```
Dd[x_, 1, y_] := Sum[(j + y)^0, {j, 0, Floor[x - y - 1]}]
Cc[x_, 1, y_] := y^(-1) Dd[x y, 1, y]
Table[{n / 7 - 1, Limit[Cc[n / 7, 1, z], z → Infinity]}, {n, 1, 20}] // TableForm
```

$-\frac{6}{7}$	$-\frac{6}{7}$
$-\frac{5}{7}$	$-\frac{5}{7}$
$-\frac{4}{7}$	$-\frac{4}{7}$
$-\frac{3}{7}$	$-\frac{3}{7}$
$-\frac{2}{7}$	$-\frac{2}{7}$
$-\frac{1}{7}$	$-\frac{1}{7}$
0	0
$\frac{1}{7}$	$\frac{1}{7}$
$\frac{2}{7}$	$\frac{2}{7}$
$\frac{3}{7}$	$\frac{3}{7}$
$\frac{4}{7}$	$\frac{4}{7}$
$\frac{5}{7}$	$\frac{5}{7}$
$\frac{6}{7}$	$\frac{6}{7}$
1	1
$\frac{8}{7}$	$\frac{8}{7}$
$\frac{9}{7}$	$\frac{9}{7}$
$\frac{10}{7}$	$\frac{10}{7}$
$\frac{11}{7}$	$\frac{11}{7}$
$\frac{12}{7}$	$\frac{12}{7}$
$\frac{13}{7}$	$\frac{13}{7}$

```
Limit[ (y^(s - 1) HurwitzZeta[ s, y + 1])^k, y → Infinity]
```

$$\left(\frac{1}{-1+s}\right)^k$$

```
Integrate[ (w z)^-s, {z, 1, n}, {w, 1, n / z}]
```

$$\text{ConditionalExpression}\left[\frac{1}{(-1+s)^2} - \frac{n^{1-s}\,(1+(-1+s)\,\text{Log}[n])}{(-1+s)^2},\ \text{Re}[n] \geq 0 \,||\, n \notin \text{Reals}\right]$$

```
Grid[Table[
  Chop[N[1 / ((s - 1)^k) - Integrate[D[y^(k (s - 1)) Zeta[s, y + 1]^k, y], {y, 1, Infinity}]] -
    N[(Zeta[s, 2])^k]], {s, 2, 4}, {k, 1, 4}]]
```

0	0	0	0
0	0	0	0
0	0	0	0

```
FullSimplify[y ((j + 1) y)^-s]
```

$$y\,((1+j)\,y)^{-s}$$

```
(1 / y) (1 / y) ^ - s
```

$\left(\frac{1}{y}\right)^{1-s}$

```
y ^ (s - 1)
```

y^{-1+s}

```
f[n_, s_, t_] :=
 Sum[ (-1) ^ (k - 1) / k (1 / (s - 1) ^ k) (Gamma[ k, 0, (s - 1) Log[n]] / Gamma[k]), {k, 1, Infinity}]
N[f[100, 0, 30]]
```

$28.0217 - 2.09386 \times 10^{-14}\, i$

```
N[-Gamma[0, -Log[100]] - Log[Log[100]] - EulerGamma]
```

$28.0217 + 3.14159\, i$

```
f[n, -1, 50]
```

$\sum_{k=1}^{\infty} \frac{(-2)^{-k} (-1)^{-1+k} \text{Gamma}[k, 0, -2 \text{Log}[n]]}{k \text{Gamma}[k]}$

```
N[-Gamma[0, -Log[10 000]]]
```

$1246.14 + 3.14159\, i$

```
N[LogIntegral[10 000]]
```

1246.14

```
f2[n_, s_, t_] := Sum[ (-1) ^ (k - 1) / (s - 1) ^ k / (k !)
   (Integrate[ r ^ (k - 1) E ^ (-r), {r, 0, (s - 1) Log[n]}]), {k, 1, Infinity}]
f2[n, -1, 50]
```

$\sum_{k=1}^{\infty} \text{ConditionalExpression}\left[\frac{(-2)^{-k} (-1)^{-1+k} (\text{Gamma}[k] - \text{Gamma}[k, -2 \text{Log}[n]])}{k \text{Gamma}[k]}, \text{Re}[k] > 0\right]$

```
FullSimplify[(-1) ^ (k - 1) / k (1 / (s - 1) ^ k) / ((k - 1) !)]
```

