```
d2[n_, k_] := d2[n, k] = Sum[d2[j, k - 1] d2[n / j, 1], {j, Divisors[n]}];
d2[n_, 1] := d2[n, 1] = 1; d2[1, 1] := 0; d2[n_, 0] := 0; d2[1, 0] := 1

d[n_, k_] := d[n, k] = Sum[d[j, k - 1] d[n / j, 1], {j, Divisors[n]}];
d[n_, 1] := 1; d[n_, 0] := d[n, 0] = 0; d[1, 0] := 1

K[n_, 0] := K[n, 0] = If[n == 1, 1, 0]
K[n_, 1] := K[n, 1] = If[n == 1, 0, FullSimplify[MangoldtLambda[n] / Log[n]]]
K[n_, k_] := K[n, k] = Sum[K[j, k - 1] K[n / j, 1], {j, Divisors[n]}]
K1[n_, k_] := K1[n, k] = Sum[Binomial[k, j] K[n, k - j], {j, 0, k}]

sc[f_, k_, t_] := SeriesCoefficient[Series[f[x], {x, 0, Floor[t]}], k]

q2[b_, f_, n_, 0] := q2[b, f, n, 0] = 1
q2[b_, f_, n_, 1] :=
 q2[b, f, n, 1] = Sum[ b[n, k] sc[f, k, N[Floor[Log[2, n]]], {k, 0, N[Log[2, n]]}]
q2[b_, f_, n_, k_] := q2[b, f, n, k] =
  Sum[q2[b, f, n / j, k - 1] q2[b, f, j, 1], {j, Divisors[n]}]

q1[b_, f_, n_, 0] := q1[b, f, n, 0] = 1
q1[b_, f_, n_, 1] := q1[b, f, n, 1] = Sum[ b[n, k] sc[f, k, 20], {k, 0, 20}]
q1[b_, f_, n_, k_] :=
 q1[b, f, n, k] = Sum[q1[b, f, n / j, k - 1] q1[b, f, j, 1], {j, Divisors[n]}]

GG[x_] := x / Log[1 + x]
GG2[x_] := x / Log[1 - x]
Mcos[x_] := -Cos[x]
Msin[x_] := -Sin[x]
Expd[x_] := E^x
Lg1[x_] := Log[x + 1]
Lg2[x_] := Log[1 - x]
ggp[n_, k_] := q2[K, GG, n, k]
ggp2[n_, k_] := q2[K, GG2, n, k]
ggd2[n_, k_] := ggd2[n, k] = q2[d2, GG2, n, k]
ggd2i[n_, k_] := ggd2i[n, k] = q2[d2, GG, n, k]
ggdx[n_, k_] := q2[ggd2, GG2, n, k]
ggdxi[n_, k_] := q2[ggd2i, GG, n, k]
lg1[n_, k_] := q2[d2, Lg1, n, k]
lg2[n_, k_] := q2[d2, Lg2, n, k]
lg2d[n_, k_] := q1[d, Lg2, n, k]
expd[n_, k_] := q1[d, Expd, n, k]
expd2[n_, k_] := q2[d2, Expd, n, k]
expk[n_, k_] := q2[K, Expd, n, k]
sind[n_, k_] := q1[K1, Sin, n, k]
cosd[n_, k_] := q1[K1, Cos, n, k]
mcosd[n_, k_] := q1[K1, Mcos, n, k]
msind[n_, k_] := q1[K1, Msin, n, k]
tand[n_, k_] := q2[d2, Tan, n, k]
asinsind[n_, k_] := q2[sind, ArcSin, n, k]
atantand[n_, k_] := q2[tand, ArcTan, n, k]
```

```
LL[f_, n_, k_] := LL[f, n, k] = Sum[ f[j, k], {j, 1, n}]

Table[{n, ggp2[n, 1], ggd2[n, 1], ggdx[n, 1]}, {n, 1, 100}] // TableForm
```

