```
Clear[dz]
bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
FI[n_] := FactorInteger[n]; FI[1] := {}
dz[n_, z_] := dz[n, z] = Product[(-1)^p[[2]] bin[-z, p[[2]]], {p, FI[n]}]
Ds[n_, s_, z_] := Sum[ j^-s dz[j, z], {j, 1, n}]
pr[n_, s_] := Sum[ FullSimplify[MangoldtLambda[j] / Log[j]] j^-s, {j, 2, n}]
ch[n_] := -Sum[MangoldtLambda[j], {j, 2, n}]


Expand[D[Ds[97, -1, z] Ds[96, -1, -z], z]]
```

$$97 - \frac{1\,464\,943\,747\,z}{600} + \frac{15\,673\,357\,z^2}{24} + \frac{69\,580\,916\,693\,z^3}{6480} + \frac{9\,356\,135\,z^4}{72} + \frac{1\,893\,723\,061\,z^5}{10\,800} + \frac{5432\,z^6}{9} + \frac{270\,703\,999\,z^7}{1080} + \frac{154\,372\,z^9}{405} + \frac{256\,z^{11}}{27}$$

```
N[D[D[Ds[100, s, z] Ds[50, s, -z], z] /. z → 0, s] /. s → 0]
```

```
-44.5599
```

```
N[D[D[Ds[100, s, z] / Ds[50, s, z], z] /. z → 0, s] /. s → 0]
```

```
-44.5599
```

```
N[D[(pr[100, s] - pr[50, s]), s] /. s → 0]
```

```
-44.5599
```

```
D[Ds[100, 0, 2 z] Ds[50, 0, -z], z] /. z → 0
```

$$\frac{779}{20}$$

```
(2 pr[100, 0] - pr[50, 0])
```

$$\frac{779}{20}$$

```
D[Ds[100, 0, 2 z] / Ds[50, 0, z], z] /. z → 0
```

$$\frac{779}{20}$$

```
Limit[((Ds[100, 0, 2 z] - 1) / z) / ((Ds[50, 0, z] - 1) / z), z → 0]
```

$$\frac{3424}{1087}$$

```
FullSimplify[Expand[(Ds[100, 0, 2 z] / Ds[50, 0, z] - 1) / z]] /. z → 0
```

$$\frac{779}{20}$$

```
Limit[ (x^z - 1) / (y^z - 1), z → 0]
```

$$\frac{\text{Log}[x]}{\text{Log}[y]}$$

```
ll[n_, z_] := LaguerreL[-z, Log[n]]
l2[n_] := LogIntegral[n] - Log[Log[n]] - EulerGamma
```

```
N@Limit[(ll[100, z] / ll[50, z] - 1) / z, z → 0]
```

```
11.4943
```

```
N@l2[100] - l2[50]
```

```
11.4943
```

```
N[D[(ll[100, z] / ll[50, z]), z] /. z → 0]
```

```
11.4943
```

```
N@Limit[(ll[100, z] ll[50, z] - 1) / z, z → 0]
```

```
44.5492
```

```
N@l2[100] + l2[50]
```

```
44.5492
```

```
N[D[(ll[100, 2 z] / ll[50, z]), z] /. z → 0]
```

```
39.5161
```

```
N@2 l2[100] - l2[50]
```

```
39.5161
```

```
binomial[z_, k_] := binomial[z, k] = Product[z - j, {j, 0, k - 1}] / k!
Ds[n_, 0, s_, a_] := UnitStep[n - 1]
Ds[n_, 1, s_, a_] := Ds[n, 1, s, a] = HarmonicNumber[Floor[n], s] - HarmonicNumber[a, s]
Ds[n_, 2, s_, a_] := Ds[n, 2, s, a] =
  Sum[(m^(-2 s)) + 2 (m^-s) (Ds[Floor[n / m], 1, s, m]), {m, a + 1, Floor[n^(1 / 2)]}]
Ds[n_, k_, s_, a_] := Ds[n, k, s, a] =
  Sum[(m^(-s k)) + k (m^(-s (k - 1))) Ds[Floor[n / (m^(k - 1))], 1, s, m] +
    Sum[binomial[k, j] (m^-s)^j Ds[Floor[n / (m^j)], k - j, s, m], {j, 1, k - 2}],
   {m, a + 1, Floor[n^(1 / k)]}]
Dnsz[n_, s_, z_] := Expand@Sum[binomial[z, k] Ds[n, k, s, 1], {k, 0, Log2@n}]
Dns112z[n_, s_, z_] :=
 Expand@Sum[(-1)^j binomial[z, j] 2^(j (1 - s)) Dnsz[n / (2^j), s, z], {j, 0, Log2@n}]
```

```
D[Expand[Dns112z[150, -2, z] Dns112z[50, -1, -z]], z] /. z → 0
```

