```
E1[n_, k_, x_] := Sum[E1[n / j, k - 1, x], {j, 1, n}] - x Sum[E1[n / (x j), k - 1, x], {j, 1, n / x}];
E1[n_, 0, x_] := 1
E2[n_, k_, x_] :=
 E2[n, k, x] = Sum[E2[n / j, k - 1, x], {j, 2, n}] - x Sum[E2[n / (x j), k - 1, x], {j, 1, n / x}];
E2[n_, 0, x_] := 1
bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
E1Alt[n_, z_, x_] := Sum[bin[z, k] E2[n, k, x], {k, 0, Log[If[x < 2, x, 2], n]}]
zeros[n_, x_] := zeros[n, x] = List @@ NRoots[E1Alt[n, z, x] == 0, z][[All, 2]]
```

```
Expand[E1Alt[10, z, 1.1]]
```

$$1 - 3.27641\,z + 5.48472\,z^2 - 5.97008\,z^3 + 4.86101\,z^4 - 3.06271\,z^5 + 1.48555\,z^6 - 0.557369\,z^7 + 0.169716\,z^8 - 0.0413481\,z^9 + 0.00803162\,z^{10} - 0.00124643\,z^{11} + 0.00015447\,z^{12} - 0.0000153414\,z^{13} + 1.22862\times10^{-6}\,z^{14} - 7.97563\times10^{-8}\,z^{15} + 4.20639\times10^{-9}\,z^{16} - 1.80066\times10^{-10}\,z^{17} + 6.22029\times10^{-12}\,z^{18} - 1.7114\times10^{-13}\,z^{19} + 3.66415\times10^{-15}\,z^{20} - 5.88043\times10^{-17}\,z^{21} + 6.65205\times10^{-19}\,z^{22} - 4.72792\times10^{-21}\,z^{23} + 1.58752\times10^{-23}\,z^{24}$$

```
zeros[10, 1.1]
```