1	-1	-1	-1
2	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{3}{4}$
3	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{3}{4}$
4	$\frac{1}{3}$	$\frac{7}{12}$	$-\frac{113}{144}$
5	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{5}{6}$
6	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{53}{72}$
7	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{5}{6}$
8	$\frac{7}{24}$	$\frac{17}{24}$	$-\frac{25}{36}$
9	$\frac{1}{3}$	$\frac{7}{12}$	$-\frac{107}{144}$
10	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{103}{144}$
11	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{37}{48}$
12	$\frac{5}{24}$	$\frac{23}{24}$	$-\frac{43}{72}$
13	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{37}{48}$
14	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{103}{144}$
15	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{103}{144}$
16	$\frac{209}{720}$	$\frac{649}{720}$	$-\frac{303\,769}{518\,400}$
17	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{593}{720}$
18	$\frac{5}{24}$	$\frac{23}{24}$	$-\frac{2333}{4320}$
19	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{593}{720}$
20	$\frac{5}{24}$	$\frac{23}{24}$	$-\frac{2333}{4320}$
21	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{763}{1080}$
22	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{763}{1080}$
23	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{593}{720}$
24	$\frac{103}{360}$	$\frac{533}{360}$	$-\frac{35\,479}{129\,600}$
25	$\frac{1}{3}$	$\frac{7}{12}$	$-\frac{661}{864}$
26	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{763}{1080}$
27	$\frac{7}{24}$	$\frac{17}{24}$	$-\frac{2981}{4320}$
28	$\frac{5}{24}$	$\frac{23}{24}$	$-\frac{2333}{4320}$
29	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{593}{720}$
30	$\frac{1}{4}$	$\frac{5}{4}$	$-\frac{269}{720}$
31	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{593}{720}$
32	$\frac{113}{360}$	$\frac{1739}{1440}$	$-\frac{245\,431}{518\,400}$
33	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{3187}{4320}$
34	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{3187}{4320}$
35	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{3187}{4320}$
36	$\frac{13}{40}$	$\frac{209}{120}$	$-\frac{19\,979}{86\,400}$
37	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{2237}{2880}$

38	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{3187}{4320}$
39	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{3187}{4320}$
40	$\frac{103}{360}$	$\frac{533}{360}$	$-\frac{186\,061}{518\,400}$
41	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{2237}{2880}$
42	$\frac{1}{4}$	$\frac{5}{4}$	$-\frac{2827}{5760}$
43	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{2237}{2880}$
44	$\frac{5}{24}$	$\frac{23}{24}$	$-\frac{21\,229}{34\,560}$
45	$\frac{5}{24}$	$\frac{23}{24}$	$-\frac{21\,229}{34\,560}$
46	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{3187}{4320}$
47	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{2237}{2880}$
48	$\frac{49}{120}$	$\frac{3503}{1440}$	$\frac{62\,441}{518\,400}$
49	$\frac{1}{3}$	$\frac{7}{12}$	$-\frac{2617}{3456}$
50	$\frac{5}{24}$	$\frac{23}{24}$	$-\frac{21\,229}{34\,560}$
51	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{3187}{4320}$
52	$\frac{5}{24}$	$\frac{23}{24}$	$-\frac{21\,229}{34\,560}$
53	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{2237}{2880}$
54	$\frac{103}{360}$	$\frac{533}{360}$	$-\frac{186\,061}{518\,400}$
55	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{3187}{4320}$
56	$\frac{103}{360}$	$\frac{533}{360}$	$-\frac{186\,061}{518\,400}$
57	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{3187}{4320}$
58	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{3187}{4320}$
59	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{2237}{2880}$
60	$\frac{53}{120}$	$\frac{101}{40}$	$\frac{8743}{57\,600}$
61	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{2237}{2880}$
62	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{3187}{4320}$
63	$\frac{5}{24}$	$\frac{23}{24}$	$-\frac{21\,229}{34\,560}$
64	$\frac{21\,821}{60\,480}$	$\frac{103\,133}{60\,480}$	$-\frac{395\,646\,487}{1\,828\,915\,200}$
65	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{2377}{3456}$
66	$\frac{1}{4}$	$\frac{5}{4}$	$-\frac{2579}{6720}$
67	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{8261}{10\,080}$
68	$\frac{5}{24}$	$\frac{23}{24}$	$-\frac{129\,617}{241\,920}$
69	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{2377}{3456}$
70	$\frac{1}{4}$	$\frac{5}{4}$	$-\frac{2579}{6720}$
71	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{8261}{10\,080}$
72	$\frac{191}{360}$	$\frac{2437}{720}$	$\frac{8827}{16\,200}$
73	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{8261}{10\,080}$
74	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{2377}{3456}$
75	$\frac{5}{24}$	$\frac{23}{24}$	$-\frac{129\,617}{241\,920}$
76	$\frac{5}{24}$	$\frac{23}{24}$	$-\frac{129\,617}{241\,920}$

