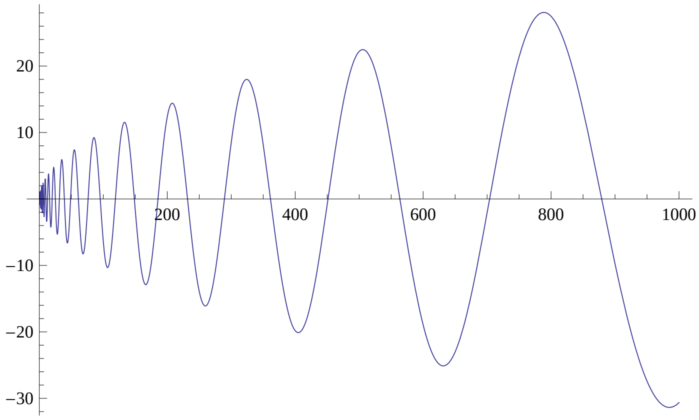
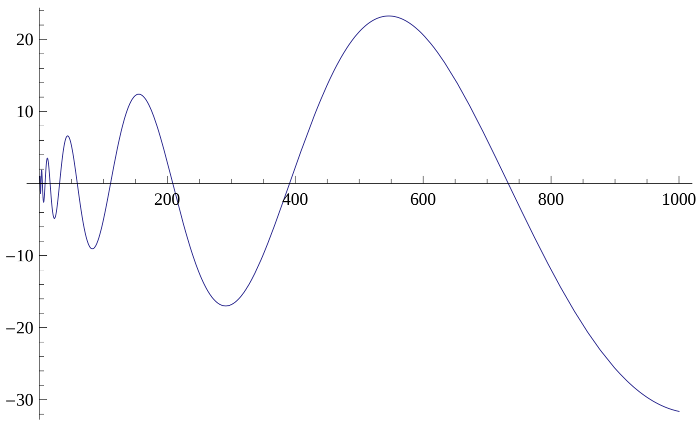
```
Plot[Re[x^(1 - N[ZetaZero[1]])], {x, 1, 1000}]
```




```
Plot[Re[x^(1 - (.5 + 5 I))], {x, 1, 1000}]
```



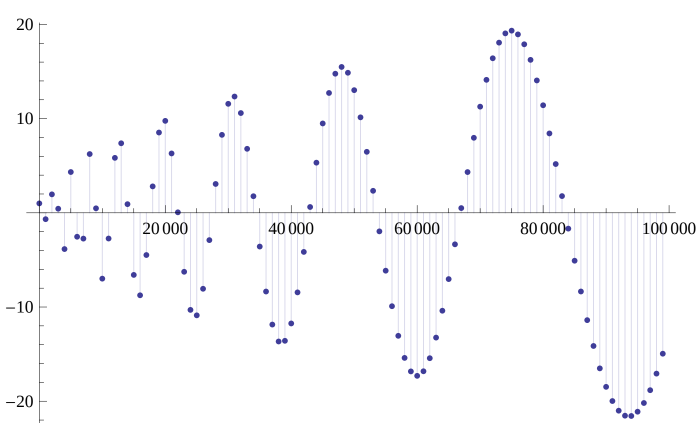

```
FullSimplify[(1 - x^(1 - s)) Zeta[s]] /. s → 3
```

$\left(1-\frac{1}{x^2}\right)$ Zeta[3]

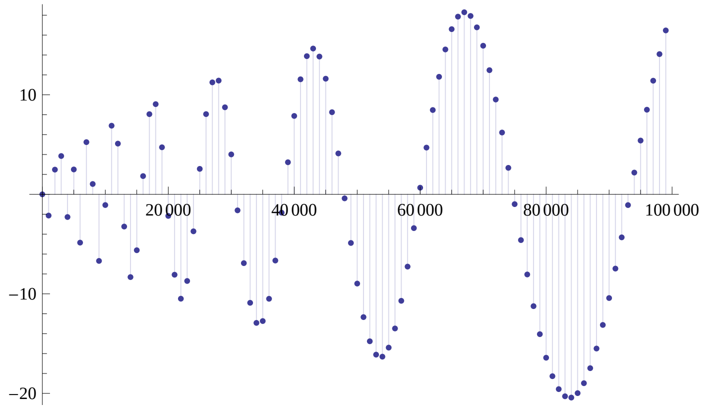
```
FullSimplify[(x^(s - 1) - 1) / x^(s - 1) zZeta[s] /. s → ZetaZero[1]]
```

$\left(1-x^{1-\text{ZetaZero}[1]}\right)$ zZeta[ZetaZero[1]]

```
par[n_, s_] := Sum[ j^-s, {j, 1, n}]
DiscretePlot[Re[par[n, N[ZetaZero[1]]]], {n, 1, 100 000, 1000}]
```



```
DiscretePlot[Im[par[n, N[ZetaZero[1]]]], {n, 1, 100 000, 1000}]
```




```
DiscretePlot[{ Re[par[n, N[ZetaZero[1]]]] , Re[n^(1 - N[ZetaZero[1]])]}, {n, 1, 1000, 1}]

f[x_] := Sum[ j^-s, {j, 1, n}] - x^(1 - s) Sum[ j^-s, {j, 1, n / x}]
f2[x_] := -x^(1 - s) Sum[ j^-s, {j, 1, n / x}]
f3[x_] := Sum[(-x^(1 - s)) ( j^-s), {j, 1, n / x}]
f4[x_] := Sum[(-x^(1 - s)) ( j^-s), {j, 1, n / x}]
f5[n_, s_, x_] := Sum[-j^-s (1 - s) x^-s, {j, 1, n / x}]
f6[x_] := Sum[-j^-s x^(1-s), {j, 1, n / x}]
```

```
FullSimplify@D[f[x], x]
```

$$x^{-1-s}\left((-1+s)\,x\,\text{HarmonicNumber}\left[\frac{n}{x},s\right]+n\,s\,\text{HurwitzZeta}\left[1+s,\frac{n+x}{x}\right]\right)$$

```
N[x^(-1-s) ((-1 + s) x HarmonicNumber[n/x, s] + n s HurwitzZeta[1 + s, (n + x)/x]) /. s → N[ZetaZero[1]] /.
   n → 1 000 000 000 000 000 000 /. x → 2]
```

$1.14871\times10^{-7} - 1.67402\times10^{-6}\,i$

```
Limit[x^(-1-s) ((-1 + s) x HarmonicNumber[n/x, s] + n s HurwitzZeta[1 + s, (n + x)/x]), n → Infinity]
```

$$\text{Limit}\left[x^{-1-s}\left((-1+s)\,x\,\text{HarmonicNumber}\left[\frac{n}{x},s\right]+n\,s\,\text{HurwitzZeta}\left[1+s,\frac{n+x}{x}\right]\right),\ n\to\infty\right]$$

```
D[f[x], x]
```

$$-(1-s)\,x^{-s}\,\text{HarmonicNumber}\left[\frac{n}{x},s\right]+n\,s\,x^{-1-s}\left(-\text{HarmonicNumber}\left[\frac{n}{x},1+s\right]+\text{Zeta}[1+s]\right)$$

```
N[- (1 - s) x^-s HarmonicNumber[n/x, s] + n s x^(-1-s) (-HarmonicNumber[n/x, 1 + s] + Zeta[1 + s])] /.
   s → N[ZetaZero[1]] /. n → 10 000 000 000 000 000 /. x → 2
```

$-0.0526776 - 0.550533\,i$

```
FullSimplify@D[f2[x], x]
```

$$x^{-1-s}\left((-1+s)\,x\,\text{HarmonicNumber}\left[\frac{n}{x},s\right]+n\,s\,\text{HurwitzZeta}\left[1+s,\frac{n+x}{x}\right]\right)$$

```
FullSimplify@D[f3[x], x]
```

$$x^{-1-s}\left((-1+s)\,x\,\text{HarmonicNumber}\left[\frac{n}{x},s\right]+n\,s\,\text{HurwitzZeta}\left[1+s,\frac{n+x}{x}\right]\right)$$

$$x^{-1-s}\left((-1+s)\,x\,\text{HarmonicNumber}\left[\frac{n}{x},s\right]+n\,s\,\text{HurwitzZeta}\left[1+s,\frac{n+x}{x}\right]\right)\ /.\ s\to N[\text{ZetaZero}[1]]\ /.\ n\to 10\,000\,000\,000\,000\,000\ /.\ x\to 2$$

$-4.42509\times 10^{-8}-3.36104\times 10^{-8}\,i$

```
FullSimplify@D[f4[x], x]
```

$$x^{-1-s}\left((-1+s)\,x\,\text{HarmonicNumber}\left[\frac{n}{x},s\right]+n\,s\,\text{HurwitzZeta}\left[1+s,\frac{n+x}{x}\right]\right)\ /.\ s\to N[\text{ZetaZero}[1]]\ /.\ n\to 10\,000\,000\,000\,000\,000\ /.\ x\to 4$$

$1.07906\times 10^{-8}+1.03187\times 10^{-8}\,i$

```
FullSimplify@D[f4[x], {x, 2}]
```

$$s\,x^{-3-s}\left((-1+s)\,x^2\,\text{HurwitzZeta}\left[s,\frac{n+x}{x}\right]-2\,n\,s\,x\,\text{HurwitzZeta}\left[1+s,\frac{n+x}{x}\right]+n^2\,(1+s)\,\text{HurwitzZeta}\left[2+s,\frac{n+x}{x}\right]-(-1+s)\,x^2\,\text{Zeta}[s]\right)\ /.\ s\to N[\text{ZetaZero}[12]]\ /.\ n\to 10\,000\,000\,000\,000\ /.\ x\to 2$$

