This is another scratch set of notes to show the derivation that starts with the partial sum equivalent of the log of the dirichlet eta function, and by generalizing the notion of alternating series sufficiently, leads to an expression that represents the difference between the Riemann Prime Counting function and the logarithmic integral – log log n – euler gamma.

I have written this out better in other papers, though.

Start with

$$\Pi(n)=\sum_{j=2}^{\lfloor n\rfloor}1-\frac{1}{2}\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}1+\frac{1}{3}\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor\frac{n}{j\cdot k}\rfloor}1-\frac{1}{4}\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor\frac{n}{j\cdot k}\rfloor}\sum_{m=2}^{\lfloor\frac{n}{j\cdot k\cdot l}\rfloor}1+\frac{1}{5}...$$

Now let's look at the alternating series version of this identity...

$$???=\sum_{j=2}^{\lfloor n\rfloor}(-1)^{j+1}-\frac{1}{2}\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}(-1)^{j+1}(-1)^{k+1}+\frac{1}{3}\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor\frac{n}{j\cdot k}\rfloor}(-1)^{j+1}(-1)^{k+1}(-1)^{k+1}-\frac{1}{4}...$$

To cut to the chase, this ends up being

$$\Pi(n)-\sum_{k=1}^{\lfloor\frac{\log n}{\log 2}\rfloor}2^k k=\sum_{j=2}^{\lfloor n\rfloor}(-1)^{j+1}-\frac{1}{2}\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}(-1)^{j+1}(-1)^{k+1}+\frac{1}{3}\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor\frac{n}{j\cdot k}\rfloor}(-1)^{j+1}(-1)^{k+1}(-1)^{k+1}-\frac{1}{4}...$$

Now, generalize the notion of alternating series with the following function, where $\frac{a}{b}$ is a rational number.

$$\alpha(n,\frac{a}{b})=b\cdot(\lfloor\frac{n}{b}\rfloor-\lfloor\frac{n-1}{b}\rfloor)-a\cdot(\lfloor\frac{n}{a}\rfloor-\lfloor\frac{n-1}{a}\rfloor)$$

Note that $\alpha(n,\frac{2}{1})=(-1)^{n+1}$, as desired.

So now suppose we have some rational number $c=\frac{a}{b}, a>b$.

$$???=b^{-1}\sum_{j=b+1}^{\lfloor n\cdot b\rfloor}\alpha(j,c)-\frac{1}{2}b^{-2}\sum_{j=b+1}^{\lfloor n\cdot b\rfloor}\sum_{k=b+1}^{\lfloor\frac{n\cdot b^2}{j}\rfloor}\alpha(j,c)\alpha(k,c)+\frac{1}{3}b^{-3}\sum_{j=b+1}^{\lfloor n\cdot b\rfloor}\sum_{k=b+1}^{\lfloor\frac{n\cdot b^2}{j}\rfloor}\sum_{l=b+1}^{\lfloor\frac{n\cdot b^3}{j\cdot k}\rfloor}\alpha(j,c)\alpha(k,c)\alpha(l,c)-\frac{1}{4}b^{-4}...$$

$$\alpha(n,\frac{a}{b})=(\lfloor n\rfloor-\lfloor n-\frac{1}{b}\rfloor)-c\cdot(\lfloor\frac{n}{c}\rfloor-\lfloor\frac{n}{c}-\frac{1}{a}\rfloor)$$

$$???=\sum_{1+b^{-1}\le j\le n}\alpha(j,c)-\frac{1}{2}\sum_{1+b^{-1}\le j\le n}\sum_{1+b^{-1}\le k\le\frac{n}{j}}\alpha(j,c)\alpha(k,c)$$
$$+\frac{1}{3}\sum_{1+b^{-1}\le j\le n}\sum_{1+b^{-1}\le k\le\frac{n}{j}}\sum_{1+b^{-1}\le l\le\frac{n}{j\cdot k}}\alpha(j,c)\alpha(k,c)\alpha(l,c)-\frac{1}{4}...$$

With the added constraint that $b^{-1}|j\,,b^{-1}|k\,,b^{-1}|l$, and so on.

$$\Pi(n)-\sum_{k=1}^{\lfloor\frac{\log n}{\log c}\rfloor} c^k k=\sum_{1+b^{-1}\le j\le n}\alpha(j,c)-\frac{1}{2}\sum_{1+b^{-1}\le j\le n}\;\sum_{1+b^{-1}\le k\le\frac{n}{j}}\alpha(j,c)\alpha(k,c)$$
$$+\frac{1}{3}\sum_{1+b^{-1}\le j\le n}\;\sum_{1+b^{-1}\le k\le\frac{n}{j}}\;\sum_{1+b^{-1}\le l\le\frac{n}{j\cdot k}}\alpha(j,c)\alpha(k,c)\alpha(l,c)-\frac{1}{4}\ldots$$

With the constraint that $b^{-1}|j\,,b^{-1}|k\,,b^{-1}|l$, and so on.

Now, it turns out that

$$\lim_{c\to 1^+}\sum_{k=1}^{\lfloor\frac{\log n}{\log c}\rfloor}\frac{c^k-1}{k}=li(n)-\log\log n-\gamma$$

Thus

$$\Pi(n)-li(n)+\log\log n+\gamma=(1+\sum_{1+b^{-1}\le j\le n}\alpha(j,c))+\frac{1}{2}(1-\sum_{1+b^{-1}\le j\le n}\;\sum_{1+b^{-1}\le k\le\frac{n}{j}}\alpha(j,c)\alpha(k,c))$$
$$+\frac{1}{3}(1+\sum_{1+b^{-1}\le j\le n}\;\sum_{1+b^{-1}\le k\le\frac{n}{j}}\;\sum_{1+b^{-1}\le l\le\frac{n}{j\cdot k}}\alpha(j,c)\alpha(k,c)\alpha(l,c))+\frac{1}{4}\ldots$$

with the constraint that $b^{-1}|j\,,b^{-1}|k\,,b^{-1}|l$, and so on.

$$\alpha(n,\frac{a}{b})=(\lfloor n\rfloor-\lfloor n-\frac{1}{b}\rfloor)-c\cdot(\lfloor\frac{n}{c}\rfloor-\lfloor\frac{n}{c}-\frac{1}{a}\rfloor)$$

$$\Pi(n)-li(n)+\log\log n+\gamma=(1+\sum_{1+b^{-1}\le j\le n}\alpha(j,c))+\frac{1}{2}(1-\sum_{j\cdot k\le n;\,j,k\ge 1+b^{-1}}\alpha(j,c)\alpha(k,c))$$
$$+\frac{1}{3}(1+\sum_{j\cdot k\cdot l\le n;\,j,k,l\ge 1+b^{-1}}\alpha(j,c)\alpha(k,c)\alpha(l,c))+\frac{1}{4}\ldots$$