x Log[j]], {j, 1, n}]
pro[n_, x_] := ((2 x Sin[x Log[n]] + Cos[x Log[n]])) Sum[ j^(-1 / 2) Sin[ x Log[j]], {j, 1, n}] +
  (2 x Cos[x Log[n]] - Sin[x Log[n]]) Sum[ j^(-1 / 2) Cos[ x Log[j]], {j, 1, n}]
pra[n_, x_] := {((2 x Sin[x Log[n]] + Cos[x Log[n]]) / (2 x Cos[x Log[n]] - Sin[x Log[n]])),
  Sum[ j^(-1 / 2) Sin[ x Log[j]], {j, 1, n}], Sum[ j^(-1 / 2) Cos[ x Log[j]], {j, 1, n}]}
prb[n_, x_] := ((2 x Sin[x Log[n]] + Cos[x Log[n]]) / (2 x Cos[x Log[n]] - Sin[x Log[n]]))
prs[n_, x_] := ((2 x Sin[x Log[n]] + Cos[x Log[n]]))
prc[n_, x_] := ((2 x Cos[x Log[n]] - Sin[x Log[n]]))
```

```
sso10d10[n_, s_] := Sum[
  j^(-1/2) (2 s Cosh[s Log[n/j]] - Sinh[s Log[n/j]]) / (2 s Cosh[s Log[n]] - Sinh[s Log[n]]),
  {j, 1, n}]
sso10d10a[n_, s_] := Sum[j^(-1/2)
   (2 s (Cosh[s Log[n] - s Log[j]]) - (Sinh[s Log[n] - s Log[j]])) /
    (2 s Cosh[s Log[n]] - Sinh[s Log[n]]), {j, 1, n}]
sso10d10b[n_, s_] := Sum[j^(-1/2)
   (2 s (Cosh[s Log[n]] Cosh[s Log[j]] - Sinh[s Log[n]] Sinh[s Log[j]]) -
      (Sinh[s Log[n]] Cosh[s Log[j]] - Cosh[s Log[n]] Sinh[s Log[j]])) /
    (2 s Cosh[s Log[n]] - Sinh[s Log[n]]), {j, 1, n}]
sso10d10c[n_, s_] := Sum[j^(-1/2) (
     2 s (Cosh[s Log[n]] Cosh[s Log[j]] - Sinh[s Log[n]] Sinh[s Log[j]]) -
      (Sinh[s Log[n]] Cosh[s Log[j]] - Cosh[s Log[n]] Sinh[s Log[j]])
    ) / (2 s Cosh[s Log[n]] - Sinh[s Log[n]]), {j, 1, n}]
sso10d10d[n_, s_] := (
   Sum[j^(-1/2) (2 s (Cosh[s Log[n]] Cosh[s Log[j]])), {j, 1, n}] +
    Sum[j^(-1/2) (2 s (-Sinh[s Log[n]] Sinh[s Log[j]]) ), {j, 1, n}] +
    Sum[j^(-1/2) (-(Sinh[s Log[n]] Cosh[s Log[j]])), {j, 1, n}] +
    Sum[j^(-1/2) (-(-Cosh[s Log[n]] Sinh[s Log[j]])), {j, 1, n}]
  ) / (2 s Cosh[s Log[n]] - Sinh[s Log[n]])
sso10d10e[n_, s_] := (
   (2 s Cosh[s Log[n]] - Sinh[s Log[n]]) Sum[j^(-1/2) Cosh[s Log[j]], {j, 1, n}] +
    (-2 s Sinh[s Log[n]] + Cosh[s Log[n]]) Sum[j^(-1/2) Sinh[s Log[j]], {j, 1, n}]
  ) / (2 s Cosh[s Log[n]] - Sinh[s Log[n]])
sso10d10f[n_, s_] := Sum[j^(-1/2) Cosh[s Log[j]], {j, 1, n}] -
  (2 s Sinh[s Log[n]] - Cosh[s Log[n]]) / (2 s Cosh[s Log[n]] - Sinh[s Log[n]])
   Sum[j^(-1/2) Sinh[s Log[j]], {j, 1, n}]
hyp[n_, s_] := sso10d10f[n, s - 1/2]
```

```
sso10d10f[10 000, .3 + 2 I]
```

0.532091 − 0.340213 ⅈ

```
Zeta[.8 + 2 I]
```

0.533102 − 0.342462 ⅈ

```
hyp[100 000, .8 + 2 I]
```

0.533592 − 0.342803 ⅈ

```
FullSimplify[(-s a + 1/2 b) / (s b - 1/2 a )]
```

$-\frac{b - 2 a s}{a - 2 b s}$

```
Sinh[I x]
```

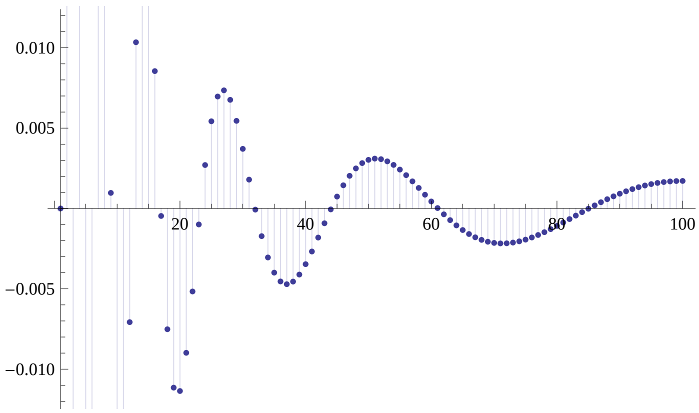
ⅈ Sin[x]

```
Cosh[I x]
```

Cos[x]

```
zetx[n_, t_] := (t Sin[t Log[n]] + Cos[t Log[n]] / 2) / ( t Cos[t Log[n]] - Sin[t Log[n]] / 2)
    Sum[j ^ (-1 / 2) Sin[t Log[j]], {j, 1, n}] + Sum[j ^ (-1 / 2) Cos[t Log[j]], {j, 1, n}]
zetj[n_, t_, j_] := (t Sin[t Log[n]] + Cos[t Log[n]] / 2) / ( t Cos[t Log[n]] - Sin[t Log[n]] / 2)
    j ^ (-1 / 2) Sin[t Log[j]] + j ^ (-1 / 2) Cos[t Log[j]]
```

```
DiscretePlot[Im@ zetj[100, 10 + 1. I, j], {j, 1, 100}]
```




```
Zeta[2.5]
```

```
1.34149
```