```mathematica
Dd[x_, 0, y_] := 1
Dd[x_, k_, y_] := Sum[Dd[x / (j + y), k - 1, y], {j, 0, Floor[x - y]}]
Cc[x_, k_, y_] := y^-k Dd[x y^k, k, y + 1]
FAlt[n_, 0, a_, y_] := 1
FAlt[n_, k_, a_, y_] := If[n < (a / y + 1)^k, 0, FAlt[n, k, a + 1, y] +
    Sum[y^-j Binomial[k, j] FAlt[n / (a / y + 1)^j, k - j, a + 1, y], {j, 1, k}]]
F2Alt[n_, 0, a_, y_] := 1
F2Alt[n_, 1, a_, y_] := y^-1 (Floor[y (n - 1) - a + 1])
F2Alt[n_, k_, a_, y_] := Sum[y^-j Binomial[k, j] F2Alt[n / (m / y + 1)^j, k - j, m + 1, y],
   {m, a, Floor[y (n^(1 / k) - 1)]}, {j, 1, k}]
F2Altx[n_, 1, a_, y_] := y^-1 Sum[1 , {m, a, Floor[y (n^(1 / 1) - 1)]}]
```

F2Alt[200, 3, 1, 4 / 3]

$$\frac{75\,735}{64}$$

Cc[200, 3, 4 / 3]

$$\frac{75\,735}{64}$$

n / ((m / y + 1) ^ j)

$$n \left(1 + \frac{m}{y}\right)^{-j}$$

F2Alt[100, 1, 1, 4.1]

98.7805

Cc[100, 1, 4.1]

98.7805

Table[F2Alt[100, k, 1, 3], {k, 0, Log[(1 / 3) + 1, 100]}]

$$\left\{1, 99, \frac{995}{3}, \frac{14\,602}{27}, \frac{15\,020}{27}, \frac{96\,524}{243}, \frac{16\,925}{81}, \frac{180\,896}{2187}, \frac{164\,248}{6561}, \frac{114\,599}{19\,683}, \right.$$
$$\left. \frac{62\,908}{59\,049}, \frac{24\,047}{177\,147}, \frac{7346}{531\,441}, \frac{1678}{1\,594\,323}, \frac{134}{4\,782\,969}, \frac{16}{14\,348\,907}, \frac{1}{43\,046\,721}\right\}$$

Binomial[2, 1]

2

ff[y_] := y^-2 Floor[y 100^(1 / 2) - y]

Cc[n_, 0, a_, y_] := 1
Cc[n_, 1, a_, y_] := y^-1 (Floor[y (n - 1) - a + 1])
Cc[n_, k_, a_, y_] := Sum[y^-j Binomial[k, j] Cc[n / (m / y + 1)^j, k - j, m + 1, y],
   {m, a, Floor[y (n^(1 / k) - 1)]}, {j, 1, k}]

Cc[100, 2, 1, 1]

283

```
Cc[x_, 0, a_, y_] := 1
Cc[x_, 1, a_, y_] := y^-1 (Floor[y (x - 1) - a + 1])
Cc[x_, k_, a_, y_] := Sum[y^-j Binomial[k, j] Cc[n / (m / y + 1) ^j, k - j, m + 1, y],
   {m, a, Floor[y (x^ (1 / k) - 1)]}, {j, 1, k}]
```

```
Sum[ Binomial[ z, k] (y^ (k (s - 1)) Zeta[ s, y + 1]^k), {k, 0, Infinity}]
```

$$\text{FullSimplify}\left[\left(\frac{y + y^s \, \text{Zeta}[s, 1 + y]}{y}\right)^z\right]$$

$$D\left[\left(1 + y^{-1+s} \, \text{Zeta}[s, 1 + y]\right)^z, \, y\right]$$

$$\text{FullSimplify}\left[z \left(1 + y^{-1+s} \, \text{Zeta}[s, 1 + y]\right)^{-1+z} \left((-1 + s) \, y^{-2+s} \, \text{Zeta}[s, 1 + y] - s \, y^{-1+s} \, \text{Zeta}[1 + s, 1 + y]\right)\right]$$

$$y^{-2+s} \, z \left(1 + y^{-1+s} \, \text{Zeta}[s, 1 + y]\right)^{-1+z} \left((-1 + s) \, \text{Zeta}[s, 1 + y] - s \, y \, \text{Zeta}[1 + s, 1 + y]\right)$$

```
FullSimplify[Sum[(-1) ^ (k + 1) / k (y^ (k (s - 1)) Zeta[ s, y + 1]^k), {k, 1, Infinity}]]
```

$$\text{Log}\left[1 + y^{-1+s} \, \text{Zeta}[s, 1 + y]\right]$$

$$\text{FullSimplify}\left[\text{Log}\left[\frac{y + y^s \, \text{Zeta}[s, 1 + y]}{y}\right]\right]$$

$$\text{Log}\left[1 + y^{-1+s} \, \text{Zeta}[s, 1 + y]\right]$$

$$D\left[\text{Log}\left[1 + y^{-1+s} \, \text{Zeta}[s, 1 + y]\right], \, y\right]$$

$$\frac{(-1 + s) \, y^{-2+s} \, \text{Zeta}[s, 1 + y] - s \, y^{-1+s} \, \text{Zeta}[1 + s, 1 + y]}{1 + y^{-1+s} \, \text{Zeta}[s, 1 + y]}$$

$$\text{ss}[s\_, z\_] := (s / (s - 1)) \wedge z - \text{Integrate}\left[y^{-2+s} \, z \left(1 + y^{-1+s} \, \text{Zeta}[s, 1 + y]\right)^{-1+z}\right.$$
$$\left.((-1 + s) \, \text{Zeta}[s, 1 + y] - s \, y \, \text{Zeta}[1 + s, 1 + y]), \{y, 1, \text{Infinity}\}\right]$$

```
N[ss[2, 2]]
```

