```
Integrate[Log[x] / x^(1 / 2), {x, 0, n}]
```

$2\sqrt{n}\ (-2 + \text{Log}[n])$

```
FullSimplify[Integrate[Log[x]^k / x^(1 / 2), {x, 1, n}], Element[k, Integers]]
```

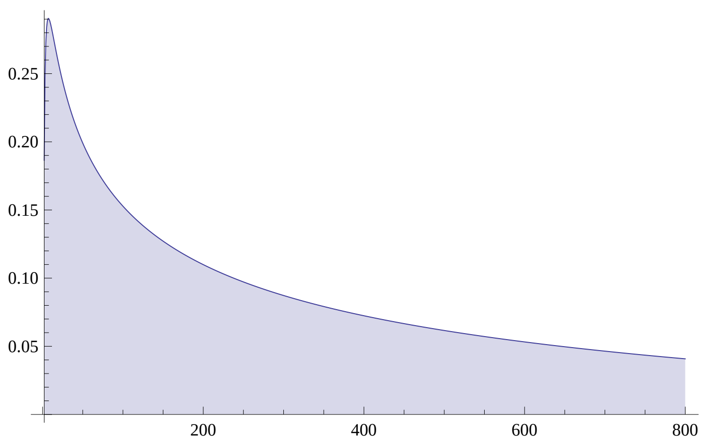
$\text{ConditionalExpression}\left[(-1)^k\, 2^{1+k} \left(-k! + \text{Gamma}\left[1 + k, -\frac{\text{Log}[n]}{2}\right]\right), k \geq 0 \,\&\&\, \text{Log}[n] > 0\right]$

$\text{intlog}[n\_, k\_] := (-1)^k\, 2^{1+k} \left(-k! + \text{Gamma}\left[1 + k, -\frac{\text{Log}[n]}{2}\right]\right)$

```
sumlog[n_, k_] := Sum[Log[x]^k / x^(1 / 2), {x, 1, n}]
dlog[n_, k_] := sumlog[n, k] - intlog[n, k]
```

$\text{dlog2}[n\_, k\_] := \text{sumlog}[n, k] - \left(-\text{ExpIntegralE}\left[-k, -\frac{\text{Log}[n]}{2}\right] \text{Log}[n]^{1+k}\right)$

```
DiscretePlot[{Re@dlog[n, 1]}, {n, 2, 800}]
```




```
FullSimplify[Integrate[Log[x]^k / x^(1 / 2), {x, 1, n}], Element[k, Integers]]
```

$\text{ConditionalExpression}\left[(-1)^k\, 2^{1+k} \left(-k! + \text{Gamma}\left[1 + k, -\frac{\text{Log}[n]}{2}\right]\right), k \geq 0 \,\&\&\, \text{Log}[n] > 0\right]$

```
Chop@dlog[1 000 000, 4.]
```

18.218

```
sumlog[n, 1]
```

$-\frac{1}{4} (2\,\text{EulerGamma} + \pi + \text{Log}[64] + 2\,\text{Log}[\pi])\, \text{Zeta}\left[\frac{1}{2}\right] + \text{Zeta}^{(1,0)}\left[\frac{1}{2}, 1 + n\right]$

```
Chop@dlog[1 800 000, 2.]
```

$-15.931 + 1.0374 \times 10^{-10}\, i$

```
dlog[n, 1]
```

$4 \left(-1 + \text{Gamma}\left[2, -\frac{\text{Log}[n]}{2}\right]\right) - \frac{1}{4} (2\,\text{EulerGamma} + \pi + \text{Log}[64] + 2\,\text{Log}[\pi])\, \text{Zeta}\left[\frac{1}{2}\right] + \text{Zeta}^{(1,0)}\left[\frac{1}{2}, 1 + n\right]$

dlog[100 000., 1.]

$3.94085 + 7.36809 \times 10^{-13}\, i$

dlog[1 000 000., 1.]

