Overview

My goal in this paper is to explain a particular point-of-view that conceptually connects the Riemann Prime Counting function, the logarithmic integral, and the harmonic number.

To that end, I'll introduce a value I'll write as $\log(1+\boldsymbol{x})$. This value will stand, variously, for the logarithmic integral, the harmonic number, and count of primes given by the Riemann Prime Counting function.

The really big idea here, the guiding concept, is that many of the relationships between x^k , $(1+x)^z$, and $\log(1+x)$, have analogs where the logarithmic integral, the harmonic number, and the Riemann Prime Counting function fill the role of $\log(1+x)$. They all satisfy a broad range of properties that $\log x$ satisfies.

To flesh out the roles these functions play, I'll also define values written as $\boldsymbol{x}^{\boldsymbol{k}}$ and $(1+\boldsymbol{x})^z$.

Section 1:
$\boldsymbol{x}^{\boldsymbol{k}}$

Overview of This Section

As a concept, $\boldsymbol{x}^k$ is the most straightforward of the terms introduced here, so I begin with it.

It's well known that $(1+x)^z$, $\log(1+x)$, and $\log^k(1+x)$ can all be expressed in terms of x^k , of course, going back to Newton's generalized binomial theorem and to Taylor series.

My big picture goal here, after I define $\boldsymbol{x}^k$, is to express some important convolution sums, which I notate as $(1+\boldsymbol{x})^z$, $\log(1+\boldsymbol{x})$, and $\log^k(1+\boldsymbol{x})$, as sums of the form $\sum_{k=0}^{\infty} C_k \cdot \boldsymbol{x}^k$. Just as a reminder, variations of $\log(1+\boldsymbol{x})$ include the Riemann Prime counting function, the logarithmic integral, and the Harmonic number. So that justifies the interest in taking this approach.

So, I start by defining $\boldsymbol{x}^k$. In particular, for developing intuition, $\boldsymbol{x}^k$ has a nice conceptual description and a relatively straightforward geometric interpretations, so that comes first.

Then I show how variations of $\boldsymbol{x}^k$ relate to each other and explain the non-standard notation used here and its justification.

Following that, I give a recursive definition for $\boldsymbol{x}^k$. This definition really hammers home how closely related the variants of $\boldsymbol{x}^k$ (and later $\log(1+\boldsymbol{x})$) are to each other. All four variants of $\boldsymbol{x}^k$, and x^k itself, can be expressed with the same recursive form, with parameter changes. I also provide a Mathematica implementation of this recursive form.

Then I write out, long-hand, just to be extremely clear, values of $\boldsymbol{x}^k$ for the first few values of *k* in integral or sum notation.

I then list closed forms for some variations of $\boldsymbol{x}^k$ and for their derivatives or finite differences with respect to x. One big point here is the relationship between $\boldsymbol{x}^k$ and these closed forms. A goal of the non-standard notation in here is to build instincts about what relationships should exist between these more standard closed forms. Think of my non-standard notation as a kind of scaffolding for finding these identities that can then be replaced with the more standard closed form notation.

And finally, because our entire goal in defining $\boldsymbol{x}^k$ is to write sums of the form $\sum_{k=0}^{\infty} C_k \cdot \boldsymbol{x}^k$ to express $(1+\boldsymbol{x})^z$, $\log(1+\boldsymbol{x})$, and $\log^k(1+\boldsymbol{x})$, I cover convergence involving sums of $\boldsymbol{x}^k$. But this is almost an afterthought; convergence on sums built on $\boldsymbol{x}^k$ is pretty straightforward compared to x^k .

Conceptual and Geometric Interpretation of $\boldsymbol{x}^k$

To cut to the chase, it's easiest to see what $\boldsymbol{x}^k$ means, conceptual, by showing a few examples. So first, here are a few variants of $\boldsymbol{x}^2$:

$$x^2$$

x^2 : Square

Real-valued area bounded by $t>0$, $u>0$, $t\le x$, and $u\le x$.

$\boldsymbol{x}^{2+\int}$: Triangle

Real-valued area bounded by $t>0$, $u>0$, and $t+u\le x$.

$\boldsymbol{x}^{2+\Sigma}$ Discrete Triangle

Count of whole number pairs (t,u) satisfying $t>0$, $u>0$, and $t+u\le x$.

$\boldsymbol{x}^{2*\int}$: Hyperbola

Real-valued area bounded by $t > 0$, $u > 0$, and $(1+t)\cdot(1+u) \le x$.

