An Investigation of Certain Kinds of Polynomials and their Zeros, Used to Represent Many Number-Theoretic Functions

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Summary:The Riemann Prime counting function can be expressed in terms of the zeros of the generalized divisor summatory function.

Introduction

There is a certain type of polynomial, intimately connected to prime counting functions, that I have a number of questions about.

Refresher of Well-Known Mathematics: The Riemann Prime Counting Function

As a quick refresher, the Riemann Prime counting function can be defined as

$$\Pi(n)=\pi(n)+\frac{1}{2}\pi(n^{\frac{1}{2}})+\frac{1}{3}\pi(n^{\frac{1}{3}})...$$

where $\pi(n)$ is the number of primes less than or equal to n. So, for example, $\pi(10)=4$ because there are four primes less than or equal to 10 (those being 2,3,5,7). Meanwhile $\Pi(10)=5\frac{1}{3}$, because there are four primes less than or equal to 10, two primes less than or equal to $10^{\frac{1}{2}}$ (2,3), and finally one prime less than or equal to $10^{\frac{1}{3}}$ (2).

The most obvious way to compute this function in Mathematica is

```
RiemannPrimeCountingFunctionReference[n_] := Sum[ PrimePi[ n^(1/k)]/k,{k,1,Log2@n}]
```

This will be our main point of reference for this function.

Refresher of Well-Known Mathematics: The Explicit Formula for the Riemann Prime Counting Function Using the Zeros of the Riemann Zeta Function

There are other interesting ways to express the Riemann Prime Counting function, however. In particular, the zeros of certain other functions can be used in interesting ways.

The most noteworthy of these, of course, is the zeros of the Riemann Zeta function. Riemann originally introduced the Riemann Prime counting function (which he called f(x)) in his famous paper that also introduced the Riemann Zeta function, $\zeta(s)$, and its non-trivial zeros, generally denoted as ρ . In fact, his paper showed that the Riemann Prime counting function can be expressed explicitly in terms of the non-trivial zeros of the Riemann Zeta function, more or less, as

$$\Pi(n)=li(n)-\sum_{\rho} li(n^{\rho})-\log 2+\int_{x}^{\infty} \frac{dt}{t(t^2-1)\log t}$$

where $li(n)$ is the logarithmic integral function. I'm glossing over a number of important topics about order of summation, convergence, and some other tricky issues. This function is covered very well in Edwards *adsfasdfasdf.* This representation of $\Pi(n)$ is justly famous for tying the long term behavior of $\Pi(n)$ to properties of the non-trivial zeta zeros, leading to the importance of the Riemann hypothesis regarding those zeros.

Okay. That's all quite well known stuff. Enough overview. Now, I want to present a curious alternative way to define the Riemann Prime counting function, using a very different set of function zeros.

Expressing the Riemann Prime Counting Function with the Generalized Divisor Summatory Function

First, I'll start by defining the generalized divisor summatory function, which I will label $[\zeta(0)^z]_n$. One way to express it is as

$$[\zeta(0)^z]_n = \binom{z}{0} 1 + \binom{z}{1}\sum_{j=2}^{n} 1 + \binom{z}{2}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor \frac{n}{j} \rfloor} 1 + \binom{z}{3}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor \frac{n}{j} \rfloor}\sum_{l=2}^{\lfloor \frac{n}{j\cdot k} \rfloor} 1 + \binom{z}{4}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor \frac{n}{j} \rfloor}\sum_{l=2}^{\lfloor \frac{n}{j\cdot k} \rfloor}\sum_{m=2}^{\lfloor \frac{n}{j\cdot k\cdot l} \rfloor} 1 + \ldots$$

where $\binom{z}{k}$ is the generalized binomial, allowing z to be a complex value. See section ??? of Ivic's *The Riemann Zeta-Function: Theory and Applications* for other details about this function.

$$[\zeta(0)]_n = \sum_{j \le n} (1)(j) \quad [\zeta(0)^2]_n = \sum_{j \le n} (1*1)(j) \quad [\zeta(0)^3]_n = \sum_{j \le n} (1*1*1)(j) \quad [\zeta(0)^z]_n = \sum_{j \le n} (1^{*z})(j)$$

The connections between $[\zeta(0)^z]_n$ and $\Pi(n)$ are very, very deep. But for the moment, just note that

$$\Pi(n) = \lim_{z \to 0} \frac{\partial}{\partial z}[\zeta(0)^z]_n = \lim_{z \to 0} \frac{[\zeta(0)^z]_n - 1}{z}$$

Now, for a fixed positive integer *n*, $[\zeta(0)^z]_n$ will be a polynomial of degree $\lfloor \frac{\log n}{\log 2} \rfloor$. For example, $[\zeta(0)^z]_n = \frac{1}{6}z^3 + \frac{7}{2}z^2 + \frac{16}{3}z + 1$. Because this is just a typical polynomial, it can be expressed via its zeros in the usual way. If we denote the values of *z* where $[\zeta(0)^z]_n = 0$ as ρ, then $[\zeta(0)^z]_n = \prod_{\rho} 1 - \frac{z}{\rho}$.

And now here's the connection with the Riemann Prime counting function. Because $\Pi(n) = \lim_{z \to 0} \frac{[\zeta(0)^z]_n - 1}{z}$, we can use those same zeros, after working through a bit of algebra, to say

$$\Pi(n) = \sum_{\rho} -\frac{1}{\rho}$$

Because this paper is intended to be exploratory and question-oriented, we won't rigorously show that this works, though it does. Instead, let's explore it empirically. Try experimenting with this brief Mathematica snippet:

Instead, let's explore it empirically. Try experimenting with this brief Mathematica snippet:

```
zeta0[n_,z_,k_]:=zeta0[n,z,k]=1+((z+1)/k-1) Sum[ zeta0[Floor[n/j],z,k+1],{j,2,n}]
roots[n_]:=If[(c=Exponent[f=zeta0[n,z,1],z])==0,{}, If[ c==1,List@NRoots[f==0,z][[2]],List@@NRoots[f==0,z][[All,2]]]]
RiemannPrimeCountingFunction[n_] := Chop@FullSimplify@Sum[ -rho^-1,{rho, roots[n]}]
```

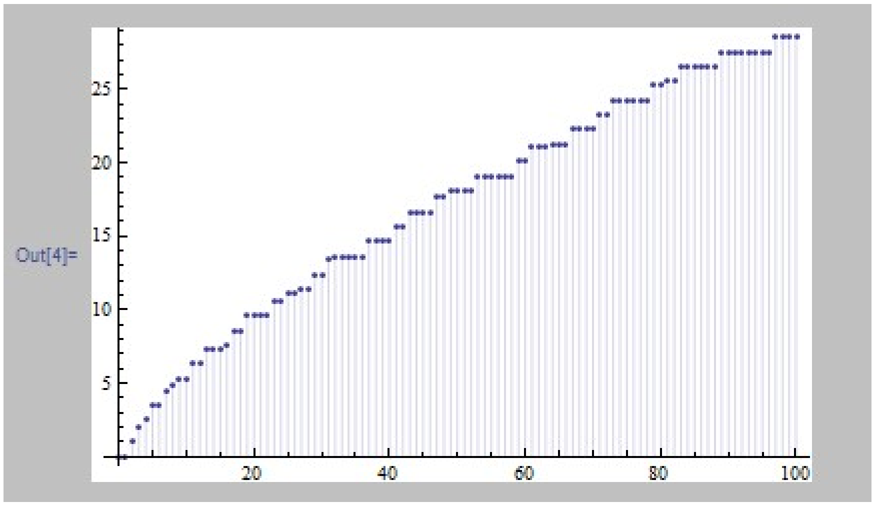
Note that the Divisor Summatory function $[\zeta(0)^z]_n$ is computed recursively here with the expression Dnz[n_, z_, 1], and that roots[n_] will compute the roots of $[\zeta(0)^z]_n$ for some constant positive integer n. A bit of experimentation should show that RiemannPrimeCountingFunction[n] and RiemannPrimeCountingFunctionReference[n] do indeed produce the same values. So as promised, the count of prime powers, $\Pi(n)$, is encoded in those zeros of $[\zeta(0)^z]_n$.

Let's explore a bit more.

Here's a graph of the Riemann Prime counting function, going from 0 to 100.

```
DiscretePlot[RiemannPrimeCountingFunction[n],{n,0,100}]
```

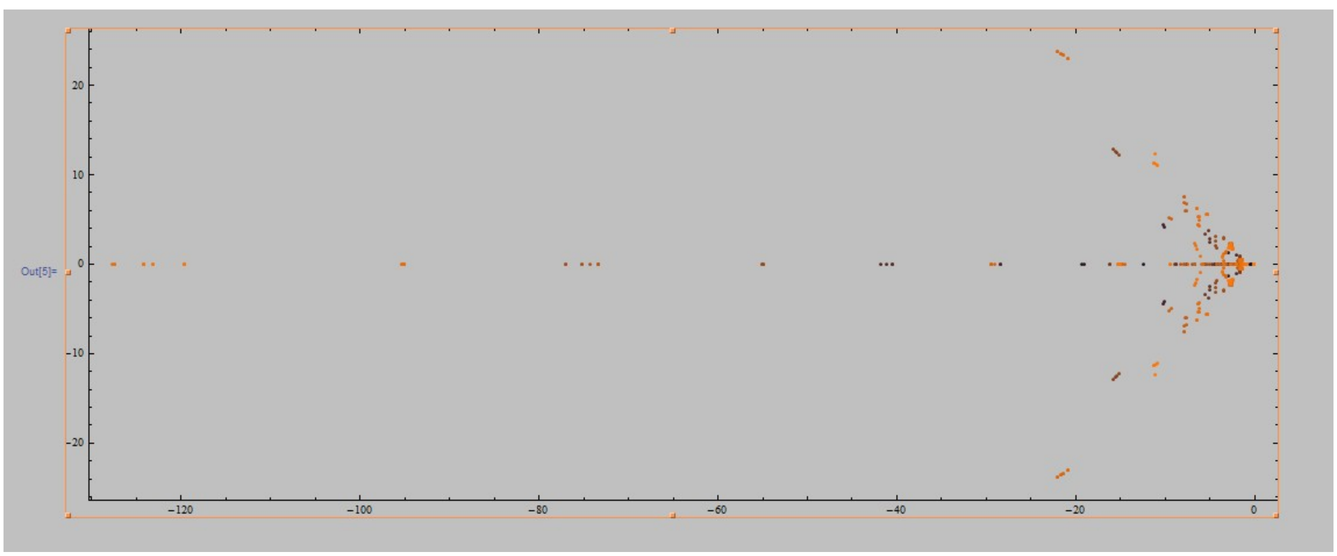
This Mathematica code yields



As advertised, it does indeed increase only at prme power values.

And the following Mathematica code is a plot of those zeros for values of *n* from 0 to 100, with a color gradient applied.

```
Graphics[Table[{ColorData["RustTones"][n/100],Point[{Re[#],Im[#]}]}&/@roots[n],{n,0,100}],Frame->True]
```



This table shows $[\zeta(0)^z]_n$ expanded as finite term polynomials for values of *n* going from 0 to 100.

```
Table[{n,Expand@zeta0[n,z,1]},{n,1,30}]//TableForm
```

```
{1, 1},
{2, 1+z},
{3, 1+2 z},
{4, 1+(5 z)/2+z²/2},
{5, 1+(7 z)/2+z²/2},
{6, 1+(7 z)/2+(3 z²)/2},
```

```
{7, 1+(9 z)/2+(3 z^2)/2},
{8, 1+(29 z)/6+2 z^2+z^3/6},
{9, 1+(16 z)/3+(5 z^2)/2+z^3/6},
{10, 1+(16 z)/3+(7 z^2)/2+z^3/6},
{11, 1+(19 z)/3+(7 z^2)/2+z^3/6},
{12, 1+(19 z)/3+4 z^2+(2 z^3)/3},
{13, 1+(22 z)/3+4 z^2+(2 z^3)/3},
{14, 1+(22 z)/3+5 z^2+(2 z^3)/3},
{15, 1+(22 z)/3+6 z^2+(2 z^3)/3},
{16, 1+(91 z)/12+(155 z^2)/24+(11 z^3)/12+z^4/24},
{17, 1+(103 z)/12+(155 z^2)/24+(11 z^3)/12+z^4/24},
{18, 1+(103 z)/12+(167 z^2)/24+(17 z^3)/12+z^4/24},
{19, 1+(115 z)/12+(167 z^2)/24+(17 z^3)/12+z^4/24},
{20, 1+(115 z)/12+(179 z^2)/24+(23 z^3)/12+z^4/24},
{21, 1+(115 z)/12+(203 z^2)/24+(23 z^3)/12+z^4/24},
{22, 1+(115 z)/12+(227 z^2)/24+(23 z^3)/12+z^4/24},
{23, 1+(127 z)/12+(227 z^2)/24+(23 z^3)/12+z^4/24},
{24, 1+(127 z)/12+(235 z^2)/24+(29 z^3)/12+(5 z^4)/24},
{25, 1+(133 z)/12+(247 z^2)/24+(29 z^3)/12+(5 z^4)/24},
{26, 1+(133 z)/12+(271 z^2)/24+(29 z^3)/12+(5 z^4)/24},
{27, 1+(137 z)/12+(283 z^2)/24+(31 z^3)/12+(5 z^4)/24},
{28, 1+(137 z)/12+(295 z^2)/24+(37 z^3)/12+(5 z^4)/24},
{29, 1+(149 z)/12+(295 z^2)/24+(37 z^3)/12+(5 z^4)/24},
{30, 1+(149 z)/12+(295 z^2)/24+(49 z^3)/12+(5 z^4)/24}
```

Finally, here are numeric approximations of those zeros for values of *n* from 0 to 100.

```
Grid@Table[{n,Table[j,{j,roots[n]}]},{n,1,30}]
```

```
{1, {}},
{2, {-1.}},
{3, {-0.5}},
{4, {-4.56155,-0.438447}},
{5, {-6.70156,-0.298438}},
{6, {-2.,-0.333333}},
{7, {-2.75831,-0.241694}},
{8, {-8.772,-3.,-0.227998}},
{9, {-12.473,-2.31959,-0.207381}},
{10, {-19.3634,-1.41809,-0.218507}},
{11, {-19.0185,-1.80686,-0.174602}},
{12, {-3.,-2.82288,-0.177124}},
{13, {-2.92599-1.25394 i,-2.92599+1.25394 i,-0.14802}},
{14, {-5.57525,-1.773,-0.151746}},
{15, {-7.57377,-1.27033,-0.155907}},
{16, {-10.2913-4.44128 i,-10.2913+4.44128 i,-1.2665,-0.150831}},
{17, {-10.1642-4.18445 i,-10.1642+4.18445 i,-1.54283,-0.128751}},
{18, {-28.368,-3.77618,-1.72599,-0.129805}},
{19, {-28.4047,-3.,-2.4818,-0.113483}},
{20, {-41.8542,-2.01579-0.978644 i,-2.01579+0.978644 i,-0.114201}},
{21, {-41.209,-3.,-1.67512,-0.115891}},
{22, {-40.5402,-4.12173,-1.22037,-0.117694}},
{23, {-40.557,-3.86667,-1.47239,-0.10394}},
{24, {-5.00304-2.42184 i,-5.00304+2.42184 i,-1.48963,-0.104295}},
{25, {-5.03548-2.91604 i,-5.03548+2.91604 i,-1.4299,-0.0991419}},
{26, {-5.16558-3.78035 i,-5.16558+3.78035 i,-1.16859,-0.100246}},
{27, {-5.56913-3.37906 i,-5.56913+3.37906 i,-1.16462,-0.0971295}},
{28, {-8.79004,-4.72927,-1.1831,-0.097597}},
{29, {-8.93869,-4.38108,-1.39219,-0.0880418}},
{30, {-16.1801,-1.66598-0.772391 i,-1.66598+0.772391 i,-0.0879758}}
```

And remember, we can express the Riemann Prime Counting function with the zeros as $\Pi(n)=\sum_{\rho} -\frac{1}{\rho}$

So... this is all true, but so what? What do these zeros tell us, if anything, about the Riemann Prime Counting function, $\Pi(n)$?

The Mathematica Code

What we've just done here isn't just a quirk of the Riemann Prime Counting function. If we extend the Divisor Summatory function $D_z(n)$ with just a few more parameters, it turns out lots of important functions can be expressed in terms of similar zeros. This includes

- the Mertens function $M(n)$
- the Divisor summatory function $D(n)$
- the divisor sigma summatory function $\sum_{j=1}^{n} \sigma_a(n)$
- the Euler Phi summatory function $\sum_{j=1}^{n} \varphi(j)$
- the Liouville Lambda summatory function $\sum_{j=1}^{n} \lambda(j)$
- the Chebyshev psi function $\psi(n)$
- Harmonic numbers H_n
- the modulus function $n \mod c$

and many, many weighted sums of prime powers. These identities will be listed at great length in later sections.

One major question in this paper is whether these zeros converge in cases where they represent converging infinite series. If so, then they can also be used to represent

- the Riemann Zeta function $\zeta(s)$ for some fixed s where $\Re(s)>1$ (and $\zeta(s)^z$ and $\log\zeta(s)$ and $\frac{\zeta'(s)}{\zeta(s)}$)
- the Laguerre L polynomial $L_z(n)$ for some fixed n
- the logarithmic integral $li(n)$ for some fixed n
- the logarithm of a positive real number $\log x$ for some fixed x
- the difference between the Riemann prime counting function and the logarithmic integral $\Pi(n)-li(n)$ for some fixed n
- the difference between the Chebyshev function and n $\psi(n)-n$ for some fixed n
- the Dirichlet eta function $\eta(s)$ for some fixed s with $\Re(s)>0$ (and $\eta(s)^z$ and $\log\eta(s)$ and $\frac{\eta'(s)}{\eta(s)}$)

and several other related functions that will also be listed at very great length below.