$-1.66366\times 10^{-7}+2.9153\times 10^{-7}\,i$

```
D[(-x^(1-s)) (j^-s), x]
```

$-j^{-s}\,(1-s)\,x^{-s}$

```
f5[10000000000000000, N[ZetaZero[1]], 2]
```

$-3.61776\times 10^{7}-3.45135\times 10^{7}\,i$

$$N\left[x^{-1-s}\left((-1+s)\,x\,\text{HarmonicNumber}\left[\frac{n}{x},s\right]+n\,s\,\text{HurwitzZeta}\left[1+s,\frac{n+x}{x}\right]\right)\ /.\ s\to N@\text{ZetaZero}[1]\ /.\ n\to 10\,000\,000\,000\,000\,000\ /.\ x\to 5\right]$$

$3.90634\times 10^{-9}+3.64325\times 10^{-9}\,i$

$$N\left[\left((-1+s)\,x\,\text{HarmonicNumber}\left[\frac{n}{x},s\right]+n\,s\,\text{HurwitzZeta}\left[1+s,\frac{n+x}{x}\right]\right)\ /.\ s\to N@\text{ZetaZero}[1]\ /.\ n\to 10\,000\,000\,000\,000\,000\ /.\ x\to 5\right]$$

$-3.72529\times 10^{-9}-5.96046\times 10^{-8}\,i$

$$\text{FullSimplify}\left[(-1+s)\,x\,\text{HarmonicNumber}\left[\frac{n}{x},s\right]+n\,s\,\text{HurwitzZeta}\left[1+s,\frac{n+x}{x}\right]\right]$$

$$(-1+s)\,x\,\text{HarmonicNumber}\left[\frac{n}{x},s\right]+n\,s\,\text{HurwitzZeta}\left[1+s,\frac{n+x}{x}\right]$$