$\boldsymbol{x}^{2 *\Sigma}$: Discrete Hyperbola

Count of ordered whole number pairs (t,u) satisfying $t > 0$, $u > 0$, and $(1+t)\cdot(1+u) \le x$.

And here are a few variants of $\boldsymbol{x}^{3}$:

$$x^3$$

x^3 : Cube

Real-valued volume bounded by $t > 0$, $u > 0$, $v > 0$, $t \le x$, $u \le x$, and $v \le x$.

$\boldsymbol{x}^{3 +\int}$: Triangular Pyramid

Real-valued volume bounded by $t > 0$, $u > 0$, $v > 0$, and $t+u+v \le x$.

$\boldsymbol{x}^{3 +\Sigma}$: Discrete Triangular Pyramid

Count of whole number pairs (t,u,v) satisfying $t > 0$, $u > 0$, $v > 0$, and $t+u+v \le x$.

$\boldsymbol{x}^{3 *\int}$: 3-dimensional Hyperbola

Real-valued volume bounded by $t > 0$, $u > 0$, $v > 0$, and $(1+t)\cdot(1+u)\cdot(1+v) \le x$.

$\boldsymbol{x}^{3 *\Sigma}$: Discrete 3-dimensional hyperbola.

Count of ordered whole number pairs (t,u,v) satisfying $t > 0$, $u > 0$, $v > 0$, and $(1+t)\cdot(1+u)\cdot(1+v) \le x$.

So this defines squaring and cubing. Hopefully it's clear how to generalize these to arbitrary larger whole number powers in k dimensions, although I'll include an explicit recursive formula for that below.

How These Variations Relate to Each Other

Let me point out a few straight-forward relationships between these different terms.

* $\boldsymbol{x}^{k +\Sigma}$ is the discrete equivalent of $\boldsymbol{x}^{k +\int}$
* $\boldsymbol{x}^{k *\Sigma}$ is the discrete equivalent of $\boldsymbol{x}^{k *\int}$
* $\boldsymbol{x}^{k *\int}$ is the multiplicative equivalent of the additive $\boldsymbol{x}^{k +\int}$
* $\boldsymbol{x}^{k *\Sigma}$ is the multiplicative equivalent of the additive $\boldsymbol{x}^{k +\Sigma}$

To emphasize these relationships, visually, I'll generally list identities in a grid, like this:

$$\boldsymbol{x}^{k} =$$

	$\int$	Σ
+	$\boldsymbol{x}^{k +\int}$	$\boldsymbol{x}^{k +\Sigma}$
*	$\boldsymbol{x}^{k *\int}$	$\boldsymbol{x}^{k *\Sigma}$

A Few Quick Notes on This Nonstandard Notation

* A quick explanation for this notation: normally with multiplication, the output of one multiplication can be calculated and then used as input in another multiplication. For example, when computing x^3 , we could say $v = x^2$, compute *v* as an intermediate step, then compute $v \cdot x$ and arrive at a correct answer for x^3 . This works because the constraints on the 3 terms are all independent of one another. This isn't valid for any of the four variations of $\boldsymbol{x}^{k}$. I use this notation to make clear that all multiplication and exponentiation of bolded terms needs to be considered as one big atomic convolution operation. But addition and scalar multiplication can be performed first in the usual ways. I've tried a number of notations – the discrete multiplicative case can be notated as Dirichlet convolutions, for example – but ultimately I settled on cribbing from vector space notation as being the most straightforward. In particular, I wanted a way to show the notational symmetries for the four variants.

* If I don't specify $+/*$ or $\int / \sum$ – so like this $\boldsymbol{x}^k$ - it means I'm saying something that applies to all four variations.

* This is the only non-standard notation I'll include here, I promise! This is it.

Recursive Definition for $\boldsymbol{x}^k$

So, one might think, from the descriptions of the variations of $\boldsymbol{x}^k$, these terms must have certain symmetries with one another. And they do.

In fact, with just a few simple parameters, a recursive function can be written that calculates both x^k and any of the four variations of $\boldsymbol{x}^k$, with x a real number >= 1 and k a whole number >= 0.

