2. $[\zeta(s)^z]_n$: Partial Sums of the Zeta Function and Exponential-Style Dirichlet Convolutions

So this chapter collects a number of ways of expressing $[\zeta(s)^z]_n$, which, remember, is the partial sum of the zeta function, with an exponential-style Dirichlet convolution to some arbitrary complex value z.

Later chapters detail in greater length some consequences of this.

```
ri[]:=RandomInteger[{1,40}]; rr[]:=RandomReal[{-5,5}]+RandomReal[{-5,5}]I
bin[ z_, k_ ] := Product[ z-j, { j, 0, k-1 } ]/k!
FI[n_]:=FactorInteger[n];FI[1]:={}
dz[n_,s_,z_]:=n^-s Product[(-1)^p[[2]]Binomial[-z,p[[2]]],{p,FI[n]}]
referencezeta[n_,s_, z_]:=Sum[dz[j,s,z],{j,1,n}]
```

2.1 Using the Generalized Divisor Function $d_z(n)$ to define $[\zeta(s)^z]_n$

Our main goal is to show the degree to which power series ideas can be applied to partial sums of the zeta function. It's well-known that the binomial coefficients can handle complex arguments when defined like so:

$$\binom{z}{k}=\frac{z(z-1)...(z-k+1)}{k!} \tag{2.1.1}$$

```
ri[]:=RandomInteger[{1,40}]; rr[]:=RandomReal[{-5,5}]+RandomReal[{-5,5}]I
bin[ z_, k_ ] := Product[ z-j, { j, 0, k-1 } ]/k!
Table[ Chop[Binomial[ v = rr[], u = ri[] ] - bin[ v, u ]], {j,0,300}]
```

Using (2.1.1), we can take an idea from pg. 421 of A. Ivic's “The Riemann Zeta-Function: Theory and Applications”, and define $[\zeta(s)^z]_n$ in the following way.

First, if we have some number *n* prime factored as $n=\prod_{p^k|n} p^k$, the number of divisors function $d_z(n)$ from (1.3) can be defined as

$$d_z(n)=\prod_{p^k|n}\frac{z^{(k)}}{k!}$$

```
dk[ n_, k_ ] := Sum[ dk[ j, k-1 ] dk[ n/j, 1 ], { j, Divisors[ n ] } ]
dk[ n_, 1 ] := 1;dk[ n_, 0 ] := 0;dk[ 1, 0 ] := 1
FI[ n_ ] := FactorInteger[ n ];FI[ 1 ] := { }
dz[ n_, z_ ] := Product[ (-1)^p[ [ 2 ] ] Binomial[ -z, p[ [ 2 ] ] ], { p, FI[ n ] } ]
Grid[ Table[ dk[ n, k ]- dz[ n, k ] , { n, 1, 100 }, { k, 1, 10 } ] ]
```

So what can we do with $d_z(n)$? Well, if we define $[\nabla\zeta(s)^z]n$ as

$$[\nabla\zeta(s)^z]_n=[\zeta(s)^z]_n-[\zeta(s)^z]_{n-1}$$

we find that

$$[\nabla\zeta(s)^z]_n=n^{-s}d_z(n)$$

Which is to say, if we want to compute $[\zeta(s)^z]_n$, one approach using known techniques is

$$[\zeta(s)^z]_n=\sum_{j=1}^{n}[\nabla\zeta(s)^z]_n$$
$$[\zeta(s)^z]_n=\sum_{j=1}^{n}\frac{d_z(j)}{j^s}$$
$$[\zeta(s)^z]_n=\sum_{j=1}^{n} j^{-s}\prod_{p^k|j}\frac{z^{(k)}}{k!}$$

```
FI[ n_ ] := FactorInteger[ n ];FI[ 1 ] := { }
dzeta[ j_, s_, z_ ] := j^-s Product[ (-1)^p[ [ 2 ] ] Binomial[ -z, p[ [ 2 ] ] ], { p, FI[ j ] } ]
zeta[ n_, s_, z_ ] := Sum[ dzeta[ j, s, z ], { j, 1, n } ]
Grid[ Table[ zeta[ 100, 0, s+t I ], { s, -1.3, 4, .7 }, { t, -1.3, 4, .7 } ] ]
```

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\zeta(s)^z=\sum_{j=1}^{\infty}\frac{d_z(j)}{j^s}$$

So, on the one hand, this approach to computing $[\zeta(s)^z]_n$ works. On the other hand, it requires factoring all numbers from 1 to n, which in most cases is difficult and unenlightening, compared to some alternatives we'll cover below that are, essentially, the convolution equivalents of power series.