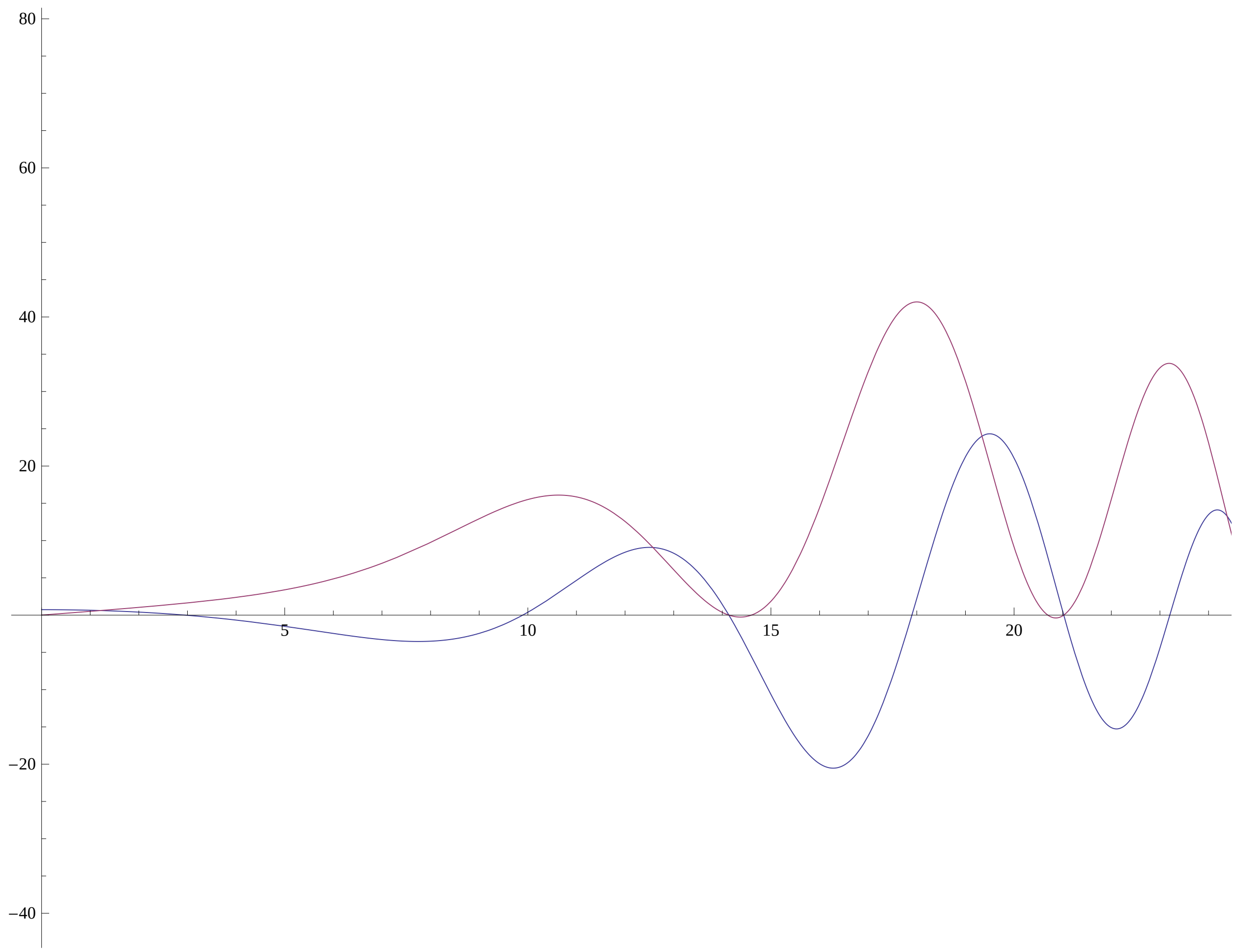
```
FullSimplify[(-x^(1-s)) (j^-s)]
```

$-j^{-s}\,x^{1-s}$

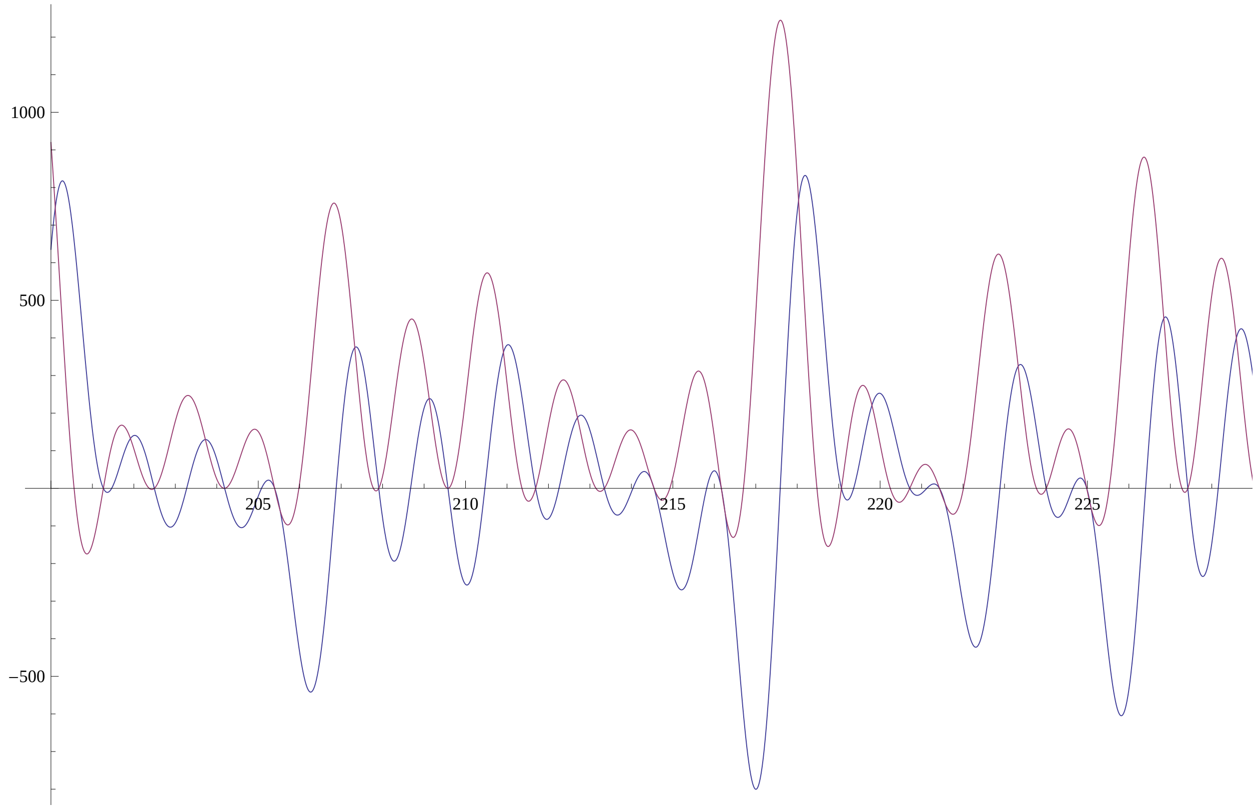
```
FullSimplify@D[f6[x], x]

N[((-1 + s) x HarmonicNumber[n/x, s] + n s HurwitzZeta[1 + s, (n + x)/x]) /. s → 2 /.
   n → 10 000 000 000 /. x → 2]

3.28987

Pi^2 / 3.

3.28987


FullSimplify[D[Sum[-j^-s x^(1-s), {j, 1, n / x}], x]]

x^(-1-s) ((-1 + s) x HarmonicNumber[n/x, s] + n s HurwitzZeta[1 + s, (n + x)/x])

FullSimplify[Sum[D[-j^-s x^(1-s), x], {j, 1, n / x}]]

(-1 + s) x^-s HarmonicNumber[n/x, s]

N[(-1 + s) x^-s HarmonicNumber[n/x, s] /. s → N@ZetaZero[1] /. n → 10 000 000 000 /. x → 3]

-10 144.3 - 31 752.2 i

N[x^(-1-s) ((-1 + s) x HarmonicNumber[n/x, s] + n s HurwitzZeta[1 + s, (n + x)/x]) /. s → N@ZetaZero[2] /.
   n → 100 000 000 000 /. x → 2]

7.04541 × 10^-8 - 1.57978 × 10^-6 i

N[ ((-1 + s) x HarmonicNumber[n/x, s]) /. s → N@ZetaZero[12] /. n → 1 000 000 000 /. x → 4]

12 810.6 - 61 934.5 i

N[ (n s HurwitzZeta[1 + s, (n + x)/x]) /. s → N@ZetaZero[12] /. n → 1 000 000 000 /. x → 4]

-12 810.6 + 61 934.5 i

N[ x^(-1-s) (Sum[ ((-1 + s) x ) / j^s, {j, 1, n / x}] +
        (Sum[(n s) / (j + (n / x) + 1)^(s + 1), {j, 0, Infinity}])) /.
    s → N[ZetaZero[3]] /. n → 100 000 000 000 /. x → 1]

-6.94563 × 10^-7 - 1.41968 × 10^-6 i

FullSimplify[D[f6[x], {x, 2}] /. x → 1]

s ((-1 + s) HurwitzZeta[s, 1 + n] - 2 n s HurwitzZeta[1 + s, 1 + n] +
   n^2 (1 + s) HurwitzZeta[2 + s, 1 + n] + Zeta[s] - s Zeta[s])

FullSimplify[D[f6[x], {x, 3}] /. x → 1]

s (1 + s) (-(-1 + s) HurwitzZeta[s, 1 + n] + 3 n s HurwitzZeta[1 + s, 1 + n] -
   3 n^2 (1 + s) HurwitzZeta[2 + s, 1 + n] + n^3 (2 + s) HurwitzZeta[3 + s, 1 + n] + (-1 + s) Zeta[s])
```

```
fr[s_] := N[x^(-1-s) ((-1 + s) x HarmonicNumber[n/x, s] + n s HurwitzZeta[1 + s, (n + x)/x]) /.
    n → 100 000 000 000 000 /. x → 1]
fr2[s_] := N[s ((-1 + s) HurwitzZeta[s, 1 + n] - 2 n s HurwitzZeta[1 + s, 1 + n] + n^2 (1 + s)
       HurwitzZeta[2 + s, 1 + n] + Zeta[s] - s Zeta[s]) /. n → 100 000 000 000 000 /. x → 1]
fr3[s_] := N[s (1 + s) (-(-1 + s) HurwitzZeta[s, 1 + n] + 3 n s HurwitzZeta[1 + s, 1 + n] -
      3 n^2 (1 + s) HurwitzZeta[2 + s, 1 + n] + n^3 (2 + s) HurwitzZeta[3 + s, 1 + n] +
      (-1 + s) Zeta[s]) /. n → 100 000 000 000 000 /. x → 1]
```

```
Plot[{Re[ fr[.5 + t I]], Im[ fr[.5 + t I]]}, {t, 0, 30}]
```



```
Plot[{Re[fr[.5 + t I]], Im[fr[.5 + t I]]}, {t, 200, 200 + 30}]
```




```
FullSimplify[D[f4[x], {x, 0}] /. x → 1]
```

−HarmonicNumber[n, s]

```
D[(1 - x^(1 - s)) Zeta[s], x]
```

$-(1-s)\,x^{-s}\,\text{Zeta}[s]$

$\text{f6}[x\_] := \text{Sum}\left[-j^{-s}\,x^{1-s},\ \{j,\ 1,\ n/x\}\right]$

$\text{f6a}[x\_] := \text{Sum}\left[D\left[-j^{-s}\,x^{1-s},\ \{x,\ 2\}\right],\ \{j,\ 1,\ n/x\}\right]$

```
D[f6[x], x]
```

$$-(1-s)\,x^{-s}\,\text{HarmonicNumber}\left[\frac{n}{x},\ s\right] + n\,s\,x^{-1-s}\left(-\text{HarmonicNumber}\left[\frac{n}{x},\ 1+s\right] + \text{Zeta}[1+s]\right)$$

$$-(1-s)\,x^{-s}\,\text{HarmonicNumber}\left[\frac{n}{x},\ s\right] + n\,s\,x^{-1-s}\left(-\text{HarmonicNumber}\left[\frac{n}{x},\ 1+s\right] + \text{Zeta}[1+s]\right)\ /.\ x \to 1$$

(−1 + s) HarmonicNumber[n, s] + n s (−HarmonicNumber[n, 1 + s] + Zeta[1 + s])

```
N[(-1 + s) HarmonicNumber[n, s] + n s (-HarmonicNumber[n, 1 + s] + Zeta[1 + s]) /. n → 100 000 /.
  s → N[ZetaZero[1]]]
```

−0.00127692 − 0.000932455 ⅈ

```
FullSimplify[D[f6[x], x] /. x → 1]
```

```
N[(-1 + s) HarmonicNumber[n, s] + n s HurwitzZeta[1 + s, 1 + n] /. n → 100 000 000 000 /.
  s → N[ZetaZero[1]]]
```

$-1.56701\times10^{-6} - 2.06586\times10^{-7}\,ⅈ$

```
Expand[(-1 + s) HarmonicNumber[n, s] + n s (-HarmonicNumber[n, 1 + s] + Zeta[1 + s])]
```

```
-HarmonicNumber[n, s] + s HarmonicNumber[n, s] - n s HarmonicNumber[n, 1 + s] + n s Zeta[1 + s]
```

```
fr2[n_, s_] := Sum[ (s - 1) / j^s - n s / j^(s + 1), {j, 1, n}] + n s Zeta[1 + s]
```

```
fr2[100 000 000 000, N@ZetaZero[1]]
```

```
-0.0000305176 + 0.00012207 i
```

```
N[1 - ZetaZero[1]]
```

```
0.5 - 14.1347 i
```

```
N[ZetaZero[1]]
```

```
0.5 + 14.1347 i
```

```
Table[N[2^k / ((2^k + 3)^1.5)], {k, 1, 30}]
```

```
{0.178885, 0.21598, 0.219281, 0.193192, 0.154542, 0.116699, 0.0853696, 0.0614172, 0.0438086,
 0.0311132, 0.0220486, 0.0156078, 0.0110425, 0.00781035, 0.00552351, 0.00390598, 0.00276204,
 0.00195309, 0.00138106, 0.000976558, 0.000690532, 0.000488281, 0.000345267, 0.000244141,
 0.000172633, 0.00012207, 0.0000863167, 0.0000610352, 0.0000431584, 0.0000305176}
```

f6[x_] := **Sum**$\left[-j^{-s}\, x^{1-s}, \{j, 1, n/x\}\right]$
f6b[x_] := **Sum**$\left[-j^{-s}\, x^{1-s}, \{j, 1, n\}\right]$

```
D[f6[x], x] /. x → 1
```

```
-(1 - s) HarmonicNumber[n, s] + n s (-HarmonicNumber[n, 1 + s] + Zeta[1 + s])
```

```
D[f6b[x], x]
```

$-(1-s)\, x^{-s}$ HarmonicNumber[n, s]

$$\mathrm{N}\left[x^{-1-s}\left((-1+s)\, x\, \mathrm{HarmonicNumber}\left[\frac{n}{x}, s\right] + n\, s\, \mathrm{HurwitzZeta}\left[1+s, \frac{n+x}{x}\right]\right) \mathbin{/.}\ s \to \mathrm{N@ZetaZero}[2] \mathbin{/.}\ n \to 100\,000\,000\,000 \mathbin{/.}\ x \to 2\right]$$

$$x^{-1-s}\left((-1+s)\, x\, \mathrm{HarmonicNumber}\left[\frac{n}{x}, s\right] + n\, s\, \mathrm{HurwitzZeta}\left[1+s, \frac{n+x}{x}\right]\right) \mathbin{/.}\ x \to 1$$

```
(-1 + s) HarmonicNumber[n, s] + n s HurwitzZeta[1 + s, 1 + n]
```

```
N[(-1 + s) HarmonicNumber[n, s] + n s HurwitzZeta[1 + s, 1 + n] /. s → N@ZetaZero[2] /.
  n → 100 000 000 000]
```

$7.04204 \times 10^{-8} - 1.579 \times 10^{-6}\, i$

```
N[n (s) HurwitzZeta[1 + s, 1 + n] - (1 - s) HarmonicNumber[n, s] /. s → N@ZetaZero[2] /.
  n → 100 000 000 000]
```