The function, $f_k(x, d, \theta, I)$, is defined as:

$$f_k(x, d, \theta, I) = g_k(x, d+I) \quad \text{where} \quad g_k(x, t) = \begin{cases} d \cdot g_{k-1}(\theta(x, t), d+I) + g_k(x, d+t) & x \geq t \\ 0 & x < t, k > 0 \\ 1 & k = 0 \end{cases}$$

Given some value of x such that 1<x<2, x^k can be expressed as

$$x^k = \lim_{d \to 0} f_k(x, d, \theta_=, 1) \quad \text{with} \quad \theta_=(x, j) = x$$

Given two helper functions, $\theta_+(x, t) = x - t$ and $\theta_*(x, t) = \frac{x}{t}$, the four variants of $\boldsymbol{x}^k$ can likewise be expressed as

	$\int$	$\sum$
$+$	$\lim_{d \to 0} f_k(x, d, \theta_+, 0)$	$f_k(x, 1, \theta_+, 0)$
$*$	$\lim_{d \to 0} f_k(x, d, \theta_*, 1)$	$f_k(x, 1, \theta_*, 1)$

Mathematica Computation of $\boldsymbol{x}^k$

Just to make this redundantly clear, in Mathematica, this recursive function can be implemented as

```
delta=.01;
thetaAdd[x_,t_]:=x-t
thetaMul[x_,t_]:=x/t
thetaEq[x_,t_]:=x
f[x_,t_,0,d_,fn_,I_]:=1
f[x_,t_,k_,d_,fn_,I_]:= If[x<t,0, d f[fn[x,t],d+I,k-1,d,fn,I] + f[x,d+t,k,d,fn,I]]
```

With this, an approximation of x^k can be computed, if 1 < x < 2, as f[x,1+delta,k,delta,thetaEq,1], with the approximation getting closer and closer as *delta* gets closer to 0.

And $\boldsymbol{x}^k$ can be computed as

	$\int$	$\sum$
$+$	`f[x,delta,k,delta,thetaAdd,0]`	`f[x,1,k,1,thetaAdd,0]`
$*$	`f[x,1+delta,k,delta,thetaMul,1]`	`f[x,2,k,1,thetaMul,1]`

again with approximations of the integrals getting more accurate as delta gets closer to 0.

This isn't really a reasonable way to compute $\boldsymbol{x}^{k+\Sigma}$ and $\boldsymbol{x}^{k*\Sigma}$ except for small values of x, but it's especially not at all a reasonable way to compute $\boldsymbol{x}^{k+\int}$ or $\boldsymbol{x}^{k+\int}$. Fortunately, closed form values of those exist and will be mentioned shortly.

Specific Expressions of $\boldsymbol{x}^k$

I'll list a few explicit integrals and sums for computing $\boldsymbol{x}^k$ to the first few values of k, just to keep things nice and clear. Although the recursive expression in the previous section should hopefully hammer home just how closely related these various terms are to each other, the expressions in this section are easier to reason about for finding closed forms, which will be given in the next section.

So, here's the first power,

$$x=\int_0^x dt$$

and

$$\boldsymbol{x}=$$

	$\int$	Σ
+	$\int_0^x dt$	$\sum_{t=1}^{x} 1$
*	$\int_1^x dt$	$\sum_{t=2}^{x} 1$

And the second,

$$x^2=\int_0^x\int_0^x du\,dt$$

with

$$\boldsymbol{x}^2=$$

	$\int$	Σ
+	$\int_0^x\int_0^{x-t} du\,dt$	$\sum_{t=1}^{x}\sum_{u=1}^{x-t} 1$
*	$\int_1^x\int_1^{\frac{x}{t}} du\,dt$	$\sum_{t=2}^{x}\sum_{u=2}^{\lfloor\frac{x}{t}\rfloor} 1$

And the third power,

$$x^3=\int_0^x\int_0^x\int_0^x dv\,du\,dt$$

and

$$\boldsymbol{x}^3=$$

	$\int$	Σ
+	$\int_0^x\int_0^{x-t}\int_0^{x-t-u} dv\,du\,dt$	$\sum_{t=1}^{x}\sum_{u=1}^{x-t}\sum_{v=1}^{x-t-u} 1$

*	$\int_1^x \int_1^{\frac{x}{t}} \int_1^{\frac{x}{tu}} dv\,du\,dt$	$\sum_{t=2}^{x} \sum_{u=2}^{\lfloor \frac{x}{t} \rfloor} \sum_{v=2}^{\lfloor \frac{x}{t \cdot u} \rfloor} 1$

and so on. This should be pretty simple.

Closed Form Values for $\boldsymbol{x^k}$

With a bit of work, most variations of $\boldsymbol{x^k}$ can be expressed in concise closed forms. This is significant; later sections are going to show how variants of $\boldsymbol{x^k}$ behave similarly to x^k. This will then let us find interesting identities involving the following closed forms, inspired by identities involving x^k.

