```
binomial[z_, k_] := binomial[z, k] = Product[z - j, {j, 0, k - 1}] / k!
Dnka[n_, 0, a_] := UnitStep[n - 1]
Dnka[n_, 1, a_] := Dnka[n, 1, a] = Floor[n] - a
Dnka[n_, 2, a_] :=
 Dnka[n, 2, a] = Sum[1 + 2 (Dnka[Floor[n / m], 1, m]), {m, a + 1, Floor[n^(1 / 2)]}]
Dnka[n_, k_, a_] := Dnka[n, k, a] =
  Sum[1 + k Dnka[Floor[n / (m^(k - 1))], 1, m] + Sum[binomial[k, j]
      Dnka[Floor[n / (m^j)], k - j, m], {j, 1, k - 2}], {m, a + 1, Floor[n^(1 / k)]}]

Dnz[n_, z_] := Expand@Sum[binomial[z, k] Dnka[n, k, 1], {k, 0, Log2@n}]
dz[n_, z_] := dz[n, z] = Product[(-1)^p[[2]] bin[-z, p[[2]]], {p, FI[n]}]
dnz[n_, z_] := Dnz[n, z] - Dnz[n - 1, z]
DnzRoots[n_] := If[(c = Exponent[f = Dnz[n, z], z]) == 0, {},
  If[c == 1, List@NRoots[f == 0, z][[2]], List @@ NRoots[f == 0, z][[All, 2]]]]
dnzRoots[n_] := If[(c = Exponent[f = dz[n, z], z]) == 0, {},
  If[c == 1, List@NRoots[f == 0, z][[2]], List @@ NRoots[f == 0, z][[All, 2]]]]
DnzR[n_, z_] := Chop@Expand@Product[1 - z / rho, {rho, DnzRoots[n]}]


DnzRoots[100 000 000]
```

$$\{-7494.07, -1440.44, -281.526, -37.2362 - 148.713\,i, -37.2362 + 148.713\,i, -35.5639,$$
$$-13.9034 - 54.5839\,i, -13.9034 + 54.5839\,i, -13.5241 - 33.6412\,i, -13.5241 + 33.6412\,i,$$
$$-11.5772 - 12.0093\,i, -11.5772 + 12.0093\,i, -6.16326 - 16.2264\,i, -6.16326 + 16.2264\,i,$$
$$-4.59605 - 8.12566\,i, -4.59605 + 8.12566\,i, -4.13439 - 4.72012\,i, -4.13439 + 4.72012\,i,$$
$$-3.44066 - 2.91838\,i, -3.44066 + 2.91838\,i, -2.68701 - 1.52872\,i, -2.68701 + 1.52872\,i,$$
$$-1.94007 - 0.419915\,i, -1.94007 + 0.419915\,i, -0.99477, -1.73548 \times 10^{-7}\}$$

```
a2 := {-7494.069914345595`, -1440.4409044583288`, -281.5257697989761`,
  -37.23624188545817` - 148.7128809232404` i, -37.23624188545817` + 148.7128809232404` i,
  -35.563903002932584`, -13.903356146346544` - 54.58386377231285` i,
  -13.903356146346544` + 54.58386377231285` i, -13.524093734436974` - 33.6412254304076` i,
  -13.524093734436974` + 33.6412254304076` i, -11.577243234325424` - 12.009344141541174` i,
  -11.577243234325424` + 12.009344141541174` i, -6.1632575936044525` - 16.22644747537662` i,
  -6.1632575936044525` + 16.22644747537662` i, -4.596047565793488` - 8.125664873762274` i,
  -4.596047565793488` + 8.125664873762274` i, -4.13439045189022` - 4.720122122665344` i,
  -4.13439045189022` + 4.720122122665344` i, -3.4406566978797453` - 2.918382537307335` i,
  -3.4406566978797453` + 2.918382537307335` i, -2.6870149652152633` - 1.5287204893576303` i,
  -2.6870149652152633` + 1.5287204893576303` i, -1.9400667098251676` - 0.419915062049149` i,
  -1.9400667098251676` + 0.419915062049149` i, -0.9947702510510221`}

Product[1 - 1 / p, {p, a2}]
```