$3.92955 + 2.89397 \times 10^{-12}\, i$

Table[dlog[1 000 000., k], {k, 1, 20}]

$\{3.92955 + 2.89397 \times 10^{-12}\, i,\ -15.9129 + 7.03468 \times 10^{-11}\, i,\ 97.3218 + 1.30447 \times 10^{-9}\, i,$
$-749.782 + 2.17774 \times 10^{-8}\, i,\ 7931.66 + 3.44158 \times 10^{-7}\, i,\ -88\,683.3 + 6.13983 \times 10^{-6}\, i,$
$1.33827 \times 10^{6} + 0.0000899973\, i,\ -1.99802 \times 10^{7} + 0.00130568\, i,\ 3.80757 \times 10^{8} + 0.0187965\, i,$
$-7.30512 \times 10^{9} + 0.978346\, i,\ 1.65249 \times 10^{11} + 3.83083\, i,\ -3.89981 \times 10^{12} - 66.936\, i,$
$1.02358 \times 10^{14} + 768.713\, i,\ -2.85204 \times 10^{15} + 31\,816.5\, i,\ 8.57635 \times 10^{16} + 152\,640.\, i,$
$-2.74151 \times 10^{18} - 1.51509 \times 10^{6}\, i,\ 9.32535 \times 10^{19} + 3.00833 \times 10^{7}\, i,$
$-3.35652 \times 10^{21} + 1.06486 \times 10^{9}\, i,\ 1.27556 \times 10^{23} + 5.89641 \times 10^{9}\, i,\ -5.10213 \times 10^{24} - 3.1436 \times 10^{10}\, i\}$

N@20!

2.4329×10^{18}

dlog2[1 000 000., 11] / 11!

$4139.84 + 9.59704 \times 10^{-8}\, i$

dlog2[100 000., 11] / 11!

$4114.66 + 3.62024 \times 10^{-9}\, i$

Chop@Table[dlog2[10 000., k] / k!, {k, 1, 20}] // TableForm

3.9687
-7.7921
16.6516
-30.5007
66.7616
-123.761
261.578
-505.578
$1030.57 + 1.03684 \times 10^{-10}\, i$
$-2041.95 + 3.56646 \times 10^{-10}\, i$
4101.07
-8188.11
16 386.8
-32 766.2
65 537.1
-131 071.
262 144.
-524 288.
1.04858×10^{6}
-2.09715×10^{6}

```
Chop@Table[dlog2[100 000., k] / k !, {k, 1, 20}] // TableForm
```

3.94085
-7.89939
16.4027
$-30.8424 + 1.27754 \times 10^{-10}\, i$
$66.6651 + 4.00106 \times 10^{-10}\, i$
$-122.886 + 8.19979 \times 10^{-10}\, i$
$264.411 + 1.42186 \times 10^{-9}\, i$
$-499.896 + 2.13721 \times 10^{-9}\, i$
$1039.48 + 2.83552 \times 10^{-9}\, i$
$-2030.17 + 1.22486 \times 10^{-8}\, i$
$4114.66 + 3.62024 \times 10^{-9}\, i$
$-8174.1 - 4.37862 \times 10^{-9}\, i$
$16\,399.9 + 3.21408 \times 10^{-9}\, i$
$-32\,755. + 7.89763 \times 10^{-9}\, i$
$65\,546. + 2.09994 \times 10^{-9}\, i$
$-131\,065. - 1.08327 \times 10^{-9}\, i$
$262\,149. + 1.05227 \times 10^{-9}\, i$
$-524\,285. + 1.72127 \times 10^{-9}\, i$
$1.04858 \times 10^{6} + 4.17326 \times 10^{-10}\, i$
-2.09715×10^{6}

```
Chop@Table[dlog2[1 000 000., k] / k !, {k, 1, 20}] // TableForm
```