So, those closed form expressions are

$$\boldsymbol{x^k} =$$

	$\int$	$\sum$
+	$\frac{x^k}{k!}$	$\binom{x}{k}$
*	$(-1)^{-k} \cdot P(k, -\log x)$	$D_k'(x)$

Here,

$\binom{x}{k}$: binomial coefficient defined as $\binom{x}{k} = \frac{x!}{k!(x-k)!} = \frac{(x)_k}{k!}$, with $(x)_k = x(x-1)(x-2)...(x-k+1)$

$P(k,x)$: regularized incomplete gamma function, defined as $P(k,x) = \frac{\gamma(k,x)}{\Gamma(k)}$ and

$$(-1)^{-k} \cdot P(k, -\log x) = \int_1^x \frac{\log^{k-1} t}{(k-1)!} dt$$

$D_k'(x)$: the strict divisor summatory function, defined recursively as $D_k'(x) = \sum_{j=2}^{\lfloor x \rfloor} D_{k-1}'(\frac{x}{j})$ with $D_0'(x) = 1$

which is not a standard function.

So, with the exception of $\boldsymbol{x^{k*\Sigma}}$, all of these closed forms are for fairly standard functions. Not surprisingly, then, they often have many other representations. Several of those representations suggest some interesting symmetries between these forms, so I want to note a few of those below.

$\boldsymbol{x^{k*\int}}$ can also be expressed, for suitable values of x and k, as

$$= \frac{\log^k x}{k!} \cdot {}_1F_1(k; k+1; \log x)$$

$$= \frac{\log^k x}{k!} \cdot \sum_{t=0}^{\infty} \left(1 + \frac{t}{k}\right)^{-1} \cdot \frac{\log^t x}{t!}$$

$$= (-1)^k \left(1 - x \sum_{j=0}^{k-1} \frac{(-\log x)^j}{j!}\right)$$

$$(-1)^k \left(1 - x \sum_{j=0}^{k-1} \frac{(-\log x)^j}{j!}\right)$$

Because I'm interested in stressing symmetries, note the following relationships between these closed forms:

$$\boldsymbol{x^{k+\int}} = \int_0^x \frac{t^{k-1}}{(k-1)!} dt$$

$$\boldsymbol{x}^{k*\int}=\int_1^x \frac{\log^{k-1} t}{(k-1)!}\,dt$$

as well as

$$\boldsymbol{x}^{k+\int}=\frac{x}{k}\cdot\frac{x}{k-1}\cdot\ldots\cdot\frac{x}{k-(k-1)}$$
$$\boldsymbol{x}^{k+\Sigma}=\frac{x}{k}\cdot\frac{x-1}{k-1}\cdot\ldots\cdot\frac{x-(k-1)}{k-(k-1)}$$

Note that, if we notate the difference we're measuring as d (so $d=1$ in the discrete case and $\lim_{d\to 0}$ in the continuous case), then $\boldsymbol{x}^{k+\int}$ and $\boldsymbol{x}^{k+\Sigma}$ can both be written identically as

$$\frac{x}{k}\cdot\frac{x-1\cdot d}{k-1}\cdot\ldots\cdot\frac{x-(k-1)\cdot d}{k-(k-1)}$$

Another very significant symmetry is as follows. If we notate the multiset coefficient as $\left(\!\binom{x}{k}\!\right)=\frac{x(x+1)(x+2)\ldots(x+k-1)}{k!}$, then we can express our two discrete sums $\boldsymbol{x}^{k+\Sigma}$ and $\boldsymbol{x}^{k*\Sigma}$ as

$$\boldsymbol{x}^{k+\Sigma}=\binom{x}{k}=\sum_{a=1}^{x}\sum_{j=0}^{k}(-1)^{k-j}\cdot\binom{k}{j}\cdot\left(\!\binom{j}{a}\!\right)$$
$$\boldsymbol{x}^{k*\Sigma}=D_k{}'(x)=\sum_{a=1}^{x}\sum_{j=0}^{k}(-1)^{k-j}\cdot\binom{k}{j}\prod_{p^\alpha|a}\left(\!\binom{j}{\alpha}\!\right)$$

Here, $\prod_{p^\alpha|a} f(\alpha)$ denotes a product over the powers of the prime factorization of a.

Closed Form Values for Derivatives / Finite Differences

We can also take inspiration from identities involving x^k to find interesting identities involving the derivatives or finite differences of $\boldsymbol{x}^k$, the closed forms of which will be listed here.

