```
Clear[ee, ff]
ee[n_, j_, k_, z_] := ee[n, j, k, z] =
  If[n < j, 0, z / k DivisorSigma[1, j] / j (1 + ee[n - j, 1, k + 1, z]) + ee[n, j + 1, k, z]]
ff[n_, j_, k_, z_] := ff[n, j, k, z] =
  If[n < j, 0, z / k / j (1 + ff[n - j, 1, k + 1, z]) + ff[n, j + 1, k, z]]
roots[n_] := If[(c = Exponent[f = (ee[n, 1, 1, z] - ee[n - 1, 1, 1, z]), z]) == 0, {},
  If[c == 1, List@NRoots[f == 0, z][[2]], List @@ NRoots[f == 0, z][[All, 2]]]]
roots2[n_] := If[(c = Exponent[f = (ee[n, 1, 1, z] - ee[n - 1, 1, 1, z]), z]) == 0,
  {}, If[c == 1, List@Roots[f == 0, z][[2]], List @@ Roots[f == 0, z][[All, 2]]]]
roots3[n_] := If[(c = Exponent[f = (ff[n, 1, 1, z] - ff[n - 1, 1, 1, z]), z]) == 0,
  {}, If[c == 1, List@Roots[f == 0, z][[2]], List @@ Roots[f == 0, z][[All, 2]]]]
bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
Clear[bn]
bn[n_, k_] := bn[n, k] = Sum[bn[n - j, k - 1], {j, 1, n}]
bn[n_, 0] := UnitStep[n]
bz[n_, z_] := Sum[bin[z, k] bn[n, k], {k, 0, n}]
bzz[n_, z_] := ff[n, 1, 1, z] + 1
ez[n_, z_] := ee[n, 1, 1, z] + 1
roots4[n_] := If[(c = Exponent[f = (ez[n, z]), z]) == 0, {},
  If[c == 1, List@Roots[f == 0, z][[2]], List @@ Roots[f == 0, z][[All, 2]]]]
roots4a[n_] := If[(c = Exponent[f = (ez[n, z] - ez[n - 1, z]), z]) == 0, {},
  If[c == 1, List@Roots[f == 0, z][[2]], List @@ Roots[f == 0, z][[All, 2]]]]
roots4ap[n_] := Expand@FullSimplify@
   Product[If[r == 0, (D[(ez[n, z] - ez[n - 1, z]), z] /. z → 0) z, 1 - z / r], {r, roots4a[n]}]
```

```
DivisorSigma[1, 10]
```

18

```
Divisors[10]
```

{1, 2, 5, 10}

```
Table[Expand@(ee[n, 1, 1, z] - ee[n - 1, 1, 1, z]), {n, 1, 8}]
```

$$\left\{z, \frac{3z}{2}+\frac{z^2}{2}, \frac{4z}{3}+\frac{3z^2}{2}+\frac{z^3}{6}, \frac{7z}{4}+\frac{59z^2}{24}+\frac{3z^3}{4}+\frac{z^4}{24}, \frac{6z}{5}+\frac{15z^2}{4}+\frac{43z^3}{24}+\frac{z^4}{4}+\frac{z^5}{120}, 2z+\frac{1697z^2}{360}+\frac{55z^3}{16}+\frac{113z^4}{144}+\frac{z^5}{16}+\frac{z^6}{720}, \frac{8z}{7}+\frac{92z^2}{15}+\frac{2021z^3}{360}+\frac{89z^4}{48}+\frac{35z^5}{144}+\frac{z^6}{80}+\frac{z^7}{5040}, \frac{15z}{8}+\frac{8147z^2}{1120}+\frac{4049z^3}{480}+\frac{21127z^4}{5760}+\frac{11z^5}{16}+\frac{167z^6}{2880}+\frac{z^7}{480}+\frac{z^8}{40320}\right\}$$

```
Expand@ee[10, 1, 1, z] /. z → 1
```

