```
D[Log[Zeta[s]], s]
```

$$\frac{\text{Zeta}'[s]}{\text{Zeta}[s]}$$

```
Integrate[Zeta'[s]/Zeta[s], {s, 0, Infinity}]
```

$-i\pi + \text{Log}[2]$

```
Log[Zeta[0]]
```

$i\pi - \text{Log}[2]$

```
N[Integrate[Zeta'[s]/Zeta[s], {s, -I, I}]]
```

$0. - 3.12581\, i$

```
N[Log[Zeta[I]] - Log[Zeta[-I]]]
```

$0. - 3.12581\, i$

```
FI[n_] := FactorInteger[n]; FI[1] := {}
dzeta[j_, s_, z_] := j^-s Product[(-1)^p[[2]] Binomial[-z, p[[2]]], {p, FI[j]}]
zeta[n_, s_, z_] := Sum[dzeta[j, s, z], {j, 1, n}]

FullSimplify[Sum[ dzeta[j, s, -1] D[ zeta[10 / j, s, 1], s], {j, 1, 10}]]
```

$$-\text{Integrate}[2520^{-s} (-315^s (1 + 2^s + 4^s) \text{Log}[2] - 8^s (35^s (1 + 3^s) \text{Log}[3] + 63^s \text{Log}[5] + 45^s \text{Log}[7])), \{s, 0, \text{Infinity}\}]$$

$$\frac{16}{3}$$

```
Expand[D[(1 - 2^(1 - s)) Zeta[s], s] / ((1 - 2^(1 - s)) Zeta[s])] /. s → 0
```

$-2\,\text{Log}[2] + \text{Log}[2\pi]$

```
Log[ (1 - 2^(1 - s)) Zeta[s]]
```

$\text{Log}\left[\left(1 - 2^{1-s}\right) \text{Zeta}[s]\right]$

```
Log[1 - 2^(1 - s)] + Log[Zeta[s]] /. s → 0
```

$2 i\pi - \text{Log}[2]$

```
Log[1 - 2^(1 - s)] + Log[Zeta[s]]
```

$\text{Log}\left[1 - 2^{1-s}\right] + \text{Log}[\text{Zeta}[s]]$

```
FullSimplify[D[(1 - 2^(1 - s)) Zeta[s], s] / ((1 - 2^(1 - s)) Zeta[s])]
```

$$\frac{\text{Log}[4]}{-2 + 2^s} + \frac{\text{Zeta}'[s]}{\text{Zeta}[s]}$$

```
FullSimplify[D[Log[1 - 2^(1-s)] + Log[Zeta[s]], s]]
```

$$\frac{\text{Log}[4]}{-2 + 2^s} + \frac{\text{Zeta}'[s]}{\text{Zeta}[s]}$$

```
Log[4]/(-2 + 2^s) /. s → 2
```

$$\frac{\text{Log}[4]}{2}$$

```
D[LogIntegral[n^(1 - s)] - Log[Log[n^(1 - s)]] - EulerGamma, s]
```

$$\frac{\text{Log}[n]}{\text{Log}\left[n^{1-s}\right]} - \frac{n^{1-s}\,\text{Log}[n]}{\text{Log}\left[n^{1-s}\right]}$$

```
FullSimplify[Log[n]/Log[1/n] - Log[n]/(n Log[1/n])]
```

```
Limit[((-1 + n) Log[n])/(n Log[1/n]), n → Infinity]
```

-1

```
N[Log[4]/2]
```

0.693147

```
D[f[j] j^-s, s]
```

$-j^{-s}\,f[j]\,\text{Log}[j]$

```
D[f[j] f[k] (j k)^-s, s]
```

$-(j\,k)^{-s}\,f[j]\,f[k]\,\text{Log}[j\,k]$

```
bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
tz[ n_, s_, z_] := Sum[ bin[ z, k] ((1 - 2^(1 - s)) Zeta[s] - 1)^k, {k, 0, n}]
zeros[n_, s_] := List @@ Roots[tz[n, s, z] == 0, z][[All, 2]]
```

```
Expand[tz[10, ZetaZero[1], z]]
```

$$1 - \frac{7381\,z}{2520} + \frac{177\,133\,z^2}{50\,400} - \frac{84\,095\,z^3}{36\,288} + \frac{341\,693\,z^4}{362\,880} - \frac{8591\,z^5}{34\,560} + \frac{7513\,z^6}{172\,800} - \frac{121\,z^7}{24\,192} + \frac{11\,z^8}{30\,240} - \frac{11\,z^9}{725\,760} + \frac{z^{10}}{3\,628\,800}$$

```
N[zeros[30, -5]]
```