Through the usual rules of calculus, the derivative of x^k with respect to *x* is easily found to be, of course,

$$\frac{\partial}{\partial x}x^k=k\cdot x^{k-1}$$

Given the closed forms above, we can also express the derivatives of $\boldsymbol{x}^{k+\int}$ and $\boldsymbol{x}^{k+\int}$ with respect to x, and the finite differences of $\boldsymbol{x}^{k+\Sigma}$ and $\boldsymbol{x}^{k*\Sigma}$ with respect to x. Those values are

$$\frac{\partial}{\partial x}\boldsymbol{x}^k=\ /\ \nabla_x \boldsymbol{x}^k=$$

	$\int$	Σ
+	$\frac{x^{k-1}}{(k-1)!}$	$\binom{x-1}{k-1}$
*	$\frac{\log^{k-1} x}{(k-1)!}$	$d_k{}'(x)$

Note that, if we notate the difference we're measuring as d (so $d=1$ in the finite difference case and $\lim_{d\to 0}$ in the case of the derivatives), then $\frac{\partial}{\partial x}\boldsymbol{x}^{k+\int}$ and $\nabla_x \boldsymbol{x}^{k+\Sigma}$ can both be written identically as

$$\frac{x-1\cdot d}{k-1}\cdot\frac{x-2\cdot d}{k-2}\cdot\ldots\cdot\frac{x-(k-2)\cdot d}{(k-1)-(k-2)}$$

$d_k'(x)$ is another function without a standard name, but with a handful of interesting properties worth noting. $d_k'(x)$ can be defined as

$$d_k'(x)=D_k'(x)-D_k'(x-1)$$

$$d_k'(x)=\sum_{j|x;\ 1<j<x} d_{k-1}'\left(\frac{x}{j}\right) \text{ with } d_1'(x)=1$$

$$d_k'(x)=\sum_{j=0}^{k}(-1)^{k-j}\cdot\binom{k}{j}\prod_{p^\alpha|x}\left(\binom{j}{\alpha}\right)$$

Here, $\prod_{p^\alpha|x} f(\alpha)$ denotes a product over the powers of the prime factorization of x.

Additionally, if we have x in its prime factored form as $x=p_1^{a_1}\cdot p_2^{a_2}\cdot\ldots p_t^{a_t}$, and ${}_pF_q$ is the generalized hypergeometric function, then

$$d_k'(p_1^{a_1}\cdot p_2^{a_2}\cdot\ldots p_t^{a_t})=(-1)^{k+1}\cdot k\cdot {}_pF_q\left\{\begin{matrix}1+a_1, 1+a_2, \ldots 1+a_t, 1-k\\ 1\,(\text{t-1 times}), 2\end{matrix};1\right\}$$

Two special cases of $d_k'(x)$ are worth noting. First, if the prime factorization of $x=p^a$ - so x is a prime raised to a whole number power – then $d_k'(x)$ is

$$d_k'(p^a)=\binom{a-1}{k-1}$$

which is $\nabla_a \boldsymbol{a}^{k+\Sigma}$ from the above table, showing another nice bit of symmetry. The other special case is when $x=p_1\cdot p_2\cdot\ldots p_a$, which is to say, the prime factorization of x consists only of primes not raised to any powers. In that case,

$$d_k'(p_1\cdot p_2\cdot\ldots p_a)=S(a,k)\cdot k!$$

where $S(a,k)$ are Stirling numbers of the second kind.

These derivatives and finite differences are going to prove to be important for the following identities in just a second.

Meanwhile, just as, given $\frac{\partial}{\partial x}x^k=k\cdot x^{k-1}$, we can integrate to get $x^k=\int_0^x k\cdot t^{k-1}dt$, with our convolutions we likewise have the following:

	$\int$	$\sum$
+	$\frac{x^k}{k!}=\int_0^x \frac{t^{k-1}}{(k-1)!}dt$	$\binom{x}{k}=\sum_{t=1}^{x}\binom{t-1}{k-1}$
*	$(-1)^{-k}\cdot P(k,-\log x)=\int_1^x \frac{\log^{k-1} t}{(k-1)!}dt$	$D_k'(x)=\sum_{t=2}^{x} d_k'(t)$

Multiplication: $x^{a+b}=x^a\cdot x^b$

One basic property of exponents is they're added together when two terms with the same base are multiplied together, like so.

