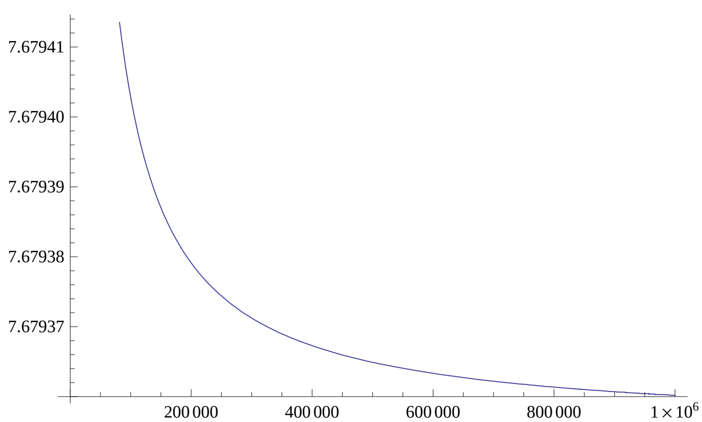
```
pt[n_, s_] := ((1 - s) n^s HarmonicNumber[n, s] - s n^(1 - s) HarmonicNumber[n, 1 - s]) /
  ((1 - s) n^s - s n^(1 - s))
pt2[n_, s_] := ((1 - s) n^s (Zeta[s] - Zeta[s, n + 1]) -
    s n^(1 - s) (Zeta[1 - s] - Zeta[1 - s, n + 1])) / ((1 - s) n^s - s n^(1 - s))
pt2a[n_, s_] := ((1 - s) n^s (Zeta[s]) - s n^(1 - s) (Zeta[1 - s])) / ((1 - s) n^s - s n^(1 - s))
pt2a1[n_, s_] := ((1 - s) n^s (Zeta[s])) / ((1 - s) n^s - s n^(1 - s))
pt2a2[n_, s_] := (-s n^(1 - s) (Zeta[1 - s])) / ((1 - s) n^s - s n^(1 - s))
pt2a2b[n_, s_] := (-Zeta[1 - s]) / ((1 - s) / s n^(2 s - 1) - 1)
pt2a2c[n_, s_] := Zeta[1 - s] / (1 - (1 - s) / s n^(2 s - 1))
pt2a2d[n_, s_] := 1 / (1 - (1 - s) / s n^(2 s - 1))
pt2b[n_, s_] :=
 ((1 - s) n^s (-Zeta[s, n + 1]) - s n^(1 - s) (-Zeta[1 - s, n + 1])) / ((1 - s) n^s - s n^(1 - s))
pt2bx[n_, s_] := ((1 - s) n^s (-Zeta[s, n + 1]) - s n^(1 - s) (-Zeta[1 - s, n + 1]))
pt2b1[n_, s_] := ((1 - s) n^s (-Zeta[s, n + 1])) / ((1 - s) n^s - s n^(1 - s))
pt2b2[n_, s_] := (-s n^(1 - s) (-Zeta[1 - s, n + 1])) / ((1 - s) n^s - s n^(1 - s))
pt2b1a[n_, s_] := ((1 - s) n^s (-Zeta[s, n + 1]))
pt2b2a[n_, s_] := (-s n^(1 - s) (-Zeta[1 - s, n + 1]))
pt2b2as[n_, s_, t_] := pt2b1a[n, s] - pt2b1a[n, t]
pt2b2at[n_, s_, t_] := pt2b1a[n, s] + pt2b2a[n, t]
pt2b2ax[n_, s_, t_] := ((1 - s) n^s (-Zeta[s, n + 1])) - ((1 - t) n^t (-Zeta[t, n + 1]))
pt2bz[n_, s_] := ((1 - s) n^s (-Zeta[s, n + 1]) - s n^(1 - s) (-Zeta[1 - s, n + 1]))
pt2by[n_, s_] := ((1 - s) n^s (-Zeta[s, n + 1]) - s n^(1 - s) (-Zeta[1 - s, n + 1]))
pt2ay[n_, s_] := n^(-.5) ((1 - s) n^s (Zeta[s]) - s n^(1 - s) (Zeta[1 - s]))
ff[n_, s_] := (1 - s) n^s
pto[n_, s_] := (ff[n, s] HarmonicNumber[n, s] - ff[n, 1 - s] HarmonicNumber[n, 1 - s]) /
  (ff[n, s] - ff[n, 1 - s])
pto2[n_, s_, t_] := (ff[n, s] HarmonicNumber[n, s] - ff[n, t] HarmonicNumber[n, t]) /
  (ff[n, s] - ff[n, t])
```

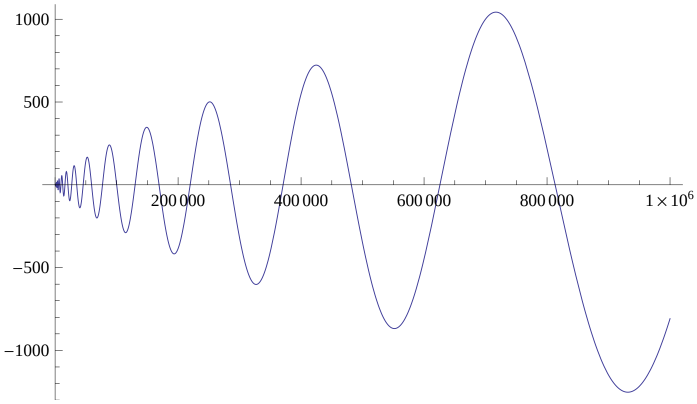
```
Plot[Abs@pt2b2as[n, -3 + 15 I, .3], {n, 1, 1000000}]
```