Some Notes on Notation and a Justification

1 The Generalized Divisor Function

Let's start with the most basic relationship. I'll then generalize it, step-by-step, over the course of the next few sections. Here we'll begin with our function $[\zeta(0)^z]_n$. What I'll do here is show how the zeros of this polynomial, $[\zeta(0)^z]_n$, can express the Riemann Prime Counting function (as shown in the introduction), the Mertens function, and the Divisor summatory function.

So let's define $[\zeta(0)^z]_n$ a couple different ways.

Definitions

There is a set of four different representations for functions like $[\zeta(0)^z]_n$ that I will be relying on over and over in the course of this paper. Three of them have very close analogs to complex exponentiation, namely the familiar identities $x^z=\sum_{k=0}^{\infty}\binom{z}{k}(x-1)^k$, $x^z=\sum_{k=0}^{\infty}\frac{z^k}{k!}(\log x)^k$, and $x^z=\sum_{k=0}^{\infty}\binom{z}{k}(x-1)^{z-k}$. The fourth identity will take advantage of $\nabla[\zeta(0)^z]_n$ as a multiplicative function. All of these will then be implemented in Mathematica immediately following their descriptions.

Identity Style 1: Via Newton's Generalized Binomial Expansion

Here's the first definition, mentioned already in the introduction. It can be written most concisely as a recurrence relation:

$$[\zeta(0)^z]_n=1+f(n,2,1) \quad \text{where} \quad f(n,j,k)=\begin{cases}(\frac{z+1}{k}-1)(1+f(\frac{n}{j},2,k+1))+f(n,j+1,k) & \text{if } n\geq j\\ 0 & \text{if } n<j\end{cases}$$

The recurrence leads to our first identity:

$$\boxed{[\zeta(0)^z]_n=\binom{z}{0}1+\binom{z}{1}\sum_{j=2}^{n}1+\binom{z}{2}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}1+\binom{z}{3}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor\frac{n}{j\cdot k}\rfloor}1+\binom{z}{4}\ldots} \tag{1.1}$$

This style of identity is something like a Dirichlet convolution analog of the Newton generalized binomial theorem $x^z=\sum_{k=0}^{\infty}\binom{z}{k}(x-1)^k$ which, when it converges, provides one approach to complex exponentiation.

Identity Style 2: As Exponentiation

Here's a second expression for $[\zeta(0)^z]_n$. First, define the following prime-identifying function.

$$\boxed{\kappa(n)=\begin{cases}\frac{1}{a} & \text{if } n=p^a\\ 0 & \text{otherwise}\end{cases}} \tag{1.2a}$$

From the sake of symmetry, it would be more natural to notate this as $\nabla[\log\zeta(0)]_n$ but as we'll be using this function a lot, we'll rely on $\kappa(n)$ for concision. If you're familiar with the Von Mangoldt function, it is also the case that $\kappa(n)=\frac{\Lambda(n)}{\log n}$

at positive integers > 1. I'll list a few more identities for $\kappa(n)$ at the very end of this section.

With $\kappa(n)$, we can express can use it to express $[\zeta(0)^z]_n$ with the recurrence relation

$$[\zeta(0)^z]_n = 1 + p(n,2,1) \quad \text{where} \quad p(n,j,k) = \begin{cases} \frac{z}{k}\cdot\kappa(j)(1+p(\frac{n}{j},2,k+1)) + p(n,j+1,k) & \text{if } n \geq j \\ 0 & \text{if } n < j \end{cases}$$

which leads to the form

$$\boxed{[\zeta(0)^z]_n = 1 + \frac{z^1}{1!}\sum_{j=2}^{n}\kappa(j) + \frac{z^2}{2!}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\kappa(j)\cdot\kappa(k) + \frac{z^3}{3!}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor\frac{n}{j\cdot k}\rfloor}\kappa(j)\cdot\kappa(k)\cdot\kappa(l) + \frac{z^4}{4!}\ldots} \tag{1.2}$$

This is a Dirichlet convolution analog of the familiar $x^z = \sum_{k=0}^{\infty}\frac{z^k}{k!}(\log x)^k$, another method of performing complex exponentiation. This expression in particular should also make the connection between $[\zeta(0)^z]_n$ and the Riemann prime counting function $\Pi(n)$ clear – it is trivial to see here that $\lim_{z\to 0}\frac{\partial}{\partial z}[\zeta(0)^z]_n = \lim_{z\to 0}\frac{[\zeta(0)^z]_n - 1}{z} = \sum_{j=2}^{n}\kappa(j) = \Pi(n)$.

Identity Style 3: Via The Hyperbola Method

Here's a third expression for $[\zeta(0)^z]_n$. Although it might not be at all obvious, this approach is something like a generalization of the Dirichlet Hyperbola method.

$$\boxed{[\zeta(0)^z]_n = [(1+\zeta(0,2))^z]_n \text{ where } [(1+\zeta(0,y))^z]_n = \begin{cases} \sum_{k=0}^{\lfloor\frac{\log n}{\log y}\rfloor}\binom{z}{k}\cdot[(1+\zeta(0,y+1))^{z-k}]_{n/y^k} & \text{if } n \geq y \\ 1 & \text{if } n < y \end{cases}} \tag{1.3}$$

This style of identity is something like a Dirichlet convolution analog of $x^z = \sum_{k=0}^{\infty}\binom{z}{k}(x-1)^{z-k}$.

A little clarification might make the notation $[(1+\zeta(0,y))^z]_n$ at least a bit less opaque. Combinatorially, if *z* is a positive integer, $[(1+\zeta(0,y))^z]_n$ is the count of solutions to $a_1\cdot a_2\cdot\ldots a_z \leq n$ where either $a_z \geq y$ or $a_z = 1$. If *n < y,* there will only be one solution, the case $1\cdot 1\cdot\ldots 1 \leq n$ - thus the end condition.

If we apply the recurrence in (1.3) to itself until it goes away, we are left with the entirely unwieldy expression

$$[\zeta(0)^z]_n =$$
$$1 + \sum_{a=2}^{n}\sum_{j=1}^{\lfloor\frac{\log n}{\log a}\rfloor}\binom{z}{j}(1 + \sum_{b=a+1}^{\lfloor\frac{n}{a^j}\rfloor}\sum_{k=1}^{\lfloor\frac{\log n - j\log a}{\log b}\rfloor}\binom{z-j}{k}\cdot(1 + \sum_{c=b+1}^{\lfloor\frac{n}{a^j b^k}\rfloor}\sum_{l=1}^{\lfloor\frac{\log n - j\log a - k\log b}{\log c}\rfloor}\binom{z-j-k}{l}(1 + \sum_{d=c+1}^{\lfloor\frac{n}{a^j b^k c^l}\rfloor}\sum_{m=1}^{\lfloor\frac{\log n - j\log a - k\log b - l\log c}{\log d}\rfloor}\binom{z-j-k-l}{m}(1+\ldots))))$$

Identity Style 4: As a Sum of Multiplicative Functions

Let's end with one last identity, which takes advantage of the fact that $[\nabla\zeta(0)^z]_n$ is a multiplicative function.

First, let $z^{(k)}$ be the rising factorial, defined as $z^{(k)} = z(z+1)\ldots(z+k-1)$. Also, let's say that the prime factorization of a number n is expressed as $n = \prod_{p^k|n} p^k$.

Ivic gives $d_z(n)=\prod_{p^\alpha|n}(-1)^\alpha\binom{-z}{\alpha}$.

Rewriting this a bit, because $d_z(n)=\nabla[\zeta(0)^z]_n$ and $(-1)^k\binom{-z}{k}=\frac{z^{(k)}}{k!}$, we have $\nabla[\zeta(0)^z]_n=\prod_{p^k|n}\frac{z^{(k)}}{k!}$. So, another important expression for $[\zeta(0)^z]_n$ is

$$\boxed{[\zeta(0)^z]_n=\sum_{j=1}^{n}\prod_{p^k|j}\frac{z^{(k)}}{k!}} \tag{1.4}$$

We can, if so inclined, take the core idea from (1.4) to write the following recurrence relation, where p_j means the jth prime:

$$[\zeta(0)^z]_n=f_1(n,1) \quad \text{where} \quad f_k(n,j)=\begin{cases}(\frac{z-1}{k}+1)\cdot f_{k+1}(\frac{n}{p_j},j)+f_1(n,j+1) & \text{if } n\geq p_j\\ 1 & \text{otherwise}\end{cases} \tag{1.4a}$$

If we apply this recurrence to itself until it disappears, we are left with the following expression which shows something about the internal structure of $[\zeta(0)^z]_n$.

$$[\zeta(0)^z]_n=\sum_{a=0}^{\lfloor\frac{\log n}{\log 2}\rfloor}\frac{z^{(a)}}{a!}\cdot\sum_{b=0}^{\lfloor\frac{\log n-a\log 2}{\log 3}\rfloor}\frac{z^{(b)}}{b!}\cdot\sum_{c=0}^{\lfloor\frac{\log n-a\log 2-b\log 3}{\log 5}\rfloor}\frac{z^{(c)}}{c!}\cdot\sum_{d=0}^{\lfloor\frac{\log n-a\log 2-b\log 3-c\log 5}{\log 7}\rfloor}\frac{z^{(d)}}{d!}\cdot\ldots$$

As will be more clear in the next section, this is basically a partial sum version of an Euler product.

So now we've collected several interesting ways to express $[\zeta(0)^z]_n$. Let's turn to some code to compute these ideas.

The Mathematica Code

Here are implementations of these identities for $[\zeta(0)^z]_n$ in Mathematica. (1.1) is implemented as `zeta0V1[n_, z_]`, (1.2) is implemented as `zeta0V2[n_, z_]`, (1.3) is implemented as `zeta0V3[n_, z_]`, (1.4) is implemented as `zeta0V4[n_, z_]`. and (1.4a) is implemented as `zeta0V4a[n_, z_]`.

```
binomial[z_,k_]:=binomial[z,k]=Product[z-j,{j,0,k-1}]/k!

zeta0Minus1[n_,0]:=1
zeta0Minus1[n_,k_]:=zeta0Minus1[n,k]=Sum[zeta0Minus1[Floor[n/j],k-1],{j,2,n}]
zeta0V1[n_,z_]:=Sum[binomial[z,k] zeta0Minus1[n,k],{k,0,Log2@n}]

kappa[ n_ ] := kappa[n]=If[ n < 1 || Floor[n] != n, 0, FullSimplify[ MangoldtLambda[n]/Log[n]]]
logzeta0[ n_, 0 ] := 1
logzeta0[ n_, k_ ] := logzeta0[n,k]=Sum[ kappa[j] logzeta0[ Floor[n/j], k-1 ], { j, 2,  n } ]
zeta0V2[ n_, z_ ] := Sum[ z^k/k! logzeta0[ n, k ], { k, 0, Log2@n } ]

zeta0V3Hurwitz[n_,y_,z_] := zeta0V3Hurwitz[n,y,z] = If[ n<y, 1, Sum[ binomial[z,k] zeta0V3Hurwitz[Floor[n/y^k], y+1, z-k],{k,0,Log[y,n]}]]
zeta0V3[n_,z_]:= zeta0V4Hurwitz[n,2,z]

FI[ n_ ] := FactorInteger[ n ]
FI[ 1 ] := { }
zeta0V4[n_,z_]:= Sum[ Product[ Pochhammer[z,p[[2]]]/p[[2]]! , { p, FI[ j ] } ], {j,1,n}]

eulerProduct0[n_,j_,k_,z_]:=eulerProduct0[n,j,k,z]=If[Prime[j]>n||n<=1,1,(1+(z-1)/k)eulerProduct0[Floor[n/Prime[j]],j,k+1,z]+eulerProduct0[n,j+1,1,z]]
zeta0V4a[n_,z_]:=eulerProduct0[n,1,1,z]
```

These vary in terms of how fast they are. For more practical experimentation, here's one final implementation, `zeta0[n_,` z_]. I won't describe how it works, but it is much, much faster to compute.

```
binomial[z_,k_]:=binomial[z,k]=Product[z-j,{j,0,k-1}]/k!
zeta0Hurwitz[n_,y_,0]:=UnitStep[n-1]
zeta0Hurwitz[n_,y_,1]:=zeta0Hurwitz[n,y,1]=Floor[n]-y
zeta0Hurwitz[n_,y_,2]:=zeta0Hurwitz[n,y,2]=Sum[ 1+2 (zeta0Hurwitz[Floor[n/m],m,1]),{m,y+1,Floor[n^(1/2)]}]
zeta0Hurwitz[n_,y_,k_]:=zeta0Hurwitz[n,y,k]=Sum[1 +k zeta0Hurwitz[Floor[n/(m^(k-1))], m,1]+Sum[binomial[k,j]  zeta0Hurwitz[Floor[n/(m^j)],m,k-j],
{j,1,k-2}],{m,y+1,Floor[n^(1/k)]}]

zeta0[n_,  z_] := Expand@Sum[binomial[z,k]zeta0Hurwitz[n,1,k],{k,0,Log2@n}]
zeta0Roots[n_]:=If[(c=Exponent[f=zeta0[n,z],z])==0,{}, If[ c==1,List@NRoots[f==0,z][[2]],List@@NRoots[f==0,z][[All,2]]]]
zeta0R[n_, z_] := Chop@Expand@Product[ 1-z/rho,{rho,zeta0Roots[n]}]
```

For seeing the zeros in action, the two really important functions to note here are `zeta0[n_,  z_]` and `zeta0R[n_, z_]`. Both of these functions compute the same value, the latter using the roots of the function.

Also note the function `zeta0Roots[n_]`, which displays those zeros of $[\zeta(0)^z]_n$ for a fixed n.

Special Values of $[\zeta(0)^z]_n$

So what can we do with $[\zeta(0)^z]_n$? Well, for several values of z, $[\zeta(0)^z]_n$ expresses important number theoretic functions, and if we can say interesting things about $[\zeta(0)^z]_n$, then perhaps that might shed light on those functions. In the table below, $D(n)$ is the Divisor Summatory function, $M(n)$ is the Mertens function, and $\Pi(n)$ is the Riemann Prime Counting function.

Table 1

Variant of $[\zeta(0)^z]_n$	*Value*	*Via Roots, in Mathematica*	*Value in Mathematica*
$[\zeta(0)]_n$	n	zeta0R[n,1]	n
$[\zeta(0)^2]_n$	$D(n)$	zeta0R[n,2]	Sum[1,{j,1,n},{k,1,Floor[n/j]}]
$[\zeta(0)^{-1}]_n$	$M(n)$	zeta0R[n,-1]	Sum[MoebiusMu[j],{j,1,n}]
$[\log\zeta(0)]_n$	$\Pi(n)$	Limit[D[zeta0R[n,z],z],z->0]	Sum[PrimePi[n^(1/k)]/k,{k,1,Log2@n}]

Try out `DiscretePlot[Limit[D[zeta0R[n,z],z],z->0], {n,4,100}]` and you should indeed see a graph of a function that only increases at prime numbers raised to integer powers.

Some Examples of $[\zeta(0)^z]_n$

To get a better feel for $[\zeta(0)^z]_n$, here are the polynomials generated by the function for a few fixed values of n:

$$[\zeta(0)^z]_{12}=1+\frac{19z}{3}+4z^2+\frac{2z^3}{3}$$

$$[\zeta(0)^z]_{100}=1+\frac{428z}{15}+\frac{16289z^2}{360}+\frac{331z^3}{16}+\frac{611z^4}{144}+\frac{67z^5}{240}+\frac{7z^6}{720}$$

$$\begin{aligned}[\zeta(0)^z]_{10000}=1&+\frac{56175529z}{45045}+\frac{5304616687z^2}{1663200}+\frac{64238883431z^3}{19958400}+\frac{3688608229z^4}{2177280}+\\ &\frac{11603252491z^5}{21772800}+\frac{4483862353z^6}{43545600}+\frac{557009347z^7}{43545600}+\frac{2872319z^8}{2903040}+\\ &\frac{688397z^9}{14515200}+\frac{58651z^{10}}{43545600}+\frac{8339z^{11}}{479001600}+\frac{17z^{12}}{95800320}+\frac{z^{13}}{6227020800}\end{aligned}$$

For a fixed value of n, $[\zeta(0)^z]_n$ is a quite unremarkable polynomial of finite degree. In fact, it is of order $\lfloor \frac{\log n}{\log 2} \rfloor$, which examination of both (1.1) and (1.2) should suggest.

The Roots of $[\zeta(0)^z]_n$

The observations of the previous section lead trivially to the fact that, as entirely unremarkable polynomials, $[\zeta(0)^z]_n$ for a fixed value of n should have $\lfloor \frac{\log n}{\log 2} \rfloor$ roots. And that is indeed the case. For instance, for our previous examples, if we use the notation that ρ is the label for the roots of our function, then

$$[\zeta(0)^\rho]_{12}=0 \text{ when } \rho=\begin{Bmatrix} 1/2(-3-\sqrt{7}) \\ 1/2(-3+\sqrt{7}) \\ -3 \end{Bmatrix}$$

$$[\zeta(0)^\rho]_{100}=0 \text{ when } \rho\approx\begin{Bmatrix} -0.933809 & -0.0372047 & -11.1997-12.3982i \\ -11.1997+12.3982i & -2.67195-1.86184i & -2.67195+1.86184i \end{Bmatrix}$$

$$[\zeta(0)^\rho]_{10000}=0 \text{ when } \rho\approx\begin{Bmatrix} -1005.17 & -25.9197-61.2147i & -25.9197+61.2147i \\ -12.5619 & -9.95084-13.237i & -9.95084+13.237i \\ -4.34989-4.84639i & -4.34989+4.84639i & -2.23696-1.84432i \\ -2.23696+1.84432i & -1.17804-0.181571i & -1.17804+0.181571i \\ & -0.000803511 & \end{Bmatrix}$$

And because these are standard, unobjectionable polynomials, that means, for a fixed value of n, $[\zeta(0)^z]_n=\prod_\rho (1-\frac{z}{\rho})$. In particular, we have these special values:

Table 1b

n	$\prod_\rho (1-\frac{1}{\rho})$
$D(n)$	$\prod_\rho (1-\frac{2}{\rho})$
$M(n)$	$\prod_\rho (1+\frac{1}{\rho})$
$\Pi(n)$	$-\sum_\rho \rho^{-1}$

Using Roots to Demonstrate Computing Special Values of $[\zeta(0)^z]_{12}$

To belabor the point, let's work through a simple example.

