6: Computing $[\zeta(s)^z]_n$ and $\Pi(n)$ with the Dirichlet Hyperbola Method

6. Computing $\Pi(n)$ with the Hyperbola Method

Here we compute $[\zeta(0)^z]_n$ and $\Pi(n)$ pretty quickly, using the Dirichlet Hyperbola Method to $[(\zeta(0)-1)^k]_n$. This approach computes $[\zeta(0)^z]_n$ for any complex z in faster than $O(n)$ time and $O(\log n)$ space. It's especially well suited to computing $\Pi(n)$.

6.1 The Function $[f(s,y)^k]_n$

As a reminder, we previously defined exponential convolutions of the partial sum of the Hurwitz Zeta function as

$$\boxed{[\zeta(s,y+1)^k]_n = \sum_{j=1} (j+y)^{-s} [\zeta(s,y+1)^{k-1}]_{n(j+y)^{-1}}} \tag{6.1.1}$$

and generalized it, for some function $f(n)$, as

$$[f(y+1)^k]_n = \sum_{j=1} f(j+y) [f(y+1)^{k-1}]_{n(j+y)^{-1}} \tag{6.1.2}$$

```
F[f_, n_, 0, a_] := UnitStep[n-1]
F[f_, n_, k_, a_] := Sum[ f[j] F[f, n/j, k-1, a],{j,a+1,Floor[n]}]
```

Now we're going to use some properties of it to compute the Riemann Prime counting function rather quickly.

Our approach is use internal symmetries of $[\zeta(s,y)^k]_n$ to compute it more quickly. To uncover those symmetries, we'll start with the following identity.

6.2 $[\zeta(s,y)^k]_n$ in terms of $[\zeta(s,y+1)^k]_n$

$[\zeta(s,y)^k]_n$ can be expressed in terms of $[\zeta(s,y+1)^k]_n$ as

$$[\zeta(s,y)^k]_n = \sum_{j=0}^{k} \binom{k}{j} y^{-sj} [\zeta(s,y+1)^{k-j}]_{n\cdot(y+1)^{-1}} \tag{6.2.1}$$

The limit of this as *n* approaches infinity, if $\Re(s) > 1$, is

$$\zeta(s,y)^k=\sum_{j=0}^{k}\binom{k}{j}y^{-sj}\zeta(s,y+1)^{k-j} \tag{6.2.2}$$

This can be generalized as

$$[f(y)^k]_n=\sum_{j=0}^{k}\binom{k}{j}f(y+1)^j[f(y+1)^{k-j}]_{n(y+1)^{-j}} \tag{6.2.3}$$

```
FullSimplify[Table[ Zeta[s,a]^k-Sum[a^(-s j)Binomial[k,j] Zeta[s, a+1]^(k-j), {j,0,k}],{k,1,5},{a,2,5},{s,2,4}]]
```

6.3 $[\zeta(s,y+1)^k]_n$ in terms of $[\zeta(s,y)^k]_n$

Although we won't rely on it here, the idea from 6.2 can be inverted as

$$[\zeta(s,y+1)^k]_n=\sum_{j=0}^{k}(-1)^j\binom{k}{j}y^{-j\cdot s}[\zeta(s,y)^{k-j}]_{n(y+1)^{-j}} \tag{6.3.1}$$

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\zeta(s,y+1)^k=\sum_{j=0}^{k}(-1)^j\binom{k}{j}y^{-sj}\zeta(s,y)^{k-j} \tag{6.3.2}$$

