$$f(p)=\frac{1}{1-z}\cdot\frac{\sin(\pi z)}{\pi}$$

$$f(p^2)=\frac{1}{(2-z)(1-z)}\cdot\frac{\sin(\pi z)}{\pi}$$

$$f(p^3)=\frac{2}{(3-z)(2-z)(1-z)}\cdot\frac{\sin(\pi z)}{\pi}$$

$$f(p^4)=\frac{6}{(4-z)(3-z)(2-z)(1-z)}\cdot\frac{\sin(\pi z)}{\pi}$$

$$f(p^k)=\frac{(k-1)!}{(k-z)_k}\cdot\frac{\sin(\pi z)}{\pi}=\binom{k-1}{z-1}$$

...

$$f(p)=\frac{1}{1-z}\cdot\frac{\sin(\pi z)}{\pi}$$

$$f(p_1\cdot p_2)=\frac{z}{(2-z)(1-z)}\cdot\frac{\sin(\pi z)}{\pi}$$

$$f(p_1^2\cdot p_2)=\frac{2z}{(3-z)(2-z)(1-z)}\cdot\frac{\sin(\pi z)}{\pi}$$

$$f(p_1^3\cdot p_2)=\frac{6z}{(4-z)(3-z)(2-z)(1-z)}\cdot\frac{\sin(\pi z)}{\pi}$$

$$f(p_1^4\cdot p_2)=\frac{24z}{(5-z)(4-z)(3-z)(2-z)(1-z)}\cdot\frac{\sin(\pi z)}{\pi}$$

$$f(p_1^k\cdot p_2)=\frac{k!\cdot z}{(k+1-z)_{k+1}}\cdot\frac{\sin(\pi z)}{\pi}\ (?)$$

...

$$f(p)=\frac{1}{1-z}\cdot\frac{\sin(\pi z)}{\pi}$$

$$f(p_1\cdot p_2)=\frac{z}{(2-z)(1-z)}\cdot\frac{\sin(\pi z)}{\pi}$$

$$f(p_1\cdot p_2\cdot p_3)=\frac{z(1+z)}{(3-z)(2-z)(1-z)}\cdot\frac{\sin(\pi z)}{\pi}$$

(but pattern seems not to hold) Actually, what it is, is the Stirling numbers of the second kind.

$$f_z'(p_1\cdot p_2\cdot\ldots p_a)=\frac{\sin(\pi z)}{\pi}\cdot\sum_{k=0}^{a}\frac{(-1)^k}{(z-k)}\cdot S(a,k)\cdot k!$$

1

$1 \to 1$

2

$2 \to 1$
$1,1 \to z$

3

$3 \to 2$
$2,1 \to 2z$
$1,1,1 \to z(1+z)$

4

$4 \to 6$
$3,1 \to 6z$
$2,2 \to z(z+5)$
$2,1,1 \to 2z(1+2z)$
$1,1,1,1 \to z^2(z+5)$

5

$5 \to 24$
$4,1 \to 24z$
$3,2 \to 6z(3+z)$
$3,1,1 \to 6z(1+3z)$
$2,2,1 \to 2z(z^2+9z+2)$
$2,1,1,1 \to 2z(4z^2+9z-1)$
$1,1,1,1,1 \to z(1+z)(z^2+15z-4)$

6

$6 \to 120$
$5,1 \to 120z$
$4,2 \to 12z(7+3z)$
$4,1,1 \to 24z(1+4z)$
$3,3 \to 2z(2+z)(19+z)$
$3,2,1 \to 2z(z^2+9z+2)$
$3,1,1,1 \to 6z(9z^2+13z-2)$
$2,2,2 \to z(z+3)(z^2+27z+2)$
$2,2,1,1 \to 2z(2z^3+33z^2+31z-6)$
$2,1,1,1,1 \to 2z(8z^3+51z^2+7z-6)$
$1,1,1,1,1,1 \to z^2(z^3+42z^2+119z-42)$