$$x^{a+b}=x^a\cdot x^b$$

This is also true for our convolutions, as

$$\boldsymbol{x^{a+b}}=\boldsymbol{x^a}\cdot\boldsymbol{x^b}$$

Thus, by analogy to

$$x^{a+b}=\int_0^x\int_0^x\frac{\partial}{\partial t}t^a\cdot\frac{\partial}{\partial u}u^b\,du\,dt$$

we have

	$\int$	$\sum$
$+$	$\boldsymbol{x^{a+b}}=\int_0^x\int_0^{x-t}\frac{\partial}{\partial t}\boldsymbol{t^a}\cdot\frac{\partial}{\partial u}\boldsymbol{u^b}\,du\,dt$	$\boldsymbol{x^{a+b}}=\sum_{t=1}^{x}\sum_{u=1}^{x-t}\nabla_t\boldsymbol{t^a}\cdot\nabla_u\boldsymbol{u^b}$
$*$	$\boldsymbol{x^{a+b}}=\int_1^x\int_1^{\frac{x}{t}}\frac{\partial}{\partial t}\boldsymbol{t^a}\cdot\frac{\partial}{\partial u}\boldsymbol{u^b}\,du\,dt$	$\boldsymbol{x^{a+b}}=\sum_{t=2}^{x}\sum_{u=2}^{\frac{x}{t}}\nabla_t\boldsymbol{t^a}\cdot\nabla_u\boldsymbol{u^b}$

Finally, if we insert our actual closed form values, we have

$$x^{a+b}=\int_0^x\int_0^x(a\,t^{a-1})\cdot(b\,u^{b-1})\,du\,dt$$

In our convolution cases, this leads to the following identities, which are essentially analogs to $x^{a+b}=x^a\cdot x^b$:

	$\int$	$\sum$
$+$	$\frac{x^{a+b}}{(a+b)!}=\int_0^x\int_0^{x-t}\frac{t^{a-1}}{(a-1)!}\cdot\frac{u^{b-1}}{(b-1)!}\,du\,dt$	$\binom{x}{a+b}=\sum_{t=1}^{x}\sum_{u=1}^{x-t}\binom{t-1}{a-1}\cdot\binom{u-1}{b-1}$
$*$	$(-1)^{-a-b}\cdot P(a+b,-\log x)=\int_1^x\int_1^{\frac{t}{t}}\frac{\log^{a-1}t}{(a-1)!}\cdot\frac{\log^{b-1}u}{(b-1)!}\,du\,dt$	$D_{a+b}'(x)=\sum_{t=2}^{x}\sum_{u=2}^{\lfloor\frac{x}{t}\rfloor}d_a'(t)\cdot d_b'(u)$

Multiplication: $\frac{\partial}{\partial x}x^{a+b}=\frac{\partial}{\partial x}x^a\cdot x^b$

We can also take the derivative of the above identity, of course, as

$$\frac{\partial}{\partial x}x^{a+b}=\frac{\partial}{\partial x}x^a\cdot x^b$$

In the convolution case, the analogous relationship is

$$\frac{\partial}{\partial x}\boldsymbol{x^{a+b}}=\frac{\partial}{\partial x}\boldsymbol{x^a}\cdot\boldsymbol{x^b}\ /\ \nabla_x\boldsymbol{x^{a+b}}=\nabla_x\boldsymbol{x^a}\cdot\boldsymbol{x^b}$$

One way to view this is as

$$\frac{\partial}{\partial x}x^{a+b}=\frac{\partial}{\partial x}\left(\int_0^x\int_0^x \frac{\partial}{\partial t}t^a\cdot\frac{\partial}{\partial u}u^b\,du\,dt\right)$$

which is

$$\frac{\partial}{\partial x}x^{a+b}=\int_0^x \frac{\partial}{\partial t}t^a\cdot\frac{\partial}{\partial x}x^b+\frac{\partial}{\partial x}x^a\cdot\frac{\partial}{\partial t}t^b\,dt$$

For our convolutions, the analogy is

	$\int$	$\sum$
$+$	$\frac{\partial}{\partial x}\boldsymbol{x}^{\boldsymbol{a+b}}=\int_0^x \frac{\partial}{\partial t}\boldsymbol{t}^{\boldsymbol{a}}\cdot\frac{\partial}{\partial t}(\boldsymbol{x}-\boldsymbol{t})^{\boldsymbol{b}}\,dt$	$\nabla_x \boldsymbol{x}^{\boldsymbol{a+b}}=\sum_{t=1}^{x-1}\nabla_t \boldsymbol{t}^{\boldsymbol{a}}\cdot\nabla_t(\boldsymbol{x}-\boldsymbol{t})^{\boldsymbol{b}}$
$*$	$\frac{\partial}{\partial x}\boldsymbol{x}^{\boldsymbol{a*b}}=\int_1^x \frac{\partial}{\partial t}\boldsymbol{t}^{\boldsymbol{a}}\cdot\frac{\partial}{\partial t}\left(\frac{\boldsymbol{x}}{\boldsymbol{t}}\right)^{\boldsymbol{b}}\,dt$	$\nabla_x \boldsymbol{x}^{\boldsymbol{a*b}}=\sum_{t\mid x,\,1<t<x}\nabla_t \boldsymbol{t}^{\boldsymbol{a}}\cdot\nabla_t\left(\frac{\boldsymbol{x}}{\boldsymbol{t}}\right)^{\boldsymbol{b}}$