77	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{2377}{3456}$
78	$\frac{1}{4}$	$\frac{5}{4}$	$-\frac{2579}{6720}$
79	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{8261}{10\,080}$
80	$\frac{49}{120}$	$\frac{3503}{1440}$	$\frac{883\,667}{7\,257\,600}$
81	$\frac{209}{720}$	$\frac{649}{720}$	$-\frac{22\,507}{38\,400}$
82	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{2377}{3456}$
83	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{8261}{10\,080}$
84	$\frac{53}{120}$	$\frac{101}{40}$	$\frac{202\,589}{1\,209\,600}$
85	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{2377}{3456}$
86	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{2377}{3456}$
87	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{2377}{3456}$
88	$\frac{103}{360}$	$\frac{533}{360}$	$-\frac{13\,501}{44\,800}$
89	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{8261}{10\,080}$
90	$\frac{53}{120}$	$\frac{101}{40}$	$\frac{202\,589}{1\,209\,600}$
91	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{2377}{3456}$
92	$\frac{5}{24}$	$\frac{23}{24}$	$-\frac{129\,617}{241\,920}$
93	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{2377}{3456}$
94	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{2377}{3456}$
95	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{2377}{3456}$
96	$\frac{6001}{10\,080}$	$\frac{10\,567}{2520}$	$\frac{281\,881\,763}{304\,819\,200}$
97	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{8261}{10\,080}$
98	$\frac{5}{24}$	$\frac{23}{24}$	$-\frac{129\,617}{241\,920}$
99	$\frac{5}{24}$	$\frac{23}{24}$	$-\frac{129\,617}{241\,920}$
100	$\frac{13}{40}$	$\frac{209}{120}$	$-\frac{55\,687}{302\,400}$

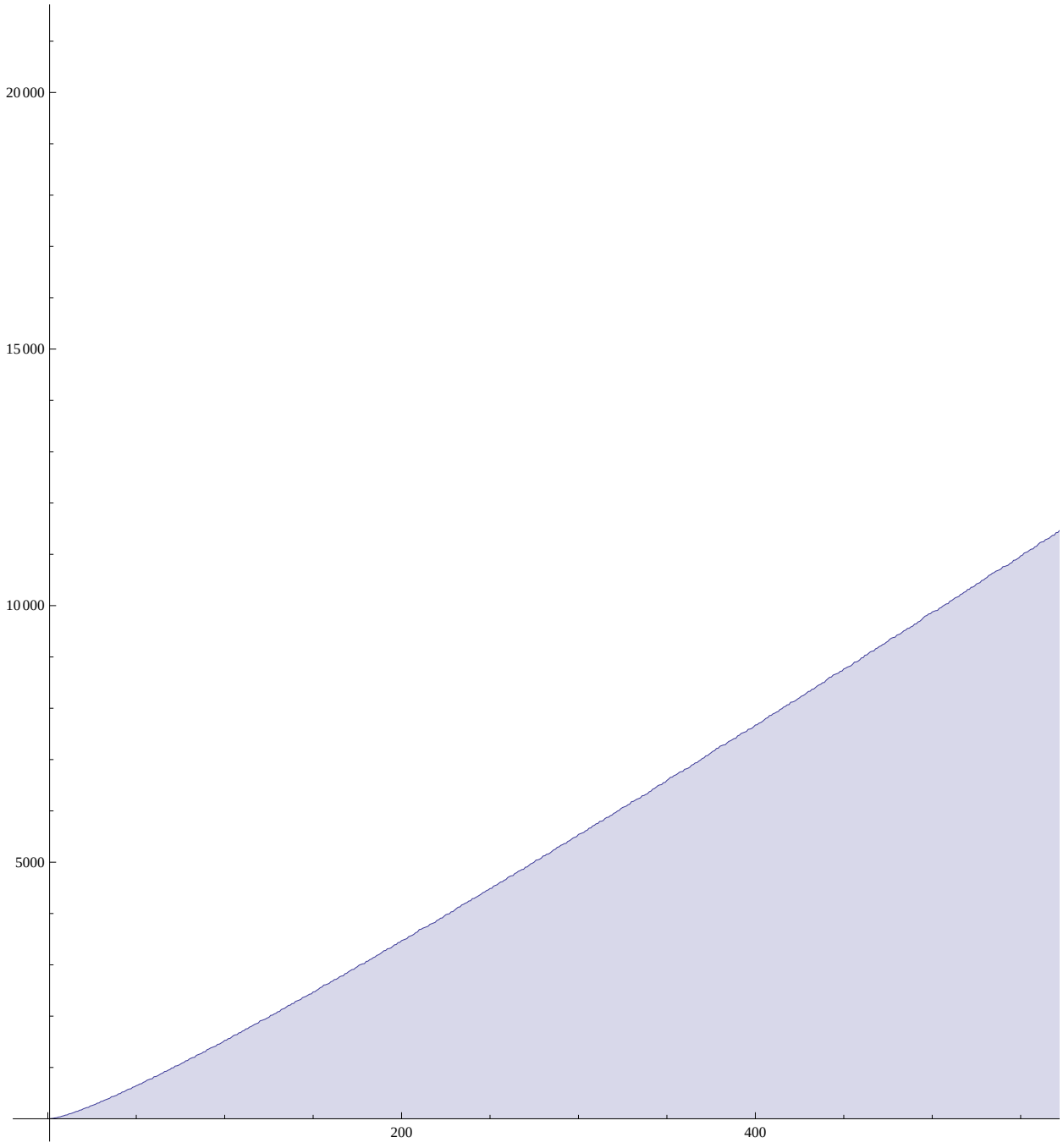
```
Table[{n, N[LL[ggp2, n, 1]], N[LL[ggd2, n, 1]], N[LL[ ggdx, n, 1]]}, {n, 1, 100}] // TableForm
```

