$$D_z(n)=L_{-z}(\log n)-\sum_{k=0}^{\infty}\binom{z}{k}\left(\int_1^{\infty}\frac{\partial}{\partial y}(y^{-k}D_{k,y+1}(n y^k))dy\right)$$

$$D_z(n)=L_{-z}(\log n)-\int_1^{\infty}\frac{\partial}{\partial y}\left(\sum_{k=0}^{\infty}\binom{z}{k}y^{-k}D_{k,y+1}(n y^k)\right)dy$$

$$D_z(n)=L_{-z}(\log n)-\int_1^{\infty}\frac{\partial}{\partial y}\left(\binom{z}{0}+\binom{z}{1}y^{-1}D_{1,y+1}(n y)+\binom{z}{2}y^{-2}D_{2,y+1}(n y^2)+\binom{z}{3}y^{-3}D_{3,y+1}(n y^3)+\ldots\right)dy$$

$$\begin{gathered}\binom{z}{0}+\\ \binom{z}{1}y^{-1}D_{1,y+1}(n y)+\\ \binom{z}{2}y^{-2}D_{2,y+1}(n y^2)+\\ \binom{z}{3}y^{-3}D_{3,y+1}(n y^3)+\\ \binom{z}{4}y^{-4}D_{4,y+1}(n y^4)+\\ \binom{z}{5}y^{-5}D_{5,y+1}(n y^5)+\ldots\end{gathered}$$

$$\begin{aligned}&1+\\ &z\,y^{-1}D_{1,y+1}(n y)+\\ &\frac{z(z-1)}{2}y^{-2}D_{2,y+1}(n y^2)+\\ &\frac{z(z-1)(z-2)}{6}y^{-3}D_{3,y+1}(n y^3)+\\ &\frac{z(z-1)(z-2)(z-3)}{24}y^{-4}D_{4,y+1}(n y^4)+\\ &\frac{z(z-1)(z-2)(z-3)(z-4)}{120}y^{-5}D_{5,y+1}(n y^5)+\ldots\end{aligned}$$

$$D_{0,y}(x)=1;\; D_{k,y}(x)=\sum_{j=0}^{\lfloor x-y\rfloor}D_{k-1,y}\left(\frac{x}{j+y}\right)$$

$$D_{1,y}(x)=\sum_{j=0}^{\lfloor x-y\rfloor}1=\lfloor x-y\rfloor+1$$

$$D_{2,y}(x)=\sum_{j=0}^{\lfloor x-y\rfloor}\sum_{k=0}^{\lfloor \frac{x}{j+y}-y\rfloor}1$$

$$D_{3,y}(x)=\sum_{j=0}^{\lfloor x-y\rfloor}\sum_{k=0}^{\lfloor \frac{x}{j+y}-y\rfloor}\sum_{l=0}^{\lfloor \frac{x}{(j+y)(k+y)}-y\rfloor} 1$$

$$D_{4,y}(x)=\sum_{j=0}^{\lfloor x-y\rfloor}\sum_{k=0}^{\lfloor \frac{x}{j+y}-y\rfloor}\sum_{l=0}^{\lfloor \frac{x}{(j+y)(k+y)}-y\rfloor}\sum_{m=0}^{\lfloor \frac{x}{(j+y)(k+y)(l+y)}-y\rfloor} 1$$

$$D_{5,y}(x)=\sum_{j=0}^{\lfloor x-y\rfloor}\sum_{k=0}^{\lfloor \frac{x}{j+y}-y\rfloor}\sum_{l=0}^{\lfloor \frac{x}{(j+y)(k+y)}-y\rfloor}\sum_{m=0}^{\lfloor \frac{x}{(j+y)(k+y)(m+y)}-y\rfloor}\sum_{o=0}^{\lfloor \frac{x}{(j+y)(k+y)(l+y)(m+y)}-y\rfloor} 1$$

$$\begin{aligned}
&1+\\
&z\,y^{-1}D_{1,y+1}(n\,y)+\\
&\frac{z(z-1)}{2}y^{-2}D_{2,y+1}(n\,y^2)+\\
&\frac{z(z-1)(z-2)}{6}y^{-3}D_{3,y+1}(n\,y^3)+\\
&\frac{z(z-1)(z-2)(z-3)}{24}y^{-4}D_{4,y+1}(n\,y^4)+\\
&\frac{z(z-1)(z-2)(z-3)(z-4)}{120}y^{-5}D_{5,y+1}(n\,y^5)\\
&+\ldots
\end{aligned}$$