If we fill in our closed form representations, this becomes

$$(a+b)\,x^{a+b-1}=\int_0^x a\,t^{a-1}\cdot b\,x^{b-1}+a\,x^{a-1}\cdot b\,t^{b-1}\,dt$$

And so, finally, we arrive where we want to be - in the convolution case, this is

	$\int$	$\sum$
$+$	$\frac{x^{a+b-1}}{(a+b-1)!}=\int_0^x \frac{t^{a-1}}{(a-1)!}\cdot\frac{(x-t)^{b-1}}{(b-1)!}\,dt$	$\binom{x-1}{a+b-1}=\sum_{t=1}^{x-1}\binom{t-1}{a-1}\cdot\binom{(x-t)-1}{b-1}$
$*$	$\frac{\log^{a+b-1}x}{(a+b-1)!}=\int_1^x \frac{\log^{a-1}t}{(a-1)!}\cdot\left(\frac{\log^{b-1}\left(\frac{x}{t}\right)}{(b-1)!}\cdot\frac{1}{t}\right)dt$	$d_{a+b}{}'(x)=\sum_{t\mid x,\,1<t<x}d_a{}'(t)\cdot d_b{}'\left(\frac{x}{t}\right)$

On Convergence

Let's talk about convergence. In the next few sections, I'll write sums of the form $\sum_{k=0}^{\infty}C_k\cdot\boldsymbol{x}^{\boldsymbol{k}}$. It's important to know if those sums converge, given that $\sum_{k=0}^{\infty}C_k\cdot x^k$ often only converges for certain values of x. I'm only concerned about cases where x is an integer value greater than or equal to 1, here.

Look at the closed forms just given for $\boldsymbol{x}^{\boldsymbol{k}+\int}$ or $\boldsymbol{x}^{\boldsymbol{k}+\int}$. As k grows, regardless of x, the factorial in the denominator dwarfs the exponential in the numerator. So those sums will eventually converge.

The cases of $\boldsymbol{x}^{\boldsymbol{k}+\Sigma}$ and $\boldsymbol{x}^{\boldsymbol{k}*\Sigma}$ are even easier.

It happens to be the case that, if k>=x, $\boldsymbol{x}^{\boldsymbol{k}+\Sigma}=\binom{x}{k}=0$, so for a given whole number x, the sum will consist of x non-zero terms.

And it is likewise the case that, if $k>\lfloor\frac{\log x}{\log 2}\rfloor$, $\boldsymbol{x}^{\boldsymbol{k}*\Sigma}=0$, so for a given whole number x, the sum will consist of $\lfloor\frac{\log x}{\log 2}\rfloor$ non-zero terms.

General Series Reversion and Inverse Functions

Here's one immediate use of these convergence properties.

First, I want to mention how inverse functions normally work, for Taylor series. Suppose we have some function defined as a power series in x with a sequence of coefficients a_k , as

$$f(x)=\sum_{k=1}^{\infty} a_k \cdot x^k$$

There is a second sequence of coefficients b_k , determined by a_k , that will invert this relationship, as

$$x=\sum_{k=1}^{\infty} b_k \cdot f(x)^k$$

Let's quickly look at three examples of such pairs. One very simple one is $a_k=1, b_k=(-1)^{k+1}$. This gives us

$$\frac{x}{1-x}=\sum_{k=1}^{\infty} x^k$$

$$x=\sum_{k=1}^{\infty} (-1)^{k+1} \cdot \left(\frac{x}{1-x}\right)^k$$

Another example includes $a_k=\frac{(-1)^{k+1}}{k}, b_k=\frac{1}{k!}$, giving us

$$\log(1+x)=\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k$$

$$x=\sum_{k=1}^{\infty} \frac{1}{k!} \cdot \log^k(1+x)$$

Let's look at somewhat more esoteric coefficients. If $a_k=0$ and $b_k=0$ when k is even, and $a_k=\frac{(-1)^{\frac{k-1}{2}}}{k!}$ and $b_k=\frac{(-1)^{\frac{k-1}{2}}}{k} \cdot \binom{-1/2}{k/2-1/2}$ when k is odd (the second term in b_k is a binomial coefficient), then we have

