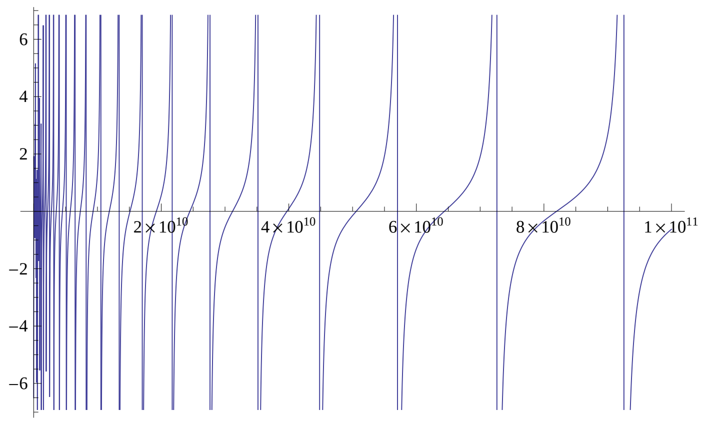
```
ss[n_, s_] := Tan[s Log[n] + ArcTan[1 / (2 s)]]

Plot[Re@ss[n, 13], {n, 0, 100 000 000 000}]
```

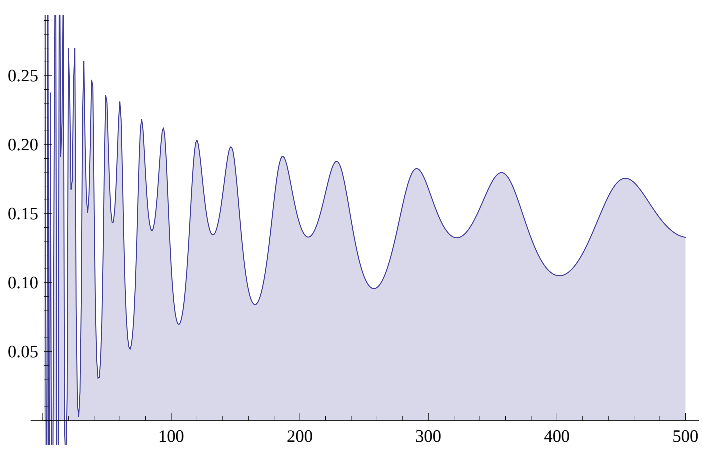



```
Limit[ ss[n, .01 I + 13], n → Infinity]

0. + 1. ⅈ

tc[n_, s_] := Sum[ j^(-1 / 2) Cos[s Log[j]], {j, 1, n}]
tc2[n_, s_] := Sum[ j^(-1 / 2) Sin[s Log[j]], {j, 1, n}]
tc3[n_, s_] :=  tc[n, s] + ss[n, s] tc2[n, s]
tc4[n_, s_] := Sum[ j^(-1 / 2) Cos[s Log[j]], {j, 1, n}] +
  Tan[s Log[n] + ArcTan[1 / (2 s)]] Sum[ j^(-1 / 2) Sin[s Log[j]], {j, 1, n}]

DiscretePlot[Re@tc3[n, N@Im@ZetaZero@1 + .2 I], {n, 1, 500}]
```




```
tc4[10 000, .2 I + 5]

0.736957 - 0.21064 ⅈ

Zeta[.7 + 5 I]

0.726329 + 0.209087 ⅈ

1 / 2 + (.2 I + 5) I

0.3 + 5. ⅈ

((.7 + 5 I) - 1 / 2) / I

5. - 0.2 ⅈ
```

```
sr[t_] := 1 / 2 + t I

sr[.2 I + 5]

0.3 + 5. i

FullSimplify[(s + I / 2) / I]

1
— - i s
2

1 / 2 - (.7 + 5 I) I

5.5 - 0.7 i
```

```
rr[n_, s_, t_] := ((1 - s) n^s HarmonicNumber[n, s] - (1 - t) n^t HarmonicNumber[n, t]) /
  ((1 - s) n^s - (1 - t) n^t )
rr2[n_, s_, t_] := ((1 - s) n^s HarmonicNumber[n, s] - (1 - t) n^t HarmonicNumber[n, t]) /
  ((1 - s) n^s )
rr3[n_, s_, t_] := HarmonicNumber[n, s] - (1 - t) n^t / ((1 - s) n^s ) HarmonicNumber[n, t]
rr4a[n_, m_, d_] := ((1 - (m - d)) n^(m - d) HarmonicNumber[n, (m - d)] -
     (1 - (m + d)) n^(m + d) HarmonicNumber[n, (m + d)]) /
  ((1 - (m - d)) n^(m - d) - (1 - (m + d)) n^(m + d) )
rr4[n_, s_, t_] := rr4a[n, (s + t) / 2, (s - t) / 2]
rr5a[n_, m_, d_] := ((1 - m + d) E^((m - d) Log[n]) HarmonicNumber[n, m - d] -
     (1 - m - d) E^((m + d) Log[n]) HarmonicNumber[n, m + d]) /
  ((1 - m + d) E^((m - d) Log[n]) - (1 - m - d) E^((m + d) Log[n]) )
rr5[n_, s_, t_] := rr5a[n, (s + t) / 2, (s - t) / 2]
rr6a[n_, m_, d_] :=
 (((1 - m + d) / (1 - m - d))^(1 / 2) E^((m - d) Log[n]) HarmonicNumber[n, m - d] -
     ((1 - m + d) / (1 - m - d))^(-1 / 2) E^((m + d) Log[n]) HarmonicNumber[n, m + d]) /
  (((1 - m + d) / (1 - m - d))^(1 / 2) E^((m - d) Log[n]) -
     ((1 - m + d) / (1 - m - d))^(-1 / 2) E^((m + d) Log[n]) )
rr6[n_, s_, t_] := rr6a[n, (s + t) / 2, (s - t) / 2]
rr7a[n_, m_, d_] :=
 (E^Log[((1 - m + d) / (1 - m - d))^(1 / 2)] E^((m - d) Log[n]) HarmonicNumber[n, m - d] -
    E^Log[((1 - m + d) / (1 - m - d))^(-1 / 2)] E^((m + d) Log[n]) HarmonicNumber[n, m + d]) /
  (E^Log[((1 - m + d) / (1 - m - d))^(1 / 2)] E^((m - d) Log[n]) -
    E^Log[((1 - m + d) / (1 - m - d))^(-1 / 2)] E^((m + d) Log[n]) )
rr7[n_, s_, t_] := rr7a[n, (s + t) / 2, (s - t) / 2]
rr8a[n_, m_, d_] := (E^ArcTanh[d / (1 - m)] E^((m - d) Log[n]) HarmonicNumber[n, m - d] -
    E^-ArcTanh[d / (1 - m)] E^((m + d) Log[n]) HarmonicNumber[n, m + d]) /
  (E^ArcTanh[d / (1 - m)] E^((m - d) Log[n]) - E^-ArcTanh[d / (1 - m)] E^((m + d) Log[n]) )
rr8[n_, s_, t_] := rr8a[n, (s + t) / 2, (s - t) / 2]
rr9a[n_, m_, d_] := (E^((m - d) Log[n] + ArcTanh[d / (1 - m)]) HarmonicNumber[n, m - d] -
    E^((m + d) Log[n] - ArcTanh[d / (1 - m)]) HarmonicNumber[n, m + d]) /
  (E^((m - d) Log[n] + ArcTanh[d / (1 - m)]) - E^((m + d) Log[n] - ArcTanh[d / (1 - m)]) )
rr9[n_, s_, t_] := rr9a[n, (s + t) / 2, (s - t) / 2]
rr10a[n_, m_, d_] :=
 Sum[ (E^((m - d) Log[n / j] + ArcTanh[d / (1 - m)]) - E^((m + d) Log[n / j] - ArcTanh[d / (1 - m)]) ) /
   (E^((m - d) Log[n] + ArcTanh[d / (1 - m)]) - E^((m + d) Log[n] - ArcTanh[d / (1 - m)]) ), {j,
   1, n}]
rr10[n_, s_, t_] := rr9a[n, (s + t) / 2, (s - t) / 2]
```

```
rr11a[n_, m_, d_] :=
 Sum[j^-m Sinh[ArcTanh[d/(-1 + m)] + d Log[n/j]] / Sinh[ArcTanh[d/(-1 + m)] + d Log[n]], {j, 1, n}]
rr11[n_, s_, t_] := rr11a[n, (s + t) / 2, (s - t) / 2]
rr12a[n_, m_, d_] :=
 Sum[j^-m (Sinh[ArcTanh[d/(-1 + m)] + d Log[n]] Cosh[d Log[j]] - Cosh[ArcTanh[d/(-1 + m)] + d Log[n]]
      Sinh[d Log[j]]) / Sinh[ArcTanh[d/(-1 + m)] + d Log[n]], {j, 1, n}]
rr12[n_, s_, t_] := rr12a[n, (s + t) / 2, (s - t) / 2]
rr13a[n_, m_, d_] := (1 / Sinh[ArcTanh[d/(-1 + m)] + d Log[n]])
  (Sinh[ArcTanh[d/(-1 + m)] + d Log[n]] Sum[j^-m Cosh[d Log[j]], {j, 1, n}] -
    Cosh[ArcTanh[d/(-1 + m)] + d Log[n]] Sum[j^-m Sinh[d Log[j]], {j, 1, n}])
rr13[n_, s_, t_] := rr13a[n, (s + t) / 2, (s - t) / 2]
rr14a[n_, m_, d_] := Sum[j^-m Cosh[d Log[j]], {j, 1, n}] -
  Coth[ArcTanh[d/(-1 + m)] + d Log[n]] Sum[j^-m Sinh[d Log[j]], {j, 1, n}]
rr14[n_, s_, t_] := rr14a[n, (s + t) / 2, (s - t) / 2]
rr14[100 000, .2 + 6 I, .2 - 6 I]
```

0.820972 + 0. i

```
rr14[n, s + t I, s - t I]
```

$$\sum_{j=1}^{n} j^{-s} \operatorname{Cos}[t \operatorname{Log}[j]] + i \operatorname{Cot}\left[\operatorname{ArcTan}\left[\frac{t}{-1+s}\right] + t \operatorname{Log}[n]\right] \sum_{j=1}^{n} i\, j^{-s} \operatorname{Sin}[t \operatorname{Log}[j]]$$

```
Log[((1 + d - m) / (1 - m - d))^(1 / 2)] /. m → .4 /. d → .2
```

0.346574

```
ArcTanh[d / (1 - m)] /. m → .4 /. d → .2
```

0.346574

```
(1 / 2) Log[((1 + d - m) / (1 - m - d))] /. m → .4 /. d → .2
```

0.346574

```
(1 / 2) (Log[((1 - m) + d)] - Log[(1 - m) - d]) /. m → .4 /. d → .2
```

0.346574

```
(1 / 2) (Log[(1 + d / (1 - m))] - Log[1 - d / (1 - m)]) /. m → .4 /. d → .2
```