The zeros of $[\zeta(0)^z]_{12}$ are, as just noted,

$$[\zeta(0)^\rho]_{12}=0 \text{ when } \rho=\begin{Bmatrix} 1/2(-3-\sqrt{7}) \\ 1/2(-3+\sqrt{7}) \\ -3 \end{Bmatrix}$$

Using Table 1b, we can use these zeros to compute the Divisor function $D(12)$, the Mertens function $M(12)$, and the Riemann Prime Counting function $\Pi(12)$ like so:

$$\begin{array}{llll}
n= & \prod_\rho (1-\frac{1}{\rho})= & (1-\frac{1}{1/2(-3-\sqrt{7})})(1-\frac{1}{1/2(-3+\sqrt{7})})(1-\frac{1}{-3})= & 12 \\
D(12)= & \prod_\rho (1-\frac{2}{\rho})= & (1-\frac{2}{1/2(-3-\sqrt{7})})(1-\frac{2}{1/2(-3+\sqrt{7})})(1-\frac{2}{-3})= & 35 \\
M(12)= & \prod_\rho (1+\frac{1}{\rho})= & (1+\frac{1}{1/2(-3-\sqrt{7})})(1+\frac{1}{1/2(-3+\sqrt{7})})(1+\frac{1}{-3})= & -2 \\
\Pi(12)= & -\sum_\rho \frac{1}{\rho}= & -(\frac{1}{1/2(-3-\sqrt{7})}+\frac{1}{1/2(-3+\sqrt{7})}+\frac{1}{-3})= & \frac{19}{3}
\end{array}$$

A quick check in something like Mathematica shows that, indeed,

```
Sum[1,{j,1,12},{k,1,Floor[12/j]}]=35

Sum[MoebiusMu[j],{j,1,12}] = -2

Sum[ PrimePi[12^(1/k)]/k,{k,1,Log2@12}] = 19/3
```

as expected.

Questions

1. Can anything interesting be said about these roots as n changes? What is their long term behavior? Do they have any meaningful properties of their own?

2. There are discontinuities in the roots when $n=2^k$, with k an integer, because the number of roots increases by one in such a case. There are also other instances of discontinuities in other cases. Is that significant?

3. Are there tools from the general study of polynomial roots that can tell us anything interesting about this approach, more broadly?

So those are a few questions. But let's move on – there are more productive and interesting questions if we generalize our function a bit.

Addendum: Other Identities for $\kappa(n)$

Using any of our definitions for $[\zeta(0)^z]_n$,

$$\kappa(n)=\lim_{z\to 0}\frac{\partial}{\partial z}([\zeta(0)^z]_n-[\zeta(0)^z]_{n-1})=\lim_{z\to 0}\frac{\partial}{\partial z}\nabla[\zeta(0)^z]_n$$

If we apply this to (1.1) above, we end up with an identity first discovered by Yuri V. Linnik and independently discovered by this paper's author,

$$\kappa(n)=\sum_{j=n;\, j>1} 1-\frac{1}{2}\sum_{j\cdot k=n;\, j,k>1} 1+\frac{1}{3}\sum_{j\cdot k\cdot l=n;\, j,k,l>1} 1-\frac{1}{4}\sum_{j\cdot k\cdot l\cdot m=n;\, j,k,l,m>1} 1+\ldots$$

If we do likewise with (1.3), we will have

$$\kappa(n)=\sum_{y|n,\, y>1}\sum_{k=1}^{y^k|n}\frac{(-1)^{k+1}}{k}\cdot\nabla[(1+\zeta(0,y+1))^{-k}]_{\frac{n}{y^k}} \quad \text{where} \quad \nabla[(1+\zeta(0,y))^z]_n=\begin{cases}\sum_{m|n;\, m\geq y}\sum_{k=1}^{m^k|n}\binom{z}{k}\cdot\nabla[(1+\zeta(0,m+1))^{z-k}]_{n/m^k} & \text{if } n\neq 1\\ 1 & \text{if } n=1\end{cases}$$

And, finally, (1.4) gave $\nabla[\zeta(0)^z]_n$ as a multiplicative function, expressed as a product over its prime factorization, as $\nabla[\zeta(0)^z]_n=\prod_{p^k|n}\frac{z^{(k)}}{k!}$, where $z^{(k)}=z(z+1)...(z+k-1)$ is the rising factorial. Thus

$$\kappa(n)=\lim_{z\to 0}\frac{\partial}{\partial z}\prod_{p^k|n}\frac{z^{(k)}}{k!}=\lim_{z\to 0}\frac{1}{z}\cdot\prod_{p^k|n}\frac{z(z+1)...(z+k-1)}{1\cdot 2\cdot ...\cdot k}$$

2 Extending the Function, Connecting with the Riemann Zeta Function

In the previous section we used the roots of $[\zeta(0)^z]_n$ to express the Mertens function, the Riemann Prime counting function, and the Dirichlet Divisor problem function. This immediately raised a few questions that I can't answer.

If we generalize $[\zeta(0)^z]_n$, though, the kind of questions we can ask immediately gets much, much deeper. The generalization we'll add add in this section is a parameter, *s*, that is intimately related to the *s* value in the Riemann Zeta function.

The upshot of doing this is as follows. The complex value *s* is going to be a weighting factor, mirroring its role in Dirichlet series and providing convergence for $\lim_{n\to\infty}[\zeta(s)^z]_n$ when $\Re(s)>1$. We will also be able to take the derivative of *s*, which lets us express other interesting functions with these zeros (specifically the Chebyshev function $\psi(n)$).

So let's go ahead and define $[\zeta(s)^z]_n$ a few different ways.

Definitions

We are going to mirror, here, the identities from Section 1. In all cases, the generalizations will be trivial alterations requiring no extra comment.

Identity Style 1: Via Newton's Generalized Binomial Expansion

Here is one definition for $[\zeta(s)^z]_n$, corresponding to (1.1). We have the recurrence relation

$$[\zeta(s)^z]_n=1+f(n,2,1) \quad \text{where} \quad f(n,j,k)=\begin{cases} j^{-s}\cdot(\frac{z+1}{k}-1)(1+f(\frac{n}{j},2,k+1))+f(n,j+1,k) & \text{if } n\geq j \\ 0 & \text{if } n<j \end{cases}$$

and if we apply that relation to itself until it disappears, we have

$$\boxed{[\zeta(s)^z]_n=\binom{z}{0}1+\binom{z}{1}\sum_{j=2}^{n}j^{-s}+\binom{z}{2}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}j^{-s}\cdot k^{-s}+\binom{z}{3}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor\frac{n}{j\cdot k}\rfloor}j^{-s}\cdot k^{-s}\cdot l^{-s}+\binom{z}{4}...}\tag{2.1}$$

Note that this definition is very similar to (1.1). The only difference is the sums were formerly uniformly over the value 1.

Identity Style 2: As Exponentiation

A second definition, generalizing (1.2) and using the $\kappa(n)$ function from (1.2a), is the recurrence relation

$$[\zeta(s)^z]_n=1+p(n,2,1) \quad \text{where} \quad p(n,j,k)=\begin{cases} j^{-s}\cdot\frac{z}{k}\cdot\kappa(j)(1+p(\frac{n}{j},2,k+1))+p(n,j+1,k) & \text{if } n\geq j \\ 0 & \text{if } n<j \end{cases}$$

which, when applied to itself, leads to

$$\boxed{[\zeta(s)^z]_n = 1 + \frac{z^1}{1!}\sum_{j=2}^{n}\kappa(j)\cdot j^{-s} + \frac{z^2}{2!}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\kappa(j)\cdot j^{-s}\cdot\kappa(k)\cdot k^{-s} + \frac{z^3}{3!}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor\frac{n}{j\cdot k}\rfloor}\kappa(j)\cdot j^{-s}\cdot\kappa(k)\cdot k^{-s}\cdot\kappa(l)\cdot l^{-s} + \frac{z^4}{4!}\ldots}$$

(2.2)

Identity Style 3: Via The Hyperbola Method

And

$$\boxed{[\zeta(s)^z]_n = [(1+\zeta(s,2))^z]_n \text{ where } [(1+\zeta(s,y))^z]_n = \begin{cases} \sum_{k=0}^{\lfloor\frac{\log n}{\log y}\rfloor} \binom{z}{k}\cdot(y^{-s})^k\cdot[(1+\zeta(s,y+1))^{z-k}]_{n/y^k} & \text{if } n\geq y \\ 1 & \text{if } n<y \end{cases}}$$

(2.3)

Identity Style 4: As a Sum of Multiplicative Functions

Corresponding to (1.3), is

$$\boxed{[\zeta(s)^z]_n = \sum_{j=1}^{n}\prod_{p^k|j}\frac{z^{(k)}}{k!}\cdot p^{-ks}}$$

(2.4)

$$[\zeta(s)^z]_n = f_1(n,1) \quad \text{where} \quad f_k(n,j) = \begin{cases} p_j^{-s}\left(\left(1+\frac{z-1}{k}\right)f_{k+1}\left(\frac{n}{p_j},j\right)+f_1(n,j+1)\right) & \text{if } n\geq p_j \\ 1 & \text{otherwise} \end{cases}$$

$$[\zeta(s)^z]_n = \sum_{a=0}^{\lfloor\frac{\log n}{\log 2}\rfloor}\frac{z^{(a)}}{a!}\cdot 2^{-as}\cdot\sum_{b=0}^{\lfloor\frac{\log n - a\log 2}{\log 3}\rfloor}\frac{z^{(b)}}{b!}\cdot 3^{-bs}\cdot\sum_{c=0}^{\lfloor\frac{\log n - a\log 2 - b\log 3}{\log 5}\rfloor}\frac{z^{(c)}}{c!}\cdot 5^{-cs}\cdot\sum_{d=0}^{\lfloor\frac{\log n - a\log 2 - b\log 3 - c\log 5}{\log 7}\rfloor}\frac{z^{(d)}}{d!}\cdot 7^{-ds}\cdot\ldots$$

Identity Style 5: As a Product of Zeros

For fixed values of *n* and *s,* $[\zeta(s)^z]_n$ is a typical polynomial of order $\lfloor\frac{\log n}{\log 2}\rfloor$ with $\lfloor\frac{\log n}{\log 2}\rfloor$ zeros such that $[\zeta(s)^\rho]_n = 0$. Thus,

$$\boxed{[\zeta(s)^z]_n = \prod_{\rho}\left(1-\frac{z}{\rho}\right)}$$

(2.5)

and

$$[\log(\zeta(s))]_n = -\sum_{\rho}\frac{1}{\rho}$$

The Mathematica Code

For performance reasons, we're not going to bother directly implementing (2.1), (2.2), (2.3), or (2.4). Instead, we will implement $[\zeta(s)^z]_n$ in Mathematica, here named `zeta[n_, s_,z_]`, as:

```
binomial[z_,k_]:=binomial[z,k]=Product[z-j,{j,0,k-1}]/k!
zetaHurwitz[n_,s_,y_,0]:=UnitStep[n-1]
zetaHurwitz[n_,s_,y_,1]:=zetaHurwitz[n,s,y,1]=HarmonicNumber[Floor[n],s]-HarmonicNumber[y,s]
```

```
zetaHurwitz[n_,s_,y_,2]:=zetaHurwitz[n,s,y,2]=Sum[ (m^(-2s))+2(m^-s) (zetaHurwitz[Floor[n/m],s,m,1]),{m,y+1,Floor[n^(1/2)]}]
zetaHurwitz[n_,s_,y_,k_]:=zetaHurwitz[n,s,y,k]=Sum[(m^(-s k)) +k(m^(-s(k-1))) zetaHurwitz[Floor[n/(m^(k-1))],s, m,1]+Sum[binomial[k,j] (m^-s)^j
zetaHurwitz[Floor[n/(m^j)],s, m,k-j],{j,1,k-2}],{m,y+1,Floor[n^(1/k)]}]

zeta[n_,s_,1]:=Expand@Sum[binomial[1,k]zetaHurwitz[n,s,1,k],{k,0,1}]
zeta[n_, s_, z_] := Expand@Sum[binomial[z,k]zetaHurwitz[n,s,1,k],{k,0,Log2@n}]

zetaRoots[n_,s_]:=If[(c=Exponent[f=zeta[n,s,z],z])==0,{}, If[ c==1,List@NRoots[f==0,z][[2]],List@@NRoots[f==0,z][[All,2]]]]
zetaR[n_, s_,z_] := Chop@Expand@Product[ 1-z/rho,{rho,zetaRoots[n,s]}]

DzetaRoots[n_,s_]:=If[(c=Exponent[f=N[D[zeta[n,t,z],t]/.t->s]-1,z])==0,{}, If[ c==1,List@NRoots[f==0,z][[2]],List@@NRoots[f==0,z][[All,2]]]]
DzetaR[n_,s_,z_]:=Chop@Expand[-1+Product[1-z/rho,{rho,DzetaRoots[n,s]}]]
```

Two important functions to note here are `zeta[n_, s_, z_]` and `zetaR[n_, s_, z_]`. Both of these functions compute the same value, the latter using the roots of the function. Also note the function `zetaRoots[n_, s_]`, which displays the roots of $[\zeta(s)^z]_n$ for a fixed *n* and *s*.

Special Values of $[\zeta(s)^z]_n$

So, now that we have the mechanics of computation out of the way, what's so interesting about $[\zeta(s)^z]_n$? Well, some noteworthy values of $[\zeta(s)^z]_n$ are included in Table 2, and all of them can be expressed as the roots we're looking at. For reference here, $\psi(n)$ is the Chebyshev function, H_n is a Harmonic number, B_j is a Bernoulli number, and $\kappa(n)$ is our prime power identifying function from (1.1b).

Table 2

Variant of $[\zeta(s)^z]_n$	*Value*	*Via Roots, in Mathematica*	*Value in Mathematica*
$[\log\zeta(s)]_n$	$\sum_{j=1}^{n} \kappa(j)\, j^{-s}$	D[zetaR[n,s,z],z]/.z->0	Sum[FullSimplify[MangoldtLambda[j]/Log[j]] j^-s, {j,2,n}]
$-\lim_{s\to 0} \frac{\partial}{\partial s}[\zeta(s)]_n$	$\log n!$	DzetaR [n,0,1]	Log[n!]
$-\lim_{s\to 0} \frac{\partial}{\partial s}[\log\zeta(s)]_n$	$\psi(n)$	D[DzetaR [n,0,z],z]/.z->0	Sum[MangoldtLambda[j], {j,2,n}]
$[\zeta(1)]_n$	H_n	zetaR[n,1,1]	HarmonicNumber[n]
$[\zeta(-1)]_n$	$\frac{n(n-1)}{2}$	zetaR[n,-1,1]	n(n-1)/2
$[\zeta(-2)]_n$	$\frac{n(n+1)(2n+1)}{6}$	zetaR[n,-2,1]	n(n+1)(2n+1)/6
$[\zeta(-3)]_n$	$n^2(n+1)^2 4$	zetaR[n,-3,1]	n^2(n+1)^2 / 4
$[\zeta(-k)]_n, k\in\mathbb{N}$	$n^k+\sum_{j=0}^{k} \frac{B_j k!\, n^{k-j+1}}{j!(k-j+1)!}$	zetaR[n,-k,1]	n^k+Sum[BernoulliB[j] k! n^(k-j+1)/(j!((k-j+1)!)),{j,0,k}]

Several other important functions emerge as *n* approaches infinity, shown here in Table 2b. Note that the bolded expressions contain limits that can't actually be evaluated in Mathematica as written; replace ∞ with some suitable finite approximation as needs dictate. Here, $\zeta(s)$ is the Riemann Zeta function.

Table 2b

Variant of $[\zeta(s)^z]_n$	*Value*	*Via Roots, in Mathematica*	*Value in Mathematica*
$\lim_{n\to\infty} [\zeta(2)]_n$	$\frac{\pi^2}{6}$	**zetaR[∞,2,1]**	Pi^2/6

$\lim_{n\to\infty}[\zeta(4)]_n$	$\frac{\pi^4}{90}$	**zetaR[∞,4,1]**	Pi^4/90
$\lim_{n\to\infty}[\zeta(6)]_n$	$\frac{\pi^6}{945}$	**zetaR[∞,6,1]**	Pi^6/945
$\lim_{n\to\infty}[\zeta(s)]_n, \Re(s)>1$	$\zeta(s)$	**zetaR[∞,s,1]**	Zeta[s]
$\lim_{n\to\infty}[\zeta(s)^z]_n, \Re(s)>1$	$\zeta(s)^z$	**zetaR[∞,s,z]**	Zeta[s]^z
$\lim_{n\to\infty}[\log\zeta(s)^z]_n, \Re(s)>1$	$\log\zeta(s)$	**D[zetaR[∞,s,z],z]/.z->0**	Log[Zeta[s]]
$\frac{\partial}{\partial s}\lim_{n\to\infty}[\log\zeta(s)^z]_n, \Re(s)>1$	$\frac{\zeta'(s)}{\zeta(s)}$	**D[DzetaR[∞,s,z],z]/.z->0**	Zeta'[s]/Zeta[s]

Some Examples of $[\zeta(s)^z]_n$

To get a taste of this function, here are a few examples of it.

$$[\zeta(1)^z]_{50}=1+\frac{36227089580823978984163\,z}{18594267025475980238400}+\frac{2722987611283\,z^2}{2248776129600}+\frac{1770229\,z^3}{5702400}+\frac{41\,z^4}{1440}+\frac{13\,z^5}{11520}$$

$$[\zeta(-1)^z]_{500}=1+\frac{1878019\,z}{84}+\frac{120118007\,z^2}{2520}+\frac{6961123\,z^3}{180}+\frac{1657477\,z^4}{120}+\frac{45367\,z^5}{18}+\frac{3131\,z^6}{15}+\frac{3571\,z^7}{315}+\frac{26\,z^8}{315}$$

$$[\zeta(0.5+14.1347\,i)^z]_{20}\approx 1-(2.06564-0.208919\,I)z+(1.0225-0.278071\,I)z^2$$
$$-(0.268006-0.181271\,I)z^3+(0.000833781-0.0103832\,I)z^4$$

$$\lim_{s\to 0}\frac{\partial}{\partial s}[\zeta(s)^z]_{80}\approx -79.4645\,z-120.818\,z^2-62.7616\,z^3-9.80744\,z\,4-0.815798\,z^5-0.00577623\,z^6$$

Questions

Now we are ready to asking some more interesting questions.