2.70581

```
N[ Zeta[2] ^ 2]
```

2.70581

$$\text{s2}[s\_] := \text{Log}[s / (s - 1)] -$$
$$\text{Integrate}\left[\frac{(-1 + s) \, y^{-2+s} \, \text{Zeta}[s, 1 + y] - s \, y^{-1+s} \, \text{Zeta}[1 + s, 1 + y]}{1 + y^{-1+s} \, \text{Zeta}[s, 1 + y]}, \, \{y, 1, \text{Infinity}\}\right]$$

```
N[s2[ 2]]
```

0.4977

```
Log[Zeta[2.]]
```

0.4977

```
Limit[ (Zeta[s] - 1) ^ 2 - (1 / (s - 1)) ^ 2, {s → 1}]
```

{−∞}

`Integrate[` $\dfrac{(-1+s)\, y^{-2+s}\, \text{Zeta}[s,\, 1+y] - s\, y^{-1+s}\, \text{Zeta}[1+s,\, 1+y]}{1+y^{-1+s}\, \text{Zeta}[s,\, 1+y]}$ `, {y, 1, Infinity}]`

$$\int_1^\infty \frac{-\dfrac{2\,\text{Zeta}[-1,1+y]}{y^3}+\dfrac{\text{Zeta}[0,1+y]}{y^2}}{1+\dfrac{\text{Zeta}[-1,1+y]}{y^2}}\ \mathbb{d}y$$

`Integrate[` $y^{-2+s}\, z\, \left(1+y^{-1+s}\, \text{Zeta}[s,\, 1+y]\right)^{-1+z}$
  $\left((-1+s)\, \text{Zeta}[s,\, 1+y] - s\, y\, \text{Zeta}[1+s,\, 1+y]\right),$ `{y, 1, Infinity}]`

$$\int_1^\infty y^{-2+s}\, z\, \left(1+y^{-1+s}\, \text{Zeta}[s,\, 1+y]\right)^{-1+z} \left((-1+s)\, \text{Zeta}[s,\, 1+y] - s\, y\, \text{Zeta}[1+s,\, 1+y]\right)\mathbb{d}y$$

`FullSimplify[Sum[ Binomial[ z, k] (y^(k (s-1)) Zeta[ s, y+1]^k), {k, 0, Infinity}]]`

$\left(1+y^{-1+s}\, \text{Zeta}[s,\, 1+y]\right)^z$

`D[` $\left(\dfrac{y+y^s\, \text{Zeta}[s,\, 1+y]}{y}\right)^z$ `, y]`

$z\,\left(\dfrac{y+y^s\, \text{Zeta}[s,\, 1+y]}{y}\right)^{-1+z}$

$\left(-\dfrac{y+y^s\, \text{Zeta}[s,\, 1+y]}{y^2}+\dfrac{1+s\, y^{-1+s}\, \text{Zeta}[s,\, 1+y] - s\, y^s\, \text{Zeta}[1+s,\, 1+y]}{y}\right)$

`D[FullSimplify[` $\left(\dfrac{y+y^s\, \text{Zeta}[s,\, 1+y]}{y}\right)^z$ `], y]`

$z\,\left(1+y^{-1+s}\, \text{Zeta}[s,\, 1+y]\right)^{-1+z}\left((-1+s)\, y^{-2+s}\, \text{Zeta}[s,\, 1+y] - s\, y^{-1+s}\, \text{Zeta}[1+s,\, 1+y]\right)$

`D[` $\left(1+y^{-1+s}\, \text{Zeta}[s,\, 1+y]\right)^z$ `, y]`

$z\,\left(1+y^{-1+s}\, \text{Zeta}[s,\, 1+y]\right)^{-1+z}\left((-1+s)\, y^{-2+s}\, \text{Zeta}[s,\, 1+y] - s\, y^{-1+s}\, \text{Zeta}[1+s,\, 1+y]\right)$

`D[` $\left(1+y^{-1+s}\, \text{Zeta}[s,\, 1+y]\right)^z$ `, y]`

`D[ y^(s-1) Zeta[ s, y+1], y]`

$(-1+s)\, y^{-2+s}\, \text{Zeta}[s,\, 1+y] - s\, y^{-1+s}\, \text{Zeta}[1+s,\, 1+y]$

`Integrate[` $(-1+s)\, y^{-2+s}\, \text{Zeta}[s,\, 1+y] - s\, y^{-1+s}\, \text{Zeta}[1+s,\, 1+y],$ `{y, 1, Infinity}]`

$$\int_1^\infty \left((-1+s)\, y^{-2+s}\, \text{Zeta}[s,\, 1+y] - s\, y^{-1+s}\, \text{Zeta}[1+s,\, 1+y]\right)\mathbb{d}y$$