{21.3331, 21.928, 23.0053, 23.9996, 25., 26., 27., 28., 29., 30.,
-2.27531 - 7.41594 i, -2.27531 + 7.41594 i, 2.33424 - 7.70546 i, 2.33424 + 7.70546 i,
5.76559 - 7.33338 i, 5.76559 + 7.33338 i, 8.61325 - 6.6668 i, 8.61325 + 6.6668 i,
11.0822 - 5.84299 i, 11.0822 + 5.84299 i, 13.2721 - 4.92967 i, 13.2721 + 4.92967 i,
15.241 - 3.96622 i, 15.241 + 3.96622 i, 17.0268 - 2.97847 i, 17.0268 + 2.97847 i,
18.6566 - 1.98505 i, 18.6566 + 1.98505 i, 20.1506 - 1.001 i, 20.1506 + 1.001 i}

```
Product[ 1 - 1 / j, {j, zeros[30, N[ZetaZero[1]]]}]
```

-1.16733×10^{-14}

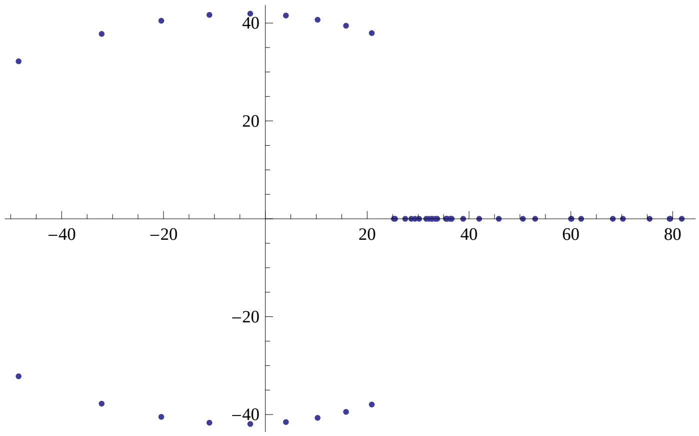
```
N[(1 - 2^(1 - s)) Zeta[s] /. s → 2]
```

0.822467

```
zeros[30, -2.01]
```

{0.939965, 1.99652, 2.99989, 4., 5., 6.00021, 6.99928, 8.06403, 8.43623, 9.27978, 9.42386, 11.0145, 11.1107, 11.7585, 13.2187, 13.5727, 15.6937, 16.1859, 18.425, 19.0522, 22.036, 22.2628, 24.7523, 25.8874, 27.0282, 28.6258, 29.2042, 30.9752, 31.5193, 32.4819}

```
ListPlot[Table[{Re[n], Im[n]}, {n, zeros[50, .5]}]]
```




```
FI[n_] := FactorInteger[n]; FI[1] := {}
dzeta[j_, s_, z_] := j^-s Product[(-1)^p[[2]] Binomial[-z, p[[2]]], {p, FI[j]}]
zeta[n_, s_, z_] := Sum[dzeta[j, s, z], {j, 1, n}]
p[j_, s_, k_] := D[dzeta[j, s, z], {z, k}] /. z → 0
pz[j_, s_, z_] := Sum[bin[z, k] p[j, s, k], {k, 1, Log[2, j]}]
```

```
pz[32, 0, z]
```

$$\frac{z}{5} + \frac{5}{12}(-1+z)z + \frac{7}{24}(-2+z)(-1+z)z + \frac{1}{12}(-3+z)(-2+z)(-1+z)z + \frac{1}{120}(-4+z)(-3+z)(-2+z)(-1+z)z$$

```
pz[16, 0, z]
```

$$\frac{z}{4} + \frac{11}{24}(-1+z)z + \frac{1}{4}(-2+z)(-1+z)z + \frac{1}{24}(-3+z)(-2+z)(-1+z)z$$

```
pz[8, 0, z]
```

$$\frac{z}{3} + \frac{1}{2}(-1+z)z + \frac{1}{6}(-2+z)(-1+z)z$$

```
Expand[pz[36, 0, z]]
```