3.92955
-7.95646
$16.2203 + 2.17412 \times 10^{-10}\, i$
$-31.2409 + 9.07393 \times 10^{-10}\, i$
$66.0971 + 2.86798 \times 10^{-9}\, i$
$-123.171 + 8.52754 \times 10^{-9}\, i$
$265.53 + 1.78566 \times 10^{-8}\, i$
$-495.542 + 3.23828 \times 10^{-8}\, i$
$1049.26 + 5.17981 \times 10^{-8}\, i$
$-2013.1 + 2.69606 \times 10^{-7}\, i$
$4139.84 + 9.59704 \times 10^{-8}\, i$
$-8141.53 - 1.39741 \times 10^{-7}\, i$
$16\,437.6 + 1.23448 \times 10^{-7}\, i$
$-32\,715.1 + 3.64958 \times 10^{-7}\, i$
$65\,584.7 + 1.16726 \times 10^{-7}\, i$
$-131\,030. - 7.24136 \times 10^{-8}\, i$
$262\,178. + 8.45779 \times 10^{-8}\, i$
$-524\,262. + 1.66323 \times 10^{-7}\, i$
$1.0486 \times 10^{6} + 4.84722 \times 10^{-8}\, i$
$-2.09714 \times 10^{6} - 1.29212 \times 10^{-8}\, i$

```
Chop@Table[(-1)^(k+1) 2^(k+1), {k, 1, 20}] // TableForm
```

```
4
-8
16
-32
64
-128
256
-512
1024
-2048
4096
-8192
16 384
-32 768
65 536
-131 072
262 144
-524 288
1 048 576
-2 097 152
```

```
zv[k_] := (-1)^(k+1) 2^(k+1)
```

```
zz := {3.929553892847673`, -7.956461430038549`, 16.220296829406422`,
  -31.24091502455548`, 66.09712576771827`, -123.17120161528628`, 265.5303322565017`,
  -495.5416983733711`, 1049.2644258860655`, -2013.0959043772134`,
  4139.8379919593535`, -8141.529649237544`, 16437.63643590042`, -32715.070374004863`,
  65584.74998887476`, -131029.90587445711`, 262178.20893203217`,
  -524261.74367646256`, 1.0485950918186842`*^6, -2.097138811838005`*^6}
zs[z_, n_] := Zeta[1/2] + Sum[z^k zv[k], {k, 0, n}]
zs2[z_, n_] := Zeta[1/2] + Sum[z^k zz[[k]], {k, 1, n}]
```

```
zs2[0.1, 20]
```

```
-1.13331
```

```
Zeta[.6]
```

```
-1.95266
```

```
zz[[6]]
```

```
-123.171
```

```
zv[6]
```

```
-128
```

```
sl[n_, k_] :=
 (Sum[Log[x]^k / x^(1/2), {x, 1, n}] - Integrate[Log[x]^k / x^(1/2), {x, 0, n}]) k!
```

```
N[sl[100 000, 2]]
```

```
-31.5976
```

```
N[-2 Sqrt[n] (-2 + Log[n]) - 1/4 (2 EulerGamma + π + Log[64] + 2 Log[π]) Zeta[1/2] + Zeta^(1,0)[1/2, 1 + n] /.
  n → 10 000 000 000 000 000]
```

$3.92265 - 8.53367 \times 10^{-7}\, i$

```
Table[(N@Im@ZetaZero@1 I)^k, {k, 0, 10}]
```