$$-\frac{48\,548\,033}{420}$$

```
(D[Dns112z[150, -2, z], z] /. z → 0) - (D[Dns112z[50, -1, z], z] /. z → 0)
```

$$-\frac{48\,548\,033}{420}$$

```
N[D[D[Dns112z[150, s, z] Dns112z[50, s, -z], z] /. z → 0, s] /. s → 0]
```

$$35.9046 - 5.4744 \times 10^{-14}\, i$$

```
N[D[(D[Dns112z[150, s, z], z] /. z → 0) - (D[Dns112z[50, s, z], z] /. z → 0), s] /. s → 0]
```

$$35.9046 - 5.4744 \times 10^{-14}\, i$$

```
Expand[Ds[100, 0, z] Ds[50, 0, -z]] /. z → 1
```

$$-300$$

```
(Expand[D[Ds[100, 0, z], {z, 2}]] /. z → 0) - (Expand[D[Ds[50, 0, z], {z, 2}]] /. z → 0)
```

$$\frac{4267}{90}$$

```
Expand[Ds[100, 0, z] Ds[100, 0, -z]] /. z → 1
```

$$100$$

```
Table[Expand[Ds[100, 0, z] Ds[n, 0, -z]] /. z → 1, {n, 1, 100}]
```

$$\{100, 0, -100, -100, -200, -100, -200, -200, -200, -100, -200, -200, -300, -200, -100, -100, -200, -200, -300, -300, -200, -100, -200, -200, -200, -100, -100, -100, -200, -300, -400, -400, -300, -200, -100, -100, -200, -100, 0, 0, -100, -200, -300, -300, -300, -300, -200, -300, -300, -300, -300, -200, -200, -300, -300, -200, -200, -100, 0, -100, -100, -200, -100, -100, -100, 0, -100, -200, -200, -100, -200, -300, -300, -400, -300, -300, -300, -200, -300, -400, -400, -400, -300, -400, -400, -300, -200, -100, -100, -200, -200, -100, -100, 0, 100, 200, 200, 100, 100, 100, 100\}$$

```
Table[D[Ds[100, 0, z] / Ds[n, 0, z], z] /. z → 0, {n, 1, 100}]
```

$$\left\{\frac{428}{15}, \frac{413}{15}, \frac{398}{15}, \frac{781}{30}, \frac{751}{30}, \frac{751}{30}, \frac{721}{30}, \frac{237}{10}, \frac{116}{5}, \frac{116}{5}, \frac{111}{5}, \frac{111}{5}, \frac{106}{5}, \frac{106}{5}, \frac{106}{5}, \frac{419}{20}, \frac{399}{20}, \frac{399}{20}, \frac{379}{20}, \frac{379}{20}, \frac{379}{20}, \frac{379}{20}, \frac{359}{20}, \frac{359}{20}, \frac{349}{20}, \frac{349}{20}, \frac{1027}{60}, \frac{1027}{60}, \frac{967}{60}, \frac{967}{60}, \frac{907}{60}, \frac{179}{12}, \frac{179}{12}, \frac{179}{12}, \frac{179}{12}, \frac{179}{12}, \frac{167}{12}, \frac{167}{12}, \frac{167}{12}, \frac{167}{12}, \frac{155}{12}, \frac{155}{12}, \frac{143}{12}, \frac{143}{12}, \frac{143}{12}, \frac{143}{12}, \frac{131}{12}, \frac{131}{12}, \frac{125}{12}, \frac{125}{12}, \frac{125}{12}, \frac{125}{12}, \frac{113}{12}, \frac{113}{12}, \frac{113}{12}, \frac{113}{12}, \frac{113}{12}, \frac{101}{12}, \frac{101}{12}, \frac{89}{12}, \frac{89}{12}, \frac{89}{12}, \frac{29}{4}, \frac{29}{4}, \frac{29}{4}, \frac{25}{4}, \frac{25}{4}, \frac{25}{4}, \frac{25}{4}, \frac{21}{4}, \frac{21}{4}, \frac{17}{4}, \frac{17}{4}, \frac{17}{4}, \frac{17}{4}, \frac{17}{4}, \frac{13}{4}, \frac{13}{4}, 3, 3, 2, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0\right\}$$

```
Table[D[Ds[100, 0, z] - Ds[n, 0, z], z] /. z → 0, {n, 1, 100}]
```

