```
ClearAll["Global`*"]
```

```
D2[n_, k_] := D2[n, k] = If[n < 1, 0, Sum[D2[Floor[n / j], k - 1], {j, 2, n}]];
D2[n_, 0] := If[n < 1, 0, 1]
DD2[n_, 0] := If[n ≥ 1, 1, 0]
D22[n_, a_] := If[n < 2^a, 0, DD2[n, a]]
D1[n_, z_] := D1[n, z] = Sum[FactorialPower[z, a] / a! D22[n, a], {a, 0, Log[2, n]}];
D1[0, z_] := 0
E1[n_, a_, b_] := E1[n, a, b] = Sum[(-1)^j Binomial[a, j] b^j D1[Floor[n / b^j], a], {j, 0, a}]
E2[n_, a_, b_] := E2[n, a, b] = Sum[(-1)^(a - j) Binomial[a, j] E1[n, j, b], {j, 0, a}]
D2E2[n_, k_, b_] :=
 Sum[ (-1)^j b^j Binomial[k, j] Sum[ Binomial[j, m] D22[n / b^j, k - m], {m, 0, j}], {j, 0, k}]
lin[n_, b_] := Sum[(-1)^(k + 1) / k E2[n, k, b], {k, 1, Log[2, n]}]
lind[n_, b_] := Sum[(-1)^(k + 1) / k D22[n, k], {k, 1, Log[2, n]}]
lind2[n_, b_] := Sum[(-1)^(k + 1) / k D2E2[n, k, b], {k, 1, Log[2, n]}]
lind2k[n_, b_, k_] := (-1)^(k + 1) / k D2E2[n, k, b]
lindk[n_, b_, k_] := (-1)^(k + 1) / k D22[n, k]
linAdd[n_, b_] := Sum[ b^k / k, {k, 1, Log[b, n]}]
linref[n_] := Sum[ FullSimplify[MangoldtLambda[j] / Log[j]], {j, 2, n}]
ldif[n_, b_, k_] := FullSimplify[Expand[lind2k[n, b, k]] - lindk[n, b, k]]
```

```
lin[1000, 6 / 3] + linAdd[1000, 6 / 3]
```

$$\frac{445\,273}{2520}$$

```
lind2[100, -6 / 3] - (lin[100, 6 / 3] + linAdd[100, 6 / 3])
```

$$-\frac{50}{3}$$

```
Expand[E2[10 000, 3, 2]]
```

```
-8 D22[1250, 0] - 24 D22[1250, 1] - 24 D22[1250, 2] - 8 D22[1250, 3] + 12 D22[2500, 1] +
 24 D22[2500, 2] + 12 D22[2500, 3] - 6 D22[5000, 2] - 6 D22[5000, 3] + D22[10 000, 3]
```

```
Table[ { n,  Expand[E2[10 000, n, 2]]}, {n, 1, 7}] // TableForm
```

```
1    -2 D22[5000, 0] - 2 D22[5000, 1] + D22[10 000, 1]
2    4 D22[2500, 0] + 8 D22[2500, 1] + 4 D22[2500, 2] - 4 D22[5000, 1] - 4 D22[5000, 2] + D22[10 000, 2
3    -8 D22[1250, 0] - 24 D22[1250, 1] - 24 D22[1250, 2] - 8 D22[1250, 3] + 12 D22[2500, 1] + 24 D22[2
4    16 D22[625, 0] + 64 D22[625, 1] + 96 D22[625, 2] + 64 D22[625, 3] + 16 D22[625, 4] - 32 D22[1250,
5    -32 D22[312, 0] - 160 D22[312, 1] - 320 D22[312, 2] - 320 D22[312, 3] - 160 D22[312, 4] - 32 D22[
6    64 D22[156, 0] + 384 D22[156, 1] + 960 D22[156, 2] + 1280 D22[156, 3] + 960 D22[156, 4] + 384 D22
7    -128 D22[78, 0] - 896 D22[78, 1] - 2688 D22[78, 2] - 4480 D22[78, 3] - 4480 D22[78, 4] - 2688 D22
```

```
D2E2[n_, k_, b_] :=
 Sum[ (-1)^j b^j Binomial[k, j] Sum[ Binomial[j, m] D2[n / b^j, k - m], {m, 0, j}], {j, 0, k}]
```

```
D2E2[10, 3, 3]
```

```
-26
```

```
E2[10, 3, 3]
```

```
1
```

```
DD2E2[n_, k_, b_] :=
 Sum[ (-1)^j b^j Binomial[k, j] Sum[ Binomial[j, m] DD2[n / b^j, k - m], {m, 0, j}], {j, 0, k}]
```

```
Expand[DD2E2[10 000, 1, -2]]
```

```
2 DD2[-5000, 0] + 2 DD2[-5000, 1] + DD2[10 000, 1]
```

```
Expand[DD2E2[10 000, 1, 2]]
```

```
-2 DD2[5000, 0] - 2 DD2[5000, 1] + DD2[10 000, 1]
```

```
Expand[E2[10, 3, 3]]
```

```
1
```

```
Table[ {n, D2E2[10, n, 3] - E2[10, n, 3]}, {n, 1, 20}] // TableForm
```

```
1     0
2     0
3     0
4     0
5     0
6     0
7     0
8     0
9     0
10    0
11    0
12    0
13    0
14    0
15    0
16    0
17    0
18    0
19    0
20    0
```

```
linAdd2[n_, b_] := (1 / 2) Sum[ (b^2)^k / k, {k, 1, Log[b^2, n]}]
```

```
Table[ {n, lin[n, -2] + linAdd2[n, 2] - (lin[n, 2] + linAdd[n, 2])}, {n, 2, 100}] // TableForm
```

```
2      0
3      0
4      0
5      0
6      0
7      0
8      0
9      0
10     0
11     0
```

```
12    0
13    0
14    0
15    0
16    0
17    0
18    0
19    0
20    0
21    0
22    0
23    0
24    0
25    0
26    0
27    0
28    0
29    0
30    0
31    0
32    0
33    0
34    0
35    0
36    0
37    0
38    0
39    0
40    0
41    0
42    0
43    0
44    0
45    0
46    0
47    0
48    0
49    0
50    0
51    0
52    0
53    0
54    0
55    0
56    0
57    0
58    0
59    0
60    0
61    0
62    0
63    0
64    0
65    0
66    0
67    0
```