`Integrate[` $(-1+s)\, y^{-2+s}\, \text{Zeta}[s,\, 1+y],$ `{y, 1, Infinity}]`

$$\int_1^\infty (-1+s)\, y^{-2+s}\, \text{Zeta}[s,\, 1+y]\ \mathbb{d}y$$

`Integrate[` $-s\, y^{-1+s}\, \text{Zeta}[1+s,\, 1+y],$ `{y, 1, Infinity}]`

$$\int_1^\infty -s\, y^{-1+s}\, \text{Zeta}[1+s,\, 1+y]\ \mathbb{d}y$$

```
c[s_, y_] := y^(s-1) Zeta[s, y+1]
Table[{Zeta[s] - 1, 1 / (s - 1) - Integrate[D[c[s, y], y], {y, 1, Infinity}]}, {s, 2, 6}]
```

$$\left\{\left\{-1 + \frac{\pi^2}{6}, -1 + \frac{\pi^2}{6}\right\}, \{-1 + \text{Zeta}[3], -1 + \text{Zeta}[3]\},\right.$$

$$\left.\left\{-1 + \frac{\pi^4}{90}, -1 + \frac{\pi^4}{90}\right\}, \{-1 + \text{Zeta}[5], -1 + \text{Zeta}[5]\}, \left\{-1 + \frac{\pi^6}{945}, -1 + \frac{\pi^6}{945}\right\}\right\}$$

```
c[s_, y_] := y^(s-1) HurwitzZeta[s, y+1]
 {c[s, 1]^z, (Zeta[s] - 1)^z}
```

$$\{(-1 + \text{Zeta}[s])^z, (-1 + \text{Zeta}[s])^z\}$$

```
c[s_, y_] := y^(s-1) HurwitzZeta[s, y+1]
{Limit[c[s, y]^z, y -> Infinity], 1 / (s - 1)^z}
```

$$\left\{\left(\frac{1}{-1+s}\right)^z, (-1+s)^{-z}\right\}$$

```
c[s_, y_] := y^(s-1) HurwitzZeta[s, y+1]
{c[s, 1]^z, (Zeta[s] - 1)^z}
```

$$\{(-1 + \text{Zeta}[s])^z, (-1 + \text{Zeta}[s])^z\}$$

```
c[s_, y_] := y^(s-1) HurwitzZeta[s, y+1]
Grid[Table[{N[(Zeta[s] - 1)^z],
   N[1 / ((s - 1)^z) - Integrate[D[c[s, y]^z, y], {y, 1, Infinity}]]}, {s, 2, 4}, {z, 1, 4}]]
```

{0.644934, 0.644934}	{0.41594, 0.41594}	{0.268254, 0.268254}	{0.173006, 0.173006}
{0.202057, 0.202057}	{0.040827, 0.040827}	{0.00824938, 0.00824938}	{0.00166684, 0.00166684}
{0.0823232, 0.0823232}	{0.00677711, 0.00677711}	{0.000557914, 0.000557914}	{0.0000459293, 0.0000459293}

```
{(s / (s - 1))^z, Sum[ Binomial[ z, k] (s - 1)^-k, {k, 0, Infinity}]}
```

$$\left\{\left(\frac{s}{-1+s}\right)^z, \left(\frac{s}{-1+s}\right)^z\right\}$$

```
Sum[ Binomial[z, k] cc[s, y]^k, {k, 0, Infinity}]
```

$$(1 + \text{cc}[s, y])^z$$

```
FullSimplify[D[ (1 + c[s, y])^z, y]]
```

$$y^{-2+s} z \left(1 + y^{-1+s} \text{HurwitzZeta}[s, 1+y]\right)^{-1+z}$$
$$((-1 + s) \text{HurwitzZeta}[s, 1+y] - s y \text{HurwitzZeta}[1+s, 1+y])$$

```
Sum[ (-1)^(k+1) / k cc[s, y]^k, {k, 1, Infinity}]
```

$$\text{Log}[1 + \text{cc}[s, y]]$$

```
C2[ x_, 0, y_] := 1
C2[ x_, k_, y_] := Sum[C2[ x y / (j + y), k - 1, y], {j, 1, Floor[x y - y]}]
C1Alt[ x_, z_, y_] :=
 Sum[ Binomial[ z, k] C2[x, k, y], {k, 0, Floor[Log[ x] / (Log[y + 1] - Log[y])]}]
```

```
C2[ 100, 2, 1]
```

283

```
C1[ x_, 0, y_] := 1
C1[ x_, k_, y_] := Sum[ C1[ x y / (j + y), k - 1, y], {j, 0, Floor[x y - y]}]
```

```
C1[ 100, 3, 2]
```

8213

```
C1Alt[ 100, 4, 1]
```

3575

```
Limit[ (C1Alt[ 100, z, 2] - 1) / z, z → 0]
```

$$\frac{65\,437}{5544}$$

```
Cc[x_, 0, a_, y_] := 1
Cc[x_, 1, a_, y_] := y^-1 (Floor[y (x - 1) - a + 1])
Cc[x_, k_, a_, y_] :=
 Cc[x, k, a, y] = Sum[y^-j Binomial[k, j] Cc[x / (m / y + 1) ^j, k - j, m + 1, y],
    {m, a, Floor[y (x^(1 / k) - 1)]}, {j, 1, k}]
C1Alta[ x_, z_, y_ ] := Sum[Binomial[ z, k] Cc[x, k, 1, y], {k, 0, Floor[Log[ (y + 1) / y, x]]}]
```

```
Cc[ 100, 4, 0, 1]
```

3575

```
C1Alta[4, 2, 40.]
```

9.47625

```
N[LaguerreL[-2, Log[4]]]
```

9.54518

```
d1Alt[n_, z_] := Product[(-1)^p[[2]] Binomial[-z, p[[2]]], {p, FI[n]}];
FI[n_] := FactorInteger[n]; FI[1] := {}
c1[ n_, z_, y_ ] := y^-1 Sum[ d1Alt[ j, z], {j, Floor[(n-1) y] + 1, Floor[n y]}]
Table[{n, c1[n, 3, 2] - (C1Alta[n, 3, 2] - C1Alta[ n-1, 3, 2])}, {n, 10, 40}] // TableForm
```