$$17.3548 + 3.33067 \times 10^{-16}\,i$$

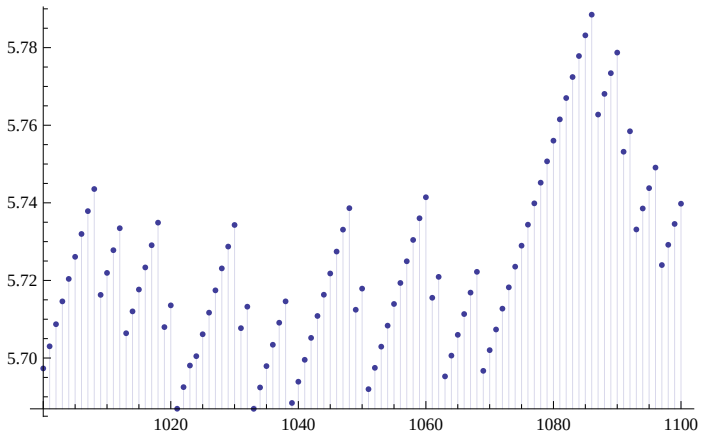
```
DnzRoots[1000]
```

$$\{-145.722, -8.80186 - 14.3448\,i, -8.80186 + 14.3448\,i, -4.45483 - 3.16845\,i,$$
$$-4.45483 + 3.16845\,i, -2.04875 - 1.06859\,i, -2.04875 + 1.06859\,i, -0.961602, -0.00572997\}$$

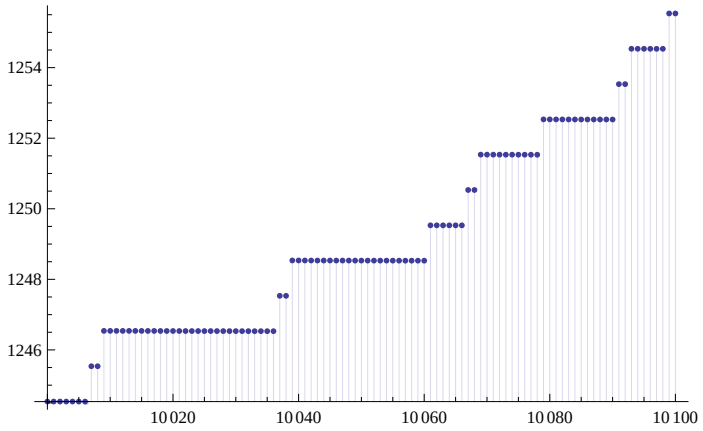
```
Table[Chop@Product[If[Abs[p] < .3, 1, 1 - 1 / p], {p, DnzRoots[10^k]}], {k, 1, 8}] // TableForm
```

```
1.79323
3.58701
5.69732
8.02866
10.3821
12.7234
15.0416
17.3548
```