$$d_z'(n)=\frac{\sin(\pi z)}{\pi}\cdot\sum_{k=0}\frac{(-1)^k}{z-k}\cdot d_k'(n)$$

$$d_z'(p_1^{a_1}\cdot p_2^{a_2}\cdot\ldots p_k^{a_k})=\frac{\sin(\pi z)}{\pi}\cdot\frac{1}{(a_1+a_2+\ldots a_k-z)_{a_1+a_2+\ldots a_k}}\cdot P(z)$$

$$\frac{1}{(a_1+a_2+\ldots a_k-z)_{a_1+a_2+\ldots a_k}}\cdot P(z)=\sum_{k=0}\frac{(-1)^k}{z-k}\cdot d_k'(n)$$

$$P(z)=(a_1+a_2+\ldots a_k-z)_{a_1+a_2+\ldots a_k}\cdot\sum_{k=0}\frac{(-1)^k}{z-k}\cdot d_k'(n)$$

$$d_z'(p_1\cdot p_2\cdot\ldots p_a)=\frac{\sin(\pi z)}{\pi}\cdot\sum_{k=0}^{\infty}\frac{(-1)^k}{(z-k)}\cdot S(a,k)\cdot k!$$

$$d_k'(p^a)=\binom{a-1}{k-1}$$

$$d_k'(p_1\cdot p_2\cdot\ldots p_a)=S(a,k)\cdot k!$$

where $S(a,k)$ are Stirling numbers of the second kind.

$$d_k'(p_1^{a_1}\cdot p_2^{a_2}\cdot\ldots p_t^{a_t})=(-1)^{k+1}\cdot k\cdot {}_pF_q(\{1+a_{1,}1+a_{2,}\ldots 1+a_t,1-k\},\{1\,\text{(t-1 times)},2\},1)$$

Now,

$$S(a,k)\cdot k!=\sum_{j=0}^{k}(-1)^{k-j}\cdot j^a\cdot\binom{k}{j}$$

$$d_k'(n)=\sum_{j=0}^{k}(-1)^{k-j}\cdot\binom{k}{j}d_j(n)$$

$$d_k'(p_1^2\cdot p_2^2)=\sum_{j=0}^{k}(-1)^{k-j}\cdot\binom{k}{j}d_j(p_1^2\cdot p_2^2)=\sum_{j=0}^{k}(-1)^{k-j}\cdot\binom{k}{j}\cdot\frac{j^{(2)}}{2!}\cdot\frac{j^{(2)}}{2!}$$

$$d_k'(p_1^2\cdot p_2^2\cdot p_3)=\sum_{j=0}^{k}(-1)^{k-j}\cdot\binom{k}{j}\cdot\frac{j^{(2)}}{2!}\cdot\frac{j^{(2)}}{2!}\cdot\frac{j^{(1)}}{1!}$$

$$d_k'(p_1\cdot p_2\cdot p_3)=\sum_{j=0}^{k}(-1)^{k-j}\cdot\binom{k}{j}\cdot\frac{j^{(1)}}{1!}\cdot\frac{j^{(1)}}{1!}\cdot\frac{j^{(1)}}{1!}=\sum_{j=0}^{k}(-1)^{k-j}\cdot\binom{k}{j}\cdot j^3$$

$$d_k'(p_1^2\cdot p_2^2\cdot p_3^2)=\sum_{j=0}^{k}(-1)^{k-j}\cdot\binom{k}{j}\cdot\left(\frac{j(j+1)}{2}\right)^3$$

$$d_k'(p_1^3\cdot p_2\cdot p_3)=\sum_{j=0}^{k}(-1)^{k-j}\cdot\binom{k}{j}\cdot\frac{j^3(j+1)(j+2)}{6}$$