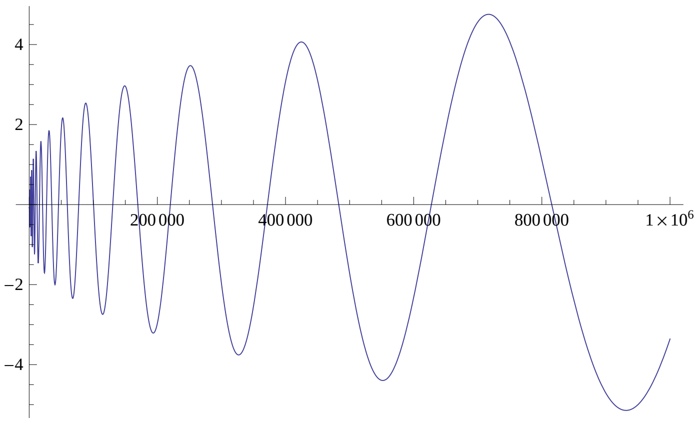

```
pt2a1[1000000000000, .9]
```

-9.43011

```
Plot[{Re@Zeta[.3 + 12 I, n + 1]}, {n, 1, 1000000}]
```




```
Plot[{Re@Zeta[.7 - 12 I, n + 1]}, {n, 1, 1000000}]
```




```
pto2[10000000000, -.4, -1]
```

−2143.1

```
Zeta[-.4]
```

−0.247165

```
HarmonicNumber[n, 0]
```

n

```
FullSimplify[pto2[n, s, -1]]
```

$$\frac{n\,(1 + n + n^{s}\,(-1 + s)\,\text{HarmonicNumber}[n, s])}{2 + n^{1+s}\,(-1 + s)}$$

$$\text{FullSimplify}\left[\frac{n + n^{s}\,(-1 + s)}{1 + n^{s}\,(-1 + s)}\right]$$

$$1 + \frac{-1 + n}{1 + n^{s}\,(-1 + s)}$$

```
Zeta[.3]
```

−0.904559

```
pr[n_, s_] := (n + n^s (-1 + s) HarmonicNumber[n, s])/(1 + n^s (-1 + s))
pra[n_, s_] := {(n^s (-1 + s) HarmonicNumber[n, s])/(1 + n^s (-1 + s)), n/(1 + n^s (-1 + s))}
prb[n_, s_] := n/(1 + n^s (-1 + s)) + Sum[(n^s (-1 + s) j^-s)/(1 + n^s (-1 + s)), {j, 1, n}]
prc[n_, s_] := n/(1 + n^s (-1 + s)) + Sum[1/(j^s + (j^s n^-s)/(-1 + s)), {j, 1, n}]
prk[n_, s_] := (n (1 + n + n^s (-1 + s) HarmonicNumber[n, s]))/(2 + n^(1 + s) (-1 + s))

FullSimplify[prc[n, s]]

n/(1 + n^s (-1 + s)) + Sum[1/(j^s + (j^s n^-s)/(-1 + s)), {j, 1, n}]

Zeta[.7 + 1000 I]

0.784054 + 0.374829 I

FullSimplify[(n + n^s (-1 + s) HarmonicNumber[n, s])/(1 + n^s (-1 + s))]

(n + n^s (-1 + s) HarmonicNumber[n, s])/(1 + n^s (-1 + s))

FullSimplify[n/(1 + n^s (-1 + s))]

n/(1 + n^s (-1 + s))

FullSimplify[(n^s (-1 + s) j^-s)/(1 + n^s (-1 + s))]

1/(j^s + (j^s n^-s)/(-1 + s))

N@prk[10 000 000 000, 1/2]

-1.46037

Zeta[.5]

-1.46035

FullSimplify[HarmonicNumber[n, -1]]

1/2 n (1 + n)

pr[n, s]

(n + n^s (-1 + s) HarmonicNumber[n, s])/(1 + n^s (-1 + s))
```

```
prx[n_, s_, a_] := (n^a + n^s (-1 + s) HarmonicNumber[n, s]^a) / (1 + n^s (-1 + s))
```

```
prk[10 000 000, -.5]
```

-527.477

```
pto3[n_, s_, t_] := (ff[n, s] HarmonicNumber[n, s] - ff[n, t] HarmonicNumber[n, t]) / ff[n, s]
```

```
FullSimplify[pto3[n, s, -1]]
```

$$\frac{n^{-s}(1+n)}{-1+s} + \mathrm{HarmonicNumber}[n, s]$$

```
Zeta[.7]
```

-2.77839

```
pto3[1 000 000, .7, -4]
```

-2.77888

```
pt[n_, s_] := ((1 - s) n^s HarmonicNumber[n, s] - s n^(1 - s) HarmonicNumber[n, 1 - s]) /
  ((1 - s) n^s - s n^(1 - s))
et[n_, s_] := ((1 - s) n^s HarmonicNumber[n, s] - (1 - s) n^s HarmonicNumber[n, 1 - s]) /
  ((1 - s) n^s - s n^(1 - s))
pt2b2ax[n_, s_, t_] := ((1 - s) n^s (-Zeta[s, n + 1])) - ((1 - t) n^t (-Zeta[t, n + 1]))
```

```
et[100 000, .9]
```

