Overview

My goal in this paper is to explain a particular point-of-view that conceptually connects the Riemann Prime Counting function, the logarithmic integral, and the harmonic number.

To that end, I'll introduce a value I'll write as $\log(1+\boldsymbol{x})$. This value will stand, variously, for the logarithmic integral, the harmonic number, and count of primes given by the Riemann Prime Counting function.

The really big idea here, the guiding concept, is that many of the relationships between x^k , $(1+x)^z$, and $\log(1+x)$, have analogs where the logarithmic integral, the harmonic number, and the Riemann Prime Counting function fill the role of $\log(1+x)$. They all satisfy a broad range of properties that $\log x$ satisfies.

To flesh out the roles these functions play, I'll also define values written as $\boldsymbol{x}^{\boldsymbol{k}}$ and $(1+\boldsymbol{x})^z$.

Section 1:
$\boldsymbol{x}^{\boldsymbol{k}}$

Overview of This Section

As a concept, $\boldsymbol{x}^k$ is the most straightforward of the terms introduced here, so I begin with it.

It's well known that $(1+x)^z$, $\log(1+x)$, and $\log^k(1+x)$ can all be expressed in terms of x^k , of course, going back to Newton's generalized binomial theorem and to Taylor series.

My overarching goal here, after I define $\boldsymbol{x}^k$, is to express some important convolution sums, which I notate as $(1+\boldsymbol{x})^z$, $\log(1+\boldsymbol{x})$, and $\log^k(1+\boldsymbol{x})$, as sums of the form $\sum_{k=0}^{\infty} C_k \cdot \boldsymbol{x}^k$. Just as a reminder, variations of $\log(1+\boldsymbol{x})$ include the Riemann Prime counting function, the logarithmic integral, and the Harmonic number. So that justifies the interest in taking this approach.

So, I start by defining $\boldsymbol{x}^k$. In particular, for developing intuition, $\boldsymbol{x}^k$ has a nice conceptual description and a relatively straightforward geometric interpretations, so that comes first.

Then I show how variations of $\boldsymbol{x}^k$ relate to each other and explain the non-standard notation used here and its justification.

Following that, I give a recursive definition for $\boldsymbol{x}^k$. This definition really hammers home how closely related the variants of $\boldsymbol{x}^k$ (and later $\log(1+\boldsymbol{x})$) are to each other. All four variants of $\boldsymbol{x}^k$, and x^k itself, can be expressed with the same recursive form, with parameter changes. I also provide a Mathematica implementation of this recursive form.

Then I write out, long-hand, just to be extremely clear, values of $\boldsymbol{x}^k$ for the first few values of k in integral or sum notation.

I then list closed forms for some variations of $\boldsymbol{x}^k$ and for their derivatives or finite differences with respect to x. One big point here is the relationship between $\boldsymbol{x}^k$ and these closed forms. A goal of the non-standard notation in here is to build instincts about what relationships should exist between these more standard closed forms. Think of my non-standard notation as a kind of scaffolding for finding these identities that can then be replaced with the more standard closed form notation.

And finally, because our entire goal in defining $\boldsymbol{x}^k$ is to write sums of the form $\sum_{k=0}^{\infty} C_k \cdot \boldsymbol{x}^k$ to express $(1+\boldsymbol{x})^z$, $\log(1+\boldsymbol{x})$, and $\log^k(1+\boldsymbol{x})$, I cover convergence involving sums of $\boldsymbol{x}^k$. But this is almost an afterthought; convergence on sums built on $\boldsymbol{x}^k$ is pretty straightforward compared to x^k .

Conceptual and Geometric Interpretation of $\boldsymbol{x}^k$

To cut to the chase, it's easiest to see what $\boldsymbol{x}^k$ means, conceptual, by showing a few examples. So first, here are a few variants of $\boldsymbol{x}^2$:

$$x^2$$

x^2 : Square

Real-valued area bounded by $t>0$, $u>0$, $t\le x$, and $u\le x$.

$\boldsymbol{x}^{2+\int}$: Triangle

Real-valued area bounded by $t>0$, $u>0$, and $t+u\le x$.

$\boldsymbol{x}^{2+\Sigma}$ Discrete Triangle

Count of whole number pairs (t,u) satisfying $t>0$, $u>0$, and $t+u\le x$.

$\boldsymbol{x}^{2*\int}$: Hyperbola

Real-valued area bounded by $t > 0$, $u > 0$, and $(1+t)\cdot(1+u) \leq x$.

$\boldsymbol{x}^{2*\Sigma}$: Discrete Hyperbola

Count of ordered whole number pairs (t,u) satisfying $t > 0$, $u > 0$, and $(1+t)\cdot(1+u) \leq x$.

And here are a few variants of $\boldsymbol{x}^3$:

$$x^3$$

x^3 : Cube

Real-valued volume bounded by $t > 0$, $u > 0$, $v > 0$, $t \leq x$, $u \leq x$, and $v \leq x$.

$\boldsymbol{x}^{3+\int}$: Triangular Pyramid

Real-valued volume bounded by $t > 0$, $u > 0$, $v > 0$, and $t+u+v \leq x$.

$\boldsymbol{x}^{3+\Sigma}$: Discrete Triangular Pyramid

Count of whole number pairs (t,u,v) satisfying $t > 0$, $u > 0$, $v > 0$, and $t+u+v \leq x$.

$\boldsymbol{x}^{3*\int}$: 3-dimensional Hyperbola

Real-valued volume bounded by $t > 0$, $u > 0$, $v > 0$, and $(1+t)\cdot(1+u)\cdot(1+v) \leq x$.

$\boldsymbol{x}^{3*\Sigma}$: Discrete 3-dimensional hyperbola.

Count of ordered whole number pairs (t,u,v) satisfying $t > 0$, $u > 0$, $v > 0$, and $(1+t)\cdot(1+u)\cdot(1+v) \leq x$.

So this defines squaring and cubing. Hopefully it's clear how to generalize these to arbitrary larger whole number powers in k dimensions, although I'll include an explicit recursive formula for that below.

How These Variations Relate to Each Other

Let me point out a few straight-forward relationships between these different terms.

* $\boldsymbol{x}^{k+\Sigma}$ is the discrete equivalent of $\boldsymbol{x}^{k+\int}$
* $\boldsymbol{x}^{k*\Sigma}$ is the discrete equivalent of $\boldsymbol{x}^{k*\int}$
* $\boldsymbol{x}^{k*\int}$ is the multiplicative equivalent of the additive $\boldsymbol{x}^{k+\int}$
* $\boldsymbol{x}^{k*\Sigma}$ is the multiplicative equivalent of the additive $\boldsymbol{x}^{k+\Sigma}$

To emphasize these relationships, visually, I'll generally list identities in a grid, like this:

$$\boldsymbol{x}^k =$$

	$\int$	$\sum$
+	$\boldsymbol{x}^{k+\int}$	$\boldsymbol{x}^{k+\Sigma}$
*	$\boldsymbol{x}^{k*\int}$	$\boldsymbol{x}^{k*\Sigma}$

A Few Quick Notes on This Nonstandard Notation

* A quick explanation for the use of the enclosing curly braces { and }: normally with multiplication, the output of one multiplication can be calculated and then used as input in another multiplication. For example, when computing x^3 , we could say $v = x^2$, compute v as an intermediate step, then compute $v \cdot x$ and arrive at a correct answer for x^3 . This works because the constraints on the 3 terms are all independent of one another. This isn't valid for any of the four variations of $\boldsymbol{x}^k$. The braces mean that all multiplication and exponentiation contained within needs to be considered as one big atomic operation.

* If I don't specify $+ / *$ or $\int / \sum$ outside of the curly braces – so like this $\boldsymbol{x}^k$ - it means I'm saying something

that applies to all four variations.

* This is the only non-standard notation I'll include here, I promise! This is it.

Recursive Definition for x^k

So, one might think, from the descriptions of the variations of x^k, these terms must have certain symmetries with one another. And they do.

In fact, with just a few simple parameters, a recursive function can be written that calculates both x^k and any of the four variations of x^k, with x a real number >= 1 and k a whole number >= 0.

The function, $f_k(x,d,\theta,I)$, is defined as:

$$f_k(x,d,\theta,I)=g_k(x,d+I) \quad \text{where} \quad g_k(x,t)=\begin{cases} d\cdot g_{k-1}(\theta(x,t),d+I)+g_k(x,d+t) & x\geq t \\ 0 & x<t,k>0 \\ 1 & k=0 \end{cases}$$

Given some value of x such that 1<x<2, x^k can be expressed as

$$x^k=\lim_{d\to 0} f_k(x,d,\theta_=,1) \quad \text{with} \quad \theta_=(x,j)=x$$

Given two helper functions, $\theta_+(x,t)=x-t$ and $\theta_*(x,t)=\frac{x}{t}$, the four variants of x^k can likewise be expressed as

	$\int$	$\sum$
+	$\lim_{d\to 0} f_k(x,d,\theta_+,0)$	$f_k(x,1,\theta_+,0)$
*	$\lim_{d\to 0} f_k(x,d,\theta_*,1)$	$f_k(x,1,\theta_*,1)$

Mathematica Computation of x^k

Just to make this redundantly clear, in Mathematica, this recursive function can be implemented as

```
delta=.01;
thetaAdd[x_,t_]:=x-t
thetaMul[x_,t_]:=x/t
thetaEq[x_,t_]:=x
f[x_,t_,0,d_,fn_,I_]:=1
f[x_,t_,k_,d_,fn_,I_]:= If[x<t,0, d f[fn[x,t],d+I,k-1,d,fn,I] + f[x,d+t,k,d,fn,I]]
```

With this, an approximation of x^k can be computed, if 1 < x < 2, as f[x,1+delta,k,delta,thetaEq,1], with the approximation getting closer and closer as *delta* gets closer to 0.

And x^k can be computed as

	$\int$	$\sum$
+	`f[x,delta,k,delta,thetaAdd,0]`	`f[x,1,k,1,thetaAdd,0]`
*	`f[x,1+delta,k,delta,thetaMul,1]`	`f[x,2,k,1,thetaMul,1]`

again with approximations of the integrals getting more accurate as delta gets closer to 0.

