```
(*For the following block of code,
all you really need to know are what the following 2 functions
 do.-"Use[n]" is 1 if n is not divisible by the first WheelEntries primes,
0 otherwise-"Coprimes[n]" is the sum of Use[1] through Use[n]*)
Clear[d]
WheelEntries = 4;
FirstNonWheel = 11;
WheelSize := Product[Prime[j], {j, 1, WheelEntries}];
CoprimeCache := CoprimeCache = Table[CoprimeQ[WheelSize, n], {n, 1, WheelSize}]
Use[n_] := Use[n] = If[CoprimeCache[[Mod[n - 1, WheelSize] + 1]] == True, 1, 0]
LegendrePhi[x_, a_] :=
 LegendrePhi[x, a] = LegendrePhi[x, a - 1] - LegendrePhi[x / Prime[a], a - 1]
LegendrePhi[x_, 0] := Floor[x]
LegPhiCache := LegPhiCache = Table[LegendrePhi[n, WheelEntries], {n, 1, WheelSize}]
FullWheel := LegendrePhi[WheelSize, WheelEntries];
Coprimes[n_] :=
 LegPhiCache[[Mod[n - 1, WheelSize] + 1]] + Floor[(n - 1) / WheelSize] FullWheel

(*There are only 3 changes from the previous version of this code-
 the use of "Coprimes" in d[n,1,a]-the use of "Use" in d[n,k,a]-
 the addition of "WheelEntries" in CountPrimes.*)
d[n_, 0, a_] := 1
d[n_, 1, a_] := d[n, 1, a] = Coprimes[n] - Coprimes[a]
d[n_, k_, a_] :=
 d[n, k, a] = Sum[If[Use[m] == 0, 0, Binomial[k, j] d[Floor[n / (m^(k - j))], j, m]],
   {m, a + 1, n^(1 / k)}, {j, 0, k - 1}]
RiemannPrimeCounting[n_] := Sum[(-1)^(k + 1) / k d[Floor[n], k, FirstNonWheel - 1],
  {k, 1, Log[FirstNonWheel, n]}]
CountPrimes[n_] := WheelEntries +
  Sum[MoebiusMu[k] RiemannPrimeCounting[n^(1 / k)] / k, {k, 1, Log[2, n]}]
```

```
Timing[CountPrimes[10^10]]
```

```
{55.459, 417828337}
```

```
Sum[If[ CoprimeCache[[j]] == True, 1, 0], {j, WheelSize}]
```

```
480
```

```
WheelSize
```

```
2310
```

```
(* What I want here is a map from index space to wheel space.

  so, 1→17
      2→19


 *)
```

```
CoprimeCache
```

```
{True, False, False, False, False, False, False, False, False, False, True, False, True,
 False, False, False, True, False, True, False, False, False, True, False, False, False,
 False, False, True, False, True, False, False, False, False, False, True, False, False,
 False, True, False, True, False, False, False, True, False, False, False, False,
 False, True, False, False, False, False, False, True, False, True, False, False,
 False, False, False, True, False, False, False, True, False, True, False, False,
 False, False, False, True, False, False, False, True, False, False, False, False,
 False, True, False, False, False, False, False, False, False, True, False, False,
 False, True, False, True, False, False, False, True, False, True, False, False, False,
 True, False, False, False, False, False, False, False, True, False, False, False,
 False, False, True, False, False, False, True, False, False, False, False, False,
 True, False, True, False, False, False, True, False, False, False, False, False,
 True, False, True, False, False, False, False, False, True, False, False, False,
 False, False, True, False, False, False, True, False, True, False, False, False, True,
 False, False, False, False, False, True, False, True, False, False, False, False,
 False, True, False, False, False, True, False, True, False, False, False, True, False,
 True, False, False, False, False, False, False, False, False, False, True, False}
```

```
x := 1
```

```
x2 := Prepend[{1, 2, 3}, x2]
```

```
Position[{2, 4, 6, 8, 10}, _?(# > 7 &)]
```

```
{{4}, {5}}
```

```
CoprimeOffsets := Flatten[Position[CoprimeCache, _?(# == True &)]]
```

```
CoprimeOffsets
```

```
{1, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71,
 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 121, 127, 131, 137, 139, 143,
 149, 151, 157, 163, 167, 169, 173, 179, 181, 187, 191, 193, 197, 199, 209}
```

```
FullWheel
```

```
48
```

```
WheelSize
```

```
210
```

```
doid[n_] := CoprimeOffsets[[Mod[n - 1, FullWheel] + 1]] + Floor[(n - 1) / FullWheel] WheelSize
```

```
doid[50]
```

```
221
```

```
Clear[ CoprimeOffsets, CoprimeCache, WheelForID]
bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
WheelEntries = 6;
WheelSize := Product[Prime[j], {j, 1, WheelEntries}];
CoprimeCache := CoprimeCache = Table[If[CoprimeQ[WheelSize, n], 1, 0], {n, 1, WheelSize}]
LegendrePhi[x_, a_] :=
 LegendrePhi[x, a] = LegendrePhi[x, a - 1] - LegendrePhi[x / Prime[a], a - 1]
LegendrePhi[x_, 0] := Floor[x]
LegPhiCache3 := LegPhiCache3 = Accumulate[CoprimeCache]
FullWheel := LegendrePhi[WheelSize, WheelEntries];
Coprimes[n_] :=
 LegPhiCache3[[Mod[n - 1, WheelSize] + 1]] + Floor[(n - 1) / WheelSize] FullWheel
CoprimeOffsets := CoprimeOffsets = Flatten[Position[CoprimeCache, _?(# == 1 &)]]
WheelForID[n_] := WheelForID[n] =
  CoprimeOffsets[[Mod[n - 1, FullWheel] + 1]] + Floor[(n - 1) / FullWheel] WheelSize
FirstNonWheel := WheelForID[2]

Clear[DB]
DB[n_, 0, a_] := 1
DB[n_, 1, a_] := DB[n, 1, a] = Coprimes[n] - Coprimes[WheelForID[a]]
DB[n_, k_, a_] := DB[n, k, a] = Sum[bin[k, j] DB[Floor[n / (WheelForID[m]^(k - j))], j, m],
    {m, a + 1, Coprimes[Floor[n^(1 / k)]]}, {j, 0, k - 1}]
DBz[n_, z_] := Expand[Sum[bin[z, k] DB[Floor[n], k, 1], {k, 0, Log[FirstNonWheel, n]}]]
RiemannPrimeCounting[n_] :=
 Sum[(-1)^(k + 1) / k DB[Floor[n], k, 1], {k, 1, Log[FirstNonWheel, n]}]
CountPrimes[n_] := WheelEntries +
  Sum[MoebiusMu[k] RiemannPrimeCounting[n^(1 / k)] / k, {k, 1, Log[2, n]}]
DBzeros[n_] := List @@ NRoots[DBz[n, z] == 0, z][[All, 2]]
DBzAlt[n_, z_] := Expand[Product[1 - z / rho, {rho, DBzeros[n]}]]

AbsoluteTiming[DBz[1 000 000 000, z]]
```