-35 145.4

```
Zeta[.9]
```

-9.43011

```
pt2b2ax[100 000, .9, .8]
```

-0.0499999

```
fa[n_, s_] := (1 - s) n^s
prt[n_, s_, t_] :=
 (ff[n, s] HarmonicNumber[n, s] - ff[n, t] HarmonicNumber[n, t]) / (ff[n, s])
prt2[n_, s_, t_] := HarmonicNumber[n, s] - ff[n, t] / (ff[n, s]) HarmonicNumber[n, t]
```

```
prt2[1 000 000 000, .6, .3]
```

-1.9495

```
Zeta[.6]
```

-1.95266

```
FullSimplify[fa[n, 1 - s] / fa[n, s]]
```

$$\frac{n^{1-2s}\, s}{1-s}$$

```
pb[n_, s_] :=
 n^-s / (s - 1 / 2) Sum[ j^(-1 / 2) (2 s Cosh[s Log[n / j]] - Sinh[s Log[n / j]]), {j, 1, n}]
```

```
pb[1 000 000, .3 + 7 I]
```

1.02523 + 0.338313 i

```
Zeta[.8 + 7 I]
```

1.02505 + 0.338122 i

```
prt[1 000 000 000, .8 + 7 I, 1 - (.8 + 7 I)]
```

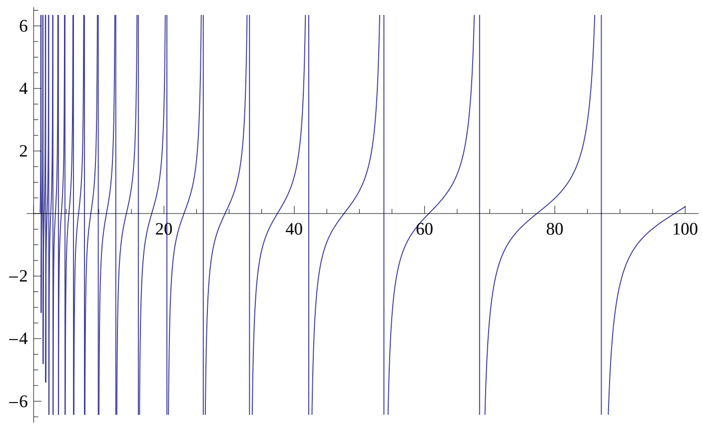
1.02505 + 0.338117 i

```
FullSimplify[Cos[ t Log[j]] +
  ((t Sin[t Log[n]] + (1 / 2) Cos[t Log[n]]) / (t Cos[t Log[n]] - (1 / 2) Sin[t Log[n]]))
   Sin[t Log[j]]]
```

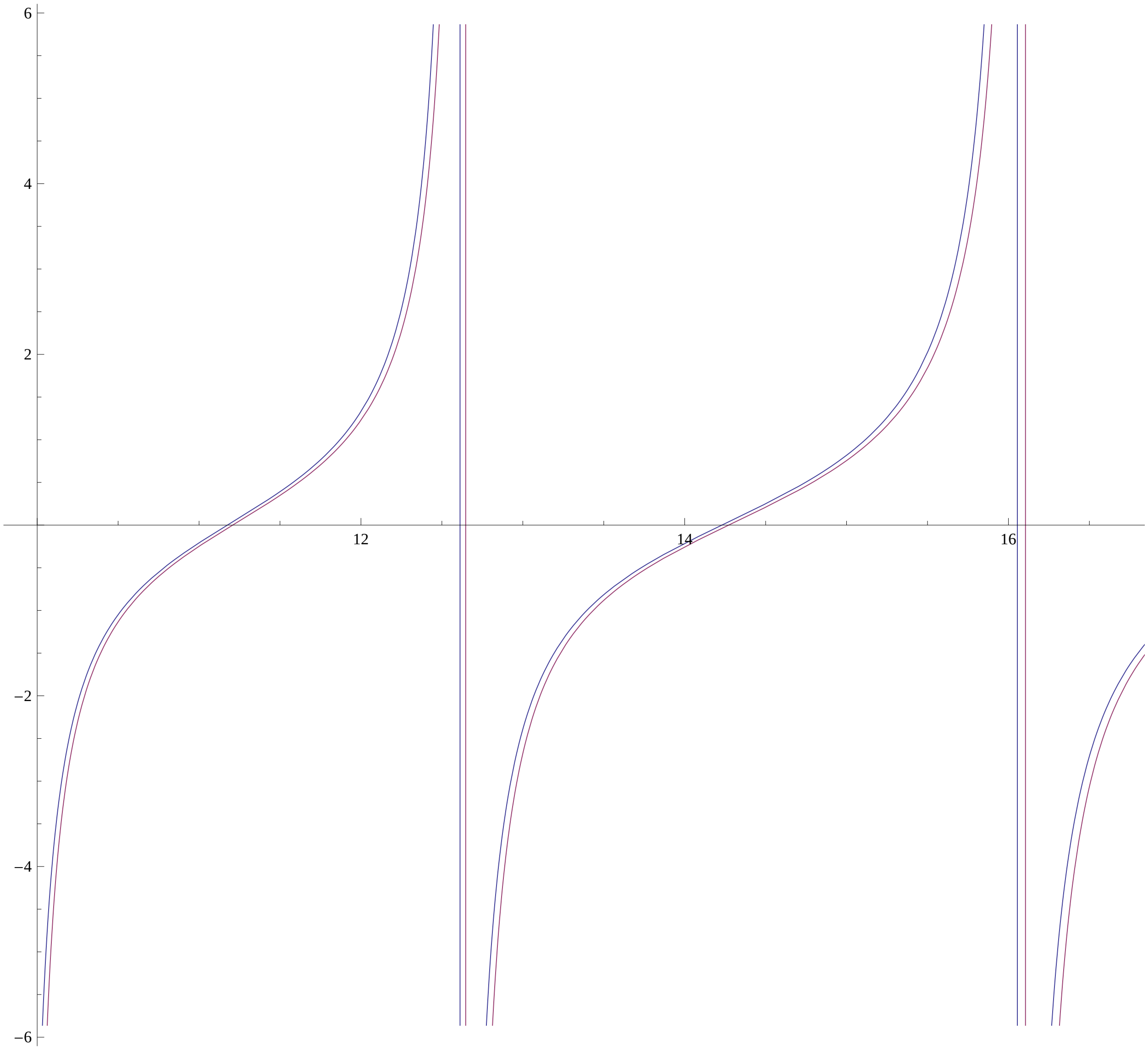
$$\frac{2\, t \operatorname{Cos}[t\, (\operatorname{Log}[j] - \operatorname{Log}[n])] + \operatorname{Sin}[t\, (\operatorname{Log}[j] - \operatorname{Log}[n])]}{2\, t \operatorname{Cos}[t \operatorname{Log}[n]] - \operatorname{Sin}[t \operatorname{Log}[n]]}$$

```
pr[n_, t_] := ((t Sin[t Log[n]] + (1 / 2) Cos[t Log[n]]) / (t Cos[t Log[n]] - (1 / 2) Sin[t Log[n]]))
```

```
Plot[ pr[n, 13], {n, 1, 100}]
```



```
Plot[{pr[n, 13], Tan[13 Log[n]]}, {n, 10, 20}]
```




```
pr[n_, t_] :=
 ((t Sin[t Log[n]] + (1 / 2) Cos[t Log[n]]) / (t Cos[t Log[n]] - (1 / 2) Sin[t Log[n]]))
prs[n_, t_] := ((t / (2 I) (n^(I t ) - n^(-I t )) + (1 / 2) Cos[t Log[n]]) /
   (t Cos[t Log[n]] - (1 / 2) Sin[t Log[n]]))
prst[n_, t_] := ((2 t / (I) (n^(I t ) - n^(-I t )) + (n^(I t) + n^(-I t))) /
   (2 t (n^(I t) + n^(-I t)) - (1 / I) (n^(I t) - n^(-I t))))
prst1[n_, t_] := ((-2 t I (n^(I t ) - n^(-I t )) + (n^(I t) + n^(-I t))) /
   (2 t (n^(I t) + n^(-I t)) + I (n^(I t) - n^(-I t))))
```

```
prst1[100, .3]
```

```
-0.894314 + 0. i
```

```
((-2 t I (n^(I t) - n^(-I t)) + (n^(I t) + n^(-I t))) /
  (2 t (n^(I t) + n^(-I t)) + I (n^(I t) - n^(-I t))))
```