$$\Pi(n)=li(n)-\log\log n-\gamma-\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\int_{1}^{\infty}\frac{\partial}{\partial y}(y^{-k}D_{k,y+1}(n y^{k}))\,dy$$

$$\Pi(n)=li(n)-\log\log n-\gamma-\int_{1}^{\infty}\frac{\partial}{\partial y}(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}y^{-k}D_{k,y+1}(n y^{k}))\,dy$$

$$\boxed{\begin{array}{l} y^{-1}D_{1,y+1}(n y)+ \\ -\frac{1}{2}y^{-2}D_{2,y+1}(n y^{2})+ \\ \frac{1}{3}y^{-3}D_{3,y+1}(n y^{3})+ \\ -\frac{1}{4}y^{-4}D_{4,y+1}(n y^{4})+ \\ \frac{1}{5}y^{-5}D_{5,y+1}(n y^{5}) \\ +\ldots \end{array}}$$

$$\boxed{C_{0,y}(x)=1\,;\ C_{k,y}(x)=\frac{1}{y}\sum_{j=0}^{\lfloor y x-y-1\rfloor}C_{k-1,y}(\frac{y x}{j+y+1})}$$

$$\boxed{C_{k,y}(x)=\frac{1}{y}\sum_{j=0}^{\lfloor y x-y-1\rfloor}\frac{1}{k}-C_{k-1,y}(\frac{y x}{j+y+1})}$$

$$\boxed{C_{0,y}(x)=1\,;\ C_{k,y}(x)=\frac{1}{y}\sum_{j=1}^{\lfloor x y-y\rfloor}C_{k-1,y}(\frac{x y}{j+y})}$$

$$\Pi(n)=li(n)-\log\log n-\gamma-\int_1^\infty \frac{\partial}{\partial y}\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}C_{k,y}(n)\right)dy$$

$$\boxed{P_{k,y}(x)=\frac{1}{y}\sum_{j=1}^{\lfloor xy-y\rfloor}\frac{1}{k}-P_{k+1,y}\left(\frac{xy}{j+y}\right)}$$

$$\Pi(n)=li(n)-\log\log n-\gamma-\int_1^\infty \frac{\partial}{\partial y}P_{1,y}(n)\,dy$$

$$P_{k,y}(x)=$$

$$\frac{1}{y}\sum_{j=1}^{\lfloor xy-y\rfloor}1-$$

$$\frac{1}{2y^2}\sum_{j=1}\sum_{k=1}^{\lfloor\frac{xy^2}{j+y}-y\rfloor}1+$$

$$\frac{1}{3y^3}\sum_{j=1}\sum_{k=1}\sum_{l=1}^{\lfloor\frac{xy^3}{(j+y)(k+y)}-y\rfloor}1-$$

$$\frac{1}{4y^4}\sum_{j=1}\sum_{k=1}\sum_{l=1}\sum_{m=1}^{\lfloor\frac{xy^4}{(j+y)(k+y)(l+y)}-y\rfloor}1+\ldots$$

$$4!\sum_{j=1}y^{-1}\sum_{k=j+1}y^{-1}\sum_{l=k+1}y^{-1}\sum_{m=l+1}^{\lfloor\frac{x}{(\frac{j}{y}+1)(\frac{k}{y}+1)(\frac{l}{y}+1)}-y\rfloor}y^{-1}$$

$$F_{k,a}(n)=0 \text{ when } n^{\frac{1}{k}}<a$$

$$F_{k,a}(n)=0 \text{ when } n^{\frac{1}{k}}<a$$

$$F_{k,a}(n)=0 \text{ when } n<\left(\frac{a}{y}+1\right)^k$$

$$F_{k,a}(n)=0 \text{ when } y(n^{\frac{1}{k}}-1)<a$$

$$\boxed{C_{0,y}(x)=1;\ C_{k,y}(x)=\frac{1}{y}\sum_{j=1}^{\lfloor xy-y\rfloor}C_{k-1,y}\left(\frac{xy}{j+y}\right)}$$

```
Dd[x_,0,y_]:=1
Dd[x_,k_,y_]:=Sum[Dd[x/(j+y),k-1,y],{j,0,Floor[x-y]}]
Cc[x_,k_,y_]:=y^-k Dd[x y^k,k,y+1]
FAlt[n_,0,a_, y_]:=1
FAlt[n_,k_,a_, y_]:=If[n<(a/y+1)^k,0,FAlt[n,k,a+1, y]+Sum[ y^-j  Binomial[k,j] FAlt[n/(a/y+1)^j,k-j,a+1, y],{j,1,k}]]
F2Alt[n_,0,a_, y_]:=1
F2Alt[n_,k_,a_, y_]:=Sum[ y^-j  Binomial[k,j] F2Alt[n/(m/y+1)^j,k-j,m+1, y],{m,a,Floor[ y(n^(1/k)-1)]},{j,1,k}]
```