This isn't really a reasonable way to compute $\boldsymbol{x}^{k+\Sigma}$ and $\boldsymbol{x}^{k*\Sigma}$ except for small values of x, but it's especially not at all a reasonable way to compute $\boldsymbol{x}^{k+\int}$ or $\boldsymbol{x}^{k+\int}$. Fortunately, closed form values of those exist and will be mentioned shortly.

Specific Expressions of $\boldsymbol{x}^k$

I'll list a few explicit integrals and sums for computing $\boldsymbol{x}^k$ to the first few values of k, just to keep things nice and clear. Although the recursive expression in the previous section should hopefully hammer home just how closely related these various terms are to each other, the expressions in this section are easier to reason about for finding closed forms, which will be given in the next section.

So, here's the first power,

$$x=\int_0^x dt$$

and

$$\boldsymbol{x}=$$

	$\int$	Σ
$+$	$\int_0^x dt$	$\sum_{t=1}^{x} 1$
$*$	$\int_1^x dt$	$\sum_{t=2}^{x} 1$

And the second,

$$x^2=\int_0^x \int_0^x du\, dt$$

with

$$\boldsymbol{x}^2=$$

	$\int$	Σ
$+$	$\int_0^x \int_0^{x-t} du\, dt$	$\sum_{t=1}^{x} \sum_{u=1}^{x-t} 1$
$*$	$\int_1^x \int_1^{\frac{x}{t}} du\, dt$	$\sum_{t=2}^{x} \sum_{u=2}^{\lfloor \frac{x}{t} \rfloor} 1$

And the third power,

$$x^3=\int_0^x \int_0^x \int_0^x dv\, du\, dt$$

and

$$\boldsymbol{x}^3=$$

	$\int$	Σ
$+$	$\int_0^x \int_0^{x-t} \int_0^{x-t-u} dv\, du\, dt$	$\sum_{t=1}^{x} \sum_{u=1}^{x-t} \sum_{v=1}^{x-t-u} 1$
$*$	$\int_1^x \int_1^{\frac{x}{t}} \int_1^{\frac{x}{tu}} dv\, du\, dt$	$\sum_{t=2}^{x} \sum_{u=2}^{\lfloor \frac{x}{t} \rfloor} \sum_{v=2}^{\lfloor \frac{x}{t \cdot u} \rfloor} 1$

and so on. This should be pretty simple.

Closed Form Values for $\boldsymbol{x^k}$

With a bit of work, most variations of $\boldsymbol{x^k}$ can be expressed in concise closed forms. This is significant; later sections are going to show how variants of $\boldsymbol{x^k}$ behave similarly to x^k. This will then let us find interesting identities involving the following closed forms, inspired by identities involving x^k.

So, those closed form expressions are

$$\boldsymbol{x^k} =$$

	$\int$	$\sum$
+	$\frac{x^k}{k!}$	$\binom{x}{k}$
*	$(-1)^{-k}\cdot P(k,-\log x)$	$D_k'(x)$

$\binom{x}{k}$: binomial coefficient defined as $\binom{x}{k}=\frac{x!}{k!(x-k)!}=\frac{(x)_k}{k!}$, with $(x)_k = x(x-1)(x-2)...(x-k+1)$

$P(k,x)$: regularized incomplete gamma function, defined as $P(k,x)=\frac{\gamma(k,x)}{\Gamma(k)}$ and $(-1)^{-k}\cdot P(k,-\log x)=\int_1^x \frac{\log^{k-1} t}{(k-1)!}dt$

$D_k'(x)$: the strict divisor summatory function, defined recursively as $D_k'(x)=\sum_{j=2}^{\lfloor x\rfloor} D_{k-1}'(\frac{x}{j})$ with $D_0'(x)=1$

$\boldsymbol{x^{k*\int}}$ can also be expressed, for suitable values of x and k, as

$$=\frac{\log^k x}{k!}\cdot {}_1F_1(k;k+1;\log x)$$
$$=\frac{\log^k x}{k!}\cdot\sum_{t=0}^{\infty}\left(1+\frac{t}{k}\right)^{-1}\cdot\frac{\log^t x}{t!}$$
$$=(-1)^k\left(1-x\sum_{j=0}^{k-1}\frac{(-\log x)^j}{j!}\right)$$
$$(-1)^k\left(1-x\sum_{j=0}^{k-1}\frac{(-\log x)^j}{j!}\right)$$

Because I'm interested in stressing symmetries, note the following relationships between these closed forms:

$$\boldsymbol{x^{k+\int}}=\int_0^x \frac{t^{k-1}}{(k-1)!}dt$$
$$\boldsymbol{x^{k*\int}}=\int_1^x \frac{\log^{k-1} t}{(k-1)!}dt$$

and

$$\boldsymbol{x^{k+\int}}=\frac{x}{k}\cdot\frac{x}{k-1}\cdot\ldots\cdot\frac{x}{k-k+1}$$
$$\boldsymbol{x^{k+\Sigma}}=\frac{x}{k}\cdot\frac{x-1}{k-1}\cdot\ldots\cdot\frac{x-k+1}{k-k+1}$$

If we notate the rising factorial as $x^{(k)}=x(x+1)(x+2)...(x+k-1)$, then we can express

$$\boldsymbol{x}^{\boldsymbol{k}+\Sigma}=\binom{x}{k}=\sum_{a=1}^{x}\sum_{j=0}^{k}(-1)^{k-j}\cdot\binom{k}{j}\cdot\left(\!\binom{j}{a}\!\right)$$

and we can likewise express its multiplicative equivalent as the very closely related

$$\boldsymbol{x}^{\boldsymbol{k}*\Sigma}=D_k'(x)=\sum_{a=1}^{x}\sum_{j=0}^{k}(-1)^{k-j}\cdot\binom{k}{j}\prod_{p^{\alpha}|a}\left(\!\binom{j}{\alpha}\!\right)$$

(here, $\prod_{p^{\alpha}|a} f(\alpha)$ denotes a product over the powers of the prime factorization of a).

	$\int$	Σ
$+$	$\frac{x^k}{k!}$	$\frac{(x)_k}{k!}$
$*$	$\frac{\log^k x}{k!}\cdot\sum_{t=0}^{\infty}\left(1+\frac{t}{k}\right)^{-1}\cdot\frac{\log^t x}{t!}$	$D_k'(x)$

Closed Form Values for Derivatives / Finite Differences

We can also take inspiration from identities involving x^k to find interesting identities involving the derivatives or finite differences of $\boldsymbol{x}^{\boldsymbol{k}}$, the closed forms of which will be listed here.

Through the usual rules of calculus, the derivative of x^k with respect to x is easily found to be

$$\frac{\partial}{\partial x}x^k=k\cdot x^{k-1}$$

Given the closed forms above, we can also express the derivatives of $\boldsymbol{x}^{\boldsymbol{k}+\int}$ and $\boldsymbol{x}^{\boldsymbol{k}+\int}$ with respect to x, and the finite differences of $\boldsymbol{x}^{\boldsymbol{k}+\Sigma}$ and $\boldsymbol{x}^{\boldsymbol{k}*\Sigma}$ with respect to x. Those values are

$$\frac{\partial}{\partial x}\boldsymbol{x}^{\boldsymbol{k}}=\ /\ \nabla_x\boldsymbol{x}^{\boldsymbol{k}}=$$

	$\int$	Σ
$+$	$\frac{x^{k-1}}{(k-1)!}$	$\frac{(x-1)_{k-1}}{(k-1)!}$
$*$	$\frac{\log^{k-1}x}{(k-1)!}$	$d_k'(x)$

Here, $d_k'(x)$ is another function without a standard name, but with a handful of interesting properties worth noting.

Multiplication: $x^{a+b}=x^a\cdot x^b$

$$x^{a+b}=x^a\cdot x^b$$

$$\boldsymbol{x}^{\boldsymbol{a+b}}=\boldsymbol{x}^{\boldsymbol{a}}\cdot\boldsymbol{x}^{\boldsymbol{b}}$$

$$x^{a+b}=\int_0^x\int_0^x\frac{\partial}{\partial t}t^a\cdot\frac{\partial}{\partial u}u^b\,du\,dt$$

	$\int$	$\sum$
$+$	$\boldsymbol{x}^{\boldsymbol{a+b}}=\int_0^x\int_0^{x-t}\frac{\partial}{\partial t}\boldsymbol{t}^{\boldsymbol{a}}\cdot\frac{\partial}{\partial u}\boldsymbol{u}^{\boldsymbol{b}}\,du\,dt$	$\boldsymbol{x}^{\boldsymbol{a+b}}=\sum_{t=1}^{x}\sum_{u=1}^{x-t}\nabla_t\boldsymbol{t}^{\boldsymbol{a}}\cdot\nabla_u\boldsymbol{u}^{\boldsymbol{b}}$
$*$	$\boldsymbol{x}^{\boldsymbol{a+b}}=\int_1^x\int_1^{\frac{x}{t}}\frac{\partial}{\partial t}\boldsymbol{t}^{\boldsymbol{a}}\cdot\frac{\partial}{\partial u}\boldsymbol{u}^{\boldsymbol{b}}\,du\,dt$	$\boldsymbol{x}^{\boldsymbol{a+b}}=\sum_{t=2}^{x}\sum_{u=2}^{\frac{x}{t}}\nabla_t\boldsymbol{t}^{\boldsymbol{a}}\cdot\nabla_u\boldsymbol{u}^{\boldsymbol{b}}$

$$x^{a+b}=\int_0^x\int_0^x(a\,t^{a-1})\cdot(b\,u^{b-1})\,du\,dt$$

	$\int$	$\sum$
$+$	$\frac{x^{a+b}}{(a+b)!}=\int_0^x\int_0^{x-t}\frac{t^{a-1}}{(a-1)!}\cdot\frac{u^{b-1}}{(b-1)!}\,du\,dt$	$\binom{x}{a+b}=\sum_{t=1}^{x}\sum_{u=1}^{x-t}\binom{t-1}{a-1}\cdot\binom{u-1}{b-1}$
$*$	$(-1)^{-a-b}\cdot P(a+b,-\log x)=\int_1^x\int_1^{\frac{t}{t}}\frac{\log^{a-1}t}{(a-1)!}\cdot\frac{\log^{b-1}u}{(b-1)!}\,du\,dt$	$D_{a+b}'(x)=\sum_{t=2}^{x}\sum_{u=2}^{\lfloor\frac{x}{t}\rfloor}d_a'(t)\cdot d_b'(u)$

$$\frac{\partial}{\partial x}x^{a+b}=\frac{\partial}{\partial x}x^a\cdot x^b$$

$$\frac{\partial}{\partial x}\boldsymbol{x}^{\boldsymbol{a+b}}=\frac{\partial}{\partial x}\boldsymbol{x}^{\boldsymbol{a}}\cdot\boldsymbol{x}^{\boldsymbol{b}}$$

$$???\ x^{a+b}=\int_0^x\int_0^x\frac{\partial}{\partial t}t^a\cdot\frac{\partial}{\partial u}u^b\,du\,dt\ ???$$