$-\frac{(-1)^{k} (-1 + s)^{-k}}{\text{Gamma}[1 + k]}$

```
Integrate[ r ^ (k - 1) E ^ (-r), {r, 0, (s - 1) Log[n]}]
```

ConditionalExpression[Gamma[k] - Gamma[k, (-1 + s) Log[n]], Re[k] > 0]

```
f3[n_, s_] :=
 Integrate[Sum[ (-1) ^ (k - 1) / (s - 1) ^ k / (k !) ( r ^ (k - 1) E ^ (-r)), {k, 1, Infinity}],
  {r, 0, (s - 1) Log[n]}]
f3[n, -1]
```

ConditionalExpression[-ExpIntegralEi[Log[n]] + ExpIntegralEi[2 Log[n]] - Log[2], Log[n] < 0]

```
f3[n, -2]
```

ConditionalExpression[
 $-\text{ExpIntegralEi}[2 \text{Log}[n]] + \text{ExpIntegralEi}[3 \text{Log}[n]] - \text{Log}\left[\frac{3}{2}\right], \text{Log}[n] < 0$]

```
f3[n, -2.5]
```

$$\int_0^{-3.5\,\mathrm{Log}[n]} -0.285714\, e^{-r}\, \mathrm{Hypergeometric1F1}[1., 2., 0.285714\, r]\, dr$$

```
f3[n, 1]
```

0

```
f3[n, 2]
```

-Gamma[0, Log[n]] + Gamma[0, 2 Log[n]] + Log[2]

```
f3[n, 3]
```

$$-\mathrm{Gamma}[0, 2\,\mathrm{Log}[n]] + \mathrm{Gamma}[0, 3\,\mathrm{Log}[n]] + \mathrm{Log}\left[\frac{3}{2}\right]$$

```
f3[n, s]
```

-Gamma[0, (-1 + s) Log[n]] + Gamma[0, s Log[n]] - Log[(-1 + s) Log[n]] + Log[s Log[n]]

```
Expand[Limit[
  -Gamma[0, (-1 + s) Log[n]] + Gamma[0, s Log[n]] - Log[(-1 + s) Log[n]] + Log[s Log[n]], s → 0]]
```

$$-\mathrm{EulerGamma} + \mathrm{ExpIntegralEi}[\mathrm{Log}[n]] + \frac{1}{2}\mathrm{Log}\left[\frac{1}{\mathrm{Log}[n]}\right] - \frac{1}{2}\mathrm{Log}[\mathrm{Log}[n]]$$

```
FullSimplify[
 -Gamma[0, (-1 + s) Log[n]] + Gamma[0, s Log[n]] - Log[(-1 + s) Log[n]] + Log[s Log[n]]]
```

-Gamma[0, (-1 + s) Log[n]] + Gamma[0, s Log[n]] - Log[(-1 + s) Log[n]] + Log[s Log[n]]

```
-Gamma[0, (-1 + s) Log[n]] + Gamma[0, s Log[n]] - Log[(-1 + s) Log[n]] + Log[s Log[n]]
```

-Gamma[0, (-1 + s) Log[n]] + Gamma[0, s Log[n]] - Log[(-1 + s) Log[n]] + Log[s Log[n]]

```
f4[n_, s_] :=
 Integrate[Sum[ ((-1)^(k - 1) / k) / (s - 1)^k / ((k - 1)!) ( r^(k - 1) E^(-r)), {k, 1, Infinity}],
  {r, 0, (s - 1) Log[n]}]
f4[100, 0]
```

-EulerGamma + ExpIntegralEi[Log[100]] - Log[Log[100]]

```
bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
f5[n_, s_, z_] :=
 Integrate[Sum[ (bin[z, k]) / (s - 1)^k / ((k - 1)!) ( r^(k - 1) E^(-r)), {k, 0, Infinity}],
  {r, 0, (s - 1) Log[n]}]
FullSimplify[f5[n, s, z]]
```

$$\int_0^{(-1+s)\,\mathrm{Log}[n]} \frac{e^{-r}\, z\, \mathrm{Hypergeometric1F1}\left[1 - z, 2, -\frac{r}{-1+s}\right]}{-1 + s}\, dr$$

f5[n, s, -2]

$$-\frac{n^{-s}\,((-1+n^{s})\,(-1+2\,s)+s\,\mathrm{Log}[n])}{s^{2}}$$

f5[n, 0, z]