$\{1. + 0.\, i,\ 0. + 14.1347\, i,\ -199.79 + 0.\, i,\ 0. - 2823.98\, i,$
$39\,916.2 + 0.\, i,\ 0. + 564\,205.\, i,\ -7.97488 \times 10^{6} + 0.\, i,\ 0. - 1.12723 \times 10^{8}\, i,$
$1.59331 \times 10^{9} + 0.\, i,\ 0. + 2.25209 \times 10^{10}\, i,\ -3.18327 \times 10^{11} + 0.\, i\}$

```
Cz1[n_, s_] :=
 Sum[Cos[s Log[j]] / j^(1 / 2), {j, 1, n}] - Integrate[Cos[s Log[j]] / j^(1 / 2), {j, 0, n}]
Cz2[n_, s_] := Sum[(1 + Sum[(-1)^k (s Log[j])^(2 k) / ((2 k)!), {k, 1, Infinity}]) / j^(1 / 2),
    {j, 1, n}] - Integrate[
    (1 + Sum[(-1)^k (s Log[j])^(2 k) / ((2 k)!), {k, 1, Infinity}]) / j^(1 / 2), {j, 0, n}]
Cz3[n_, s_] := (Sum[(1) / j^(1 / 2), {j, 1, n}] - Integrate[(1) / j^(1 / 2), {j, 0, n}]) +
   Sum[(Sum[(-1)^k (s Log[j])^(2 k) / ((2 k)!), {k, 1, Infinity}]) / j^(1 / 2), {j, 1, n}] -
   Integrate[(Sum[(-1)^k (s Log[j])^(2 k) / ((2 k)!), {k, 1, Infinity}]) / j^(1 / 2), {j, 0, n}]
Cz4[n_, s_] := (-2 √n + HarmonicNumber[n, 1/2]) +
   Sum[(Sum[(-1)^k (s Log[j])^(2 k) / ((2 k)!), {k, 1, Infinity}]) / j^(1 / 2), {j, 1, n}] -
   Integrate[(Sum[(-1)^k (s Log[j])^(2 k) / ((2 k)!), {k, 1, Infinity}]) / j^(1 / 2), {j, 0, n}]
Cz5[n_, s_] := (-2 √n + HarmonicNumber[n, 1/2]) +
   Sum[Sum[((-1)^k (s Log[j])^(2 k) / ((2 k)!)) / j^(1 / 2), {j, 1, n}] -
     Integrate[((-1)^k (s Log[j])^(2 k) / ((2 k)!)) / j^(1 / 2), {j, 0, n}], {k, 1, Infinity}]
Cz6[n_, s_] := (-2 √n + HarmonicNumber[n, 1/2]) +
   Sum[s^(2 k) (Sum[((-1)^k ( Log[j])^(2 k) / ((2 k)!)) / j^(1 / 2), {j, 1, n}] -
       Integrate[((-1)^k ( Log[j])^(2 k) / ((2 k)!)) / j^(1 / 2), {j, 0, n}]), {k, 1, Infinity}]
Cz7[n_, s_] := (-2 √n + HarmonicNumber[n, 1/2]) +
   Sum[(-1)^k s^(2 k) / ((2 k)!) (Sum[(( Log[j])^(2 k)) / j^(1 / 2), {j, 1, n}] -
       Integrate[(( Log[j])^(2 k)) / j^(1 / 2), {j, 0, n}]), {k, 1, Infinity}]
Cz8[n_, s_] := (-2 √n + HarmonicNumber[n, 1/2]) + Sum[(-1)^k s^(2 k) / ((2 k)!)
     (Sum[(( Log[j])^(2 k)) / j^(1 / 2), {j, 1, n}] - (2^(1+2 k) Gamma[1 + 2 k, -Log[n]/2])), {k,
     1, Infinity}]
Cz9[n_, s_] := Zeta[1 / 2] + Sum[(-1)^k s^(2 k) / ((2 k)!) zv[2 k], {k, 1, Infinity}]
Cz9a[n_, s_] := Zeta[1 / 2] + Sum[s^k / (k!) zv[k], {k, 1, Infinity}]
Ex7[n_, s_] :=
 (-2 √n + HarmonicNumber[n, 1/2]) + Sum[ s^k / (k!) (Sum[(( Log[j])^k) / j^(1 / 2), {j, 1, n}] -
       Integrate[(( Log[j])^k) / j^(1 / 2), {j, 0, n}]), {k, 1, Infinity}]
Ex7a[n_, s_] := (-2 √n + HarmonicNumber[n, 1/2]) +
   Sum[ s^k / (k!) (Sum[(( Log[j])^k) / j^(1 / 2), {j, 1, n}] -
        (-ExpIntegralE[-k, -Log[n]/2] Log[n]^(1+k))), {k, 1, Infinity}]

Ez1[n_, s_] :=
 Sum[Cos[s Log[j]] / j^(1 / 2), {j, 1, n}] - Integrate[Cos[s Log[j]] / j^(1 / 2), {j, 0, n}] +
  I (Sum[Sin[s Log[j]] / j^(1 / 2), {j, 1, n}] - Integrate[Sin[s Log[j]] / j^(1 / 2), {j, 0, n}])
```

```
N@Ex7[100, 1 / 10 I + 1 / 10]
```

$-1.01878 + 0.297658\, i$

```
N@Cz9a[10 000, (240 / 10) + (1 / 10) I]
```

$0.539645 + 5.55112 \times 10^{-17}\, i$

```
Zeta[.6 + .1 I]
```

$-1.80556 - 0.580585\, i$

```
N@Ez1[100, .1 I + .1]
```

$-1.77841 + 0.590908\, i$

```
Cz1[100, .4 + .1 I]
```

$-0.67567 + 0.216651\, i$

```
(Zeta[.5 + (.4 + .1 I) I] + Zeta[.5 - (.4 + .1 I) I]) / 2
```