	$\int$	$\sum$
$+$	$\frac{\partial}{\partial x}\boldsymbol{x}^{\boldsymbol{a+b}}=\int_0^x\frac{\partial}{\partial t}\boldsymbol{t}^{\boldsymbol{a}}\cdot\frac{\partial}{\partial t}(\boldsymbol{x}-\boldsymbol{t})^{\boldsymbol{b}}\,dt$	$\boldsymbol{x}^{\boldsymbol{a+b}}=\sum_{t=1}^{x-1}\nabla_t\boldsymbol{t}^{\boldsymbol{a}}\cdot\nabla_t(\boldsymbol{x}-\boldsymbol{t})^{\boldsymbol{b}}$
$*$	$\frac{\partial}{\partial x}\boldsymbol{x}^{\boldsymbol{a+b}}=\int_1^x\frac{\partial}{\partial t}\boldsymbol{t}^{\boldsymbol{a}}\cdot\frac{\partial}{\partial t}\left(\frac{\boldsymbol{x}}{\boldsymbol{t}}\right)^{\boldsymbol{b}}\,dt$	$\boldsymbol{x}^{\boldsymbol{a+b}}=\sum_{t\mid x,\,1<t<x}\nabla_t\boldsymbol{t}^{\boldsymbol{a}}\cdot\nabla_t\left(\frac{\boldsymbol{x}}{\boldsymbol{t}}\right)^{\boldsymbol{b}}$

$$???\ x^{a+b}=\int_0^x\int_0^x(a\,t^{a-1})\cdot(b\,u^{b-1})\,du\,dt\ ???$$

	$\int$	$\sum$
$+$	$\frac{x^{a+b-1}}{(a+b-1)!}=\int_0^x \frac{t^{a-1}}{(a-1)!}\cdot\frac{(x-t)^{b-1}}{(b-1)!}dt$	$\binom{x-1}{a+b-1}=\sum_{t=1}^{x-1}\binom{t-1}{a-1}\cdot\binom{(x-t)-1}{b-1}$
$*$	$\frac{\log^{a+b-1} x}{(a+b-1)!}=\int_1^x \frac{\log^{a-1} t}{(a-1)!}\cdot(\frac{\log^{b-1}(\frac{x}{t})}{(b-1)!}\cdot\frac{1}{t})dt$	$d_{a+b}'(x)=\sum_{t\mid x,\, 1<t<x} d_a'(t)\cdot d_b'(\frac{x}{t})$

On Convergence

Let's talk about convergence. In the next few sections, I'll write sums of the form $\sum_{k=0}^{\infty} C_k\cdot \boldsymbol{x^k}$. It's important to know if those sums converge, given that $\sum_{k=0}^{\infty} C_k\cdot x^k$ often only converges for certain values of x. I'm only concerned about cases where x is an integer value greater than or equal to 1, here.

Look at the closed forms just given for $\boldsymbol{x}^{k+\int}$ or $\boldsymbol{x}^{k+\int}$. As k grows, regardless of x, the factorial in the denominator dwarfs the exponential in the numerator. So those sums will eventually converge.

The cases of $\boldsymbol{x}^{k+\Sigma}$ and $\boldsymbol{x}^{k*\Sigma}$ are even easier.

It happens to be the case that, if k>=x, $\boldsymbol{x}^{k+\Sigma}=\binom{x}{k}=0$, so for a given whole number x, the sum will consist of x non-zero terms.

And it is likewise the case that, if $k>\lfloor\frac{\log x}{\log 2}\rfloor$, $\boldsymbol{x}^{k*\Sigma}=0$, so for a given whole number x, the sum will consist of $\lfloor\frac{\log x}{\log 2}\rfloor$ non-zero terms.

Series Reversion

Now let's move on to the first major identity.

Section 2:
$(1+\boldsymbol{x})^z$

Overview

N

Newton's Generalized Binomial Theorem

Newton's generalized binomial theorem provides a very well known way to express complex exponentiation. If $0<x<2$, then

$$(1+x)^z=\sum_{k=0}^{\infty}\binom{z}{k}x^k$$

where $\binom{z}{k}$ is defined as $\dfrac{z\cdot(z-1)\cdot\ldots\cdot(z-k+1)}{k!}$ and z can be any complex value. Now here's something crucially important; with this definition, we can take a derivative of x^z with respect to z.

It turns out we can define an analogous identity for any of our variations of $\boldsymbol{x}^k$, like so.

$$(1+\boldsymbol{x})^z=\sum_{k=0}^{\infty}\binom{z}{k}\boldsymbol{x}^k$$

This will converge for any x that is a positive whole number.

Explicit Definition of Generalized Binomial

Let me write this out more explicitly. Newton's Generalized Binomial expression, and then the four convolution equivalents, is

$$\begin{aligned}
(1+x)^z &= 1 + \binom{z}{1}\int_0^x dt + \binom{z}{2}\int_0^x\int_0^x du\,dt + \binom{z}{3}\int_0^x\int_0^x\int_0^x dv\,du\,dt + \ldots \\
(1+\boldsymbol{x})^{z+\int} &= 1 + \binom{z}{1}\int_0^x dt + \binom{z}{2}\int_0^x\int_0^{x-t} du\,dt + \binom{z}{3}\int_0^x\int_0^{x-t}\int_0^{x-t-u} dv\,du\,dt + \ldots \\
(1+\boldsymbol{x})^{z+\Sigma} &= 1 + \binom{z}{1}\sum_{t=1}^{x}1 + \binom{z}{2}\sum_{t=1}^{x}\sum_{u=1}^{x-t}1 + \binom{z}{3}\sum_{t=1}^{x}\sum_{u=1}^{x-t}\sum_{v=1}^{x-t-u}1 + \ldots \\
(1+\boldsymbol{x})^{z*\int} &= 1 + \binom{z}{1}\int_1^x dt + \binom{z}{2}\int_1^x\int_1^{\frac{x}{t}} du\,dt + \binom{z}{3}\int_1^x\int_1^{\frac{x}{t}}\int_1^{\frac{x}{tu}} dv\,du\,dt + \ldots \\
(1+\boldsymbol{x})^{z*\Sigma} &= 1 + \binom{z}{1}\sum_{t=2}^{x}1 + \binom{z}{2}\sum_{t=2}^{x}\sum_{u=2}^{\lfloor\frac{x}{t}\rfloor}1 + \binom{z}{3}\sum_{t=2}^{x}\sum_{u=2}^{\lfloor\frac{x}{t}\rfloor}\sum_{v=2}^{\lfloor\frac{x}{t\cdot u}\rfloor}1 + \ldots
\end{aligned}$$

Recursive Definition for $(1+\boldsymbol{x})^z$

To highlight the symmetry between these expressions, I note they all can be written as differently parameterized versions of the following recursive expression:

$$f_k(x)=1+d\cdot\left(\frac{z+1}{k}-1\right)\sum_{t=1}^{\lfloor\frac{x-I}{d}\rfloor} f_{k+1}(\theta(x,d\cdot t+I))$$
$$f(x,d,\theta,I)=f_1(x)$$

Given this, Newton's generalized binomial theorem (for 1 < x < 2) can be written as

$$(1+x)^z=\lim_{d\to 0} f(x,d,\theta_=,1) \quad \text{with} \quad \theta_=(x,t)=x$$

If two helper functions, $\theta_+(x,t)=x-t$ and $\theta_*(x,t)=\frac{x}{t}$, are defined, we can also express our four variants of $(1+\boldsymbol{x})^z$ as

$$(1+\boldsymbol{x})^z=$$

	$\int$	Σ
+	$\lim_{d\to 0} f(x,d,\theta_+)$	$f(x,1,\theta_+)$
*	$\lim_{d\to 0} f(x,d,\theta_*)$	$f(x,1,\theta_*)$

Mathematica Implementation of $(1+\boldsymbol{x})^z$

This recursive definition, for the 4 variants of $(1+\boldsymbol{x})^z$, can be implemented in Mathematica as

```
delta=.01;
thetaAdd[x_,t_]:=x-t
thetaMul[x_,t_]:=x/t
thetaEq[x_,t_]:=x
f[x_,z_,k_,d_,fn_,I_]:=1+d ((z+1)/k-1) Sum[ f[fn[x,d t+I],z,k+1,d,fn,I],{t,1,(x-I)/d}]
```

$(1+\boldsymbol{x})^z$ can then be computed as

	$\int$	Σ
+	`f[x,z,1,delta,thetaAdd,0]`	`f[x,z,1,1,thetaAdd,0]`
*	`f[x,z,1,delta,thetaMul,1]`	`f[x,z,1,1,thetaMul,1]`

with the values on the left side of the table getting closer to their actual values as delta gets closer to 0.