138

```
Sum[PartitionsP[j], {j, 1, 10}]
```

138

```
PartitionsP[2]
```

2

```
FullSimplify@Expand@(ee[7, 1, 1, z] - ee[6, 1, 1, z])
```

$$\frac{z (2+z) (3+z) (8+z) (120+z (529+z (50+z)))}{5040}$$

```
FullSimplify@Expand@Product[(1 - z / r), {r, roots2a[7]}]
```

$$\frac{(2+z) (3+z) (8+z) (120+z (529+z (50+z)))}{5760}$$

```
D[Expand@ee[12, 1, 1, z], z] /. z → 0
```

$$\frac{102\,397}{5544}$$

```
roots3[10]
```

{0, -9, -8, -7, -6, -5, -4, -3, -2, -1}

```
Expand@bzz[10, z] /. z → 2
```

66

```
Pochhammer[11, z] / (z!) /. z → 2
```

66

```
FullSimplify@Expand[bz[10, z] - bz[9, z]]
```

$$\frac{z (1+z) (2+z) (3+z) (4+z) (5+z) (6+z) (7+z) (8+z) (9+z)}{3\,628\,800}$$

```
Expand@ez[10, z] /. z → 1
```

139

```
Sum[PartitionsP[j], {j, 0, 10}]
```

139

```
FullSimplify@Product[1 - z / r, {r, roots4[10]}]
```

$$\frac{1}{3\,628\,800} (3\,628\,800+z (1+z) (54\,597\,600+z (107\,560\,296+ z (66\,853\,484+z (18\,554\,166+z (2\,618\,139+z (195\,714+z (7656+z (144+z)))))))))$$

```
Expand@FullSimplify@Product[If[r == 0, 9 / 5 z, 1 - z / r], {r, roots4a[10]}] /. z → 1
```

42

```
PartitionsP[10]
```

42

```
Expand[ez[10, z] - ez[9, z]]
```

$$\frac{9 z}{5}+\frac{252\,019 z^2}{25\,200}+\frac{64\,193 z^3}{4032}+\frac{59\,453 z^4}{5670}+\frac{7457 z^5}{2304}+\frac{88\,453 z^6}{172\,800}+\frac{49 z^7}{1152}+\frac{221 z^8}{120\,960}+\frac{z^9}{26\,880}+\frac{z^{10}}{3\,628\,800}$$

```
D[Expand[ez[10, z] - ez[9, z]], z] /. z → 0
```

$$\frac{9}{5}$$

```
D[roots4ap[10], z] /. z → 0
```

$$\frac{9}{5}$$

```
DivisorSigma[1, 10] / 10
```

$$\frac{9}{5}$$

```
Clear[bq, bb]
bq[n_, k_] := bq[n, k] = Sum[PartitionsQ[j] bq[n - j, k - 1], {j, 1, n}]
bq[n_, 0] := UnitStep[n]
bqz[n_, z_] := Sum[ bin[z, k] bq[n, k], {k, 0, n}]
dbqz[n_, z_] := bqz[n, z] - bqz[n - 1, z]
a[n_] := If[n < 1, 0, Sum[Mod[d, 2] d, {d, Divisors[n]}]]
bb[n_, j_, k_, z_] :=
 bb[n, j, k, z] = If[n < j, 0, z / k a[j] / j (1 + bb[n - j, 1, k + 1, z]) + bb[n, j + 1, k, z]]
bbz[n_, z_] := 1 + bb[n, 1, 1, z]
Table[n D[Expand@dbqz[n, z], z] /. z → 0, {n, 1, 15}]
```

{1, 1, 4, 1, 6, 4, 8, 1, 13, 6, 12, 4, 14, 8, 24}

```
Sum[PartitionsQ[j], {j, 0, 10}]
```