ConditionalExpression[-1 + LaguerreL[-z, Log[n]], -1 ≤ Re[Log[n]] ≤ 1 || Log[n] ∉ Reals]

$$\int_{0}^{n} e^{-r}\,\mathrm{Hypergeometric1F1}\left[1-z,\,2,\,-\frac{r}{-1+s}\right]\,dr$$

$$\int_{0}^{n} e^{-r}\,\mathrm{Hypergeometric1F1}\left[1-z,\,2,\,-\frac{r}{-1+s}\right]\,dr$$

Limit[Gamma[0, s Log[n]] + Log[s Log[n]], s → 0]

-EulerGamma

-∞

-Log[-Log[10]]

$-i\pi$ - Log[Log[10]]

Expand[Limit[
-Gamma[0, (-1 + s) Log[n]] + Gamma[0, s Log[n]] - Log[(-1 + s) Log[n]] + Log[s Log[n]], s → 0]]

$$-\mathrm{EulerGamma}+\mathrm{ExpIntegralEi}[\mathrm{Log}[n]]+\frac{1}{2}\,\mathrm{Log}\left[\frac{1}{\mathrm{Log}[n]}\right]-\frac{1}{2}\,\mathrm{Log}[\mathrm{Log}[n]]$$

Infinity::indet : Indeterminate expression -∞ + ∞ - Gamma[0, -Log[n]] - Log[-Log[n]] encountered. ≫

Indeterminate

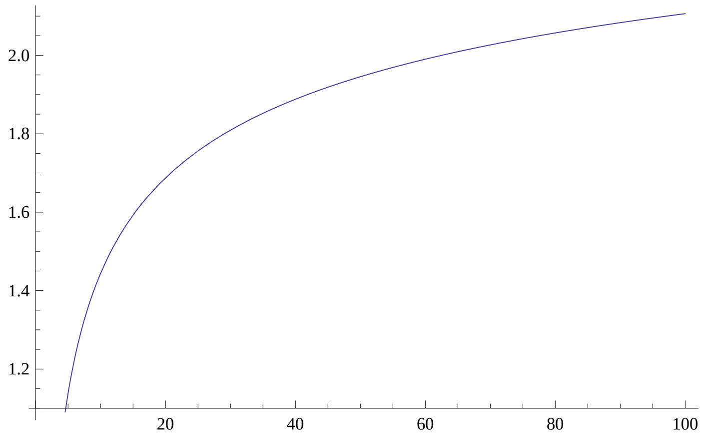
$$\frac{1}{2}\,\mathrm{Log}\left[\frac{1}{\mathrm{Log}[n]}\right]-\frac{1}{2}\,\mathrm{Log}[\mathrm{Log}[n]]\ /.\ n\to 100$$

-Log[Log[100]]

Expand[Limit[
-Gamma[0, (-1 + s) Log[n]] + Gamma[0, s Log[n]] - Log[(-1 + s) Log[n]] + Log[s Log[n]], s → 0]]