1	-1.	-1.	-1.
2	-0.5	-0.5	-1.75
3	0.	0.	-2.5
4	0.333333	0.583333	-3.28472
5	0.833333	1.08333	-4.11806
6	1.	1.75	-4.85417
7	1.5	2.25	-5.6875
8	1.79167	2.95833	-6.38194
9	2.125	3.54167	-7.125
10	2.29167	4.20833	-7.84028
11	2.79167	4.70833	-8.61111
12	3.	5.66667	-9.20833
13	3.5	6.16667	-9.97917
14	3.66667	6.83333	-10.6944
15	3.83333	7.5	-11.4097
16	4.12361	8.40139	-11.9957
17	4.62361	8.90139	-12.8193
18	4.83194	9.85972	-13.3594
19	5.33194	10.3597	-14.183
20	5.54028	11.3181	-14.723

21	5.70694	11.9847	−15.4295
22	5.87361	12.6514	−16.136
23	6.37361	13.1514	−16.9596
24	6.65972	14.6319	−17.2333
25	6.99306	15.2153	−17.9984
26	7.15972	15.8819	−18.7049
27	7.45139	16.5903	−19.3949
28	7.65972	17.5486	−19.935
29	8.15972	18.0486	−20.7586
30	8.40972	19.2986	−21.1322
31	8.90972	19.7986	−21.9558
32	9.22361	21.0063	−22.4292
33	9.39028	21.6729	−23.167
34	9.55694	22.3396	−23.9047
35	9.72361	23.0063	−24.6424
36	10.0486	24.7479	−24.8737
37	10.5486	25.2479	−25.6504
38	10.7153	25.9146	−26.3881
39	10.8819	26.5813	−27.1259
40	11.1681	28.0618	−27.4848
41	11.6681	28.5618	−28.2615
42	11.9181	29.8118	−28.7523
43	12.4181	30.3118	−29.5291
44	12.6264	31.2701	−30.1433
45	12.8347	32.2285	−30.7576
46	13.0014	32.8951	−31.4953
47	13.5014	33.3951	−32.2721
48	13.9097	35.8278	−32.1516
49	14.2431	36.4111	−32.9088
50	14.4514	37.3694	−33.5231
51	14.6181	38.0361	−34.2608
52	14.8264	38.9944	−34.8751
53	15.3264	39.4944	−35.6518
54	15.6125	40.975	−36.0107
55	15.7792	41.6417	−36.7485
56	16.0653	43.1222	−37.1074
57	16.2319	43.7889	−37.8451
58	16.3986	44.4556	−38.5829
59	16.8986	44.9556	−39.3596
60	17.3403	47.4806	−39.2078
61	17.8403	47.9806	−39.9845
62	18.0069	48.6472	−40.7223
63	18.2153	49.6056	−41.3365
64	18.5761	51.3108	−41.5529
65	18.7427	51.9775	−42.2407
66	18.9927	53.2275	−42.6244
67	19.4927	53.7275	−43.444
68	19.7011	54.6858	−43.9798
69	19.8677	55.3525	−44.6676
70	20.1177	56.6025	−45.0513
71	20.6177	57.1025	−45.8709
72	21.1483	60.4872	−45.326
73	21.6483	60.9872	−46.1455
74	21.815	61.6539	−46.8333
75	22.0233	62.6122	−47.3691
76	22.2316	63.5705	−47.9049