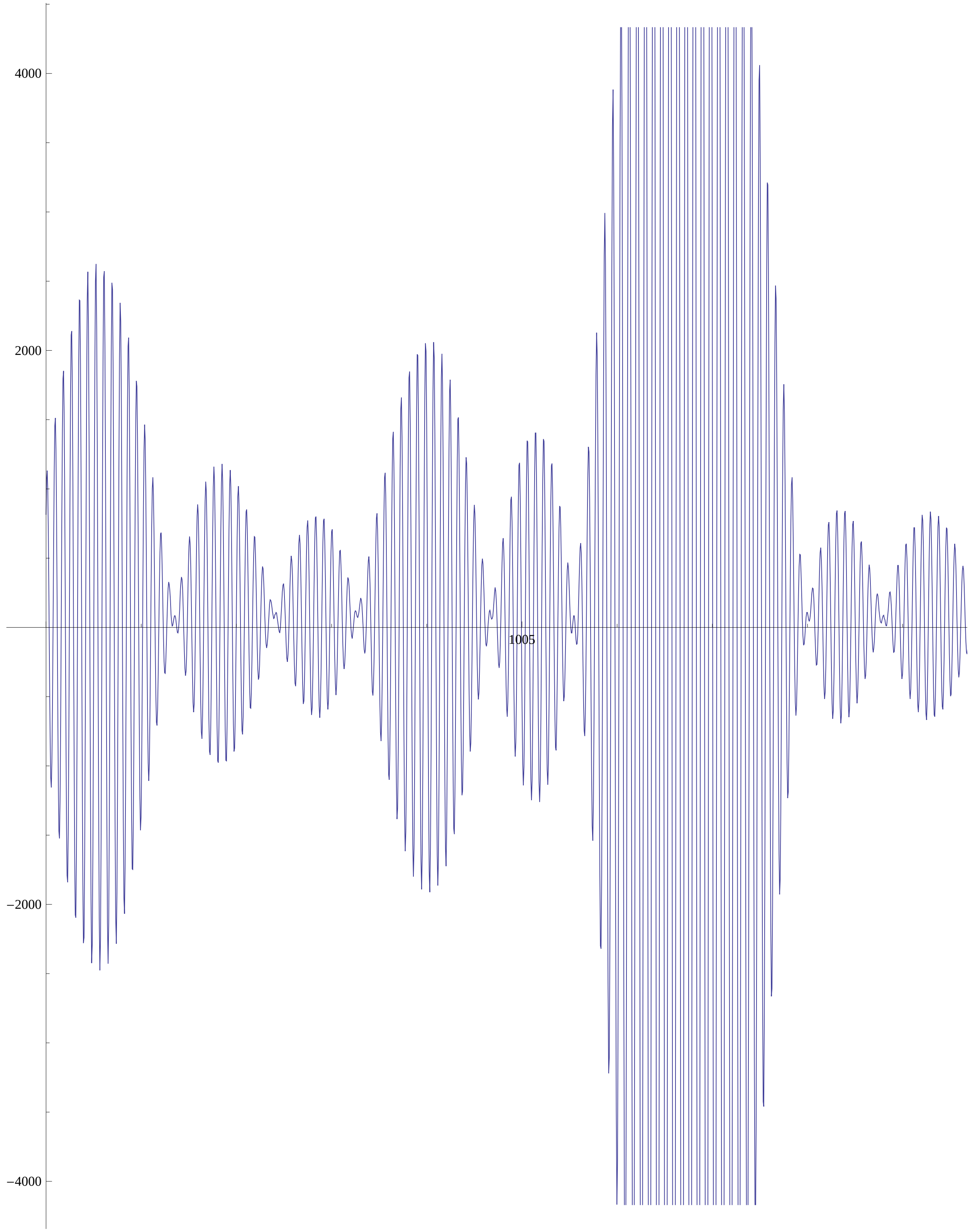
$7.04204 \times 10^{-8} - 1.579 \times 10^{-6}\, i$

```
N[ (s) (n HurwitzZeta[1 + s, 1 + n]) - (1 - s) HarmonicNumber[n, s] /. s → N@ZetaZero[2] /.
  n → 100 000 000 000]
```

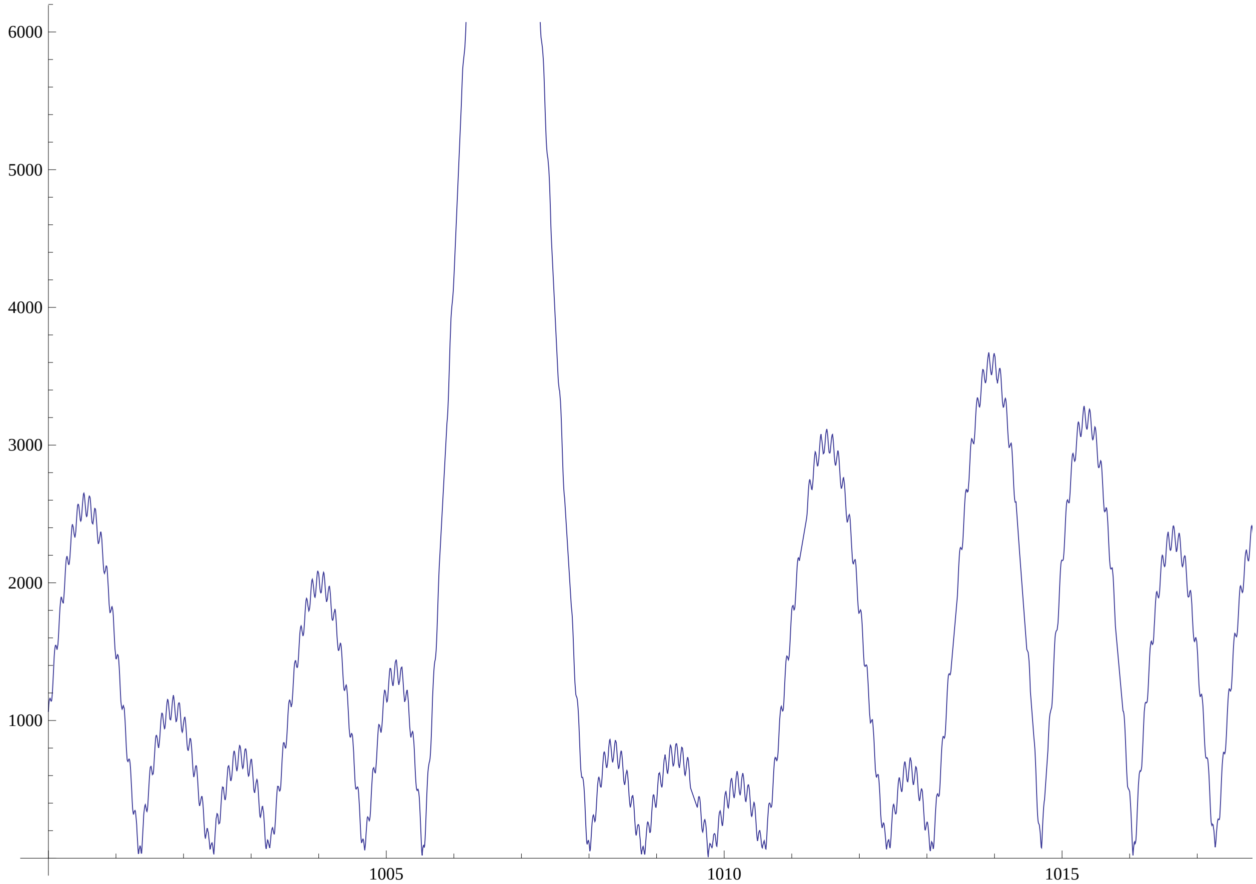
$7.04204 \times 10^{-8} - 1.579 \times 10^{-6}\, i$

```
N[ (s) (n HurwitzZeta[1 + s, 1 + n]) - (1 - s) HarmonicNumber[n, s] /. s → N@ZetaZero[2]]

(-0.5 + 21.022 i) HarmonicNumber[n, 0.5 + 21.022 i] +
 (0.5 + 21.022 i) n HurwitzZeta[1.5 + 21.022 i, 1. + n]

N[ {(s) (n HurwitzZeta[1 + s, 1 + n]), (1 - s) HarmonicNumber[n, s]} /. s → N@ZetaZero[2]] /.
 n → 100 000 000 000

{-14 089.2 + 315 914. i, -14 089.2 + 315 914. i}

N[ { (s) (n HurwitzZeta[1 + s, 1 + n]), (1 - s) HarmonicNumber[n, s]} /. s → N@ZetaZero[1]] /.
 n → 100 000 000 000

{313 515. + 41 332.9 i, 313 515. + 41 332.9 i}

N@ZetaZero[1]

0.5 + 14.1347 i

N[ { Abs[(s) (n HurwitzZeta[1 + s, 1 + n])], Abs[(1 - s) HarmonicNumber[n, s]]} /.
   s → N@ZetaZero[1]] /. n → 100 000 000 000

{316 228., 316 228.}

N[ { Abs[(n HurwitzZeta[1 + s, 1 + n])], Abs[HarmonicNumber[n, s]]} /. s → N@ZetaZero[1]] /.
 n → 100 000 000 000

{22 358.4, 22 358.4}

N[ { Abs[(n HurwitzZeta[1 + s, 1 + n])], Abs[HarmonicNumber[n, s]]} /. s → .2 + N@ZetaZero[1]] /.
 n → 100 000 000 000

{140.988, 141.133}

N[ { Abs[(n HurwitzZeta[1 + s, 1 + n])], Abs[HarmonicNumber[n, s]]} /.
   s → 13.2 I + N@ZetaZero[1]] /. n → 100 000 000 000 000 000

{1.15668 × 10^7, 1.15668 × 10^7}

N[ { Abs[(n HurwitzZeta[1 + s, 1 + n])], Abs[HarmonicNumber[n, s]]} /. s → .3 + N@ZetaZero[1]] /.
 n → 100 000 000 000 000 000

{177.426, 177.797}
```

```
Plot[ { Abs[(s) (n HurwitzZeta[1 + s, 1 + n])] - Abs[(1 - s) HarmonicNumber[n, s]]} /.
   n → 1 000 000 000 000 000 000 000 000 000 000 000 /. s → .5 + t I, {t, 1000, 1020}]
```



```
Plot[ { Abs[(s) (n HurwitzZeta[1 + s, 1 + n]) - (1 - s) HarmonicNumber[n, s]]} /.
   n → 1 000 000 000 000 000 000 000 000 000 000 000 /. s → .5 + t I, {t, 1000, 1020}]
```




```
D[(1 - x^(1 - s)) Zeta[s], x] /. x → 1

-(1 - s) Zeta[s] /. s → .3 + N[ZetaZero[2]]

1.16198 + 5.90511 i

N[n (s) HurwitzZeta[1 + s, 1 + n] - (1 - s) HarmonicNumber[n, s] /. s → .3 + N@ZetaZero[2] /.
  n → 100 000 000 000]

1.16198 + 5.90511 i

D[(1 - x^(1 - s)) Zeta[s], x]

-(1 - s) x^-s Zeta[s]

D[(1 - x^(1 - s)) Zeta[s], x] /. x → 1

-(1 - s) Zeta[s]

Chop[N[ (n HurwitzZeta[1 + s, 1 + n]) /. s → 0] /. n → 100 000 000 000 000 000 000]

ComplexInfinity

Chop[N[{n (s) HurwitzZeta[1 + s, 1 + n], (1 - s) HarmonicNumber[n, s]} /. s → -.1 /.
   n → 100 000 000 000 000]]

{2.51189 × 10^15, 2.51189 × 10^15}
```

```
N[(n (s) HurwitzZeta[1 + s, 1 + n]) - ((1 - s) HarmonicNumber[n, s]) /. s → .8 + 3 I /.
  n → 100 000 000 000 000]
```