This can be generalized as

$$[f(y+1)^k]_n=\sum_{j=0}^{k}(-1)^j\binom{k}{j}f(y)^j[f(y)^{k-j}]_{n(y+1)^{-j}} \tag{6.3.3}$$

```
F[f_,n_,0,a_]:=UnitStep[n-1]
F[f_,n_,k_,a_]:=Sum[f[j] F[f,n/j,k-1,a],{j,a+1,Floor[n]}]
FAlt[f_,n_,k_,a_]:=If[n<(a+1)^k,0,Sum[Binomial[k,j] f[a+1]^j F[f,n/(a+1)^j,k-j,a+1],{j,0,k}]]
Grid[Table[{F[MoebiusMu,n,k,2]-FAlt[MoebiusMu,n,k,2]},{n,10,500,10},{k,1,5}]]
Grid[Table[{F[LiouvilleLambda,n,k,4]-FAlt[LiouvilleLambda,n,k,4]},{n,10,500,10},{k,1,5}]]
Grid[Table[{F[MoebiusMu,n,k,2]-FAlt[MoebiusMu,n,k,2]},{n,10,500,10},{k,1,5}]]
Grid[Table[{F[LiouvilleLambda,n,k,4]-Falt[LiouvilleLambda,n,k,4]},{n,10,500,10},{k,1,5}]]
```

```
FullSimplify[Table[ Zeta[s,a]^k-Sum[(-1)^j (a-1)^(-s j)Binomial[k,j] Zeta[s, a-1]^(k-j), {j,0,k}],{k,1,5},{a,2,5},{s,2,4}]]
```

6.4 Convergence for $[\zeta(s,y)^k]_n$

Before we continue, we'll need to note the following straightforward property of $[\zeta(s,y)^k]_n$.

$$[\zeta(s,y)^k]_n=0 \text{ when } n<y^k \tag{6.4.1}$$

This can be generalized as

$$[f(y)^k]_n = 0 \text{ when } n < y^k \tag{6.4.2}$$

```
F[f_,n_,0,a_]:=UnitStep[n-1]
F[f_,n_,k_,a_]:=Sum[f[j] F[f,n/j,k-1,a],{j,a+1,Floor[n]}]
Grid[Table[ F[MoebiusMu,n,k,2],{n,1,64},{k,1,6}]]
Grid[Table[ F[LiouvilleLambda,n,k,3],{n,1,81},{k,1,6}]]
```

6.5 Unrolling $[\zeta(s,y)^k]_n$ and $[f(y)^k]_n$

Let's take advantage of (6.4.1) to rewrite (6.2.1) so the $[\zeta(s,y)^k]_n$ term on right hand side only uses values of *k* smaller than the *k* on the left hand side.

Here's how we do this: let's recursively replace the right hand side reference to $[\zeta(s,y)^k]_n$ with the identity (6.2.1) itself, but only when *j* is 0. Let's keep doing that until $[\zeta(s,y)^k]_n$ is 0, at which point we'll have the definition we want. It will look like this:

$$\boxed{[\zeta(s,y)^k]_n = \sum_{j=1}^{k} \binom{k}{j} \sum_{m=y+1}^{\lfloor n^{\frac{1}{k}} \rfloor} m^{-j\cdot s} [\zeta(s,m)^{k-j}]_{n\cdot m^{-j}}} \tag{6.5.1}$$

The limit of this as *n* approaches infinity, if $\Re(s) > 1$, is

$$\zeta(s,y)^k = \sum_{j=1}^{k} \binom{k}{j} \sum_{m=y+1} m^{-j\cdot s} \zeta(s,m)^{k-j} \tag{6.5.2}$$

This can be generalized as

$$\boxed{[f(y)^k]_n = \sum_{j=1}^{k} \binom{k}{j} \sum_{m=y+1}^{\lfloor n^{\frac{1}{k}} \rfloor} f(m)^j [f(m)^{k-j}]_{n\cdot m^{-j}}} \tag{6.5.3}$$

```
F[f_,n_,0,a_]:=UnitStep[n-1]
F[f_,n_,k_,a_]:=Sum[f[j] F[f,n/j,k-1,a],{j,a+1,Floor[n]}]
FAlt[fn_, n_,0,a_]:=UnitStep[n-1]
FAlt[fn_, n_,k_,a_]:=Sum[Binomial[k,j] fn[m]^j FAlt[fn, n/(m^j),k-j,m],{j,1,k}, {m,a+1,Floor[n^(1/k)]}]
id[n_] := 1
Grid[Table[{F[id, n,k, 2]-FAlt[id, n,k, 2]},{n,10,500,10},{k,1,7}]]
Grid[Table[{F[MoebiusMu, n,k,3]-FAlt[MoebiusMu, n,k, 3]},{n,10,500,10},{k,1,7}]]
```