1. What happens to the roots of $[\zeta(s)^z]_n$ as s changes, for a fixed value of n ? Is there an easy way to reason about this?

2. What is the relationship between the roots of $[\zeta(s)^z]_n$ and the roots of $\frac{\partial}{\partial s}[\zeta(s)^z]_n$, for a fixed value of n? Is there any easy way to reason about this?

3. For a fixed value of s, where $\Re(s)>1$, do the roots of $[\zeta(s)^z]_n$ converge as n approaches infinity?

3 Extending the Function with a Sampling Scale Factor

For our next generalization, we will introduce a sampling / quantization scale factor, which we will label *x*. From this approach, our original function $[\zeta(s)^z]_n$ had a sampling scale factor of $x=1$. Our main concern will be what happens as x approaches 0.

We will write this new function with the sampling scale factor as $[(1+x^{1-s}\cdot\zeta(s,1+x^{-1}))^z]_n$, which is a bit mystifying. The primary justification for this notation is that, for $\Re(s)>1$, it will be the case that if we take the limit $\lim_{n\to\infty}[(1+x^{1-s}\cdot\zeta(s,1+x^{-1}))^z]_n$, we will have $(1+x^{1-s}\cdot\zeta(s,1+x^{-1}))^z$, where $\zeta(s,x)$ is the Hurwitz zeta function.

There's actually a very reasonable combinatorial explanation for this notation, but we'll skip that for now.

So now let's look at a few ways to define our new function.

Definitions

Identity Style 1: Via Newton's Generalized Binomial Expansion

Let's begin with a recurrence relation definition for $[(1+x^{1-s}\cdot\zeta(s,1+x^{-1}))^z]_n$. It looks like this:

$$[(1+x^{1-s}\cdot\zeta(s,1+x^{-1}))^z]_n=1+f_1(n,1+x) \text{ where } f_k(n,j)=\begin{cases} x\cdot(j\cdot x)^{-s}\cdot(\frac{z+1}{k}-1)(1+f_{k+1}(\frac{n}{j},1+x))+f_k(n,j+x) & \text{if } n\geq j \\ 0 & \text{if } n<j\end{cases}$$

If $x=1$, this immediately and simply returns us to our original recurrence relation from (2.1) (which was, more or less, $[(1+1\cdot\zeta(s,1+1))^z]_n=1+f_1(n,1+1)$ where $f_k(n,j)=1\cdot(j\cdot 1)^{-s}\cdot(\frac{z+1}{k}-1)(1+f_{k+1}(\frac{n}{j},1+1))+f_k(n,j+1)$ if $n\geq j, 0\, otherwise$)

If we apply the recurrence to itself until it is eliminated, we have

$$[(1+x^{1-s}\cdot\zeta(s,1+x^{-1}))^z]_n=\binom{z}{0}1+\binom{z}{1}\cdot x\cdot\sum_{(1+j\cdot x)\leq n}(1+j\cdot x)^{-s}+\binom{z}{2}\cdot x^2\cdot\sum_{(1+j\cdot x)(1+k\cdot x)\leq n}((1+j\cdot x)\cdot(1+k\cdot x))^{-s}+\binom{z}{3}\ldots \tag{3.1}$$

And if we take the limit of this as x approaches 0, we end up with the following very significant integral:

$$\boxed{\lim_{x\to 0}[(1+x^{1-s}\cdot\zeta(s,1+x^{-1}))^z]_n=\binom{z}{0}1+\binom{z}{1}\int_1^n x^{-s}\,dx+\binom{z}{2}\int_1^n\int_1^{\frac{n}{x}}(x\cdot y)^{-s}\,dy\,dx+\binom{z}{3}\int_1^n\int_1^{\frac{n}{x}}\int_1^{\frac{n}{xy}}(x\cdot y\cdot z)^{-s}\,dz\,dy\,dx+\ldots}$$

Identity Style 2: As Exponentiation

Unlike the identities from previous sections, $[(1+x^{1-s}\cdot\zeta(s,1+x^{-1}))^z]_n$ changes at non-integer values of *n*, so it doesn't have expressions quite a tidy as (1.2) or (2.2). Nevertheless, the following is the case:

$$\boxed{[(1+x^{1-s}\cdot\zeta(s,1+x^{-1}))^z]_n=\sum_{k=0}\frac{z^k}{k!}\cdot[(\log(1+x^{1-s}\cdot\zeta(s,1+x^{-1})))^k]_n}$$

(3.2)

Identity Style 3: Via The Hyperbola Method

$$\boxed{\begin{array}{l}[(1+x^{1-s}\cdot\zeta(s,1+x^{-1}))^z]_n \text{ where} \\ [(1+x^{1-s}\cdot\zeta(s,y))^z]_n=\begin{cases}\sum_{k=0}^{\lfloor\frac{\log n}{\log(x\,y)}\rfloor}\binom{z}{k}\cdot(x^{1-s}\cdot y^{-s})^k\cdot[(1+x^{1-s}\cdot\zeta(s,1+y))^{z-k}]_{n/(x\cdot y)^k} & \text{if } n\geq x\,y\\ 1 & \text{if } n<x\,y\end{cases}\end{array}}$$

(3.3)

Identity Style 4: As a Sum of Multiplicative Functions

Identity Style 5: As a Product of Zeros

For fixed values of *n, s,* and *x*, $[(1+x^{1-s}\cdot\zeta(s,1+x^{-1}))^z]_n$ is a typical polynomial of order $\lfloor\frac{\log n}{\log(1+y)}\rfloor$ with $\lfloor\frac{\log n}{\log(1+y)}\rfloor$ zeros such that $[(1+x^{1-s}\cdot\zeta(s,1+x^{-1}))^\rho]_n=0$.
Thus,

$$\boxed{[(1+x^{1-s}\cdot\zeta(s,1+x^{-1}))^z]_n=\prod_\rho\left(1-\frac{z}{\rho}\right)}$$

(3.5)

and

$$[\log(1+x^{1-s}\cdot\zeta(s,1+x^{-1}))]_n=-\sum_\rho\frac{1}{\rho}$$

The Mathematica Code

We can implement $[(1+x^{1-s}\cdot\zeta(s,1+x^{-1}))^z]_n$ in Mathematica, here named `zetaScaled[n_, s_,y_,z_]`, like so:

```
binomial[z_,k_]:=binomial[z,k]=Product[z-j,{j,0,k-1}]/k!
zetaHurwitz[n_,s_,y_,0]:=UnitStep[n-1]
zetaHurwitz[n_,s_,y_,1]:=zetaHurwitz[n,s,y,1]=HarmonicNumber[Floor[n],s]-HarmonicNumber[y,s]
zetaHurwitz[n_,s_,y_,2]:=zetaHurwitz[n,s,y,2]=Sum[ (m^(-2s))+2(m^-s) (zetaHurwitz[Floor[n/m],s,m,1]),{m,y+1,Floor[n^(1/2)]}]
zetaHurwitz[n_,s_,y_,k_]:=zetaHurwitz[n,s,y,k]=Sum[(m^(-s k)) +k(m^(-s(k-1))) zetaHurwitz[Floor[n/(m^(k-1))],s, m,1]+Sum[binomial[k,j] (m^-s)^j
zetaHurwitz[Floor[n/(m^j)],s, m,k-j],{j,1,k-2}],{m,y+1,Floor[n^(1/k)]}]

zetaMinus1Scaled[n_,s_,y_,k_]:=y^(k(1-s)) zetaHurwitz[n y^-k,s,y^-1,k]

zetaScaled[n_, s_, y_,z_] := Expand@Sum[binomial[z,k]zetaMinus1Scaled[n,s,y,k],{k,0,Log[y+1,n]}]

zetaScaledRoots[n_,s_,y_]:=If[(c=Exponent[f=zetaScaled[n,s,y,z],z])==0,{}, If[ c==1,List@NRoots[f==0,z][[2]],List@@NRoots[f==0,z][[All,2]]]]
zetaScaledR[n_, s_,y_, z_] := Chop@Expand@Product[ 1-z/rho,{rho,zetaScaledRoots[n,s,y]}]

DzetaScaledRoots[n_,s_,y_]:=If[(c=Exponent[f=N[D[zetaScaled[n,t,y,z],t]/.t->s]-1,z])==0,{}, If[ c==1,List@NRoots[f==0,z][[2]],List@@NRoots[f==0,z]
[[All,2]]]]
DzetaScaledR[n_, s_,y_, z_] := Chop@Expand[-1+Product[ 1-z/rho,{rho,DzetaScaledRoots[n,s,y]}]]
```

Two important functions to note here are `zetaScaled[n_, s_, y_, z_]` and `zetaScaledR[n_, s_, y_, z_]`. Both of these functions compute the same value, the latter using the roots of the function. Also note the function `zetaScaledRoots[n_, s_, y_]`, which displays those zeros of $[(1+y^{1-s}\cdot\zeta(s,1+y^{-1}))^z]_n$ for a fixed *n* and *s*.

Another function to note here is `DzetaScaledR[n_, s_, y_, z_]`, which computes the same value as the derivative of $[(1+y^{1-s}\cdot\zeta(s,1+y^{-1}))^z]_n$ with respect to *s* at *s*, `D[zetaScaled[n_, t_, y_, z_], t]/.t->s`, for some fixed value of *n* and *s*. The zeros corresponding to those two functions can be computed with `DzetaScaledRoots[n_, s_, y_]`.

Special Values of $[(1+x^{1-s}\cdot\zeta(s,1+x^{-1}))^z]_n$

Some noteworthy values of $[(1+x^{1-s}\cdot\zeta(s,1+x^{-1}))^z]_n$ are included in Table 3. Note that the bolded limits can't actually be evaluated in Mathematica as written; replace infinity with some suitable finite approximation as needs dictate, as well as the cases where x approachs 0. Here, $\gamma(k,n)$ is the lower incomplete gamma function, $L_z(n)$ is the Laguerre L polynomial, $\Gamma(k,n)$ is the upper incomplete gamma function, $li(n)$ is the logarithmic integral, γ is Euler's constant gamma, and $\zeta(s,y)$ is the Hurwitz zeta function.

Table 3

Variant of $[(1+y^{1-s}\cdot\zeta(s,1+y^{-1}))^z]_n$	*Value*	*Via Roots, in Mathematica*	*Value in Mathematica*
$\lim_{x\to 0}[(1+x^{1-s}\cdot\zeta(s,1+x^{-1}))^z]_n$	$\sum_{k=0}^{\infty}\binom{z}{k}\frac{\gamma(k,(s-1)\log n)}{(s-1)^k\cdot\Gamma(k)}$	**zetaScaledR[n,s,0,z]**	Sum[Binomial[z,k] (s-1)^(-k) (Gamma[k,0,(s-1)Log[n]]/Gamma[k]), {k,0,Infinity}]
$\lim_{x\to\infty}[(1+x^{1-s}\cdot\zeta(0,1+x^{-1}))^z]_n$	$L_{-z}(\log n)$	**zetaScaledR[n,0,0,z]**	LaguerreL[-z,Log[n]]
$\lim_{x\to 0}[\log(1+x^{1-s}\cdot\zeta(s,1+x^{-1}))]_n$	$-\Gamma(0,(s-1)\log n)+\Gamma(0,s\log n)+\log(\frac{s}{s-1})$	**D[zetaScaledR[n,s,0,z],z]/. z->0**	Gamma[0,s Log[n]]-Gamma[0,(s-1)Log[n]] +Log[s/(s-1)]
$\lim_{x\to 0}[\log(1+x^{1-s}\cdot\zeta(0,1+x^{-1}))]_n$	$li(n)-\log\log n-\gamma$	**D[zetaScaledR[n,0,0,z],z]/. z->0**	LogIntegral[n] – Log[Log[n]]-EulerGamma
$\lim_{s\to 0}\frac{\partial}{\partial s}\lim_{x\to 0}[\log(1+x^{1-s}\cdot\zeta(s,1+x^{-1}))]_n$	$n-\log n-1$	**D[DzetaScaledR[n,0,0,z],z]/ .z->0**	N – Log[n] - 1
$\lim_{n\to\infty}[(1+x^{1-s}\cdot\zeta(s,1+x^{-1}))^z]_n, \Re(s)>1$	$(1+x^{1-s}\cdot\zeta(s,1+x^{-1}))^z$	**zetaScaledR[∞,s,y,z]**	(1+y^(1-s)Zeta[s,1+y^-1])^z
$\lim_{n\to\infty}\lim_{x\to 0}[(1+x^{1-s}\cdot\zeta(s,1+x^{-1}))^z]_n, \Re(s)>1$	$(\frac{s}{s-1})^z$	**zetaScaledR[∞,s,0,z]**	(s/(s-1))^z

Questions

1.

4 Extending the Function to Support Alternating Signs

asdf

The Definition

Now we're going to perform our last generalization.

Suppose x is some rational fraction of the form $x=\frac{x_n}{x_d}$, where both the numerator, x_n , and the denominator, x_d , are whole numbers > 0. Then let's define the following function:

$$t_x(m)=(x_d\cdot(\lfloor\frac{m}{x_d}\rfloor-\lfloor\frac{m-1}{x_d}\rfloor)-x_n\cdot(\lfloor\frac{m}{x_n}\rfloor-\lfloor\frac{m-1}{x_n}\rfloor)) \tag{4.1}$$

Identity Style 1: Via Newton's Generalized Binomial Expansion

$$[((1-x^{1-s})\zeta(s))^z]_n=1+f_1(n,1+\frac{1}{x_d}) \text{ where } f_k(n,j)=\begin{cases} t_x(j)\cdot\frac{1}{x_d}\cdot(\frac{j}{x_d})^{-s}\cdot(\frac{z+1}{k}-1)(1+f_{k+1}(\frac{n}{j},1+\frac{1}{x_d}))+f_k(n,j+\frac{1}{x_d}) & \text{if } n\geq j \\ 0 & \text{if } n<j \end{cases}$$

Some examples of the function include

$$\boxed{\begin{aligned}&[((1-x^{1-s})\zeta(s))^z]_n=\\&\quad 1+\\&\quad \binom{z}{1}\cdot\frac{1}{x^d}\cdot\sum_{1+\frac{j}{x_d}\leq n} t(j)(1+\frac{j}{x_d})^{-s}\\&\quad +\binom{z}{2}\cdot(\frac{1}{x^d})^2\cdot\sum_{(1+\frac{j}{x_d})(1+\frac{k}{x_d})\leq n} t_x(j)t_x(k)(1+\frac{j}{x_d})^{-s}(1+\frac{k}{x_d})^{-s}\\&\quad +\binom{z}{3}\ldots\end{aligned}} \tag{4.2}$$

Identity Style 2: As Exponentiation

$$\boxed{[((1-x^{1-s})\zeta(s))^z]_n=\sum_{k=0}\frac{z^k}{k!}\cdot[(\log((1-x^{1-s})\zeta(s)))^k]_n}$$

Identity Style 3: Via The Hyperbola Method

Identity Style 5: As a Product of Zeros

For fixed values of *n, s,* and *x*, $[((1-x^{1-s})\zeta(s))^z]_n$ is a typical polynomial of order $\lfloor \frac{\log n}{\log(max(x,2))} \rfloor$ with $\lfloor \frac{\log n}{\log(max(x,2))} \rfloor$ zeros such that $[((1-x^{1-s})\zeta(s))^\rho]_n=0$.
Thus,

$$\boxed{[((1-x^{1-s})\zeta(s))^z]_n=\prod_\rho (1-\frac{z}{\rho})} \tag{3.5}$$

and

$$[\log((1-x^{1-s})\zeta(s))]_n=-\sum_\rho \frac{1}{\rho}$$

Relationships to Other Functions

This function generalizes (3.1) as

$$D_z(n,s,y,n+1)=D_z(n,s,y)$$

and (2.1) as

$$[((1-(n+1)^{1-s})\zeta(s))^z]_n=D_z(n,s,1)=[\zeta(s)^z]_n$$

It can also be expressed in terms of $[\zeta(s)^z]_n$ from (2.1) as

$$\boxed{[((1-x^{1-s})\zeta(s))^z]_n=\sum_{k=0} \frac{(-z)^{(k)}}{k!}\cdot x^{k(1-s)}[\zeta(s)^z]_{n\cdot x^{-k}}} \tag{4.3}$$

and, conversely, $[\zeta(s)^z]_n$ can be expressed in terms of it as

$$\boxed{[\zeta(s)^z]_n=\sum_{k=0} \frac{z^{(k)}}{k!}\cdot x^{k(1-s)}[((1-x^{1-s})\zeta(s))^z]_{n\cdot x^{-k}}} \tag{4.4}$$

The Mathematica Code

We can implement $D_z(n,s,y,\frac{a}{b})$ in Mathematica, here named `Dnsyabz[n_, s_,a_,b_,z_]`, like so:

```
binomial[z_,k_]:=binomial[z,k]=Product[z-j,{j,0,k-1}]/k!
zetaHurwitz[n_,s_,y_,0]:=UnitStep[n-1]
zetaHurwitz[n_,s_,y_,1]:=zetaHurwitz[n,s,y,1]=HarmonicNumber[Floor[n],s]-HarmonicNumber[y,s]
zetaHurwitz[n_,s_,y_,2]:=zetaHurwitz[n,s,y,2]=Sum[ (m^(-2s))+2(m^-s) (zetaHurwitz[Floor[n/m],s,m,1]),{m,y+1,Floor[n^(1/2)]}]
zetaHurwitz[n_,s_,y_,k_]:=zetaHurwitz[n,s,y,k]=Sum[(m^(-s k)) +k(m^(-s(k-1))) zetaHurwitz[Floor[n/(m^(k-1))],s, m,1]+Sum[binomial[k,j] (m^-s)^j
zetaHurwitz[Floor[n/(m^j)],s, m,k-j],{j,1,k-2}],{m,y+1,Floor[n^(1/k)]}]
zeta[n_, s_, z_] := Expand@Sum[binomial[z,k]zetaHurwitz[n,s,1,k],{k,0,Log2@n}]

zetaAlt[n_,s_,x_,z_]:=Expand@Sum[(-1)^j binomial[z,j] x^(j(1-s))zeta[n/(x^j),s,z],{j,0,Log[x,n]}]

zetaAltZeros[n_,s_,x_]:=If[(c=Exponent[f=zetaAlt[n,s,x,z],z])==0,{},If[c==1,List@NRoots[f==0,z][[2]],List@@NRoots[f==0,z][[All,2]]]]
zetaAltR[n_,s_,x_,z_]:=Chop@Expand@Product[1-z/rho,{rho,zetaAltZeros[n,s,x]}]
```

```
DzetaAltZeros[n_,s_,x_]:=If[(c=Exponent[f=(D[zetaAlt[n,t,x,z],t]/.t->s)-1,z])==0,{},If[c==1,List@NRoots[f==0,z][[2]],List@@NRoots[f==0,z][[All,2]]]]
DzetaAltR[n_,s_,x_,z_]:=-1+Chop@Expand@Product[1-z/rho,{rho,DzetaAltZeros[n,s,x]}]
```

Two important functions to note here are `Dnsz[n_, s_, z_]` and `DnszR[n_, s_, z_]`. Both of these functions compute the same value, the latter using the roots of the function. Also note the function `DnszZeros[n_, s_]`, which displays those zeros of $[((1-x^{1-s})\zeta(s))^z]_n$ for a fixed n and s.