2.2 $[\zeta(s)^z]_n$ from $[(\zeta(s)-1)^k]_n$

For example, we can express $[\zeta(s)^z]_n$ with $[(\zeta(s)-1)^k]_n=\sum_{j=1}(j+1)^{-s}\cdot[(\zeta(s)-1)^k]_{n(j+1)^{-1}}$ from (1.4) as

$$\boxed{[\zeta(s)^z]_n=\sum_{k=0}^{\infty}\binom{z}{k}[(\zeta(s)-1)^k]_n}$$

```
ri[]:=RandomInteger[{10,100}]; rr[]:=RandomReal[{-4,4}]+RandomReal[{-4,4}]I
FI[n_]:=FactorInteger[n];FI[1]:={}
dzeta[n_,s_,z_]:=n^-s Product[(-1)^p[[2]]Binomial[-z,p[[2]]],{p,FI[n]}]
zeta[n_,s_, z_]:=Sum[dzeta[j,s,z],{j,1,n}]
zetam1[n_,s_,0]:=UnitStep[n-1]
zetam1[n_,s_,k_]:=Sum[j^-s zetam1[n/j,s,k-1],{j,2,n}]
zetaalt[n_,s_ ,z_]:=Sum[Binomial[z,k] zetam1[n,s,k],{k,0,Log[2,n]}]
Table[Chop[zeta[a=ri[],s=rr[],t=rr[]]-zetaalt[a,s,t]],{j,0,300}]
```

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\zeta(s)^z=\sum_{k=0}^{\infty}\binom{z}{k}(\zeta(s)-1)^k$$

```
{Zeta[s]^z,Sum[Binomial[z,k] (Zeta[s]-1)^k,{k,0,Infinity}]}
```

This can be generalized as

$$\boxed{[f^z]_n=\sum_{k=0}^{\infty}\binom{z}{k}[(f-1)^k]_n}$$

Treating z as a complex, continuous variable opens up a lot of possibilities. It means that z can take negative values. We can apply limits to z. We can take derivatives with respect to it. (D1) might have zeros for z that contain valuable information. And so on.

It's particularly useful because $[\zeta(0)^{-1}]_n$ is $M(n)$, the Mertens function, and $\lim_{z\to 0}\frac{[\zeta(0)^z]_n-1}{z}$ is $\Pi(n)$, the Riemann Prime counting function. Hence, observations we make generally about $[\zeta(s)^z]_n$ might have interesting implications for those two functions.

2.3 $[\zeta(s)^z]_n$ as Explicit Sums

(2.1.1) expresses $[\zeta(s)^z]_n$ concisely and usefully, but it is interesting to write out the idea more explicitly. Doing so gives us

$$[\zeta(s)^z]_n = \binom{z}{0}1+\binom{z}{1}\sum_{j=2}^{n} j^{-s}+\binom{z}{2}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor \frac{n}{j}\rfloor}(j\cdot k)^{-s}+\binom{z}{3}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor \frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor \frac{n}{j\cdot k}\rfloor}(j\cdot k\cdot l)^{-s}+\ldots \tag{2.3.1}$$

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\begin{aligned}\zeta(s)^z &= \binom{z}{0}1+\binom{z}{1}\sum_{j=2}^{\infty} j^{-s}+\binom{z}{2}\sum_{j=2}^{\infty}\sum_{k=2}^{\infty}(j\cdot k)^{-s}+\binom{z}{3}\sum_{j=2}^{\infty}\sum_{k=2}^{\infty}\sum_{l=2}^{\infty}(j\cdot k\cdot l)^{-s}+\ldots = \\ &\binom{z}{0}(\zeta(s)-1)^0+\binom{z}{1}(\zeta(s)-1)^1+\binom{z}{2}(\zeta(s)-1)^2+\binom{z}{3}(\zeta(s)-1)^3+\ldots\end{aligned} \tag{2.3.2}$$

This can be generalized as

$$[f^z]_n = 1+\binom{z}{1}\sum_{j=2}^{\lfloor n\rfloor} f(j)+\binom{z}{2}\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor \frac{n}{j}\rfloor} f(j)f(k)+\binom{z}{3}\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor \frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor \frac{n}{j\cdot k}\rfloor} f(j)f(k)f(l)+\ldots \tag{2.3.3}$$

2.4 Recursive Expressions for $[\zeta(s)^z]_n$

Although it might not be obvious on casual inspection, (2.3.1) can be recursively as

$$\boxed{[\zeta(s)^z]_n = 1+f_1(n,2) \quad \text{where} \quad f_k(n,j)=\begin{cases} j^{-s}\left(\frac{z+1}{k}-1\right)\left(1+f_{k+1}\left(\frac{n}{j},2\right)\right)+f_k(n,j+1) & \text{if } n\ge j \\ 0 & \text{otherwise}\end{cases}}$$

```
ri[]:=RandomInteger[{10,100}]; rr[]:=RandomReal[{-4,4}]+RandomReal[{-4,4}]I
FI[n_]:=FactorInteger[n];FI[1]:={}
dzeta[n_,s_,z_]:=n^-s Product[(-1)^p[[2]]Binomial[-z,p[[2]]],{p,FI[n]}]
zeta[n_,s_, z_]:=Sum[dzeta[j,s,z],{j,1,n}]
F[ n_, s_, j_, k_, z_] := If[ n<j,0, j^-s((z+1)/k-1)(1+F[ n/j, s, 2, k+1, z]) + F[ n, s, j+1, k, z]]
zetaalt[ n_, s_, z_] := 1 + F[ n, s, 2, 1, z ]
Table[ Chop[zeta[ a=ri[], b=rr[], c=rr[] ]- zetaalt[ a, b, c ]], { j,1,100} ]
```