$$-\mathrm{EulerGamma}+\mathrm{ExpIntegralEi}[\mathrm{Log}[n]]+\frac{1}{2}\,\mathrm{Log}\left[\frac{1}{\mathrm{Log}[n]}\right]-\frac{1}{2}\,\mathrm{Log}[\mathrm{Log}[n]]$$

```
Plot[ EulerGamma - ExpIntegralEi[-Log[n]] - 1/2 Log[-1/Log[n]] + 1/2 Log[-Log[n]], {n, 1, 100}]
```




```
Limit[-ExpIntegralEi[-3 Log[n]] + ExpIntegralEi[-2 Log[n]] + 1/2 Log[-3/(2 Log[n])] -
  1/2 Log[-1/(3 Log[n])] - 1/2 Log[-2 Log[n]] + 1/2 Log[-Log[n]], n → Infinity]
```

```
Log[3/2]
```

```
N[Log[Zeta[3]]]
```

```
0.184034
```

```
N[Log[3 / 2]]
```

```
0.405465
```

```
N[-Gamma[0, -Log[100]]]
```

```
30.1261 + 3.14159 i
```

```
Expand[Limit[
  -Gamma[0, (-1 + s) Log[n]] + Gamma[0, s Log[n]] - Log[(-1 + s) Log[n]] + Log[s Log[n]], s → 0]]
```

```
-EulerGamma + ExpIntegralEi[Log[n]] + 1/2 Log[1/Log[n]] - 1/2 Log[Log[n]]
```

```
Expand[Limit[Gamma[0, s Log[n]] - Log[(-1 + s) Log[n]] + Log[s Log[n]], s → 0]] /. n → 100
```

```
-EulerGamma - i π - Log[Log[100]]
```

```
Expand[Limit[-Gamma[0, (-1 + s) Log[n]], s → 0]] /. n → 100
```

```
i π + ExpIntegralEi[Log[100]]
```

```
-EulerGamma - Log[-Log[n]] + ExpIntegralEi[Log[n]] +
  1/2 Log[1/Log[n]] + Log[-Log[n]] - 1/2 Log[Log[n]] /. n → 100
```

```
-EulerGamma + ExpIntegralEi[Log[100]] - Log[Log[100]]
```

```
tt[n_, s_, x_] := Sum[ (x^(k (1 - s)) - 1) / k, {k, 1, Log[x, n]}]
```

```
N[tt[100, 3 - I, 1.000001]]
```

```
-2.90912 + 0.463656 i
```

```
N[LogIntegral[100] - Log[Log[100]] - EulerGamma]

28.0217

N[Expand[Limit[-Gamma[0, (-1 + s) Log[n]] +
      Gamma[0, s Log[n]] - Log[(-1 + s) Log[n]] + Log[s Log[n]], s → -1]] /. n → 100]

1215.32

N[ExpIntegralEi[(-2 + I) Log[100]] - Log[(-2 + I) Log[100]] - EulerGamma]

-2.90912 + 0.463656 i

zeta[n_, s_, z_, k_] := 1 + ((z + 1) / k - 1) Sum[j^-s zeta[n / j, s, z, k + 1], {j, 2, n}]

N[D[zeta[100, s, -1, 1], s] /. {s → 0}]

-10.0174

bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
Lm1[n_, k_] := Sum[Lm1[n / j, k - 1], {j, 2, n}]
Lm1[n_, 1] := Sum[Log[j], {j, 2, n}]
Lm1[n_, 0] := UnitStep[n - 1]
Lz[n_, z_] := Sum[bin[z, k] Lm1[n, k], {k, 1, Log[2, n]}]
Zm1[n_, k_, s_] := Sum[j^-s Zm1[n / j, k - 1, s], {j, 2, n}]
Zm1[n_, 0, s_] := UnitStep[n - 1]
Lz2[n_, z_] := -Sum[bin[z, k] (D[Zm1[n, k, s], s] /. s → 0) / k, {k, 1, Log[2, n]}]
Lz3[n_, z_] := D[-Sum[bin[z, k] / k Zm1[n, k, s], {k, 1, Log[2, n]}], s] /. s → 0


N[Lm1[100, 5]]

44.4803

-(N[D[Zm1[100, 5, s], s]] /. s → 0) / 5

44.4803

N[Lz2[100, -1]]

-94.0453

N[Lz[100, -1]]

-94.0453

N[Lz3[100, -1]]

-94.0453

Limit[D[-Gamma[0, (-1 + s) Log[n]] +
    Gamma[0, s Log[n]] - Log[(-1 + s) Log[n]] + Log[s Log[n]], s], s → 0]

1 - n + Log[n]

L2[n_, 1, b_] := L2[n, 1, b] = Sum[Log[j], {j, 2, n}] - b Sum[Log[j b], {j, 1, n / b}]
L2[n_, k_, b_] := Sum[L2[n / j, k - 1, b], {j, 2, n}] - b Sum[L2[n / (j b), k - 1, b], {j, 1, n}]
L1[n_, z_, c_] := Sum[Binomial[z, k] L2[n, k, c], {k, 0, Floor[Log[n] / Log[c]]}]
```

```
D1xD[n_, k_, x_] :=
 D1xD[n, k, x] = Sum[D1xD[n / (j + 1), k - 1, x] - x D1xD[n / (x j), k - 1, x], {j, 1, n}]
D1xD[n_, 0, x_] := UnitStep[n - 1]
DxD[n_, z_, x_] := Sum[Binomial[z, k] D1xD[n, k, x], {k, 0, Log[x, n]}]
```

```
N[L2[100, 1, 2]]
```

-2.53088

```
D1xD[100, 1, 2]
```

-1

```
Limit[D[Gamma[0, s Log[n]], s], s → 0]
```

$-\infty$

```
Limit[D[-Log[(-1 + s) Log[n]], s], s → 0]
```

1

```
Limit[D[Log[s Log[n]], s], s → 0]
```

∞

```
Limit[D[-Gamma[0, (-1 + s) Log[n]] +
   Gamma[0, s Log[n]] - Log[(-1 + s) Log[n]] + Log[s Log[n]], s], s → 0]
```

