$$[f(s)]_n = \sum_{j=1}^{n} f(j) \cdot j^{-s}$$

$$\nabla[f(s)]_n = [f(s)]_n - [f(s)]_{n-1} = f(n) \cdot n^{-s}$$

$$[f(s)^{a+b}]_n = \sum_{j \cdot k \leq n} \nabla[f(s)^a]_j \cdot \nabla[f(s)^b]_k$$

$$\nabla[f(s)^{a+b}]_n = \sum_{j \cdot k = n} \nabla[f(s)^a]_j \cdot \nabla[f(s)^b]_k$$

Identity Style 1: Via Newton's Generalized Binomial Expansion

$$[f(s)^z]_n = 1 + b(n, 2, 1) \quad \text{where} \quad b(n, j, k) = \begin{cases} \nabla[f(s)]_j \cdot \left(\frac{z+1}{k} - 1\right)\left(1 + b\left(\frac{n}{j}, 2, k+1\right)\right) + b(n, j+1, k) & \text{if } n \geq j \\ 0 & \text{if } n < j \end{cases}$$

$$\boxed{[f(s)^z]_n = \binom{z}{0} 1 + \binom{z}{1} \sum_{j=2}^{n} \nabla[f(s)]_j + \binom{z}{2} \sum_{j=2}^{n} \sum_{k=2}^{\lfloor \frac{n}{j} \rfloor} \nabla[f(s)]_j \cdot \nabla[f(s)]_k + \binom{z}{3} \sum_{j=2}^{n} \sum_{k=2}^{\lfloor \frac{n}{j} \rfloor} \sum_{l=2}^{\lfloor \frac{n}{j \cdot k} \rfloor} \nabla[f(s)]_j \cdot \nabla[f(s)]_k \cdot \nabla[f(s)]_l + \ldots}$$

$$[(f(s)-1)^k]_n = \sum_{j=0}^{k} (-1)^{k-j} \cdot \binom{k}{j} [f(s)^j]_n \quad \text{and} \quad \nabla[(f(s)-1)^k]_n = \sum_{j=0}^{k} (-1)^{k-j} \cdot \binom{k}{j} \nabla[f(s)^j]_n$$

$$[f(s)^z]_n = \sum_{k=0} \binom{z}{k} [(f(s)-1)^k]_n \quad \text{and} \quad \nabla[f(s)^z]_n = \sum_{k=0} \binom{z}{k} \nabla[(f(s)-1)^k]_n$$

Identity Style 2: As Exponentiation

$$[(\log f(s))^k]_n = \lim_{z \to 0} \frac{\partial^k}{\partial z^k} [f(s)^z]_n \quad \text{and} \quad \nabla[(\log f(s))^k]_n = \lim_{z \to 0} \frac{\partial^k}{\partial z^k} \nabla[f(s)^z]_n$$

$$[f(s)^z]_n = 1 + p(n, 2, 1) \quad \text{where} \quad p(n, j, k) = \begin{cases} \frac{z}{k} \cdot \nabla[\log f(s)]_j \cdot \left(1 + p\left(\frac{n}{j}, 2, k+1\right)\right) + p(n, j+1, k) & \text{if } n \geq j \\ 0 & \text{if } n < j \end{cases}$$

$$\boxed{[f(s)^z]_n = 1 + \frac{z^1}{1!} \sum_{j=2}^{n} \nabla[\log f(s)]_j + \frac{z^2}{2!} \sum_{j=2}^{n} \sum_{k=2}^{\lfloor \frac{n}{j} \rfloor} \nabla[\log f(s)]_j \cdot \nabla[\log f(s)]_k + \frac{z^3}{3!} \ldots}$$

$$[f(s)^z]_n = \sum_{k=0} \frac{z^k}{k!} \cdot [(\log f(s))^k]_n \quad \text{and} \quad \nabla[f(s)^z]_n = \sum_{k=0} \frac{z^k}{k!} \cdot \nabla[(\log f(s))^k]_n$$

Identity Style 3: Via The Hyperbola Method

$$\boxed{[f(s)^z]_n = h(n, 2, z) \text{ where } h(n, y, z) = \begin{cases} \displaystyle\sum_{k=0}^{\lfloor \frac{\log n}{\log y} \rfloor} \binom{z}{k} \cdot (\nabla[f(s)]_y)^k \cdot h\left(\frac{n}{y^k}, y+1, z-k\right) & \text{if } n \geq y \\ 1 & \text{if } n < y \end{cases}}$$