or, most compactly, as

$$\boxed{[\zeta(s)^z]_n = f_1(n) \quad \text{where} \quad f_k(n)=1+\left(\frac{z+1}{k}-1\right)\sum_{j=2}^{\lfloor n\rfloor} j^{-s}\cdot f_{k+1}\left(\frac{n}{j}\right)}$$

```
ri[]:=RandomInteger[{10,100}]; rr[]:=RandomReal[{-4,4}]+RandomReal[{-4,4}]I
FI[n_]:=FactorInteger[n];FI[1]:={}
```

```
dzeta[n_,s_,z_]:=n^-s Product[(-1)^p[[2]]Binomial[-z,p[[2]]],{p,FI[n]}]
zeta[n_,s_, z_]:=Sum[dzeta[j,s,z],{j,1,n}]
F[n_,s_,z_,k_]:=1+((z+1)/k-1) Sum[j^-s F[n/j,s,z,k+1],{j,2,n}]
Table[ Chop[zeta[ a=ri[], b=rr[], c=rr[] ]- F[ a, b, c, 1 ]], { j,1,100} ]
```

Compare this latter expression to

$$\zeta(s)^z = f_1 \; where \; f_k = 1 + (\frac{z+1}{k} - 1)(\zeta(s) - 1) f_{k+1}$$

```
F[k_,z_, s_, t_] := If[t>200,0,1+((z+1)/k-1)(N[Zeta[s]]-1) F[k+1,z,s,t+1]]
Table[Chop[F[1,z,s,1]-Zeta[s]^z],{s,2,8,.7},{z,-3,4,.4}]
```

These can both be generalized to

$$[f^z]_n = 1 + f_1(n,2) \quad \text{where} \quad f_k(n,j) = \begin{cases} f(j)(\frac{z+1}{k} - 1)(1 + f_{k+1}(\frac{n}{j}, 2)) + f_k(n, j+1) & \text{if } n \ge j \\ 0 & \text{otherwise} \end{cases}$$

and

$$[f^z]_n = f_1(n) \quad \text{where} \quad f_k(n,z) = 1 + (\frac{z+1}{k} - 1) \sum_{j=2}^{\lfloor n \rfloor} f(j) f_{k+1}(\frac{n}{j})$$

2.5 Variations of the Identity from 2.4

(D1) has a few variants. Here's another equation for $[\zeta(s)^z]_n$, this time with $\mu(n)$ the Moebius function:

$$[(\zeta(s)^{-1} - 1)^k]_n = \sum_{j=1} (j+1)^{-s} \mu(j+1) [(\zeta(s)^{-1} - 1)^{k-1}]_{n(j+1)^{-1}}$$

$$[\zeta(s)^z]_n = \sum_{k=0}^{\infty} \binom{-z}{k} [(\zeta(s)^{-1} - 1)^k]_n \tag{2.5.1}$$

```
ri[]:=RandomInteger[{10,100}]; rr[]:=RandomReal[{-4,4}]+RandomReal[{-4,4}]I
zeta[n_,s_,z_,k_]:=1+((z+1)/k-1) Sum[j^-s zeta[n/j,s,z,k+1],{j,2,n}]
Mm1[ n_, s_, 0 ] := UnitStep[n-1]
Mm1[ n_, s_, k_ ] := Sum[ j^-s MoebiusMu[ j ] Mm1[ n/j, s, k-1 ], { j, 2, n } ]
zetaalt[ n_, s_, z_ ] := Sum[ Binomial[ -z, k ] Mm1[ n, s, k ], { k, 0, Log[ 2, n ] } ]
Table[ Chop[zeta[ a=ri[], b=rr[], c=rr[],1 ]- zetaalt[ a, b, c ]] , { j, 1, 100 } ]
```

The limit of this as n approaches infinity, if $\Re(s) > 1$, is

$$\zeta(s)^z = \sum_{k=0}^{\infty} \binom{-z}{k} (\zeta(s)^{-1} - 1)^k \tag{2.5.2}$$

```
Table[{Zeta[s]^z,Sum[Binomial[-z,k] (Zeta[s]^(-1)-1)^k,{k,0,Infinity}]},{z,-3,10}]//TableForm
```