1 - n + Log[n]

```
Limit[D[Gamma[0, s Log[n]] + Log[s Log[n]], s], s → 0]
```

Log[n]

```
Limit[D[-Gamma[0, (-1 + s) Log[n]], s], s → 0]
```

-n

```
D[-Gamma[0, (-1 + s) Log[n]], s]
```

$$\frac{n^{1-s}}{-1+s}$$

```
D[-Log[(-1 + s) Log[n]], s]
```

$$-\frac{1}{-1+s}$$

```
D[Gamma[0, s Log[n]], s]
```

$$-\frac{n^{-s}}{s}$$

```
D[Log[s Log[n]], s]
```

$$\frac{1}{s}$$

```
Expand[N[D[zeta[100, s, z, 1] / z, s] /. {s → 0}]]
```

$-94.0453 - 169.15\, z - 81.6195\, z^2 - 17.6846\, z^3 - 1.19616\, z^4 - 0.0438125\, z^5$

```
zeta[n_, s_, z_, k_] := 1 + ((z + 1) / k - 1) Sum[j^-s zeta[n / j, s, z, k + 1], {j, 2, n}]
bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
Zm1[n_, s_, k_] := Sum[j^-s Zm1[n / j, s, k - 1], {j, 2, n}]
Zm1[n_, s_, 0 _] := UnitStep[n - 1]
gg[n_, z_] := Sum[ Expand[N[D[bin[z, k] Zm1[n, s, k] / z, s]]] /. s → 0, {k, 0, Log[2, n]}]
gga[n_, z_] := Sum[ Expand[N[D[bin[z, k] Zm1[n, s, k], s]]] /. s → 0, {k, 0, Log[2, n]}]
ggar[n_, z_] := -Sum[ Expand[N[D[(-1)^k / k Zm1[n, s, k], s]]] /. s → 0, {k, 1, Log[2, n]}]
ggk[n_, z_, k_] := D[bin[z, k] Zm1[n, s, k] / z, s] /. s → 0
ggk2[n_, k_] := Expand[N[D[Zm1[n, s, k], s]]] /. s → 0
zeros[n_] := List @@ NRoots[gga[n, z] - 1 == 0, z][[All, 2]]
prod[n_, z_] := 1 - Product[1 - z / r, {r, zeros[n]}]
sum[n_] := Sum[-1 / r, {r, zeros[n]}]
zeros2[n_] := List @@ NRoots[gg[n, z] == 0, z][[All, 2]]
prod2[n_, z_] := Product[1 - z / r, {r, zeros2[n]}]
pggad[n_] :=
 Limit[D[Sum[ Expand[N[bin[z, k] Zm1[n, s, k]]] /. s → 0, {k, 0, Log[2, n]}], z], z → 0]
ggad[n_] := Limit[
  D[Sum[ Expand[N[D[bin[z, k] Zm1[n, s, k], s]]] /. s → 0, {k, 0, Log[2, n]}], z], z → 0]
Lm1[n_, k_] := (k / (k - 1)) Sum[Lm1[n / j, k - 1], {j, 2, n}]
Lm1[n_, 1] := Sum[-Log[j], {j, 2, n}]
Lm1[n_, 0] := UnitStep[n - 1]
zeros3[n_] := List @@ NRoots[N[Expand[D[zeta[n, s, z, 1], s] /. s → 0]] - 1 == 0, z][[All, 2]]
zeros3a[n_] := List @@ NRoots[N[Expand[zeta[n, s, z, 1] /. s → 0]] == 0, z][[All, 2]]
```

```
ggar[100, z]
```

-94.0453

```
gga[100, z]
```

$0. - 94.0453 z - 169.15 z^2 - 81.6195 z^3 - 17.6846 z^4 - 1.19616 z^5 - 0.0438125 z^6$

```
Expand[zeta[100, 0, z, 1]]
```

$$1 + \frac{428 z}{15} + \frac{16289 z^2}{360} + \frac{331 z^3}{16} + \frac{611 z^4}{144} + \frac{67 z^5}{240} + \frac{7 z^6}{720}$$

```
N[Expand[D[zeta[100, s, z, 1], s] /. s → 0]]
```

$-94.0453 z - 169.15 z^2 - 81.6195 z^3 - 17.6846 z^4 - 1.19616 z^5 - 0.0438125 z^6$

```
sum[100]
```