$$\left\{\frac{428}{15}, \frac{413}{15}, \frac{398}{15}, \frac{781}{30}, \frac{751}{30}, \frac{751}{30}, \frac{721}{30}, \frac{237}{10}, \frac{116}{5}, \frac{116}{5}, \frac{111}{5}, \frac{111}{5}, \frac{106}{5}, \frac{106}{5}, \frac{106}{5}, \frac{419}{20}, \frac{399}{20}, \frac{399}{20}, \frac{379}{20}, \frac{379}{20}, \frac{379}{20}, \frac{379}{20}, \frac{359}{20}, \frac{359}{20}, \frac{349}{20}, \frac{349}{20}, \frac{1027}{60}, \frac{1027}{60}, \frac{967}{60}, \frac{967}{60}, \frac{907}{60}, \frac{179}{12}, \frac{179}{12}, \frac{179}{12}, \frac{179}{12}, \frac{179}{12}, \frac{167}{12}, \frac{167}{12}, \frac{167}{12}, \frac{167}{12}, \frac{155}{12}, \frac{155}{12}, \frac{143}{12}, \frac{143}{12}, \frac{143}{12}, \frac{143}{12}, \frac{131}{12}, \frac{131}{12}, \frac{125}{12}, \frac{125}{12}, \frac{125}{12}, \frac{125}{12}, \frac{113}{12}, \frac{113}{12}, \frac{113}{12}, \frac{113}{12}, \frac{113}{12}, \frac{101}{12}, \frac{101}{12}, \frac{89}{12}, \frac{89}{12}, \frac{89}{12}, \frac{29}{4}, \frac{29}{4}, \frac{29}{4}, \frac{25}{4}, \frac{25}{4}, \frac{25}{4}, \frac{25}{4}, \frac{21}{4}, \frac{21}{4}, \frac{17}{4}, \frac{17}{4}, \frac{17}{4}, \frac{17}{4}, \frac{17}{4}, \frac{13}{4}, \frac{13}{4}, 3, 3, 2, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0\right\}$$

```
Table[D[Ds[100, 0, z] Ds[n, 0, z], z] /. z → 0, {n, 1, 100}]
```