These ideas generalize to

$$[(\zeta_n(s)^a-1)^k]=\sum_{j=1}(j+1)^{-s}(d_a(j+1))[(\zeta(s)^a-1)^{k-1}]_{n(j+1)^{-1}}$$

$$[\zeta_n(s)^z]=\sum_{k=0}^{\infty}\binom{z/a}{k}[(\zeta_n(s)^a-1)^k]$$

(2.5.3)

```
ri[]:=RandomInteger[{10,100}]; rr[]:=RandomReal[{-4,4}]+RandomReal[{-4,4}]I
zeta[n_,s_,z_,k_]:=1+((z+1)/k-1) Sum[j^-s zeta[n/j,s,z,k+1],{j,2,n}]
FI[n_]:=FactorInteger[n];FI[1]:={}
dz[n_,z_]:=Product[(-1)^p[[2]]Binomial[-z,p[[2]]],{p,FI[n]}]
Am1[ n_, s_, a_, 0 ] := UnitStep[n-1]
Am1[ n_, s_, a_, k_ ] := Sum[ j^-s dz[ j, a ] Am1[ n/j, s, a, k-1 ], { j, 2, n } ]
zetaalt[ n_, s_, z_, a_ ] := Sum[ Binomial[ z/a, k ] Am1[ n, s, a, k ], { k, 0, Log[ 2, n ] } ]
Table[ Chop[zeta[ a=ri[], b=rr[], c=rr[],1 ]-zetaalt[ a, b, c, rr[] ]], { j, 1, 100 } ]
```

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\zeta(s)^z=\sum_{k=0}^{\infty}\binom{z/a}{k}(\zeta(s)^a-1)^k$$

(2.5.4)

```
Grid[Table[{Zeta[s]^z,Sum[Binomial[z/a,k] (Zeta[s]^a-1)^k,{k,0,Infinity}]},{z,-3,10},{a,1,6}]]
```

2.6 $[\zeta(s)^z]_n$ from $[(\log\zeta(s))^k]_n$

(D1), above, is one of two very important ways to express $[\zeta(s)^z]_n$ with z a complex variable. The second way, in terms of

$$[(\log\zeta(s))^k]_n=\sum_{j=2} j^{-s}\,\kappa(j)\cdot[(\log\zeta(s))^{k-1}]_{n\,j^{-1}}$$

(2.6.1)

from (1.6), is

$$[\zeta(s)^z]_n=\sum_{k=0}^{\infty}\frac{z^k}{k!}[(\log\zeta(s))^k]_n$$

(2.6.2)

```
ri[]:=RandomInteger[{10,100}]; rr[]:=RandomReal[{-4,4}]+RandomReal[{-4,4}]I
zeta[n_,s_,z_,k_]:=1+((z+1)/k-1) Sum[j^-s zeta[n/j,s,z,k+1],{j,2,n}]
logzeta[ n_, s_, 0 ] := UnitStep[ n -1 ]
logzeta[ n_, s_, k_ ] := Sum[ j^-s MangoldtLambda[ j ] / Log[ j ]logzeta[ n/j, s, k-1 ], { j, 2,  n } ]
zetaalt[ n_, s_, z_ ] := Sum[ z^k/k! logzeta[ n, s, k ], { k, 0, Log[ 2, n ] } ]
Table[ Chop[zeta[ a=ri[], b=rr[], c=rr[],1 ]-zetaalt[ a, b, c ] ], { j,1,100 } ]
```

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\zeta(s)^z=\sum_{k=0}^{\infty}\frac{z^k}{k!}(\log\zeta(s))^k$$

(2.6.3)

```
{ Zeta[s]^z, Sum[ z^k/k! Log[ Zeta[s] ]^k, { k, 0, Infinity } ] }
```

This generalizes to

$$\boxed{[f^z]_n = \sum_{k=0}^{\infty} \frac{z^k}{k!} [(\log f)^k]_n}$$ (2.6.4)

This can be written more explicitly as

$$[\zeta(s)^z]_n = 1 + \frac{z^1}{1!} \sum_{j=2}^{n} \kappa(j) j^{-s} + \frac{z^2}{2!} \sum_{j=2}^{n} \sum_{k=2}^{\lfloor \frac{n}{j} \rfloor} \kappa(j)\kappa(k)(j\cdot k)^{-s} + \frac{z^3}{3!} \sum_{j=2}^{n} \sum_{k=2}^{\lfloor \frac{n}{j} \rfloor} \sum_{l=2}^{\lfloor \frac{n}{j\cdot k} \rfloor} \kappa(j)\kappa(k)\kappa(l)(j\cdot k\cdot l)^{-s} + \frac{z^4}{4!} \ldots$$ (2.6.5)

The limit of this as n approaches infinity, if $\Re(s) > 1$, is

$$\zeta_n(s)^z = 1 + \frac{z^1}{1!} \sum_{j=2}^{\infty} \kappa(j) j^{-s} + \frac{z^2}{2!} \sum_{j=2}^{\infty} \sum_{k=2}^{\infty} \kappa(j)\kappa(k)(j\cdot k)^{-s} + \frac{z^3}{3!} \sum_{j=2}^{\infty} \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} \kappa(j)\kappa(k)\kappa(l)(j\cdot k\cdot l)^{-s} + \frac{z^4}{4!} \ldots$$ (2.6.6)