0.346574

ExpToTrig[(E^((m - d) Log[n / j] + ArcTanh[d / (1 - m)]) - E^((m + d) Log[n / j] - ArcTanh[d / (1 - m)])) / (E^((m - d) Log[n] + ArcTanh[d / (1 - m)]) - E^((m + d) Log[n] - ArcTanh[d / (1 - m)]))]

$$\left(\text{Cosh}\left[\text{ArcTanh}\left[\frac{d}{1-m}\right]+(-d+m)\,\text{Log}\left[\frac{n}{j}\right]\right]-\text{Cosh}\left[\text{ArcTanh}\left[\frac{d}{1-m}\right]-(d+m)\,\text{Log}\left[\frac{n}{j}\right]\right]+\text{Sinh}\left[\text{ArcTanh}\left[\frac{d}{1-m}\right]+(-d+m)\,\text{Log}\left[\frac{n}{j}\right]\right]+\text{Sinh}\left[\text{ArcTanh}\left[\frac{d}{1-m}\right]-(d+m)\,\text{Log}\left[\frac{n}{j}\right]\right]\right)\Big/\left(\text{Cosh}\left[\text{ArcTanh}\left[\frac{d}{1-m}\right]+(-d+m)\,\text{Log}[n]\right]-\text{Cosh}\left[\text{ArcTanh}\left[\frac{d}{1-m}\right]-(d+m)\,\text{Log}[n]\right]+\text{Sinh}\left[\text{ArcTanh}\left[\frac{d}{1-m}\right]+(-d+m)\,\text{Log}[n]\right]+\text{Sinh}\left[\text{ArcTanh}\left[\frac{d}{1-m}\right]-(d+m)\,\text{Log}[n]\right]\right)$$

$$\textbf{FullSimplify}\left[\left(\text{Cosh}\left[\text{ArcTanh}\left[\frac{d}{1-m}\right]+(-d+m)\,\text{Log}\left[\frac{n}{j}\right]\right]-\text{Cosh}\left[\text{ArcTanh}\left[\frac{d}{1-m}\right]-(d+m)\,\text{Log}\left[\frac{n}{j}\right]\right]+\text{Sinh}\left[\text{ArcTanh}\left[\frac{d}{1-m}\right]+(-d+m)\,\text{Log}\left[\frac{n}{j}\right]\right]+\text{Sinh}\left[\text{ArcTanh}\left[\frac{d}{1-m}\right]-(d+m)\,\text{Log}\left[\frac{n}{j}\right]\right]\right)\Big/\left(\text{Cosh}\left[\text{ArcTanh}\left[\frac{d}{1-m}\right]+(-d+m)\,\text{Log}[n]\right]-\text{Cosh}\left[\text{ArcTanh}\left[\frac{d}{1-m}\right]-(d+m)\,\text{Log}[n]\right]+\text{Sinh}\left[\text{ArcTanh}\left[\frac{d}{1-m}\right]+(-d+m)\,\text{Log}[n]\right]+\text{Sinh}\left[\text{ArcTanh}\left[\frac{d}{1-m}\right]-(d+m)\,\text{Log}[n]\right]\right)\right]$$

$$n^{-m}\left(\frac{n}{j}\right)^m \text{Csch}\left[\text{ArcTanh}\left[\frac{d}{-1+m}\right]+d\,\text{Log}[n]\right]\,\text{Sinh}\left[\text{ArcTanh}\left[\frac{d}{-1+m}\right]+d\,\text{Log}\left[\frac{n}{j}\right]\right]$$

$$\text{Cosh}\left[\text{ArcTanh}\left[\frac{d}{-1+m}\right]+d\,\text{Log}[n]\right]\Big/\,\text{Sinh}\left[\text{ArcTanh}\left[\frac{d}{-1+m}\right]+d\,\text{Log}[n]\right]$$

$$\text{Coth}\left[\text{ArcTanh}\left[\frac{d}{-1+m}\right]+d\,\text{Log}[n]\right]$$

Sum[j^{-m} Cosh[d Log[j]], {j, 1, n}] - Coth[ArcCoth[(m - 1) / d] + d Log[n]] Sum[j^{-m} Sinh[d Log[j]], {j, 1, n}] /. m → 0

$$\sum_{j=1}^{n}\text{Cosh}[d\,\text{Log}[j]]+\text{Coth}\left[\text{ArcCoth}\left[\frac{1}{d}\right]-d\,\text{Log}[n]\right]\sum_{j=1}^{n}\text{Sinh}[d\,\text{Log}[j]]$$

ArcTanh[d / (m - 1)] /. m → .4 /. d → .3

-0.549306

ArcCoth[(m - 1) / d] /. m → .4 /. d → .3

-0.549306

Limit[Tanh[0 Log[n]], n → Infinity]

0

$$\textbf{FullSimplify}\left[\sum_{j=1}^{n} j^{-s}\,\text{Cos}[t\,\text{Log}[j]]+i\,\text{Cot}\left[\text{ArcTan}\left[\frac{t}{-1+s}\right]+t\,\text{Log}[n]\right]\sum_{j=1}^{n} i\,j^{-s}\,\text{Sin}[t\,\text{Log}[j]]\right]$$

$$\sum_{j=1}^{n} j^{-s}\,\text{Cos}[t\,\text{Log}[j]]+i\,\text{Cot}\left[\text{ArcTan}\left[\frac{t}{-1+s}\right]+t\,\text{Log}[n]\right]\sum_{j=1}^{n} i\,j^{-s}\,\text{Sin}[t\,\text{Log}[j]]$$

$$\text{ax}[n\_, s\_, t\_] := \sum_{j=1}^{n} j^{-s}\,\text{Cos}[t\,\text{Log}[j]] + i\,\text{Cot}\left[\text{ArcTan}\left[\frac{t}{-1+s}\right] + t\,\text{Log}[n]\right] \sum_{j=1}^{n} i\, j^{-s}\,\text{Sin}[t\,\text{Log}[j]]$$

$$\text{ax2}[n\_, s\_, t\_] := \sum_{j=1}^{n} N[j^{-s}\,\text{Cos}[t\,\text{Log}[j]]] - \text{Cot}\left[\text{ArcTan}\left[\frac{t}{-1+s}\right] + t\,\text{Log}[n]\right] \sum_{j=1}^{n} N[j^{-s}\,\text{Sin}[t\,\text{Log}[j]]]$$

$$\text{ax3}[n\_, s\_, t\_] := \left((1 - s + i\,t)\,\text{HarmonicNumber}[n, s - i\,t] + n^{2\,i\,t}\,(-1 + s + i\,t)\,\text{HarmonicNumber}[n, s + i\,t]\right) \Big/ \left(1 - s + n^{2\,i\,t}\,(-1 + s + i\,t) + i\,t\right)$$

```
ax3[1 000 000 000 000, .5, N@Im@ZetaZero@1 + 3]
```

$3.04084 + 3.67284 \times 10^{-12}\, i$

```
Zeta[N@ZetaZero@1 + 3 I]
```

$2.053 + 0.7817\, i$

$$\text{TrigToExp}\left[\sum_{j=1}^{n} j^{-s}\,\text{Cos}[t\,\text{Log}[j]] + i\,\text{Cot}\left[\text{ArcTan}\left[\frac{t}{-1+s}\right] + t\,\text{Log}[n]\right] \sum_{j=1}^{n} i\, j^{-s}\,\text{Sin}[t\,\text{Log}[j]]\right]$$

$$\text{FullSimplify}\Bigg[\text{Sum}\left[\left(\frac{1}{2} j^{-s-i\,t} + \frac{1}{2} j^{-s+i\,t}\right), \{j, 1, n\}\right] + \left(\left(e^{\frac{1}{2}\left(\text{Log}\left[1-\frac{i\,t}{-1+s}\right]-\text{Log}\left[1+\frac{i\,t}{-1+s}\right]\right)} n^{-i\,t} + e^{\frac{1}{2}\left(-\text{Log}\left[1-\frac{i\,t}{-1+s}\right]+\text{Log}\left[1+\frac{i\,t}{-1+s}\right]\right)} n^{i\,t}\right) \sum_{j=1}^{n}\left(-\frac{1}{2} j^{-s-i\,t} + \frac{1}{2} j^{-s+i\,t}\right)\right) \Big/ \left(e^{\frac{1}{2}\left(\text{Log}\left[1-\frac{i\,t}{-1+s}\right]-\text{Log}\left[1+\frac{i\,t}{-1+s}\right]\right)} n^{-i\,t} - e^{\frac{1}{2}\left(-\text{Log}\left[1-\frac{i\,t}{-1+s}\right]+\text{Log}\left[1+\frac{i\,t}{-1+s}\right]\right)} n^{i\,t}\right)\Bigg]$$

$$\left((1 - s + i\,t)\,\text{HarmonicNumber}[n, s - i\,t] + n^{2\,i\,t}\,(-1 + s + i\,t)\,\text{HarmonicNumber}[n, s + i\,t]\right) \Big/ \left(1 - s + n^{2\,i\,t}\,(-1 + s + i\,t) + i\,t\right)$$

$$\text{rr14a}[n\_, m\_, d\_] := \text{Sum}[j^{-m}\,\text{Cosh}[d\,\text{Log}[j]], \{j, 1, n\}] - \text{Coth}\left[\text{ArcTanh}\left[\frac{d}{-1+m}\right] + d\,\text{Log}[n]\right] \text{Sum}[j^{-m}\,\text{Sinh}[d\,\text{Log}[j]], \{j, 1, n\}]$$

$$\text{rr14a2}[n\_, m\_, d\_] := \left\{\text{Sum}[j^{-m}\,\text{Cosh}[d\,\text{Log}[j]], \{j, 1, n\}], -\text{Coth}\left[\text{ArcTanh}\left[\frac{d}{-1+m}\right] + d\,\text{Log}[n]\right], \text{Sum}[j^{-m}\,\text{Sinh}[d\,\text{Log}[j]], \{j, 1, n\}]\right\}$$

```
rr14[n_, s_, t_] := rr14a[n, (s + t) / 2, (s - t) / 2]
rr14a2[1000., 0, N@ZetaZero@1 + .4]
```

$\{-6696.21 + 16\,263.8\, i,\ -0.999994 - 4.80011 \times 10^{-6}\, i,\ -6696.44 + 16\,263.9\, i\}$

```
rr14a[n, 0, s]
```

$$\sum_{j=1}^{n} \text{Cosh}[s\,\text{Log}[j]] + \text{Coth}[\text{ArcTanh}[s] - s\,\text{Log}[n]] \sum_{j=1}^{n} \text{Sinh}[s\,\text{Log}[j]]$$

$$\text{so}[n\_, s\_] := \sum_{j=1}^{n} \text{Cosh}[s\,\text{Log}[j]] + \text{Coth}[\text{ArcTanh}[s] - s\,\text{Log}[n]] \sum_{j=1}^{n} \text{Sinh}[s\,\text{Log}[j]]$$

$$\text{so2}[n\_, s\_] := \sum_{j=1}^{n} \left(\frac{j^{-s}}{2} + \frac{j^{s}}{2} + \frac{e^{\frac{1}{2}(-\text{Log}[1-s]+\text{Log}[1+s])} n^{-s} + e^{\frac{1}{2}(\text{Log}[1-s]-\text{Log}[1+s])} n^{s}}{e^{\frac{1}{2}(-\text{Log}[1-s]+\text{Log}[1+s])} n^{-s} - e^{\frac{1}{2}(\text{Log}[1-s]-\text{Log}[1+s])} n^{s}} \left(-\frac{j^{-s}}{2} + \frac{j^{s}}{2}\right) \right)$$

$$\text{so3}[n\_, s\_] := \sum_{j=1}^{n} \left(\frac{j^{-s} \left(n^{2s}(-1+s) + j^{2s}(1+s)\right)}{1 + n^{2s}(-1+s) + s} \right)$$

FullSimplify[so3[n, s] / so3[n, 1 - s]]

$$\left(\left(n^{2s}(-2+s) + n^{2} s\right) \left((1+s)\,\text{HarmonicNumber}[n, -s] + n^{2s}(-1+s)\,\text{HarmonicNumber}[n, s]\right)\right) \Big/ \left(\left(1 + n^{2s}(-1+s) + s\right) \left(n^{2} s\,\text{HarmonicNumber}[n, 1-s] + n^{2s}(-2+s)\,\text{HarmonicNumber}[n, -1+s]\right)\right)$$

FullSimplify[so3[n, s] / so3[n, 1 - s / 2]]

$$\left(\left(n^{s}(-4+s) + n^{2} s\right) \left((1+s)\,\text{HarmonicNumber}[n, -s] + n^{2s}(-1+s)\,\text{HarmonicNumber}[n, s]\right)\right) \Big/ \left(\left(1 + n^{2s}(-1+s) + s\right) \left(n^{2} s\,\text{HarmonicNumber}\left[n, 1-\frac{s}{2}\right] + n^{s}(-4+s)\,\text{HarmonicNumber}\left[n, -1+\frac{s}{2}\right]\right)\right)$$

$$\text{rr14b}[n\_, s\_] := \sum_{j=1}^{n} \text{Cosh}[s\,\text{Log}[j]] + \text{Coth}[\text{ArcTanh}[s] - s\,\text{Log}[n]] \sum_{j=1}^{n} \text{Sinh}[s\,\text{Log}[j]]$$

$$\text{rr14c}[n\_, s\_] := \sum_{j=1}^{n} \left(\frac{j^{-s}}{2} + \frac{j^{s}}{2}\right) + \text{Coth}[\text{ArcTanh}[s] - s\,\text{Log}[n]] \sum_{j=1}^{n} \left(-\frac{j^{-s}}{2} + \frac{j^{s}}{2}\right)$$

$$\text{rr14d}[n\_, s\_] := \sum_{j=1}^{n} \left(\frac{j^{-s}}{2} + \frac{j^{s}}{2}\right) + \left(-1 + \frac{2(1+s)}{1 + n^{2s}(-1+s) + s}\right) \sum_{j=1}^{n} \left(-\frac{j^{-s}}{2} + \frac{j^{s}}{2}\right)$$

$$\text{rr14d2}[n\_, s\_] := \sum_{j=1}^{n} \left(\frac{j^{-s}}{2} + \frac{j^{s}}{2}\right) + \left(-1 + \frac{2}{1 + n^{2s}(-1+s)/(1+s)}\right) \sum_{j=1}^{n} \left(-\frac{j^{-s}}{2} + \frac{j^{s}}{2}\right)$$

$$\text{rr14e}[n\_, s\_] := \frac{(1+s)\,\text{HarmonicNumber}[n, -s] + n^{2s}(-1+s)\,\text{HarmonicNumber}[n, s]}{1 + n^{2s}(-1+s) + s}$$

$$\text{rr14f}[n\_, s\_] := \frac{(1+s)\,\text{HarmonicNumber}[n, -s]}{(s+1) + n^{2s}(-1+s)} + \frac{n^{2s}(-1+s)\,\text{HarmonicNumber}[n, s]}{(s+1) + n^{2s}(-1+s)}$$

$$\text{rr14g}[n\_, s\_] := \frac{\text{HarmonicNumber}[n, -s]}{1 + n^{2s}(-1+s)/(s+1)} + \frac{\text{HarmonicNumber}[n, s]}{n\hat{}(-2s)(s+1)/(s-1) + 1}$$

$$\text{rr14ga}[n\_, s\_] := \left\{\frac{\text{HarmonicNumber}[n, -s]}{1 + n^{2s}(s-1)/(s+1)}, \frac{\text{HarmonicNumber}[n, s]}{n\hat{}(-2s)(s+1)/(s-1) + 1}\right\}$$