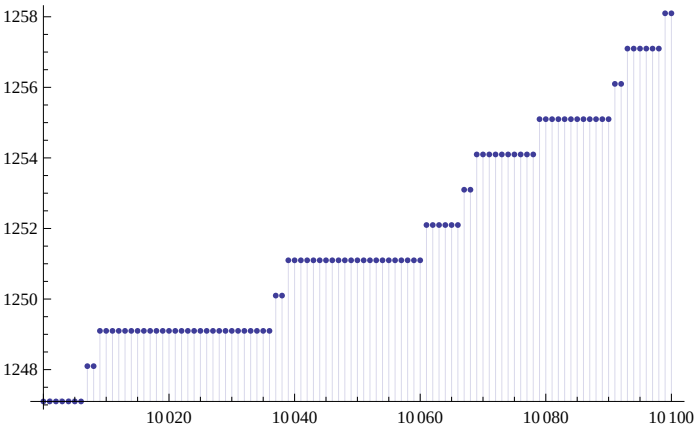
```
DiscretePlot[Chop@Product[If[Abs[p] < .3, 1, 1 - 1 / p], {p, DnzRoots[k]}], {k, 1000, 1100}]
```




```
DiscretePlot[Product[If[Abs[p] > .3, 1, -1 / p], {p, DnzRoots[k]}], {k, 10 000, 10 100}]
```



```
DiscretePlot[Sum[-1 / p, {p, DnzRoots[k]}], {k, 10 000, 10 100}]
```




```
binomial[z_, k_] := binomial[z, k] = Product[z - j, {j, 0, k - 1}] / k!
Ds[n_, 0, s_, a_] := UnitStep[n - 1]
Ds[n_, 1, s_, a_] := Ds[n, 1, s, a] = HarmonicNumber[Floor[n], s] - HarmonicNumber[a, s]
Ds[n_, 2, s_, a_] := Ds[n, 2, s, a] =
  Sum[(m^(-2 s)) + 2 (m^-s) (Ds[Floor[n / m], 1, s, m]), {m, a + 1, Floor[n^(1 / 2)]}]
Ds[n_, k_, s_, a_] := Ds[n, k, s, a] =
  Sum[(m^(-s k)) + k (m^(-s (k - 1))) Ds[Floor[n / (m^(k - 1))], 1, s, m] +
    Sum[binomial[k, j] (m^-s)^j Ds[Floor[n / (m^j)], k - j, s, m], {j, 1, k - 2}],
   {m, a + 1, Floor[n^(1 / k)]}]
Dnsz[n_, s_, z_] := Expand@Sum[binomial[z, k] Ds[n, k, s, 1], {k, 0, Log2@n}]
Dnsza[n_, s_, z_] := Expand@Sum[j^-s dz[j, z], {j, 1, n}]

Dns112z[n_, s_, z_] :=
 Expand@Sum[(-1)^j binomial[z, j] 2^(j (1 - s)) Dnsz[n / (2^j), s, z], {j, 0, Log2@n}]
Dns112za[n_, s_, z_] :=
 Expand@Sum[(-1)^j binomial[z, j] 2^(j (1 - s)) Dnsza[n / (2^j), s, z], {j, 0, Log2@n}]

Dns112zZeros[n_, s_] := If[(c = Exponent[f = Dns112z[n, s, z], z]) == 0, {},
  If[c == 1, List@NRoots[f == 0, z][[2]], List @@ NRoots[f == 0, z][[All, 2]]]]
Dns112zR[n_, s_, z_] := Chop@Expand@Product[1 - z / rho, {rho, Dns112zZeros[n, s]}]
```

```
Dns112zZeros[100 000 000, N@ZetaZero[1]]
```

$$\{-23.4433 + 115.492\,i,\ -3.00749 + 7.24046\,i,\ -0.00399172 - 0.0024432\,i,$$
$$1. + 3.6082 \times 10^{-6}\,i,\ 1.59804 + 38.0367\,i,\ 1.61973 - 1.50287\,i,\ 2.00164 - 0.0052567\,i,$$
$$2.70904 + 0.134875\,i,\ 3.0931 - 8.44949\,i,\ 3.1986 - 1.21641\,i,\ 3.56358 + 1.39473\,i,$$
$$4.17529 - 3.86039\,i,\ 4.4097 + 16.3853\,i,\ 4.60627 + 3.92814\,i,\ 5.18591 + 0.00864523\,i,$$
$$9.63811 - 0.0407831\,i,\ 12.2774 - 154.546\,i,\ 12.4618 - 21.1794\,i,$$
$$21.8501 + 3.95519\,i,\ 24.4498 - 2.96747\,i,\ 32.8255 - 0.284314\,i,\ 66.7753 - 26.1226\,i,$$
$$68.0094 + 67.8245\,i,\ 87.9623 - 5862.85\,i,\ 133.035 - 68.5173\,i,\ 820.076 - 843.833\,i\}$$

```
dnzRoots[100 000 000]
```

$$\{-7.,\ -7.,\ -6.,\ -6.,\ -5.,\ -5.,\ -4.,\ -4.,\ -3.,\ -3.,\ -2.,\ -2.,\ -1.,\ -1.,\ 0.,\ 0.\}$$

```
dz[1 000 000, z]
```

$$\frac{(-5-z)^2\,(-4-z)^2\,(-3-z)^2\,(-2-z)^2\,(-1-z)^2\,z^2}{518\,400}$$

```
dzz[n_, s_, z_] := n^-s Product[(-1)^p[[2]] bin[-z, p[[2]]], {p, FI[n]}]
dzzRoots[n_, s_] := If[(c = Exponent[f = dzz[n, s, z], z]) == 0, {},
  If[c == 1, List@NRoots[f == 0, z][[2]], List @@ NRoots[f == 0, z][[All, 2]]]]
```

```
dzz[100, 0, z]
```