$$d_k'(p_1^a)=\sum_{j=0}^{k}(-1)^{k-j}\cdot\binom{k}{j}\cdot\frac{j^{(a)}}{a!}$$

$$d_k'(p_1^{a_1}\cdot p_2^{a_2}\cdot\ldots p_t^{a_t})=(-1)^{k+1}\cdot k\cdot {}_{t+1}F_t(\{1+a_1, 1+a_2, \ldots 1+a_t, 1-k\}, \{1\,(\text{t-1 times}), 2\}, 1)$$

$$d_k'(p_1^{a_1})=(-1)^{k+1}\cdot k\cdot {}_pF_q(\{1+a_1, 1-k\}, \{2\}, 1)=\binom{a-1}{k-1}$$

$$d_k'(p_1\cdot p_2\cdot\ldots p_t)=(-1)^{k+1}\cdot k\cdot {}_pF_q(\{2\,(\text{t times}), 1-k\}, \{1\,(\text{t-1 times}), 2\}, 1)=S(t,k)\cdot k!$$

$$d_k'(p_1^{a_1}\cdot p_2^{a_2}\cdot\ldots p_t^{a_t})=\sum_{j=0}(-1)^{k+j-1}\cdot\frac{1}{j!}\cdot\binom{k}{j+1}\frac{(a_1+j)!}{(a_1!\,j!)}\cdot\frac{(a_2+j)!}{(a_2!\,j!)}\cdots\frac{(a_t+j)!}{(a_t!\,j!)}$$

$$d_k'(n)=\sum_{j=0}(-1)^{k+j-1}\cdot\frac{1}{j!}\cdot\binom{k}{j+1}\prod_{p^a|n}\frac{(a+j)!}{a!\,j!}$$

...

$$\binom{x-1}{k-1}=\sum_{j=0}^{k}(-1)^{k-j}\cdot\binom{k}{j}\frac{j^{(x)}}{x!}$$

$$d_k'(x)=\sum_{j=0}^{k}(-1)^{k-j}\cdot\binom{k}{j}\prod_{p^a|x}\frac{j^{(a)}}{a!}$$

$$d_k'(x)=\sum_{j=0}^{k}(-1)^{k-j}\cdot\binom{k}{j}\prod_{p^a|x}\frac{(a+j-1)!}{a!(j-1)!}$$

...

$$\binom{x-1}{k-1}=(-1)^{k+1}\cdot k\cdot {}_pF_q(\{1+x, 1-k\}, \{2\}, 1)$$

$$d_k'(p_1^{a_1}\cdot p_2^{a_2}\cdot\ldots p_t^{a_t})=(-1)^{k+1}\cdot k\cdot {}_pF_q(\{1+a_1, 1+a_2, \ldots 1+a_t, 1-k\}, \{1\,(\text{t-1 times}), 2\}, 1)$$

...

$$\binom{x-1}{k-1}=(-1)^{k+1}\cdot k\cdot {}_2F_1(1+x, 1-k; 2, 1)$$

…

$$\binom{x}{k}=\sum_{a\le x}\sum_{j=0}^{k}(-1)^{k-j}\cdot\binom{k}{j}\frac{j^{(a)}}{a!}$$

$$D_k'(x)=\sum_{a\le x}\sum_{j=0}^{k}(-1)^{k-j}\cdot\binom{k}{j}\prod_{p^\alpha|a}\frac{j^{(\alpha)}}{\alpha!}$$

$$D_k'(x)=\sum_{2^{a_1}\cdot 3^{a_2}\cdot\ldots\le x}\sum_{j=0}^{k}(-1)^{k-j}\cdot\binom{k}{j}\prod_{a}\frac{j^{(a)}}{a!}$$

...

$$D_k'(x)=\sum_{j=0}^{k}(-1)^{k-j}\cdot\binom{k}{j}\sum_{2^{a_1}\cdot 3^{a_2}\cdot\ldots\le x}\prod_{a}\frac{j^{(a)}}{a!}$$