Should we recursively apply (6.5.1) to itself until $[f(y)^k]_n$ is eliminated entirely (we can stop the recursion when k is 0, given that $[f(y)^0]_n = 1_{[1,\infty)}(n)$), we'll be left with nested sums and have, essentially, expressed $[\zeta(s,y)^k]_n$ with the Dirichlet Hyperbola method. For example, repeating this process gives us

$$[\zeta(s,y+1)]_n = \sum_{b=y+1}^{\lfloor n \rfloor} b^{-s} \tag{6.5.4}$$

$$[\zeta(s,y+1)^2]_n = \sum_{b=y+1}^{\lfloor n^{\frac{1}{2}} \rfloor} b^{-2s} + 2 \sum_{b=a+1}^{\lfloor n^{\frac{1}{2}} \rfloor} b^{-s} \cdot \sum_{c=b+1}^{\lfloor \frac{n}{b} \rfloor} c^{-s} \tag{6.5.5}$$

$$[\zeta(s,y+1)^3]_n = \sum_{b=y+1}^{\lfloor n^{\frac{1}{3}}\rfloor} b^{-3s} + 3\sum_{b=y+1}^{\lfloor n^{\frac{1}{3}}\rfloor}\sum_{c=b+1}^{\lfloor \frac{n}{b^2}\rfloor} b^{-2s}\cdot c^{-s} + 3\sum_{b=y+1}^{\lfloor n^{\frac{1}{3}}\rfloor}\sum_{c=b+1}^{\lfloor (\frac{n}{b})^{\frac{1}{2}}\rfloor} b^{-s}\cdot c^{-2s} + 6\sum_{b=y+1}^{\lfloor n^{\frac{1}{3}}\rfloor}\sum_{c=b+1}^{\lfloor (\frac{n}{b})^{\frac{1}{2}}\rfloor}\sum_{d=c+1}^{\lfloor \frac{n}{b\cdot c}\rfloor} b^{-s}\cdot c^{-s}\cdot d^{-s} \tag{6.5.6}$$

This can be generalized as

$$[f(y+1)]_n = \sum_{b=y+1}^{\lfloor n\rfloor} f(b) \tag{6.5.7}$$

$$[f(y+1)^2]_n = \sum_{b=y+1}^{\lfloor n^{\frac{1}{2}}\rfloor} f(b)^2 + 2\sum_{b=a+1}^{\lfloor n^{\frac{1}{2}}\rfloor} f(b)\cdot \sum_{c=b+1}^{\lfloor \frac{n}{b}\rfloor} f(c) \tag{6.5.8}$$