10	$\frac{9}{4}$
11	$\frac{3}{8}$
12	$\frac{15}{4}$
13	$-\frac{3}{2}$
14	$\frac{7}{8}$
15	$\frac{15}{8}$
16	$\frac{17}{8}$
17	$-\frac{15}{8}$
18	$\frac{45}{8}$
19	-3
20	$\frac{21}{4}$
21	$-\frac{3}{8}$
22	$\frac{3}{8}$
23	$-\frac{9}{4}$
24	$\frac{69}{8}$
25	$-\frac{21}{8}$
26	3
27	$-\frac{11}{4}$
28	$\frac{45}{8}$
29	-3
30	6
31	$-\frac{15}{8}$
32	$\frac{31}{8}$
33	$\frac{9}{8}$
34	$-\frac{21}{4}$
35	$\frac{3}{2}$
36	$\frac{93}{8}$
37	$-\frac{33}{8}$
38	$\frac{3}{4}$
39	$-\frac{21}{8}$
40	$\frac{69}{8}$

```
c1Alt[n_, z_, y_] := Product[(-1)^p[[2]] Binomial[-z, p[[2]]], {p, FI[n y]}];
FI[n_] := FactorInteger[n]; FI[1] := {}
C1Alt[ n_, z_, y_] := y^-z Sum[ c1Alt[ j, z, y], {j, 1, n y}]
```

```
C1Alt[ 100, 3, 2]
```

$$\frac{7317}{8}$$

```
Dd[n_, 0, a_] := 1; Dd[n_, 1, a_] := Floor[n] - a + 1
Dd[n_, k_, a_] :=
 Sum[Binomial[k, j] Dd[n / (m^(k - j)), j, m + 1], {m, a, n^(1 / k)}, {j, 0, k - 1}]
CcAlt[ n_, k_, y_] := y^-k Dd[ n y^k, k, 1]
Cc[x_, 0, a_, y_] := 1
Cc[x_, 1, a_, y_] := y^-1 (Floor[y (x - 1) - a + 1])
Cc[x_, k_, a_, y_] :=
 Cc[x, k, a, y] = Sum[y^-j Binomial[k, j] Cc[x / (m / y + 1)^j, k - j, m + 1, y],
    {m, a, Floor[y (x^(1 / k) - 1)]}, {j, 1, k}]
C1Alta[ x_, z_, y_] := Sum[ Binomial[ z, k] Cc[x, k, 1, y], {k, 0, Floor[Log[ (y + 1) / y, x]]}]
C1[ x_, 0, y_] := 1
C1[ x_, k_, y_] := Sum[ C1[ x y / (j + y), k - 1, y], {j, 0, 0}] +
  y^-1 Sum[ C1[ x y / (j + y), k - 1, y], {j, 1, Floor[x y - y]}]
C1a[ x_, 0, y_] := 1
C1a[ x_, k_, y_] :=
 C1a[ x, k - 1, y] + y^-1 Sum[ C1a[ x y / (j + y), k - 1, y], {j, 1, Floor[x y - y]}]
C2[x_, 0, y_] := 1
C2[x_, k_, y_] := y^-1 Sum[ C2[ x y / (j + y), k - 1, y], {j, 1, Floor[x y - y]}]
C1Altb[ x_, z_, y_ ] := Sum[ Binomial[ z, k] C2[x, k, y], {k, 0, Floor[Log[ (y + 1) / y, x]]}]
C1a[ x_, y_] := 1 + 3 y^-1 Sum[1, {j, 1, Floor[x y - y]}] + 3 C2[x, 2, y] + C2[x, 3, y]
C1b[ x_, y_] := 1 + 3 y^-1 Sum[1, {j, 1, Floor[x y - y]}] +
   3 y^-2 Sum[1, {j, 1, Floor[x y - y]}, {k, 1, Floor[(x y / (j + y)) y - y]}] +
   y^-3 Sum[1, {j, 1, Floor[x y - y]}, {k, 1, Floor[(x y / (j + y)) y - y]},
    {m, 1, Floor[(x y^2 / ((j + y) (k + y))) y - y]}]
C1c[ x_, y_] := 1 + 3 y^-1 Sum[1, {j, 1, Floor[x y - y]}] +
   3 y^-2 Sum[1, {j, 1, Floor[x y - y]}, {k, 1, Floor[(x y / (j + y)) y - y]}] +
   y^-3 Sum[1, {j, 1, Floor[x y - y]}, {k, 1, Floor[(x y / (j + y)) y - y]},
    {m, 1, Floor[(x y^2 / ((j + y) (k + y))) y - y]}]
```

```
C1a[20, 3, 3]
```

$$\frac{5309}{27}$$

```
C1c[20, 3]
```

$$\frac{5309}{27}$$

```
Cc[ 20, 3, 0, 3]
```

$$\frac{665}{9}$$

```
C1Alta[ 20, 3, 3]
```

$$\frac{5309}{27}$$

```
C1Altb[ 20, 3, 3]
```

C2c[20, 0, 3] + 3 C2c[20, 1, 3] + 3 C2c[20, 2, 3] + C2c[20, 3, 3]

```
CcAlt[ 20, 2, 2]
```