Another function to note here is `DDnszR[n_, s_, z_]`, which computes the same value as the derivative of $[((1-x^{1-s})\zeta(s))^z]_n$ with respect to s at s, `D[Dnsz[n_, t_, z_], t]/.t->s`, for some fixed value of n and s. The zeros corresponding to those two functions can be computed with `DDnszZeros[n_, s_]`.

Special Values of $[((1-x^{1-s})\zeta(s))^z]_n$

Some noteworthy values of $[((1-x^{1-s})\zeta(s))^z]_n$ are included in Table 4. Here $n \bmod x$ is the remainder of n divided by x, $\Pi(n)$ is the Riemann Prime Counting function, and $\psi(n)$ is the Chebyshev function.

Table 4

Variant of $[((1-x^{1-s})\zeta(s))^z]_n$	*Value*	*Via Roots, in Mathematica*	*Value in Mathematica*
$[(1-x^{1-0})\zeta(0)]_n, x\in\mathbb{N}$	$n \bmod x$	zetaAltR[n,0,x,1]	Mod[n,x]
$[\log((1-x^{1-s})\zeta(s))]_n$	$\sum_{j=1}^{n}\frac{\kappa(j)}{j^s}-\sum_{k=1}^{\lfloor\frac{\log n}{\log x}\rfloor}\frac{x^{k(1-s)}}{k}$	D[zetaAltR[n,s,x,z],z]/.z->0	Sum[FullSimplify[MangoldtLambda[j]/Log[j]] j^(-s),{j,2,n}]-Sum[(x/y)^(k(1-s)) k^-1,{k,1,Floor[Log[x/y,n]]}]
$[\log((1-x^{1-0})\zeta(0))]_n$	$\Pi(n)-\sum_{k=1}^{\lfloor\frac{\log n}{\log x}\rfloor}\frac{x^k}{k}$	D[zetaAltR[n,0,x,z],z]/.z->0	Sum[PrimePi[n^(1/k)]/k,{k,1,Log2@n}]-Sum[(x/y)^k k^-1,{k,1,Floor[Log[x/y,n]]}]
$\lim_{s\to 0}\frac{\partial}{\partial s}[\log((1-x^{1-s})\zeta(s))]_n$	$\psi(n)-\sum_{k=1}^{\lfloor\frac{\log n}{\log x}\rfloor}x^k\log x$	D[DzetaAltR [n,0,x,z],z]/.z->0	Sum[MangoldtLambda[j],{j,2,n}]-Sum[(x/y)^k Log[x/y],{k,1,Floor[Log[x/y,n]]}]

Here H_n is a Harmonic number, $\Pi(n)$ is the Riemann Prime Counting function, $li(n)$ is the logarithmic integral, γ is Euler's constant gamma, and $\psi(n)$ is the Chebyshev function. Note that the bolded terms involve limits that can't actually be evaluated in Mathematica as written; replace infinity with some suitable finite approximation as needs dictate.

Table 4b

Variant of $[((1-x^{1-s})\zeta(s))^z]_n$	*Value*	*Via Roots, in Mathematica*	*Value in Mathematica*
$\lim_{n\to\infty}[((1-x^{1-s})\zeta(s))^z]_n, \Re(s)>0$	$((1-x^{1-s})\zeta(s))^z$	**zetaAltR[∞,s,y,x,z]**	((1-(x/y)^(1-s))Zeta[s])^z
$\lim_{n\to\infty}\lim_{s\to 1}[((1-x^{1-s})\zeta(s))^z]_n$	$\log x$	**zetaAltR[∞,s,y,x,1]**	Log[x/y]
$\lim_{n\to\infty}[\log((1-x^{1-s})\zeta(s))]_n, \Re(s)>0$	$\log((1-x^{1-s})\zeta(s))$	**D[zetaAltR[∞,s,x,z],z]/.z->0**	Log[(1-2^(1-s))Zeta[s]]
$\lim_{n\to\infty}[\log((1-x^{1-s})\zeta(s))]_n, \Re(s)>1$	$\log\zeta(s)-\sum_{k=1}^{\infty}\frac{x^{k(1-s)}}{k}$	**D[zetaAltR[∞,s,x,z],z]/.z->0**	Log[Zeta[s]]-Sum[2^(k(1-s))/k,{k,1,Infinity}]
$\lim_{x\to 1^+}([\log((1-x^{1-0})\zeta(0))]_n+H_{\lfloor\frac{\log n}{\log x}\rfloor})$	$\Pi(n)-li(n)+\log\log n+\gamma$	**(D[zetaAltR[n,0,#,z],z]/.z->0)+HarmonicNumber[Floor[Log[#,n]]]&[∞]**	Sum[PrimePi[n^(1/k)]/k,{k,1,Log2@n}]-LogIntegral[n]+Log[Log[n]]+EulerGamma
$\lim_{x\to 1^+}\lim_{s\to 0}\frac{\partial}{\partial s}[\log((1-x^{1-s})\zeta(s))]_n$	$\psi(n)-n+1$	**(D[DzetaAltR[n,0,#,z],z]/.z->0)&[∞]**	Sum[MangoldtLambda[j],{j,2,n}]-n+1

Questions

5 A Very Special Case of $[((1-x^{1-s})\zeta(s))^z]$: Connecting to the Dirichlet $\eta(s)$ Function

$[((1-x^{1-s})\zeta(s))^z]$ in section 4 is the culmination of a process of generalization, but its motivation might be a bit difficult to follow. Here we're going to look at one very special case of that function, $[((1-2^{1-s})\zeta(s))^z]_n$, that is easier to understand, has a few special identities, and is closely connected to the Dirichlet $\eta(s)$ function.

The Definition

The function (4.1), $t(m)=(b\cdot(\lfloor\frac{m}{b}\rfloor-\lfloor\frac{m-1}{b}\rfloor)-a\cdot(\lfloor\frac{m}{a}\rfloor-\lfloor\frac{m-1}{a}\rfloor))$, when a is 2 and b is 1, simplifies to $t(m)=(-1)^{m+1}$. Thus, (4.2) simplifies to our primary definition for $[\eta(s)^z]_n$,

Identity Style 1: Via Newton's Generalized Binomial Expansion

$$[\eta(s)^z]_n=1+f(n,2,1) \text{ where } f(n,j,k)=\begin{cases}(-1)^{j+1}\cdot j^{-s}\cdot(\frac{z+1}{k}-1)(1+f(\frac{n}{j},2,k+1))+f(n,j+1,k) & \text{if } n\geq j\\ 0 & \text{if } n< j\end{cases}$$

$$[\eta(s)^z]_n=\binom{z}{0}1+\binom{z}{1}\sum_{j=2}^{n}(-1)^{j+1}\cdot j^{-s}+\binom{z}{2}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}(-1)^{j+1}\cdot j^{-s}\cdot(-1)^{k+1}\cdot k^{-s}+\binom{z}{3}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor\frac{n}{j\cdot k}\rfloor}(-1)^{j+1}\cdot j^{-s}\cdot(-1)^{k+1}\cdot k^{-s}\cdot(-1)^{l+1}\cdot l^{-s}+\binom{z}{4}...$$

(5.1)

Identity Style 2: As Exponentiation

Another way to express this function, if we define the helper function $\kappa_\eta(n)$ as

$$\kappa_\eta(n)=\begin{cases}\kappa(n)-\frac{2^k}{k} & \text{if } n=2^k\\ \kappa(n) & \text{otherwise}\end{cases}$$

is

$$[\eta(s)^z]_n=1+p(n,2,1) \quad\text{where}\quad p(n,j,k)=\begin{cases}j^{-s}\cdot\frac{z}{k}\cdot\kappa_\eta(j)(1+p(\frac{n}{j},2,k+1))+p(n,j+1,k) & \text{if } n\geq j\\ 0 & \text{if } n< j\end{cases}$$

$$[\eta(s)^z]_n=1+\frac{z^1}{1!}\sum_{j=2}^{n}\kappa_\eta(j)\cdot j^{-s}+\frac{z^2}{2!}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\kappa_\eta(j)\cdot j^{-s}\cdot\kappa_\eta(k)\cdot k^{-s}+\frac{z^3}{3!}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor\frac{n}{j\cdot k}\rfloor}\kappa_\eta(j)\cdot j^{-s}\cdot\kappa_\eta(k)\cdot k^{-s}\cdot\kappa_\eta(l)\cdot l^{-s}+\frac{z^4}{4!}...$$

(5.2)

Identity Style 3: Via The Hyperbola Method

$$[\eta(s)^z]_n=[(1+\eta(s,2))^z]_n \text{ where } [(1+\eta(s,y))^z]_n=\begin{cases}\sum_{k=0}^{\lfloor\frac{\log n}{\log y}\rfloor}\binom{z}{k}(y^{-s}\cdot(-1)^{y+1})^k\cdot[(1+\eta(s,y+1))^{z-k}]_{n/y^k} & \text{if } n\geq y\\ 1 & \text{if } n<y\end{cases}$$

Identity Style 4: As a Sum of Multiplicative Functions

A final way to express this same function, with ${}_2F_1(a;b;c;d)$ the confluent hypergeometric function of the second kind, is

$$\boxed{[\eta(s)^z]_n = \sum_{j=1}^{n} \prod_{p^k | j} \begin{cases} p^{-sk} \cdot (-z) \cdot {}_2F_1(1-k;1-z;2;-1) & \text{if } p=2 \\ p^{-sk} \cdot \dfrac{z^{(k)}}{k!} & \text{if } p \neq 2 \end{cases}}$$

(5.3)

Identity Style 5: As a Product of Zeros

For fixed values of *n, s,* and *x*, $[\eta(s)^z]_n$ is a typical polynomial of order $\lfloor \frac{\log n}{\log 2} \rfloor$ with $\lfloor \frac{\log n}{\log 2} \rfloor$ zeros such that. $[\eta(s)^\rho]_n = 0$ Thus,

$$\boxed{[\eta(s)^z]_n = \prod_\rho \left(1 - \frac{z}{\rho}\right)}$$

(3.5)

and

$$[\log \eta(s)]_n = -\sum_\rho \frac{1}{\rho}$$

The Mathematica Code

We can implement $[\eta(s)^z]_n$ in Mathematica, here named `eta[n_, s_,z_]`, like so:

```
binomial[z_,k_]:=binomial[z,k]=Product[z-j,{j,0,k-1}]/k!
zetaHurwitz[n_,s_,y_,0]:=UnitStep[n-1]
zetaHurwitz[n_,s_,y_,1]:=zetaHurwitz[n,s,y,1]=HarmonicNumber[Floor[n],s]-HarmonicNumber[y,s]
zetaHurwitz[n_,s_,y_,2]:=zetaHurwitz[n,s,y,2]=Sum[ (m^(-2s))+2(m^-s) (zetaHurwitz[Floor[n/m],s,m,1]),{m,y+1,Floor[n^(1/2)]}]
zetaHurwitz[n_,s_,y_,k_]:=zetaHurwitz[n,s,y,k]=Sum[(m^(-s k)) +k(m^(-s(k-1))) zetaHurwitz[Floor[n/(m^(k-1))],s, m,1]+Sum[binomial[k,j] (m^-s)^j
zetaHurwitz[Floor[n/(m^j)],s, m,k-j],{j,1,k-2}],{m,y+1,Floor[n^(1/k)]}]
zeta[n_, s_, z_] := Expand@Sum[binomial[z,k]zetaHurwitz[n,s,1,k],{k,0,Log2@n}]

eta[n_, s_,z_] := Expand@Sum[ (-1)^j binomial[z,j] 2^(j(1-s))zeta[ n/(2^j),s,z],{j,0,Log2@n}]

etaZeros[n_,s_]:=If[(c=Exponent[f=eta[n,s,z],z])==0,{}, If[ c==1,List@NRoots[f==0,z][[2]],List@@NRoots[f==0,z][[All,2]]]]
etaR[n_, s_, z_] := Chop@Expand@Product[ 1-z/rho,{rho,etaZeros[n,s]}]

DetaZeros[n_,s_]:=If[(c=Exponent[f=(D[eta[n,t,z],t]/.t->s)-1,z])==0,{}, If[ c==1,List@NRoots[f==0,z][[2]],List@@NRoots[f==0,z][[All,2]]]]
DetaR[n_, s_, z_] := -1+Chop@Expand@Product[ 1-z/rho,{rho,DetaZeros[n,s]}]
```

Two important functions to note here are `eta[n_, s_, z_]` and `etaR[n_, s_, z_]`. Both of these functions compute the same value, the latter using the roots of the function. Also note the function `etaZeros[n_, s_]`, which displays those zeros of $[\eta(s)^z]_n$ for a fixed *n* and *s*.

Another function to note here is `DetaR[n_, s_, z_]`, which computes the same value as the derivative of $[\eta(s)^z]_n$ with respect to *s* at *s*, `D[eta[n_, t_, z_], t]/.t->s`, for some fixed value of *n* and *s*. The zeros corresponding to those two functions can be computed with `DetaZeros[n_, s_]`.

Special Values of $[\eta(s)^z]_n$

Some noteworthy values of $[\eta(s)^z]_n$ are included in Table 5. Here $\Pi(n)$ is the Riemann Prime Counting function, and $\psi(n)$ is the Chebyshev function.

Table 5

Variant of $[\eta(s)^z]_n$	*Value*	*Via Roots, in Mathematica*	*Value in Mathematica*
$[\eta(0)]_n$	$\frac{1}{2}+\frac{1}{2}(-1)^{n+1}$	etaR[n, 0,1]	1/2+1/2(-1)^(n+1)
$[\log\eta(s)]_n$	$\sum_{j=1}^{n}\frac{\kappa(j)}{j^s}-\sum_{k=1}^{\lfloor\frac{\log n}{\log 2}\rfloor}\frac{2^{k(1-s)}}{k}$	D[etaR[n,s,z],z]/.z->0	Sum[FullSimplify[MangoldtLambda[j]/Log[j]] j^(-s),{j,2,n}]-Sum[2^(k(1-s)) k^-1, {k,1,Log2@n}]
$[\log\eta(0)]_n$	$\Pi(n)-\sum_{k=1}^{\lfloor\frac{\log n}{\log 2}\rfloor}\frac{2^k}{k}$	D[etaR[n,0,z],z]/.z->0	Sum[PrimePi[n^(1/k)]/k, {k,1,Log2@n}]-Sum[2^k k^-1, {k,1,Log2@n}]
$\lim_{s\to 0}\frac{\partial}{\partial s}[\log\eta(s)]_n$	$\psi(n)-\sum_{k=1}^{\lfloor\frac{\log n}{\log 2}\rfloor}2^k\log 2$	D[DetaR[n,0,z],z]/.z->0	Sum[MangoldtLambda[j], {j,2,n}]-Sum[2^k Log[2], {k,1,Log2@n}]

More values of $[\eta(s)^z]_n$ are included in Table 5a. Here $\zeta(s)$ is the Riemann Zeta function and $\eta(s)$ is the Dirichlet Eta function. Note that the bolded terms involve limits that can't actually be evaluated in Mathematica as written; replace infinity with some suitable finite approximation as needs dictate.

