```
Clear[dz]
bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
FI[n_] := FactorInteger[n]; FI[1] := {}
dz[n_, z_] := dz[n, z] = Product[(-1)^p[[2]] bin[-z, p[[2]]], {p, FI[n]}]
DZ[n_, s_, z_] := Sum[ j^-s dz[j, z], {j, 1, n}]
Dz[n_, s_, z_, k_] := 1 + ((z + 1) / k - 1) Sum[j^-s Dz[n / j, s, z, k + 1], {j, 2, n}]
D1xD[n_, s_, k_, x_] :=
 Sum[(j + 1)^-s D1xD[n / (j + 1), s, k - 1, x] - x (j x)^-s D1xD[n / (x j), s, k - 1, x], {j, 1, n}]
D1xD[n_, s_, 0, x_] := UnitStep[n - 1]
DxD[n_, s_, z_, x_] :=
 Sum[bin[z, k] D1xD[n, s, k, x], {k, 0, If[x < 2, Log[x, n], Log[2, n]]}]
DxDAlt[n_, s_, z_, x_] :=
 Sum[(-1)^j bin[z, j] x^(j (1 - s)) DZ[n / x^j, s, z], {j, 0, Log[If[x < 2, x, 2], n]}]


Expand@DxD[100, 0, z, 3 / 2]
```

$$1-\frac{8\,149\,753\,z}{2\,365\,440}+\frac{44\,308\,889\,z^2}{6\,451\,200}-\frac{3\,885\,851\,z^3}{1\,075\,200}-\frac{47\,910\,697\,z^4}{41\,287\,680}+\frac{32\,326\,801\,z^5}{13\,762\,560}-\frac{159\,089\,057\,z^6}{117\,964\,800}+\frac{110\,725\,357\,z^7}{275\,251\,200}-\frac{490\,059\,z^8}{9\,175\,040}+\frac{21\,519\,z^9}{9\,175\,040}+\frac{2673\,z^{10}}{91\,750\,400}-\frac{2187\,z^{11}}{1\,009\,254\,400}$$

```
Expand@DxDAlt[100, 0, z, 3 / 2]
```

$$1-\frac{8\,149\,753\,z}{2\,365\,440}+\frac{44\,308\,889\,z^2}{6\,451\,200}-\frac{3\,885\,851\,z^3}{1\,075\,200}-\frac{47\,910\,697\,z^4}{41\,287\,680}+\frac{32\,326\,801\,z^5}{13\,762\,560}-\frac{159\,089\,057\,z^6}{117\,964\,800}+\frac{110\,725\,357\,z^7}{275\,251\,200}-\frac{490\,059\,z^8}{9\,175\,040}+\frac{21\,519\,z^9}{9\,175\,040}+\frac{2673\,z^{10}}{91\,750\,400}-\frac{2187\,z^{11}}{1\,009\,254\,400}$$

```
DxDAlt2[n_, s_, z_, x_] :=
 Sum[(-1)^j bin[z, j] x^(j (1 - s)) Sum[ j^-s dz[j, z], {j, 1, n / x^j}],
   {j, 0, Log[If[x < 2, x, 2], n]}]
Expand@DxDAlt2[100, 0, z, 3 / 2]
```

$$1-\frac{8\,149\,753\,z}{2\,365\,440}+\frac{44\,308\,889\,z^2}{6\,451\,200}-\frac{3\,885\,851\,z^3}{1\,075\,200}-\frac{47\,910\,697\,z^4}{41\,287\,680}+\frac{32\,326\,801\,z^5}{13\,762\,560}-\frac{159\,089\,057\,z^6}{117\,964\,800}+\frac{110\,725\,357\,z^7}{275\,251\,200}-\frac{490\,059\,z^8}{9\,175\,040}+\frac{21\,519\,z^9}{9\,175\,040}+\frac{2673\,z^{10}}{91\,750\,400}-\frac{2187\,z^{11}}{1\,009\,254\,400}$$

```
DxDAlt2[n_, s_, z_, x_] :=
 Sum[ Sum[(-1)^j bin[z, j] x^(j (1 - s)) r^-s dz[r, z], {r, 1, n / x^j}],
   {j, 0, Log[If[x < 2, x, 2], n]}]
Expand@DxDAlt2[100, 0, z, 3 / 2]
```

$$1-\frac{8\,149\,753\,z}{2\,365\,440}+\frac{44\,308\,889\,z^2}{6\,451\,200}-\frac{3\,885\,851\,z^3}{1\,075\,200}-\frac{47\,910\,697\,z^4}{41\,287\,680}+\frac{32\,326\,801\,z^5}{13\,762\,560}-\frac{159\,089\,057\,z^6}{117\,964\,800}+\frac{110\,725\,357\,z^7}{275\,251\,200}-\frac{490\,059\,z^8}{9\,175\,040}+\frac{21\,519\,z^9}{9\,175\,040}+\frac{2673\,z^{10}}{91\,750\,400}-\frac{2187\,z^{11}}{1\,009\,254\,400}$$

```
DxDAlt2[n_, s_, z_, x_] := Sum[(-1)^j bin[z, j] x^(j (1 - s)) r^-s dz[r, z],
   {j, 0, Log[If[x < 2, x, 2], n]}, {r, 1, n / x^j}]
Expand@DxDAlt2[100, 0, z, 3 / 2]
```