$$\frac{1}{4}\,(-1-z)^2\,z^2$$

```
Dns112zR[100, 1, z] - Dns112zR[99, 1, z]
```

$$0. - 2.77556\times10^{-16}\,z - 0.0075\,z^2 - 0.005\,z^3 + 0.0025\,z^4$$

```
FullSimplify@(Dns112z[100, ZetaZero[1], z] - Dns112z[99, ZetaZero[1], z])
```

$$4^{-1-\text{ZetaZero}[1]}\,25^{-\text{ZetaZero}[1]}\,(-3+z)\,z^2\,(1+z)$$

```
Dns112zZeros[100, 1]
```

```
{-2.77849, 1.65264 - 2.69167 i, 1.65264 + 2.69167 i,
 4.3873, 9.82074 - 5.46414 i, 9.82074 + 5.46414 i}
```

```
Animate[Graphics[Table[{ColorData["RustTones"][n / 100], Point[{Re[#], Im[#]}]} & /@
     Dns112zZeros[n, .3 + j I], {n, 10 000, 10 000}], Frame → True,
  PlotRange → {{0 - 4, 2 + 4}, {-1 - 4, 1 + 4}}], {j, 14, 32, .01}]
```

```
N@ZetaZero[1]
```

```
0.5 + 14.1347 i
```

```
Dns112zZeros[100, 1]
```

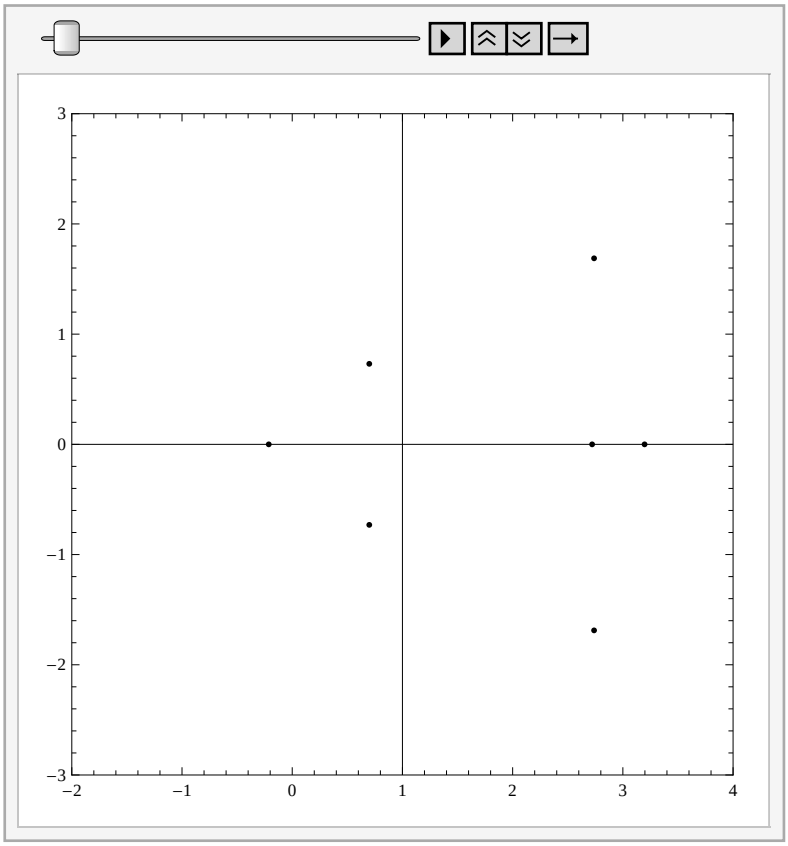
```
{-2.77849, 1.65264 - 2.69167 i, 1.65264 + 2.69167 i,
 4.3873, 9.82074 - 5.46414 i, 9.82074 + 5.46414 i}
```

```
pts = Table[{Point[{Re[#], Im[#]}]} & /@ Dns112zZeros[1000, .5 + n I], {n, 14, 32, .03}];
```

```
pts[[2]]
```

```
{{Point[{-2.46695, 1.07716}]}, {Point[{0.529164, -0.230124}]},
 {Point[{1.12859, 0.204044}]}, {Point[{1.98993, -0.74285}]}, {Point[{2.13135, 0.95397}]},
 {Point[{4.89599, -0.648908}]}, {Point[{11.7524, -1.75238}]},
 {Point[{17.1052, 15.4425}]}, {Point[{90.9615, -53.0174}]}}
```

```
ListAnimate[Table[Graphics[ptsrf[[k]], Frame → True, Axes → True, AxesOrigin → {1, 0},
   PlotRange → {{0 - 2, 2 + 2}, {-1 - 2, 1 + 2}}], {k, 1, Length[ptsrf]}]]
```




```
Length[ptsa]
```