$$[f(y+1)^3]_n = \sum_{b=y+1}^{\lfloor n^{\frac{1}{3}}\rfloor} f(b)^3 + 3\sum_{b=y+1}^{\lfloor n^{\frac{1}{3}}\rfloor}\sum_{c=b+1}^{\lfloor \frac{n}{b^2}\rfloor} f(b)^2\cdot f(c) + 3\sum_{b=y+1}^{\lfloor n^{\frac{1}{3}}\rfloor}\sum_{c=b+1}^{\lfloor (\frac{n}{b})^{\frac{1}{2}}\rfloor} f(b)\cdot f(c)^2 + 6\sum_{b=y+1}^{\lfloor n^{\frac{1}{3}}\rfloor}\sum_{c=b+1}^{\lfloor (\frac{n}{b})^{\frac{1}{2}}\rfloor}\sum_{d=c+1}^{\lfloor \frac{n}{b\cdot c}\rfloor} f(b)\cdot f(c)\cdot f(d) \tag{6.5.9}$$

```
F[f_,n_,0,a_]:=UnitStep[n-1]
F[f_,n_,k_,a_]:=Sum[f[j] F[f,n/j,k-1,a],{j,a+1,n}]
F1[f_,n_,a_]:=Sum[f[b],{b,a+1,n}]
F2[f_,n_,a_]:=Sum[f[b]^2,{b,a+1,Floor[n^(1/2)]}]+2 Sum[f[b] f[c],{b,a+1,n^(1/2)},{c,b+1,n/b}]
F3[f_,n_,a_]:=Sum[f[b]^3,{b,a+1,Floor[n^(1/3)]}]+3 Sum[f[b]^2 f[c],{b,a+1,n^(1/3)},{c,b+1,n/b^2}]+3 Sum[f[b] f[c]^2,
{b,a+1,n^(1/3)},{c,b+1,(n/b)^(1/2)}]+6 Sum[f[b] f[c] f[d],{b,a+1,n^(1/3)},{c,b+1,(n/b)^(1/2)},{d,c+1,n/(b c)}]
l[n_] := LiouvilleLambda[n];m[n_]:=MoebiusMu
Table[{F[l,n,1,2]-F1[l,n,2],F[l,n,2,2]-F2[l,n,2],F[l,n,3,2]-F3[l,n,2]},{n,10,500,10}]
Table[{F[m,n,1,3]-F1[m,n,3],F[m,n,2,3]-F2[m,n,3],F[m,n,3,3]-F3[m,n,3]},{n,10,500,10}]
```

and so on.

In general, if $\sum_{j=a}^{\lfloor n\rfloor} f(j)$ can be computed in constant time, this appears to lets us compute $[f^k]_n$ in faster than $O(n)$ time and essentially constant space.

6.6 Improving the Technique for $[\zeta(s,2)^k]_n$

Now, it happens to be the case that we can compute $\sum_{j=a}^{\lfloor n\rfloor} j^{-s}$ in constant time with Faulhaber's formula if s is 0 or a positive integer. s=0 is particularly simple, of course, and in fact, we can trivially specialize (6.5.1) as

$$\boxed{\begin{gathered}[\zeta(0,y+1)^k]_n = \sum_{j=1}^{k}\binom{k}{j}\sum_{m=y+1}^{\lfloor n^{\frac{1}{k}}\rfloor}[\zeta(0,m+1)^{k-j}]_{n\cdot m^{-j}}\\ [\zeta(0,y+1)]_n = \lfloor n\rfloor - y\\ [\zeta(0,y+1)^0]_n = 1_{[1,\infty)}(n)\end{gathered}} \tag{6.6.1}$$

```
Da[n_,0,a_]:=UnitStep[n-1]; Da[n_,1,a_]:=Floor[n]-a
Da[n_,k_,a_]:=Sum[Binomial[k,j] Da[n/(m^(k-j)),j,m],{m,a+1,n^(1/k)},{j,0,k-1}]
refD1[n_, k_] := Sum[ refD1[n/j, k-1],{j,1,n}]; refD1[n_, 0 ] := UnitStep[n-1]
refD2[n_, k_] := Sum[ refD2[n/j, k-1],{j,2,n}]; refD2[n_, 0 ] := UnitStep[n-1]
Grid[Table[Da[n,k,0]-refD1[n,k],{n,7,100,5},{k,1,7}]]
Grid[Table[Da[n,k,1]-refD2[n,k],{n,7,100,5},{k,1,7}]]
```

where $[\zeta(0,1)^k]_n$ is $[\zeta(0)^k]_n$ from (1.1) and $[\zeta(s,2)^k]_n$ is $[(\zeta(0)-1)^k]_n$ from (1.4).