68	0
69	0
70	0
71	0
72	0
73	0
74	0
75	0
76	0
77	0
78	0
79	0
80	0
81	0
82	0
83	0
84	0
85	0
86	0
87	0
88	0
89	0
90	0
91	0
92	0
93	0
94	0
95	0
96	0
97	0
98	0
99	0
100	0

4 / 2 + 16 / 4 + 64 / 6

$$\frac{50}{3}$$

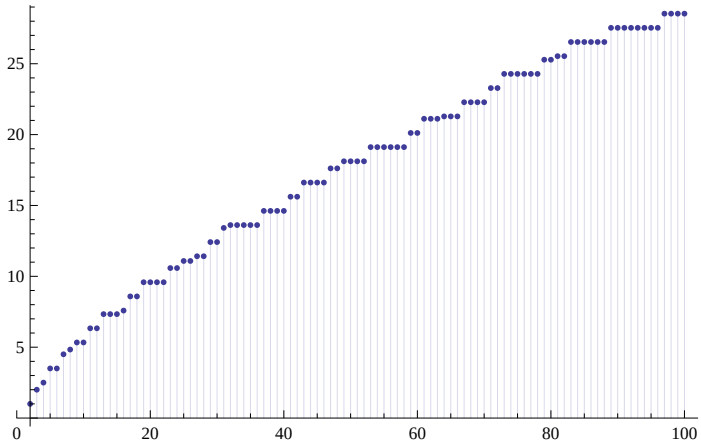
2 / 1 + 4 / 2 + 8 / 3

$$\frac{20}{3}$$

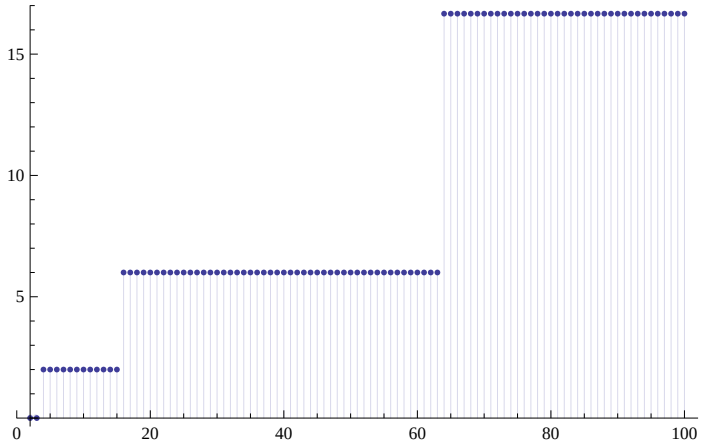
4 / 1 + 16 / 2 + 64 / 3

$$\frac{100}{3}$$

```
DiscretePlot[lind2[n, -8 / 3] + linAdd2[n, 8 / 3], {n, 2, 100}]
```




```
DiscretePlot[linAdd2[n, 2], {n, 2, 100}]
```




```
2^3^4
```