1667

```
pts4 = Table[{Point[{Re[#], Im[#]}]} & /@ Dns112zZeros[1000, .4 + n I], {n, 14, 32, .03}];
pts6 = Table[{Point[{Re[#], Im[#]}]} & /@ Dns112zZeros[1000, .6 + n I], {n, 14, 32, .03}];
pts1 = Table[{Point[{Re[#], Im[#]}]} & /@ Dns112zZeros[1000, 1 + n I], {n, 14, 32, .03}];


pts2 = Table[{Point[{Re[#], Im[#]}]} & /@ Dns112zZeros[1000, .2 + n I], {n, 14, 32, .03}];


ptsa = Table[{Point[{Re[#], Im[#]}]} & /@ Dns112zZeros[1000, .5 + n I], {n, 0, 50, .03}];

ptsr = Table[{Point[{Re[#], Im[#]}]} & /@ Dns112zZeros[100 000, .5 + n I], {n, 13, 32, .05 }];

ptst = Table[{Point[{Re[#], Im[#]}]} & /@
    Dns112zZeros[100 000, n + 14.134725141734695` i], {n, 0, 5, .03}];

N@ZetaZero[29]
```

0.5 + 98.8312 i

```
ptsg = Table[{Point[{Re[#], Im[#]}]} & /@ Dns112zZeros[100 000, n + 20 i], {n, 0, 3, .03}];
```

```
ptsr[[1]]
```

```
{{Point[{-0.755668, 2.76242}]}, {Point[{0.0914007, -0.0752522}]},
 {Point[{0.450741, 0.349267}]}, {Point[{1.53138, -0.821849}]},
 {Point[{2.31499, -7.52807}]}, {Point[{2.44116, 2.37914}]}, {Point[{2.9159, -1.76289}]},
 {Point[{4.31139, -0.986764}]}, {Point[{6.08043, 0.263259}]}, {Point[{9.83916, 12.1791}]},
 {Point[{14.3549, 42.6005}]}, {Point[{17.499, -6.14424}]}, {Point[{18.2166, 2.53785}]},
 {Point[{47.1514, -22.9204}]}, {Point[{123.952, -122.607}]}, {Point[{140.822, 187.882}]}}
```

```
ptsrf = Table[{Point[{Re[#], Im[#]}]} & /@ Dns112zZeros[100 000, .5 + n I], {n, 0, 100, .05 }];
```