Note that this is a reasonable way to compute small values of $(1+\boldsymbol{x})^{z\ +\Sigma}$ and $(1+\boldsymbol{x})^{z\ *\Sigma}$, but not at all reasonable to compute $(1+\boldsymbol{x})^{z\ +\int}$ or $(1+\boldsymbol{x})^{z\ +\int}$. Fortunately, straightforward closed form values of those exist.

Closed Form Values for $(1+\boldsymbol{x})^{z}$

In fact, the closed form values of $(1+\boldsymbol{x})^{z}$ are as follows:

$$(1+\boldsymbol{x})^{z}=$$

	$\int$	$\sum$
+	$L_z(-x)$	$\left(\!\binom{z+1}{x}\!\right)$
*	$x\cdot L_{z-1}(-\log x)$	$D_z(x)$

Here, $L_z(x)$ is the Laguerre polynomials and $D_z(x)$ is the seldom seen generalized divisor summatory function, also defined as $D_z(x)=\sum_{n=1}^{x}\prod_{p^k\mid n}\frac{z^{(k)}}{k!}$, where the product is over the prime factorization of n and $x^{(z)}$ is the rising factorial defined as $z^{(k)}=z(z+1)(z+2)...(z+k-1)$.

Closed Form Values for Derivatives / Finite Differences of $(1+\boldsymbol{x})^{z}$

Note

$$\frac{\partial}{\partial x}(1+x)^{z}=z\cdot(1+x)^{z-1}$$

Note

$$\frac{\partial}{\partial x}(1+\boldsymbol{x})^{z}=$$

OR

$$\nabla_x(1+\boldsymbol{x})^{z}=$$

	$\int$	$\sum$
+	$L_{z-1}^{1}(-x)$	$\left(\!\binom{z}{x}\!\right)$
*	$L_{z-1}^{1}(-\log x)$	$\prod_{p^k\mid x}\left(\!\binom{z}{k}\!\right)$

Here, $L_z^{(k)}(x)$ is the generalized Laguerre polynomials, $x^{(z)}$ is the rising factorial, defined as $z^{(k)}=z(z+1)(z+2)...(z+k-1)$, and $d_z(x)$ is the seldom seen generalized divisor function, also defined as $d_z(x)=\prod_{p^k\mid x}\left(\!\binom{z}{k}\!\right)$, where the product is over the prime factorization of x.

Note the visually appealing fact that, in the notation of this paper, this last identity can be rewritten as $\nabla_x(1+\boldsymbol{x})^{z\ *\Sigma}=\prod_{p^k\mid x}\nabla_k(1+\boldsymbol{k})^{z\ +\Sigma}$.

$$(1+\boldsymbol{x})^{z\ +\Sigma}=\sum_{a=1}^{x}\nabla_a(1+\boldsymbol{a})^{z\ +\Sigma}$$

$$(1+\boldsymbol{x})^{z\,*\Sigma}=\sum_{a=1}^{x}\prod_{p^{\alpha}|a}\nabla_{\alpha}(1+\boldsymbol{\alpha})^{z\,+\Sigma}$$

$$\frac{(x+z)!}{x!\,z!}=\sum_{a=1}^{x}\frac{z^{(a)}}{a!}$$

$$D_z(x)=\sum_{a=1}^{x}\prod_{p^{\alpha}|a}\frac{z^{(\alpha)}}{\alpha!}$$

$$(1+\boldsymbol{x})^{z\,+\Sigma}=\sum_{a=1}^{x}(1+\boldsymbol{a})^{z-1\,+\Sigma}$$

$$(1+\boldsymbol{x})^{z\,*\Sigma}=\sum_{a=1}^{x}\prod_{p^{\alpha}|a}(1+\boldsymbol{\alpha})^{z-1\,+\Sigma}$$

$$\frac{(x+z)!}{x!\,z!}=\sum_{a=1}^{x}\frac{(a+z-1)!}{a!(z-1)!}$$

$$D_z(x)=\sum_{a=1}^{x}\prod_{p^{\alpha}|a}\frac{(\alpha+z-1)!}{\alpha!(z-1)!}$$

Closed Forms of $(1+\boldsymbol{x})^z$ In Terms of $\boldsymbol{x}^k$

Now that I have listed closed form expressions for the various forms of $(1+\boldsymbol{x})^z$, I can take the closed form expressions for $\boldsymbol{x}^k$ from the previous chapter to write rewrite the binomial theorem in more standard notation. Specifically,

$$(1+\boldsymbol{x})^z=\sum_{k=0}^{\infty}\binom{z}{k}\boldsymbol{x}^k$$

	$\int$	$\sum$
+	$L_z(-x)=\sum_{k=0}^{\infty}\binom{z}{k}\frac{x^k}{k!}$	$\left(\!\binom{z+1}{x}\!\right)=\sum_{k=0}^{\infty}\binom{z}{k}\binom{x}{k}$
*	$x\cdot L_{z-1}(-\log x)=\sum_{k=0}^{\infty}\binom{z}{k}(-1)^{-k}\cdot P(k,-\log x)$	$D_z(x)=\sum_{k=0}^{\infty}\binom{z}{k}D_k{}'(x)$

Closed Forms of $\boldsymbol{x}^k$ in Terms of $(1+\boldsymbol{x})^z$

The binomial expression can also be inverted, like so:

$$x^k=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(1+x)^j$$

This idea also holds for the variants of $(1+\boldsymbol{x})^z$, giving

$$\mathbf{x}^k = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (1+\mathbf{x})^j$$

	$\int$	$\sum$
$+$	$\frac{x^k}{k!} = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} L_j(-x)$	$\binom{x}{k} = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \cdot \left(\binom{x+1}{j}\right)$
$*$	$(-1)^{-k} \cdot P(k, -\log x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} x \cdot L_{j-1}(-\log x)$	$D_k{}'(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} D_j(x)$

Multiplication: $(1+x)^{a+b} = (1+x)^a \cdot (1+x)^b$

The usual rule of multiplication of exponents says that

$$(1+x)^{a+b} = (1+x)^a \cdot (1+x)^b$$

$$(1+\mathbf{x})^{a+b} = (1+\mathbf{x})^a \cdot (1+\mathbf{x})^b$$

Another, more elaborate, way to say this, is

$$(1+x)^{a+b} = 1 + \int_0^x \frac{\partial}{\partial t}(1+t)^a \, dt + \int_0^x \frac{\partial}{\partial u}(1+u)^b \, du + \int_0^x \int_0^x \frac{\partial}{\partial t}(1+t)^a \cdot \frac{\partial}{\partial u}(1+u)^b \, du \, dt$$

	$\int$	$\sum$
$+$	$(1+\mathbf{x})^{a+b} = 1 + \int_0^x \frac{\partial}{\partial t}(1+\mathbf{t})^a \, dt + \int_0^x \frac{\partial}{\partial u}(1+\mathbf{u})^b \, du$ $+ \int_0^x \int_0^{x-t} \frac{\partial}{\partial t}(1+\mathbf{t})^a \cdot \frac{\partial}{\partial u}(1+\mathbf{u})^b \, du \, dt$	$(1+\mathbf{x})^{a+b} = 1 + \sum_{t=1}^{x} \nabla_t (1+\mathbf{t})^a + \sum_{u=1}^{x} \nabla_u (1+\mathbf{u})^b$ $+ \sum_{t=1}^{x} \sum_{u=1}^{x-t} \nabla_t (I+\mathbf{t})^a \cdot \nabla_u (1+\mathbf{u})^b$
$*$	$(1+\mathbf{x})^{a+b} = 1 + \int_1^x \frac{\partial}{\partial t}(1+\mathbf{t})^a \, dt + \int_1^x \frac{\partial}{\partial u}(1+\mathbf{u})^b \, du$ $+ \int_1^x \int_1^{\frac{x}{t}} \frac{\partial}{\partial t}(1+\mathbf{t})^a \cdot \frac{\partial}{\partial u}(1+\mathbf{u})^b \, du \, dt$	$(1+\mathbf{x})^{a+b} = 1 + \sum_{t=2}^{x} \nabla_t (1+\mathbf{t})^a + \sum_{u=2}^{x} \nabla_u (1+\mathbf{u})^b$ $+ \sum_{t=2}^{x} \sum_{u=1}^{\frac{x}{t}} \nabla_t (1+\mathbf{t})^a \cdot \nabla_u (1+\mathbf{u})^b$

which, with the derivatives applied, is

$$(1+x)^{a+b} = 1 + \int_0^x a(1+t)^{a-1} \, dt + \int_0^x b(1+u)^{b-1} \, du + \int_0^x \int_0^x a(1+t)^{a-1} \cdot b(1+u)^{b-1} \, du \, dt$$

An analogous approach can be applied to the four variants of $(1+\mathbf{x})^z$, giving

	$\int$	$\sum$

$+$	$L_{a+b}(-x)=1+\int_0^x L^{(1)}_{a-1}(-t)\,dt+\int_0^x L^{(1)}_{b-1}(-t)\,dt$ $+\int_0^x\int_0^{x-t} L^{(1)}_{a-1}(-t)\cdot L^{(1)}_{b-1}(-u)\,du\,dt$	$\left(\!\binom{a+b+1}{x}\!\right)=\sum_{t=0}^{x}\sum_{u=0}^{x-t}\left(\!\binom{a}{t}\!\right)\cdot\left(\!\binom{b}{u}\!\right)$
$*$	$x\cdot L_{a+b-1}(-\log x)=1+\int_1^x L^{(1)}_{a-1}(-\log t)\,dt+\int_1^x L^{(1)}_{b-1}(-\log t)\,dt$ $+\int_1^x\int_1^{\frac{x}{t}} L^{(1)}_{a-1}(-\log t)\cdot L^{(1)}_{b-1}(-\log u)\,du\,dt$	$D_{a+b}(x)=\sum_{t=1}^{x}\sum_{u=1}^{\lfloor\frac{x}{t}\rfloor} d_a(t)\cdot d_b(u)$

and

$$\frac{\partial}{\partial x}(1+x)^{a+b}=\frac{\partial}{\partial x}\left(1+\int_0^x\frac{\partial}{\partial t}(1+t)^a\,dt+\int_0^x\frac{\partial}{\partial u}(1+u)^b\,du+\int_0^x\int_0^x\frac{\partial}{\partial t}(1+t)^a\cdot\frac{\partial}{\partial u}(1+u)^b\,du\,dt\right)$$

$$\frac{\partial}{\partial x}(1+x)^{a+b}=\frac{\partial}{\partial x}(1+x)^a+\frac{\partial}{\partial x}(1+x)^b+\int_0^x\frac{\partial}{\partial t}(1+t)^a\cdot\frac{\partial}{\partial x}(1+x)^b+\frac{\partial}{\partial x}(1+x)^a\cdot\frac{\partial}{\partial t}(1+t)^b\,dt$$

$$(a+b)(1+x)^{a+b-1}=a(1+x)^{a-1}+b(1+x)^{b-1}+\int_0^x a(1+t)^{a-1}\cdot b(1+x)^{b-1}+a(1+x)^{a-1}\cdot b(1+t)^{b-1}\,dt$$