43

```
Table[a[n], {n, 1, 15}]
```

{1, 1, 4, 1, 6, 4, 8, 1, 13, 6, 12, 4, 14, 8, 24}

```
Expand@bqz[10, z]
```

$$1+\frac{20821\,z}{2520}+\frac{811393\,z^2}{50400}+\frac{2129287\,z^3}{181440}+\frac{1728701\,z^4}{362880}+\frac{34271\,z^5}{34560}+\frac{23593\,z^6}{172800}+\frac{1277\,z^7}{120960}+\frac{17\,z^8}{30240}+\frac{11\,z^9}{725760}+\frac{z^{10}}{3628800}$$

```
Expand@bbz[10, z]
```

$$1+\frac{20821\,z}{2520}+\frac{811393\,z^2}{50400}+\frac{2129287\,z^3}{181440}+\frac{1728701\,z^4}{362880}+\frac{34271\,z^5}{34560}+\frac{23593\,z^6}{172800}+\frac{1277\,z^7}{120960}+\frac{17\,z^8}{30240}+\frac{11\,z^9}{725760}+\frac{z^{10}}{3628800}$$

```
Clear[pq]
pq[n_, k_] := pq[n, k] = Sum[IntegerPartitions[j, 1] pq[n - j, k - 1], {j, 1, n}]
pq[n_, 0] := UnitStep[n]
pqz[n_, z_] := Sum[ bin[z, k] pq[n, k], {k, 0, n}]
dpqz[n_, z_] := pqz[n, z] - pqz[n - 1, z]
Clear[pqe]
pqe[n_, j_, k_, z_] := pqe[n, j, k, z] =
  If[n < j, 0, z / k DivisorSigma[0, j] / j (1 + pqe[n - j, 1, k + 1, z]) + pqe[n, j + 1, k, z]]
ppqe[n_, z_] := 1 + pqe[n, 1, 1, z]


Table[ n ! Expand[ppqe[n, z] - ppqe[n - 1, z]] /. z → 1, {n, 1, 15}]
```

{1, 3, 11, 59, 339, 2629, 20 677, 202 089, 2 066 201, 24 322 931,
 296 746 251, 4 193 572 723, 59 806 188 571, 954 679 763 829, 15 845 349 818 789}

```
(* A028342 http://oeis.org/A028342 *)
nmax = 15; CoefficientList[
  Series[Product[1 / (1 - x^k) ^ (1 / k), {k, 1, nmax}], {x, 0, nmax}], x] * Range[0, nmax] !
```

```
{1, 1, 3, 11, 59, 339, 2629, 20 677, 202 089, 2 066 201, 24 322 931,
 296 746 251, 4 193 572 723, 59 806 188 571, 954 679 763 829, 15 845 349 818 789}

Clear[pqo]
pqo[n_, j_, k_, z_] := pqo[n, j, k, z] =
  If[n < j, 0, z / k DivisorSigma[2, j] / j (1 + pqo[n - j, 1, k + 1, z]) + pqo[n, j + 1, k, z]]
ppqo[n_, z_] := 1 + pqo[n, 1, 1, z]

Table[ Expand[ppqo[n, z] - ppqo[n - 1, z]] /. z → 1, {n, 1, 15}]

{1, 3, 6, 13, 24, 48, 86, 160, 282, 500, 859, 1479, 2485, 4167, 6879}

(* http://oeis.org/A000219 *)
CoefficientList[Series[Product[(1 - x^k)^-k, {k, 1, 64}], {x, 0, 15}], x]

{1, 1, 3, 6, 13, 24, 48, 86, 160, 282, 500, 859, 1479, 2485, 4167, 6879}

Clear[pqs]
jordanTotient[n_Integer?Positive, k_: 1] := DivisorSum[n, #^k * MoebiusMu[n / #] &]
pqs[n_, j_, k_, z_] := pqs[n, j, k, z] =
  If[n < j, 0, z / k jordanTotient[j, 1] / j (1 + pqs[n - j, 1, k + 1, z]) + pqs[n, j + 1, k, z]]
ppqs[n_, z_] := 1 + pqs[n, 1, 1, z]

Table[Expand[ppqs[n, z] - ppqs[n - 1, z]] /. z → 1, {n, 1, 10}]
```