$$\frac{n^{-i t}+n^{i t}-2 i \left(-n^{-i t}+n^{i t}\right) t}{i \left(-n^{-i t}+n^{i t}\right)+2 \left(n^{-i t}+n^{i t}\right) t}$$

```
1 / I
```

$-i$

```
ba[n_, s_] := Sum[ (1 - s) (n / j)^s - s (j / n)^(s - 1), {j, 1, n}]
ba2[n_, s_] := (1 - s) n^s HarmonicNumber[ n, s] - s n^(1 - s) HarmonicNumber[ n, 1 - s]
```

$$\text{ba3}[n\_, s\_] := -n^{\frac{1}{2}-s}\left(\frac{1}{2}+s\right)\text{HarmonicNumber}\left[n, \frac{1}{2}-s\right]+n^{\frac{1}{2}+s}\left(\frac{1}{2}-s\right)\text{HarmonicNumber}\left[n, \frac{1}{2}+s\right]$$

$$\text{ba4}[n\_, s\_] := \text{Sum}\left[\left(\frac{n}{j}\right)^{\frac{1}{2}+s}\left(\frac{1}{2}-s\right)-\left(\frac{j}{n}\right)^{-\frac{1}{2}+s}\left(\frac{1}{2}+s\right), \{j, 1, n\}\right]$$

$$\text{ba5}[n\_, s\_] := -I\left(n^{\frac{1}{2}-i s}\left(\frac{1}{2}+i s\right)\text{HarmonicNumber}\left[n, \frac{1}{2}-i s\right]-n^{\frac{1}{2}+i s}\left(\frac{1}{2}-i s\right)\text{HarmonicNumber}\left[n, \frac{1}{2}+i s\right]\right)$$

$$\text{ba5a}[n\_, s\_] := -I\left(n^{\frac{1}{2}-i s}\left(\frac{1}{2}+i s\right)\text{HarmonicNumber}\left[n, \frac{1}{2}-i s\right]\right)$$

$$\text{ba5b}[n\_, s\_] := -I\left(n^{\frac{1}{2}+i s}\left(\frac{1}{2}-i s\right)\text{HarmonicNumber}\left[n, \frac{1}{2}+i s\right]\right)$$

$$\text{ba5ax}[n\_, s\_] := -I\left(n^{\frac{1}{2}-i s}\left(\frac{1}{2}+i s\right)\text{HarmonicNumber}\left[n, \frac{1}{2}-i s\right]\right)\Big/ n\text{^}(1/2)$$

$$\text{ba5bx}[n\_, s\_] := -I\left(n^{\frac{1}{2}+i s}\left(\frac{1}{2}-i s\right)\text{HarmonicNumber}\left[n, \frac{1}{2}+i s\right]\right)\Big/ n\text{^}(1/2)$$