Table 5a

Variant of $[\eta(s)^z]_n$	*Value*	*Via Roots, in Mathematica*	*Value in Mathematica*
$\lim_{n\to\infty}[\eta(s)^z]_n,\ \Re(s)>0$	$\eta(s)^z$	**etaR[∞, s,z]**	((1-2^(1-s))Zeta[s])^z
$\lim_{n\to\infty}[\eta(s)^z]_n,\ \Re(s)>0$	$((1-2^{(1-s)})\zeta(s))^z$	**etaR[∞, s,z]**	((1-2^(1-s))Zeta[s])^z
$\lim_{n\to\infty}[\log\eta(s)]_n,\ \Re(s)>0$	$\log((1-2^{(1-s)})\zeta(s))$	**D[etaR[∞, s,z],z]/.z->0**	Log[(1-2^(1-s))Zeta[s]]
$\lim_{n\to\infty}[\log\eta(s)]_n,\ \Re(s)>1$	$\log\zeta(s)-\sum_{k=1}^{\infty}\frac{2^{k(1-s)}}{k}$	**D[etaR[∞, s,z],z]/.z->0**	Log[Zeta[s]]-Sum[2^(k(1-s))/k, {k,1,Infinity}]
$\lim_{n\to\infty}[\eta(1)]_n$	$\log 2$	**etaR[∞, 1,1]**	Log[2]
$\lim_{n\to\infty}[\eta(s)^{\rho_k}]_n$ where $\zeta(\rho)=0$ and $0<\Re(\rho)<1$	0	**etaR[∞, ZetaZero@k,1]**	0

Some Examples of $[\eta(s)^z]_n$

$$
\begin{aligned}
&[\eta(0.5+14.134725141734695\,i)^{\rho}]_{10000000000} \approx \\
&\quad (1.+0.\,i)+(1349.62100658792+286.8498080505834\,i)\,z-(4933.767046855022+1194.8380079941549\,i)\,z^{2} \\
&\quad +(7938.419402922736+2153.6869074308997\,i)\,z^{3}-(7599.078329205213+2235.79395754807\,i)\,z^{4} \\
&\quad +(4913.4973898049075+1501.7397359818626\,i)\,z^{5}-(2304.6819900456385+693.1534636360454\,i)\,z^{6} \\
&\quad +(821.1453803767897+226.350174587896\,i)\,z^{7}-(229.44721333805904+52.21892235943409\,i)\,z^{8} \\
&\quad +(51.435605030068615+7.989794094186166\,i)\,z^{9}-(9.400404740033807+0.5597068774652687\,i)\,z^{10} \\
&\quad +(1.4166113819011805-0.08010284922221\,i)\,z^{11}-(0.1775076598568166-0.03296531689764337\,i)\,z^{12} \\
&\quad +(0.01862425722739902-0.005877834137370738\,i)\,z^{13}-(0.0016461853950728023-0.0007080207152759359\,i)\,z^{14} \\
&\quad +(0.00012307633055110706-0.00006344681279928446\,i)\,z^{15}-(7.78139931694767*^{-6}-4.431497723468613*^{-6}\,i)\,z^{16} \\
&\quad +(4.134977156509175*^{-7}-2.502639441025558*^{-7}\,i)\,z^{17}-(1.826578605342965*^{-8}-1.1772850887173012*^{-8}\,i)\,z^{18} \\
&\quad +(6.620677133647833*^{-10}-4.667926818641298*^{-10}\,i)\,z^{19}-(1.943819834853632*^{-11}-1.541175678603278*^{-11}\,i)\,z^{20} \\
&\quad +(4.561558255872053*^{-13}-4.130415185401319*^{-13}\,i)\,z^{21}-(8.426156210930456*^{-15}-8.765038522692641*^{-15}\,i)\,z^{22} \\
&\quad +(1.2057046641948994*^{-16}-1.440308240380748*^{-16}\,i)\,z^{23}-(1.3204925393185037*^{-18}-1.7950086833630073*^{-18}\,i)\,z^{24} \\
&\quad +(1.1063972297236223*^{-20}-1.6758494023649118*^{-20}\,i)\,z^{25}-(7.216928070205532*^{-23}-1.1831642881169047*^{-22}\,i)\,z^{26} \\
&\quad +(3.7819549796583117*^{-25}-6.559484430096852*^{-25}\,i)\,z^{27}-(1.590619378093596*^{-27}-2.89897061167791*^{-27}\,i)\,z^{28} \\
&\quad +(5.049860536556247*^{-30}-9.771668999741509*^{-30}\,i)\,z^{29}-(1.1163246671792968*^{-32}-2.1033167439037223*^{-32}\,i)\,z^{30} \\
&\quad +(1.1168665700750259*^{-35}-2.3390868213424994*^{-35}\,i)\,z^{31}-(1.7319600603546155*^{-39}-1.5405404480503026*^{-38}\,i)\,z^{32} \\
&\quad +(8.071479154510946*^{-43}+5.853223685484761*^{-42}\,i)\,z^{33}
\end{aligned}
$$

$[\eta(0.5+14.134725141734695\,i)^{\rho}]_{10000000000}=0$ when

$\rho=\{$

$-3864.73-1018.79\,i$
$-26.2144+176.538\,i$
$-23.4619-181.371\,i$
$-4.12866+7.34176\,i$
$-3.72207-18.2745\,i$
$-1.57817+30.8767\,i$
$-0.000707144+0.000149944\,i$
$1.-7.40352*10^{-8}\,i$
$1.99997-0.0012656\,i$
$2.05938-5.29269\,i$
$2.49167+5.68823\,i$
$2.494-1.73396\,i$
$2.95185+0.0543244\,i$
$3.22142-0.46806\,i$
$3.37038+0.822981\,i$
$4.08412-2.42114\,i$
$4.29356+2.34112\,i$
$7.72879-0.00991479\,i$
$8.12663-5.15547\,i$
$8.14801+5.31789\,i$
$16.3646+6.26176\,i$
$16.4991-0.0467327\,i$
$17.145-6.17218\,i$
$20.2719-33.6102\,i$
$26.1258+44.9315\,i$
$40.7551-3.18062\,i$
$49.9124+24.7706\,i$
$58.4182-27.7668\,i$
$92.6007+108.398\,i$
$120.861-70.0834\,i$
$228.429-79.127\,i$
$311.514-371.285\,i$
$330.177+764.904\,i$

$\}$

Questions

6 Using a Wheel

Here's another variation on the theme.

Instead of working, generally, with all whole numbers, let's only work with whole numbers that aren't divisible by the first few primes. For computational purposes, if we implement a wheel, this can execute much more quickly while still saying valid things about the primes.

The Definition

More formally,

Let P_K *be the set of the first K distinct primes,* $\{2,3,5,...p_K\}$.

Let B_K *be the set of all natural numbers > 1 not divisible by any members of* P_K .

Identity Style 1: Via Newton's Generalized Binomial Expansion

Then we can define $[\zeta(s)^z]_{n\in B_K}$ as

$$[\zeta(s)^z]_{n\in B_K}=\binom{z}{0}1+\binom{z}{1}\sum_{j\le n;\, j\in B_K} j^{-s}+\binom{z}{2}\sum_{j\cdot k\le n;\, j,k\in B_K}(j\cdot k)^{-s}+\binom{z}{3}\sum_{j\cdot k\cdot l\le n;\, j,k,l\in B_K}(j\cdot k\cdot l)^{-s}+...$$

(6.1)

Identity Style 2: As Exponentiation

This can also be expressed as

$$[\zeta(s)^z]_{n\in B_K}=1+\frac{z^1}{1!}\sum_{j\le n;\, j\in B_K}\kappa(j)\cdot j^{-s}+\frac{z^2}{2!}\sum_{j\cdot k\le n;\, j,k\in B_K}\kappa(j)\kappa(k)\cdot(jk)^{-s}+\frac{z^3}{3!}\sum_{j\cdot k\cdot l\le n;\, j,k,l\in B_K}\kappa(j)\kappa(k)\kappa(l)(jkl)^{-s}+\frac{z^4}{4!}...$$

(6.2)

and as

Identity Style 3: Via The Hyperbola Method

Identity Style 4: As a Sum of Multiplicative Functions

$$[\zeta(s)^z]_{n\in B_K}=\sum_{j\le n;\, j\in B_K}^{n}\prod_{p^k|j}\frac{z^{(k)}}{k!}\cdot p^{-as}$$

(6.3)

Identity Style 5: As a Product of Zeros

For fixed values of *n, s,* and *x*, $[\zeta(s)^z]_{n\in B_K}$ is a typical polynomial of order $\lfloor\frac{\log n}{\log p_{k+1}}\rfloor$ with $\lfloor\frac{\log n}{\log p_{k+1}}\rfloor$ zeros such that $[\zeta(s)^\rho]_{n\in B_K}=0$. Thus,

$$[\zeta(s)^z]_{n\in B_K}=\prod_\rho (1-\frac{z}{\rho}) \tag{3.5}$$

and

$$[\log\zeta(s)^z]_{n\in B_K}=-\sum_\rho \frac{1}{\rho}$$

The Mathematica Code

We can implement $[\zeta(s)^z]_{n\in B_K}$ in Mathematica, here named `zetaWheel[n_, s_,z_]`, like so:

```
binomial[z_,k_]:=binomial[z,k]=Product[z-j,{j,0,k-1}]/k!
WheelEntries=7;
WheelSize:=WheelSize=Product[Prime[j],{j,1,WheelEntries}];
CoprimeCache:=CoprimeCache=Table[If[CoprimeQ[WheelSize,n],1,0],{n,1,WheelSize}]
LegPhiCache:=LegPhiCache=Accumulate[CoprimeCache]
FullWheel:=FullWheel = LegPhiCache[[WheelSize]]
CoprimeOffsets:=CoprimeOffsets=Flatten[Position[CoprimeCache,_?(#==1&)]]
Coprimes[n_]:=Coprimes[n]=LegPhiCache[[Mod[n-1,WheelSize]+1]]+Floor[(n-1)/WheelSize] FullWheel
WheelForID[n_]:=WheelForID[n]=CoprimeOffsets[[Mod[n-1,FullWheel]+1]]+Floor[(n-1)/FullWheel]WheelSize
FirstNonWheel :=FirstNonWheel=WheelForID[2]

FI[n_]:=FactorInteger[n]
FI[1]:={}

zetaHurwitzWheel[n_,y_,0]:=1
zetaHurwitzWheel[n_,y_,1]:=zetaHurwitzWheel[n,y,1]=Coprimes[n]-Coprimes[WheelForID[y]]
zetaHurwitzWheel[n_,y_,2]:=zetaHurwitzWheel[n,y,2]=(Coprimes[Floor[n^(1/2)]]-y)+2Sum[zetaHurwitzWheel[Floor[n/WheelForID[m]],m,1],
{m,y+1,Coprimes[Floor[n^(1/2)]]}]
zetaHurwitzWheel[n_,y_,k_]:=zetaHurwitzWheel[n,y,k]=(Coprimes[Floor[n^(1/k)]]-y)+k Sum[zetaHurwitzWheel[Floor[n/(WheelForID[m]^(k-1))],m,1],
{m,y+1,Coprimes[Floor[n^(1/k)]]}]+Sum[binomial[k,j]zetaHurwitzWheel[Floor[n/(WheelForID[m]^(k-j))],m,j],{m,y+1,Coprimes[Floor[n^(1/k)]]},{j,2,k-
1}]

zetaWheel[n_,0,z_]:=Expand[Sum[binomial[z,k] zetaHurwitzWheel[Floor[n],1,k],{k,0,Log[FirstNonWheel,n]}]]
zetaWheel[n_,s_,z_]:=Expand[Sum[j^-s Product[If[p[[1]]<FirstNonWheel,0,(-1)^p[[2]] binomial[-z,p[[2]]]],{p,FI[j]}],{j,1,n}]]

DzetaWheelZeros[n_,s_]:=If[(c=Exponent[f=zetaWheel[n,s,z],z])==0,{},If[c==1,List@NRoots[f==0,z][[2]],List@@NRoots[f==0,z][[All,2]]]]
DzetaWheelR[n_,s_,z_]:=Expand[Product[1-z/rho,{rho,DzetaWheelZeros[n,s]}]]

DzetaWheelZeros[n_,s_]:=If[(c=Exponent[f=(D[zetaWheel[n,t,z],t]/.t->s)-1,z])==0,{},If[c==1,List@NRoots[f==0,z][[2]],List@@NRoots[f==0,z][[All,2]]]]
DzetaWheelR[n_,s_,z_]:=Expand[Product[1-z/rho,{rho,DzetaWheelZeros[n,s]}]]
```

Two important functions to note here are `zetaWheel[n_, s_, z_]` and `zetaWheelR[n_, s_, z_]`. Both of these functions compute the same value, the latter using the roots of the function. Also note the function `zetaWheelZeros[n_, s_]`, which displays those zeros of $[\zeta(s)^z]_{n\in B_K}$ for a fixed *n* and *s*.

Special Values of $[\zeta(s)^z]_{n\in B_K}$

Noteworthy values of $[\zeta(s)^z]_{n\in B_K}$ are included in Table 7. Here $\Pi(n)$ is the Riemann Prime Counting function, and $\psi(n)$ is the Chebyshev function.

Table 6

Variant of $[\zeta(s)^z]_{n\in B_K}$	*Value*	*Via Roots, in Mathematica*	*Value in Mathematica*

$[\log\zeta(s)]_{n\in B_K}$	$\sum_{j=1}^{n}\kappa(j)\,j^{-s}-\sum_{j=1}^{K}\sum_{m=1}^{\lfloor\log_{P_j} n\rfloor} m^{-1}P_j^{-ms}$	D[zetaWheelR[n,s,z],z]/.z->0	Sum[FullSimplify[MangoldtLambda[j]/Log[j]]j^-s, {j,2,n}]-Sum[m^-1 Prime[j]^(-m s), {j,1,WheelEntries}, {m,1,Log[Prime[j], n]}]
$[\log\zeta(0)]_{n\in B_K}$	$\Pi(n)-\sum_{j=1}^{K}\sum_{m=1}^{\lfloor\log_{P_j} n\rfloor} m^{-1}$	D[zetaWheelR[n,0,z],z]/.z->0	Sum[PrimePi[n^(1/k)]/k, {k,1,Log2@n}]-Sum[m^-1, {j,1,WheelEntries}, {m,1,Log[Prime[j], n]}]
$-\lim_{s\to 0}\frac{\partial}{\partial s}[\log\zeta(s)]_{n\in B_K}$	$\psi(n)-\sum_{j=1}^{K}\log P_j\cdot\lfloor\log_{P_j} n\rfloor$	D[DzetaWheelR[n,0,z],z]/.z->0	N@Sum[MangoldtLambda[j], {j,2,n}] - Sum[Log[Prime@j] Floor[Log[Prime@j, n]], {j,1,WheelEntries}]

Questions

7 Other Related Functions and Values: $\lambda(n)$, $\sigma_a(n)$, $\varphi(n)$, $a(n)$, $\pi(n)$, $\frac{\pi}{4}$

Summatory functions for several other important multiplicative functions are closely related to $[\zeta(s)]_n$. Here we will look at the Liouville $\lambda(n)$ function, the divisor $\sigma_a(n)$ function, and the Jordan Totient function $J_a(n)$, of which the Euler Totient function $\varphi(n)$ is the most familiar variant. Weighted summatory functions for all these functions can be expressed in terms of $[\zeta(s)]_n$, and in turn they can all be expressed by zeros similar to the ones in other sections of this paper.

The Definitions

$$\nabla[f]_n = [f]_n - [f]_{n-1}$$

$$[f\cdot g]_n = \sum_{j\cdot k\le n} \nabla[f]_j\cdot\nabla[g]_k \quad\text{and}\quad [\frac{f}{g}]_n = \sum_{j\cdot k\le n} \nabla[f]_j\cdot\nabla[g^{-1}]_k$$

$$[f_{1/a}\cdot g_{1/b}]_n = \sum_{j^a\cdot k^b\le n} \nabla[f]_j\cdot\nabla[g]_k \quad\text{and}\quad [\frac{f_{1/a}}{g_{1/b}}]_n = \sum_{j^a\cdot k^b\le n} \nabla[f]_j\cdot\nabla[g^{-1}]_k$$

$$[f\cdot g_{\log_n m}]_n = [f_{\log_m n}\cdot g]_m = [f_{\log n}\cdot g_{\log m}]_e$$

$$n\cdot m = n\cdot n^{\log_n m} = m\cdot m^{\log_m n} = e^{\log n}\cdot e^{\log m}$$

$$[f_{\log n}\cdot g_{\log m}]_e = \sum_{\frac{\log j}{\log n}+\frac{\log k}{\log m}\le 1} \nabla[f]_j\cdot\nabla[g]_k$$

Defining Arbitrary Multiplication and Division

$$[(\zeta_{\log n}(s)\cdot\zeta_{\log m}(s))^z] = \sum_{\frac{\log j}{\log n}+\frac{\log k}{\log m}\le 1} \nabla[\zeta(s)^z]_j\cdot\nabla[\zeta(s)^z]_k$$

$$[(\frac{\zeta_{\log n}(s)}{\zeta_{\log m}(s)})^z] = \sum_{\frac{\log j}{\log n}+\frac{\log k}{\log m}\le 1} \nabla[\zeta(s)^z]_j\cdot\nabla[\zeta(s)^{-z}]_k$$

$$[\log(\zeta_{\log n}(s)\cdot\zeta_{\log m}(s))]_e = [\log\zeta(s)]_n + [\log\zeta(s)]_m$$

$$[\log\frac{\zeta_{\log n}(s)}{\zeta_{\log m}(s)}]_e = [\log\zeta(s)]_n - [\log\zeta(s)]_m$$

In particular,

$$[\log \frac{\zeta_{\log n}(0)}{\zeta_{\log m}(0)}] = \Pi(n) - \Pi(m)$$

and

$$\lim_{s \to 0} \frac{\partial}{\partial s}[\log \frac{\zeta_{\log n}(s)}{\zeta_{\log m}(s)}] = \psi(n) - \psi(m)$$