$$\left\{\frac{428}{15}, \frac{443}{15}, \frac{458}{15}, \frac{931}{30}, \frac{961}{30}, \frac{961}{30}, \frac{991}{30}, \frac{1001}{30}, \frac{508}{15}, \frac{508}{15}, \frac{523}{15}, \frac{523}{15}, \frac{538}{15}, \frac{538}{15}, \frac{538}{15}, \frac{2167}{60},\right.$$
$$\frac{2227}{60}, \frac{2227}{60}, \frac{2287}{60}, \frac{2287}{60}, \frac{2287}{60}, \frac{2287}{60}, \frac{2347}{60}, \frac{2347}{60}, \frac{2377}{60}, \frac{2377}{60}, \frac{799}{20}, \frac{799}{20}, \frac{819}{20},$$
$$\frac{819}{20}, \frac{839}{20}, \frac{843}{20}, \frac{843}{20}, \frac{843}{20}, \frac{843}{20}, \frac{843}{20}, \frac{863}{20}, \frac{863}{20}, \frac{863}{20}, \frac{863}{20}, \frac{883}{20}, \frac{883}{20}, \frac{903}{20}, \frac{903}{20},$$
$$\frac{903}{20}, \frac{903}{20}, \frac{923}{20}, \frac{923}{20}, \frac{933}{20}, \frac{933}{20}, \frac{933}{20}, \frac{933}{20}, \frac{953}{20}, \frac{953}{20}, \frac{953}{20}, \frac{953}{20}, \frac{953}{20}, \frac{953}{20}, \frac{973}{20},$$
$$\frac{973}{20}, \frac{993}{20}, \frac{993}{20}, \frac{993}{20}, \frac{2989}{60}, \frac{2989}{60}, \frac{2989}{60}, \frac{3049}{60}, \frac{3049}{60}, \frac{3049}{60}, \frac{3049}{60}, \frac{3109}{60}, \frac{3109}{60}, \frac{3109}{60},$$
$$\frac{3169}{60}, \frac{3169}{60}, \frac{3169}{60}, \frac{3169}{60}, \frac{3169}{60}, \frac{3169}{60}, \frac{3229}{60}, \frac{3229}{60}, \frac{811}{15}, \frac{811}{15}, \frac{826}{15}, \frac{826}{15}, \frac{826}{15},$$
$$\left.\frac{826}{15}, \frac{826}{15}, \frac{826}{15}, \frac{841}{15}, \frac{841}{15}, \frac{841}{15}, \frac{841}{15}, \frac{841}{15}, \frac{841}{15}, \frac{841}{15}, \frac{856}{15}, \frac{856}{15}, \frac{856}{15}, \frac{856}{15}\right\}$$

```
Table[D[Ds[100, 0, z] + Ds[n, 0, z], z] /. z → 0, {n, 1, 100}]
```

$$\left\{\frac{428}{15}, \frac{443}{15}, \frac{458}{15}, \frac{931}{30}, \frac{961}{30}, \frac{961}{30}, \frac{991}{30}, \frac{1001}{30}, \frac{508}{15}, \frac{508}{15}, \frac{523}{15}, \frac{523}{15}, \frac{538}{15}, \frac{538}{15}, \frac{538}{15}, \frac{2167}{60},\right.$$
$$\frac{2227}{60}, \frac{2227}{60}, \frac{2287}{60}, \frac{2287}{60}, \frac{2287}{60}, \frac{2287}{60}, \frac{2347}{60}, \frac{2347}{60}, \frac{2377}{60}, \frac{2377}{60}, \frac{799}{20}, \frac{799}{20}, \frac{819}{20},$$
$$\frac{819}{20}, \frac{839}{20}, \frac{843}{20}, \frac{843}{20}, \frac{843}{20}, \frac{843}{20}, \frac{843}{20}, \frac{863}{20}, \frac{863}{20}, \frac{863}{20}, \frac{863}{20}, \frac{883}{20}, \frac{883}{20}, \frac{903}{20}, \frac{903}{20},$$
$$\frac{903}{20}, \frac{903}{20}, \frac{923}{20}, \frac{923}{20}, \frac{933}{20}, \frac{933}{20}, \frac{933}{20}, \frac{933}{20}, \frac{953}{20}, \frac{953}{20}, \frac{953}{20}, \frac{953}{20}, \frac{953}{20}, \frac{953}{20}, \frac{973}{20},$$
$$\frac{973}{20}, \frac{993}{20}, \frac{993}{20}, \frac{993}{20}, \frac{2989}{60}, \frac{2989}{60}, \frac{2989}{60}, \frac{3049}{60}, \frac{3049}{60}, \frac{3049}{60}, \frac{3049}{60}, \frac{3109}{60}, \frac{3109}{60}, \frac{3109}{60},$$
$$\frac{3169}{60}, \frac{3169}{60}, \frac{3169}{60}, \frac{3169}{60}, \frac{3169}{60}, \frac{3169}{60}, \frac{3229}{60}, \frac{3229}{60}, \frac{811}{15}, \frac{811}{15}, \frac{826}{15}, \frac{826}{15}, \frac{826}{15},$$
$$\left.\frac{826}{15}, \frac{826}{15}, \frac{826}{15}, \frac{841}{15}, \frac{841}{15}, \frac{841}{15}, \frac{841}{15}, \frac{841}{15}, \frac{841}{15}, \frac{841}{15}, \frac{856}{15}, \frac{856}{15}, \frac{856}{15}, \frac{856}{15}\right\}$$

```
D[x^z, z] /. z → 0
```

```
Log[x]
```

```
D[LogIntegral[x], x]
```