	$\int$	$\sum$
$+$	$L^{(1)}_{a+b-1}(-x)=L^{(1)}_{a-1}(-x)+L^{(1)}_{b-1}(-x)$ $+\int_0^x L^{(1)}_{a-1}(-t)\cdot L^{(1)}_{b-1}(-(x-t))\,dt$	$\left(\!\binom{a+b}{x}\!\right)=\sum_{t=0}^{x}\left(\!\binom{a}{t}\!\right)\cdot\left(\!\binom{b}{x-t}\!\right)$
$*$	$L^{(1)}_{-(a+b)-1}(\log x)=\frac{1}{x}\cdot L^{(1)}_{-a-1}(\log x)+\frac{1}{x}\cdot L^{(1)}_{-b-1}(\log x)$ $+\int_1^x dt$???	$d_{a+b}(x)=\sum_{t\mid x} d_a(t)\cdot d_b(\frac{x}{t})$

General Transformations Using $(1+x)^z$

It

$$\hat{f}(x)=f(x)+\int_0^x\left(\frac{\partial}{\partial t}(1+t)^z\right)\cdot f(x)\,dt$$

$$f(x)=\hat{f}(x)+\int_0^x\left(\frac{\partial}{\partial t}(1+t)^{-z}\right)\cdot\hat{f}(x)\,dt$$

	$\int$	$\sum$

+	$\hat{f}(x)=f(x)+\int_0^x \frac{\partial}{\partial t}(1+\boldsymbol{t})^z\cdot f(x-t)\,dt$ $f(x)=\hat{f}(x)+\int_0^x \frac{\partial}{\partial t}(1+\boldsymbol{t})^{-z}\cdot \hat{f}(x-t)\,dt$	$\hat{f}(x)=f(x)+\sum_{t=1}^{x} \nabla_t(1+\boldsymbol{t})^z\cdot f(x-t)$ $f(x)=\hat{f}(x)+\sum_{t=1}^{x} \nabla_t(1+\boldsymbol{t})^{-z}\cdot \hat{f}(x-t)$
*	$\hat{f}(x)=f(x)+\int_1^x \frac{\partial}{\partial t}(1+\boldsymbol{t})^z\cdot f(\frac{x}{t})\,dt$ $f(x)=\hat{f}(x)+\int_1^x \frac{\partial}{\partial t}(1+\boldsymbol{t})^{-z}\cdot \hat{f}(\frac{x}{t})\,dt$	$\hat{f}(x)=f(x)+\sum_{t=2}^{x} \nabla_t(1+\boldsymbol{t})^z\cdot f(\frac{x}{t})$ $f(x)=\hat{f}(x)+\sum_{t=2}^{x} \nabla_t(1+\boldsymbol{t})^{-z}\cdot \hat{f}(\frac{x}{t})$

It

$$\hat{f}(x)=f(x)+\int_0^x (z\cdot(1+t)^{z-1})\cdot f(x)\,dt$$

$$f(x)=\hat{f}(x)+\int_0^x (-z\cdot(1+t)^{-z-1})\cdot \hat{f}(x)\,dt$$

	$\int$	$\sum$
+	$\hat{f}(x)=f(x)+\int_0^x L_{z-1}^{(1)}(-t)\cdot f(x-t)\,dt$ $f(x)=\hat{f}(x)+\int_0^x L_{-z-1}^{(1)}(-t)\cdot \hat{f}(x-t)\,dt$	$\hat{f}(x)=f(x)+\sum_{t=1}^{x}\left(\binom{z}{t}\right)\cdot f(x-t)$ $f(x)=\hat{f}(x)+\sum_{t=1}^{x}\left(\binom{-z}{t}\right)\cdot \hat{f}(x-t)$
*	$\hat{f}(x)=f(x)+\int_1^x L_{z-1}^{(1)}(-\log t)\cdot f(\frac{x}{t})\,dt$ $f(x)=\hat{f}(x)+\int_1^x L_{-z-1}^{(1)}(-\log t)\cdot \hat{f}(\frac{x}{t})\,dt$	$\hat{f}(x)=f(x)+\sum_{t=2}^{x} d_z(t)\cdot f(\frac{x}{t})$ $f(x)=\hat{f}(x)+\sum_{t=2}^{x} d_{-z}(t)\cdot \hat{f}(\frac{x}{t})$

It

$$\hat{f}(x)=f(x)+\int_0^x f(x)\,dt$$

$$f(x)=\hat{f}(x)-\int_0^x \frac{1}{(1+t)^2}\cdot \hat{f}(x)\,dt$$

It

	$\int$	$\sum$
+	$\hat{f}(x)=f(x)+\int_0^x f(x-t)\,dt$ $f(x)=\hat{f}(x)-\int_0^x e^{-t}\cdot \hat{f}(x-t)\,dt$	$\hat{f}(x)=f(x)+\sum_{t=1}^{x} f(x-t)$ $f(x)=\hat{f}(x)-\hat{f}(x-1)$

*	$\hat{f}(x)=f(x)+\int_1^x f(\frac{x}{t})\,dt$ $f(x)=\hat{f}(x)-\int_1^x \frac{1}{t}\cdot\hat{f}(\frac{x}{t})\,dt$	$\hat{f}(x)=f(x)+\sum_{t=2}^{x} f(\frac{x}{t})$ $f(x)=\hat{f}(x)+\sum_{t=2}^{x} \mu(t)\cdot\hat{f}(\frac{x}{t})$

Alright. I finally have enough defined for me to talk about $\log(1+\boldsymbol{x})$.

Section 3:
$\log(1+\boldsymbol{x})$

Overview

T

The Mercator Series Equivalent

The Mercator series for the logarithm is very well known to be, for 0 < x < 2,

$$\log(1+x)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}x^k$$

It turns out that a very useful analogous identity holds for the four variants of $\boldsymbol{x}^{\boldsymbol{k}}$ as

$$\log(1+\boldsymbol{x})=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\boldsymbol{x}^{\boldsymbol{k}}$$

This Series for $\log(1+\boldsymbol{x})$ Written Out Explicitly

Written out more explicitly, what I'm saying here is

$$\begin{array}{lclclclc}
\log(1+x) & = & \int_0^x dt & - & \frac{1}{2}\int_0^x\int_0^x du\,dt & + & \frac{1}{3}\int_0^x\int_0^x\int_0^x dv\,du\,dt & -\ldots \\
\log(1+\boldsymbol{x})^{+\int} & = & \int_0^x dt & - & \frac{1}{2}\int_0^x\int_0^{x-t} du\,dt & + & \frac{1}{3}\int_0^x\int_0^{x-t}\int_0^{x-t-u} dv\,du\,dt & -\ldots \\
\log(1+\boldsymbol{x})^{+\Sigma} & = & \sum_{t=1}^{x}1 & - & \frac{1}{2}\sum_{t=1}^{x}\sum_{u=1}^{x-t}1 & + & \frac{1}{3}\sum_{t=1}^{x}\sum_{u=1}^{x-t}\sum_{v=1}^{x-t-u}1 & -\ldots \\
\log(1+\boldsymbol{x})^{*\int} & = & \int_1^x dt & - & \frac{1}{2}\int_1^x\int_1^{\frac{x}{t}} du\,dt & + & \frac{1}{3}\int_1^x\int_1^{\frac{x}{t}}\int_1^{\frac{x}{tu}} dv\,du\,dt & -\ldots \\
\log(1+\boldsymbol{x})^{*\Sigma} & = & \sum_{t=2}^{x}1 & - & \frac{1}{2}\sum_{t=2}^{x}\sum_{u=2}^{\lfloor\frac{x}{t}\rfloor}1 & + & \frac{1}{3}\sum_{t=2}^{x}\sum_{u=2}^{\lfloor\frac{x}{t}\rfloor}\sum_{v=2}^{\lfloor\frac{x}{t\cdot u}\rfloor}1 & -\ldots
\end{array}$$

It's hard not to notice the general symmetry between these different statements, I think.

Recursive Definition for $\log(1+\boldsymbol{x})$

All five of these can be expressed with the same recursive form. If we define the following recursive function,

$$f_k(x) = d \cdot \sum_{j=1}^{\lfloor \frac{x-I}{d} \rfloor} \frac{1}{k} - f_{k+1}(\theta(x, j \cdot d + I))$$
$$f(x, d, \theta, I) = f_1(x)$$

then we can compute $\log(1+x)$, for 1 < x < 2, as

$$\log(1+x) = \lim_{d \to 0} f(x, d, \theta_=, 1) \quad \text{with} \quad \theta_=(x, t) = x$$

And if we define two helper functions, $\theta_+(x, t) = x - t$ and $\theta_*(x, t) = \frac{x}{t}$, we can also express our four variants of $\log(1+\boldsymbol{x})$ as

$$\log(1+\boldsymbol{x}) =$$

	$\int$	$\sum$
+	$\lim_{d \to 0} f(x, d, \theta_+, 0)$	$f(x, 1, \theta_+, 0)$
*	$\lim_{d \to 0} f(x, d, \theta_*, 1)$	$f(x, 1, \theta_*, 1)$

Mathematica Implementation of $\log(1+\boldsymbol{x})$

Just to keep things clear, this means that in Mathematica, given the following code,

```
delta=.01;
thetaAdd[x_,t_]:=x-t
thetaMul[x_,t_]:=x/t
thetaEq[x_,t_]:=x
f[x_,k_,d_,fn_,I_]:=d Sum[1/k-f[fn[x,d t+I],k+1,d,fn,I],{t,1,(x-I)/d}]
```

we can then compute $\log(1+\boldsymbol{x})$ as

	$\int$	$\sum$
+	`f[x,1,delta,thetaAdd, 0]`	`f[x,1,1,thetaAdd, 0]`
*	`f[x,1,delta,thetaMul, 1]`	`f[x,1,1,thetaMul, 1]`

again with the values on the left side of the table getting closer as delta gets closer to 0.