0.176104 + 1.79124 ⅈ

```
(-.2 + 3 I) Zeta[.8 + 3 I]
```

0.176104 + 1.79124 ⅈ

```
-x^(1 - s) Sum[ j^-s, {j, 1, n / x}]
```

$-x^{1-s}\,\text{HarmonicNumber}\left[\frac{n}{x}, s\right]$

$\mathbf{D}\left[-x^{1-s}\,\textbf{HarmonicNumber}\left[\frac{n}{x}, s\right], x\right]$ **/. x → 1**

-(1 - s) HarmonicNumber[n, s] + n s (-HarmonicNumber[n, 1 + s] + Zeta[1 + s])

```
lz[n_, s_] := -(1 - s) HarmonicNumber[n, s] + n s (-HarmonicNumber[n, 1 + s] + Zeta[1 + s])
lz2[n_, s_] := HarmonicNumber[n, s] + n ( s / (s - 1)) (Zeta[1 + s] - HarmonicNumber[n, 1 + s])
```

```
N[lz[100 000 000 000, 2]]
```

1.64493

```
FullSimplify[-(1 - s) / (s - 1)]
```

1

```
N[lz2[1 000 000 000 000, .5]]
```

-1.46019

```
Zeta[.5]
```

-1.46035

```
Limit[ HarmonicNumber[n, s] + n ( s / (s - 1)) (Zeta[1 + s] - HarmonicNumber[n, 1 + s]),
  n → Infinity] /. s → 1 / 2
```

$\text{Zeta}\left[\frac{1}{2}\right]$

```
Limit[FullSimplify[n ( s / (s - 1)) (Zeta[1 + s] - HarmonicNumber[n, 1 + s])], n → Infinity]
```

$\text{Limit}\left[\frac{n\,s\,\text{HurwitzZeta}[1+s, 1+n]}{-1+s}, n \to \infty\right]$

```
Limit[ HarmonicNumber[n, s] - n ((s + 1) / (s)) (Zeta[s - 1] - HarmonicNumber[n, s - 1]),
  n → Infinity] /. s → 5 / 2
```

$-\infty$

```
(* *)
```

```
D[Sum[ j^-s, {j, 1, n}] + x^(1 - s) Sum[ j^-s, {j, 1, n / x}], x] /. x → 1
```

(1 - s) HarmonicNumber[n, s] - n s (-HarmonicNumber[n, 1 + s] + Zeta[1 + s])

```
D[(1 + x^(1 - s)) Zeta[s], x] /. x → 1
```

(1 - s) Zeta[s]

```
FullSimplify[D[Sum[ j^-s, {j, 1, n}] - x^(-s) Sum[ j^-s, {j, 1, n / x}], x] /. x → 1]

s (HarmonicNumber[n, s] + n HurwitzZeta[1 + s, 1 + n])

D[(1 - x^(-s)) Zeta[s], x] /. x → 1

s Zeta[s]

D[Sum[ j^-s, {j, 1, n}] - x^(1 + s) Sum[ j^-s, {j, 1, n / x}], x] /. x → 1

-(1 + s) HarmonicNumber[n, s] + n s (-HarmonicNumber[n, 1 + s] + Zeta[1 + s])

D[(1 - x^(1 + s)) Zeta[s], x] /. x → 1

-(1 + s) Zeta[s]

(* These don't converge for re(s) ≤ 1, so they are worth nothing *)
```

```
(* *)
```

```
D[Sum[ j^-s, {j, 1, n}] - x^(1 - s) Sum[ j^-s, {j, 1, n / x}], {x, 2}] /. x → 1

(1 - s) s HarmonicNumber[n, s] - 2 n s (-HarmonicNumber[n, 1 + s] + Zeta[1 + s]) +
 2 n (1 - s) s (-HarmonicNumber[n, 1 + s] + Zeta[1 + s]) +
 n^2 s (1 + s) (-HarmonicNumber[n, 2 + s] + Zeta[2 + s])

D[(1 - x^(1 - s)) Zeta[s], {x, 2}] /. x → 1

(1 - s) s Zeta[s]

FullSimplify[D[Sum[ j^-s, {j, 1, n}] - x^(1 - s) Sum[ j^-s, {j, 1, n / x}], {x, 2}] /. x → 1]

s ((-1 + s) HurwitzZeta[s, 1 + n] - 2 n s HurwitzZeta[1 + s, 1 + n] +
    n^2 (1 + s) HurwitzZeta[2 + s, 1 + n] + Zeta[s] - s Zeta[s])

ts[n_, s_] := s ((-1 + s) HurwitzZeta[s, 1 + n] -
     2 n s HurwitzZeta[1 + s, 1 + n] + n^2 (1 + s) HurwitzZeta[2 + s, 1 + n] + Zeta[s] - s Zeta[s])

ts[10 000 000, sss = -.5] / sss / (1 - sss)

-0.207856

Zeta[-.5]

-0.207886
```

```
(* *)
D[(1 - x^(1 - s)) Zeta[s, y], x] /. x → 1

-(1 - s) Zeta[s, y]

D[Sum[ (j + y)^-s, {j, 0, n}] - x^(1 - s) Sum[ (j + y)^-s, {j, 0, n / x}], x] /. x → 1

-(1 - s) (HurwitzZeta[s, y] - HurwitzZeta[s, 1 + n + y]) + n s HurwitzZeta[1 + s, 1 + n + y]
```

```
N[-(1-s) Zeta[s, y] /. s → .5 /. y → 1]
```

0.730177

```
N[-(1-s) (HurwitzZeta[s, y] - HurwitzZeta[s, 1+n+y]) + n s HurwitzZeta[1+s, 1+n+y] /.
    s → .5 /. y → 1 /. n → 100 000 000]
```

0.730027

```
(.5 - 1) Zeta[.5, 1]
```

0.730177

```
N[-(1-s) (Zeta[s] - HurwitzZeta[s, n+2]) + n s HurwitzZeta[1+s, n+2] /. s → .5 /.
  n → 100 000 000]
```

0.730027

```
N[ s n HurwitzZeta[1+s, n+2] - (1-s) (Zeta[s] - HurwitzZeta[s, n+2]) /. s → .5 /.
  n → 10 000 000 000]
```

0.730162

```
Sum[ (j + y)^-s, {j, 0, n}]
```

HurwitzZeta[s, y] - HurwitzZeta[s, 1 + n + y]

```
x^(1-s) Sum[ (j + y)^-s, {j, 0, n/x}]
```

$x^{1-s}\left(\text{HurwitzZeta}[s, y] - \text{HurwitzZeta}\left[s, 1+\frac{n}{x}+y\right]\right)$

```
(* *)
Sum[ j^-s, {j, 1, n}] - x^(1-s) Sum[ (j)^-s, {j, 1, n/x}]
```