TrigToExp[Sinh[s Log[j]]]

$$-\frac{j^{-s}}{2} + \frac{j^{s}}{2}$$

```
rr14d2[100000, .3]
```

-0.919274

```
Zeta[.3]
```

-0.904559

```
FullSimplify[TrigToExp[Coth[ArcTanh[s] - s Log[n]]]]
```

$$-1+\frac{2\,(1+s)}{1+n^{2s}\,(-1+s)+s}$$

$$\text{FullSimplify}\left[\sum_{j=1}^{n}\left(\frac{j^{-s}}{2}+\frac{j^{s}}{2}\right)+\left(-1+\frac{2\,(1+s)}{1+n^{2s}\,(-1+s)+s}\right)\sum_{j=1}^{n}\left(-\frac{j^{-s}}{2}+\frac{j^{s}}{2}\right)\right]$$

$$\frac{(1+s)\,\text{HarmonicNumber}[n,-s]+n^{2s}\,(-1+s)\,\text{HarmonicNumber}[n,s]}{1+n^{2s}\,(-1+s)+s}$$

$$\text{FullSimplify}\left[\frac{(1+s)\,\text{HarmonicNumber}[n,-s]}{(s+1)+n^{2s}\,(-1+s)}\right]$$

$$\frac{(1+s)\,\text{HarmonicNumber}[n,-s]}{1+n^{2s}\,(-1+s)+s}$$

$$\text{FullSimplify}\left[\frac{n^{2s}\,(-1+s)\,\text{HarmonicNumber}[n,s]}{(s+1)+n^{2s}\,(-1+s)}\right]$$

$$\frac{n^{2s}\,(-1+s)\,\text{HarmonicNumber}[n,s]}{1+n^{2s}\,(-1+s)+s}$$

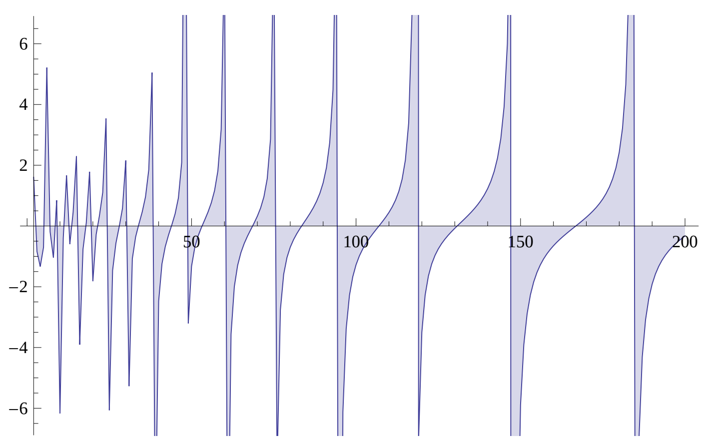
$$\text{rr14a}[n\_,m\_,d\_]:=\text{Sum}[j^{-m}\,\text{Cosh}[d\,\text{Log}[j]],\{j,1,n\}]-\text{Coth}\left[\text{ArcTanh}\left[\frac{d}{-1+m}\right]+d\,\text{Log}[n]\right]\text{Sum}[j^{-m}\,\text{Sinh}[d\,\text{Log}[j]],\{j,1,n\}]$$

```
rr14a[n, s, t]
```

$$\sum_{j=1}^{n}j^{-s}\,\text{Cosh}[t\,\text{Log}[j]]-\text{Coth}\left[\text{ArcTanh}\left[\frac{t}{-1+s}\right]+t\,\text{Log}[n]\right]\sum_{j=1}^{n}j^{-s}\,\text{Sinh}[t\,\text{Log}[j]]$$

$$\text{pl}[n\_,s\_,t\_]:=\left(\sum_{j=1}^{n}j^{-s}\,\text{Cosh}[t\,\text{Log}[j]]\right)\Big/\left(\sum_{j=1}^{n}j^{-s}\,\text{Sinh}[t\,\text{Log}[j]]\right)$$

```
DiscretePlot[ Im@pl[n, .6, N@ZetaZero@1 - 1 / 2], {n, 2, 200}]
```



```
pl[100, .5, N@ZetaZero@1 - 1 / 2]
```

0. − 0.978974 ⅈ

$$\text{rr14g}[n\_, s\_] := \frac{\text{HarmonicNumber}[n, -s]}{1 + n^{2s}(-1+s)/(s+1)} + \frac{\text{HarmonicNumber}[n, s]}{n\char`^(-2s)(s+1)/(s-1)+1}$$

$$\text{rr14gx}[n\_, s\_] := \left\{\frac{1}{1 + n^{2s}(-1+s)/(s+1)}, \text{HarmonicNumber}[n, -s], \frac{1}{n\char`^(-2s)(s+1)/(s-1)+1}, \text{HarmonicNumber}[n, s]\right\}$$

```
FullSimplify[rr14g[n, s] / rr14g[n , 1 - s]]
```

$$\left(\left(n^{2s}(-2+s)+n^2 s\right)\left((1+s)\,\text{HarmonicNumber}[n,-s]+n^{2s}(-1+s)\,\text{HarmonicNumber}[n,s]\right)\right)\Big/\left(\left(1+n^{2s}(-1+s)+s\right)\left(n^2 s\,\text{HarmonicNumber}[n,1-s]+n^{2s}(-2+s)\,\text{HarmonicNumber}[n,-1+s]\right)\right)$$

```
fa[s_] := 2^s Pi^(s - 1) Sin[ Pi s / 2] Gamma[ 1 - s]
fb[n_, s_] :=
```

$$\left(\left(n^{2s}(-2+s)+n^2 s\right)\left((1+s)\,\text{HarmonicNumber}[n,-s]+n^{2s}(-1+s)\,\text{HarmonicNumber}[n,s]\right)\right)\Big/\left(\left(1+n^{2s}(-1+s)+s\right)\left(n^2 s\,\text{HarmonicNumber}[n,1-s]+n^{2s}(-2+s)\,\text{HarmonicNumber}[n,-1+s]\right)\right)$$

$$\text{fc}[n\_, s\_] := \left(\left(n^{2s}(-2+s)+n^2 s\right)(1+s)\,\text{HarmonicNumber}[n,-s]\right)\Big/\left(\left(1+n^{2s}(-1+s)+s\right)\left(n^2 s\,\text{HarmonicNumber}[n,1-s]+n^{2s}(-2+s)\,\text{HarmonicNumber}[n,-1+s]\right)\right) + \left(\left(n^{2s}(-2+s)+n^2 s\right)n^{2s}(-1+s)\,\text{HarmonicNumber}[n,s]\right)\Big/\left(\left(1+n^{2s}(-1+s)+s\right)\left(n^2 s\,\text{HarmonicNumber}[n,1-s]+n^{2s}(-2+s)\,\text{HarmonicNumber}[n,-1+s]\right)\right)$$

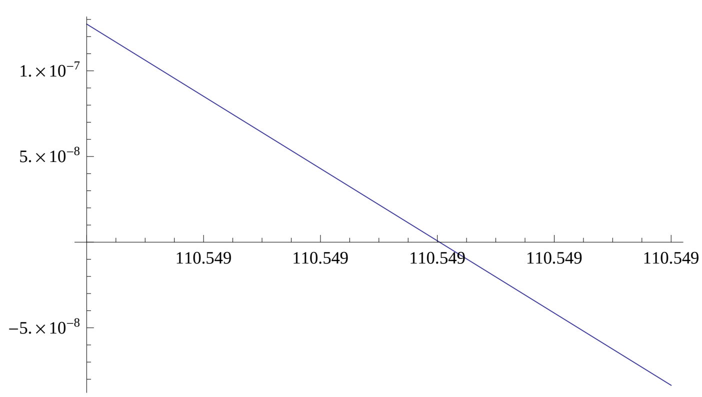
```
fc[10 000, .3]
```

0.33656

```
fb[10 000, .3]
```

0.33656

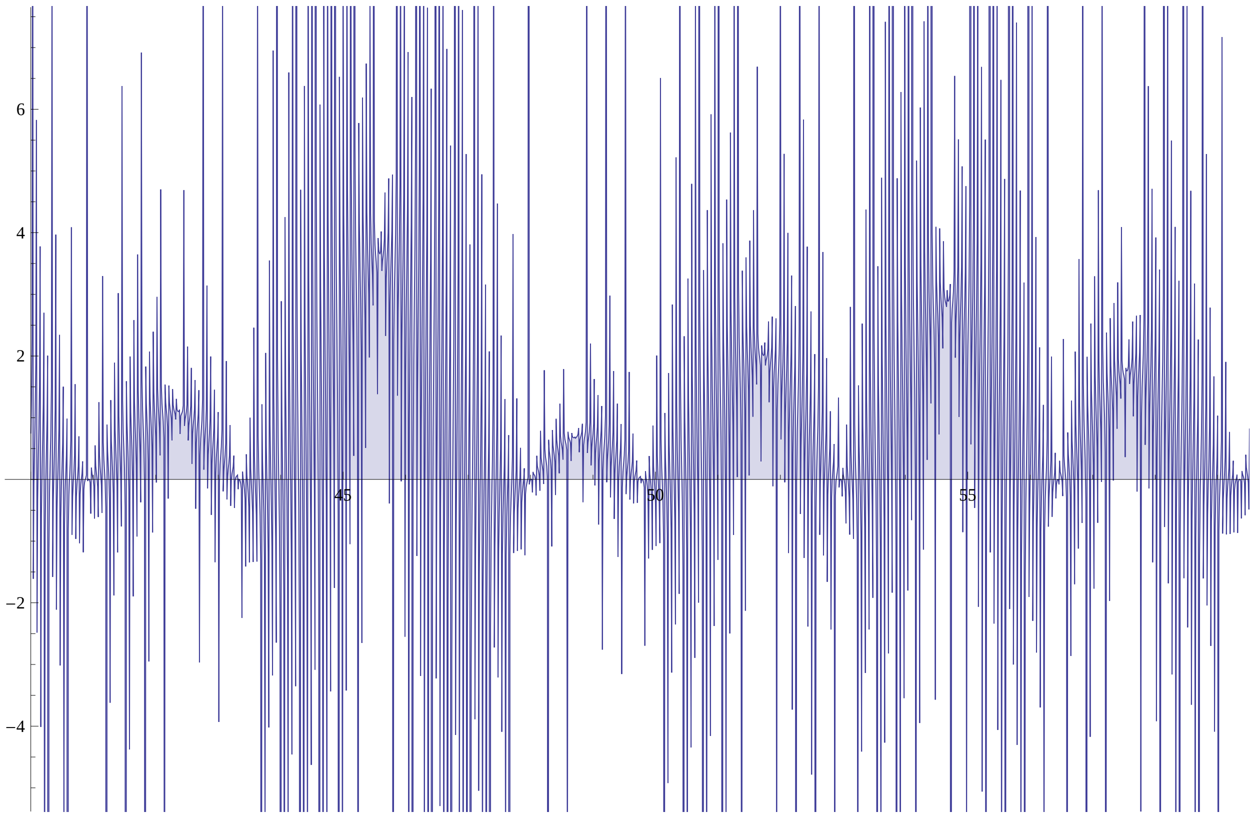
```
Plot[ Re@Zeta[.8 + s I], {s, 110.548996, 110.548997}]
```




```
Zeta[.8 + 110.548996 I]
```