If f is a multiplicative function, this can be generalized as

$$[f^z]_n = 1 + \frac{z^1}{1!} \sum_{j=2}^{n} \kappa(j) f(j) + \frac{z^2}{2!} \sum_{j=2}^{n} \sum_{k=2}^{\lfloor \frac{n}{j} \rfloor} \kappa(j)\kappa(k) f(j) f(k) + \frac{z^3}{3!} \sum_{j=2}^{n} \sum_{k=2}^{\lfloor \frac{n}{j} \rfloor} \sum_{l=2}^{\lfloor \frac{n}{j\cdot k} \rfloor} \kappa(j)\kappa(k)\kappa(l) f(j) f(k) f(l) + \frac{z^4}{4!} \ldots$$ (2.6.7)

2.7 Variations of the Identity from 2.6

Take a look at (D3). Because z is a complex, continuous variable, we can take the derivative of $[\zeta(s)^z]_n$ with respect to z, giving us

$$\boxed{\frac{\partial}{\partial z} [\zeta(s)^z]_n = \sum_{k=0}^{\infty} \frac{z^k}{k!} [(\log \zeta(s))^{k+1}]_n}$$ (2.7.1)

```
ri[]:=RandomInteger[{10,100}];rr[]:=RandomReal[{-4,4}]+RandomReal[{-4,4}]I
zeta[n_,s_,z_,k_]:=1+((z+1)/k-1) Sum[j^-s zeta[n/j,s,z,k+1],{j,2,n}]
logzeta[n_,s_,0]:=UnitStep[n-1]
logzeta[n_,s_,k_]:=Sum[j^-s FullSimplify[MangoldtLambda[j]/Log[j]]logzeta[n/j,s,k-1],{j,2,n}]
zetaalt[n_,s_,z_]:=Sum[z^k/k! logzeta[n,s, k+1],{k,0,Log[2,n]-1}]
Table[Chop[Expand[D[zeta[a=ri[],b=rr[],z,1],z]]-zetaalt[a,b,z]],{n,1,100}]
```

The limit of this as n approaches infinity, if $\Re(s) > 1$, is

$$\frac{\partial}{\partial z} \zeta(s)^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} (\log \zeta(s))^{k+1}$$ (2.7.2)

```
{D[Zeta[s]^z,z],Sum[ z^k/k! Log[Zeta[s]]^(k+1),{k,0,Infinity}]}
```

It generalizes to

$$\boxed{\frac{\partial}{\partial z} [f^z]_n = \sum_{k=0}^{\infty} \frac{z^k}{k!} [(\log f)^{k+1}]_n}$$ (2.7.3)

$$\boxed{\frac{\partial}{\partial z}[\zeta(s)^z]_n=[\log\zeta(s)\cdot\zeta(s)^z]_n}$$

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\frac{\partial}{\partial z}\zeta(s)^z=\log\zeta(s)\zeta(s)^z$$

(2.7.2)

```
{D[Zeta[s]^z,z],Sum[ z^k/k! Log[Zeta[s]]^(k+1),{k,0,Infinity}]}
```

It generalizes to

$$\boxed{\frac{\partial}{\partial z}[f^z]_n=[\log f\cdot f^z]_n}$$

(2.7.3)

$$\boxed{[\zeta(s)^z]_n=1+\int_0^z[\log\zeta(s)\cdot\zeta(s)^y]_n\,dy}$$

$$(\boxed{D_z(n)=1+\sum_{j=2}^n\kappa(j)\int_0^z D_y(\frac{n}{j})dy}\quad\boxed{n=1+\sum_{j=1}^n\Pi(\frac{n}{j})\int_0^1 d_y(j)\,dy}\quad\boxed{n=1+\sum_{j=2}^n\kappa(j)\int_0^1 D_y(\frac{n}{j})\,dy})$$

and also

$$(\boxed{[\zeta(s)^z]_n=1+\int_0^z\sum_{k=0}^\infty\frac{y^k}{k!}[(\log\zeta(s))^{k+1}]_n\,dy})$$

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\zeta(s)^z=\int\log\zeta(s)\zeta(s)^z\,dz$$

(2.7.2)

```
{D[Zeta[s]^z,z],Sum[ z^k/k! Log[Zeta[s]]^(k+1),{k,0,Infinity}]}
```

It generalizes to

$$\boxed{[f^z]_n=\int[\log f\cdot f^z]_n dz}$$

(2.7.3)

In fact, we can take its derivative an infinite number of times, although, as a polynomial of finite terms, most of those derivatives will be 0.