2 417 851 639 229 258 349 412 352

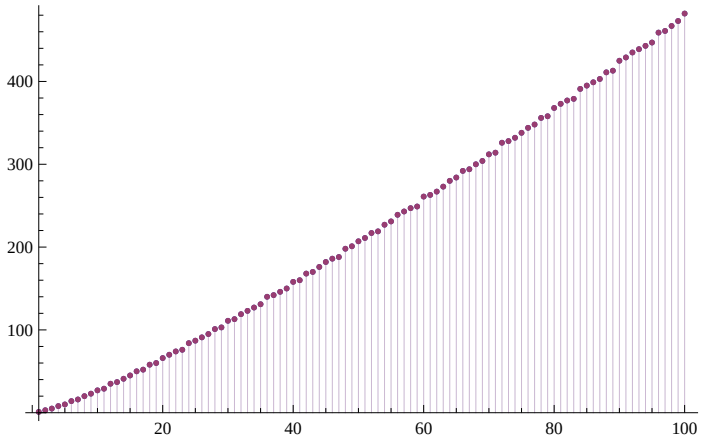
```
2^12
```

4096

```
(2^3)^4
```

4096

```
DiscretePlot[{E1[n, 2, -19], D1[n, 2]}, {n, 1, 100}]
```




```
Table[{E1[n, 2, -19], D1[n, 2]}, {n, 1, 100}] // TableForm
```

1	1
3	3
5	5
8	8
10	10
14	14
16	16
20	20
23	23
27	27
29	29
35	35
37	37
41	41
45	45
50	50
52	52
58	58
60	60
66	66
70	70
74	74
76	76
84	84
87	87
91	91
95	95
101	101
103	103
111	111
113	113
119	119
123	123
127	127
131	131
140	140
142	142
146	146

```
150   150
158   158
160   160
168   168
170   170
176   176
182   182
186   186
188   188
198   198
201   201
207   207
211   211
217   217
219   219
227   227
231   231
239   239
243   243
247   247
249   249
261   261
263   263
267   267
273   273
280   280
284   284
292   292
294   294
300   300
304   304
312   312
314   314
326   326
328   328
332   332
338   338
344   344
348   348
356   356
358   358
368   368
373   373
377   377
379   379
391   391
395   395
399   399
403   403
411   411
413   413
425   425
429   429
435   435
439   439
443   443
```

```
447    447
459    459
461    461
467    467
473    473
482    482
```

```
Table[{n, Expand[E2[10 000, n, 2]]}, {n, 1, 7}] // TableForm
```

```
1    -1
2    5
3    -6
4    411
5    1139
6    -794
7    -1253
```

```
lin[10 000, 2] + linAdd[10 000, 2]
```

$$\frac{56\,175\,529}{45\,045}$$

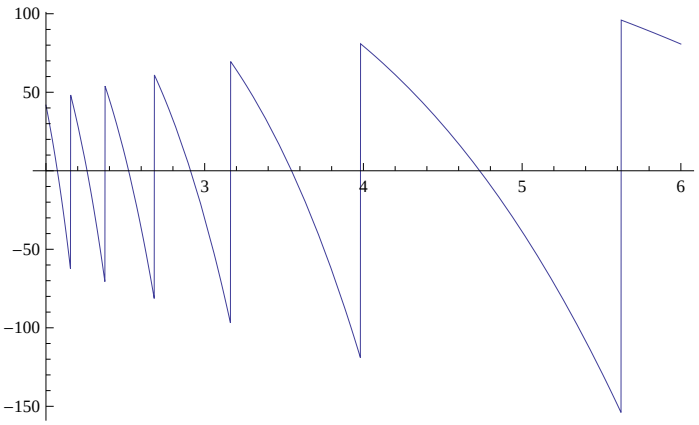
$$\mathbf{\frac{56\,175\,529}{45\,045}}$$

$$\frac{56\,175\,529}{45\,045}$$

```
linref[100 000]
```

$$\frac{991\,892\,879}{102\,960}$$

```
Plot[lin[1000, n], {n, 2, 6}]
```



lind2[100, 2]