$1.27293 \times 10^{-7} - 0.929295\,ⅈ$

```
ps[n_, s_] := Re[(1 - Tanh[ArcTanh[1 / (2 s)] - s Log[n]]) HarmonicNumber[n, 1 / 2 + s]]
psf[n_, s_] := Re[(1 - Tanh[ArcTanh[1 / (2 s - 1)] - (s - 1 / 2) Log[n]]) HarmonicNumber[n, s]]
psx[n_, s_] := (1 - Tanh[ArcTanh[1 / (2 s)] - s Log[n]]) HarmonicNumber[n, 1 / 2 + s]
psx2[n_, s_] := ((1 - Tanh[ArcTanh[1 / (2 s)] - s Log[n]]) HarmonicNumber[n, 1 / 2 + s] +
    (1 + Tanh[ArcTanh[1 / (2 s)] - s Log[n]]) HarmonicNumber[n, 1 / 2 - s]) / 2
psx2a[n_, s_] := {(1 - Tanh[ArcTanh[1 / (2 s)] - s Log[n]]) HarmonicNumber[n, 1 / 2 + s] / 2,
  (1 + Tanh[ArcTanh[1 / (2 s)] - s Log[n]]) HarmonicNumber[n, 1 / 2 - s] / 2}
DiscretePlot[psf[10 000 000 000 000 000 000 000, .5 + s I], {s, 40, 60, .01}]
```




```
psx2[10 000 000 000, .8 + 110.548996 I - 1 / 2]
```

$1.3437 \times 10^{-6} - 0.929293\, i$

```
psx3[n_, s_] := ((1 - Tanh[ArcTanh[1 / (2 s)] - s Log[n]]) HarmonicNumber[n, 1 / 2 + s] +
    (1 + Tanh[ArcTanh[1 / (2 s)] - s Log[n]]) HarmonicNumber[n, 1 / 2 - s]) / 2
```

$$\text{f2[n\_, s\_] :=} \left(1 + \text{Tanh}\left[\text{ArcTanh}\left[\frac{1}{1-2s}\right] + \left(s - \frac{1}{2}\right)\text{Log[n]}\right]\right) \Big/ 2\,\text{HarmonicNumber[n, s]}$$

$$\text{f3[n\_, s\_] :=} \left(1 - n^{1-2s}\frac{s}{1-s}\right)\text{\^{}-1 HarmonicNumber[n, s]}$$

$$\text{f3a[n\_, s\_] :=} \left\{\left(1 - n^{1-2s}\frac{s}{1-s}\right)\text{\^{}-1, HarmonicNumber[n, s]}\right\}$$

```
psx4[n_, s_] := f2[n, s] + f2[n, 1 - s]
psx5[n_, s_] := f3[n, s] + f3[n, 1 - s]
```

```
psx3[n, 1 / 2 - s]
```

$$\frac{1}{2}\left(\text{HarmonicNumber}[n,1-s]\left(1-\text{Tanh}\left[\text{ArcTanh}\left[\frac{1}{2\left(\frac{1}{2}-s\right)}\right]-\left(\frac{1}{2}-s\right)\text{Log}[n]\right]\right)+\text{HarmonicNumber}[n,s]\left(1+\text{Tanh}\left[\text{ArcTanh}\left[\frac{1}{2\left(\frac{1}{2}-s\right)}\right]-\left(\frac{1}{2}-s\right)\text{Log}[n]\right]\right)\right)$$

```
f2[n, 1 - s]
```

$$\frac{1}{2}\text{HarmonicNumber}[n,1-s]\left(1+\text{Tanh}\left[\text{ArcTanh}\left[\frac{1}{1-2(1-s)}\right]+\left(\frac{1}{2}-s\right)\text{Log}[n]\right]\right)$$

```
f3a[10 000 000 000 000, N@ZetaZero@1]
```

{0.5 - 0.739634 ⅈ, 185 231. - 125 218. ⅈ}

```
psx5[100 000 000 000, N@ZetaZero@1 + 2 I]
```

0.710167 + 0. ⅈ

```
Zeta[.8 + 12 I]
```

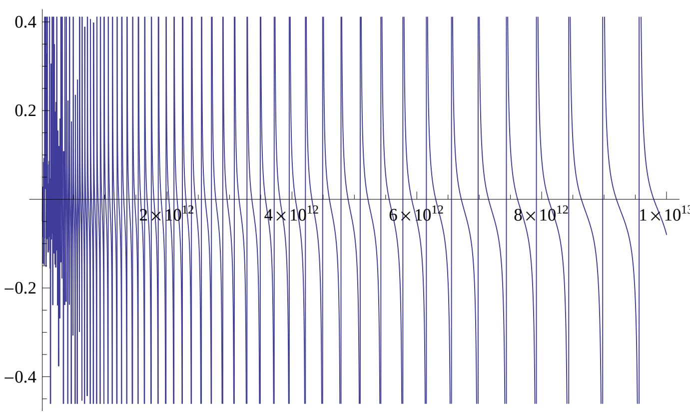
0.987421 - 0.602699 ⅈ

```
FullSimplify[TrigToExp[(1 + Tanh[ArcTanh[1/(1 - 2 s)] + (s - 1/2) Log[n]]) / 2]]
```

$$\frac{1}{1+\frac{n^{1-2s}\,s}{-1+s}}$$

$$\frac{(1-s)\,n\text{^}s}{(1-s)\,n\text{^}s-n^{1-s}\,s}$$

```
Plot[Re[f3[n, N@ZetaZero@10 + .1 I]], {n, 1, 10 000 000 000 000}]
```




```
al[s_] := 1 / (1 / 2 Pi^(-s / 2) Gamma[s / 2] s (s + 1))
al2[s_] := (1 / 2) s (s - 1) Pi^(-s / 2) Gamma[1 / 2 s]
f4[n_, s_] := (al[s] - al[1 - s] n^(1 - 2 s) s / (1 - s))^-1 HarmonicNumber[n, s]
xi[n_, s_] := f4[n, s] + f4[n, 1 - s]
zt[n_, s_] := al[s] xi[n, s]
```

```
zt[10 000 000 000 000 000, .6 + 7 I]
```

1.02336 + 0.376035 ⅈ

```
Zeta[.6 + 7 I]
```

1.02297 + 0.375953 i

```
xi[100 000 000 000 000, .6 + 7 I]
```

0.151263 - 0.0378792 i

```
Zeta[.6 + 7 I] al2[.6 + 7 I]
```

0.152156 + 0.00540746 i

```
f4[10 000 000 000, N@ZetaZero@2]
```

$-2.73865 \times 10^{-10} - 0.260547\, i$

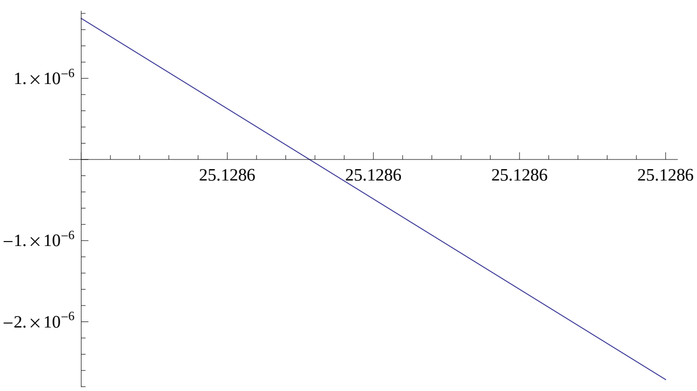
$$\texttt{f6[n\_, s\_] :=} \left(1 - n^{1-2s} \frac{s}{1-s}\right)\texttt{\^{}-1 HarmonicNumber[n, s]}$$

```
f6a[n_, s_] := n^(s - 1 / 2) (1 - s) HarmonicNumber[n, s]
f6b[n_, s_] := n^(s - 3 / 5) (1 - s) HarmonicNumber[n, s]
```

```
f6b[1 000 000 000 000 000 000 000, N@ZetaZero@300]
```

$2.51189 \times 10^{8} + 7.45058 \times 10^{-9}\, i$

```
Plot[ Re@Zeta[.53 + s I], {s, 25.12858, 25.1286}]
```



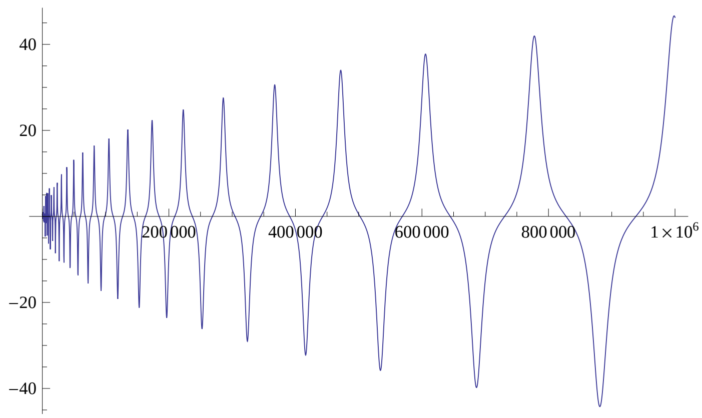

```
Zeta[.53 + 25.12858 I]
```

$1.73886 \times 10^{-6} + 0.165551\, i$

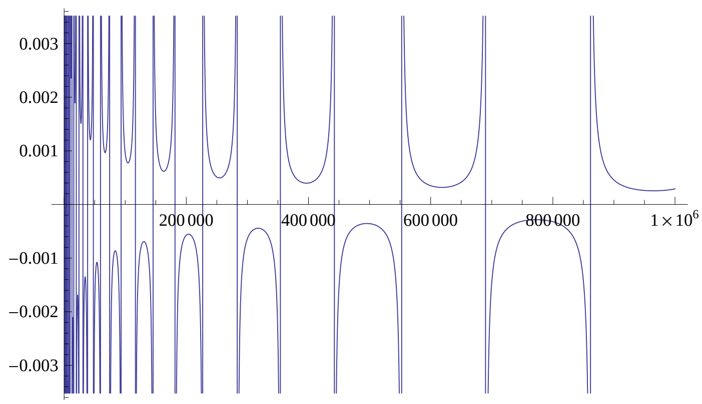
```
f6[1 000 000 000, .53 + 25.12858 I]
```

-285.103 + 565.901 i

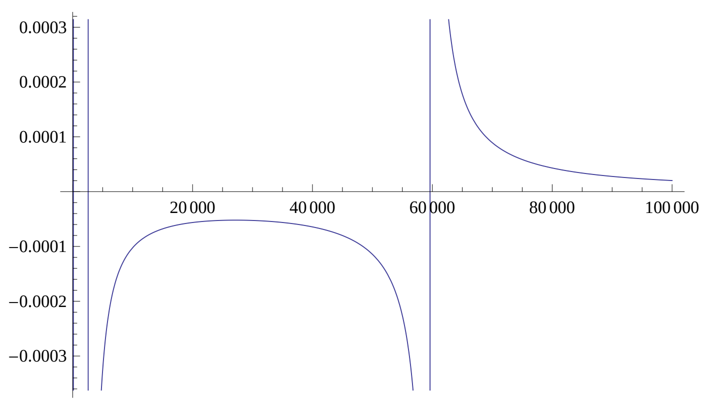
```
Plot[ Re@f6[n, .53 + 25.12858 I], {n, 1, 1 000 000}]
```




```
Plot[ Re@f6[n, N@ZetaZero@1], {n, 1, 1 000 000}]
```




```
Plot[1 / (x Cos[Log@x] ), {x, 1, 100 000}]
```




```
Limit[ 1 / (x Cos[Log[x]]), x → Infinity]
```