$$C_{0,y}(x)=1\,;\; C_{k,y}(x)=y^{-1}\sum_{j=1}^{\lfloor xy-y\rfloor} C_{k-1,y}\left(\frac{x\,y}{j+y}\right)$$

$$C_{k,a,y}(n)=\sum_{j=1}^{k} y^{-j}\binom{k}{j}\sum_{m=a}^{\lfloor (x^{\frac{1}{k}})y-y\rfloor} C_{k-j,m+1,y}\left(x\cdot\left(1+\frac{m}{y}\right)^{-j}\right)$$

$$C_{1,a,y}(x)=y^{-1}\lfloor (x-1)y-a+1\rfloor$$

$$C_{0,a,y}(x)=1$$

$$\boxed{D_{k,2}(x)=(-1)^k\left(1-\frac{\Gamma(k,-\log x)}{\Gamma(k)}\right)-\int_1^\infty \frac{\partial}{\partial y}C_{k,y}(x)\,dy}$$

$$\boxed{D_z(n)=L_{-z}(\log n)-\int_1^\infty \frac{\partial}{\partial y}\left(\sum_{k=0}^{\lfloor\frac{\log n}{\log(y+1)-\log y}\rfloor}\binom{z}{k}C_{k,y}(n)\right)dy}$$

$$\Pi(n)=li(n)-\log\log n-\gamma-\int_1^\infty \frac{\partial}{\partial y}\left(\sum_{k=1}^{\lfloor\frac{\log n}{\log(y+1)-\log y}\rfloor}\frac{(-1)^{k-1}}{k}C_{k,y}(n)\right)dy$$

$$\boxed{C_{z,k,y}(x) = \frac{z-k+1}{y\,k} \sum_{j=1}^{\lfloor xy-y \rfloor} 1 + C_{z,k+1,y}\left(\frac{x\,y}{j+y}\right)}$$

$$\boxed{C_{z,k}(n) = \frac{z-k+1}{k} \sum_{j=2}^{\lfloor n \rfloor} 1 + C_{z,k+1}\left(\frac{n}{j}\right);\, D_z(n) = 1 + C_{z,1}(n)}$$

$$\boxed{D_z(n) = L_{-z}(\log n) - \int_1^{\infty} \frac{\partial}{\partial\, y}\, 1 + C_{z,1,y}(n)\, dy}$$

$$C_0(x,y)=1;\ C_k(x,y)=C_{k-1}(x,y)+y^{-1}\sum_{j=1}^{\lfloor xy-y\rfloor} C_{k-1}\left(\frac{xy}{j+y},y\right)$$

$$C_1(x,y)=1+y^{-1}\sum_{j=1}^{\lfloor xy-y\rfloor} 1$$

$$C_2(x,y)=1+y^{-1}\sum_{j=1}^{\lfloor xy-y\rfloor} 2+y^{-1}\sum_{k=1}^{\lfloor \frac{xy^2}{j+y}-y\rfloor} 1$$

$$C_3(x,y)=1+y^{-1}\sum_{j=1}^{\lfloor xy-y\rfloor} 3+y^{-1}\sum_{k=1}^{\lfloor \frac{xy^2}{j+y}-y\rfloor} 3+y^{-1}\sum_{l=1}^{\lfloor \frac{xy^3}{(j+y)(k+y)}-y\rfloor} 1$$

$$C_0(x,y)=1;\ C_k(x,y)=(1-y^{-1})C_{k-1}(x,y)+y^{-1}\sum_{j=0}^{\lfloor xy-y\rfloor} C_{k-1}\left(\frac{xy}{j+y},y\right)$$

$$C_1(x,y)=y^{-1}\sum_{j=0}^{\lfloor xy-y\rfloor} 1-(y^{-1}-1)$$

$$C_3(x,y)=(1-y^{-1})^3 D_0(x)+3(1-y^{-1})^2 y^{-1} D_1(xy)+3(1-y^{-1})y^{-2} D_2(xy^2)+y^{-3} D_3(xy^3)$$