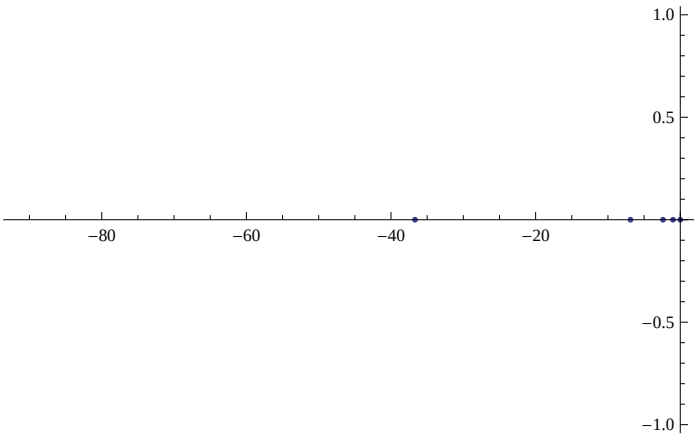
$$\left\{3.5322020,\ 1+\frac{20\,328\,191\,839\,z}{420}+\frac{29\,384\,294\,573\,z^2}{360}+\frac{3\,096\,475\,919\,z^3}{72}+\frac{1\,028\,280\,641\,z^4}{144}+\frac{253\,213\,241\,z^5}{720}+\frac{1\,961\,089\,z^6}{720}+\frac{199\,z^7}{1008}\right\}$$