$$\text{Limit}\left[\frac{\text{Sec}[\text{Log}[x]]}{x}, x \to \infty\right]$$

```
rr14g[n_, s_] := HarmonicNumber[n, -s] / (1 + n^(2 s) (-1 + s) / (s + 1)) + HarmonicNumber[n, s] / (n^(-2 s) (s + 1) / (s - 1) + 1)
rr14gx[n_, s_] := {1 / (1 + n^(2 s) (-1 + s) / (s + 1)),
  HarmonicNumber[n, -s], 1 / (n^(-2 s) (s + 1) / (s - 1) + 1), HarmonicNumber[n, s]}
FullSimplify[rr14g[n, s] / rr14g[n , 1 - s]]
((n^(2 s) (-2 + s) + n^2 s) ((1 + s) HarmonicNumber[n, -s] + n^(2 s) (-1 + s) HarmonicNumber[n, s])) /
 ((1 + n^(2 s) (-1 + s) + s) (n^2 s HarmonicNumber[n, 1 - s] + n^(2 s) (-2 + s) HarmonicNumber[n, -1 + s]))
fa[s_] := 2^s Pi^(s - 1) Sin[ Pi s / 2] Gamma[ 1 - s]
fas[s_] := 1 / 2 s (s - 1) Pi^(-s / 2) Gamma[s / 2]
fax[s_] := fas[1 - s] / fas[s]
fba[n_, s_] :=
 ((n^(2 s) (s - 2) + n^2 s) ((s + 1) HarmonicNumber[n, -s] + n^(2 s) (s - 1) HarmonicNumber[n, s])) /
  (((s + 1) + n^(2 s) (s - 1)) (n^2 s HarmonicNumber[n, 1 - s] + n^(2 s) (s - 2) HarmonicNumber[n, s - 1]))
fba2[n_, s_] := ((n^(2 s) (s - 2) + n^2 s) Sum[ ((s + 1) j^s + n^(2 s) (s - 1) j^-s), {j, 1, n}]) /
  (((s + 1) + n^(2 s) (s - 1)) Sum[(n^2 s j^-(1 - s) + n^(2 s) (s - 2) j^-(s - 1)), {j, 1, n}])
fba3[n_, s_] := ((n^(s-1) (s - 2) + n^(1-s) s)
    Sum[ (n^(-s - 1) (s + 1) j^s + n^(s-1) (s - 1) j^-s), {j, 1, n}]) /
  ((n^(-s - 1) (s + 1) + n^(s-1) (s - 1)) Sum[(n^(1-s) s j^-(1 - s) + n^(s-1) (s - 2) j^-(s - 1)), {j, 1, n}])
fbb[n_, s_] := ((n^(2 s) (s - 2) + n^2 s) HarmonicNumber[n, 0 - s]) / ((1 + n^(2 s) (s - 1) / (s + 1))
     (n^2 s HarmonicNumber[n, 1 - s] + n^(2 s) (s - 2) HarmonicNumber[n, s - 1])) +
  ((n^(2 s) (s - 2) + n^2 s) ( HarmonicNumber[n, s - 0])) / ((1 + (s + 1) / (s - 1) n^(-2 s))
     (n^2 s HarmonicNumber[n, 1 - s] + n^(2 s) (s - 2) HarmonicNumber[n, s - 1]))
fbc[n_, s_] := ((n^(s-1) (s - 2) + n^(1-s) s) HarmonicNumber[n, 0 - s]) / ((1 + n^(2 s) (s - 1) / (s + 1))
     (n^(1-s) s HarmonicNumber[n, 1 - s] + n^(s-1) (s - 2) HarmonicNumber[n, s - 1])) +
  ((n^(s-1) (s - 2) + n^(1-s) s) ( HarmonicNumber[n, s - 0])) / ((1 + (s + 1) / (s - 1) n^(-2 s))
     (n^(1-s) s HarmonicNumber[n, 1 - s] + n^(s-1) (s - 2) HarmonicNumber[n, s - 1]))
fbd[n_, s_] := (n^(s-1) (s - 2) + n^(1-s) s) ( HarmonicNumber[n, -s] / ((1 + n^(2 s) (s - 1) / (s + 1))
       (n^(1-s) s HarmonicNumber[n, 1 - s] + n^(s-1) (s - 2) HarmonicNumber[n, s - 1])) +
    HarmonicNumber[n, s] / ((1 + (s + 1) / (s - 1) n^(-2 s))
       (n^(1-s) s HarmonicNumber[n, 1 - s] + n^(s-1) (s - 2) HarmonicNumber[n, s - 1])))

fa[.3 + I]
-0.610073 - 0.275269 i
fax[.3 + I]
-0.610073 - 0.275269 i
fba[100 000 000 000, .3 + I]
-0.610409 - 0.275615 i
fba3[10, .5 + 3 I]
0.719881 - 0.694098 i
```

```
FullSimplify[(n^(s-1) (s - 2) + n^(1-s) s) / (n^(-s - 1) (s + 1) + n^(s-1) (s - 1))]
```

$$\frac{n^{2s}(-2+s)+n^2 s}{1+n^{2s}(-1+s)+s}$$

```
FullSimplify[rr14g[n, s] / rr14g[n , 1 - s / 2]]
```

$$\left(\left(n^s(-4+s)+n^2 s\right)\left((1+s)\,\text{HarmonicNumber}[n,-s]+n^{2s}(-1+s)\,\text{HarmonicNumber}[n,s]\right)\right)\Big/\left(\left(1+n^{2s}(-1+s)+s\right)\left(n^2 s\,\text{HarmonicNumber}\left[n,1-\frac{s}{2}\right]+n^s(-4+s)\,\text{HarmonicNumber}\left[n,-1+\frac{s}{2}\right]\right)\right)$$

```
ab[n_, s_] :=
 ((n^s (-4 + s) + n^2 s) ((1 + s) HarmonicNumber[n, -s] + n^(2 s) (-1 + s) HarmonicNumber[n, s])) /
  ((1 + n^(2 s) (-1 + s) + s) (n^2 s HarmonicNumber[n, 1 - s/2] + n^s (-4 + s) HarmonicNumber[n, -1 + s/2]))
```

```
ab[1 000 000 000 000 000 000 000, .9 + 2 I]
```

−0.274712 − 0.813137 i

```
fas[1 - s / 2] / fas[s] /. s → .9 + 2 I
```

−0.327492 − 0.856247 i

```
FullSimplify[rr14g[n, s] / rr14g[n , 1 - s / 3]]
```

$$\left(\left(n^{2s/3}(-6+s)+n^2 s\right)\left((1+s)\,\text{HarmonicNumber}[n,-s]+n^{2s}(-1+s)\,\text{HarmonicNumber}[n,s]\right)\right)\Big/\left(\left(1+n^{2s}(-1+s)+s\right)\left(n^2 s\,\text{HarmonicNumber}\left[n,1-\frac{s}{3}\right]+n^{2s/3}(-6+s)\,\text{HarmonicNumber}\left[n,-1+\frac{s}{3}\right]\right)\right)$$

```
ac[n_, s_] :=
 ((n^(2 s/3) (-6 + s) + n^2 s) ((1 + s) HarmonicNumber[n, -s] + n^(2 s) (-1 + s) HarmonicNumber[n, s])) /
  ((1 + n^(2 s) (-1 + s) + s) (n^2 s HarmonicNumber[n, 1 - s/3] + n^(2 s/3) (-6 + s) HarmonicNumber[n, -1 + s/3]))
```

```
ac[10 000 000 000, .2 + 12 I]
```

1.70478 − 1.21934 i

```
fas[1 - s / 3] / fas[s] /. s → .2 + 12 I
```

72.7714 − 33.6692 i

```
fa[s_] := 2^s Pi^(s - 1) Sin[Pi s / 2] Gamma[1 - s]
fa2[s_] := 2^(1-s) π^(-s) Sin[1/2 π (1 - s)] Gamma[s]
fba[n_, s_] :=
 ((n^(2 s) (s - 2) + n^2 s) ((s + 1) HarmonicNumber[n, -s] + n^(2 s) (s - 1) HarmonicNumber[n, s])) /
  (((s + 1) + n^(2 s) (s - 1)) (n^2 s HarmonicNumber[n, 1 - s] + n^(2 s) (s - 2) HarmonicNumber[n, s - 1]))
ffba[n_, s_] := ((n^(2 (1-s)) (-1 - s) + n^2 (1 - s))
    (-n^(2 (1-s)) s HarmonicNumber[n, 1 - s] + (2 - s) HarmonicNumber[n, -1 + s])) /
  ((2 - s - n^(2 (1-s)) s) (n^(2 (1-s)) (-1 - s) HarmonicNumber[n, -s] + n^2 (1 - s) HarmonicNumber[n, s]))
fa3[s_] := Gamma[s]
ffbb[n_, s_] := ((n^(2 (1-s)) (-1 - s) + n^2 (1 - s))
    (-n^(2 (1-s)) s HarmonicNumber[n, 1 - s] + (2 - s) HarmonicNumber[n, -1 + s])) /
  (2^(1-s) π^(-s) Sin[1/2 π (1 - s)] (2 - s - n^(2 (1-s)) s)
    (n^(2 (1-s)) (-1 - s) HarmonicNumber[n, -s] + n^2 (1 - s) HarmonicNumber[n, s]))
```

```
fa3[.3]
```

2.99157

```
ffbb[100 000 000, .3]
```

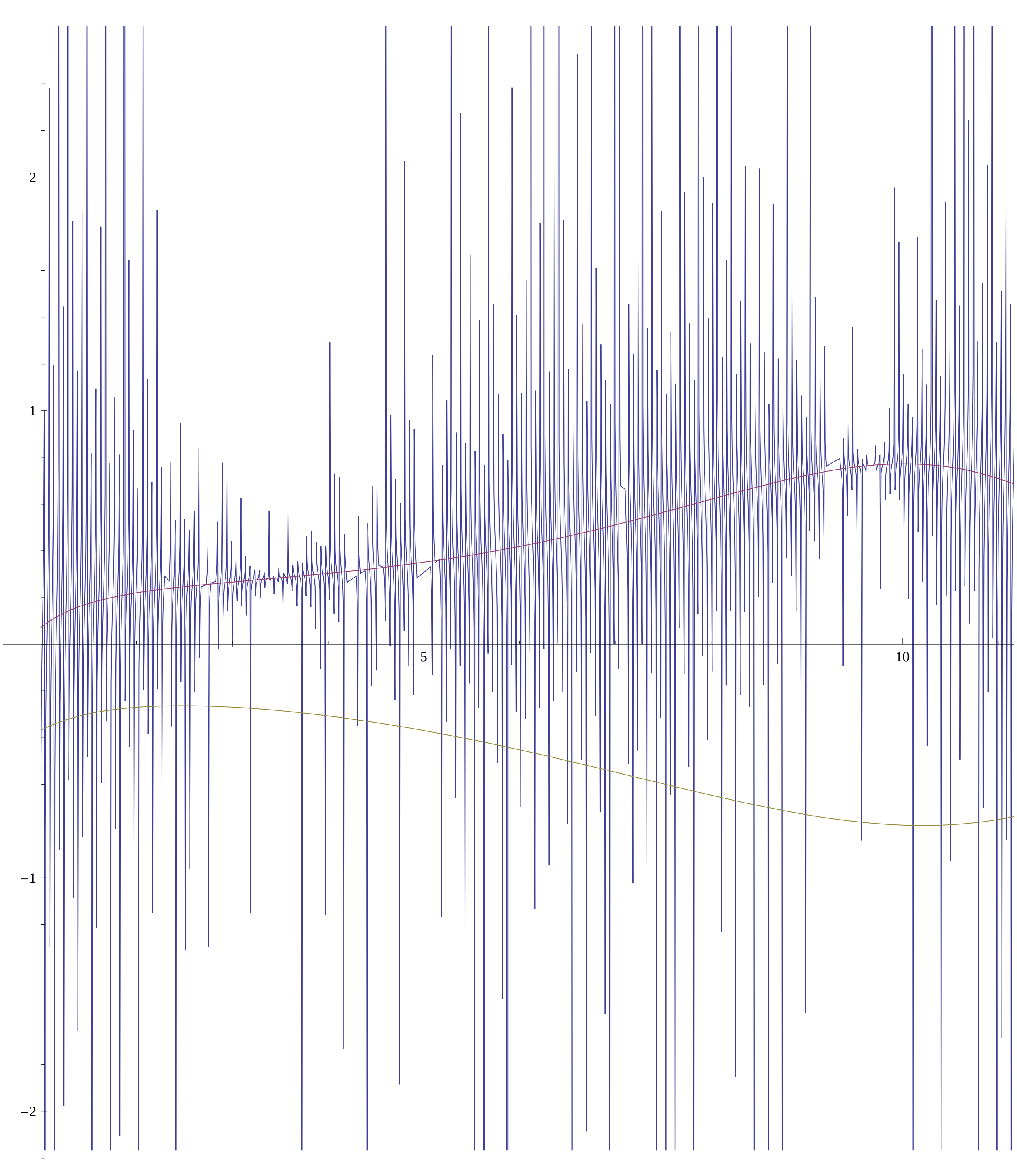
2.98588

```
FullSimplify[((n^(2 (1-s)) (-1 - s) + n^2 (1 - s))
    (-n^(2 (1-s)) s HarmonicNumber[n, 1 - s] + (2 - s) HarmonicNumber[n, -1 + s])) /
  (2^(1-s) π^(-s) Sin[1/2 π (1 - s)] (2 - s - n^(2 (1-s)) s)
    (n^(2 (1-s)) (-1 - s) HarmonicNumber[n, -s] + n^2 (1 - s) HarmonicNumber[n, s]))]
```

```
(2^(-1+s) π^s (1 + n^(2 s) (-1 + s) + s)
   (n^2 s HarmonicNumber[n, 1 - s] + n^(2 s) (-2 + s) HarmonicNumber[n, -1 + s]) Sec[π s / 2]) /
 ((n^(2 s) (-2 + s) + n^2 s) ((1 + s) HarmonicNumber[n, -s] + n^(2 s) (-1 + s) HarmonicNumber[n, s]))
```

```
f6[n_, s_] := (1 - n^(1-2 s) s/(1 - s))^-1 HarmonicNumber[n, s]
f6a[n_, s_] := (1 - n^(1-2 s) s/(1 - s))^-1
```

```
Plot[{Re@f6[10 000 000 000 000 000 000 000 000 000, .5 + s I],
  Re@Zeta[.5 + s I] / 2,  RiemannSiegelZ[s] / 2}, {s, 1, 20}]
```




```
f6[10 000 000 000 000 000 000, .5 + 17.8455995404 I]
```