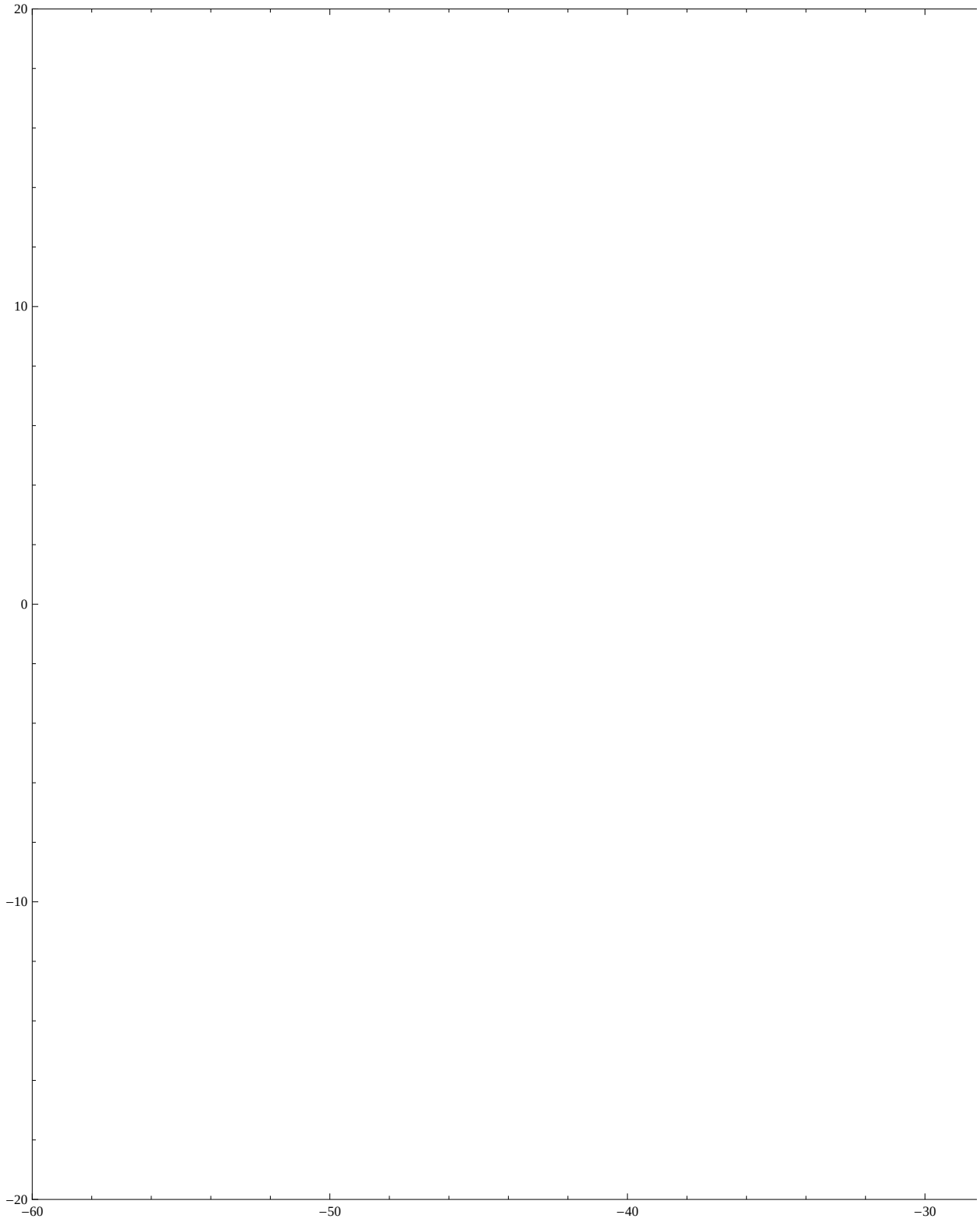
```
{0.0444587 - 3.42988 i, 0.0444587 + 3.42988 i, 0.0546023 - 1.48706 i,
 0.0546023 + 1.48706 i, 0.795269 - 0.646218 i, 0.795269 + 0.646218 i, 1.46544 - 0.713603 i,
 1.46544 + 0.713603 i, 2.36265 - 6.22156 i, 2.36265 + 6.22156 i, 5.85488 - 14.1717 i,
 5.85488 + 14.1717 i, 6.20902 - 23.4675 i, 6.20902 + 23.4675 i, 8.1197, 12.8107 - 5.87752 i,
 12.8107 + 5.87752 i, 23.7113 - 13.883 i, 23.7113 + 13.883 i, 28.9035 - 27.9333 i,
 28.9035 + 27.9333 i, 29.1542, 48.0603 - 12.3423 i, 48.0603 + 12.3423 i}
```

```
zeros[100, 1.2]
```

```
{-5.86998 - 7.85834 i, -5.86998 + 7.85834 i, -1.9926 - 3.1597 i, -1.9926 + 3.1597 i,
 -1.96676 - 2.07497 i, -1.96676 + 2.07497 i, 0.35708 - 1.03283 i, 0.35708 + 1.03283 i,
 0.468404, 0.752205, 3.64989 - 0.978101 i, 3.64989 + 0.978101 i, 5.92397 - 5.95277 i,
 5.92397 + 5.95277 i, 6.80322 - 16.2372 i, 6.80322 + 16.2372 i, 17.4628 - 1.98367 i,
 17.4628 + 1.98367 i, 21.8002, 26.2825 - 25.0348 i, 26.2825 + 25.0348 i,
 41.406 - 29.4914 i, 41.406 + 29.4914 i, 56.8501 - 41.1016 i, 56.8501 + 41.1016 i}
```

```
colfunc = ColorData["AvocadoColors"]; aa = 1.2; bb = 2 - aa;
pts2 = Table[
   {colfunc[(n - aa) / bb], Point[{Re[#], Im[#]}]} & /@ zeros[100, n], {n, aa, aa + bb, .01}];
Graphics[pts2, Frame → True, PlotRange → {{-60, 0}, {-20, 20}}]
```

```
$Aborted
```




```
E2a[n_, k_, x_] := E2a[n + 1, k - 1, x] - x E2a[2 n + 1, k - 1, x]; E2a[n_, 0, x_] := "wit"
```

```
Expand[E2a[109, 4, x]]
```

wit − 4 wit x + 6 wit x^2 − 4 wit x^3 + wit x^4

```
E2[1000, 2, 11 / 10]
```

$$\frac{83}{5}$$

```
aa = (11 / 10); nn = 1000;
Sum[ 1, {j, 2, Floor[nn]}, {k, 2, Floor[nn / j]}] -
 2 aa Sum[ 1, {j, 1, Floor[nn / aa]}, {k, 2, Floor[nn / ( aa j )]}] +
 aa^2 Sum[ 1, {j, 1, Floor[nn / aa]}, {k, 1, Floor[nn / (j aa^2 )]}]
Sum[ 1, {j, 2, Floor[nn]}, {k, 2, Floor[nn / j]}] -
 2 aa Sum[ 1, {j, 2, nn}, {k, 1, Floor[nn / ( aa j )]}] +
 aa^2 Sum[ 1, {j, 1, Floor[nn / aa]}, {k, 1, Floor[nn / (j aa^2 )]}]
```