$94.0453 + 0. i$

```
prod[100, 1]
```

$-363.739 + 7.10543 \times 10^{-15} i$

```
gga[100, 1]
```

-363.739

```
zeros[100]
```

{-10.6971 - 12.1993 i, -10.6971 + 12.1993 i,
 -2.54005 - 1.8272 i, -2.54005 + 1.8272 i, -0.816685, -0.0108436}

```
zeros3[100]
```

{-10.6971 - 12.1993 ⅈ, -10.6971 + 12.1993 ⅈ,
 -2.54005 - 1.8272 ⅈ, -2.54005 + 1.8272 ⅈ, -0.816685, -0.0108436}

```
Sum[ -1 / j, {j, zeros3[100]}]
```

94.0453 + 0. ⅈ

```
zeta[ 100, -0.010843552541250006`, 0, 1] - 1
```

0.

```
D[gga[100, z], z] /. z → 0
```

-94.0453

```
zeros3[100]
```

{-10.6971 - 12.1993 ⅈ, -10.6971 + 12.1993 ⅈ,
 -2.54005 - 1.8272 ⅈ, -2.54005 + 1.8272 ⅈ, -0.816685, -0.0108436}

```
zeros3a[100]
```

{-11.1997 - 12.3982 ⅈ, -11.1997 + 12.3982 ⅈ,
 -2.67195 - 1.86184 ⅈ, -2.67195 + 1.86184 ⅈ, -0.933809, -0.0372047}

```
5 / 4 × 4 / 3 × 3 / 2 × 2 / 1
```

5

```
N[Lm1[100, 4]]
```

-779.857

```
ggk2[100, 4]
```

-779.857

```
zeta[10, s, 2, 1]
```

$$1 + 2 \left(6^{-s} + 7^{-s} + 8^{-s} + 9^{-s} + 10^{-s} + 4^{-s} \left(1 + 2^{-1-s}\right) + 5^{-s} \left(1 + 2^{-1-s}\right) + 3^{-s} \left(1 + \frac{1}{2} \left(2^{-s} + 3^{-s}\right)\right) + 2^{-s} \left(1 + \frac{1}{2} \left(2^{-s} + 3^{-s} + 4^{-s} + 5^{-s}\right)\right)\right)$$

```
Integrate[t^(k - 1) E^-t, {t, 0, x}] bin[z, k] 1 / (s - 1)^k ((k - 1)!)^-1 /. k → 3
```

$$\frac{(1 - z)(2 - z)\, z\, (2 - \text{Gamma}[3, x])}{12\, (-1 + s)^3}$$

```
Sum[Integrate[t^(k - 1) E^-t bin[z, k] 1 / (s - 1)^k ((k - 1)!)^-1, {t, 0, x}], {k, 0, Infinity}]
```

$$\sum_{k=0}^{\infty} \text{ConditionalExpression}\left[-\frac{1}{(-1 + k)!\, k!} (-1)^k (-1 + s)^{-k} z\, (\text{Gamma}[k] - \text{Gamma}[k, x])\, \text{Pochhammer}[1 - z, -1 + k],\ \text{Re}[k] > 0\right]$$

```
Sum[t^(k - 1) E^-t bin[z, k] 1 / (s - 1)^k ((k - 1)!)^-1, {k, 0, Infinity}]
```

$$\frac{e^{-t} z\, \text{Hypergeometric1F1}\left[1 - z, 2, -\frac{t}{-1+s}\right]}{-1 + s}$$

```
bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k !
zeta1mxm1[n_, s_, k_,  x_] := Sum[ j^(-s) zeta1mxm1[n / j, s, k - 1, x], {j, 2, n}] -
  x Sum[ (j x)^(-s) zeta1mxm1[n / (x j), s, k - 1, x], {j, 1, n / x}]
zeta1mxm1[n_, s_, 0, x_] := UnitStep[n - 1]
zeta1mx[n_, s_, z_, x_] :=
 Sum[ bin[z, k] zeta1mxm1[n, s, k, x], {k, 0, If[x < 2, Log[x, n], Log[2, n]]}]
logzeta1mx[n_, s_, x_] := D[zeta1mx[n, s, z, x], z] /. z → 0
L2[n_, 1, x_] := - (Sum[Log[j], {j, 2, n}] - x Sum[Log[j x], {j, 1, n / x}])
L2[n_, k_, x_] :=
 (k / (k - 1)) (Sum[L2[n / j, k - 1, x], {j, 2, n}] - x Sum[L2[n / (j x), k - 1, x], {j, 1, n}])
```

```
Expand[zeta1mx[100, 0, z, 101]]
logzeta1mx[100, 0, 2]
```