92

```
Sum[ 1 / 4, {j, 1 / 2, 40, 1 / 2}, {k, 1 / 2, 20 / j, 1 / 2}]
```

92

```
Dd[ 80, 2, 1] / 4
```

92

```
C1[ 20, 3, 3]
```

$$\frac{5309}{27}$$

```
Sum[ 1 / 4, {j, 1, 80}, {k, 1, 80 / j}]
```

92

```
C2[100, 2, 2]
```

318

```
C1Altb[ 20, 3, 3]
```

$$\frac{5309}{27}$$

```
C2[ x_, 0, y_] := 1
C2[ x_, k_, y_] :=
 C2[x, k, y] = y^-1 Sum[ C2[ x y / (j + y), k - 1, y], {j, 1, Floor[x y - y]}]
C1a[ x_, 0, y_] := 1
C1a[ x_, k_, y_] :=  C1a[x, k, y] =
  C1a[ x, k - 1, y] + y^-1 Sum[ C1a[ x y / (j + y), k - 1, y], {j, 1, Floor[x y - y]}]
C1b[ x_, 0, y_] := 1
C1b[ x_, k_, y_] :=
 (1 - y^-1) C1b[x, k - 1, y] + y^-1 Sum[ C1b[ x y / (j + y), k - 1, y], {j, 0, Floor[x y - y]}]
C1Altb[ x_, z_, y_ ] := Sum[ Binomial[ z, k] C2[x, k, y], {k, 0, Floor[Log[ (y + 1) / y, x]]}]
```

```
Grid[Table[ {C1a[100, z, y], C1Altb[100, z, y]}, {z, 1, 5}, {y, 1, 4, .3}]]
```

{100., 100.}	{99.4615, 99.4615}	{99.75, 99.75}	{99.9474, 99.9474}	{99.6364, 99.6364}	{99.8, 99.8}	{99.9286, 99.9286}	{99.7097, 99.7097}	{99.8235, 99.8235}	{99.9189, 99.9189}	{100., 100.}
{482., 482.}	{493.189, 493.189}	{505.531, 505.531}	{513.299, 513.299}	{518.314, 518.314}	{523.72, 523.72}	{527.046, 527.046}	{530.053, 530.053}	{532.73, 532.73}	{534.923, 534.923}	{537.625, 537.625}
{1471., 1471.}	{1558.15, 1558.15}	{1645.1, 1645.1}	{1698.56, 1698.56}	{1747.17, 1747.17}	{1785.02, 1785.02}	{1811.01, 1811.01}	{1835.89, 1835.89}	{1855.24, 1855.24}	{1871.47, 1871.47}	{1889.53, 1889.53}
{3575., 3575.}	{3937.11, 3937.11}	{4288.08, 4288.08}	{4511.62, 4511.62}	{4729.78, 4729.78}	{4893.58, 4893.58}	{5012.04, 5012.04}	{5124.26, 5124.26}	{5210.47, 5210.47}	{5286.41, 5286.41}	{5362.69, 5362.69}
{7537., 7537.}	{8640.9, 8640.9}	{9714.53, 9714.53}	{10432., 10432.}	{11142.2, 11142.2}	{11682.6, 11682.6}	{12092., 12092.}	{12472.2, 12472.2}	{12769.9, 12769.9}	{13038., 13038.}	{13292.3, 13292.3}

```
C1a[ 29, 3, 3.2]
```

336.061

```
C1b[ 29, 3, 3.2]
```

336.061

```
C1Alta[29, 3, 3.2]
```

336.061

```
Expand[y^-1 (1 - y^-1)]
```

$$-\frac{1}{y^2} + \frac{1}{y}$$

```
C1b2[ x_, y_] := (1 - y^-1) ^2 + 2 (1 - y^-1) y^-1 Sum[1, {j, 0, Floor[x y - y]}] +
  y^-2 Sum[1, {j, 0, Floor[x y - y]}, {k, 0, Floor[ x y^2 / (j + y) - y]}]
```

```
C1b2[ 31, 2]
```

$$\frac{495}{4}$$

```
Clb[31, 2, 2]
```

$$\frac{495}{4}$$

```
Clb3[x_, y_] := (1 - y^-1)^3 + 3 (1 - y^-1)^2 y^-1 Sum[1, {j, 0, Floor[x y - y]}] +
   3 (1 - y^-1) y^-2 Sum[1, {j, 0, Floor[x y - y]}, {k, 0, Floor[x y^2 / (j + y) - y]}] +
   y^-3 Sum[1, {j, 0, Floor[x y - y]},
     {k, 0, Floor[x y^2 / (j + y) - y]}, {l, 0, Floor[x y^3 / ((j + y) (k + y)) - y]}]
```

```
Clb3[31, 2]
```

```
339
```

```
Cla[31, 3, 2]
```

```
339
```

```
tt[x_, y_] := y^-3 Sum[1, {j, 0, Floor[x y - y]},
    {k, 0, Floor[x y^2 / (j + y) - y]}, {l, 0, Floor[x y^3 / ((j + y) (k + y)) - y]}]
tt2[x_, y_] := y^-2 Sum[1, {j, 0, Floor[x y - y]}, {k, 0, Floor[x y^2 / (j + y) - y]}]
tt3[x_, y_] := y^-1 Sum[1, {j, 0, x y - y]}]
```

```
tt[30, 3.5]
```

```
135.79
```

```
3.5^-3 Dd[30 × 3.5^3, 3, 3.5]
```

```
135.72
```

```
tt2[30, 3.5]
```

```
81.7959
```

```
3.5^-2 Dd[30 × 3.5^2, 2, 3.5]
```

```
81.9592
```

```
tt3[30, 3.5]
```

```
29.1429
```

```
3.5^-1 Dd[30 × 3.5^1, 1, 3.5]
```

```
29.2857
```

```
Clb[x_, 0, y_] := 1
Clb[x_, k_, y_] :=
 (1 - y^-1) Clb[x, k - 1, y] + y^-1 Sum[Clb[x y / (j + y), k - 1, y], {j, 0, Floor[x y - y]}]
```

```
d1Alt[n_, z_] := Product[(-1)^p[[2]] Binomial[-z, p[[2]]], {p, FI[n]}];
FI[n_] := FactorInteger[n]; FI[1] := {}
c1[n_, z_, y_] := Sum[d1Alt[j, z], {j, Floor[(n - 1)] + 1, Floor[n]}]
Table[{n, c1[n, 2, 2], (Clb[n, 2, 2] - Clb[n - 1 / 2, 2, 2])}, {n, 2, 40, 1 / 2}] // TableForm
```