$$\frac{83}{5}$$

$$\frac{83}{5}$$

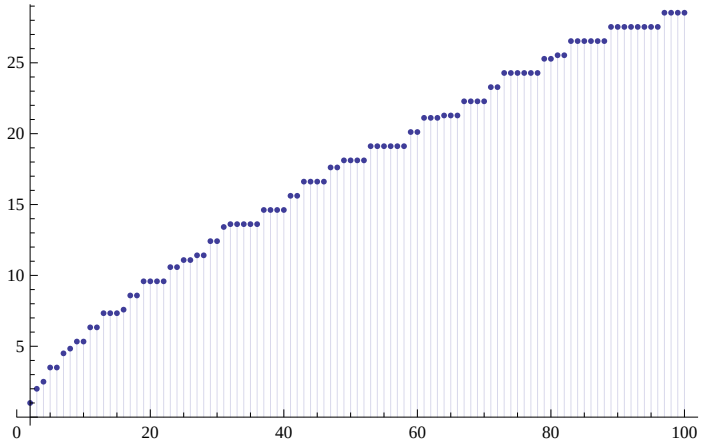
```
E2[1000, 3, 1.73]
```

$$-526.961$$

```
aa = 1.73; nn = 1000;
Sum[ 1, {j, 2, Floor[nn]}, {k, 2, Floor[nn / j]}, { l, 2, Floor[nn / (j k )]}] -
  3 aa
   Sum[ 1, {j, 1, Floor[nn / aa]}, {k, 2, Floor[nn / ( aa j )]}, {l, 2, Floor[nn / ( aa j k)]}] +
  3 aa^2 Sum[ 1, {j, 1, Floor[nn / aa]}, {k, 1, Floor[nn / ( aa^2 j )]},
    {l, 2, Floor[nn / (aa^2 j k )]}] -
  aa^3 Sum[ 1, {j, 1, Floor[nn / aa]}, {k, 1, Floor[nn / (j aa^2 )]},
    {l, 1, Floor[nn / (j k aa^3)]}]
```

$$-526.961$$

```
cc = 6 / 5;
DiscretePlot[Sum[cc^j / j, {j, 1, Log[cc, n]}] + D[ E1Alt[ n, z, cc], z] /. z → 0, {n, 2, 100}]
```



```
E2b2[n_, 0, x_] := 1
E2b2[n_, k_, x_] := E2b2[n, k, x] = Sum[ E2b2[ n / j, k - 1, x], {j, 2, Floor[n]}]
E2b1[n_, 0, 0, x_] := 1
E2b1[n_, 0, k2_, x_] := E2b2[n, k2, x]
E2b1[n_, k1_, k2_, x_] :=
 E2b1[n, k1, k2, x] = Sum[ E2b1[n / (j x), k1 - 1, k2, x], {j, 1, Floor[n / x]}]
E2b[ n_, k_, x_] := Sum[ (-1)^(j) Binomial[ k, j] x^j E2b1[n, j, k - j, x], {j, 0, k}]
E2bm[ n_, k_, x_] := Sum[ (-1)^(k - j) Binomial[ k, k - j] x^(k - j) E2b1[n, k - j, j, x],
  {j, 0, Min[k, Floor[Log[2, n]]]}]
aa^3 Sum[ 1, {j, 1, Floor[nn / aa]},
  {k, 1, Floor[nn / (j aa^2 )]}, {l, 1, Floor[nn / (j k aa^3)]}]
```

```
18 277.3
```

```
E2[120, 64, 103 / 99]
```

```
832 524 536 290 077 543 216 399 922 016 413 481 063 815 467 129 946 006 359 415 091 052 674 576 889 803 ↘
   239 695 237 129 441 446 868 023 154 173 958 830 762 763 945 721 338 859 /
 17 519 882 917 518 744 496 479 284 793 694 675 989 649 338 050 187 388 530 654 455 364 919 677 998 074 ↘
   409 004 683 948 399 828 593 600 887 618 898 319 662 125 318 717 867
```

```
E2b[4000, 8, 4 / 3]
```