$$1 + \frac{428 z}{15} + \frac{16289 z^2}{360} + \frac{331 z^3}{16} + \frac{611 z^4}{144} + \frac{67 z^5}{240} + \frac{7 z^6}{720}$$

$$\frac{4}{5}$$

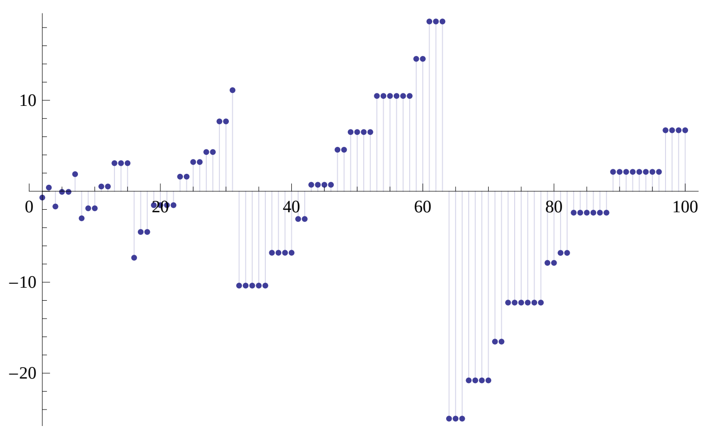
```
N[D[zeta1mxm1[100, s, 3, 2], s] /. s → 0]
```

17.9431

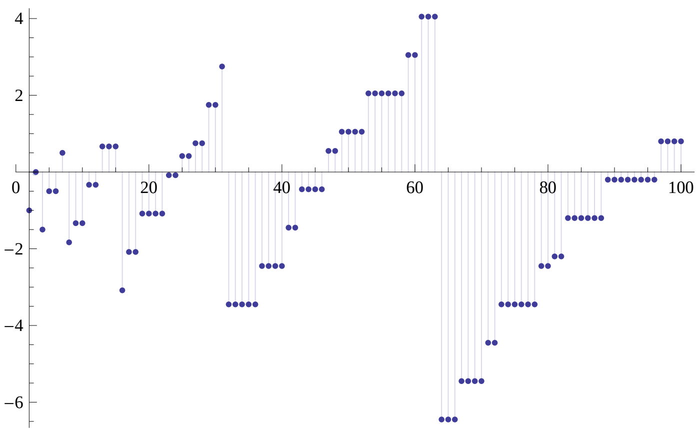
```
N[L2[100, 3, 2]]
```

17.9431

```
DiscretePlot[-N[D[logzeta1mx[n, s, 2], s] /. s → 0], {n, 2, 100}]
```



```
DiscretePlot[logzeta1mx[n, 0, 2], {n, 2, 100}]
```




```
D[(x^(k (1 - s)) - 1) / k, s] /. s → 0
```

$-x^k \text{Log}[x]$

```
Limit[ Sum[ -x^k Log[x], {k, 1, Log[x, n]}], x → 1]
```

1 - n

```
D[ LogIntegral[n^(1 - s)] - Log[Log[n^(1 - s)]] - EulerGamma, s] /. s → 0
```

1 - n

```
Table[ (-1)^(j) Limit[bin[-z, j] / z, z → 0], {j, 1, 10}]
```

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}\right\}$$

```
(-1)^(j) Limit[bin[z, j] / z, z → 0]
```

$$-\frac{(-1)^{2j}\,\text{Gamma}[j]}{j!}$$

```
bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
zeta[n_, s_, z_, k_] := 1 + ((z + 1) / k - 1) Sum[j^-s zeta[n / j, s, z, k + 1], {j, 2, n}]
zeta1mxm1[n_, s_, k_, x_] := Sum[ j^(-s) zeta1mxm1[n / j, s, k - 1, x], {j, 2, n}] -
  x Sum[ (j x)^(-s) zeta1mxm1[n / (x j), s, k - 1, x], {j, 1, n / x}]
zeta1mxm1[n_, s_, 0, x_] := UnitStep[n - 1]
zeta1mx[n_, s_, z_, x_] :=
 Sum[ bin[z, k] zeta1mxm1[n, s, k, x], {k, 0, If[x < 2, Log[x, n], Log[2, n]]}]
logzeta1mx[n_, s_, x_] := D[zeta1mx[n, s, z, x], z] /. z → 0
```

```
Table[FullSimplify[
  {n, 7 z Expand[zeta1mx[Floor[n / 7], 0, z, aa = 101] - zeta1mx[Floor[(n - 1) / 7], 0, z, aa]],
   (Expand[zeta1mx[n, 0, z, aa = 101] - zeta1mx[n - 1, 0, z, aa]]) -
    (Expand[zeta1mx[n, 0, z, bb = 7] - zeta1mx[n - 1, 0, z, bb]])}], {n, 41, 80}] // TableForm
```