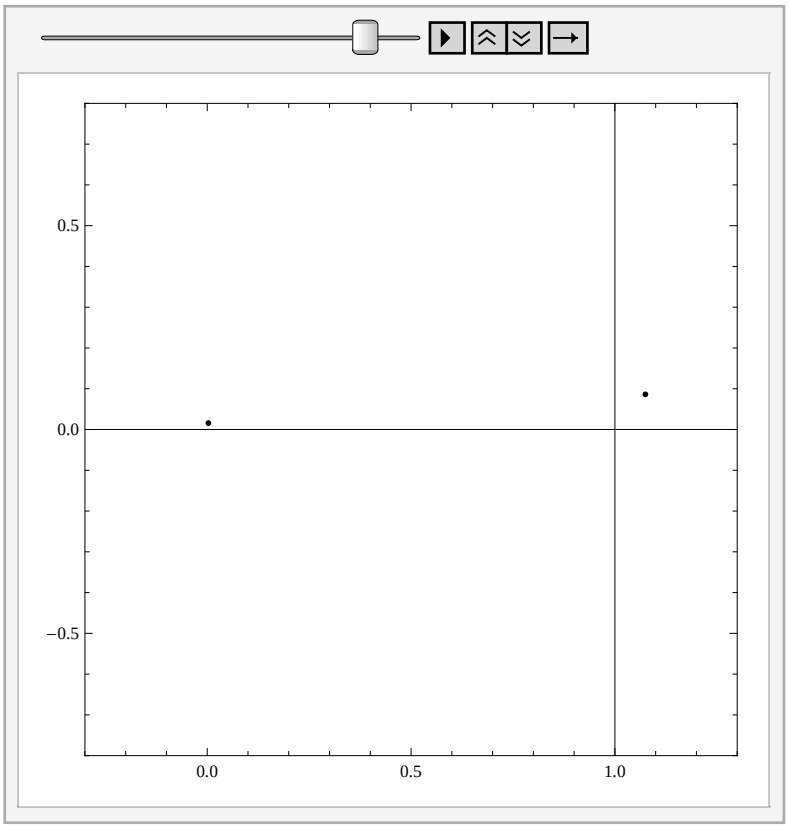
```
Expand@Sum[ n^(.5 + 3 I) dz[n, z], {n, 1, 100 000}]
```

$$(1. + 0.\,i) - (339\,784. - 755\,827.\,i)\,z - \left(1.01599\times10^{6} - 2.22564\times10^{6}\,i\right)z^{2} - \left(1.27993\times10^{6} - 2.70071\times10^{6}\,i\right)z^{3} - \left(888\,053. - 1.8117\times10^{6}\,i\right)z^{4} - (392\,502. - 759\,690.\,i)\,z^{5} - (114\,201. - 210\,752.\,i)\,z^{6} - (23\,607.5 - 40\,139.4\,i)\,z^{7} - (3309.46 - 5303.18\,i)\,z^{8} - (343.29 - 489.395\,i)\,z^{9} - (23.2221 - 31.114\,i)\,z^{10} - (1.13919 - 1.33696\,i)\,z^{11} - (0.0343179 - 0.0377698\,i)\,z^{12} - (0.000708362 - 0.000633775\,i)\,z^{13} - \left(8.96127\times10^{-6} - 6.53149\times10^{-6}\,i\right)z^{14} - \left(3.96572\times10^{-8} - 2.46872\times10^{-8}\,i\right)z^{15} - \left(2.42609\times10^{-10} - 2.85282\times10^{-11}\,i\right)z^{16}$$

```
Expand@Dns112za[100 000, .5 + I, z]
```

$$(1. + 0.\,i) + (1.13474 + 2.10739\,i)\,z - (6.07616 + 7.95427\,i)\,z^{2} + (9.72613 + 13.1162\,i)\,z^{3} - (8.15909 + 11.2965\,i)\,z^{4} + (4.05033 + 5.67129\,i)\,z^{5} - (1.25553 + 1.75194\,i)\,z^{6} + (0.247138 + 0.336905\,i)\,z^{7} - (0.0307918 + 0.0396298\,i)\,z^{8} + (0.00243475 + 0.0027785\,i)\,z^{9} - (0.000125607 + 0.000113519\,i)\,z^{10} + \left(4.23162\times10^{-6} + 2.29327\times10^{-6}\,i\right)z^{11} - \left(1.00326\times10^{-7} + 1.78131\times10^{-8}\,i\right)z^{12} + \left(2.21115\times10^{-9} + 9.15285\times10^{-10}\,i\right)z^{13} - \left(3.85328\times10^{-11} + 3.77593\times10^{-11}\,i\right)z^{14} + \left(3.04746\times10^{-13} + 4.03536\times10^{-13}\,i\right)z^{15} - \left(1.15218\times10^{-15} + 1.95429\times10^{-15}\,i\right)z^{16}$$

```
ListAnimate[Table[Graphics[ptsrf[[k]], Frame → True, Axes → True,
    AxesOrigin → {1, 0}, PlotRange → {{-.3, 1.3}, {-.8, .8}}], {k, 1, Length[ptsrf]}]]
```




```
ptsre = Table[{Point[{Re[#], Im[#]}]} & /@ Dns112zZeros[100 000, .5 + n I], {n, 95, 105, .005 }];
```

```
$Aborted
```