77	22.3983	64.2372	−48.5927
78	22.6483	65.4872	−48.9765
79	23.1483	65.9872	−49.796
80	23.5566	68.4198	−49.6743
81	23.8469	69.3212	−50.2604
82	24.0136	69.9879	−50.9482
83	24.5136	70.4879	−51.7677
84	24.9552	73.0129	−51.6002
85	25.1219	73.6795	−52.288
86	25.2886	74.3462	−52.9758
87	25.4552	75.0129	−53.6636
88	25.7414	76.4934	−53.965
89	26.2414	76.9934	−54.7845
90	26.683	79.5184	−54.617
91	26.8497	80.1851	−55.3048
92	27.058	81.1434	−55.8406
93	27.2247	81.8101	−56.5284
94	27.3914	82.4768	−57.2162
95	27.558	83.1434	−57.904
96	28.1534	87.3367	−56.9792
97	28.6534	87.8367	−57.7987
98	28.8617	88.795	−58.3345
99	29.07	89.7534	−58.8703
100	29.395	91.495	−59.0545

```
DiscretePlot[LL[ggp, n, 8], {n, 1, 1000}]
```



```
Series[x / Log[1 + x], {x, 0, 20}]
```

$$1+\frac{x}{2}-\frac{x^2}{12}+\frac{x^3}{24}-\frac{19\,x^4}{720}+\frac{3\,x^5}{160}-\frac{863\,x^6}{60\,480}+\frac{275\,x^7}{24\,192}-\frac{33\,953\,x^8}{3\,628\,800}+$$
$$\frac{8183\,x^9}{1\,036\,800}-\frac{3\,250\,433\,x^{10}}{479\,001\,600}+\frac{4671\,x^{11}}{788\,480}-\frac{13\,695\,779\,093\,x^{12}}{2\,615\,348\,736\,000}+\frac{2\,224\,234\,463\,x^{13}}{475\,517\,952\,000}-$$
$$\frac{132\,282\,840\,127\,x^{14}}{31\,384\,184\,832\,000}+\frac{2\,639\,651\,053\,x^{15}}{689\,762\,304\,000}-\frac{111\,956\,703\,448\,001\,x^{16}}{32\,011\,868\,528\,640\,000}+\frac{50\,188\,465\,x^{17}}{15\,613\,165\,568}-$$
$$\frac{2\,334\,028\,946\,344\,463\,x^{18}}{786\,014\,494\,949\,376\,000}+\frac{301\,124\,035\,185\,049\,x^{19}}{109\,285\,437\,800\,448\,000}-\frac{12\,365\,722\,323\,469\,980\,029\,x^{20}}{4\,817\,145\,976\,189\,747\,200\,000}+O[x]^{21}$$

```
Series[Log[1 + x] / x, {x, 0, 20}]
```

$$1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\frac{x^4}{5}-\frac{x^5}{6}+\frac{x^6}{7}-\frac{x^7}{8}+\frac{x^8}{9}-\frac{x^9}{10}+\frac{x^{10}}{11}-$$
$$\frac{x^{11}}{12}+\frac{x^{12}}{13}-\frac{x^{13}}{14}+\frac{x^{14}}{15}-\frac{x^{15}}{16}+\frac{x^{16}}{17}-\frac{x^{17}}{18}+\frac{x^{18}}{19}-\frac{x^{19}}{20}+\frac{x^{20}}{21}+O[x]^{21}$$