2	2	1
$\frac{5}{2}$	2	$\frac{5}{4}$
3	2	$\frac{3}{2}$
$\frac{7}{2}$	2	1

4	3	$\frac{7}{4}$
$\frac{9}{2}$	3	$\frac{3}{2}$
5	2	$\frac{3}{2}$
$\frac{11}{2}$	2	$\frac{3}{2}$
6	4	2
$\frac{13}{2}$	4	$\frac{5}{4}$
7	2	2
$\frac{15}{2}$	2	2
8	4	$\frac{3}{2}$
$\frac{17}{2}$	4	$\frac{3}{2}$
9	3	$\frac{11}{4}$
$\frac{19}{2}$	3	1
10	4	$\frac{5}{2}$
$\frac{21}{2}$	4	2
11	2	$\frac{3}{2}$
$\frac{23}{2}$	2	2
12	6	$\frac{5}{2}$
$\frac{25}{2}$	6	$\frac{7}{4}$
13	2	2
$\frac{27}{2}$	2	2
14	4	$\frac{5}{2}$
$\frac{29}{2}$	4	$\frac{3}{2}$
15	4	3
$\frac{31}{2}$	4	1
16	5	$\frac{11}{4}$
$\frac{33}{2}$	5	$\frac{5}{2}$
17	2	$\frac{3}{2}$
$\frac{35}{2}$	2	$\frac{5}{2}$
18	6	3
$\frac{37}{2}$	6	1
19	2	$\frac{5}{2}$
$\frac{39}{2}$	2	$\frac{5}{2}$
20	6	$\frac{5}{2}$
$\frac{41}{2}$	6	$\frac{7}{4}$
21	4	3
$\frac{43}{2}$	4	$\frac{3}{2}$
22	4	$\frac{5}{2}$
$\frac{45}{2}$	4	3
23	2	2
$\frac{47}{2}$	2	$\frac{3}{2}$
24	8	$\frac{7}{2}$

$\frac{49}{2}$	8	$-\frac{3}{2}$
25	3	$\frac{13}{4}$
$\frac{51}{2}$	3	2
26	4	2
$\frac{53}{2}$	4	$\frac{5}{2}$
27	4	3
$\frac{55}{2}$	4	2
28	6	3
$\frac{57}{2}$	6	2
29	2	2
$\frac{59}{2}$	2	2
30	8	$\frac{9}{2}$
$\frac{61}{2}$	8	$\frac{5}{4}$
31	2	2
$\frac{63}{2}$	2	$\frac{7}{2}$
32	6	2
$\frac{65}{2}$	6	$\frac{5}{2}$
33	4	3
$\frac{67}{2}$	4	$\frac{3}{2}$
34	4	$\frac{7}{2}$
$\frac{69}{2}$	4	2
35	4	3
$\frac{71}{2}$	4	$\frac{3}{2}$
36	9	$\frac{17}{4}$
$\frac{73}{2}$	9	$\frac{3}{2}$
37	2	$\frac{5}{2}$
$\frac{75}{2}$	2	3
38	4	2
$\frac{77}{2}$	4	3
39	4	$\frac{7}{2}$
$\frac{79}{2}$	4	1
40	8	$\frac{7}{2}$

```
referenceChebyshev[n_] := Sum[MangoldtLambda[j], {j, 2, n}]
num[c_] := Numerator[c]; den[c_] := Denominator[c]
alpha[n_, c_] := den[c] (Floor[n / den[c]] - Floor[(n - 1) / den[c]]) -
  num[c] (Floor[n / num[c]] - Floor[(n - 1) / num[c]])
L2[n_, 0, c_] := 1
L2[n_, 1, c_] :=
 L2[n, 1, c] = (1 / den[c]) Sum[alpha[j, c] Log[j / den[c]], {j, den[c] + 1, den[c] n}]
L2[n_, k_, c_] := L2[n, k, c] = (1 / den[c]) Sum[
    If[alpha[j, c] == 0, 0, alpha[j, c] L2[den[c] n / j, k - 1, c]], {j, den[c] + 1, den[c] n}]
ChebAlt[n_, c_] := Sum[(-1) ^ (k - 1) L2[n, k, c], {k, 0, Floor[Log[n] / Log[If[c < 2, c, 2]]]}] +
  Sum[c^k Log[c], {k, 1, Floor[Log[n] / Log[c]]}]
Grid[Table[{N[referenceChebyshev[n]], N[ChebAlt[n, (b + 1) / b]]}, {n, 5, 100, 5}, {b, 1, 4}]]
```