$$1+\frac{345\,725\,687\,z}{60}+\frac{410\,984\,831\,z^2}{45}+\frac{61\,241\,237\,z^3}{16}+\frac{64\,556\,533\,z^4}{144}+\frac{2\,360\,137\,z^5}{240}+\frac{5399\,z^6}{720}$$

```
DBzeros[100 000 000]
```

$$\left\{-1264.47,\ -36.6817,\ -6.88326,\ -2.39112,\ -1.00658,\ -1.73548\times10^{-7}\right\}$$

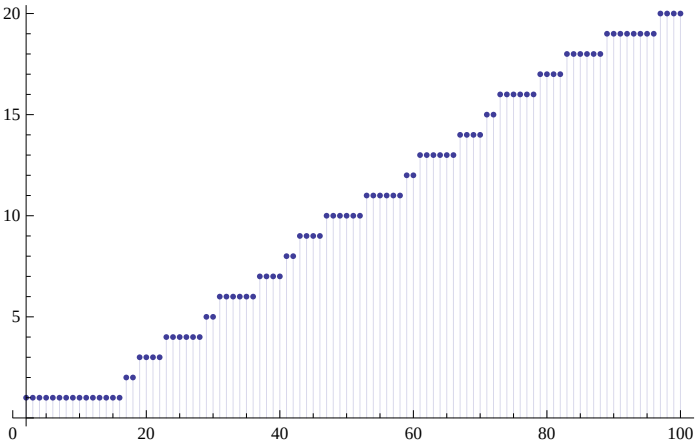
```
ListPlot[Table[{Re[n], Im[n]}, {n, DBzeros[100 000 000]}]]
```




```
DBz[1000, z]
```

$$1 + \frac{329\,z}{2} + \frac{49\,z^2}{2}$$

```
DiscretePlot[DBz[n, 1], {n, 2, 100}]
```




```
Clear[Dk]
Dk[n_, s_, 0] := UnitStep[n - 1]
Dk[n_, s_, k_] :=
 Dk[n, k] = Sum[WheelForID[j]^-s Dk[Floor[n / WheelForID[j]], s, k - 1], {j, 2, Coprimes[n]}]
Dkz[n_, s_, z_] := Expand[Sum[bin[z, k] Dk[Floor[n], s, k], {k, 0, Log[FirstNonWheel, n]}]]
```

```
DBz[100 000, z]
```

$$1 + \frac{115\,421\,z}{12} + \frac{209\,983\,z^2}{24} + \frac{9745\,z^3}{12} + \frac{5\,z^4}{24}$$

```
-N[D[D[Dkz[10 000, s, z], z] /. z → 0, s] /. s → 0] +
 Sum[ Log[Prime[j]] * Floor[Log[Prime[j], 10 000]], {j, 1, WheelEntries}]
```

```
10 013.4
```

```
Sum[ N[MangoldtLambda[j]], {j, 2, 10 000}]
```

```
10 013.4
```

```
Prime[1
]
```

```
2
```

```
(D[Dkz[10 000, -1, z], z] /. z → 0) +
 Sum[k^-1 Prime[j], {j, 1, WheelEntries}, {k, 1, Log[Prime[j], 10 000]}]
```

$$\frac{2\,080\,384\,591\,267}{360\,360}$$

```
Sum[ FullSimplify[ j MangoldtLambda[j] / Log[j]], {j, 2, 10 000}]
```

$$\frac{2\,082\,395\,892\,559}{360\,360}$$

```
Sum[k^-1, {j, 1, WheelEntries}, {k, 1, Log[Prime[j], 100]}]
```