8 Other Related Functions and Values: $\lambda(n)$, $\sigma_a(n)$, $\varphi(n)$, $a(n)$, $\pi(n)$, $\frac{\pi}{4}$

$$\nabla[\zeta(s)^z]_n = \prod_{p^k|n} \frac{z^{(k)}}{k!} \cdot p^{-ks}$$

$$\nabla[\zeta(s)]_n = n^{-s}$$

$$\nabla[\zeta(s)^{-1}]_n = \mu(n) \cdot n^{-s}$$

The Liouville $\lambda(n)$ Function

Let's start by defining a function we'll notate as $[(\frac{\zeta_{1/2}(2s)}{\zeta(s)})^z]_n$. Its primary definition will be

$$\boxed{[(\frac{\zeta_{1/2}(2s)}{\zeta(s)})^z]_n = \sum_{j^2 \cdot k \leq n} \nabla[\zeta(2s)^z]_j \cdot \nabla[\zeta(s)^{-z}]_k}$$

$$[\log \frac{\zeta_{1/2}(2s)}{\zeta(s)}]_n = [\log \zeta_{1/2}(2s)]_n - [\log \zeta(s)]_n = [\log \zeta(s)]_{n^{\frac{1}{2}}} - [\log \zeta(s)]_n$$

$$\lambda(n) \cdot n^{-s} = \nabla[\frac{\zeta_{1/2}(s)}{\zeta(s)}]_n$$

The Divisor $\sigma_a(n)$ Function

We can take a similar approach with the divisor $\sigma_a(n)$ function. We begin by defining $[(\zeta(s-a) \cdot \zeta(s))]_n$ as

$$\boxed{[(\zeta(s-a) \cdot \zeta(s))^z]_n = \sum_{j \cdot k \leq n} \nabla[\zeta(s-a)^z]_j \cdot \nabla[\zeta(s)^z]_k}$$

$$[\log(\zeta(s-a) \cdot \zeta(s))]_n = [\log \zeta(s-a)]_n + [\log \zeta(s)]_n$$

$$\sigma_a(n) \cdot n^{-s} = \nabla[\zeta(s-a) \cdot \zeta(s)]_n$$

The Euler and Jordan Totient Functions $\varphi(n)$ and $J_a(n)$

$$\boxed{[(\frac{\zeta(s-a)}{\zeta(s)})^z]_n = \sum_{j \cdot k \leq n} \nabla[\zeta(s-a)^z]_j \cdot \nabla[\zeta(s)^{-z}]_k}$$

$$[\log\frac{\zeta(s-a)}{\zeta(s)}]_n=[\log\zeta(s-a)]_n-[\log\zeta(s)]_n$$

$$J_a(n)\cdot n^{-s}=\nabla[\frac{\zeta(s-a)}{\zeta(s)}]_n$$

$$\varphi(n)\cdot n^{-s}=\nabla[\frac{\zeta(s-1)}{\zeta(s)}]_n$$

The Euler Totient Function $a(n)$ and $[(\prod_{k=1}\zeta_{1/k}(k\,s))^z]_n$

$$\boxed{[(\prod_{k=1}\zeta_{1/k}(k\,s))^z]_n=\sum_{a\cdot b^2\cdot c^3\cdot d^4\cdot\ldots\le n}\nabla[\zeta(s)^z]_a\cdot\nabla[\zeta(2s)^z]_b\cdot\nabla[\zeta(3s)^z]_c\cdot\nabla[\zeta(4s)^z]_d\cdot\ldots.}$$

$$[\log(\prod_{k=1}\zeta_{1/k}(k\,s))]_n=\sum_{k=1}[\log(\zeta_{1/k}(k\,s))]_n=\sum_{k=1}[\log\zeta(s)]_{n^{\frac{1}{k}}}$$

$$a(n)\cdot n^{-s}=\nabla[\prod_{k=1}\zeta_{1/k}(k\,s)]_n$$

The Count of Primes $\pi(n)$

$$\boxed{[(\prod_{k=1}\zeta_{1/k}(k\,s)^{\frac{\mu(k)}{k}})^z]_n=\sum_{a\cdot b^2\cdot c^3\cdot d^5\cdot e^6\ldots\le n}\nabla[\zeta(s)^z]_a\cdot\nabla[\zeta(2s)^{-\frac{z}{2}}]_b\cdot\nabla[\zeta(3s)^{-\frac{z}{3}}]_c\cdot\nabla[\zeta(5s)^{-\frac{z}{5}}]_d\cdot\nabla[\zeta(6s)^{\frac{z}{6}}]_e\cdot\ldots.}$$

$$[\log(\prod_{k=1}\zeta_{1/k}(k\,s)^{\frac{\mu(k)}{k}})]_n=\sum_{k=1}\frac{\mu(k)}{k}\cdot[\log\zeta_{1/k}(k\,s)]_n=\sum_{k=1}\frac{\mu(k)}{k}\cdot[\log\zeta(s)]_{n^{\frac{1}{k}}}$$

$$b(n)\cdot n^{-s}=\nabla[(\prod_{k=1}\zeta_{1/k}(k\,s)^{\frac{\mu(k)}{k}})^z]_n$$

$$b(n)=\prod_{p^k\| n}\frac{1}{k\,!}$$

Leibniz's Formula for $\frac{\pi}{4}$

Leibniz's formula for π is the series $\frac{\pi}{4}=\sum_{j=0}^{\infty}\frac{(-1)^j}{2\,j+1}$. For our partial sums, it is better to rewrite this as $\frac{\pi}{4}=\sum_{j=1}^{\infty}\cos(\frac{\pi}{2}\cdot(j-1))\cdot j^{-1}$. We can then construct the following sum.

$$[2^{-2s}(\zeta(s,\frac{1}{4})-\zeta(s,\frac{3}{4}))]_n=\sum_{j=1}^{n}\cos(\frac{\pi}{2}\cdot(j-1))\cdot j^{-s}$$

$$s(n) = \cos\left(\frac{\pi}{2}\cdot(n-1)\right)$$

The Mathematica Code

We can implement $D_z(n, s)$ in Mathematica, here named `Dnsyz[n_, s_,z_]`, like so:

```
FI[n_]:=FactorInteger[n]
FI[1]:={}
dzeta[n_,s_,z_]:=dzeta[n,s,z]=n^-s Product[Pochhammer[z,p[[2]]]/p[[2]]!,{p,FI[j]}]

binomial[z_,k_]:=binomial[z,k]=Product[z-j,{j,0,k-1}]/k!
zetaHurwitz[n_,s_,y_,0]:=UnitStep[n-1]
zetaHurwitz[n_,s_,y_,1]:=zetaHurwitz[n,s,y,1]=HarmonicNumber[Floor[n],s]-HarmonicNumber[y,s]
zetaHurwitz[n_,s_,y_,2]:=zetaHurwitz[n,s,y,2]=Sum[(m^(-2s))+2(m^-s) (zetaHurwitz[Floor[n/m],s,m,1]),{m,y+1,Floor[n^(1/2)]}]
zetaHurwitz[n_,s_,y_,k_]:=zetaHurwitz[n,s,y,k]=Sum[(m^(-s k))+k(m^(-s(k-1))) zetaHurwitz[Floor[n/(m^(k-1))],s,m,1]+Sum[binomial[k,j] (m^-s)^j
zetaHurwitz[Floor[n/(m^j)],s,m,k-j],{j,1,k-2}],{m,y+1,Floor[n^(1/k)]}]
zeta[n_,s_,z_]:=Expand@Sum[binomial[z,k]zetaHurwitz[n,s,1,k],{k,0,Log2@n}]

zetaLiouville[n_,s_,z_]:=Sum[dzeta[j,2s,z] zeta[Floor[n/j^2],s,-z],{j,1,n^(1/2)}]
zetaDivisor[n_,s_,z_,a_]:=Sum[dzeta[j,s-a,z] zeta[Floor[n/j],s,z],{j,1,n}]
zetaJordan[n_,s_,z_,a_]:=Sum[dzeta[j,s-a,z] zeta[Floor[n/j],s,-z],{j,1,n}]

zetaLiouvilleZeros[n_,s_]:=If[(c=Exponent[f=zetaLiouville[n,s,z],z])==0,{},If[c==1,List@NRoots[f==0,z][[2]],List@@NRoots[f==0,z][[All,2]]]]
zetaLiouvilleR[n_,s_,z_]:=Chop@Expand@Product[1-z/rho,{rho,zetaLiouvilleZeros[n,s]}]

zetaDivisorZeros[n_,s_,a_]:=If[(c=Exponent[f=zetaDivisor[n,s,z,a],z])==0,{},If[c==1,List@NRoots[f==0,z][[2]],List@@NRoots[f==0,z][[All,2]]]]
zetaDivisorR[n_,s_,z_,a_]:=Chop@Expand@Product[1-z/rho,{rho,zetaDivisorZeros[n,s,a]}]

zetaJordanZeros[n_,s_,a_]:=If[(c=Exponent[f=zetaJordan[n,s,z,a],z])==0,{},If[c==1,List@NRoots[f==0,z][[2]],List@@NRoots[f==0,z][[All,2]]]]
zetaJordanR[n_,s_,z_,a_]:=Chop@Expand@Product[1-z/rho,{rho,zetaJordanZeros[n,s,a]}]
```

Two important functions to note here are `Dnsz[n_, s_, z_]` and `DnszR[n_, s_, z_]`. Both of these functions compute the same value, the latter using the roots of the function. Also note the function `DnszZeros[n_, s_]`, which displays those zeros of $D_z(n, s)$ for a fixed *n* and *s*.

Another function to note here is `DDnszR[n_, s_, z_]`, which computes the same value as the derivative of $D_z(n, s)$ with respect to *s* at *s*, `D[Dnsz[n_, t_, z_], t]/.t->s`, for some fixed value of *n* and *s*. The zeros corresponding to those two functions can be computed with `DDnszZeros[n_, s_]`.

Special Values of $[\zeta(s)^z]_n$

Some noteworthy values of $[\zeta(s)^z]_n$ included in Table 2. For reference, $\lambda(n)$ is the Liouville lambda function, $\mu(n)$ is the Moebius function, $\sigma_a(n)$ is the divisor function, and $\varphi(n)$ is the Euler Totient function, and $J_a(n)$ is the Jordan Totient function.

Table 7

Variant of $[\zeta(s)]_n$	*Value*	*Via Roots, in Mathematica*	*Value in Mathematica*
$[(\frac{\zeta_{1/2}(0)}{\zeta(0)})]_n$	$\sum_{j=1}^{n} \lambda(j)$	zetaLiouvilleR[n,0,1]	Sum[LiouvilleLambda[j], {j,1,n}]
$[(\frac{\zeta_{1/2}(0)}{\zeta(0)})^{-1}]_n$	$\sum_{j=1}^{n} \lvert\mu(j)\rvert$	zetaLiouvilleR[n,0,-1]	Sum[Abs[MoebiusMu[j]], {j,1,n}]

$[\log \frac{\zeta_{1/2}(0)}{\zeta(0)}]_n$	$\sum_{j=1}^{n} \kappa(j) - \sum_{j=1}^{\lfloor n^{\frac{1}{2}} \rfloor} \kappa(j)$	D[zetaLiouvilleR[n,0,z],z]/.z->0	Sum[MangoldtLambda[j]/Log[j] ,{j,2,n}]- Sum[MangoldtLambda[j]/Log[j] ,{j,2,n^(1/2)}]
$[\zeta(-a) \cdot \zeta(0)]_n$	$\sum_{j=1}^{n} \sigma_a(j)$	zetaDivisorR[n,0,1,a]	Sum[DivisorSigma[a,j],{j,1,n}]
$[\log \zeta(-a) \cdot \zeta(0)]_n$	$\sum_{j=1}^{n} \kappa(j)\, j^a + \sum_{j=1}^{n} \kappa(j)$	D[zetaDivisorR[n,0,z,a],z]/.z->0	Sum[MangoldtLambda[j]/Log[j] j^a,{j,2,n}] +Sum[MangoldtLambda[j]/Log[j] ,{j,2,n}]
$[\frac{\zeta(-1)}{\zeta(0)}]_n$	$\sum_{j=1}^{n} \varphi(j)$	zetaJordanR[n,0,1,1]	Sum[EulerPhi[j],{j,1,n}]
$[\log \frac{\zeta(-a)}{\zeta(0)}]_n$	$\sum_{j=1}^{n} \kappa(j)\, j^a - \sum_{j=1}^{n} \kappa(j)$	D[zetaJordanR[n,0,z,a],z]/.z->0	Sum[MangoldtLambda[j]/Log[j] j^a,{j,2,n}]- Sum[MangoldtLambda[j]/Log[j] ,{j,2,n}]
$[\prod_{k=1} \zeta_{1/k}(0)]_n$	$\sum_{j=1}^{n} a(j)$		Sum[FiniteAbelianGroupCount[j],{j,1,n}]
$[\log(\prod_{k=1} \zeta_{1/k}(0))]_n$	$\sum_{k=1}^{\lfloor \log_2 n \rfloor} \Pi(n^{\frac{1}{k}})$		
$[\log(\prod_{k=1} \zeta_{1/k}(0)^{\frac{\mu(k)}{k}})]_n$	$\pi(n)$		PrimePi[n]

Several other important functions emerge as n approaches infinity, shown here in Table 2b. Note that the bolded limits can't actually be evaluated in Mathematica as written; replace infinity with some suitable finite approximation as needs dictate. Here, $\zeta(s)$ is the Riemann Zeta function.

Table 7b

Variant of $[\zeta(s)]_n$	*Value*	*Via Roots, in Mathematica*	*Value in Mathematica*
$\lim_{n\to\infty} [\frac{\zeta_{1/2}(2s)}{\zeta(s)}]_n$, $\Re(s)>1$	$\frac{\zeta(2s)}{\zeta(s)}$	**zetaLiouvilleR[∞,s,1]**	Zeta[2 s]/ Zeta[s]
$\lim_{n\to\infty} [(\frac{\zeta_{1/2}(2s)}{\zeta(s)})^z]_n$, $\Re(s)>1$	$(\frac{\zeta(2s)}{\zeta(s)})^z$	**zetaLiouvilleR[∞,s,z]**	(Zeta[2 s]/ Zeta[s])^z
$\lim_{n\to\infty} [\log \frac{\zeta_{1/2}(2s)}{\zeta(s)}]_n$, $\Re(s)>1$	$\log \zeta(2s) - \log \zeta(s)$	**D[zetaLiouvilleR[∞,s,z],z]/.z->0**	Log[Zeta[2 s]] - Log[Zeta[s]]
$\lim_{n\to\infty} [\zeta(s-a) \cdot \zeta(s)]_n$, $\Re(s)>1$	$\zeta(s-a) \cdot \zeta(s)$	**zetaDivisorR[∞,s,1]**	Zeta[s-a]/ Zeta[s]
$\lim_{n\to\infty} [(\zeta(s-a) \cdot \zeta(s))^z]_n$, $\Re(s)>1$	$(\zeta(s-a) \cdot \zeta(s))^z$	**zetaDivisorR[∞,s,z]**	(Zeta[s-a]/ Zeta[s])^z
$\lim_{n\to\infty} [\log(\zeta(s-a) \cdot \zeta(s))]_n$, $\Re(s)>1$	$\log \zeta(s-a) + \log \zeta(s)$	**D[zetaDivisorR[∞,s,z],z]/.z->0**	Log[Zeta[s-a]] + Log[Zeta[s]]
$\lim_{n\to\infty} [\frac{\zeta(s-a)}{\zeta(s)}]_n$, $\Re(s)>1$	$\frac{\zeta(s-a)}{\zeta(s)}$	**zetaJordanR[∞,s,1]**	Zeta[s-a]/ Zeta[s]
$\lim_{n\to\infty} [(\frac{\zeta(s-a)}{\zeta(s)})^z]_n$, $\Re(s)>1$	$(\frac{\zeta(s-a)}{\zeta(s)})^z$	**Limit[zetaJordanR[∞,s,z],n->Infinity]**	(Zeta[s-a]/ Zeta[s])^z
$\lim_{n\to\infty} [\log(\frac{\zeta(s-a)}{\zeta(s)})]_n$, $\Re(s)>1$	$\log \zeta(s-a) - \log \zeta(s)$	**D[zetaJordanR[∞,s,z],z]/.z->0**	Log[Zeta[s-a]] - Log[Zeta[s]]

$[2^{-2s}(\zeta(1,\frac{1}{4})-\zeta(1,\frac{3}{4}))]_n$	$\frac{\pi}{4}$		Pi / 4
$[(2^{-2s}(\zeta(3,\frac{1}{4})-\zeta(3,\frac{3}{4})))]_n$	$\frac{\pi^3}{32}$		Pi^3 / 32

Questions

8 Generalizing this Approach with Arbitrary Series

Suppose, for some *x,* we have the identity $x=\sum_{j=0}^{\infty} f(j)$.