$$\text{ba6}[n\_, s\_] := -I\,\text{Sum}\left[\left(\frac{j}{n}\right)^{-\frac{1}{2}+I s}\left(\frac{1}{2}+I s\right)-\left(\frac{n}{j}\right)^{\frac{1}{2}+I s}\left(\frac{1}{2}-I s\right), \{j, 1, n\}\right]$$

$$\text{ba7}[n\_, s\_] := -I\,\text{Sum}\left[\left(\frac{n}{j}\right)^{\frac{1}{2}-I s}\left(\frac{1}{2}+I s\right)-\left(\frac{n}{j}\right)^{\frac{1}{2}+I s}\left(\frac{1}{2}-I s\right), \{j, 1, n\}\right]$$

$$\text{ba7a}[n\_, s\_] := \text{Sum}\left[\left(\frac{n}{j}\right)^{\frac{1}{2}+I s}\left(\frac{1}{2}I+s\right)-\left(\frac{n}{j}\right)^{\frac{1}{2}-I s}\left(\frac{1}{2}I-s\right), \{j, 1, n\}\right]$$

$$\text{ba8}[n\_, s\_, a\_] := \text{Sum}\left[\left(\frac{n}{j}\right)^{a+I s}(a\,I+s)-\left(\frac{n}{j}\right)^{a-I s}(a\,I-s), \{j, 1, n\}\right]$$

```
ba7a[100 000 000, N@Im@ZetaZero@20]
```

$77.1448 + 0.\,i$

```
ba2[n, s + 1 / 2]
```

$$-n^{\frac{1}{2}-s}\left(\frac{1}{2}+s\right)\text{HarmonicNumber}\left[n, \frac{1}{2}-s\right]+n^{\frac{1}{2}+s}\left(\frac{1}{2}-s\right)\text{HarmonicNumber}\left[n, \frac{1}{2}+s\right]$$

```
(1 - s) (n / j)^s - s (j / n)^(s - 1) /. s → s + 1 / 2
```

$$\left(\frac{n}{j}\right)^{\frac{1}{2}+s}\left(\frac{1}{2}-s\right)-\left(\frac{j}{n}\right)^{-\frac{1}{2}+s}\left(\frac{1}{2}+s\right)$$

ba2[n, -s I + 1 / 2]

$$n^{\frac{1}{2}-i s}\left(\frac{1}{2}+i s\right)\text{HarmonicNumber}\left[n,\frac{1}{2}-i s\right]-n^{\frac{1}{2}+i s}\left(\frac{1}{2}-i s\right)\text{HarmonicNumber}\left[n,\frac{1}{2}+i s\right]$$

N@Im@ZetaZero@1 - 1 / 2 I

$14.1347 - 0.5\,i$

(N@ZetaZero@1 - 1 / 2) I

$-14.1347 + 0.\,i$

ba7[n, x]

$Aborted

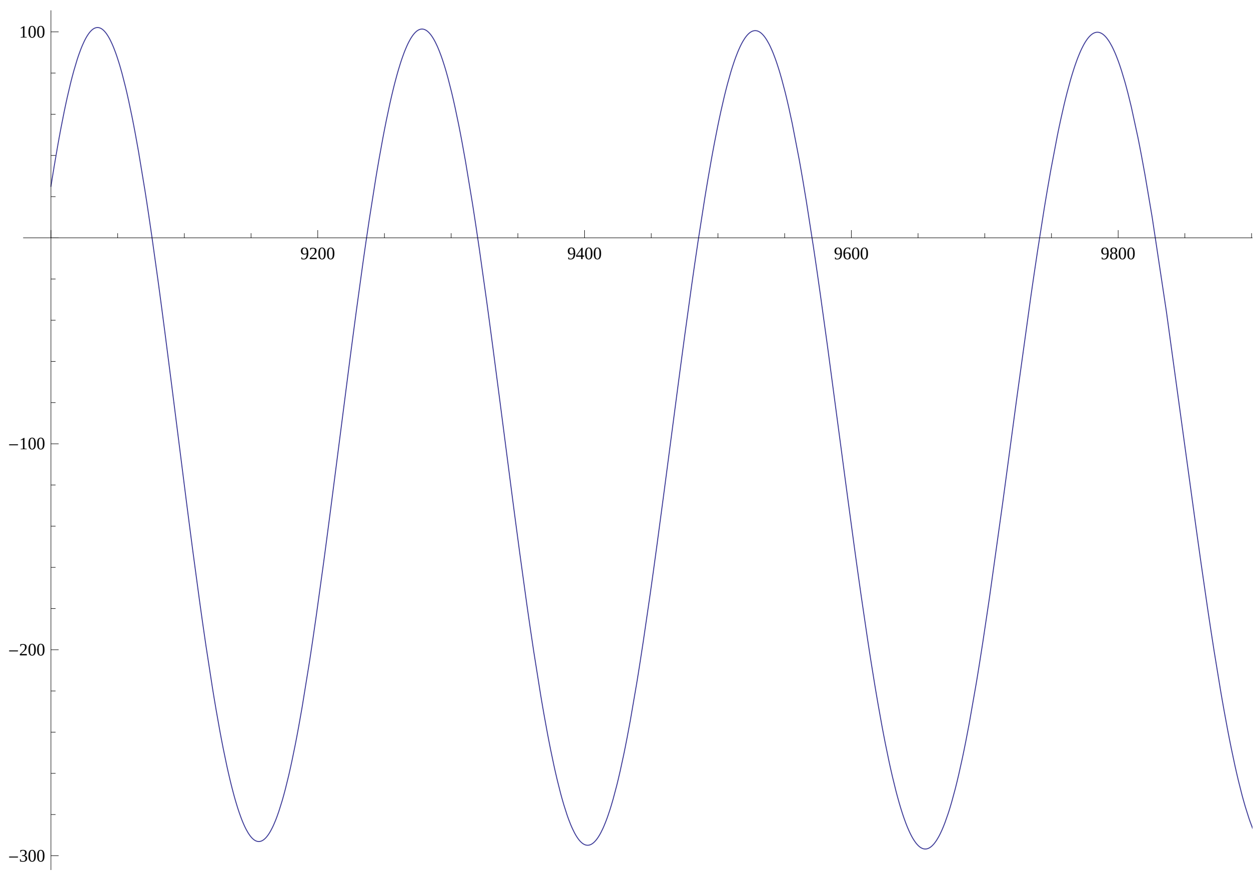
ba8[10 000 000, N@Im@ZetaZero@1, 0]