If we take this unrolling further, we find, following the pattern of (6.5.6) and then doing some aggressive simplifying, that

$$
\begin{aligned}
&[\zeta(0,y+1)^0]=1_{[1,\infty)}(n)\\
&[\zeta(0,y+1)]_n=\lfloor n\rfloor-y\\
&[\zeta(0,y+1)^2]_n=y^2-\lfloor n^{\frac{1}{2}}\rfloor^2+2\sum_{b=y+1}^{\lfloor n^{\frac{1}{2}}\rfloor}\lfloor\frac{n}{b}\rfloor\\
&[\zeta(0,y+1)^3]_n=-y^3+\lfloor n^{\frac{1}{3}}\rfloor^3+3\sum_{b=y+1}^{\lfloor n^{\frac{1}{3}}\rfloor}\lfloor\frac{n}{b^2}\rfloor-3\sum_{b=y+1}^{\lfloor n^{\frac{1}{3}}\rfloor}\lfloor(\frac{n}{b})^{\frac{1}{2}}\rfloor^2+6\sum_{b=y+1}^{\lfloor n^{\frac{1}{3}}\rfloor}\sum_{c=b+1}^{\lfloor(\frac{n}{b})^{\frac{1}{2}}\rfloor}\lfloor\frac{n}{b\,c}\rfloor
\end{aligned}
\tag{6.6.2}
$$

```
refD2[n_,k_]:=Sum[refD2[n/j,k-1],{j,2,n}];refD2[n_,0]:=UnitStep[n-1]
D1[n_,a_]:=-a+Floor[n]
D2[n_,a_]:= a^2+-Floor[n^(1/2)]^2+2 Sum[Floor[(n/b)],{b,a+1,n^(1/2)}]
D3[n_,a_]:=- a^3+ Floor[n^(1/3)]^3+3 Sum[Floor[n/(b^2)],{b,a+1,n^(1/3)}]+-3 Sum[Floor[(n/b)^(1/2)]^2,{b,a+1,n^(1/3)}]+6
Sum[Floor[n/(b c)],{b,a+1,n^(1/3)},{c,b+1,(n/b)^(1/2)}]
Table[refD2[n,1]-D1[n,1],{n,10,500,10}]
Table[refD2[n,2]-D2[n,1],{n,10,500,10}]
Table[refD2[n,3]-D3[n,1],{n,10,500,10}]
```

and so on, with the number of terms growing exponentially.

6.7 Using $[\zeta(0,2)^k]_n$ to Compute Other Functions

We can use (6.6.1) to compute the partial sum of the zeta function convolved exponential to some complex value z, $[\zeta(0)^z]_n$, as

$$[\zeta(0)^z]_n=\sum_{k=0}^{\lfloor\log_2 n\rfloor}\binom{z}{k}[\zeta(0,2)^k]_n \tag{6.7.1}$$

```
Dz[n_,z_,k_]:=1+((z+1)/k-1)Sum[Dz[n/j,z,k+1],{j,2,n}]
Da[n_,k_,a_]:=Sum[Binomial[k,j] Da[n/(m^(k-j)),j,m],{m,a+1,n^(1/k)},{j,0,k-1}]
Da[n_,0,a_]:=UnitStep[n-1]
Da[n_,1,a_]:=Floor[n]-a
DzAlt[n_,z_]:=Sum[Binomial[z,k] Da[n,k,1],{k,0,Log[2,n]}]
Grid[Table[Chop[Dz[721,s+t I,1]-DzAlt[721,s+t I]],{s,-1.3,4,.7},{t,-1.3,4,.7}]]
```