$1.17009 - 1.24053 \times 10^{9}\, i$

```
Zeta[.5 + 17.8455995404 I] / 2
```

$1.17009 + 6.63112\times10^{-12}\, i$

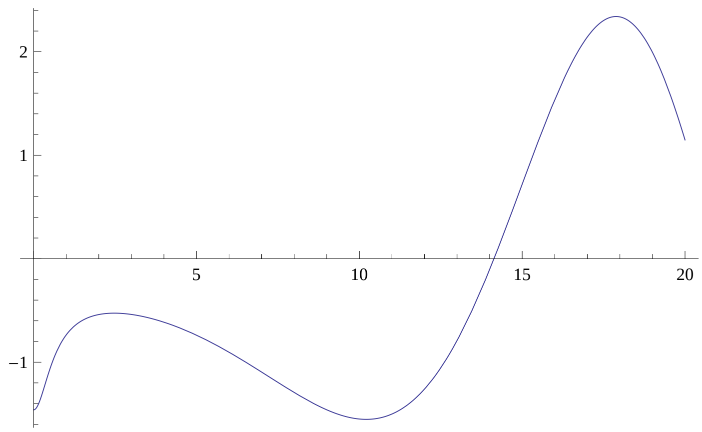
```
f6[10 000 000 000 000 000 000, 1 - (.5 + 17.8455995404 I)]
```

$1.17009 + 1.24053\times10^{9}\, i$

```
f6[n, .5 + 17.8455995404 I]
```

$$\frac{\text{HarmonicNumber}[n, 0.5 + 17.8456\, i]}{1 + (0.998431 - 0.0559923\, i)\, n^{0. - 35.6912\, i}}$$

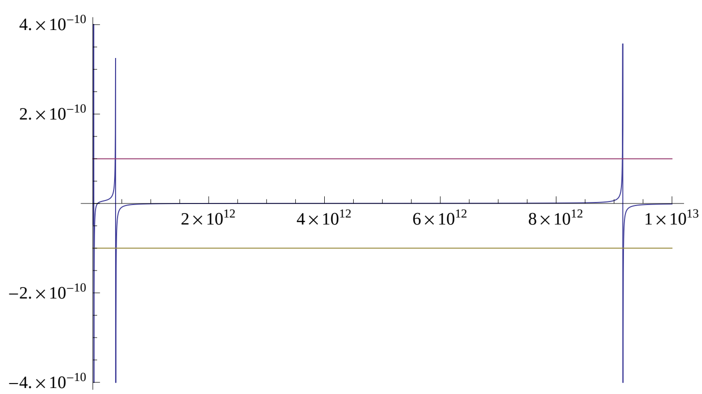
```
Plot[ RiemannSiegelZ[t], {t, 0, 20}]
```




```
RiemannSiegelTheta[27.6701822178]
```

6.28319

```
Plot[{ Tan[Log@x] / x, .0000000001, -.0000000001}, {x, 1 000 000, 10 000 000 000 000}]
```




```
D[Sec[x] / x, x]
```

$$-\frac{\text{Sec}[x]}{x^2} + \frac{\text{Sec}[x]\,\text{Tan}[x]}{x}$$

$$\textbf{Limit}\left[-\frac{\textbf{Sec}[x]}{x^2} + \frac{\textbf{Sec}[x]\,\textbf{Tan}[x]}{x},\ x \to \textbf{Infinity}\right]$$

$$\text{Limit}\left[-\frac{\text{Sec}[x]}{x^2} + \frac{\text{Sec}[x]\,\text{Tan}[x]}{x},\ x \to \infty\right]$$

```
Clear[tsa]
ts[n_] := (1 / n) Sum[Tan[Log@x] / x, {x, 1, n}]
tsa[n_] := tsa[n] = Sum[Tan[Log@x] / x, {x, 1, n}]
tsb[t_] := E^-t Sum[ t^n / (n!) tsa[n], {n, 0, Infinity}]
```

```
ts[1000000.]
```

-0.0000383088

```
N@tsb[10.]
```

-4.05019

rr14a[n_, m_, d_] := Sum[j^{-m} Cosh[d Log[j]], {j, 1, n}] -
Coth$\left[\text{ArcTanh}\left[\frac{d}{-1+m}\right] + \text{d Log[n]}\right]$ Sum[j^{-m} Sinh[d Log[j]], {j, 1, n}]

```
rr14[n_, s_, t_] := rr14a[n, (s + t) / 2, (s - t) / 2]
```

```
rr14a[n, 1 / 3, s]
```

$$\sum_{j=1}^{n} \frac{\text{Cosh}[s\,\text{Log}[j]]}{j^{1/3}} + \text{Coth}\left[\text{ArcTanh}\left[\frac{3\,s}{2}\right] - s\,\text{Log}[n]\right] \sum_{j=1}^{n} \frac{\text{Sinh}[s\,\text{Log}[j]]}{j^{1/3}}$$

$$\text{TrigToExp}\left[\frac{\text{Sinh}[s\,\text{Log}[j]]}{j^{1/3}}\right]$$

$$-\frac{1}{2}\,j^{-\frac{1}{3}-s} + \frac{1}{2}\,j^{-\frac{1}{3}+s}$$

$$\text{TrigToExp}\left[\frac{\text{Cosh}[s\,\text{Log}[j]]}{j^{1/3}}\right]$$

$$\frac{1}{2}\,j^{-\frac{1}{3}-s} + \frac{1}{2}\,j^{-\frac{1}{3}+s}$$

$$\text{x2}[\text{n\_}, \text{s\_}] := \sum_{j=1}^{n}\left(\frac{1}{2} j^{-\frac{1}{3}-s} + \frac{1}{2} j^{-\frac{1}{3}+s}\right) + \text{Coth}\left[\text{ArcTanh}\left[\frac{3\,s}{2}\right] - s\,\text{Log}[n]\right] \sum_{j=1}^{n}\left(-\frac{1}{2} j^{-\frac{1}{3}-s} + \frac{1}{2} j^{-\frac{1}{3}+s}\right)$$

$$\text{x3}[\text{n\_}, \text{s\_}] := \frac{1}{2}\,\text{Coth}\left[\text{ArcTanh}\left[\frac{3\,s}{2}\right] - s\,\text{Log}[n]\right]\left(\text{HarmonicNumber}\left[n, \frac{1}{3} - s\right] - \text{HarmonicNumber}\left[n, \frac{1}{3} + s\right]\right) + \frac{1}{2}\left(\text{HarmonicNumber}\left[n, \frac{1}{3} - s\right] + \text{HarmonicNumber}\left[n, \frac{1}{3} + s\right]\right)$$

$$\text{x4}[\text{n\_}, \text{s\_}] := \frac{1}{2}\left(\left(1 + \text{Coth}\left[\text{ArcTanh}\left[\frac{3\,s}{2}\right] - s\,\text{Log}[n]\right]\right)\text{HarmonicNumber}\left[n, \frac{1}{3} - s\right] + \left(1 - \text{Coth}\left[\text{ArcTanh}\left[\frac{3\,s}{2}\right] - s\,\text{Log}[n]\right]\right)\text{HarmonicNumber}\left[n, \frac{1}{3} + s\right]\right)$$

$$\text{x4a}[\text{n\_}, \text{s\_}] := \frac{1}{2}\left(1 + \text{Coth}\left[\text{ArcTanh}\left[\frac{3\,s}{2}\right] - s\,\text{Log}[n]\right]\right)\text{HarmonicNumber}\left[n, \frac{1}{3} - s\right]$$

$$\text{x4b}[\text{n\_}, \text{s\_}] := \text{x4a}[n, s] + \text{x4a}[n, -s]$$

$$\text{x4ax}[\text{n\_}, \text{s\_}] := \frac{1}{2}\left(1 + \text{Coth}\left[\text{ArcTanh}\left[\frac{1}{2} - \frac{3\,s}{2}\right] - \frac{\text{Log}[n]}{3} + s\,\text{Log}[n]\right]\right)\text{HarmonicNumber}[n, s]$$

$$\text{x4ay}[\text{n\_}, \text{s\_}] := \frac{1}{1 + \frac{n^{\frac{2}{3}-2\,s}\,(1+3\,s)}{3\,(-1+s)}}\,\text{HarmonicNumber}[n, s]$$

x2[n, s]

$$\frac{1}{2}\,\text{Coth}\left[\text{ArcTanh}\left[\frac{3\,s}{2}\right] - s\,\text{Log}[n]\right]\left(\text{HarmonicNumber}\left[n, \frac{1}{3} - s\right] - \text{HarmonicNumber}\left[n, \frac{1}{3} + s\right]\right) + \frac{1}{2}\left(\text{HarmonicNumber}\left[n, \frac{1}{3} - s\right] + \text{HarmonicNumber}\left[n, \frac{1}{3} + s\right]\right)$$

x4b[1 000 000, .7 - 1 / 3]

-2.77878

Zeta[.7]

-2.77839

$$\sum_{j=1}^{n}\frac{\text{Cosh}[s\,\text{Log}[j]]}{\sqrt{j}} + \text{Coth}[\text{ArcTanh}[2\,s] - s\,\text{Log}[n]]\sum_{j=1}^{n}\frac{\text{Sinh}[s\,\text{Log}[j]]}{\sqrt{j}}\ /.\ n \to 10\,000\ /.\ s \to .7 - .5$$

-2.89618

$$\sum_{j=1}^{n}\frac{\text{Cosh}[s\,\text{Log}[j]]}{j^{1/3}} + \text{Coth}\left[\text{ArcTanh}\left[\frac{3\,s}{2}\right] - s\,\text{Log}[n]\right]\sum_{j=1}^{n}\frac{\text{Sinh}[s\,\text{Log}[j]]}{j^{1/3}}\ /.\ n \to 10\,000\ /.\ s \to .7 - (1/3)$$

-2.78964

FullSimplify[x4a[n, 1 / 3 - s]]

$$\frac{1}{2}\left(1 + \text{Coth}\left[\text{ArcTanh}\left[\frac{1}{2} - \frac{3\,s}{2}\right] - \frac{\text{Log}[n]}{3} + s\,\text{Log}[n]\right]\right)\text{HarmonicNumber}[n, s]$$

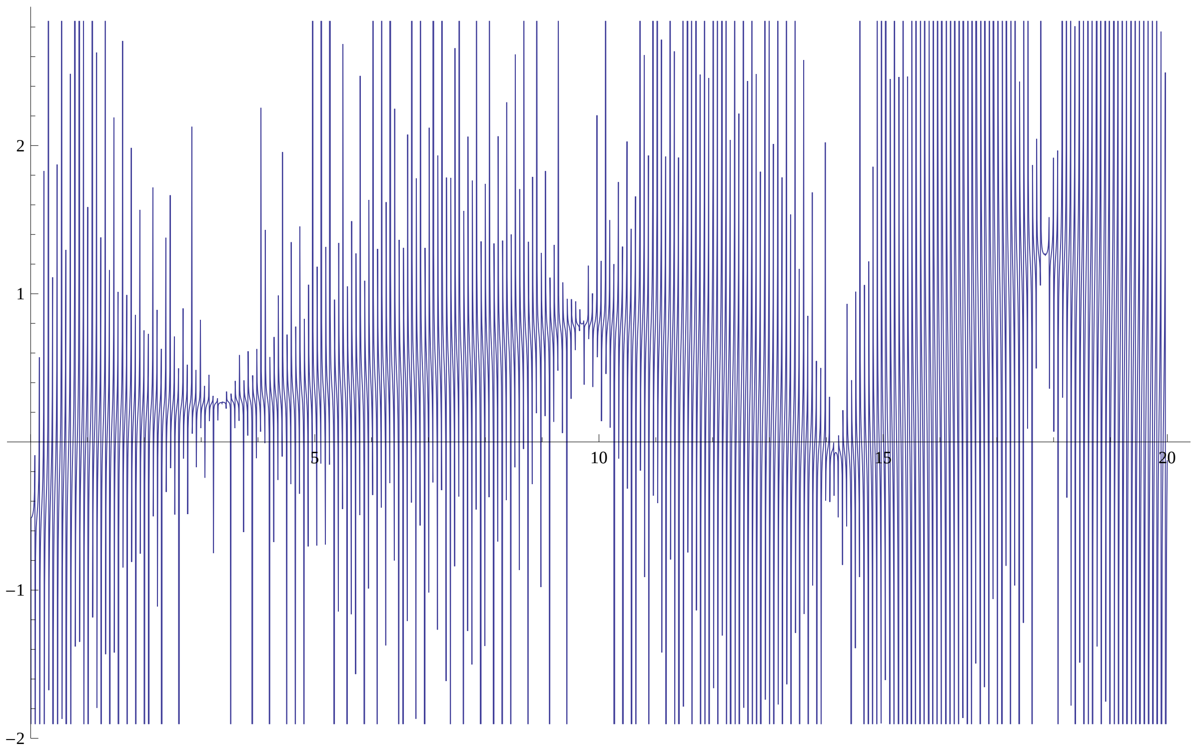
x4a[n, s + 1 / 3]

$$\frac{1}{2}\left(1+\text{Coth}\left[\text{ArcTanh}\left[\frac{3}{2}\left(\frac{1}{3}+s\right)\right]-\left(\frac{1}{3}+s\right)\text{Log}[n]\right]\right)\text{HarmonicNumber}[n,-s]$$