Note that this is a reasonable way to compute small values of $\log(1+\boldsymbol{x})^{+\Sigma}$ and $\log(1+\boldsymbol{x})^{*\Sigma}$, but not at all reasonable to compute $\log(1+\boldsymbol{x})^{+\int}$ or $\log(1+\boldsymbol{x})^{+\int}$.

Closed form for $\log(1+\boldsymbol{x})$

$\log(1+\boldsymbol{x})$ has some *really* interesting closed forms. As a matter of fact, the closed forms of $\log(1+\boldsymbol{x})$ are the central point of this entire paper. They are, in fact,

$$\log(1+\boldsymbol{x}) =$$

	$\int$	$\sum$
+	$-(Ei(-x)-\log(x)-\gamma)$	H_x
*	$li(x)-\log\log x-\gamma$	$\Pi(x)$

Here, $\Gamma(k,x)$ is the incomplete gamma function, $li(x)$ is the logarithmic integral, H_x is the Harmonic Number, $\Pi(x)$ is the Riemann Prime Counting function, and γ is Euler's constant gamma.

What These Closed Forms Mean Conceptually

asdf

Closed Form Values for Derivatives / Finite Differences

The derivatives and finite differences of $\log(1+\boldsymbol{x})$ are important to note, and well as perhaps a bit illuminating. So let's list those here.

Just as we can easily see that

$$\frac{\partial}{\partial x}\log(1+x)=\frac{1}{1+x}$$

we have have the following four closely related terms for our four variants.

$$\frac{\partial}{\partial x}\log(1+\boldsymbol{x})=$$
OR
$$\nabla_x\log(1+\boldsymbol{x})=$$

	$\int$	$\sum$
+	$\frac{1}{x}-\frac{e^{-x}}{x}$	$\frac{1}{x}$
*	$\frac{1}{\log x}-\frac{1}{x\log x}$	$\kappa(x)$

Here, once again, $\kappa(x)=\frac{1}{a}$ if x is a prime to the whole number power a, and $\kappa(x)=0$ if x has two or more prime factors.

$\log(1+\boldsymbol{x})$ in Terms of $\boldsymbol{x}^k$

With these various closed forms, this Mercator series equivalent can be rewritten as

	$\int$	$\sum$
+	$-(Ei(-x)-\log(x)-\gamma)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\cdot\frac{x^k}{k!}$	$H_x=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\binom{x}{k}$
*	$li(x)-\log\log x-\gamma=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}(-1)^{-k}\cdot P(k,-\log x)$	$\Pi(x)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}D_k'(x)$

$\log(1+\boldsymbol{x})$ in Terms of $(1+\boldsymbol{x})^z$

N

$$\log(1+x)=\lim_{z\to 0}\frac{\partial}{\partial z}(1+x)^z$$

N

$$\log(1+\boldsymbol{x})=\lim_{z\to 0}\frac{\partial}{\partial z}(1+\boldsymbol{x})^z$$

	$\int$	$\sum$
+	$-(Ei(-x)-\log(x)-\gamma)=\lim_{z\to 0}\frac{\partial}{\partial z}L_z(-x)$	$H_x=\lim_{z\to 0}\frac{\partial}{\partial z}\cdot\left(\left(\binom{x+1}{z}\right)\right)$
*	$li(x)-\log\log x-\gamma=\lim_{z\to 0}\frac{\partial}{\partial z}x\cdot L_{z-1}(-\log x)$	$\Pi(x)=\lim_{z\to 0}\frac{\partial}{\partial z}D_z(x)$

$\log(1+\boldsymbol{x})$ in Terms of $(1+\boldsymbol{x})^z$, Version 2

N

$$\log(1+x)=\lim_{z\to 0}\frac{(1+x)^z-1}{z}$$

N

$$\log(1+\boldsymbol{x})=\lim_{z\to 0}\frac{(1+\boldsymbol{x})^z-1}{z}$$

	$\int$	$\sum$
+	$-(Ei(-x)-\log(x)-\gamma)=\lim_{z\to 0}\frac{L_z(-x)-1}{z}$	$H_x=\lim_{z\to 0}\frac{\left(\left(\binom{x+1}{z}\right)\right)-1}{z}$
*	$li(x)-\log\log x-\gamma=\lim_{z\to 0}\frac{x\cdot L_{z-1}(-\log x)-1}{z}$	$\Pi(x)=\lim_{z\to 0}\frac{D_z(x)-1}{z}$

Section 4:
$\log^k(1+\boldsymbol{x})$

Overview

This section details expressions of the form $\log^k(1+\boldsymbol{x})$, which will function rather similarly to $\log^k(1+x)$.

$\log^k(1+\boldsymbol{x})$ Written Out Explicitly

N

$$\log(1+x)=\int_0^x \frac{1}{1+t}\,dt$$

N

$$\log(1+\boldsymbol{x})=$$

	$\int$	$\sum$
+	$\int_0^x \frac{1}{t}-\frac{e^{-t}}{t}\,dt$	$\sum_{t=1}^{x}\frac{1}{t}$
*	$\int_1^x \frac{1}{\log t}-\frac{1}{t\log t}\,dt$	$\sum_{t=2}^{x}\kappa(t)$

N

$$\log^2(1+x)=\int_0^x\int_0^x \frac{1}{1+t}\cdot\frac{1}{1+u}\,du\,dt$$

N

$$\log^2(1+\boldsymbol{x})=$$

	$\int$	$\sum$
+	$\int_0^x\int_0^{x-t}(\frac{1}{t}-\frac{e^{-t}}{t})\cdot(\frac{1}{u}-\frac{e^{-u}}{u})\,du\,dt$	$\sum_{t=1}^{x}\sum_{u=1}^{x-t}\frac{1}{t}\cdot\frac{1}{u}$
*	$\int_1^x\int_1^{\frac{x}{t}}(\frac{1}{\log t}-\frac{1}{t\log t})\cdot(\frac{1}{\log u}-\frac{1}{u\log u})\,du\,dt$	$\sum_{t=2}^{x}\sum_{u=2}^{\lfloor\frac{x}{t}\rfloor}\kappa(t)\cdot\kappa(u)$

N

$$\log^3(1+x)=\int_0^x\int_0^x\int_0^x \frac{1}{1+t}\cdot\frac{1}{1+u}\cdot\frac{1}{1+v}\,dv\,du\,dt$$

N

$$\log^3(1+\boldsymbol{x})=$$

	$\int$	$\sum$
+	$\int_0^x \int_0^{x-t} \int_0^{x-t-u} (\frac{1}{t}-\frac{e^{-t}}{t})\cdot(\frac{1}{u}-\frac{e^{-u}}{u})\cdot(\frac{1}{v}-\frac{e^{-v}}{v})\,dv\,du\,dt$	$\sum_{t=1}^{x}\sum_{u=1}^{x-t}\sum_{v=1}^{x-t-u}\frac{1}{t}\cdot\frac{1}{u}\cdot\frac{1}{v}$
*	$\int_1^x \int_1^{\frac{x}{t}} \int_1^{\frac{x}{t\,u}} (\frac{1}{\log t}-\frac{1}{t\log t})\cdot(\frac{1}{\log u}-\frac{1}{u\log u})\cdot(\frac{1}{\log v}-\frac{1}{v\log v})\,dv\,du\,dt$	$\sum_{t=2}^{x}\sum_{u=2}^{\lfloor\frac{x}{t}\rfloor}\sum_{v=2}^{\lfloor\frac{x}{t\cdot u}\rfloor}\kappa(t)\cdot\kappa(u)\cdot\kappa(v)$

N

Recursive Definition for $\log^k(1+\boldsymbol{x})$

N

$$\log^k(1+x)=\int_0^x \frac{1}{1+t}\cdot\log^{k-1}(1+x)\,dt$$

N

$$\log^k(1+\boldsymbol{x})=$$

	$\int$	$\sum$
+	$\int_0^x (\frac{1}{t}-\frac{e^{-t}}{t})\cdot\log^{k-1}(1+(\boldsymbol{x}-\boldsymbol{t}))\,dt$	$\sum_{t=1}^{x}\frac{1}{t}\cdot\log^{k-1}(1+(\boldsymbol{x}-\boldsymbol{t}))$
*	$\int_1^x (\frac{1}{\log t}-\frac{1}{t\log t})\cdot\log^{k-1}(1+\frac{\boldsymbol{x}}{\boldsymbol{t}})\,dt$	$\sum_{t=2}^{x}\kappa(t)\cdot\log^{k-1}(1+\frac{\boldsymbol{x}}{\boldsymbol{t}})$

Multiplication: $\log^{a+b}(1+x)=\log^a(1+x)\cdot\log^b(1+x)$

N

$$\log^{a+b}(1+x)=\log^{a}(1+x)\cdot\log^{b}(1+x)$$

N

$$\log^{a+b}(1+\boldsymbol{x})=\log^{a}(1+\boldsymbol{x})\cdot\log^{b}(1+\boldsymbol{x})$$

N

$$\log^{a+b}(1+\boldsymbol{x})=$$

	$\int$	$\sum$
+	$\int_0^x\int_0^{x-t}\frac{\partial}{\partial t}\log^a(1+\boldsymbol{t})\cdot\frac{\partial}{\partial u}\log^b(1+\boldsymbol{u})\,du\,dt$	$\sum_{t=1}^{x}\sum_{u=1}^{x-t}\nabla_t\log^a(1+\boldsymbol{t})\cdot\nabla_u\log^b(1+\boldsymbol{u})$
*	$\int_1^x\int_1^{\frac{x}{t}}\frac{\partial}{\partial t}\log^a(1+\boldsymbol{t})\cdot\frac{\partial}{\partial u}\log^b(1+\boldsymbol{u})\,du\,dt$	$\sum_{t=2}^{x}\sum_{u=2}^{\lfloor\frac{x}{t}\rfloor}\nabla_t\log^a(1+\boldsymbol{t})\cdot\nabla_u\log^b(1+\boldsymbol{u})$

Closed Form Values for $\log^k(1+\boldsymbol{x})$

Unfortunately, to the best of my knowledge, there aren't any. Although there are closed forms for all four variants of $\log(1+\boldsymbol{x})$ **,** there are no closed form expressions for any of the more general variants of $\log^k(1+\boldsymbol{x})$.