$\text{HarmonicNumber}[n, s] - x^{1-s}\,\text{HarmonicNumber}\left[\frac{n}{x}, s\right]$

$\mathbf{D}\left[-x^{1-s}\,\textbf{HarmonicNumber}\left[\frac{n}{x}, s\right], x\right]$

$-(1-s)\,x^{-s}\,\text{HarmonicNumber}\left[\frac{n}{x}, s\right] + n\,s\,x^{-1-s}\left(-\text{HarmonicNumber}\left[\frac{n}{x}, 1+s\right] + \text{Zeta}[1+s]\right)$

```
HarmonicNumber[33.3, 2]
```

1.61535

```
HurwitzZeta[2, 1] - HurwitzZeta[2, 33.3 + 1]
```

1.61535

$\mathbf{D}\left[-x^{1-s}\,(\textbf{HurwitzZeta}[s, 1] - \textbf{HurwitzZeta}[s, n/x+1]), x\right] \;/.\; x \to 1$

n s HurwitzZeta[1 + s, 1 + n] - (1 - s) (-HurwitzZeta[s, 1 + n] + Zeta[s])

```
(* *)
D[s/(s-1) n Zeta[1+s, 1+n] + HarmonicNumber[n, s], n]
```

```
FullSimplify[D[s / (s - 1) n Zeta[1 + s, 1 + n], n]]
```

$$\frac{s\,(\text{Zeta}[1+s,1+n]-n\,(1+s)\,\text{Zeta}[2+s,1+n])}{-1+s}$$

```
FullSimplify[D[HarmonicNumber[n, s], n]]

s HurwitzZeta[1 + s, 1 + n]

(* *)


FullSimplify[D[s n Zeta[1 + s, 1 + n] - (1 - s) HarmonicNumber[n, s], n]]

s ((-1 + s) HurwitzZeta[1 + s, 1 + n] + Zeta[1 + s, 1 + n] - n (1 + s) Zeta[2 + s, 1 + n])

FullSimplify[s ((-1 + s) Zeta[1 + s, 1 + n] + Zeta[1 + s, 1 + n] - n (1 + s) Zeta[2 + s, 1 + n])]

s (s Zeta[1 + s, 1 + n] - n (1 + s) Zeta[2 + s, 1 + n])


fa[n_, s_] := s n Zeta[s + 1, n + 1] - (1 - s) (Zeta[s] - Zeta[s, n + 1])
faa[n_, s_] := s n Zeta[s + 1, n + 1] - (1 - s) (-Zeta[s, n + 1])
fab[n_, s_] := {s n Zeta[s + 1, n + 1] , (1 - s) (-Zeta[s, n + 1])}

fab[10 000 000 000, N[ZetaZero[1]]]

{30 432.9 + 95 256.7 i, 30 432.9 + 95 256.7 i}

((.5 + I) - 1) Zeta[.5 + I]

0.650132 + 0.504986 i

Limit[Sum[ (s n - (s - 1) j) / j^(s + 1), {j, n, Infinity}], n → Infinity]
```

$$\text{Limit}\left[\sum_{j=n}^{\infty} j^{-1-s}\,(-j\,(-1+s)+n\,s),\ n\to\infty\right]$$

```
(* *)
```

$$\text{D}\left[-x^{1-s}\,\text{HarmonicNumber}\left[\frac{n}{x},\ s\right],\ \{x,1\}\right]\ /.\ x\to 1$$

```
-(1 - s) HarmonicNumber[n, s] + n s (-HarmonicNumber[n, 1 + s] + Zeta[1 + s])
```

$$\text{FullSimplify}\left[\text{D}\left[-x^{1-s}\,\text{HarmonicNumber}\left[\frac{n}{x},\ s\right],\ \{x,1\}\right]\ /.\ x\to 1\right]$$

```
(-1 + s) HarmonicNumber[n, s] + n s HurwitzZeta[1 + s, 1 + n]

D[(1 - x^(1 - s)) Zeta[s], x] /. x → 1

-(1 - s) Zeta[s]
```

$$\text{D}\left[-x^{1-s}\,\text{HarmonicNumber}\left[\frac{n}{x},\ s\right],\ \{x,2\}\right]\ /.\ x\to 1$$

```
(1 - s) s HarmonicNumber[n, s] - 2 n s (-HarmonicNumber[n, 1 + s] + Zeta[1 + s]) +
 2 n (1 - s) s (-HarmonicNumber[n, 1 + s] + Zeta[1 + s]) +
 n^2 s (1 + s) (-HarmonicNumber[n, 2 + s] + Zeta[2 + s])
```

FullSimplify[D[-x^(1-s) HarmonicNumber[n/x, s], {x, 2}] /. x → 1]

```
s ((-1 + s) HurwitzZeta[s, 1 + n] - 2 n s HurwitzZeta[1 + s, 1 + n] +
   n^2 (1 + s) HurwitzZeta[2 + s, 1 + n] + Zeta[s] - s Zeta[s])
```

D[(1 - x^(1 - s)) Zeta[s], {x, 2}] /. x → 1

```
(1 - s) s Zeta[s]
```

**FullSimplify[((1 - s) s HarmonicNumber[n, s] - 2 n s (-HarmonicNumber[n, 1 + s] + Zeta[1 + s]) +
2 n (1 - s) s (-HarmonicNumber[n, 1 + s] + Zeta[1 + s]) +
n^2 s (1 + s) (-HarmonicNumber[n, 2 + s] + Zeta[2 + s])) / ((1 - s) s)]**

```
1/(-1 + s) ((-1 + s) HarmonicNumber[n, s] +
   n (2 s HurwitzZeta[1 + s, 1 + n] - n (1 + s) HurwitzZeta[2 + s, 1 + n]))
```

D[-x^(1-s) HarmonicNumber[n/x, s], {x, 3}] /. x → 1

```
(-1 - s) (1 - s) s HarmonicNumber[n, s] + 6 n s (-HarmonicNumber[n, 1 + s] + Zeta[1 + s]) -
 3 n (1 - s) s^2 (-HarmonicNumber[n, 1 + s] + Zeta[1 + s]) -
 6 n^2 s (1 + s) (-HarmonicNumber[n, 2 + s] + Zeta[2 + s]) -
 3 (1 - s) (2 n s (-HarmonicNumber[n, 1 + s] + Zeta[1 + s]) -
     n^2 s (1 + s) (-HarmonicNumber[n, 2 + s] + Zeta[2 + s])) +
 n^3 s (1 + s) (2 + s) (-HarmonicNumber[n, 3 + s] + Zeta[3 + s])
```

FullSimplify[D[-x^(1-s) HarmonicNumber[n/x, s], {x, 3}] /. x → 1]

```
s (1 + s) (-(-1 + s) HurwitzZeta[s, 1 + n] + 3 n s HurwitzZeta[1 + s, 1 + n] -
   3 n^2 (1 + s) HurwitzZeta[2 + s, 1 + n] + n^3 (2 + s) HurwitzZeta[3 + s, 1 + n] + (-1 + s) Zeta[s])
```