$-0.661649 + 0.240722\, i$

```
Sum[(-1)^k (s Log[j])^(2 k) / ((2 k)!), {k, 0, Infinity}]
```

Cos[s Log[j]]

```
(Sum[(1) / j^(1 / 2), {j, 1, n}] - Integrate[(1) / j^(1 / 2), {j, 0, n}])
```

$-2\sqrt{n} + \text{HarmonicNumber}\left[n, \frac{1}{2}\right]$

```
N[dlog2[1 000 000, 2] / dlog2[1 000 000, 1]]
```

$-4.04955 + 2.08843 \times 10^{-11}\, i$

```
N[dlog2[100 000, 2] / dlog2[100 000, 1]]
```

$-4.00898 + 4.46399 \times 10^{-12}\, i$

```
FullSimplify[Integrate[( Log[j])^(2 k) / j^(1 / 2), {j, 0, n}], Element[k, Integers]]
```

$\text{ConditionalExpression}\left[2^{1+2k}\, \text{Gamma}\left[1+2k, -\frac{\text{Log}[n]}{2}\right], k > -\frac{1}{2}\right]$

$\mathbf{2^{1+2k}\, Gamma\left[1+2k, -\frac{Log[n]}{2}\right]}$

$Aborted

```
FullSimplify[Integrate[Log[j]^k / j^(1 / 2), {j, 0, n}], Element[k, Integers]]
```

$\text{ConditionalExpression}\left[(-1)^k\, 2^{1+k}\, \text{Gamma}\left[1+k, -\frac{\text{Log}[n]}{2}\right], k > -1\right]$

$\mathbf{-ExpIntegralE\left[-k, -\frac{Log[n]}{2}\right] Log[n]^{1+k}}$

$\mathbf{(-1)^k\, 2^{1+k}\, Gamma\left[1+k, -\frac{Log[n]}{2}\right]\ /.\ k \to 3\ /.\ n \to 100.}$

$658.862 - 3.69776 \times 10^{-13}\, i$

$$-\text{ExpIntegralE}\left[-k, -\frac{\text{Log}[n]}{2}\right] \text{Log}[n]^{1+k} \text{ /. } k \to 3 \text{ /. } n \to 100.$$

$658.862 - 3.69776 \times 10^{-13}\, i$

```
Integrate[Log[x]^3 / x^(1 / 2), {x, 1, n}]
```

$$\text{ConditionalExpression}\left[96 + 2\sqrt{n}\left(-48 + 24\,\text{Log}[n] - 6\,\text{Log}[n]^2 + \text{Log}[n]^3\right),\ \text{Re}[n] \geq 0 \,||\, n \notin \text{Reals}\right]$$

$$\text{CForm}\left[2\sqrt{n}\ (-48 + \text{Log}[n]\ (24 + (-6 + \text{Log}[n])\ \text{Log}[n]))\right]$$

```
2*Sqrt(n)*(-48 + Log(n)*(24 + (-6 + Log(n))*Log(n)))
```

$$\text{CForm}\left[96 + 2\sqrt{n}\left(-48 + 24\,\text{Log}[n] - 6\,\text{Log}[n]^2 + \text{Log}[n]^3\right)\right]$$

```
96 + 2*Sqrt(n)*(-48 + 24*Log(n) - 6*Power(Log(n),2) + Power(Log(n),3))
```

```
FullSimplify[Integrate[Log[x]^k / x^(1 / 2), {x, 0, n}], Element[k, Integers]]
```

$$\text{ConditionalExpression}\left[(-1)^k\, 2^{1+k}\, \text{Gamma}\left[1+k, -\frac{\text{Log}[n]}{2}\right],\ k > -1\right]$$

```
FullSimplify[Integrate[Log[x]^k / x^(1 / 2), {x, 1, n}], Element[k, Integers]]
```

$$\text{ConditionalExpression}\left[(-1)^k\, 2^{1+k}\left(-k! + \text{Gamma}\left[1+k, -\frac{\text{Log}[n]}{2}\right]\right),\ k \geq 0\ \&\&\ \text{Log}[n] > 0\right]$$

$$\text{intlog}[n\_, k\_] := (-1)^k\, 2^{1+k}\left(-k! + \text{Gamma}\left[1+k, -\frac{\text{Log}[n]}{2}\right]\right)$$

```
sumlog[n_, k_] := Sum[Log[x]^k / x^(1 / 2), {x, 2, n}]
dlog[n_, k_] := sumlog[n, k] - intlog[n, k]
edlog[n_, z_, l_] := Sum[z^k / k! dlog[n, k], {k, 0, l}]
```

```
edlog[1000, N@Im@ZetaZero@1, 100]
```