$$\boxed{\frac{\partial^{\alpha}}{\partial z^{\alpha}}[\zeta(s)^{z}]_{n}=\sum_{k=0}^{\infty}\frac{z^{k}}{k!}[(\log\zeta(s))^{k+\alpha}]_{n}} \tag{2.7.4}$$

```
rr[]:=RandomReal[{-4,4}]+RandomReal[{-4,4}]I
zeta[n_,s_,z_,k_]:=1+((z+1)/k-1) Sum[j^-s zeta[n/j,s,z,k+1],{j,2,n}]
logzeta[n_,s_,0]:=UnitStep[n-1]
logzeta[n_,s_,k_]:=Sum[j^-s FullSimplify[MangoldtLambda[j]/Log[j]]logzeta[n/j,s,k-1],{j,2,n}]
zetaalt[n_,s_, z_,a_]:=Sum[z^k/k! logzeta[n,s,k+a],{k,0,Log[2,n]-a}]
Table[Chop[Expand[D[zeta[n,c=rr[],z,1],{z,a}]]-zetaalt[n,c,z,a]],{n,1,50},{a,0,5}]
```

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\frac{\partial^{\alpha}}{\partial z^{\alpha}}\zeta(s)^{z}=\sum_{k=0}^{\infty}\frac{z^{k}}{k!}(\log\zeta(s))^{k+\alpha} \tag{2.7.5}$$

```
Table[{D[Zeta[s]^z,{z,a}],Sum[ z^k/k! Log[Zeta[s]]^(k+a),{k,0,Infinity}]},{a,1,6}]//TableForm
```

It generalizes to

$$\boxed{\frac{\partial^{\alpha}}{\partial z^{\alpha}}[f^{z}]_{n}=\sum_{k=0}^{\infty}\frac{z^{k}}{k!}[(\log f)^{k+\alpha}]_{n}} \tag{2.7.6}$$

Using these derivatives, $[\zeta(s)^{z}]_{n}$, with respect to z, can be expressed as its Maclaurin series.

$$\boxed{[\zeta(s)^{z}]_{n}=\sum_{k=0}^{\infty}\frac{z^{k}}{k!}\lim_{y\to 0}\frac{\partial^{k}}{\partial y^{k}}[\zeta(s)^{y}]_{n}} \tag{2.7.7}$$

```
rr[]:=RandomReal[{-4,4}]+RandomReal[{-4,4}]I
zeta[n_,s_,z_,k_]:=1+((z+1)/k-1) Sum[j^-s zeta[n/j,s,z,k+1],{j,2,n}]
zetaalt[n_,s_,z_]:=Sum[z^k/k! (Limit[D[zeta[n,s,y,1],{y,k}],y->0]),{k,0,Log[2,n]}]
Table[Chop[zeta[a=87,b=rr[],s+t I,1]-zetaalt[a,b,s+t I]],{s,-1.5,4,.7},{t,-1.1,4,.7}]
```

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\zeta(s)^{z}=\sum_{k=0}^{\infty}\frac{z^{k}}{k!}\lim_{y\to 0}\frac{\partial^{k}}{\partial y^{k}}\zeta(s)^{y} \tag{2.7.8}$$

```
Grid[Table[Chop[Zeta[s]^z-N[Sum[z^k/k! (Limit[D[Zeta[s]^y,{y,k}],y->0]),{k,0,50}]]],{s,2.,5.},{n,-1.5,8},{z,1,5}]]
```

It generalizes to

$$\boxed{[f^{z}]_{n}=\sum_{k=0}^{\infty}\frac{z^{k}}{k!}\lim_{y\to 0}\frac{\partial^{k}}{\partial y^{k}}[f^{y}]_{n}} \tag{2.7.9}$$

And it can be expressed as residues too.

$$[\zeta(s)^z]_n = \sum_{k=0}^{\infty} z^k \operatorname*{Res}_{m=0} \frac{[\zeta(s)^m]_n}{m^{k+1}}$$

(2.7.10)

```
rr[]:=RandomReal[{-4,4}]+RandomReal[{-4,4}]I
zeta[n_,s_,z_,k_]:=1+((z+1)/k-1) Sum[j^-s zeta[n/j,s,z,k+1],{j,2,n}]
Table[Chop[zeta[a=45,b=rr[],s+t I,1]-N[Sum[(s+t I)^k Residue[zeta[a,b,m,1]/(m^(k+1)),{m,0}],{k,0,50}]]],{s,-1.5,5},{t,1,4}]
```

The limit of this as n approaches infinity, if $\Re(s) > 1$, is

$$\zeta(s)^z = \sum_{k=0}^{\infty} z^k \operatorname*{Res}_{m=0} \frac{\zeta(s)^m}{m^{k+1}}$$

(2.7.11)

```
Grid[Table[Chop[Zeta[s]^z-N[Sum[z^k Residue[Zeta[s]^m/(m^(k+1)),{m,0}],{k,0,50}]]],{s,2.,5.},{n,-1.5,8},{z,1,5}]]
```

It generalizes to

$$[f^z]_n = \sum_{k=0}^{\infty} z^k \operatorname*{Res}_{m=0} \frac{[f^m]_n}{m^{k+1}}$$

(2.7.12)