Convergence of Sums of $\log^k(1+\boldsymbol{x})$

The recursive definitions of $\log^k(1+\boldsymbol{x})$ and $\log^k(1+\boldsymbol{x})$ should be enough to show that $\log^k(1+\boldsymbol{x})^{+\int}=0$ when k>x, and $\log^k(1+\boldsymbol{x})^{*\Sigma}=0$ when $k>\lfloor\frac{\log x}{\log 2}\rfloor$, just as was the case with both $\boldsymbol{x}^{k+\int}$ and $\boldsymbol{x}^{k*\Sigma}$.

$\log^k\boldsymbol{x}$ in terms of $(1+\boldsymbol{x})^z$

N

$$\log^k(1+x)=\lim_{z\to 0}\frac{\partial^k}{\partial z^k}(1+x)^z$$

N

$$\log^k(1+\boldsymbol{x})=\lim_{z\to 0}\frac{\partial^k}{\partial z^k}(1+\boldsymbol{x})^z$$

$\log^k(1+\boldsymbol{x})$ in terms of $\boldsymbol{x}^k$

N

$$\log^k(1+x)=\sum_{j=0}^{\infty}\frac{1}{j!}\cdot(\lim_{t\to 1}\frac{\partial^j}{\partial t^j}(\log(t))^k)\,x^j$$

N

$$\log^k(1+\boldsymbol{x})=\sum_{j=0}^{\infty}\frac{1}{j!}\cdot\left(\lim_{t\to 1}\frac{\partial^j}{\partial t^j}(\log(t))^k\right)\boldsymbol{x}^{\boldsymbol{j}}$$

$(1+\boldsymbol{x})^z$ in terms of $\log^k(1+\boldsymbol{x})$

N

$$(1+x)^z=\sum_{k=0}^{\infty}\frac{z^k}{k!}\log^k(1+x)$$

N

$$(1+\boldsymbol{x})^z=\sum_{k=0}^{\infty}\frac{z^k}{k!}\log^k(1+\boldsymbol{x})$$

$$\begin{array}{llllll}
(1+x)^z & =1+ & z\cdot\int_0^x \frac{1}{1+t}\,dt & + & \frac{z^2}{2}\cdot\int_0^x\int_0^x \frac{1}{1+t}\cdot\frac{1}{1+u}\,du\,dt & +\frac{z^3}{6}\cdot\ldots \\
(1+\boldsymbol{x})^{z+\int} & =1+ & z\cdot\int_0^x \frac{1}{t}-\frac{e^{-t}}{t}\,dt & + & \frac{z^2}{2}\cdot\int_0^x\int_0^{x-t} \left(\frac{1}{t}-\frac{e^{-t}}{t}\right)\left(\frac{1}{u}-\frac{e^{-u}}{u}\right)du\,dt & +\frac{z^3}{6}\cdot\ldots \\
(1+\boldsymbol{x})^{z+\Sigma} & =1+ & z\cdot\sum_{t=1}^{x}\frac{1}{t} & + & \frac{z^2}{2}\cdot\sum_{t=1}^{x}\sum_{u=1}^{x-t}\frac{1}{t}\cdot\frac{1}{u} & +\frac{z^3}{6}\cdot\ldots \\
(1+\boldsymbol{x})^{z*\int} & =1+ & z\cdot\int_1^x \frac{1}{\log t}-\frac{1}{t\log t}\,dt & + & \frac{z^2}{2}\cdot\int_1^x\int_1^{\frac{x}{t}} \left(\frac{1}{\log t}-\frac{1}{t\log t}\right)\left(\frac{1}{\log u}-\frac{1}{u\log u}\right)du\,dt & +\frac{z^3}{6}\cdot\ldots \\
(1+\boldsymbol{x})^{z*\Sigma} & =1+ & z\cdot\sum_{t=2}^{x}\kappa(t) & + & \frac{z^2}{2}\cdot\sum_{t=2}^{x}\sum_{u=2}^{\lfloor\frac{x}{t}\rfloor}\kappa(t)\cdot\kappa(u) & +\frac{z^3}{6}\cdot\ldots
\end{array}$$

Here, once again, $\kappa(t)=\frac{1}{a}$ if t is a prime to the whole number power a, and $\kappa(t)=0$ if t has two or more prime factors.

$\boldsymbol{x}^{\boldsymbol{k}}$ in terms of $\log^k(1+\boldsymbol{x})$

N

$$x^k=\sum_{j=0}^{\infty}\frac{1}{j!}\cdot\left(\lim_{t\to 0}\frac{\partial^j}{\partial t^j}(e^t-1)^k\right)\log^j(1+x)$$

N

$$\boldsymbol{x}^{\boldsymbol{k}}=\sum_{j=0}^{\infty}\frac{1}{j!}\cdot\left(\lim_{t\to 0}\frac{\partial^j}{\partial t^j}(e^t-1)^k\right)\log^j(1+\boldsymbol{x})$$

N

$$x^k=\sum_{j=0}^{\infty}\frac{S(j,k)\cdot k!}{j!}\cdot\log^j(1+x)$$

where $S(j,k)$ are Stirling numbers of the second kind.

$$\boldsymbol{x}^{\boldsymbol{k}}=\sum_{j=0}^{\infty}\frac{S(j,k)\cdot k!}{j!}\cdot\log^j(1+\boldsymbol{x})$$

Section 5:
x^z

Overview

N

Closed Form Values for x^z

W

$$x^z =$$

	$\int$	$\sum$
+	$\frac{x^z}{z\,!}$	$\binom{x}{z}$
*	$(-1)^{-z} \cdot P(z, -\log x)$	$D_z{}'(x)$

H

W

$$x^z =$$

	$\int$	$\sum$
+		$\frac{\sin(\pi z)}{\pi} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{z-k} \cdot \binom{x}{k}$
*		$\frac{\sin(\pi z)}{\pi} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{z-k} \cdot D_k'(x)$

H

Closed Form Values for Derivatives / Finite Differences

W

$$\frac{\partial}{\partial x} x^z = z \cdot x^{z-1}$$

G

$$\frac{\partial}{\partial x}\boldsymbol{x}^z = \,/\, \nabla_x \boldsymbol{x}^z =$$

	$\int$	$\sum$
+	$\frac{x^{z-1}}{(z-1)!}$	$\binom{x-1}{z-1}$
*	$\frac{\log^{z-1} x}{(z-1)!}$	$d_z{}'(x)$

H

Closed Form Values for $\log \boldsymbol{x}$

W

$$\log \boldsymbol{x} =$$

	$\int$	$\sum$
+	$\log x + \gamma$	H_x
*	$li(x)$	$\Pi(x)$

H

Closed Form Values for Derivatives / Finite Differences

W

$$\frac{\partial}{\partial x}\log x = \frac{1}{x}$$

G

$$\frac{\partial}{\partial x}\log \boldsymbol{x} = \,/\, \nabla_x \log \boldsymbol{x} =$$

	$\int$	$\sum$
+	$\frac{1}{x}$	$\frac{1}{x}$
*	$\frac{1}{\log x}$	$\kappa(x)$

H

Section 6:
Various Binomial Forms

Overview

Recursive Definition for $(1+x)^z$

Section 7:
$\log(1+x)$
Turns Convolution into Addition

Overview

Recursive Definition for x^z

$$x \cdot y$$

$x \cdot y$: Rectangle

Real-valued area bounded by $t>0$, $u>0$, $\frac{t}{x} \leq 1$, and $\frac{u}{y} \leq 1$.

$x \cdot y^{+\int}$: Triangle

Real-valued area bounded by $t>0$, $u>0$, and $\frac{t}{x}+\frac{u}{y} \leq 1$.

$x \cdot y^{+\Sigma}$ Discrete Triangle

Count of whole number pairs (t,u) satisfying $t>0$, $u>0$, and $\frac{t}{x}+\frac{u}{y} \leq 1$.

$x \cdot y^{*\int}$: Hyperbola

Real-valued area bounded by $t>0$, $u>0$, and $\frac{\log(1+t)}{\log x}+\frac{\log(1+u)}{\log y} \leq 1$.

$x \cdot y^{*\Sigma}$: Discrete Hyperbola

Count of ordered whole number pairs (t,u) satisfying $t>0$, $u>0$, and $\frac{\log(1+t)}{\log x}+\frac{\log(1+u)}{\log y} \leq 1$.

$$x \cdot y \cdot z$$

$x \cdot y \cdot z$: Rectangular Prism

Real-valued volume bounded by $t>0$, $u>0$, $v>0$, $t \leq x$, $\frac{t}{x} \leq 1$, $\frac{u}{y} \leq 1$,and $\frac{v}{z} \leq 1$.

$x \cdot y \cdot z^{+\int}$: Triangular Pyramid

Real-valued volume bounded by $t>0$, $u>0$, $v>0$, and $\frac{t}{x}+\frac{u}{y}+\frac{v}{z} \leq 1$.