As Mertens function is $M(n)=[\zeta(0)^{-1}]_n$, we can use (6.6.1) to compute Mertens function as

$$M(n)=\sum_{k=0}^{\lfloor\log_2 n\rfloor}(-1)^k[\zeta(0,2)^k]_n \tag{6.7.2}$$

```
Dd[n_,k_,a_]:=Sum[Binomial[k,j] Dd[n/(m^(k-j)),j,m],{m,a+1,n^(1/k)},{j,0,k-1}];Dd[n_,0,a_]:=UnitStep[n-
1];Dd[n_,1,a_]:=Floor[n]-a
Mertens[n_]:=Sum[(-1)^k Dd[n,k,1],{k,0,Log[2,n]}]
Table[Mertens[n]-Sum[MoebiusMu[j],{j,1,n}],{n,2,100}]
```

and we can use (6.6.1) to compute Riemann's Prime counting function as

$$\Pi(n) = \sum_{k=1}^{\lfloor \log_2 n \rfloor} \frac{(-1)^{k+1}}{k} [\zeta(0,2)^k]_n$$

(6.7.3)

```
Dd[n_,k_,a_]:=Sum[Binomial[k,j] Dd[n/(m^(k-j)),j,m],{m,a+1,n^(1/k)},{j,0,k-1}];Dd[n_,0,a_]:=UnitStep[n-1];Dd[n_,1,a_]:=Floor[n]-a
logD[n_]:=Sum[(-1)^(k-1)/k Dd[n,k,1],{k,1,Log[2,n]}]
RiemannPrimeCount[n_]:=Sum[PrimePi[n^(1/j)]/j,{j,1,Log[2,n]}]
Table[RiemannPrimeCount[n]-logD[n],{n,2,100}]
```

and we can use (3.1) to compute $[\log\zeta(0)^k]_n = \sum_{j=2} \frac{\Lambda(j)}{\log j} [\log\zeta(0)^{k-1}]_{n j^{-1}}$ from (1.6) as

$$[\log\zeta(0)^j]_n = \sum_{k=0}^{\lfloor \log_2 n \rfloor} (\lim_{y\to 0} \frac{\partial^k}{\partial y^k} (\log(1+y))^j) \cdot [\zeta(0,2)^k]_n$$

(6.7.4)

```
logD[n_,k_]:=Limit[D[Dz[n,z,1],{z,k}],z->0];Dz[n_,z_,k_]:=1+((z+1)/k-1)Sum[Dz[n/j,z,k+1],{j,2,n}]
Da[n_,k_,a_]:=Sum[Binomial[k,j] Da[n/(m^(k-j)),j,m],{m,a+1,n^(1/k)},{j,0,k-1}]
Da[n_,0,a_]:=UnitStep[n-1]
Da[n_,1,a_]:=Floor[n]-a
logDAlt[n_,j_]:=Sum[1/k!(Limit[D[Log[1+y]^j,{y,k}],y->0]) Da[n,k,1],{k,0,Log[2,n]}]
Grid[Table[logD[n,k]-logDAlt[n,k],{n,10,100, 10},{k,1,5}]]
```

6.8 Notes and Implementations of This Approach

(6.6.1) lets us compute $[\zeta(0)^z]_n$, $M(n)$, and $[\log\zeta(0)^k]_n$ in faster than $O(n)$ time, and in $O(\log n)$ space. In fact, once we've computed our $\log_2 n$ values of $[(\zeta(0)-1)^k]_n$ in $O(n)$ time, we can compute *any* value of $[\zeta(0)^z]_n$ or $[\log\zeta(0)^k]_n$ in $O(\log_2 n)$ operations, via the identities above.

This idea works especially well at computing $\Pi(n)$ if we apply a wheel to (6.6.1). If the sum in (6.6.1) is only taken over m=numbers not divisible by, say, the first 8 primes, and the same wheel is applied to $[\zeta(0,y)]_n$ in (6.6.1), the algorithm runs, empirically, in something like $O(n^{\frac{4}{5}})$ time, and speeds up between x1000 and x10000 in constant time terms.

A pretty fast C implementation using this idea to count primes is at *http://icecreambreakfast.com/primes/NMPrimeCounter.cpp*

Other descriptions of this technique are in section E.2 of *http://www.icecreambreakfast.com/primecount/PrimeCountingSurvey.pdf* , and section 4-2 of *http://www.icecreambreakfast.com/primecount/LinnikVariations.pdf*