2.8 Taking Advantage of $d_z(n)$ as a Multiplicative Function

$$[\nabla \zeta(s)^{z+y}]_n = [\nabla \zeta(s)^z \cdot \nabla \zeta(s)^y]_n$$

(2.8.1)

and

$$[\zeta(s)^{z+y}]_n = [\zeta(s)^z \cdot \zeta(s)^y]_n$$

(2.8.2)

```
ri[]:=RandomInteger[{10,100}]; rr[]:=RandomReal[{-4,4}]+RandomReal[{-4,4}]I
dz[n_,z_]:=Product[(-1)^p[[2]] Binomial[-z,p[[2]]],{p,FI[n]}];FI[n_]:=FactorInteger[n];FI[1]:={}
dzMul[ n_, z_, y_] := Sum[ dz[j,z] dz[ n/j,y],{j,Divisors[n]}]
zeta[n_,s_,z_,k_]:=1+((z+1)/k-1) Sum[j^-s zeta[n/j,s,z,k+1],{j,2,n}]
DzMul[ n_, z_, y_] := Sum[ (Dz[j,z,1]-Dz[j-1,z,1])Dz[ n/j, y, 1],{j,1,n}]
[ MATHEMATICA ]
```

The limit of this as n approaches infinity, if $\Re(s) > 1$, is

$$\zeta(s)^{z+y} = \zeta(s)^z \cdot \zeta(s)^y$$

(2.8.3)

```
{Zeta[s]^(x+y), Zeta[s]^x n^y}
```

It generalizes to

$$[\nabla f^{z+y}]_n = [\nabla f^z \cdot \nabla f^y]_n$$

(2.8.4)

and

$$[f^{z+y}]_n = [f^z \cdot f^y]_n$$

(2.8.5)

2.9 Taking Advantage of $d_z(n)$ as a Multiplicative Function

Taking a break from power series parallels, IF a and b are coprime,

$$[\nabla\zeta(s)^z]_a\cdot[\nabla\zeta(s)^z]_b=[\nabla\zeta(s)^z]_{a\cdot b}$$

(2.9.1)

$$[\zeta(s)]_n=\sum_{a=0}^{\frac{\log n}{\log 2}}2^{-as}\cdot\sum_{b=0}^{\frac{\log n-a\log 2}{\log 3}}3^{-bs}\cdot\sum_{c=0}^{\frac{\log n-a\log 2-b\log 3}{\log 5}}5^{-cs}\cdot\sum_{d=0}^{\frac{\log n-a\log 2-b\log 3-c\log 5}{\log 7}}7^{-ds}\cdot\ldots$$

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\zeta(s)=\sum_{a=0}^{\infty}2^{-as}\cdot\sum_{b=0}^{\infty}3^{-bs}\cdot\sum_{c=0}^{\infty}5^{-cs}\cdot\sum_{d=0}^{\infty}7^{-ds}\cdot\ldots$$

(2.9.3)

```
{Zeta[s],Product[ Sum[Prime[j]^(-s a),{a,0,Infinity}],{j,1,Infinity}]}
```

$$[\zeta(s)^z]_n=\sum_{a=0}^{\frac{\log n}{\log 2}}\frac{z^{(a)}}{a!}\cdot 2^{-as}\cdot\sum_{b=0}^{\frac{\log n-a\log 2}{\log 3}}\frac{z^{(b)}}{b!}\cdot 3^{-bs}\cdot\sum_{c=0}^{\frac{\log n-a\log 2-b\log 3}{\log 5}}\frac{z^{(c)}}{c!}\cdot 5^{-cs}\cdot\sum_{d=0}^{\frac{\log n-a\log 2-b\log 3-c\log 5}{\log 7}}\frac{z^{(d)}}{d!}\cdot 7^{-ds}\cdot\ldots$$

(2.9.2)

```
FI[n_]:=FactorInteger[n];FI[1]:={}
dz[n_,z_,s_]:=n^-s Product[(-1)^p[[2]] Binomial[-z,p[[2]]],{p,FI[n]}]
zeta[n_,s_,z_,k_]:=1+((z+1)/k-1) Sum[j^-s zeta[n/j,s,z,k+1],{j,2,n}]
(* Note that because this is truncated, it stops working for any n > 36. *)
zetaalt[ n_, s_,z_] :=
Sum[ dz[2^a,z,s]dz[3^b,z,s]dz[5^c,z,s]dz[7^d,z,s]dz[11^e,z,s]dz[13^f,z,s]dz[17^g,z,s]dz[19^h,z,s]dz[23^i,z,s]dz[29^j,z,s]dz[31^k,z,s],{a,0,Log[2,n]},{b,0,Log[3,n/(2^a)]},{c,0,Log[5,n/(2^a 3^b)]},{d,0,Log[7,n/(2^a 3^b 5^c)]},{e,0,Log[11,n/(2^a 3^b 5^c 7^d)]},{f,0,Log[13,n/(2^a 3^b 5^c 7^d 11^e)]},{g,0,Log[17,n/(2^a 3^b 5^c 7^d 11^e 13^f)]},{h,0,Log[19,n/(2^a 3^b 5^c 7^d 11^e 13^f 17^g)]},{i,0,Log[23,n/(2^a 3^b 5^c 7^d 11^e 13^f 17^g 19^h)]},{j,0,Log[29,n/(2^a 3^b 5^c 7^d 11^e 13^f 17^g 19^h 23^i)]},{k,0,Log[31,n/(2^a 3^b 5^c 7^d 11^e 13^f 17^g 19^h 23^i 29^j)]}]
```