{4.09434, 3.09434}	{4.09434, 3.09434}	{4.09434, 3.09434}	{4.09434, 3.09434}
{7.83201, 6.83201}	{7.83201, 6.83201}	{7.83201, 6.83201}	{7.83201, 6.83201}
{12.7949, 11.7949}	{12.7949, 11.7949}	{12.7949, 11.7949}	{12.7949, 11.7949}
{19.2657, 18.2657}	{19.2657, 18.2657}	{19.2657, 18.2657}	{19.2657, 18.2657}
{24.0106, 23.0106}	{24.0106, 23.0106}	{24.0106, 23.0106}	{24.0106, 23.0106}
{28.4765, 27.4765}	{28.4765, 27.4765}	{28.4765, 27.4765}	{28.4765, 27.4765}
{32.6036, 31.6036}	{32.6036, 31.6036}	{32.6036, 31.6036}	{32.6036, 31.6036}
{36.2146, 35.2146}	{36.2146, 35.2146}	{36.2146, 35.2146}	{36.2146, 35.2146}
{43.6893, 42.6893}	{43.6893, 42.6893}	{43.6893, 42.6893}	{43.6893, 42.6893}
{49.4854, 48.4854}	{49.4854, 48.4854}	{49.4854, 48.4854}	{49.4854, 48.4854}
{53.4557, 52.4557}	{53.4557, 52.4557}	{53.4557, 52.4557}	{53.4557, 52.4557}
{57.5332, 56.5332}	{57.5332, 56.5332}	{57.5332, 56.5332}	{57.5332, 56.5332}
{62.3372, 61.3372}	{62.3372, 61.3372}	{62.3372, 61.3372}	{62.3372, 61.3372}
{66.5419, 65.5419}	{66.5419, 65.5419}	{66.5419, 65.5419}	{66.5419, 65.5419}
{75.0951, 74.0951}	{75.0951, 74.0951}	{75.0951, 74.0951}	{75.0951, 74.0951}
{79.4645, 78.4645}	{79.4645, 78.4645}	{79.4645, 78.4645}	{79.4645, 78.4645}
{84.982, 83.982}	{84.982, 83.982}	{84.982, 83.982}	{84.982, 83.982}
{89.4706, 88.4706}	{89.4706, 88.4706}	{89.4706, 88.4706}	{89.4706, 88.4706}
{89.4706, 88.4706}	{89.4706, 88.4706}	{89.4706, 88.4706}	{89.4706, 88.4706}
{94.0453, 93.0453}	{94.0453, 93.0453}	{94.0453, 93.0453}	{94.0453, 93.0453}

```
N[Sum[(c^j) / j, {j, 1, Floor[Log[10] / Log[c]]}] /. c → 1116 / 1115]
```

13.1821 + 0. i

```
N[LogIntegral[10]] - N[Log[Log[10]]] - EulerGamma
```

4.75435

```
num[c_] := Numerator[c]; den[c_] := Denominator[c]
alpha[n_, c_] := den[c] (Floor[n / den[c]] - Floor[(n - 1) / den[c]]) -
  num[c] (Floor[n / num[c]] - Floor[(n - 1) / num[c]])
L2[n_, 1, c_] := L2[n, 1, c] = (1 / den[c])
    Sum[alpha[j, c] Log[j / den[c]], {j, den[c] + 1, den[c] n}]
L2[n_, k_, c_] := L2[n, k, c] = (1 / den[c])
    Sum[If[alpha[j, c] == 0, 0, alpha[j, c] L2[den[c] n / j, k - 1, c]], {j, den[c] + 1, den[c] n}]
```

```
num[c_] := Numerator[c]; den[c_] := Denominator[c]
alpha[n_, c_] := den[c] (Floor[n / den[c]] - Floor[(n - 1) / den[c]]) -
   num[c] (Floor[n / num[c]] - Floor[(n - 1) / num[c]])
E2[n_, k_, c_] := E2[n, k, c] = (1 / den[c]) Sum[If[alpha[j, c] == 0, 0,
      alpha[j, c] E2[(den[c] n) / j, k - 1, c]], {j, den[c] + 1, den[c] n}]; E2[n_, 0, c_] := 1
L2a[n_, 0, c_] := 1
L2a[n_, k_, c_] := (1 / den[c])
   Sum[If[alpha[j, c] == 0, 0, alpha[j, c] L2a[den[c] n / j, k - 1, c]], {j, den[c] + 1, den[c] n}]
E2[n_, k_, c_] := E2[n, k, c] = (1 / den[c]) Sum[If[alpha[j, c] == 0, 0,
      alpha[j, c] E2[(den[c] n) / j, k - 1, c]], {j, den[c] + 1, den[c] n}]; E2[n_, 0, c_] := 1
L2Alt[n_, 0, c_] := 1
L2Alt[n_, k_, c_] :=
  (1 / den[c]) Sum[If[alpha[j, c] == 0, 0, alpha[j, c] Log[j / den[c]] E2[den[c] n / j, k - 1, c]],
    {j, den[c] + 1, den[c] n}]
```

```
N[L2[100, 5, 3 / 2]]
```