$$\frac{1}{\text{Log}[x]}$$

```
kap[n_] := D[Ds[n, 0, z] / Ds[n - 1, 0, z], z] /. z → 0
k2[n_, k_] := (D[Ds[n, 0, z], {z, k}] /. z → 0) - (D[Ds[n - 1, 0, z], {z, k}] /. z → 0)
k2f[n_, k_] := (D[Ds[n, 0, z], {z, k}] /. z → 0)
```

```
kap[11]
```

```
1
```

```
Table[k2f[n, 2], {n, 1, 20}]
```

$$\left\{0, 0, 0, 1, 1, 3, 3, 4, 5, 7, 7, 8, 8, 10, 12, \frac{155}{12}, \frac{155}{12}, \frac{167}{12}, \frac{167}{12}, \frac{179}{12}\right\}$$

```
Table[Sum[kap[j] kap[k], {j, 2, n}, {k, 2, n / j}], {n, 1, 20}]
```

$$\left\{0, 0, 0, 1, 1, 3, 3, 4, 5, 7, 7, 8, 8, 10, 12, \frac{155}{12}, \frac{155}{12}, \frac{167}{12}, \frac{167}{12}, \frac{179}{12}\right\}$$

```
Table[Sum[(D[Ds[j, 0, z] / Ds[j - 1, 0, z], z] /. z → 0)
    (D[Ds[k, 0, z] / Ds[k - 1, 0, z], z] /. z → 0), {j, 2, n}, {k, 2, n / j}], {n, 1, 20}]
```

$$\left\{0, 0, 0, 1, 1, 3, 3, 4, 5, 7, 7, 8, 8, 10, 12, \frac{155}{12}, \frac{155}{12}, \frac{167}{12}, \frac{167}{12}, \frac{179}{12}\right\}$$

```
Table[Sum[(D[Ds[j, 0, z] / Ds[j - 1, 0, z], z]) (D[Ds[k, 0, z] / Ds[k - 1, 0, z], z]) /. z → 0,
  {j, 2, n}, {k, 2, n / j}], {n, 1, 20}]
```

$$\left\{0, 0, 0, 1, 1, 3, 3, 4, 5, 7, 7, 8, 8, 10, 12, \frac{155}{12}, \frac{155}{12}, \frac{167}{12}, \frac{167}{12}, \frac{179}{12}\right\}$$

```
D2[n_, k_] := Ds[n, k, 0, 1]
```

```
Sum[ ((-1)^(k + 1) / k) ((-1)^j / j) D2[100, k] D2[50, j], {k, 1, Log2@100}, {j, 1, Log2@50}]
```

$$-\frac{116\,309}{225}$$

```
pr[100, 0] - pr[50, 0]
```

$$\frac{125}{12}$$

```
D[Ds[100, 0, z] Ds[50, 0, -z], z] /. z → 0
```

$$\frac{125}{12}$$

```
Sum[ ((-1)^(k + 1) / k) D2[100, k], {k, 1, Log2@100}]
 Sum[ ((-1)^j / j) D2[50, j], {j, 1, Log2@50}]
```

$$-\frac{116\,309}{225}$$

```
Expand[Sum[ (bin[z, k] bin[-z, j] / z) D2[100, k] D2[50, j],
    {k, 0, Log2@100}, {j, 0, Log2@50}]] - (1 / z) /. z → 0
```

$$\frac{125}{12}$$

```
Grid@Expand[Table[ (bin[z, k] bin[-z, j] / z), {k, 0, Log2@100}, {j, 0, Log2@50}]] /. z → 0
```

Power::infy : Infinite expression $\frac{1}{0}$ encountered. ≫

ComplexInfinity	-1	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{4}$	$-\frac{1}{5}$
1	0	0	0	0	0
$-\frac{1}{2}$	0	0	0	0	0
$\frac{1}{3}$	0	0	0	0	0
$-\frac{1}{4}$	0	0	0	0	0
$\frac{1}{5}$	0	0	0	0	0
$-\frac{1}{6}$	0	0	0	0	0

```
ll[n_, z_] := LaguerreL[-z, Log[n]]
```