$$-\frac{3z}{4} + \frac{3z^2}{2} - z^3 + \frac{z^4}{4}$$

```
pz[2, 0, z]
```

z

```
Table[pz[n, 0, z], {n, 1, 10}] // TableForm
```

0
z
z
$\frac{z}{2} + \frac{1}{2} (-1 + z) z$
z
$(-1 + z) z$
z
$\frac{z}{3} + \frac{1}{2} (-1 + z) z + \frac{1}{6} (-2 + z) (-1 + z) z$
$\frac{z}{2} + \frac{1}{2} (-1 + z) z$
$(-1 + z) z$

```
Clear[rr, zp, zeta, pa, dz, dzalt, Da, zp2]
rr[n_, s_] := rr[n, s] = If[ n == 1, 0, FullSimplify[ MangoldtLambda[n] / Log[n]] n^-s]
zp[n_, s_, z_, k_] := zp[n, s, z, k] =
  Expand[1 + z / k Sum[ If[rr[j, s] == 0, 0, rr[j, s] zp[Floor[n / j], s, z, k + 1]], {j, 2, n}]]
zp2[n_, s_, z_, k_] := zp2[n, s, z, k] =
  Expand[1 + z / k Sum[t^-1 Prime[j]^(-s t) zp2[Floor[n / (Prime[j]^t)], s, z, k + 1],
      {t, 1, Log[2, n]}, {j, 1, PrimePi[n^(1 / t)]}]]
zeta[n_, s_, z_, k_] := zeta[n, s, z, , k] = Expand[
   1 + ((z + 1) / k - 1) Sum[j^-s zeta[Floor[n / j], s, z, k + 1], {j, 2, n}]]
pa[n_, 0, a_] := UnitStep[n - 1]
rie[n_] := rie[n] = Sum[ PrimePi[ n^(1 / k)] / k, {k, 1, Log[2, n]}]
pa[n_, 1, a_] := pa[n, 1, a] = rie[n] - rie[a]
pa[n_, k_, a_] := pa[n, k, a] =
  Sum[If[rr[m, 0] == 0, 0, Sum[Binomial[k, j] (rr[m, 0])^j pa[Floor[n / (m^j)], k - j, m],
     {j, 1, k}]], {m, a + 1, Floor[n^(1 / k)]}]
dz[n_, z_] := Sum[ z^k / (k!) pa[n, k, 1], {k, 0, Log[2, n]}]
Da[n_, k_, a_] := Da[n, k, a] =
  Sum[Binomial[k, j] Da[Floor[n / (m^(k - j))], j, m], {m, a + 1, n^(1 / k)}, {j, 0, k - 1}]
Da[n_, 0, a_] := UnitStep[n - 1]
Da[n_, 1, a_] := Floor[n] - a
dzalt[n_, z_] := Sum[Binomial[z, k] Da[n, k, 1], {k, 0, Log[2, n]}]
```

```
Timing[zp[100 000, 0, z, 1]]
```

$$\left\{7.301,\ 1 + \frac{991\,892\,879\, z}{102\,960} + \frac{16\,611\,877\,533\,197\, z^2}{605\,404\,800} + \frac{27\,613\,425\,421\,567\, z^3}{864\,864\,000} + \frac{8\,883\,298\,064\,606\,291\, z^4}{435\,891\,456\,000} + \frac{82\,938\,597\,121\, z^5}{10\,264\,320} + \frac{12\,123\,475\,378\,339\, z^6}{5\,748\,019\,200} + \frac{987\,114\,594\,581\, z^7}{2\,612\,736\,000} + \frac{6\,832\,898\,553\,167\, z^8}{146\,313\,216\,000} + \frac{53\,237\,749\, z^9}{13\,063\,680} + \frac{1\,772\,592\,397\, z^{10}}{7\,315\,660\,800} + \frac{20\,466\,961\, z^{11}}{2\,052\,864\,000} + \frac{30\,323\,737\, z^{12}}{114\,960\,384\,000} + \frac{841\, z^{13}}{186\,810\,624} + \frac{9773\, z^{14}}{209\,227\,898\,880} + \frac{71\, z^{15}}{373\,621\,248\,000} + \frac{17\, z^{16}}{20\,922\,789\,888\,000}\right\}$$

Timing[zeta[10 000, 0, z, 1]]

$$\left\{75.208,\ 1+\frac{56\,175\,529\,z}{45\,045}+\frac{5\,304\,616\,687\,z^{2}}{1\,663\,200}+\frac{64\,238\,883\,431\,z^{3}}{19\,958\,400}+\frac{3\,688\,608\,229\,z^{4}}{2\,177\,280}+\frac{11\,603\,252\,491\,z^{5}}{21\,772\,800}+\frac{4\,483\,862\,353\,z^{6}}{43\,545\,600}+\frac{557\,009\,347\,z^{7}}{43\,545\,600}+\frac{2\,872\,319\,z^{8}}{2\,903\,040}+\frac{688\,397\,z^{9}}{14\,515\,200}+\frac{58\,651\,z^{10}}{43\,545\,600}+\frac{8339\,z^{11}}{479\,001\,600}+\frac{17\,z^{12}}{95\,800\,320}+\frac{z^{13}}{6\,227\,020\,800}\right\}$$