$$\frac{143}{15}$$

```
D[Dkz[100, 0, z], z] /. z → 0
```

```
19
```

```
Clear[ CoprimeOffsets, CoprimeCache, WheelForID, Coprimes, WheelSize, LegPhiCache]
bin[z_, k_] := bin[z, k] = Product[z - j, {j, 0, k - 1}] / k!
WheelEntries = 7;
WheelSize := WheelSize = Product[Prime[j], {j, 1, WheelEntries}];
CoprimeCache := CoprimeCache = Table[If[CoprimeQ[WheelSize, n], 1, 0], {n, 1, WheelSize}]
LegPhiCache := LegPhiCache = Accumulate[CoprimeCache]
FullWheel := FullWheel = LegPhiCache[[WheelSize]]
CoprimeOffsets := CoprimeOffsets = Flatten[Position[CoprimeCache, _?(# == 1 &)]]
Coprimes[n_] :=
 Coprimes[n] = LegPhiCache[[Mod[n - 1, WheelSize] + 1]] + Floor[(n - 1) / WheelSize] FullWheel
WheelForID[n_] := WheelForID[n] =
  CoprimeOffsets[[Mod[n - 1, FullWheel] + 1]] + Floor[(n - 1) / FullWheel] WheelSize
FirstNonWheel := FirstNonWheel = WheelForID[2]
```

```
Clear[DB]
DB[n_, 0, a_] := 1
DB[n_, 1, a_] := DB[n, 1, a] = Coprimes[n] - Coprimes[WheelForID[a]]
DB[n_, 2, a_] := DB[n, 2, a] = (Coprimes[Floor[n^(1 / 2)]] - a) +
   2 Sum[ DB[Floor[n / WheelForID[m]], 1, m], {m, a + 1, Coprimes[Floor[n^(1 / 2)]]}]
DB[n_, k_, a_] := DB[n, k, a] = (Coprimes[Floor[n^(1 / k)]] - a) +
   k Sum[ DB[Floor[n / (WheelForID[m]^(k - 1))], 1, m], {m, a + 1, Coprimes[Floor[n^(1 / k)]]}] +
   Sum[bin[k, j] DB[Floor[n / (WheelForID[m]^(k - j))], j, m],
    {m, a + 1, Coprimes[Floor[n^(1 / k)]]}, {j, 2, k - 1}]
DBz[n_, z_] := Expand[Sum[bin[z, k] DB[Floor[n], k, 1], {k, 0, Log[FirstNonWheel, n]}]]
RiemannPrimeCounting[n_] :=
 Sum[(-1)^(k + 1) / k DB[Floor[n], k, 1], {k, 1, Log[FirstNonWheel, n]}]
CountPrimes[n_] := WheelEntries +
  Sum[MoebiusMu[k] RiemannPrimeCounting[n^(1 / k)] / k, {k, 1, Log[2, n]}]
DBzeros[n_] := List @@ NRoots[DBz[n, z] == 0, z][[All, 2]]
DBzAlt[n_, z_] := Expand[Product[ 1 - z / rho, {rho, DBzeros[n]}]]
```

```
AbsoluteTiming[DBz[100 000 000 000, z]]
```

$$\left\{29.6836978,\ 1+\frac{86\,479\,442\,449\ z}{21}+\frac{7\,940\,004\,689\,507\ z^2}{1008}+\frac{1\,153\,363\,494\,317\ z^3}{240}+\right.$$
$$\left.\frac{3\,301\,789\,279\,217\ z^4}{2880}+\frac{4\,925\,259\,257\ z^5}{48}+\frac{783\,748\,475\ z^6}{288}+\frac{9\,638\,813\ z^7}{840}+\frac{20\,491\ z^8}{20\,160}\right\}$$

$$\mathbf{1+\frac{345\,725\,687\ z}{60}+\frac{410\,984\,831\ z^2}{45}+\frac{61\,241\,237\ z^3}{16}+\frac{64\,556\,533\ z^4}{144}+\frac{2\,360\,137\ z^5}{240}+\frac{5399\ z^6}{720}}$$

```
AbsoluteTiming[DBzAlt[100 000 000 000, 1]]
```

$$\{0.0020001,\ 1.80525\times 10^{10}\}$$

```
DBzeros[100 000 000 000]
```

$$\{-11\,047.9,\ -197.953,\ -28.2135,\ -8.55061,\ -3.75578,\ -2.04436,\ -1.00016,\ -2.42832\times 10^{-10}\}$$

```
CountPrimes[100 000 000 000]
```

4 118 054 813

```
AbsoluteTiming[PrimePi[100 000 000 000]]
```

{0.4420253, 4 118 054 813}

```
WheelSize
```

510 510

```
10^11
```

100 000 000 000