$$1-\frac{8\,149\,753\,z}{2\,365\,440}+\frac{44\,308\,889\,z^2}{6\,451\,200}-\frac{3\,885\,851\,z^3}{1\,075\,200}-\frac{47\,910\,697\,z^4}{41\,287\,680}+\frac{32\,326\,801\,z^5}{13\,762\,560}-\frac{159\,089\,057\,z^6}{117\,964\,800}+\frac{110\,725\,357\,z^7}{275\,251\,200}-\frac{490\,059\,z^8}{9\,175\,040}+\frac{21\,519\,z^9}{9\,175\,040}+\frac{2673\,z^{10}}{91\,750\,400}-\frac{2187\,z^{11}}{1\,009\,254\,400}$$

```
FullSimplify[(-1)^j bin[z, j] x^(j (1 - s)) r^-s]
```

$$-\frac{(-1)^{2j}\,r^{-s}\,x^{j-j\,s}\,z\,\text{Pochhammer}[1-z,\,-1+j]}{j!}$$

```
DxDAlt2[n_, s_, z_, x_] := Sum[-(r^-s x^(j (1-s)) z Pochhammer[1 - z, -1 + j])/j! dz[r, z],
  {j, 0, Log[If[x < 2, x, 2], n]}, {r, 1, n / x^j}]
D[Expand@DxDAlt2[100, 0, z, 2], z] /. z → 0
```

$$\frac{4}{5}$$

```
D[-(r^-s x^j (1-s) z Pochhammer[1 - z, -1 + j])/j!, z] /. z → 0
```

$$-\frac{r^{-s}\,x^{j\,(1-s)}\,\text{Pochhammer}[1,\,-1+j]}{j!}$$

```
Table[Pochhammer[1, j - 1] / (j!), {j, 1, 10}]
```

$$\left\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\frac{1}{6},\frac{1}{7},\frac{1}{8},\frac{1}{9},\frac{1}{10}\right\}$$

```
DxDAlt2[n_, s_, x_] := D[Sum[-(r^-s x^(j (1-s)) z Pochhammer[1 - z, -1 + j])/j! dz[r, z],
    {j, 0, Log[If[x < 2, x, 2], n]}, {r, 1, n / x^j}], z] /. z → 0
Expand@
 DxDAlt2[
  100,
  0,
  2]
```

$$\frac{4}{5}$$

```
DxDAlt2[n_, s_, x_] := Sum[D[-(r^-s x^(j (1-s)) z Pochhammer[1 - z, -1 + j])/j! dz[r, z], z],
   {j, 0, Log[If[x < 2, x, 2], n]}, {r, 1, n / x^j}] /. z → 0
Expand@DxDAlt2[100, 0, 2]
```

$$\frac{4}{5}$$

```
DxDAlt2[n_, s_, x_] := Sum[D[-(r^-s x^j (1-s) z Pochhammer[1 - z, -1 + j])/j! dz[r, z], z] /. z → 0,
  {j, 0, Log[If[x < 2, x, 2], n]}, {r, 1, n / x^j}]
Expand@DxDAlt2[100, 0, 2]
```

$$\frac{4}{5}$$

```
DxDAlt2[n_, s_, x_] := Sum[D[-(r^-s x^j (1-s) z Pochhammer[1 - z, -1 + j])/j! dz[r, z], z] /. z → 0,
  {j, 0, Log[If[x < 2, x, 2], n]}, {r, 1, n / x^j}]
Expand@DxDAlt2[100, 0, 2]
```

$$\frac{4}{5}$$

```
FullSimplify[D[-(r^-s x^j (1-s) z Pochhammer[1 - z, -1 + j])/j! ddz[r, z], z] /. z → 0]
```

$$-\frac{r^{-s}\, x^{j-j\,s}\, \text{ddz}[r, 0]}{j}$$

```
DxDAlt2[n_, s_, x_] := Sum[D[-(r^-s x^j (1-s) z Pochhammer[1 - z, -1 + j])/j! dz[r, z], z] /. z → 0,
  {j, 0, Log[If[x < 2, x, 2], n]}, {r, 1, n / x^j}]
Expand@DxDAlt2[100, 0, 2]
```

$$\frac{4}{5}$$

```
Sum[D[-(r^-s x^j (1-s) z Pochhammer[1 - z, -1 + j])/j! ddz[r, z], z] /. z → 0, {j, 0, Log[x, n]}]
```