Timing[dz[100 000 000, z]]

$$\left\{0.,\ 1+\frac{6\,427\,431\,691\,337\,929\,z}{1\,115\,464\,350}+\frac{2\,516\,314\,672\,020\,796\,036\,867\,z^{2}}{128\,629\,994\,613\,120}+\frac{3\,575\,746\,713\,648\,621\,345\,062\,531\,z^{3}}{124\,672\,148\,625\,024\,000}+\frac{20\,380\,394\,053\,499\,739\,865\,496\,567\,z^{4}}{831\,147\,657\,500\,160\,000}+\frac{1\,351\,629\,191\,016\,384\,695\,492\,785\,649\,z^{5}}{97\,569\,507\,619\,584\,000\,000}+\frac{240\,272\,276\,318\,313\,590\,611\,783\,219\,z^{6}}{43\,364\,225\,608\,704\,000\,000}+\frac{41\,833\,655\,627\,451\,907\,360\,857\,929\,z^{7}}{25\,545\,471\,085\,854\,720\,000}+\frac{3\,761\,102\,376\,291\,956\,378\,646\,751\,z^{8}}{10\,218\,188\,434\,341\,888\,000}+\frac{5\,175\,675\,474\,335\,907\,053\,437\,241\,z^{9}}{80\,669\,908\,692\,172\,800\,000}+\frac{1\,418\,826\,547\,912\,276\,016\,443\,177\,z^{10}}{161\,339\,817\,384\,345\,600\,000}+\frac{279\,266\,312\,199\,608\,569\,103\,z^{11}}{292\,017\,769\,021\,440\,000}+\frac{1\,386\,818\,791\,325\,659\,005\,283\,z^{12}}{16\,703\,416\,388\,026\,368\,000}+\frac{4\,814\,640\,871\,442\,135\,348\,159\,z^{13}}{835\,170\,819\,401\,318\,400\,000}+\frac{3\,482\,068\,013\,538\,497\,942\,357\,z^{14}}{10\,857\,220\,652\,217\,139\,200\,000}+\frac{1\,188\,915\,869\,941\,720\,073\,z^{15}}{83\,517\,081\,940\,131\,840\,000}+\frac{4\,192\,807\,585\,604\,237\,z^{16}}{8\,351\,708\,194\,013\,184\,000}+\frac{17\,963\,977\,627\,832\,867\,z^{17}}{1\,290\,718\,539\,074\,764\,800\,000}+\frac{390\,742\,194\,213\,977\,z^{18}}{1\,290\,718\,539\,074\,764\,800\,000}+\frac{1\,555\,110\,247\,813\,z^{19}}{306\,545\,653\,030\,256\,640\,000}+\frac{31\,437\,955\,243\,z^{20}}{490\,473\,044\,848\,410\,624\,000}+\frac{14\,797\,988\,921\,z^{21}}{24\,523\,652\,242\,420\,531\,200\,000}+\frac{976\,022\,221\,z^{22}}{269\,760\,174\,666\,625\,843\,200\,000}+\frac{786\,869\,z^{23}}{47\,726\,800\,133\,326\,110\,720\,000}+\frac{1493\,z^{24}}{38\,778\,025\,108\,327\,464\,960\,000}+\frac{727\,z^{25}}{31\,022\,420\,086\,661\,971\,968\,000\,000}+\frac{z^{26}}{403\,291\,461\,126\,605\,635\,584\,000\,000}\right\}$$

Timing[dzalt[10 000 000, 4]]

{10.764, 8 840 109 380}

```
Timing[zp2[10 000 000, 0, z, 1]]
```

$$\left\{206.686,\ 1+\frac{3559637526370229\,z}{5354228880}+\frac{1989544871269240547\,z^2}{921858537600}+\frac{2021824016451264335171\,z^3}{677566025136000}+\frac{7019677821920298561119\,z^4}{2956651746048000}+\frac{6419737164240558941381\,z^5}{5217620728320000}+\frac{355971199127948600783\,z^6}{800296713216000}+\frac{87671088330394680791\,z^7}{750278168640000}+\frac{647852562694427393\,z^8}{28245766348800}+\frac{1156246192125011873\,z^9}{337983528960000}+\frac{6208422327896021939\,z^{10}}{15817629155328000}+\frac{250225399924000051\,z^{11}}{7189831434240000}+\frac{6106970322634813\,z^{12}}{2549361475584000}+\frac{63437608022863169\,z^{13}}{497125487738880000}+\frac{47189432328823\,z^{14}}{9038645231616000}+\frac{17111105280953\,z^{15}}{105450861035520000}+\frac{120162307939\,z^{16}}{31635258310656000}+\frac{179878582253\,z^{17}}{2688996956405760000}+\frac{2349779\,z^{18}}{2845499424768000}+\frac{53393233\,z^{19}}{7298706024529920000}+\frac{94223\,z^{20}}{2919482409811968000}+\frac{13\,z^{21}}{154120489205760000}+\frac{53\,z^{22}}{224800145555521536000}+\frac{z^{23}}{25852016738884976640000}\right\}$$

```
Timing[zp2[100 000, N[ZetaZero[1]], z, 1]]
```