$1.40793 \times 10^{6} + 0.\,i$

N@Im@ZetaZero@120

269.97

Plot[Im@ba5ax[n, Im@ZetaZero@100 + .1 I], {n, 9000, 10 000}]



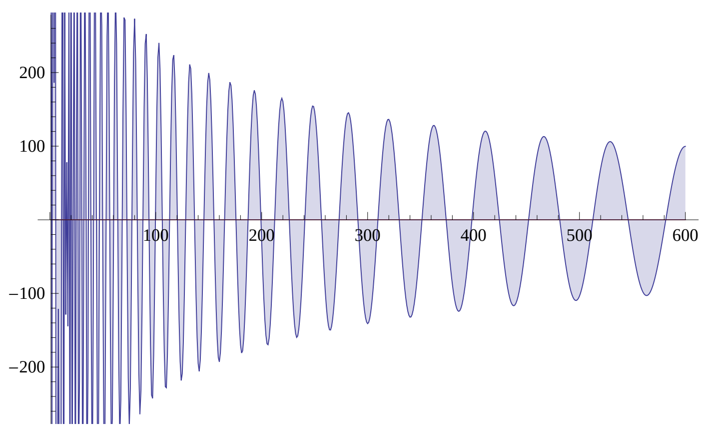
$$\mathbf{D}\left[\left(\frac{n}{j}\right)^{\frac{1}{2}+I s}\left(\frac{1}{2} I+s\right)-\left(\frac{n}{j}\right)^{\frac{1}{2}-I s}\left(\frac{1}{2} I-s\right), s\right]$$

$$\left(\frac{n}{j}\right)^{\frac{1}{2}-i s}+\left(\frac{n}{j}\right)^{\frac{1}{2}+i s}+i\left(\frac{n}{j}\right)^{\frac{1}{2}-i s}\left(\frac{i}{2}-s\right) \mathrm{Log}\left[\frac{n}{j}\right]+i\left(\frac{n}{j}\right)^{\frac{1}{2}+i s}\left(\frac{i}{2}+s\right) \mathrm{Log}\left[\frac{n}{j}\right]$$

$$\mathbf{ba7ax}[n\_, s\_] := \mathbf{Table}\left[\left\{\left(\frac{n}{j}\right)^{\frac{1}{2}+I s}\left(\frac{1}{2} I+s\right),\left(\frac{n}{j}\right)^{\frac{1}{2}-I s}\left(\frac{1}{2} I-s\right)\right\},\{j, 1, n\}\right]$$

$$\mathbf{ba7ay}[n\_, s\_] := \mathbf{DiscretePlot}\left[\left\{\mathbf{Re}\left[\left(\frac{n}{j}\right)^{\frac{1}{2}+I s}\left(\frac{1}{2} I+s\right)-\left(\frac{n}{j}\right)^{\frac{1}{2}-I s}\left(\frac{1}{2} I-s\right)\right],\right.\right.$$
$$\left.\left.\mathbf{Im}\left[\left(\frac{n}{j}\right)^{\frac{1}{2}+I s}\left(\frac{1}{2} I+s\right)-\left(\frac{n}{j}\right)^{\frac{1}{2}-I s}\left(\frac{1}{2} I-s\right)\right]\right\},\{j, 1, n\}\right]$$

ba7ay[600, N@Im@ZetaZero@10 + .1]



$$\mathbf{ba7a}[n\_, x\_] := \mathbf{Sum}\left[\left(\left(\frac{n}{j}\right)^{(1/2)+I x}\left(\frac{1}{2} I+x\right)-\left(\frac{n}{j}\right)^{(1/2)-I x}\left(\frac{1}{2} I-x\right)\right),\{j, 1, n\}\right]$$

```
ba7b[n_, x_] := Sum[ (n / j) ^ (1 / 2) (2 x Cos[ x Log[n / j]] - Sin[x Log[n / j]]), {j, 1, n}]
ba7c[n_, x_] :=
 n^ (1 / 2) ((2 x Sin[ x Log[n]] + Cos[x Log[n]]) Sum[ (j) ^ (-1 / 2) Sin[x Log[j]], {j, 1, n}] +
    (2 x Cos[ x Log[n]] - Sin[x Log[n]]) Sum[ (j) ^ (-1 / 2) Cos[x Log[j]], {j, 1, n}])
```