$$\sum_{j=0}^{\frac{\text{Log}[n]}{\text{Log}[x]}} -\frac{r^{-s}\, x^{j\,(1-s)}\, \text{ddz}[r, 0]\, \text{Pochhammer}[1, -1+j]}{j!}$$

```
Table[D[z Pochhammer[1 - z, j - 1] / j!, z] /. z → 0, {j, 0, 10}]
```

$$\left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}\right\}$$

```
DxDAlt2[n_, s_, x_] :=
 Sum[D[-r^-s z Pochhammer[1 - z, -1] dz[r, z], z] /. z → 0, {j, 0, 0}, {r, 1, n / x^j}] +
  Sum[D[-(r^-s x^j (1-s) z Pochhammer[1 - z, -1 + j])/j! dz[r, z], z] /. z → 0,
   {j, 1, Log[If[x < 2, x, 2], n]}, {r, 1, n / x^j}]
Expand@DxDAlt2[100, 0, 2]
```

$$\frac{4}{5}$$

```
D[-r^-s z Pochhammer[1 - z, -1] ddz[r, z], z] /. z → 0
```

$$r^{-s}\, \text{ddz}^{(0,1)}[r, 0]$$

```
DxDAlt2[n_, s_, x_] := Sum[r^-s D[ dz[r, z], z] /. z → 0, {j, 0, 0}, {r, 1, n / x^j}] +
  Sum[D[-r^-s x^j (1-s) z Pochhammer[1 - z, -1 + j] / j ! dz[r, z], z] /. z → 0,
   {j, 1, Log[If[x < 2, x, 2], n]}, {r, 1, n / x^j}]
Expand@DxDAlt2[100, 0, 2]
```

$$\frac{4}{5}$$

```
DxDAlt2[n_, s_, x_] := Sum[r^-s D[ dz[r, z], z] /. z → 0, {j, 0, 0}, {r, 1, n / x^j}]
Expand@DxDAlt2[100, 0, 2]
```

$$\frac{428}{15}$$

```
Table[D[(-r^-s x^j (1-s) z Pochhammer[1 - z, -1 + j] / j ! ddz[r, z]), z] /. z → 0, {j, 1, 10}]
```

$$\left\{-r^{-s}\, x^{1-s}\, \text{ddz}[r, 0],\ -\frac{1}{2} r^{-s}\, x^{2(1-s)}\, \text{ddz}[r, 0],\ -\frac{1}{3} r^{-s}\, x^{3(1-s)}\, \text{ddz}[r, 0],\ -\frac{1}{4} r^{-s}\, x^{4(1-s)}\, \text{ddz}[r, 0],\right.$$
$$-\frac{1}{5} r^{-s}\, x^{5(1-s)}\, \text{ddz}[r, 0],\ -\frac{1}{6} r^{-s}\, x^{6(1-s)}\, \text{ddz}[r, 0],\ -\frac{1}{7} r^{-s}\, x^{7(1-s)}\, \text{ddz}[r, 0],$$
$$\left.-\frac{1}{8} r^{-s}\, x^{8(1-s)}\, \text{ddz}[r, 0],\ -\frac{1}{9} r^{-s}\, x^{9(1-s)}\, \text{ddz}[r, 0],\ -\frac{1}{10} r^{-s}\, x^{10(1-s)}\, \text{ddz}[r, 0]\right\}$$

```
DxDAlt2[n_, s_, x_] := Sum[r^-s D[ dz[r, z], z] /. z → 0, {j, 0, 0}, {r, 1, n / x^j}] +
  Sum[-r^-s x^j (1-s) / j dz[r, 0], {j, 1, Log[If[x < 2, x, 2], n]}, {r, 1, n / x^j}]
Expand@DxDAlt2[100, 0, 2]
```

$$\frac{4}{5}$$

```
DxDAlt2[n_, s_, x_] := Sum[r^-s D[ dz[r, z], z] /. z → 0, {j, 0, 0}, {r, 1, n / x^j}] +
  Sum[-r^-s x^j (1-s) / j dz[r, 0], {j, 1, Log[If[x < 2, x, 2], n]}, {r, 1, 1}]
Expand@DxDAlt2[100, 0, 2]
```

$$\frac{4}{5}$$

```
DxDAlt2[n_, s_, x_] := Sum[r^-s D[ dz[r, z], z] /. z → 0, {j, 0, 0}, {r, 1, n / x^j}] +
  Sum[-1^-s x^j (1-s) / j dz[1, 0], {j, 1, Log[If[x < 2, x, 2], n]}]
Expand@DxDAlt2[100, 0, 2]
```

$$\frac{4}{5}$$

```
DxDAlt2[n_, s_, x_] := Sum[r^-s D[ dz[r, z], z] /. z → 0, {j, 0, 0}, {r, 1, n / x^j}] +
  Sum[-x^j (1-s) / j, {j, 1, Log[If[x < 2, x, 2], n]}]
Expand@DxDAlt2[100, 0, 2]
```

$$\frac{4}{5}$$