$x \cdot y \cdot z^{+\Sigma}$: Discrete Triangular Pyramid

Count of whole number pairs (t,u,v) satisfying $t>0$, $u>0$, $v>0$, and $\frac{t}{x}+\frac{u}{y}+\frac{v}{z} \leq 1$.

$x \cdot y \cdot z^{*\int}$: 3-dimensional Hyperbola

Real-valued volume bounded by $t>0$, $u>0$, $v>0$, and $\frac{\log(1+t)}{\log x}+\frac{\log(1+u)}{\log y}+\frac{\log(1+v)}{\log z} \leq 1$.

$x \cdot y \cdot z^{*\Sigma}$: Discrete 3-dimensional hyperbola.

Count of ordered whole number pairs (t,u,v) satisfying $t>0$, $u>0$, $v>0$, and $\frac{\log(1+t)}{\log x}+\frac{\log(1+u)}{\log y}+\frac{\log(1+v)}{\log z}\le 1$

.

Recursive Definition for x^z

$$\{x\cdot y\}^{+\Sigma}=\sum_{\frac{t_1}{x}+\frac{t_2}{y}\le 1} 1$$

$$\{(x\cdot y)^2\}^{+\Sigma}=\sum_{\frac{t_1}{x}+\frac{t_2}{x}+\frac{t_3}{y}+\frac{t_4}{y}\le 1} 1$$

$$\{(x\cdot y)^3\}^{+\Sigma}=\sum_{\frac{t_1}{x}+\frac{t_2}{x}+\frac{t_3}{x}+\frac{t_4}{y}+\frac{t_5}{y}+\frac{t_6}{y}\le 1} 1$$

$$\{(I+x)\cdot(I+y)\}^{+\Sigma}=\sum_{\frac{t_1}{x}+\frac{t_2}{y}\le 1} 1$$

$$\{((I+x)\cdot(I+y))^2\}^{+\Sigma}=\sum_{\frac{t_1}{x}+\frac{t_2}{x}+\frac{t_3}{y}+\frac{t_4}{y}\le 1} 1$$

$$\{((I+x)\cdot(I+y))^3\}^{+\Sigma}=\sum_{\frac{t_1}{x}+\frac{t_2}{x}+\frac{t_3}{x}+\frac{t_4}{y}+\frac{t_5}{y}+\frac{t_6}{y}\le 1} 1$$

$$\{x\cdot y\}^{*\Sigma}=\sum_{\frac{\log t_1}{\log x}+\frac{\log t_2}{\log y}\le 1} 1$$

$$\{(x\cdot y)^2\}^{*\Sigma}=\sum_{\frac{\log t_1}{\log x}+\frac{\log t_2}{\log x}+\frac{\log t_3}{\log y}+\frac{\log t_4}{\log y}\le 1} 1$$

$$\{(x\cdot y)^3\}^{*\Sigma}=\sum_{\frac{\log t_1}{\log x}+\frac{\log t_2}{\log x}+\frac{\log t_3}{\log x}+\frac{\log t_4}{\log y}+\frac{\log t_5}{\log y}+\frac{\log t_6}{\log y}\le 1} 1$$

$$\{(I+x)\cdot(I+y)\}^{*\Sigma}=\sum_{\frac{\log t_1}{\log x}+\frac{\log t_2}{\log y}\le 1} 1$$

$$\{((I+x)\cdot(I+y))^2\}^{*\Sigma}=\sum_{\frac{\log t_1}{\log x}+\frac{\log t_2}{\log x}+\frac{\log t_3}{\log y}+\frac{\log t_4}{\log y}\le 1} 1$$

$$\{((I+x)\cdot(I+y))^3\}^{*\Sigma}=\sum_{\frac{\log t_1}{\log x}+\frac{\log t_2}{\log x}+\frac{\log t_3}{\log x}+\frac{\log t_4}{\log y}+\frac{\log t_5}{\log y}+\frac{\log t_6}{\log y}\le 1} 1$$

Recursive Definition for x^z

$$((1+\boldsymbol{x})\cdot(1+\boldsymbol{y})-1)^k=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}((1+\boldsymbol{x})\cdot(1+\boldsymbol{y}))^j$$

$$((1+\boldsymbol{x})\cdot(1+\boldsymbol{y}))^z=\sum_{k=0}^{\infty}\binom{z}{k}((1+\boldsymbol{x})\cdot(1+\boldsymbol{y})-1)^k$$

$$\log((1+\boldsymbol{x})\cdot(1+\boldsymbol{y}))=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}((1+\boldsymbol{x})\cdot(1+\boldsymbol{y})-1)^{k}$$

$$\log((1+\boldsymbol{x})\cdot(1+\boldsymbol{y}))=\lim_{z\to 0}\frac{\partial}{\partial z}((1+\boldsymbol{x})\cdot(1+\boldsymbol{y}))^{z}$$

$$\log((1+\boldsymbol{x})\cdot(1+\boldsymbol{y}))=\log(1+\boldsymbol{x})+\log(1+\boldsymbol{y})$$

Section 8:
Other Convolutions

$$\left(\frac{1+\boldsymbol{x}}{1+\frac{\boldsymbol{x}}{\mathbf{2}}}\right)^{z}$$

W

$$\left(\frac{1+x}{1+\frac{x}{2}}\right)^{z}=1+\int_{0}^{x}\frac{\partial}{\partial t}(1+t)^{z}\,dt+\int_{0}^{\frac{x}{2}}\frac{\partial}{\partial u}(1+u)^{-z}\,du+\int_{0}^{x}\int_{0}^{\frac{x}{2}}\frac{\partial}{\partial t}(1+t)^{z}\cdot\frac{\partial}{\partial u}(1+u)^{-z}\,du\,dt$$

$$\left(\frac{1+\boldsymbol{x}}{1+\frac{\boldsymbol{x}}{\mathbf{2}}}\right)^{z}$$

	$\int$	$\sum$
+	$1+\int_{0}^{x}\frac{\partial}{\partial t}(1+\boldsymbol{t})^{z}\,dt+\int_{0}^{\frac{x}{2}}\frac{\partial}{\partial u}(1+\boldsymbol{u})^{-z}\,du+$ $\int_{0}^{x}\int_{0}^{\frac{x-t}{2}}\frac{\partial}{\partial t}(1+\boldsymbol{t})^{z}\cdot\frac{\partial}{\partial u}(1+\boldsymbol{u})^{-z}\,du\,dt$	$1+\sum_{t=1}^{x}\nabla_{t}(1+\boldsymbol{t})^{z}+\sum_{u=1}^{\lfloor\frac{x}{2}\rfloor}\nabla_{u}(1+\boldsymbol{u})^{-z}+$ $\sum_{t=1}^{x}\sum_{u=1}^{\lfloor\frac{x-t}{2}\rfloor}\nabla_{t}(1+\boldsymbol{t})^{z}\cdot\nabla_{u}(1+\boldsymbol{u})^{-z}$
*	$1+\int_{1}^{x}\frac{\partial}{\partial t}(1+\boldsymbol{t})^{z}\,dt+\int_{1}^{x^{\frac{1}{2}}}\frac{\partial}{\partial u}(1+\boldsymbol{u})^{-z}\,du+$ $\int_{1}^{x}\int_{1}^{(\frac{x}{t})^{\frac{1}{2}}}\frac{\partial}{\partial t}(1+\boldsymbol{t})^{z}\cdot\frac{\partial}{\partial u}(1+\boldsymbol{u})^{-z}\,du\,dt$	$1+\sum_{t=2}^{x}\nabla_{t}(1+\boldsymbol{t})^{z}+\sum_{u=2}^{\lfloor\frac{x}{2}\rfloor}\nabla_{u}(1+\boldsymbol{u})^{-z}+$ $\sum_{t=2}^{x}\sum_{u=2}^{\lfloor(\frac{x}{t})^{\frac{1}{2}}\rfloor}\nabla_{t}(1+\boldsymbol{t})^{z}\cdot\nabla_{u}(1+\boldsymbol{u})^{-z}$

H

$$\left(\frac{1+x}{1+\frac{x}{2}}\right)^{z}=1+\int_{0}^{x}z(1+t)^{z-1}\,dt+\int_{0}^{\frac{x}{2}-z}(1+u)^{-z-1}\,du+\int_{0}^{x}\int_{0}^{\frac{x}{2}}z(1+t)^{z-1}\cdot(-z(1+u)^{-z-1})\,du\,dt$$

	$\int$	$\sum$

+	$1+\int_0^x L_{z-1}^1(-t)\,dt+\int_0^{\frac{x}{2}} L_{-z-1}^1(-u)\,du+\int_0^x \int_0^{\frac{x-t}{2}} L_{z-1}^1(-t)\cdot L_{-z-1}^1(-u)\,du\,dt$	$\sum_{t=0}^{x} \sum_{u=0}^{\lfloor\frac{x-t}{2}\rfloor} \left(\binom{z}{t}\right)\cdot\left(\binom{-z}{u}\right)$
*	$1+\int_1^x L_{z-1}^1(-\log t)\,dt+\int_1^{x^{\frac{1}{2}}} L_{-z-1}^1(-\log u)\,du+\int_1^x \int_1^{(\frac{x}{t})^{\frac{1}{2}}} L_{z-1}^1(-\log t)\cdot L_{-z-1}^1(-\log u)\,du\,dt$	$\sum_{t=1}^{x} \sum_{u=1}^{\lfloor(\frac{x}{t})^{\frac{1}{2}}\rfloor} d_z(t)\cdot d_{-z}(u)$

$$\left(\frac{1+\boldsymbol{x}}{1+\frac{\boldsymbol{x}}{\boldsymbol{k}}}\right)^z$$

W

$$\prod_{k=1}\left(1+\frac{\boldsymbol{x}}{\boldsymbol{k}}\right)^z$$

W

$$\left(\prod_{k=1}\left(1+\frac{\boldsymbol{x}}{\boldsymbol{k}}\right)^{z\cdot\frac{\mu(k)}{k}}\right)$$

W