(1 / 3 + s) + (1 / 3 – s)

$$\frac{2}{3}$$

Plot[Re@x4ay[1 000 000 000 000 000 000, N[1 / 3] + s I], {s, 0, 20}]



Zeta[N@ZetaZero@1 – 1 / 2 + 1 / 3]

$-0.139457-0.0242355\,i$

$$\textbf{FullSimplify}\left[\textbf{TrigToExp}\left[\frac{1}{2}\left(1+\textbf{Coth}\left[\textbf{ArcTanh}\left[\frac{1}{2}-\frac{3\,s}{2}\right]-\frac{\textbf{Log}[n]}{3}+s\,\textbf{Log}[n]\right]\right)\right]\right]$$

$$\frac{1}{1+\frac{n^{\frac{2}{3}-2s}\,(1+3\,s)}{3\,(-1+s)}}$$

rr14a[n, t, s]

$$\sum_{j=1}^{n}\left(\frac{j^{-s-t}}{2}+\frac{j^{s-t}}{2}\right)-\text{Coth}\left[\text{ArcTanh}\left[\frac{s}{-1+t}\right]+s\,\text{Log}[n]\right]\sum_{j=1}^{n}\left(-\frac{1}{2}\,j^{-s-t}+\frac{j^{s-t}}{2}\right)$$

$$-\frac{1}{2}\,\text{Coth}\left[\text{ArcTanh}\left[\frac{s}{-1+t}\right]+s\,\text{Log}[n]\right]\,(\text{HarmonicNumber}[n,-s+t]-\text{HarmonicNumber}[n,s+t])+$$
$$\frac{1}{2}\,(\text{HarmonicNumber}[n,-s+t]+\text{HarmonicNumber}[n,s+t])$$

```
TrigToExp[j^-t Sinh[s Log[j]]]
```

$$-\frac{1}{2} j^{-s-t}+\frac{j^{s-t}}{2}$$

```
-1/2 Coth[ArcTanh[s/(-1 + t)] + s Log[n]] (HarmonicNumber[n, -s + t]) + 1/2 (HarmonicNumber[n, -s + t])
```

$$\frac{1}{2}\,\text{HarmonicNumber}[n,-s+t]-\frac{1}{2}\,\text{Coth}\left[\text{ArcTanh}\left[\frac{s}{-1+t}\right]+s\,\text{Log}[n]\right]\text{HarmonicNumber}[n,-s+t]$$

```
-1/2 Coth[ArcTanh[s/(-1 + t)] + s Log[n]] (HarmonicNumber[n, -s + t]) +
  1/2 (HarmonicNumber[n, -s + t]) /. s → -s + t
FullSimplify[
  1/2 HarmonicNumber[n, s] - 1/2 Coth[ArcTanh[(-s + t)/(-1 + t)] + (-s + t) Log[n]] HarmonicNumber[n, s] /.
   s → a + b I /. t → a]
```

$$\frac{1}{2}\left(1-i\,\text{Cot}\left[\text{ArcTan}\left[\frac{b}{-1+a}\right]+b\,\text{Log}[n]\right]\right)\text{HarmonicNumber}[n,a+i\,b]$$

```
FullSimplify[(-1/2 Coth[ArcTanh[s/(-1 + t)] + s Log[n]] (-HarmonicNumber[n, s + t]) +
      1/2 (HarmonicNumber[n, s + t]) /. s → -s + t) /. s → a + b I /. t → a]
```

$$\frac{1}{2}\left(1+i\,\text{Cot}\left[\text{ArcTan}\left[\frac{b}{-1+a}\right]+b\,\text{Log}[n]\right]\right)\text{HarmonicNumber}[n,a-i\,b]$$

```
rr[n_, a_, b_] :=
```

$$\frac{1}{2}\text{HarmonicNumber}[n, a + i b] - \frac{1}{2} i \text{Cot}\left[\text{ArcTan}\left[\frac{b}{-1+a}\right] + b \text{Log}[n]\right] \text{HarmonicNumber}[n, a + i b]$$

```
rrt[n_, a_, b_] := rr[n, a, b] + rr[n, -a, b]
```

$$\text{rrx}[n\_, a\_, b\_] := \left(1 - i \text{Cot}\left[\text{ArcTan}\left[\frac{b}{-1+a}\right] + b \text{Log}[n]\right]\right) \Big/ 2 \text{HarmonicNumber}[n, a + i b]$$

$$\text{rrxa}[n\_, a\_, b\_] := \left(1 - I \text{Cot}\left[\text{ArcTan}\left[\frac{b}{-1+a}\right] + b \text{Log}[n]\right]\right) \Big/ 2 \text{HarmonicNumber}[n, a + i b]$$

$$\text{rr2}[n\_, a\_, b\_] := \frac{1}{1 + \frac{(1-a+i b)\, n^{-2 i b}}{-1+a+i b}} \text{HarmonicNumber}[n, a + i b]$$

$$\text{rr3}[n\_, s\_] := \left(1 - n^{(\text{Conjugate}[s]-s)} \frac{\text{Conjugate}[s] - 1}{s - 1}\right) \hat{}\, -1 \text{HarmonicNumber}[n, s]$$

```
rr3x[n_, s_] := HarmonicNumber[n, s]
```

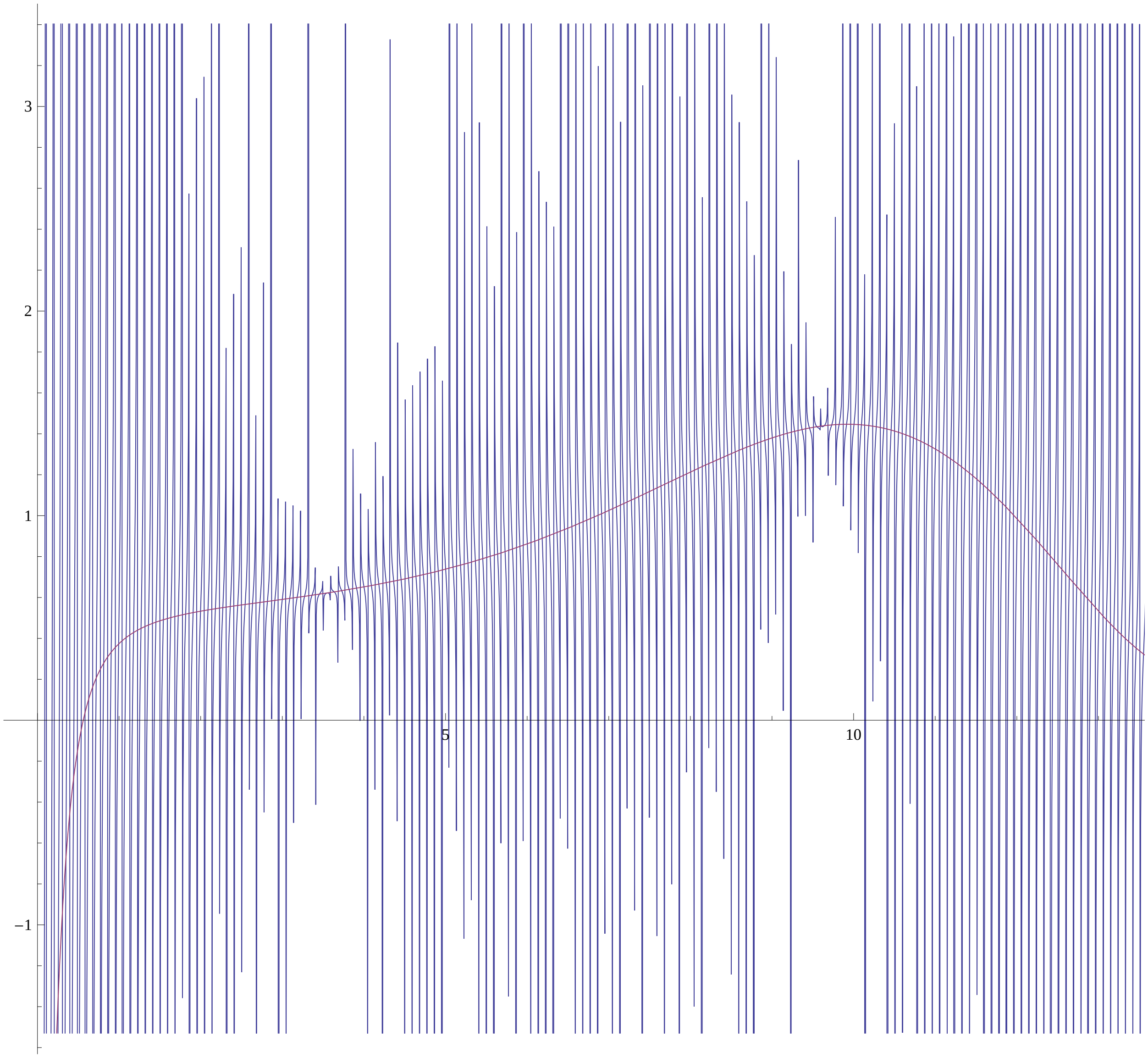
$$\text{rr3y}[n\_, s\_] := \left(1 - n^{(\text{Conjugate}[s]-s)} \frac{\text{Conjugate}[s] - 1}{s - 1}\right) \hat{}\, -1$$

$$\text{rr3z}[n\_, s\_] := \left(1 - n^{(\text{Conjugate}[s]-s)} \frac{\text{Conjugate}[s] - 1}{s - 1}\right) \hat{}\, -1 \text{HarmonicNumber}[n, s]$$

$$\text{rr4}[n\_, s\_] := \left(1 - \frac{n^{\text{Conjugate}[s]} (\text{Conjugate}[s] - 1)}{n^s (s - 1)}\right) \hat{}\, -1 \text{HarmonicNumber}[n, s]$$

$$\text{rr5}[n\_, s\_] := n^s (s - 1) \Big/ \left(n^s (s - 1) - n^{\text{Conjugate}[s]} (\text{Conjugate}[s] - 1)\right) \text{HarmonicNumber}[n, s]$$

```
Plot[{2 Re@rr3[1 000 000 000 000 000, .8 + s I], Re@Zeta[.8 + s I]}, {s, 0, 20}]
```




```
Zeta[.5 + I]
```

0.143936 − 0.7221 i

```
FullSimplify[TrigToExp[(1 - i Cot[ArcTan[b/(-1 + a)] + b Log[n]]) / 2]]
```

$$\frac{1}{1+\frac{(1-a+i\,b)\,n^{-2\,i\,b}}{-1+a+i\,b}}$$

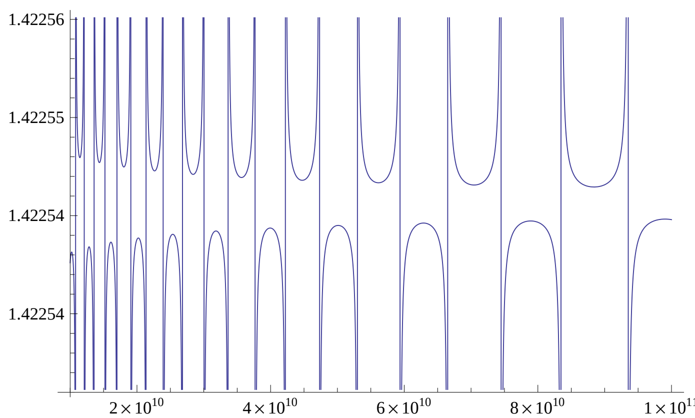
$$\text{FullSimplify}\left[\frac{1}{1+\frac{(1-a+i\,b)\,n^{-2\,i\,b}}{-1+a+i\,b}}\right]$$

$$\frac{1}{1+\frac{(1-a+i\,b)\,n^{-2\,i\,b}}{-1+a+i\,b}}$$

$$\text{FullSimplify}\left[\left(1-i\,\text{Cot}\left[\text{ArcTan}\left[\frac{b}{-1+a}\right]+b\,\text{Log}[n]\right]\right)\Big/\,2\right]$$

$$-\frac{1}{2}\,i\left(i+\text{Cot}\left[\text{ArcTan}\left[\frac{b}{-1+a}\right]+b\,\text{Log}[n]\right]\right)$$