$$\left\{1, 1, \frac{4}{3}, \frac{19}{12}, \frac{131}{60}, \frac{433}{180}, \frac{1009}{315}, \frac{38\,399}{10\,080}, \frac{415\,199}{90\,720}, \frac{2\,426\,923}{453\,600}\right\}$$

```
EulerPhi[19]

18

Clear[og]
og[n_, k_] := og[n, k] = Sum[s^j og[n - j, k - 1], {j, 1, n}]
og[n_, 0] := UnitStep[n]
ogz[n_, z_] := Sum[bin[z, k] og[n, k], {k, 0, n}]

Table[n D[Expand[ogz[n, z] - ogz[n - 1, z]], z] /. z → 0, {n, 1, 10}]
```

$$\left\{s, s^2, s^3, s^4, s^5, s^6, s^7, s^8, s^9, s^{10}\right\}$$

```
Sum[s^j / j, {j, 1, n}]
```

$$-s^{1+n}\,\text{LerchPhi}[s, 1, 1+n] - \text{Log}[1-s]$$

```
Clear[ob]
ob[n_, k_] := ob[n, k] = Sum[ ob[n - j, k - 1], {j, 2, n, 2}]
ob[n_, 0] := UnitStep[n]
obz[n_, z_] := Sum[bin[z, k] ob[n, k], {k, 0, n}]

(* Generating function for 1/(1-x^2) *)

Sum[ x^j, {j, 0, Infinity, 2}]
```

$$\frac{1}{1-x^2}$$

```
Table[n D[Expand[obz[n, z] - obz[n - 1, z]], z] /. z → 0, {n, 1, 10}]
```

{0, 2, 0, 2, 0, 2, 0, 2, 0, 2}

```
Expand@obz[10, z]
```

$$1 - \frac{12653\,z}{2520} + \frac{783473\,z^2}{50400} - \frac{217541\,z^3}{25920} + \frac{1223693\,z^4}{362880} - \frac{20059\,z^5}{34560} + \frac{14233\,z^6}{172800} - \frac{697\,z^7}{120960} + \frac{11\,z^8}{30240} - \frac{z^9}{103680} + \frac{z^{10}}{3628800}$$

```
(* This is the generating function for 1/(1-x)^2 *)
Clear[ok]
ok[n_, k_] := ok[n, k] = Sum[ (j + 1) ok[n - j, k - 1], {j, 1, n}]
ok[n_, 0] := UnitStep[n]
okz[n_, z_] := Sum[bin[z, k] ok[n, k], {k, 0, n}]

Table[n D[Expand[okz[n, z] - okz[n - 1, z]], z] /. z → 0, {n, 1, 10}]
```

{2, 2, 2, 2, 2, 2, 2, 2, 2, 2}

```
(* This is the generating function for 1/(1-x)^3 *)
Clear[ol]
ol[n_, k_] := ol[n, k] = Sum[ Binomial[j + 2, 2] ol[n - j, k - 1], {j, 1, n}]
ol[n_, 0] := UnitStep[n]
olz[n_, z_] := Sum[bin[z, k] ol[n, k], {k, 0, n}]
Table[n D[Expand[olz[n, z] - olz[n - 1, z]], z] /. z → 0, {n, 1, 10}]
```

{3, 3, 3, 3, 3, 3, 3, 3, 3, 3}

```
Expand@olz[10, z] /. z → 1
```

286

```
Sum[x^(3 k) + x^(3 k + 1) - 2 x^(3 k + 2), {k, 0, Infinity}]
```

$$\frac{1 + 2\,x}{1 + x + x^2}$$

```
Sum[x^(4 k) + x^(4 k + 1) + x^(4 k + 2) - 3 x^(4 k + 3), {k, 0, Infinity}]
```

$$\frac{1 + 2\,x + 3\,x^2}{1 + x + x^2 + x^3}$$

```
Sum[x^(5 k) + x^(5 k + 1) + x^(5 k + 2) + x^(5 k + 3) - 4 x^(5 k + 4), {k, 0, Infinity}]
```

$$\frac{1 + 2\,x + 3\,x^2 + 4\,x^3}{1 + x + x^2 + x^3 + x^4}$$