41	0	0
42	$7 z^3$	$7 z^3$
43	0	0
44	0	0
45	0	0
46	0	0
47	0	0
48	0	0
49	$7 z^2$	$-\frac{7}{2} z (-7 + 5 z)$
50	0	0
51	0	0
52	0	0
53	0	0
54	0	0
55	0	0
56	$\frac{7}{6} z^2 (1 + z) (2 + z)$	$\frac{7}{6} z^2 (1 + z) (2 + z)$
57	0	0
58	0	0
59	0	0
60	0	0
61	0	0
62	0	0
63	$\frac{7}{2} z^2 (1 + z)$	$\frac{7}{2} z^2 (1 + z)$
64	0	0
65	0	0
66	0	0
67	0	0
68	0	0
69	0	0
70	$7 z^3$	$7 z^3$
71	0	0
72	0	0
73	0	0
74	0	0
75	0	0
76	0	0
77	$7 z^2$	$7 z^2$
78	0	0
79	0	0
80	0	0

```
bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
Zm1[n_, 0, s_] := UnitStep[n - 1]
Zm1[n_, k_, s_] := Zm1[n, k, s] = Sum[j^-s Zm1[Floor[n / j], k - 1, s], {j, 2, n}]
zeta[n_, s_, z_] := Sum[ bin[ z, k] Zm1[ n, k, s], {k, 0, Log[2, n]}]
zeros[n_, s_] := List @@ NRoots[zeta[n, s, z] == 0, z][[All, 2]]
zetaalt[n_, s_, z_] := Product[1 - z / r, {r, zeros[n, s]}]
```

```
zeros[100 000, 2] // TableForm
```

$-96.3686-213.131\,i$
$-96.3686+213.131\,i$
-31.144
$-28.4883-44.9883\,i$
$-28.4883+44.9883\,i$
$-11.4956-16.6843\,i$
$-11.4956+16.6843\,i$
-10.1417
$-9.61048-4.61542\,i$
$-9.61048+4.61542\,i$
$-8.08137-9.15539\,i$
$-8.08137+9.15539\,i$
$-4.23774-12.2216\,i$
$-4.23774+12.2216\,i$
$3.95874-13.3611\,i$
$3.95874+13.3611\,i$

```
zeros[1 000 000, 2] // TableForm
```

-919.546
-98.5521
$-56.5718-89.5194\,i$
$-56.5718+89.5194\,i$
$-19.7441-38.9516\,i$
$-19.7441+38.9516\,i$
$-12.2367-14.9262\,i$
$-12.2367+14.9262\,i$
-11.7085
$-11.2775-4.30913\,i$
$-11.2775+4.30913\,i$
$-9.95445-8.90915\,i$
$-9.95445+8.90915\,i$
$-7.84051-13.7037\,i$
$-7.84051+13.7037\,i$
$-2.63047-14.9893\,i$
$-2.63047+14.9893\,i$
$6.60545-15.3389\,i$
$6.60545+15.3389\,i$

```
N[Expand[zeta[100 000, 2, z]]]
```

$$1.+0.497699\,z+0.12385\,z^2+0.0205441\,z^3+0.00255436\,z^4+0.000253471\,z^5+0.0000207963\,z^6+1.43262\times10^{-6}\,z^7+8.25343\times10^{-8}\,z^8+3.8963\times10^{-9}\,z^9+1.45011\times10^{-10}\,z^{10}+4.07809\times10^{-12}\,z^{11}+8.11705\times10^{-14}\,z^{12}+1.07464\times10^{-15}\,z^{13}+8.88423\times10^{-18}\,z^{14}+3.15853\times10^{-20}\,z^{15}+9.02612\times10^{-23}\,z^{16}$$

```
N[Log[Pi^2 / 6]^2] / 2
```

0.123853