Plot[Re[rr2[n, .5, 27.6701822178]], {n, 10 000 000 000, 100 000 000 000}]



Zeta[N@ZetaZero@1 - .5 + .2] / 2

-0.132436 - 0.0246687 i

ps[n_, s_, t_] := (1 / n) Sum[N[Re[rr2[j, s, t]]], {j, 1, n}]

ps[100 000, .2, N@Im@ZetaZero@1]

-0.0952945

$$\left(1+n^{-2\,i\,b}\,\frac{1-(a-i\,b)}{-1+a+i\,b}\right)\text{^-1 /. a}\to .7\text{ /. b}\to .3\text{ /. n}\to 20$$

0.5 - 4.39332 i

$$\left(1+n^{-(s-\text{Conjugate}[s])}\,\frac{1-\text{Conjugate}[s]}{-1+s}\right)\text{^-1 /. s}\to .7+.3\,\text{I /. n}\to 20$$

0.5 - 4.39332 i

$$\left(1-n^{(\text{Conjugate}[s]-s)}\,\frac{\text{Conjugate}[s]-1}{s-1}\right)\text{^-1 /. s}\to .7+.3\,\text{I /. n}\to 20$$

0.5 - 4.39332 i

Conjugate[s]

Conjugate[s]

$n^{-2 i b}$ /. a → .7 /. b → .3 /. n → 20

$-0.224708 - 0.974426\, i$

$n^{-(s-\mathrm{Conjugate}[s])}$ /. s → .7 + .3 I /. n → 20

$-0.224708 - 0.974426\, i$

$$\left(1 - I \,\mathrm{Cot}\left[\mathrm{ArcTan}\left[\frac{b}{-1+a}\right] + b\,\mathrm{Log}[n]\right]\right) \Big/ 2$$

$$\frac{1}{2}\left(1 - i \,\mathrm{Cot}\left[\mathrm{ArcTan}\left[\frac{b}{-1+a}\right] + b\,\mathrm{Log}[n]\right]\right) \text{ /. } a \to 1/2 \text{ /. } b \to 14.4 \text{ /. } n \to 30$$

$0.5 - 1.52274\, i$

$$\frac{1}{2}\left(1 - \mathrm{Tan}[\mathrm{ArcTan}[(a-1)/b] + b\,\mathrm{Log}[n]]\right) \text{ /. } a \to 1/2 \text{ /. } b \to 14.4 \text{ /. } n \to 30$$

2.47589

$$\mathrm{FullSimplify}\left[\left(1 - \frac{n^{\mathrm{Conjugate}[s]}\,(\mathrm{Conjugate}[s]-1)}{n^s\,(s-1)}\right)\text{^}-1\right]$$

$$\frac{1}{1 - \frac{n^{-2 i \mathrm{Im}[s]}\,(-1+\mathrm{Conjugate}[s])}{-1+s}}$$

$$\frac{n^{-2 i \mathrm{Im}[s]}\,(-1+\mathrm{Conjugate}[s])}{-1+s} \text{ /. } s \to \mathrm{N@ZetaZero@1}$$

$(-0.997501 + 0.0706593\, i)\, n^{0.-28.2695\, i}$

$$\frac{(-1+\mathrm{Conjugate}[s])}{-1+s} \text{ /. } s \to \mathrm{N@ZetaZero@1}$$

$-0.997501 + 0.0706593\, i$

(s - t I) / (s + t I) /. s → .5

$$\frac{0.5 - i\, t}{0.5 + i\, t}$$

$$\frac{1}{2}\,\text{HarmonicNumber}[n,s]-\frac{1}{2}\,\text{Coth}\left[\text{ArcTanh}\left[\frac{-s+t}{-1+t}\right]+(-s+t)\,\text{Log}[n]\right]\text{HarmonicNumber}[n,s]\;\text{/.}\;s\to\text{Re}[v]+\text{Im}[v]\;\text{/.}\;t\to\text{Re}[v]$$

$$\frac{1}{2}\,\text{HarmonicNumber}[n,\text{Im}[v]+\text{Re}[v]]+\frac{1}{2}\,\text{Coth}\left[\text{ArcTanh}\left[\frac{\text{Im}[v]}{-1+\text{Re}[v]}\right]+\text{Im}[v]\,\text{Log}[n]\right]\text{HarmonicNumber}[n,\text{Im}[v]+\text{Re}[v]]$$

$$\frac{1}{2}\,\text{HarmonicNumber}[n,\text{Im}[v]+\text{Re}[v]]+\frac{1}{2}\,\text{Coth}\left[\text{ArcTanh}\left[\frac{\text{Im}[v]}{-1+\text{Re}[v]}\right]+\text{Im}[v]\,\text{Log}[n]\right]\text{HarmonicNumber}[n,\text{Im}[v]+\text{Re}[v]]\;\text{/.}\;v\to s$$

$$\left(1+\text{Coth}\left[\text{ArcTanh}\left[\frac{\text{Im}[s]}{-1+\text{Re}[s]}\right]+\text{Im}[s]\,\text{Log}[n]\right]\right)\text{HarmonicNumber}[n,s]$$

```
FullSimplify[rr3y[n, .5 + t I]]
```

$$\frac{1}{1+\frac{n^{-2\,i\,\text{Re}[t]}\,(0.5+(0.+1.\,i)\,\text{Conjugate}[t])}{-0.5+(0.+1.\,i)\,t}}$$

$$\frac{1}{1+\frac{(1-a+i\,b)\,n^{-2\,i\,b}}{-1+a+i\,b}}\,\text{HarmonicNumber}[n,a+i\,b]\;\text{/.}\;a\to s\;\text{/.}\;b\to t$$

$$\frac{\text{HarmonicNumber}[n,s+i\,t]}{1+\frac{n^{-2\,i\,t}\,(1-s+i\,t)}{-1+s+i\,t}}$$

$$\frac{1}{1+\frac{n^{-2\,i\,t}\,(1-s+i\,t)}{-1+s+i\,t}}\,\text{Sum}[1/j\text{^}(s+I\,t),\{j,1,n\}]$$

$$\text{ExpToTrig}\left[\frac{1}{1+\frac{n^{-2\,i\,t}\,(1-s+i\,t)}{-1+s+i\,t}}\,1/j\text{^}(s+I\,t)\right]$$

$$\frac{1}{2}\,\text{HarmonicNumber}[n,s]-\frac{1}{2}\,\text{Coth}\left[\text{ArcTanh}\left[\frac{-s+t}{-1+t}\right]+(-s+t)\,\text{Log}[n]\right]\text{HarmonicNumber}[n,s]\;\text{/.}\;s\to a+b\,I\;\text{/.}\;t\to b\,I$$

$$\frac{1}{2}\,\text{HarmonicNumber}[n,a+i\,b]+\frac{1}{2}\,\text{Coth}\left[\text{ArcTanh}\left[\frac{a}{-1+i\,b}\right]+a\,\text{Log}[n]\right]\text{HarmonicNumber}[n,a+i\,b]$$

$$\text{rri}[n\_,a\_,b\_]:=\frac{1}{2}\,\text{HarmonicNumber}[n,a+i\,b]+\frac{1}{2}\,\text{Coth}\left[\text{ArcTanh}\left[\frac{a}{-1+i\,b}\right]+a\,\text{Log}[n]\right]\text{HarmonicNumber}[n,a+i\,b]$$

```
rrti[n_, a_, b_] := rri[n, a, b] + rri[n, -a, b]
rrti[10 000 000, .5, N@Im@ZetaZero@1]
```

$$-0.0000107589+1.06315\times10^{-6}\,i$$

```
Zeta[.5]
```

-1.46035

```
rr14a[n_, m_, d_] := Sum[j^-m Cosh[d Log[j]], {j, 1, n}] -
  Coth[ArcTanh[d/(-1 + m)] + d Log[n]] Sum[j^-m Sinh[d Log[j]], {j, 1, n}]
rr14[n_, s_, t_] := rr14a[n, (s + t) / 2, (s - t) / 2]
rr14[100 000, N@ZetaZero@1 - .3 + 10 I, N@ZetaZero@1 + .1]
```

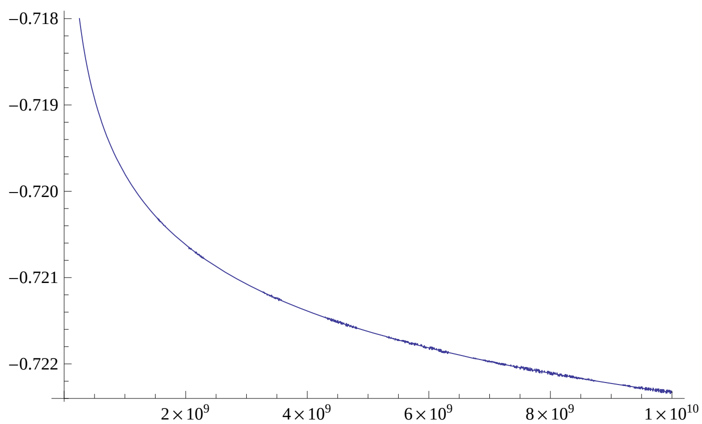
0.0655803 - 0.00824799 i

```
Zeta[N@ZetaZero@1 + .1]
```

0.0753346 + 0.0113729 i

```
rr4a[n_, m_, d_] := ((1 - (m - d)) n^(m - d) HarmonicNumber[n, (m - d)] -
    (1 - (m + d)) n^(m + d) HarmonicNumber[n, (m + d)]) /
  ((1 - (m - d)) n^(m - d) - (1 - (m + d)) n^(m + d) )
rr4[n_, s_, t_] := rr4a[n, (s + t) / 2, (s - t) / 2]

Plot[Im@rr4[n, .5 + 12 I, .5 + 12 I + .000001 I], {n, 1, 10 000 000 000}]
```




```
N@Zeta[.5 + 12 I]
```

1.01594 - 0.745112 i

```
rr[n_, a_, b_] :=
 1/2 HarmonicNumber[n, a + i b] - 1/2 i Cot[ArcTan[b/(-1 + a)] + b Log[n]] HarmonicNumber[n, a + i b]
rrt[n_, a_, b_] := rr[n, a, b] + rr[n, a, -b]

rrt[10 000, .6, N@Im@ZetaZero@1]
```

-0.00650021 + 0. i

```
FullSimplify[
 (-1/2 Coth[ArcTanh[s/(-1 + t)] + s Log[n]] (HarmonicNumber[n, -s + t] - HarmonicNumber[n, s + t]) +
     1/2 (HarmonicNumber[n, -s + t] + HarmonicNumber[n, s + t]) /.
    s -> -s + t) /. s -> a + b I /. t -> a]
```

```
1/2 (HarmonicNumber[n, a - ⅈ b] + ⅈ Cot[ArcTan[b/(-1 + a)] + b Log[n]]
    (HarmonicNumber[n, a - ⅈ b] - HarmonicNumber[n, a + ⅈ b]) + HarmonicNumber[n, a + ⅈ b])
```

```
-1/2 ⅈ Cot[ArcTan[b/(-1 + a)] + b Log[n]] (-HarmonicNumber[n, a - ⅈ b] + HarmonicNumber[n, a + ⅈ b]) +
   1/2 (HarmonicNumber[n, a - ⅈ b] + HarmonicNumber[n, a + ⅈ b]) /.
  n -> 10 000 000 000 000 000 /. a -> .5 /. b -> 3
```

```
0.60384 + 0. ⅈ
```

```
Zeta[.5 + 3 I]
```

```
0.532737 - 0.0788965 ⅈ
```

```
1/2 Coth[ArcTanh[a/(-1 + ⅈ b)] + a Log[n]] (-HarmonicNumber[n, -a + ⅈ b] + HarmonicNumber[n, a + ⅈ b]) +
   1/2 (HarmonicNumber[n, -a + ⅈ b] + HarmonicNumber[n, a + ⅈ b]) /.
  n -> 10 000 000 000 /. a -> .6 /. b -> 3
```

```
0.5 - 0.25 ⅈ
```

```
Zeta[.6 + 3 I]
```

```
0.551963 - 0.0859178 ⅈ
```

```
Limit[ArcTanh[(-a - ⅈ b + c)/(-1 + c)], c -> 1]
```

```
$Aborted
```

```
1/2 (HarmonicNumber[n, a] + ⅈ Cot[ArcTan[b/(-1 + a + ⅈ b)] + b Log[n]]
     (HarmonicNumber[n, a] - HarmonicNumber[n, a + 2 ⅈ b]) +
    HarmonicNumber[n, a + 2 ⅈ b]) /. n -> 10 000 000 000 000 /. a -> .6 /. b -> 3
```

```
0.753813 + 0.169326 ⅈ
```

```
Zeta[.6 + 6 I]
```

```
0.845771 + 0.32451 ⅈ
```