D[(1 - x^(1 - s)) Zeta[s], {x, 3}] /. x → 1

```
(-1 - s) (1 - s) s Zeta[s]
```

D[-x^(1-s) HarmonicNumber[n/x, s], {x, 4}] /. x → 1

```
(-2 - s) (-1 - s) (1 - s) s HarmonicNumber[n, s] - 24 n s (-HarmonicNumber[n, 1 + s] + Zeta[1 + s]) -
 4 n (-1 - s) (1 - s) s^2 (-HarmonicNumber[n, 1 + s] + Zeta[1 + s]) +
 36 n^2 s (1 + s) (-HarmonicNumber[n, 2 + s] + Zeta[2 + s]) +
 6 (1 - s) s (2 n s (-HarmonicNumber[n, 1 + s] + Zeta[1 + s]) -
     n^2 s (1 + s) (-HarmonicNumber[n, 2 + s] + Zeta[2 + s])) -
 12 n^3 s (1 + s) (2 + s) (-HarmonicNumber[n, 3 + s] + Zeta[3 + s]) -
 4 (1 - s) (-6 n s (-HarmonicNumber[n, 1 + s] + Zeta[1 + s]) +
     6 n^2 s (1 + s) (-HarmonicNumber[n, 2 + s] + Zeta[2 + s]) -
     n^3 s (1 + s) (2 + s) (-HarmonicNumber[n, 3 + s] + Zeta[3 + s])) +
 n^4 s (1 + s) (2 + s) (3 + s) (-HarmonicNumber[n, 4 + s] + Zeta[4 + s])
```

```
FullSimplify[D[-x^(1-s) HarmonicNumber[n/x, s], {x, 4}] /. x → 1]
```

```
s (1 + s) (2 + s) ((-1 + s) HurwitzZeta[s, 1 + n] - 4 n s HurwitzZeta[1 + s, 1 + n] +
   6 n^2 (1 + s) HurwitzZeta[2 + s, 1 + n] - 4 n^3 (2 + s) HurwitzZeta[3 + s, 1 + n] +
   n^4 (3 + s) HurwitzZeta[4 + s, 1 + n] + Zeta[s] - s Zeta[s])
```

```
D[(1 - x^(1 - s)) Zeta[s], {x, 4}] /. x → 1
```

```
(-2 - s) (-1 - s) (1 - s) s Zeta[s]
```

```
FullSimplify[((-2 - s) (-1 - s) (1 - s) s HarmonicNumber[n, s] -
    24 n s (-HarmonicNumber[n, 1 + s] + Zeta[1 + s]) -
    4 n (-1 - s) (1 - s) s^2 (-HarmonicNumber[n, 1 + s] + Zeta[1 + s]) +
    36 n^2 s (1 + s) (-HarmonicNumber[n, 2 + s] + Zeta[2 + s]) +
    6 (1 - s) s (2 n s (-HarmonicNumber[n, 1 + s] + Zeta[1 + s]) -
       n^2 s (1 + s) (-HarmonicNumber[n, 2 + s] + Zeta[2 + s])) -
    12 n^3 s (1 + s) (2 + s) (-HarmonicNumber[n, 3 + s] + Zeta[3 + s]) -
    4 (1 - s) (-6 n s (-HarmonicNumber[n, 1 + s] + Zeta[1 + s]) +
       6 n^2 s (1 + s) (-HarmonicNumber[n, 2 + s] + Zeta[2 + s]) -
       n^3 s (1 + s) (2 + s) (-HarmonicNumber[n, 3 + s] + Zeta[3 + s])) + n^4 s (1 + s) (2 + s)
     (3 + s) (-HarmonicNumber[n, 4 + s] + Zeta[4 + s])) / ((-2 - s) (-1 - s) (1 - s) s)]
```

```
1/(-1 + s) (-(-1 + s) HurwitzZeta[s, 1 + n] +
   4 n s HurwitzZeta[1 + s, 1 + n] - 6 n^2 (1 + s) HurwitzZeta[2 + s, 1 + n] +
   4 n^3 (2 + s) HurwitzZeta[3 + s, 1 + n] - n^4 (3 + s) HurwitzZeta[4 + s, 1 + n] + (-1 + s) Zeta[s])
```

```
tss[n_, s_] := 1/(-1 + s) (-(-1 + s) HurwitzZeta[s, 1 + n] +
    4 n s HurwitzZeta[1 + s, 1 + n] - 6 n^2 (1 + s) HurwitzZeta[2 + s, 1 + n] +
    4 n^3 (2 + s) HurwitzZeta[3 + s, 1 + n] - n^4 (3 + s) HurwitzZeta[4 + s, 1 + n] + (-1 + s) Zeta[s])
```

```
tss[1000, -3.000000001]
```

```
-0.00012207
```

```
N@Zeta[-3]
```

```
0.00833333
```

```
N@ZetaZero[1]
```

```
0.5 + 14.1347 i
```

```
tssa[n_, s_] := 1/(-1 + s) (-(-1 + s) HurwitzZeta[s, 1 + n] +
    4 n s HurwitzZeta[1 + s, 1 + n] - 6 n^2 (1 + s) HurwitzZeta[2 + s, 1 + n] +
    4 n^3 (2 + s) HurwitzZeta[3 + s, 1 + n] - n^4 (3 + s) HurwitzZeta[4 + s, 1 + n] + (-1 + s) Zeta[s])
```

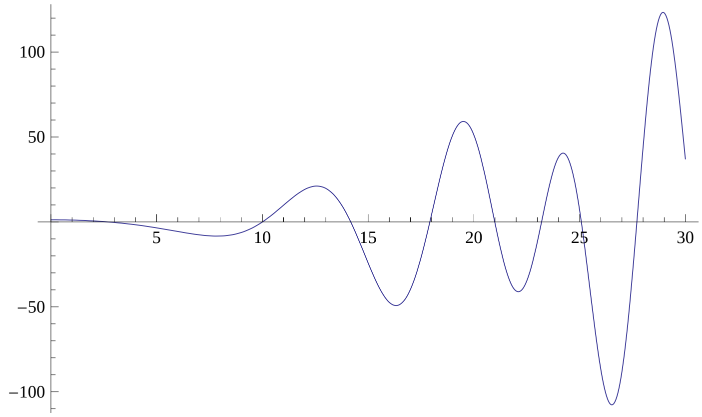
```
pp[n_] := n - n^2 Zeta[ 2, 1 + n]
pp2[n_] := n - n^2 (Zeta[2] - Sum[j^-2, {j, 1, n}])
```

```
N[pp[10 000]]

0.499983

N[pp2[1 000 000]]

0.500349


(* *)


lz[n_, s_] := -(1 - s) HarmonicNumber[n, s] + n s (-HarmonicNumber[n, 1 + s] + Zeta[1 + s])

FullSimplify[lz[n, s + t I] + lz[n, 1 - (s + t I)] - lz[n, s - t I] - lz[n, 1 - (s - t I)]]

-(s + i t) HarmonicNumber[n, 1 - s - i t] + (1 - s + i t) HarmonicNumber[n, s - i t] +
 (s - i t) HarmonicNumber[n, 1 - s + i t] + (-1 + s + i t) HarmonicNumber[n, s + i t] -
 n (-1 + s + i t) HurwitzZeta[2 - s - i t, 1 + n] - n (s - i t) HurwitzZeta[1 + s - i t, 1 + n] +
 n (-1 + s - i t) HurwitzZeta[2 - s + i t, 1 + n] + n (s + i t) HurwitzZeta[1 + s + i t, 1 + n]

FullSimplify[-(s + i t) HarmonicNumber[n, 1 - s - i t] +
   (1 - s + i t) HarmonicNumber[n, s - i t] + (s - i t) HarmonicNumber[n, 1 - s + i t] +
   (-1 + s + i t) HarmonicNumber[n, s + i t] - n (-1 + s + i t) HurwitzZeta[2 - s - i t, 1 + n] -
   n (s - i t) HurwitzZeta[1 + s - i t, 1 + n] + n (-1 + s - i t) HurwitzZeta[2 - s + i t, 1 + n] +
   n (s + i t) HurwitzZeta[1 + s + i t, 1 + n] /. s → 1 / 4]
```

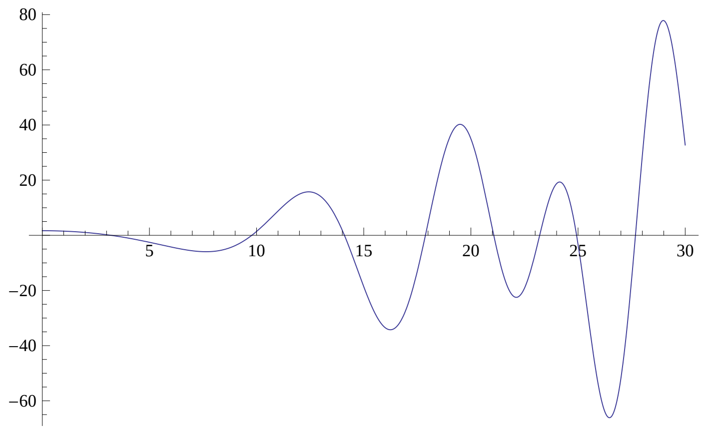
$$\left(\frac{3}{4}+i\,t\right)\text{HarmonicNumber}\left[n,\frac{1}{4}-i\,t\right]+\left(-\frac{1}{4}-i\,t\right)\text{HarmonicNumber}\left[n,\frac{3}{4}-i\,t\right]+$$
$$\left(-\frac{3}{4}+i\,t\right)\text{HarmonicNumber}\left[n,\frac{1}{4}+i\,t\right]+\left(\frac{1}{4}-i\,t\right)\text{HarmonicNumber}\left[n,\frac{3}{4}+i\,t\right]+$$
$$\frac{1}{4}n\,(-1+4\,i\,t)\,\text{HurwitzZeta}\left[\frac{5}{4}-i\,t,1+n\right]+\frac{1}{4}n\,(3-4\,i\,t)\,\text{HurwitzZeta}\left[\frac{7}{4}-i\,t,1+n\right]+$$
$$n\left(\frac{1}{4}+i\,t\right)\text{HurwitzZeta}\left[\frac{5}{4}+i\,t,1+n\right]+n\left(-\frac{3}{4}-i\,t\right)\text{HurwitzZeta}\left[\frac{7}{4}+i\,t,1+n\right]$$

```
(lz[n, s + t I] + lz[n, s - t I]) - (lz[n, 1 - s - t I] + lz[n, 1 - s + t I])

-(-s - i t) HarmonicNumber[n, 1 - s - i t] + (-1 + s - i t) HarmonicNumber[n, s - i t] -
 (-s + i t) HarmonicNumber[n, 1 - s + i t] + (-1 + s + i t) HarmonicNumber[n, s + i t] -
 n (1 - s - i t) (-HarmonicNumber[n, 2 - s - i t] + Zeta[2 - s - i t]) +
 n (s - i t) (-HarmonicNumber[n, 1 + s - i t] + Zeta[1 + s - i t]) -
 n (1 - s + i t) (-HarmonicNumber[n, 2 - s + i t] + Zeta[2 - s + i t]) +
 n (s + i t) (-HarmonicNumber[n, 1 + s + i t] + Zeta[1 + s + i t])

tc[n_, s_, t_] :=
 {(s - i t) HarmonicNumber[n, 1 - s + i t], n (-1 + s - i t) HurwitzZeta[2 - s + i t, 1 + n],
  (-1 + s + i t) HarmonicNumber[n, s + i t], n (s + i t) HurwitzZeta[1 + s + i t, 1 + n]}


lz[n, s + t I] + lz[n, s - t I]

(-1 + s - i t) HarmonicNumber[n, s - i t] + (-1 + s + i t) HarmonicNumber[n, s + i t] +
 n (s - i t) (-HarmonicNumber[n, 1 + s - i t] + Zeta[1 + s - i t]) +
 n (s + i t) (-HarmonicNumber[n, 1 + s + i t] + Zeta[1 + s + i t])
```

```
Plot[Re[lz[1000000, s + t I] + lz[1000000, s - t I] /. s → .3], {t, 0, 30}]
```




```
Plot[Re[lz[1000000, 1 - s + t I] + lz[1000000, 1 - s - t I] /. s → .3], {t, 0, 30}]
```




```
ff[ s_] := Sum[ ((1 - s) / s (j / n) + 1) / (j^(s + 1)), {j, 1, Infinity}]
```

```
(* *)

Expand[s n Zeta[s + 1, n + 1] - (1 - s) (Zeta[s] - Zeta[s, n + 1])]

N[-Zeta[s] + s Zeta[s] + Zeta[s, 1 + n] - s Zeta[s, 1 + n] + n s Zeta[1 + s, 1 + n] /.
   s → ZetaZero[1] /. n → 1000000000000]
```