MATHEMATICA

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\zeta_n(s)^z=\sum_{a=0}^{\infty}\frac{z^{(a)}}{a!}\cdot 2^{-as}\cdot\sum_{b=0}^{\infty}\frac{z^{(b)}}{b!}\cdot 3^{-bs}\cdot\sum_{c=0}^{\infty}\frac{z^{(c)}}{c!}\cdot 5^{-cs}\cdot\sum_{d=0}^{\infty}\frac{z^{(d)}}{d!}\cdot 7^{-ds}\cdot\ldots$$

(2.9.4)

```
Table[ Chop[N[Zeta[s]]^z - N[Product[ (Sum[Prime[j]^(-s a ),{a,0,Infinity}])^z,{j,1,400}]]],{s,3,8}]
```

$$[\nabla f^z]_a\cdot[\nabla f^z]_b=[\nabla f^z]_{a\cdot b}\text{ when }(a,b)=1$$

$$[f^z]_n=\sum_{a=0}^{\frac{\log n}{\log 2}}[\nabla f^z]_{2^a}\cdot\sum_{b=0}^{\frac{\log n-a\log 2}{\log 3}}[\nabla f^z]_{3^b}\cdot\sum_{c=0}^{\frac{\log n-a\log 2-b\log 3}{\log 5}}[\nabla f^z]_{5^c}\cdot\sum_{d=0}^{\frac{\log n-a\log 2-b\log 3-c\log 5}{\log 7}}[\nabla f^z]_{7^d}\cdot\ldots$$

(2.9.5)

2.10 Taking Advantage of $d_z(n)$ as a Multiplicative Function

$$[\zeta(s)^z]_n = f(n,1) \quad \text{where} \quad f(n,j) = \begin{cases} \sum_{0 \le k \le \frac{\log n}{\log p_j}} \frac{z^{(k)}}{k!} \cdot p_j^{(-sk)} f\left(\frac{n}{p_j^k}, j+1\right) & \text{if } n \ge p_j \\ 1 & \text{otherwise} \end{cases} \tag{2.10.1}$$

```
ri[]:=RandomInteger[{10,100}]; rr[]:=RandomReal[{-4,4}]+RandomReal[{-4,4}]I
zeta[n_,s_,z_,k_]:=1+((z+1)/k-1) Sum[j^-s zeta[n/j,s,z,k+1],{j,2,n}]
F[n_, i_,z_] := If[ Prime[i] > n, 1, Sum[ (-1)^a Binomial[ -z, a] F[ n/Prime[i]^a, i+1,z],{a,0,Log[Prime[i],n]}]]
Grid[Table[Chop[Dz[a=143,s+t I,1]-F[a,1, s+t I]],{s,-1.5,4,.7},{t,-1.1,4,.7}]]
```

$$[\zeta(s)^z]_n = f_1(n,1) \quad \text{where} \quad f_k(n,j) = \begin{cases} p_j^{-s}\left(\left(1+\frac{z-1}{k}\right) f_{k+1}\left(\frac{n}{p_j}, j\right) + f_1(n, j+1)\right) & \text{if } n \ge p_j \\ 1 & \text{otherwise} \end{cases} \tag{2.10.2}$$

```
ri[]:=RandomInteger[{10,100}]; rr[]:=RandomReal[{-4,4}]+RandomReal[{-4,4}]I
zeta[n_,s_,z_,k_]:=1+((z+1)/k-1) Sum[j^-s zeta[n/j,s,z,k+1],{j,2,n}]
F[n_, i_, k_, z_] := If[ Prime[i] > n || n<=1,1,(1+ (z-1)/k)F[ n/Prime[i],i,k+1, z] + F[n, i+1,1,z]]
Grid[Table[Chop[Dz[a=143,s+t I,1]-F[a,1, 1,s+t I]],{s,-1.5,4,.7},{t,-1.1,4,.7}]]
```

More generally, if f(n) is multiplicative,

$$\sum_{j=1}^{n} f(j) = f(n,1) \quad \text{where} \quad f(n,j) = \begin{cases} \sum_{0 \le k \le \frac{\log n}{\log p_j}} f(p_j^k) f\left(\frac{n}{p_j^k}, j+1\right) & \text{if } n \ge p_j \\ 1 & \text{otherwise} \end{cases}$$

…

$$\sum_{j=1}^{n} f(j) = f_1(n,1) \quad \text{where} \quad f_k(n,j) = \begin{cases} f_{k+1}\left(\frac{n}{p_j}, j\right) + f(p_j^k) \cdot (1 + f_1(n, j+1)) & \text{if } n \ge p_j \\ 1 & \text{otherwise} \end{cases}$$