Identity Style 4: As a Sum of Multiplicative Functions

$$[f(s)^z]_n=\sum_{j=1}^{n}\prod_{p^k|j} f(p,k)$$

Identity Style 5: As a Product of Zeros

For fixed values of *n,* and *s* $[f(s)^z]_n$ is a typical polynomial of order $\lfloor\frac{\log n}{\log 2}\rfloor$ and can be expressed in terms of its zeros in the usual way.

$$[\log(f(s)^x)]_n=x\cdot[\log f(s)]_n$$

$$[\log(f(s)\cdot g(t))]_n=[\log f(s)]_n+[\log g(t)]_n$$

$$[\log\frac{f(s)}{g(t)}]_n=[\log f(s)]_n-[\log g(t)]_n$$

$$[\log(\prod_{k=1} f(k\cdot s))]_n=\sum_{k=1}[\log f(k\,s)]_n$$

$$[\log(\prod_{k=1} f(k\cdot s)^k)]_n=\sum_{k=1}[k\cdot\log f(k\,s)]_n$$

Table 6

Variant of $[\zeta(s)^z]_n$	$\nabla[f(s)]_n$	$\nabla[\log f(s)]_n$	$\sum_{j=1}^{n}\prod_{p^k \mid j} f(p,k)$
$[(\frac{\zeta_{1/2}(2s)}{\zeta(s)})^z]_n$	$\lambda(n)\cdot n^{-s}$	$(\kappa(n)-\kappa(n^{1/2}))\cdot n^{-s}$	$\frac{(-1)^k\cdot(-z)^{(k)}}{k!}\cdot p^{-sk}$
$[(\zeta(s-a)\cdot\zeta(s))^z]_n$	$\sigma_a(n)\cdot n^{-s}$	$(\kappa(n)\cdot n^a+\kappa(n))\cdot n^{-s}$	$\frac{z^{(k)}}{k!}\cdot p^{-sk}\cdot {}_2F_1(-k;z;1-k-z;p^a)$
$[(\frac{\zeta(s-a)}{\zeta(s)})^z]_n$	$J_a(n)\cdot n^{-s}$	$(\kappa(n)\cdot n^a-\kappa(n))\cdot n^{-s}$	$\frac{(-z)^{(k)}}{k!}\cdot p^{-sk}\cdot {}_2F_1(-k;z;1-k+z;p^a)$
$[(\prod_{k=1}\zeta_{1/k}(ks))^z]_n$	$a(n)\cdot n^{-s}$	$\begin{cases}\frac{\sigma(k)}{k}\cdot n^{-s} & \text{if } n=p^k\\ 0 & \text{otherwise}\end{cases}$	$\sum_{j=1}^{k}\frac{z^j}{j!}\cdot p_j(k)$
$[(\prod_{k=1}\zeta_{1/k}(ks)^{\frac{\mu(k)}{k}})^z]_n$	$b(n)\cdot n^{-s}$	$\begin{cases}n^{-s} & \text{if } n \textit{ is prime}\\ 0 & \text{otherwise}\end{cases}$	$\frac{z^k}{k!}\cdot p^{-sk}$
$[(2^{-2s}(\zeta(s,\frac{1}{4})-\zeta(s,\frac{3}{4})))^z]_n$	$\cos(\frac{\pi}{2}\cdot(n-1))\cdot n^{-s}$	$\begin{cases}(-1)^{k(p-1)/2}\cdot\kappa(n) & \text{if } 2\nmid n\\ 0 & \text{otherwise}\end{cases}$	$\begin{cases}(-1)^{k(p-1)/2}\cdot p^{-sk}\cdot\frac{z^{(k)}}{k!} & \text{if } p\neq 2\\ 0 & \text{if } p=2\end{cases}$

$$p_k(n)=\begin{cases}\sum_{j=1}^{n-1}\frac{\sigma(j)}{j}\cdot p_{k-1}(n-j) & \text{if } k>1\\ \frac{\sigma(n)}{n} & \text{if } k=1\end{cases}$$

$$b(n)=\prod_{p^k \| n}\frac{1}{k!}$$