$$-2\,(D22[50,0]+D22[50,1])+D22[100,1]+\frac{1}{2}\,(-4\,(D22[25,0]+2\,D22[25,1]+D22[25,2])+4\,(D22[50,1]+D22[50,2])-D22[100,2])+\frac{1}{3}\left(-8\left(D22\left[\frac{25}{2},0\right]+3\,D22\left[\frac{25}{2},1\right]+3\,D22\left[\frac{25}{2},2\right]+D22\left[\frac{25}{2},3\right]\right)+12\,(D22[25,1]+2\,D22[25,2]+D22[25,3])-6\,(D22[50,2]+D22[50,3])+D22[100,3]\right)+\frac{1}{4}\left(-16\left(D22\left[\frac{25}{4},0\right]+4\,D22\left[\frac{25}{4},1\right]+6\,D22\left[\frac{25}{4},2\right]+4\,D22\left[\frac{25}{4},3\right]+D22\left[\frac{25}{4},4\right]\right)+32\left(D22\left[\frac{25}{2},1\right]+3\,D22\left[\frac{25}{2},2\right]+3\,D22\left[\frac{25}{2},3\right]+D22\left[\frac{25}{2},4\right]\right)-24\,(D22[25,2]+2\,D22[25,3]+D22[25,4])+8\,(D22[50,3]+D22[50,4])-D22[100,4]\right)+\frac{1}{5}\left(-32\left(D22\left[\frac{25}{8},0\right]+5\,D22\left[\frac{25}{8},1\right]+10\,D22\left[\frac{25}{8},2\right]+10\,D22\left[\frac{25}{8},3\right]+5\,D22\left[\frac{25}{8},4\right]+D22\left[\frac{25}{8},5\right]\right)+80\left(D22\left[\frac{25}{4},1\right]+4\,D22\left[\frac{25}{4},2\right]+6\,D22\left[\frac{25}{4},3\right]+4\,D22\left[\frac{25}{4},4\right]+D22\left[\frac{25}{4},5\right]\right)-80\left(D22\left[\frac{25}{2},2\right]+3\,D22\left[\frac{25}{2},3\right]+3\,D22\left[\frac{25}{2},4\right]+D22\left[\frac{25}{2},5\right]\right)+40\,(D22[25,3]+2\,D22[25,4]+D22[25,5])-10\,(D22[50,4]+D22[50,5])+D22[100,5]\right)+\frac{1}{6}\left(-64\left(D22\left[\frac{25}{16},0\right]+6\,D22\left[\frac{25}{16},1\right]+15\,D22\left[\frac{25}{16},2\right]+20\,D22\left[\frac{25}{16},3\right]+15\,D22\left[\frac{25}{16},4\right]+6\,D22\left[\frac{25}{16},5\right]+D22\left[\frac{25}{16},6\right]\right)+192\left(D22\left[\frac{25}{8},1\right]+5\,D22\left[\frac{25}{8},2\right]+10\,D22\left[\frac{25}{8},3\right]+10\,D22\left[\frac{25}{8},4\right]+5\,D22\left[\frac{25}{8},5\right]+D22\left[\frac{25}{8},6\right]\right)-240\left(D22\left[\frac{25}{4},2\right]+4\,D22\left[\frac{25}{4},3\right]+6\,D22\left[\frac{25}{4},4\right]+4\,D22\left[\frac{25}{4},5\right]+D22\left[\frac{25}{4},6\right]\right)+160\left(D22\left[\frac{25}{2},3\right]+3\,D22\left[\frac{25}{2},4\right]+3\,D22\left[\frac{25}{2},5\right]+D22\left[\frac{25}{2},6\right]\right)-60\,(D22[25,4]+2\,D22[25,5]+D22[25,6])+12\,(D22[50,5]+D22[50,6])-D22[100,6]\right)$$

lind[100, 2]

$$DD2[100,1]-\frac{1}{2}\,DD2[100,2]+\frac{1}{3}\,DD2[100,3]-\frac{1}{4}\,DD2[100,4]+\frac{1}{5}\,DD2[100,5]-\frac{1}{6}\,DD2[100,6]$$

FullSimplify[Expand[Sum[ldif[100, 2, k], {k, 1, 10}]]]

$$-\frac{416}{15}$$

```
Table[{k, ldif[1000, 3, k]}, {k, 1, 16}] // TableForm
```

1	-999
2	2526
3	-3768
4	$\frac{14\,205}{4}$
5	$-\frac{11\,238}{5}$
6	1158
7	-525
8	$\frac{201}{2}$
9	-3
10	0
11	0
12	0
13	0
14	0
15	0
16	0

```
Sum[ ldif[1000, 3, k], {k, 1, 16}]
```

$$-\frac{4137}{20}$$

```
Sum[ 3^k / k, {k, 1, Log[3, 1000]}]
```

$$\frac{4137}{20}$$