$$\mathbf{ba7d}[n\_, x\_] := \mathbf{Sum}\left[j^{\wedge}(-1/2)\left(\left(\frac{n}{j}\right)^{+I x}\left(\frac{1}{2} I+x\right)-\left(\frac{n}{j}\right)^{-I x}\left(\frac{1}{2} I-x\right)\right),\{j, 1, n\}\right]$$

```
div[n_, x_] := (1 / 2 - x I) n^ (1 / 2 + x I) -
  (1 / 2 + x I) n^ (1 / 2 - x I) 2^ (1 / 2 - x I) Pi^ (-1 / 2 - x I) Cos[ Pi / 4 + Pi x I / 2] Gamma[1 / 2 + x I]
```

$$\mathbf{div2}[n\_, x\_] := 2^{\frac{1}{2}+i x} n^{\frac{1}{2}+i x} \pi^{-\frac{1}{2}+i x}\left(\frac{1}{2} I+x\right) \mathbf{Cos}\left[\frac{1}{2} \pi\left(\frac{1}{2}-i x\right)\right] \mathbf{Gamma}\left[\frac{1}{2}-i x\right]-n^{\frac{1}{2}-i x}\left(\frac{1}{2} I-x\right)$$

ba7a[1 000 000 000 000, N@Im@ZetaZero@1]

$0.896724 + 0.\,i$

ba7b[10 000, .3 - .1 I]

$40.4575 + 167.902\,i$

ba7c[10 000, .3 - .1 I]

$40.4575 + 167.902\,i$

```
1 - (1 / 2 + x I)
```

$$\frac{1}{2} - i\, x$$

```
ba7a[1 000 000 000 000, .3 I + 10] / div2[1 000 000 000 000, .3 I + 10]
```

```
1.44628 + 0.113501 i
```

```
Zeta[.8 + 10 I]
```

```
1.44628 - 0.113501 i
```

```
pa11[n_, s_] := (n^(1 - s) s HarmonicNumber[n, 1 - s] - n^s (1 - s) HarmonicNumber[n, s])
  (2 n^(1 - s) s (2 π)^-s Cos[π s / 2] Gamma[s] - n^s (1 - s))^-1
```

```
pa11a[n_, s_] := (n^s (1 - s) HarmonicNumber[n, s] - n^(1 - s) s HarmonicNumber[n, 1 - s]) /
  (n^s (1 - s) - 2 n^(1 - s) s (2 π)^-s Cos[π s / 2] Gamma[s])
```

$$\text{pa11b[n\_, x\_] := }\left(n^{\frac{1}{2}-i x}\left(\frac{1}{2}+i x\right)\text{HarmonicNumber}\left[n,\frac{1}{2}-i x\right]-n^{\frac{1}{2}+i x}\left(\frac{1}{2}-i x\right)\text{HarmonicNumber}\left[n,\frac{1}{2}+i x\right]\right)\Big/\left(n^{\frac{1}{2}-i x}\left(\frac{1}{2}+i x\right)-2^{\frac{1}{2}+i x}n^{\frac{1}{2}+i x}\pi^{-\frac{1}{2}+i x}\left(\frac{1}{2}-i x\right)\text{Cos}\left[\frac{1}{2}\pi\left(\frac{1}{2}-i x\right)\right]\text{Gamma}\left[\frac{1}{2}-i x\right]\right)$$

$$\text{pa11c[n\_, x\_] := }\left(n^{\frac{1}{2}+i x}\left(\frac{1}{2}-i x\right)\text{HarmonicNumber}\left[n,\frac{1}{2}+i x\right]-n^{\frac{1}{2}-i x}\left(\frac{1}{2}+i x\right)\text{HarmonicNumber}\left[n,\frac{1}{2}-i x\right]\right)\Big/\left(n^{\frac{1}{2}-i x}\left(\frac{1}{2}-i x\right)-2^{\frac{1}{2}+i x}n^{\frac{1}{2}+i x}\pi^{-\frac{1}{2}+i x}\left(\frac{1}{2}+i x\right)\text{Cos}\left[\frac{1}{2}\pi\left(\frac{1}{2}-i x\right)\right]\text{Gamma}\left[\frac{1}{2}-i x\right]\right)$$

$$\text{pa11d[n\_, x\_] := }\left(n^{\frac{1}{2}-i x}\left(\frac{1}{2}I-x\right)\text{HarmonicNumber}\left[n,\frac{1}{2}-i x\right]-n^{\frac{1}{2}+i x}\left(\frac{1}{2}I+x\right)\text{HarmonicNumber}\left[n,\frac{1}{2}+i x\right]\right)\Big/\left(n^{\frac{1}{2}-i x}\left(\frac{1}{2}I-x\right)-2^{\frac{1}{2}+i x}n^{\frac{1}{2}+i x}\pi^{-\frac{1}{2}+i x}\left(\frac{1}{2}I+x\right)\text{Cos}\left[\frac{1}{2}\pi\left(\frac{1}{2}-i x\right)\right]\text{Gamma}\left[\frac{1}{2}-i x\right]\right)$$

```
zeta[n_, s_] := pa11d[n, (s I - .5 I)]
pa11da[n_, x_] :=
```

$$n^{\frac{1}{2}-i x}\left(\frac{1}{2}I-x\right)\text{HarmonicNumber}\left[n,\frac{1}{2}-i x\right]-n^{\frac{1}{2}+i x}\left(\frac{1}{2}I+x\right)\text{HarmonicNumber}\left[n,\frac{1}{2}+i x\right]$$

```
pa11e[n_, x_] :=
```