$$\{4.197,\ 1-(4.33543-1.75998\,i)\,z+(1.19578+4.25396\,i)\,z^2-(7.46539-6.53483\,i)\,z^3-(0.917224-3.46318\,i)\,z^4-(1.74755-1.97308\,i)\,z^5-(0.176183-0.396989\,i)\,z^6-(0.0860726-0.118227\,i)\,z^7-(0.00552684-0.00965199\,i)\,z^8-(0.000985228-0.00177868\,i)\,z^9-(0.0000455564-0.0000486405\,i)\,z^{10}-\left(2.29571\times10^{-6}-6.4587\times10^{-6}\,i\right)z^{11}-\left(1.31806\times10^{-7}-4.38354\times10^{-8}\,i\right)z^{12}+\left(6.2637\times10^{-10}+4.16854\times10^{-9}\,i\right)z^{13}-\left(6.24985\times10^{-11}-4.10989\times10^{-11}\,i\right)z^{14}+\left(1.94312\times10^{-13}-1.48812\times10^{-13}\,i\right)z^{15}+\left(1.74515\times10^{-15}+1.92705\times10^{-15}\,i\right)z^{16}\}$$

```
Timing[PrimePi[10^13]]
```

{0., 346 065 536 839}

```
10^8
```

100 000 000

```
CountPrimes[1000]
```

168

```
WheelEntries = 5;
WheelSize := Product[Prime[j], {j, 1, WheelEntries}];
CoprimeCache := Table[CoprimeQ[WheelSize, n], {n, 1, WheelSize}]
Use[n_] := If[CoprimeCache[[Mod[n - 1, WheelSize] + 1]] == True, 1, 0]
LegendrePhi[x_, a_] := LegendrePhi[x, a - 1] - LegendrePhi[x / Prime[a], a - 1]
LegendrePhi[x_, 0] := Floor[x]
LegPhiCache := LegPhiCache = Table[LegendrePhi[n, WheelEntries], {n, 1, WheelSize}]
FullWheel := LegendrePhi[WheelSize, WheelEntries];
Coprimes[n_] :=
 LegPhiCache[[Mod[n - 1, WheelSize] + 1]] + Floor[(n - 1) / WheelSize] FullWheel
d[n_, 0, a_] := 1
d[n_, 1, a_] := Coprimes[n] - Coprimes[a]
d[n_, k_, a_] := Sum[If[ Use[m] == 0, 0, Binomial[k, j] d[Floor[n / (m^(k - j))], j, m]],
   {m, a + 1, n^(1 / k)}, {j, 0, k - 1}]
RiemannPrimeCounting[n_] := Sum[(-1)^(k + 1) / k d[Floor[n], k, 1], {k, 1, Log[2, n]}]
CountPrimes[ n_] :=
 WheelEntries + Sum[MoebiusMu[k] RiemannPrimeCounting[n^(1 / k)] / k, {k, 1, Log[2, n]}]
```

```
CountPrimes[10 000 000]
```

664 579

```
d[n_, 0, a_] := 1
d[n_, 1, a_] := Floor[n] - a
d[n_, k_, a_] :=
 Sum[Binomial[k, j] d[Floor[n / (m^(k - j))], j, m], {m, a + 1, n^(1 / k)}, {j, 0, k - 1}]
RiemannPrimeCounting[n_] := Sum[(-1)^(k + 1) / k d[n, k, 1], {k, 1, Log[2, n]}]
CountPrimes[ n_] := Sum[MoebiusMu[k] RiemannPrimeCounting[n^(1 / k)] / k, {k, 1, Log[2, n]}]
```

```
Timing[CountPrimes[10 000 000]]
```

{32.573, 664 579}

```
Timing[PrimePi[10^13]]
```

{12.293, 346 065 536 839}

```
Limit[ (1 + y^(s - 1) HurwitzZeta[ s, 1 + y])^z, y → Infinity]
```

$$\left(\frac{s}{-1+s}\right)^z$$

```
10^7
```

10 000 000