Identity Style 1: Via Newton's Generalized Binomial Expansion

$$[F(0)^z]_n=1+f(n,2,1) \quad \text{where} \quad f(n,j,k)=\begin{cases} f(j)(\frac{z+1}{k}-1)(1+f(\frac{n}{j},2,k+1))+f(n,j+1,k) & \text{if } n\geq j \\ 0 & \text{if } n<j \end{cases}$$

$$\boxed{[F(0)^z]_n=\binom{z}{0}1+\binom{z}{1}\sum_{j=2}^{n} f(j)+\binom{z}{2}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor} f(j)\cdot f(k)+\binom{z}{3}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor\frac{n}{j\cdot k}\rfloor} f(j)\cdot f(k)\cdot f(l)+\ldots}$$

Identity Style 2: As Exponentiation

$$\kappa_F(n)=\lim_{z\to 0}\frac{\partial}{\partial z}\nabla[F(0)^z]_n$$

$$\kappa_F(n)=f(n)-\frac{1}{2}\sum_{j\cdot k=n;\, j,k>1} f(j)\cdot f(k)+\frac{1}{3}\sum_{j\cdot k\cdot l=n;\, j,k,l>1} f(j)\cdot f(k)\cdot f(l)-\ldots$$

$$[F(0)^z]_n=1+p(n,2,1) \quad \text{where} \quad p(n,j,k)=\begin{cases} \frac{z}{k}\cdot\kappa_F(j)(1+p(\frac{n}{j},2,k+1))+p(n,j+1,k) & \text{if } n\geq j \\ 0 & \text{if } n<j \end{cases}$$

$$\boxed{[F(0)^z]_n=1+\frac{z^1}{1!}\sum_{j=2}^{n}\kappa_F(j)+\frac{z^2}{2!}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\kappa_F(j)\cdot\kappa_F(k)+\frac{z^3}{3!}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor\frac{n}{j\cdot k}\rfloor}\kappa_F(j)\cdot\kappa_F(k)\cdot\kappa_F(l)+\frac{z^4}{4!}\ldots}$$

Identity Style 3: Via The Hyperbola Method

$$\boxed{[F(0)^z]_n=[(1+F(s,2))^z]_n \quad \text{where} \quad [(1+F(s,y))^z]_n=\begin{cases} \sum_{k=0}^{\lfloor\frac{\log n}{\log y}\rfloor}\binom{z}{k}\cdot f(y)^k\cdot[(1+F(s,y+1))^{z-k}]_{n/y^k} & \text{if } n\geq y \\ 1 & \text{if } n<y \end{cases}}$$

$$\lim_{n\to\infty}[F(0)]_n=x$$

$$\lim_{n\to\infty}[F(0)^z]_n=x^z$$

$$\lim_{n\to\infty}\lim_{z\to 0}\frac{\partial}{\partial z}[F(0)^z]_n=\log x$$

Roots

For a fixed value of *n*, $[F(0)^z]_n$ will have $\lfloor \frac{\log n}{\log 2} \rfloor$ values of *z*, which we label ρ , such that $[F(0)^\rho]_n = 0$. With those roots,

$$[F(0)^z]_n = \prod_\rho (1 - \frac{z}{\rho})$$

and

$$\lim_{z \to 0} \frac{\partial}{\partial z} [F(0)^z]_n = -\sum_\rho \frac{1}{\rho} .$$

In particular, if those roots converge,

$$x = \prod_\rho (1 - \frac{1}{\rho})$$

and

$$x^z = \prod_\rho (1 - \frac{z}{\rho})$$

Example: $\sqrt{2}$

Let's look at an example of this.

Using the Taylor series for $\sqrt{1+x}$ about $x=0$, and then setting *x* to 1, we can express $\sqrt{2}$ as what amounts to $\sqrt{2} = \sum_{j=1}^{\infty} \binom{1/2}{j-1}$. Then if we define the helper function

Identity Style 1: Via Newton's Generalized Binomial Expansion

$$\boxed{[S(0)^z]_n = \binom{z}{0} 1 + \binom{z}{1} \sum_{j=2}^{n} \binom{1/2}{j-1} + \binom{z}{2} \sum_{j=2}^{n} \sum_{k=2}^{\lfloor \frac{n}{j} \rfloor} \binom{1/2}{j-1} \cdot \binom{1/2}{k-1} + \binom{z}{3} \sum_{j=2}^{n} \sum_{k=2}^{\lfloor \frac{n}{j} \rfloor} \sum_{l=2}^{\lfloor \frac{n}{j \cdot k} \rfloor} \binom{1/2}{j-1} \cdot \binom{1/2}{k-1} \cdot \binom{1/2}{l-1} + \ldots}$$

Identity Style 2: As Exponentiation

$$\kappa_S(n) = \lim_{z \to 0} \frac{\partial}{\partial z} \nabla [S(0)^z]_n$$

$$\kappa_S(n) = \binom{1/2}{n-1} - \frac{1}{2} \sum_{j \cdot k = n;\, j,k > 1} \binom{1/2}{j-1} \cdot \binom{1/2}{k-1} + \frac{1}{3} \sum_{j \cdot k \cdot l = n;\, j,k,l > 1} \binom{1/2}{j-1} \cdot \binom{1/2}{k-1} \cdot \binom{1/2}{l-1} - \ldots$$

$$\boxed{[S(0)^z]_n = 1 + \frac{z^1}{1!} \sum_{j=2}^{n} \kappa_S(j) + \frac{z^2}{2!} \sum_{j=2}^{n} \sum_{k=2}^{\lfloor \frac{n}{j} \rfloor} \kappa_S(j) \cdot \kappa_S(k) + \frac{z^3}{3!} \sum_{j=2}^{n} \sum_{k=2}^{\lfloor \frac{n}{j} \rfloor} \sum_{l=2}^{\lfloor \frac{n}{j \cdot k} \rfloor} \kappa_S(j) \cdot \kappa_S(k) \cdot \kappa_S(l) + \frac{z^4}{4!} \ldots}$$

Identity Style 3: Via The Hyperbola Method

$$[S(0)^z]_n=[(1+S(0,2))^z]_n \quad \text{where} \quad [(1+S(0,y))^z]_n=\begin{cases}\displaystyle\sum_{k=0}^{\lfloor\frac{\log n}{\log y}\rfloor}\binom{z}{k}\cdot\left(\frac{1/2}{y-1}\right)^k\cdot[(1+S(0,y+1))^{z-k}]_{n/y^k} & \text{if } n\geq y\\ 1 & \text{if } n<y\end{cases}$$

The Mathematica Code

We can implement $S_z(n,s)$ in Mathematica, here named `DBnsz[n_, s_,z_]`, like so:

```
binomial[z_,k_]:=binomial[z,k]=Product[z-j,{j,0,k-1}]/k!
WheelEntries=7;
WheelSize:=WheelSize=Product[Prime[j],{j,1,WheelEntries}];
CoprimeCache:=CoprimeCache=Table[If[CoprimeQ[WheelSiz
```

Table 8

Variant of $D_z(n,s)$	*Value*	*Via Roots, in Mathematica*	*Value in Mathematica*
$S_1(n)$	$\sqrt{2}$	**SnzR[∞,1]**	2^(1/2)
$S_z(n)$	$(\sqrt{2})^z$	**SnzR[∞,z]**	2^(z/2)

9 Additive Variation

Suppose

Identity Style 1: Via Newton's Generalized Binomial Expansion

$$[[(1+F(0))^z]]_n = 1+f(n,1,1) \quad \text{where} \quad f(n,j,k)=\begin{cases} f(j)(\frac{z+1}{k}-1)(1+f(n-j,1,k+1))+f(n,j+1,k) & \text{if } n \geq j \\ 0 & \text{if } n < j \end{cases}$$

$$\boxed{[[(1+F(0))^z]]_n = \binom{z}{0}1+\binom{z}{1}\sum_{j=1}^{n} f(j)+\binom{z}{2}\sum_{j=1}^{n}\sum_{k=1}^{n-j} f(j)\cdot f(k)+\binom{z}{3}\sum_{j=1}^{n}\sum_{k=1}^{n-j}\sum_{l=1}^{n-j-k} f(j)\cdot f(k)\cdot f(l)+\ldots}$$

$$\nabla[[(1+F(0))^z]]_n = \binom{z}{1}f(n)+\binom{z}{2}\sum_{j+k=n} f(j)\cdot f(k)+\binom{z}{3}\sum_{j+k+l=n} f(j)\cdot f(k)\cdot f(l)+\ldots$$

Identity Style 2: As Exponentiation

$$\kappa_+(n)=\lim_{z\to 0}\frac{\partial}{\partial z}\nabla[[(1+F(0))^z]]_n$$

$$[[(1+F(0))^z]]_n = 1+p(n,1,1) \quad \text{where} \quad p(n,j,k)=\begin{cases} \frac{z}{k}\cdot\kappa_+(j)(1+p(n-j,1,k+1))+p(n,j+1,k) & \text{if } n \geq j \\ 0 & \text{if } n < j \end{cases}$$

$$\boxed{[[(1+F(0))^z]]_n = 1+\frac{z^1}{1!}\sum_{j=1}^{n}\kappa_+(j)+\frac{z^2}{2!}\sum_{j=1}^{n}\sum_{k=1}^{n-j}\kappa_+(j)\cdot\kappa_+(k)+\frac{z^3}{3!}\sum_{j=1}^{n}\sum_{k=1}^{n-j}\sum_{l=1}^{n-j-k}\kappa_+(j)\cdot\kappa_+(k)\cdot\kappa_+(l)+\frac{z^4}{4!}\ldots}$$

$$\nabla[[(1+F(0))^z]]_n = \frac{z^1}{1!}\kappa_+(n)+\frac{z^2}{2!}\sum_{j+k=n}\kappa_+(j)\cdot\kappa_+(k)+\frac{z^3}{3!}\sum_{j+k+l=n}\kappa_+(j)\cdot\kappa_+(k)\cdot\kappa_+(l)+\frac{z^4}{4!}\ldots$$

Identity Style 3: Via The Hyperbola Method

$$\boxed{[[(1+F(0))^z]]_n = [[(1+F(s,1))^z]]_n \quad \text{where} \quad [[(1+F(s,y))^z]]_n=\begin{cases} \sum_{k=0}^{\lfloor \frac{n}{y} \rfloor}\binom{z}{k}\cdot f(y)^k\cdot[[(1+F(s,y+1))^{z-k}]]_{n-y\cdot k} & \text{if } n \geq y \\ 1 & \text{if } n < y \end{cases}}$$

Roots

For a fixed value of n and s, $[[(1+F(0))^z]]_n$ will have n values of z, which we label ρ, such that $[[(1+F(0))^\rho]]_n=0$. With those roots,

$$[[(1+F(0))^z]]_n=\prod_\rho (1-\frac{z}{\rho})$$

and

$$\lim_{z\to 0}\frac{\partial}{\partial z}[[(1+F(0))^z]]_n=-\sum_\rho \frac{1}{\rho}.$$

Additionally, for a fixed value of n and s, $\nabla[[(1+F(0))^z]]_n$ will also have n values of z, which we label ρ, such that $\nabla[[(1+F(0))^\rho]]_n=0$. With those roots,

$$\nabla[[(1+F(0))^z]]_n=\prod_\rho (1-\frac{z}{\rho})$$

and

$$\lim_{z\to 0}\frac{\partial}{\partial z}\nabla[[(1+F(0))^z]]_n=-\sum_\rho \frac{1}{\rho}$$

Example: The Partition Function P(n)

Let's look at a few examples of this.

Identity Style 1: Via Newton's Generalized Binomial Expansion

$$[[(1+P(0))^z]]_n=1+f(n,1,1) \quad \text{where} \quad f(n,j,k)=\begin{cases} P(j)(\frac{z+1}{k}-1)(1+f(n-j,1,k+1))+f(n,j+1,k) & \text{if } n\geq j \\ 0 & \text{if } n<j \end{cases}$$

$$\boxed{[[(1+P(0))^z]]_n=\binom{z}{0}1+\binom{z}{1}\sum_{j=1}^{n}P(j)+\binom{z}{2}\sum_{j=1}^{n}\sum_{k=1}^{n-j}P(j)\cdot P(k)+\binom{z}{3}\sum_{j=1}^{n}\sum_{k=1}^{n-j}\sum_{l=1}^{n-j-k}P(j)\cdot P(k)\cdot P(l)+\dots}$$

$$\nabla[[(1+P(0))^z]]_n=\binom{z}{1}P(n)+\binom{z}{2}\sum_{j+k=n}P(j)\cdot P(k)+\binom{z}{3}\sum_{j+k+l=n}P(j)\cdot P(k)\cdot P(l)+\dots$$

Identity Style 2: As Exponentiation

$$\kappa_P(n)=\frac{\sigma(n)}{n}$$

$$[[(1+P(0))^z]]_n=1+p(n,1,1) \quad \text{where} \quad p(n,j,k)=\begin{cases} \frac{z}{k}\cdot\kappa_P(j)(1+p(n-j,1,k+1))+p(n,j+1,k) & \text{if } n\geq j \\ 0 & \text{if } n<j \end{cases}$$

$$\boxed{[[(1+P(0))^z]]_n=1+\frac{z^1}{1!}\sum_{j=1}^{n}\kappa_P(j)+\frac{z^2}{2!}\sum_{j=1}^{n}\sum_{k=1}^{n-j}\kappa_P(j)\cdot\kappa_P(k)+\frac{z^3}{3!}\sum_{j=1}^{n}\sum_{k=1}^{n-j}\sum_{l=1}^{n-j-k}\kappa_P(j)\cdot\kappa_P(k)\cdot\kappa_P(l)+\frac{z^4}{4!}\dots}$$

$$\nabla[[(1+P(0))^z]]_n = \frac{z^1}{1!}\kappa_P(n) + \frac{z^2}{2!}\sum_{j+k=n}\kappa_P(j)\cdot\kappa_P(k) + \frac{z^3}{3!}\sum_{j+k+l=n}\kappa_P(j)\cdot\kappa_P(k)\cdot\kappa_P(l) + \frac{z^4}{4!}\ldots$$

Identity Style 3: Via The Hyperbola Method

$$[[(1+P(0))^z]]_n = [[(1+P(s,1))^z]]_n \quad \text{where} \quad [[(1+P(s,y))^z]]_n = \begin{cases} \sum_{k=0}^{\lfloor \frac{n}{y} \rfloor} \binom{z}{k} \cdot P(y)^k \cdot [[(1+P(s,y+1))^{z-k}]]_{n-y\cdot k} & \text{if } n \geq y \\ 1 & \text{if } n < y \end{cases}$$

The Mathematica Code

We can implement $D_{z,B_k}(n,s)$ in Mathematica, here named `DBnsz[n_, s_,z_]`, like so:

```
binomial[z_,k_]:=binomial[z,k]=Product[z-j,{j,0,k-1}]/k!
WheelEntries=7;
WheelSize:=WheelSize=Product[Prime[j],{j,1,WheelEntries}];
CoprimeCache:=CoprimeCache=Table[If[CoprimeQ[WheelSiz
```

Table 8

Variant of $D_z(n,s)$	*Value*	*Via Roots, in Mathematica*	*Value in Mathematica*
	$1+P(n)$	**PnzR[∞,1]**	Pi / 4
	$1+P(n)$	**PnzR[∞,z]**	(Pi/4)^z
	$1+P(n)$	**SnzR[∞,1]**	2^(1/2)
	$1+P(n)$	**SnzR[∞,z]**	2^(z/2)

10 Conclusion

Some examples of the function include

Suppose we define the function $f_z(n,s)$ as

$$f_z(n,s) = 1 + \binom{z}{1}\sum_{j=2}^{n}\frac{(-1)^{j+1}}{j^{s}}+ \binom{z}{2}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor \frac{n}{j}\rfloor}\frac{(-1)^{j+1}}{j^{s}} \cdot \frac{(-1)^{k+1}}{k^{s}}$$
$$ +\binom{z}{3}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor \frac{n}{j}\rfloor}\sum_{k=2}^{\lfloor \frac{n}{j \cdot k}\rfloor}\frac{(-1)^{j+1}}{j^{s}} \cdot \frac{(-1)^{k+1}}{k^{s}} \cdot \frac{(-1)^{l+1}}{l^{s}}+...$$

Significantly, if $\eta(s)$ is the Dirichlet eta function, then, when $\Re(s)>0$, $$\displaystyle\lim_{n \rightarrow \infty} f_z(n,s) = \sum_{k=0}^{\infty}{\binom{z}{k}(\eta(s)-1)^k} = \eta(s)^z$$

Now, for any fixed values of n and s, $f_z(n,s)$ will be a polynomial of degree $\lfloor \frac{\log n}{\log 2} \rfloor $, and there will be $\lfloor \frac{\log n}{\log 2} \rfloor $ values of z, which we will label as ρ, such that $f_\rho(n,s)=0$. Further, we can express $f_z(n,s)$ over its zeros in the usual way as

$$f_z(n,s) = \prod_{\rho}(1-\frac{z}{\rho})$$

There is a bit of Mathematica code at the bottom of the post demonstrating this.

Now, here's my question(s): as n approaches infinity, and for a fixed value of s where $\Re(s) > 0$, what happens to these zeros I've named ρ? Do they converge? Can $\eta(s)$ be expressed with these (convergent) zeros as $\eta(s) = \prod_\rho (1-\frac{1}{\rho})$?

$f_\rho(100000000,0.5 + 14.1347 i) = 0 $

when
$\rho = $

$$ 1.+0.00000360842 i,
23.4432 + 115.492 i,
-3.00749 + 7.24046 i,
-0.00399172 - 0.0024432 i,
1.59804 + 38.0367 i,
1.61973 - 1.50287 i,
2.00164 - 0.0052567 i,
2.70904 + 0.134875 i,
3.0931 - 8.44949 i,
3.1986 - 1.21641 i,
3.56358 + 1.39473 i,
4.17529 - 3.86039 i,
4.4097 + 16.3853 i,
4.60627 + 3.92814 i,
5.18591 + 0.00864528 i,
9.63811 - 0.0407828 i,
12.277 - 154.547 i,
12.4618 - 21.1794 i,
21.8501 + 3.95519 i,
24.4498 - 2.96747 i,
32.8254 - 0.284303 i,
66.7752 - 26.1226 i,
68.0093 + 67.8245 i,

74.2416 - 5866. i,
133.036 - 68.5184 i,
819.951 - 843.178 i $$

```
f[n_, s_, z_, k_] := f[n, s, z, k] = 1 + ((z + 1)/k - 1) Sum[ (-1)^(j + 1) j^-s f[Floor[n/j], s, z, k + 1], {j, 2, n}]
f[n_, s_, z_] := Expand[f[n, s, z, 1]]
fzeros[n_, s_] := List @@ Roots[f[n, s, z] == 0, z][[All, 2]]
ffromzeros[n_, s_, z_] := Expand[FullSimplify[Product[1 - z/rho, {rho, fzeros[n, s]}]]]
```

11 References