-70.2469

```
N[L2b[100, 5, 3 / 2]]
```

-70.2469

```
E2[100, 3, 3 / 2]
```

$\dfrac{171}{8}$

```
L2a[ 100, 3, 3 / 2]
```

$\dfrac{171}{8}$

```
referenceChebyshev[n_] := Sum[MangoldtLambda[j], {j, 2, n}]
num[c_] := Numerator[c]; den[c_] := Denominator[c]
alpha[n_, c_] := den[c] (Floor[n / den[c]] - Floor[(n - 1) / den[c]]) -
   num[c] (Floor[n / num[c]] - Floor[(n - 1) / num[c]])
E2[n_, k_, c_] := E2[n, k, c] = (1 / den[c]) Sum[If[alpha[j, c] == 0, 0,
      alpha[j, c] E2[(den[c] n) / j, k - 1, c]], {j, den[c] + 1, den[c] n}]; E2[n_, 0, c_] := 1
L2[n_, k_, c_] := (1 / den[c]) Sum[If[alpha[j, c] == 0, 0, alpha[j, c] Log[j / den[c]]
      E2[den[c] n / j, k - 1, c]], {j, den[c] + 1, den[c] n}]; L2[n_, 0, c_] := 1
ChebAlt[n_, c_] := 1 + Sum[(-1) ^ (k - 1) L2[n, k, c],
    {k, 0, Floor[Log[n] / Log[If[c < 2, c, 2]]]}] +
  Sum[c^k Log[c], {k, 1, Floor[Log[n] / Log[c]]}]
Grid[Table[{N[referenceChebyshev[n]], N[ChebAlt[n, (b + 1) / b]]}, {n, 5, 100, 5}, {b, 1, 4}]]
```

{4.09434, 4.09434}	{4.09434, 4.09434}	{4.09434, 4.09434}	{4.09434, 4.09434}
{7.83201, 7.83201}	{7.83201, 7.83201}	{7.83201, 7.83201}	{7.83201, 7.83201}
{12.7949, 12.7949}	{12.7949, 12.7949}	{12.7949, 12.7949}	{12.7949, 12.7949}
{19.2657, 19.2657}	{19.2657, 19.2657}	{19.2657, 19.2657}	{19.2657, 19.2657}
{24.0106, 24.0106}	{24.0106, 24.0106}	{24.0106, 24.0106}	{24.0106, 24.0106}
{28.4765, 28.4765}	{28.4765, 28.4765}	{28.4765, 28.4765}	{28.4765, 28.4765}
{32.6036, 32.6036}	{32.6036, 32.6036}	{32.6036, 32.6036}	{32.6036, 32.6036}
{36.2146, 36.2146}	{36.2146, 36.2146}	{36.2146, 36.2146}	{36.2146, 36.2146}
{43.6893, 43.6893}	{43.6893, 43.6893}	{43.6893, 43.6893}	{43.6893, 43.6893}
{49.4854, 49.4854}	{49.4854, 49.4854}	{49.4854, 49.4854}	{49.4854, 49.4854}
{53.4557, 53.4557}	{53.4557, 53.4557}	{53.4557, 53.4557}	{53.4557, 53.4557}
{57.5332, 57.5332}	{57.5332, 57.5332}	{57.5332, 57.5332}	{57.5332, 57.5332}
{62.3372, 62.3372}	{62.3372, 62.3372}	{62.3372, 62.3372}	{62.3372, 62.3372}
{66.5419, 66.5419}	{66.5419, 66.5419}	{66.5419, 66.5419}	{66.5419, 66.5419}
{75.0951, 75.0951}	{75.0951, 75.0951}	{75.0951, 75.0951}	{75.0951, 75.0951}
{79.4645, 79.4645}	{79.4645, 79.4645}	{79.4645, 79.4645}	{79.4645, 79.4645}
{84.982, 84.982}	{84.982, 84.982}	{84.982, 84.982}	{84.982, 84.982}
{89.4706, 89.4706}	{89.4706, 89.4706}	{89.4706, 89.4706}	{89.4706, 89.4706}
{89.4706, 89.4706}	{89.4706, 89.4706}	{89.4706, 89.4706}	{89.4706, 89.4706}
{94.0453, 94.0453}	{94.0453, 94.0453}	{94.0453, 94.0453}	{94.0453, 94.0453}

```
num[c_] := Numerator[c]; den[c_] := Denominator[c]
alpha[n_, c_] := den[c] (Floor[n / den[c]] - Floor[(n - 1) / den[c]]) -
   num[c] (Floor[n / num[c]] - Floor[(n - 1) / num[c]])
L2[n_, 1, c_] := L2[n, 1, c] = (1 / den[c])
    Sum[alpha[j, c] Log[j / den[c]], {j, den[c] + 1, den[c] n}]
L2[n_, k_, c_] := L2[n, k, c] = (1 / den[c])
    Sum[If[alpha[j, c] == 0, 0, alpha[j, c] L2[den[c] n / j, k - 1, c]], {j, den[c] + 1, den[c] n}]
E2[n_, k_, c_] := E2[n, k, c] = (1 / den[c]) Sum[If[alpha[j, c] == 0, 0,
      alpha[j, c] E2[(den[c] n) / j, k - 1, c]], {j, den[c] + 1, den[c] n}]; E2[n_, 0, c_] := 1
L2[n_, k_, c_] := (1 / den[c]) Sum[If[alpha[j, c] == 0, 0, alpha[j, c] Log[j / den[c]]
      E2[den[c] n / j, k - 1, c]], {j, den[c] + 1, den[c] n}]; L2[n_, 0, c_] := 1
L1[n_, z_, c_] := Sum[Binomial[z, k] L2[n, k, c],
   {k, 0, Floor[Log[n] / Log[If[ c > 2, 2, c]]]}]


N[L1[100, -1, 102]]

-93.0453
```