$5.5165 \times 10^{131} + 4.16913 \times 10^{28}\, i$

```
Zeta[.5 - .4]
```

-0.603038

```
edlog[100, z]
```

$Aborted

```
gah[1.1]
```

$1.21248 \times 10^{69} + 1.16953 \times 10^{-9}\, i$

```
Zeta[.5 - .3]
```

-0.733921

```
Integrate[ Cos[s Log[j]] / j^(1 / 2), {j, 0, 1}]
```

$$\text{ConditionalExpression}\left[\frac{2}{1 + 4 s^2},\ s \in \text{Reals}\right]$$

```
Integrate[ Sin[s Log[j]] / j^(1 / 2), {j, 0, 1}]
```

$$\text{ConditionalExpression}\left[-\frac{4\,s}{1+4\,s^2},\ -\frac{1}{2}<\text{Im}[s]<\frac{1}{2}\right]$$

```
Integrate[j^(I s) / j^(1 / 2), {j, 0, 1}]
```

$$\text{ConditionalExpression}\left[\frac{2\,i}{i-2\,s},\ \text{Im}[s]<\frac{1}{2}\right]$$

$$\text{FullSimplify}\left[\frac{2}{1+4\,s^2}+I\left(-\frac{4\,s}{1+4\,s^2}\right)\right]$$

$$\frac{2\,i}{i-2\,s}$$

```
Integrate[ Cos[s Log[j]] / j^(1 / 2), {j, 0, n}]
```

$$\int_0^n \frac{\text{Cos}[s\,\text{Log}[j]]}{\sqrt{j}}\,dj$$

```
Integrate[ Cos[s Log[j]] / j^(1 / 2), {j, 1, n}]
```

$$\text{ConditionalExpression}\left[\frac{-2+2\sqrt{n}\ (\text{Cos}[s\,\text{Log}[n]]+2\,s\,\text{Sin}[s\,\text{Log}[n]])}{1+4\,s^2},\ \text{Re}[n]\geq 0\ ||\ n\notin \text{Reals}\right]$$

$$\text{pos}[n\_,\ s\_] := 1+\text{Sum}[\text{Cos}[s\,\text{Log}[j]]\,/\,j\verb|^|(1/2),\ \{j,2,n\}] - \left(\frac{-2+2\sqrt{n}\ (\text{Cos}[s\,\text{Log}[n]]+2\,s\,\text{Sin}[s\,\text{Log}[n]])}{1+4\,s^2}+2\,/\,(1+4\,s\verb|^|2)\right)$$

$$\text{pos2}[n\_,\ s\_] := \text{Sum}[\text{Cos}[s\,\text{Log}[j]]\,/\,j\verb|^|(1/2),\ \{j,1,n\}] - \left(\frac{-2+2\sqrt{n}\ (\text{Cos}[s\,\text{Log}[n]]+2\,s\,\text{Sin}[s\,\text{Log}[n]])}{1+4\,s^2}+2\,/\,(1+4\,s\verb|^|2)\right)$$

```
pos[100, 10.]
```

1.52027

```
(Zeta[.5 + 10 I] + Zeta[.5 - 10 I]) / 2
```

1.5449 + 0. i

```
FullSimplify[Integrate[ Cos[s Log[j]] / j^(1 / 2), {j, 1, n}] +
  Integrate[ Cos[s Log[j]] / j^(1 / 2), {j, 0, 1}]]
```

$$\text{ConditionalExpression}\left[\frac{2\sqrt{n}\ (\text{Cos}[s\,\text{Log}[n]]+2\,s\,\text{Sin}[s\,\text{Log}[n]])}{1+4\,s^2},\ s\in\text{Reals}\ \&\&\ (\text{Re}[n]\geq 0\ ||\ n\notin\text{Reals})\right]$$