$$\left(n^{\frac{1}{2}+i x}\left(\frac{1}{2}I+x\right)\text{HarmonicNumber}\left[n,\frac{1}{2}+i x\right]-n^{\frac{1}{2}-i x}\left(\frac{1}{2}I-x\right)\text{HarmonicNumber}\left[n,\frac{1}{2}-i x\right]\right)\Big/\left(2^{\frac{1}{2}+i x}n^{\frac{1}{2}+i x}\pi^{-\frac{1}{2}+i x}\left(\frac{1}{2}I+x\right)\text{Cos}\left[\frac{1}{2}\pi\left(\frac{1}{2}-i x\right)\right]\text{Gamma}\left[\frac{1}{2}-i x\right]-n^{\frac{1}{2}-i x}\left(\frac{1}{2}I-x\right)\right)$$

```
zetae[n_, s_] := pa11e[n, (s - .5) I]
pa11ea[n_, x_] :=
```

$$n^{\frac{1}{2}+i x}\left(\frac{1}{2}I+x\right)\text{HarmonicNumber}\left[n,\frac{1}{2}+i x\right]-n^{\frac{1}{2}-i x}\left(\frac{1}{2}I-x\right)\text{HarmonicNumber}\left[n,\frac{1}{2}-i x\right]$$

```
pa11a[n, 1 / 2 - x I]
```

$$\left(n^{\frac{1}{2}-i x}\left(\frac{1}{2}+i x\right)\text{HarmonicNumber}\left[n,\frac{1}{2}-i x\right]-n^{\frac{1}{2}+i x}\left(\frac{1}{2}-i x\right)\text{HarmonicNumber}\left[n,\frac{1}{2}+i x\right]\right)\Big/\left(n^{\frac{1}{2}-i x}\left(\frac{1}{2}+i x\right)-2^{\frac{1}{2}+i x}n^{\frac{1}{2}+i x}\pi^{-\frac{1}{2}+i x}\left(\frac{1}{2}-i x\right)\text{Cos}\left[\frac{1}{2}\pi\left(\frac{1}{2}-i x\right)\right]\text{Gamma}\left[\frac{1}{2}-i x\right]\right)$$

Zeta[.8 + 3 I]

0.590541 − 0.0980708 ⅈ

pa11d[1 000 000 000, .9 I − .5 I − 3]

0.609764 − 0.103129 ⅈ

Zeta[.55 + 113 I]

1.4668 + 0.67276 ⅈ

pa11a[n, s]

$$\frac{-n^{1-s}\, s\, \text{HarmonicNumber}[n, 1-s] + n^{s}\,(1-s)\, \text{HarmonicNumber}[n, s]}{n^{s}\,(1-s) - 2^{1-s}\, n^{1-s}\, \pi^{-s}\, s\, \text{Cos}\left[\frac{\pi s}{2}\right] \text{Gamma}[s]}$$

zetae[1 000 000 000, .55 + 113 I]

1.4668 + 0.672773 ⅈ

pa11ea[100 000, N@Im@ZetaZero@11]

52.9703 + 0. ⅈ

N@Im@ZetaZero@11

52.9703

FullSimplify$\left[2^{\frac{1}{2}+i x}\, n^{\frac{1}{2}+i x}\, \pi^{-\frac{1}{2}+i x}\right]$

$2^{\frac{1}{2}+i x}\, n^{\frac{1}{2}+i x}\, \pi^{-\frac{1}{2}+i x}$

pa11f[n_, x_] :=

$$\left(n^{i x}\left(\frac{1}{2} I + x\right) \text{HarmonicNumber}\left[n, \frac{1}{2} + i x\right] - n^{-i x}\left(\frac{1}{2} I - x\right) \text{HarmonicNumber}\left[n, \frac{1}{2} - i x\right]\right) \Big/$$
$$\left(2^{\frac{1}{2}+i x}\, n^{i x}\, \pi^{-\frac{1}{2}+i x}\left(\frac{1}{2} I + x\right) \text{Sin}[Pi / 4 + 2\, Pi\, I\, x / 4]\, \text{Gamma}\left[\frac{1}{2} - i x\right] - n^{-i x}\left(\frac{1}{2} I - x\right)\right)$$

pa11f2[n_, x_] := $n^{i x}\left(\frac{1}{2} I + x\right) \text{HarmonicNumber}\left[n, \frac{1}{2} + i x\right] -$
$n^{-i x}\left(\frac{1}{2} I - x\right) \text{HarmonicNumber}\left[n, \frac{1}{2} - i x\right]$

zetaf[n_, s_] := pa11f[n, (s − 1 / 2) I]

zetaf[1 000 000, .77 + 44 I]

0.466395 + 1.07449 ⅈ

Zeta[.77 + 44 I]

0.466395 + 1.07447 ⅈ

FullSimplify@Cos$\left[\frac{1}{2}\pi\left(\frac{1}{2} - i x\right)\right]$

$\text{Sin}\left[\frac{1}{4}(\pi + 2\, i\, \pi\, x)\right]$

FullSimplify[$\left(2^{\frac{1}{2}+i\,x}\, n^{i\,x}\, \pi^{-\frac{1}{2}+i\,x}\right)$ ^ (- I x)]

$$\left(2^{\frac{1}{2}+i\,x}\, n^{i\,x}\, \pi^{-\frac{1}{2}+i\,x}\right)^{-i\,x}$$

```
2^(1/2 + I x) / 2^(I x)
```

$\sqrt{2}$

```
Plot[Im@pa11f2[n, N@Im@ZetaZero@1 + .2 I], {n, 1, 1000000}]
```