$-2.69036\times10^{-7}+4.18397\times10^{-7}\,i$

```
N[Zeta[s, 1 + n] - s Zeta[s, 1 + n] + n s Zeta[1 + s, 1 + n] /. s → 3 /. n → 100000000000000]
```

-4.82047×10^{-43}

```
Zeta[s, 1 + n] - s Zeta[s, 1 + n] + n s Zeta[1 + s, 1 + n]

Zeta[s, 1 + n] - s Zeta[s, 1 + n] + n s Zeta[1 + s, 1 + n]

fg[n_, s_] := s n (Zeta[s + 1] - HarmonicNumber[n, s + 1]) - (1 - s) HarmonicNumber[n, s]
fg2[n_, s_] := (Zeta[s + 1] - HarmonicNumber[n, s + 1]) - (1 - s) HarmonicNumber[n, s] / (s n )
fg2a[n_, s_] := (Zeta[s + 1] - HarmonicNumber[n, s + 1])
fg2b[n_, s_] := - (1 - s) HarmonicNumber[n, s] / (s n )
```

```
fg[n, s] / (-(1 - s)) /. n → 100 000 000 000 /. s → .5
```

-1.46035

```
(fg2[n, s] / (-(1 - s))) (s n) /. n → 100 000 000 000 /. s → .5
```

-1.46034

```
fg2a[n, s] /. n → 10 000 000 000 /. s → .5
```

0.00002

```
fe[n_, s_, x_] := Sum[j^-s, {j, 1, n}] - x^(1 - s) Sum[j^-s, {j, 1, n / x}]
fe2[n_, s_, x_] :=
 Sum[j^-s, {j, 1, n}] - x^(1 - s) (Sum[j^-s - (j + n / x)^-s, {j, 1, Infinity}])
fe3[n_, s_, x_] := Sum[j^-s, {j, 1, n}] -
  (Sum[x^(1 - s) j^-s - x^(1 - s) (j + n / x)^-s, {j, 1, Infinity}])
```

```
N@fe[100 000, .5, 2]
```

0.603318

```
N[(1 - 2^(1 - .5)) Zeta[.5]]
```

0.604899

```
N@fe2[100 000, .5, 2]
```

0.60332

```
D[x^(1 - s) (Sum[j^-s - (j + n / x)^-s, {j, 1, Infinity}]), x] /. x → 1
```

-n s HurwitzZeta[1 + s, 1 + n] + (1 - s) (-HurwitzZeta[s, 1 + n] + Zeta[s])

```
D[(Sum[x^(1 - s) j^-s - x^(1 - s) (j + n / x)^-s, {j, 1, Infinity}]), x] /. x → 1
```

-n s HurwitzZeta[1 + s, 1 + n] - (1 - s) (HurwitzZeta[s, 1 + n] - Zeta[s])

```
D[(Sum[-x^(1 - s) (j + n / x)^-s, {j, 1, Infinity}]), x] /. x → 1
```

-(1 - s) HurwitzZeta[s, 1 + n] - n s HurwitzZeta[1 + s, 1 + n]

```
D[x^(1 - s) j^-s, x] /. x → 1
```

$j^{-s} (1 - s)$

```
D[-x^(1 - s) (j + n / x)^-s, x] /. x → 1
```

$-(j + n)^{-s} (1 - s) - n (j + n)^{-1-s} s$

```
D[-x^(1 - s) aa[n], x] /. x → 1
```

-(1 - s) aa[n]

```
D[bb[n] (j + n / x) ^-s, x] /. x → 1
```

$n\,(j+n)^{-1-s}\,s\,\mathrm{bb}[n]$

```
lt[n_, s_, k_] := Sum[ n^k / j^(s + k), {j, n + 1, Infinity}]
N@lt[10, -1, 3]
```

95.1663

```
Sum[ (-1)^k Binomial[t, k] (s + k) / (s - 1) n^k / j^(s + k), {k, 0, t}]
```

$$\frac{j^{-s}\left(1-\frac{n}{j}\right)^{t}(j\,s-n\,s-n\,t)}{(j-n)\,(-1+s)}$$

```
ad[n_, s_, t_] := Sum[ j^-s, {j, 1, n}] - Sum[ Sum[
    (-1)^k Binomial[t, k] (s - 1 + k) / (s - 1) n^k / j^(s + k), {k, 1, t}], {j, n + 1, Infinity}]
ad2[n_, s_] := Sum[ j^-s, {j, 1, n}] + (s) / (s - 1) Sum[n / j^(s + 1), {j, n + 1, Infinity}]
N@ad[10 000, 0, 3]
```

29 999.5

```
Zeta[.5]
```

-1.46035

```
(* *)
```

```
D[(Sum[(-x^(1 - s)) j^-s - (-x^(1 - s)) (j + n / x)^-s, {j, 1, Infinity}]), {x, 2}] /. x → 1
```

$-2\,n\,s\,\mathrm{HurwitzZeta}[1+s,1+n]+2\,n\,(1-s)\,s\,\mathrm{HurwitzZeta}[1+s,1+n]+$
$n^2\,s\,(1+s)\,\mathrm{HurwitzZeta}[2+s,1+n]-(1-s)\,s\,(\mathrm{HurwitzZeta}[s,1+n]-\mathrm{Zeta}[s])$

```
D[((-x^(1 - s)) j^-s), {x, 2}] /. x → 1
```

$j^{-s}\,(1-s)\,s$

```
D[((x^(1 - s)) (j + n / x)^-s), {x, 2}] /. x → 1
```

$-2\,n\,(j+n)^{-1-s}\,s-n^2\,(j+n)^{-2-s}\,(-1-s)\,s+2\,n\,(j+n)^{-1-s}\,(1-s)\,s-(j+n)^{-s}\,(1-s)\,s$

FullSimplify$\left[-2\,n\,(j+n)^{-1-s}\,s+2\,n\,(j+n)^{-1-s}\,(1-s)\,s\right]$

$-2\,n\,(j+n)^{-1-s}\,s^2$

```
ar[n_, s_] := ((1 - s) (s) Sum[j^-s, {j, 1, n}] - 2 s^2 Sum[ n / j^(s + 1), {j, n + 1, Infinity}] +
    s (s + 1) Sum[ n^2 / j^(s + 2), {j, n + 1, Infinity}]) / (s (1 - s))
arb[n_, s_] := ((1 - s) (s) Sum[j^-s, {j, 1, n}] +
    Sum[- 2 s^2 n / j^(s + 1) + s (s + 1) n^2 / j^(s + 2), {j, n + 1, Infinity}]) / (s (1 - s))
```

```
N@arb[100, -.5]
```

Sum::div : Sum does not converge. ≫

NIntegrate::slwcon :
Numerical integration converging too slowly; suspect one of the following: singularity, value of the integration is 0, highly oscillatory integrand, or WorkingPrecision too small. ≫

NIntegrate::ncvb :
NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in j near {j} = $\{8.16907\times10^{224}\}$.
NIntegrate obtained $-1.921970898100714\times10^{13979}$ and 1.921970898100714`15.954589770191005*^13979 for the integral and error estimates. ≫

$2.562627864134286\times10^{13979}$

```
Zeta[.5]
```

-1.46035

```
(* *)
```

```
D[(Sum[(-x^(1-s)) (j+y)^-s - (-x^(1-s)) (j+y+n/x)^-s, {j, 1, Infinity}]), x] /. x → 1
```

-(1-s) (HurwitzZeta[s, 1+y] - HurwitzZeta[s, 1+n+y]) + n s HurwitzZeta[1+s, 1+n+y]

```
D[(-x^(1-s)) (j+y)^-s, x] /. x → 1
```

$-(1-s)(j+y)^{-s}$

```
D[-(-x^(1-s)) (j+y+n/x)^-s, x] /. x → 1
```

$n\, s\, (j+n+y)^{-1-s} + (1-s)(j+n+y)^{-s}$

```
ax[n_, s_, y_] :=
 Sum[(j+y)^-s, {j, 1, n}] - s/(s-1) Sum[n/(j+y)^(-s-1), {j, n+1, Infinity}]
```