$$\frac{1\,419\,657\,991}{6561}$$

```
E2bm[120, 64, 103 / 99]
```

```
832 524 536 290 077 543 216 399 922 016 413 481 063 815 467 129 946 006 359 415 091 052 674 576 889 803 ↘
   239 695 237 129 441 446 868 023 154 173 958 830 762 763 945 721 338 859 /
 17 519 882 917 518 744 496 479 284 793 694 675 989 649 338 050 187 388 530 654 455 364 919 677 998 074 ↘
   409 004 683 948 399 828 593 600 887 618 898 319 662 125 318 717 867
```

```
Floor[Log[2, 100]]
```

```
6
```

```
E2c2[n_, 0, x_] := 1
E2c2[n_, k_, x_] := E2c2[n, k, x] = Sum[ E2c2[ n / j, k - 1, x], {j, 2, Floor[n]}]
E2c1[n_, 0, 0, x_] := 1
E2c1[n_, 0, k2_, x_] := E2c2[n, k2, x]
E2c1[n_, k1_, k2_, x_] :=
 E2c1[n, k1, k2, x] = Sum[ E2c1[n / j, k1 - 1, k2, x], {j, 1, Floor[n]}]
E2c[ n_, k_, x_] := Sum[ (-1)^j Binomial[ k, j] x^j E2c1[n / x^j, j, k - j, x], {j, 0, k}]
E2cm[ n_, k_, x_] :=
 Sum[ (-1)^(k - j) Binomial[ k, k - j] x^(k - j) E2c1[n / x^(k - j), k - j, j, x],
  {j, 0, Min[k, Floor[Log[2, n]]]}]
E2cnull[ n_, k_, x_] := Sum[ (-1)^(k - j) Binomial[ k, k - j] x^(k - j)
   E2n[n / x^(k - j), k - j, j, x], {j, 0, Min[k, Floor[Log[2, n]]]}]
E2cnullp[ n_, k_, x_] := (1 + (-1)^(k + 1) Sum[ (-1)^(k - j) Binomial[ k, k - j]
       x^(k - j) E2n[n / x^(k - j), k - j, j, x], {j, 0, Min[k, Floor[Log[2, n]]]}]) / k
pp[ n_, x_] := Sum[x^j / j, {j, 1, Log[x, n]}] + Sum[ (-1)^(k + 1) / k E2cm[ n, k, x],
   {k, 1, Log[If[x < 2, x, 2], n]}]
pp2[ n_, x_] := Sum[ (-1)^(k + 1) / k E2cm[ n, k, x], {k, 1, Log[If[x < 2, x, 2], n]}]
pp2a[ n_, x_] := Sum[x^j / j, {j, 1, Log[x, n]}]
pp2null[ n_, x_] :=
 Table[ Expand[(-1)^(k + 1) / k E2cnull[ n, k, x]], {k, 1, Log[If[x < 2, x, 2], n]}] // TableForm
```

```
E2cm[120, 64, 103 / 99]
```

832 524 536 290 077 543 216 399 922 016 413 481 063 815 467 129 946 006 359 415 091 052 674 576 889 803 ÷
239 695 237 129 441 446 868 023 154 173 958 830 762 763 945 721 338 859 /
17 519 882 917 518 744 496 479 284 793 694 675 989 649 338 050 187 388 530 654 455 364 919 677 998 074 ÷
409 004 683 948 399 828 593 600 887 618 898 319 662 125 318 717 867

```
E2c[120, 64, 103 / 99]
```

832 524 536 290 077 543 216 399 922 016 413 481 063 815 467 129 946 006 359 415 091 052 674 576 889 803 ÷
239 695 237 129 441 446 868 023 154 173 958 830 762 763 945 721 338 859 /
17 519 882 917 518 744 496 479 284 793 694 675 989 649 338 050 187 388 530 654 455 364 919 677 998 074 ÷
409 004 683 948 399 828 593 600 887 618 898 319 662 125 318 717 867

```
pp2null[5, 1.01]
```

```
E2cnullp[10000, 1, x]
```

$$1 + \text{E2n}[10\,000, 0, 1, x] - x\,\text{E2n}\left[\frac{10\,000}{x}, 1, 0, x\right]$$

```
E2cnullp[10000, 2, x]
```

$$\frac{1}{2}\left(1 - \text{E2n}[10\,000, 0, 2, x] - x^2\,\text{E2n}\left[\frac{10\,000}{x^2}, 2, 0, x\right] + 2\,x\,\text{E2n}\left[\frac{10\,000}{x}, 1, 1, x\right]\right)$$

```
E2cnullp[10000, 3, x]
```

$$\frac{1}{3}\left(1 + \text{E2n}[10\,000, 0, 3, x] - x^3\,\text{E2n}\left[\frac{10\,000}{x^3}, 3, 0, x\right] + 3\,x^2\,\text{E2n}\left[\frac{10\,000}{x^2}, 2, 1, x\right] - 3\,x\,\text{E2n}\left[\frac{10\,000}{x}, 1, 2, x\right]\right)$$

```
E2cnullp[10000, 4, x]
```

$$\frac{1}{4}\left(1 - \text{E2n}[10\,000,\ 0,\ 4,\ x] - x^4\ \text{E2n}\left[\frac{10\,000}{x^4},\ 4,\ 0,\ x\right] + 4\,x^3\ \text{E2n}\left[\frac{10\,000}{x^3},\ 3,\ 1,\ x\right] - 6\,x^2\ \text{E2n}\left[\frac{10\,000}{x^2},\ 2,\ 2,\ x\right] + 4\,x\ \text{E2n}\left[\frac{10\,000}{x},\ 1,\ 3,\ x\right]\right)$$