This was a cheat sheet of an intermediary notation I was trying out, that I was ultimately dissatisfied with.

One thing that is striking about the default notation for number theory is what a soup of symbols it is. In particular, its utter asymmetry is particularly stark when compared to the cleanly composed notation when working with the zeta function.

For example, take the three functions $D(n), \lfloor n \rfloor, M(n)$, for the Divisor Summatory function, the floor of n, and the Mertens function. Compare those three functions to the three functions $\zeta(s)^2, \zeta(s), \frac{1}{\zeta(s)}$. The relationship between these functions is much more symbolically obvious (of course)… but those first three functions are actually related in essentially the exact same way. And the thing is, whereas it is, for example, obvious from inspection that $\zeta(s)^2 \cdot \frac{1}{\zeta(s)} = \zeta(s)$, it's equally true that $D(n) \cdot M(n) = \lfloor n \rfloor$ IF the multiplication dot is interpreted here as Dirichlet convolution – which is to say, $\sum_{j=1}^{n} \sum_{k=1}^{\lfloor \frac{n}{j} \rfloor} d(j) \cdot \mu(k) = \lfloor n \rfloor$. In my later notation, that idea would be rewritten as

$$[\zeta(0)^2 \cdot \frac{1}{\zeta(0)}]_n = [\zeta(0)]_n ,$$

This intermediary version of the notation that follows in this particular note was partially inspired by notation used in Chapter 2 of Bateman and Diamond's Analytic Number Theory: An Introductory Course. Particularly, I use the same “*” notation in the exponential here to indicate a convolution power.

$$\zeta(s) = s \int_1^\infty \frac{\lfloor x \rfloor}{x^{s+1}} dx \qquad \frac{1}{\zeta(s)} = s \int_1^\infty \frac{M(x)}{x^{s+1}} dx$$

$$\lim_{x\to\infty} [\zeta_x(s)]^{*1} = s \int_1^\infty \frac{[\zeta_x(0)]^{*1}}{x^{s+1}} dx \qquad \lim_{x\to\infty} [\zeta_x(s)]^{*-1} = s \int_1^\infty \frac{[\zeta_x(0)]^{*-1}}{x^{s+1}} dx$$

$$\zeta(s)^z = \lim_{x\to\infty} [\zeta_x(0)]^{*z} = s \int_1^\infty \frac{[\zeta_x(0)]^{*z}}{x^{s+1}} dx$$

$$\log \zeta(s) = \lim_{x\to\infty} [\log \zeta_x(s)]^{*1} = s \int_1^\infty \frac{[\log \zeta_x(0)]^{*1}}{x^{s+1}} dx$$

All For Re(s) > 1, Above

$$M(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{n^s}{s\zeta(s)} ds \quad \text{For c > 1}$$

$$[\zeta_n(0)]^{*-1} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{n^s}{s} \lim_{x\to\infty} [\zeta_x(s)]^{*-1} ds \quad \text{For c > 1}$$

$$[\zeta_x(0)]^{*-1} = \frac{1}{2\pi i} \oint_C \frac{x^s}{s} \zeta(s)^{-1} ds = \sum_\rho \frac{x^\rho}{\rho \zeta'(\rho)} - 2 + \sum_{n=1}^\infty \frac{(-1)^{n-1} (2\pi)^{2n}}{(2n)!\, n \zeta(2n+1) x^{2n}}$$

$$[\zeta_n(0)]^{*z} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{n^s}{s} \lim_{x\to\infty} [\zeta_x(s)]^{*z} ds \quad \text{For c > 1}$$

$$[\log \zeta_n(0)]^{*1} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{n^s}{s} \lim_{x\to\infty} [\log \zeta_x(s)]^{*1} ds \quad \text{For c > 1}$$

$$[\log \zeta_n(0)]^{*1} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{n^s}{s} \lim_{x\to\infty} [\log \zeta_x(s)]^{*1} ds \quad \text{For c > 1}$$

$$[\log\zeta_x(0)]^{*1} = li(x) - \sum_{\rho} li(x^{\rho}) - \log 2 + \int_x^{\infty} \frac{dt}{t(t^2-1)\log t}$$

$$-\sum_{j=1}^{n}\left(\lim_{t\to 0}\frac{\partial}{\partial t}[d^t](j)^{*1}\right)[D^0]\left(\frac{n}{j}\right)^{*-1} = \frac{1}{2\pi i}\oint_C \frac{x^s}{s}\frac{-\zeta'(s)}{\zeta(s)}ds = x - \sum_{\rho}\frac{x^{\rho}}{\rho} - \log 2\pi - \frac{1}{2}\log(1-x^{-2})$$

$$\lim_{x\to\infty}[\zeta_n(s)]^{*1} = \pi^{s/2}\frac{\prod_{\rho}(1-\frac{s}{\rho})}{2(s-1)\Gamma(1+s/2)}$$

$$\lim_{x\to\infty}[\zeta_n(s)]^{*z} = \left(\pi^{s/2}\frac{\prod_{\rho}(1-\frac{s}{\rho})}{2(s-1)\Gamma(1+s/2)}\right)^z$$

$$\lim_{x\to\infty}[D^s](z)^{*z} = \left(\pi^{s/2}\prod_{\rho}(1-\frac{s}{\rho})2^{-1}(s-1)^{-1}\Gamma(1+s/2)^{-1}\right)^z$$

$$\lim_{x\to\infty}[\zeta_n(s)]^{*z} = \pi^{zs/2}\prod_{\rho}\left(1-\frac{s}{\rho}\right)^z 2^{-z}(s-1)^{-z}\Gamma(1+s/2)^{-z}$$

$$\lim_{x\to\infty}[\log\zeta_n(s)]^{*1} = \log\left(\pi^{s/2}\prod_{\rho}(1-\frac{s}{\rho})2^{-1}(s-1)^{-1}\Gamma(1+s/2)^{-1}\right)$$

$$\lim_{x\to\infty}[\log\zeta_n(s)]^{*1} = \frac{s}{2}\log(\pi) + \sum_{\rho}\log(1-\frac{s}{\rho}) - \log 2 - \log(s-1) - \log\Gamma(1+s/2)$$