$$\sin x=\sum_{k=1}^{\infty} \frac{(-1)^{\frac{k-1}{2}}}{k!} \cdot x^k$$

$$x=\sum_{k=1}^{\infty} \frac{(-1)^{\frac{k-1}{2}}}{k} \cdot \binom{-1/2}{k/2-1/2} \cdot \sin^k x$$

These latter two pairs of series are subject to the normal rules about convergence, of course.

An analogy of this approach also works for our various convolutions, which is to say, given the exact same sequence of coefficients a_k and its inverse b_k , there is a specific way to make sense of the assertion that if

$$f(\boldsymbol{x})=\sum_{k=1}^{\infty} a_k \cdot \boldsymbol{x}^{\boldsymbol{k}}$$

then

$$\boldsymbol{x} = \sum_{k=1}^{\infty} b_k \cdot f(\boldsymbol{x})^{\boldsymbol{k}}$$

$$\frac{\partial}{\partial x} f(\boldsymbol{x}) = \sum_{k=1}^{\infty} a_k \cdot \frac{\partial}{\partial x} \boldsymbol{x}^{\boldsymbol{k}}$$

$$\nabla_x f(\boldsymbol{x}) = \sum_{k=1}^{\infty} a_k \cdot \nabla_x \boldsymbol{x}^{\boldsymbol{k}}$$

$$f(\boldsymbol{x})^2 =$$

	$\int$	$\sum$
+	$\int_0^x \int_0^{x-t} \frac{\partial}{\partial t} f(\boldsymbol{t}) \cdot \frac{\partial}{\partial u} f(\boldsymbol{u})\, du\, dt$	$\sum_{t=1}^{x} \sum_{u=1}^{x-t} \nabla_t f(\boldsymbol{t}) \cdot \nabla_u f(\boldsymbol{u})$
*	$\int_1^x \int_1^{\frac{x}{t}} \frac{\partial}{\partial t} f(\boldsymbol{t}) \cdot \frac{\partial}{\partial u} f(\boldsymbol{u})\, du\, dt$	$\sum_{t=2}^{x} \sum_{u=2}^{\lfloor \frac{x}{t} \rfloor} \nabla_t f(\boldsymbol{t}) \cdot \nabla_u f(\boldsymbol{u})$

$$f(\boldsymbol{x})^3 =$$

	$\int$	$\sum$
+	$\int_0^x \int_0^{x-t} \int_0^{x-t-u} \frac{\partial}{\partial t} f(\boldsymbol{t}) \cdot \frac{\partial}{\partial u} f(\boldsymbol{u}) \cdot \frac{\partial}{\partial v} f(\boldsymbol{v})\, dv\, du\, dt$	$\sum_{t=1}^{x} \sum_{u=1}^{x-t} \sum_{v=1}^{x-t-u} \nabla_t f(\boldsymbol{t}) \cdot \nabla_u f(\boldsymbol{u}) \cdot \nabla_v f(\boldsymbol{v})$
*	$\int_1^x \int_1^{\frac{x}{t}} \int_1^{\frac{x}{tu}} \frac{\partial}{\partial t} f(\boldsymbol{t}) \cdot \frac{\partial}{\partial u} f(\boldsymbol{u}) \cdot \frac{\partial}{\partial v} f(\boldsymbol{v})\, dv\, du\, dt$	$\sum_{t=2}^{x} \sum_{u=2}^{\lfloor \frac{x}{t} \rfloor} \sum_{v=2}^{\lfloor \frac{x}{t \cdot u} \rfloor} \nabla_t f(\boldsymbol{t}) \cdot \nabla_u f(\boldsymbol{u}) \cdot \nabla_v f(\boldsymbol{v})$

and so on.

$$f(x)^{a+b} = f(x)^a \cdot f(x)^b$$

$$f(\boldsymbol{x})^{\boldsymbol{a}+\boldsymbol{b}} = f(\boldsymbol{x})^{\boldsymbol{a}} \cdot f(\boldsymbol{x})^{\boldsymbol{b}}$$

$$f(x)^{a+b} = \int_0^x \int_0^x \frac{\partial}{\partial t} f(t)^a \cdot \frac{\partial}{\partial u} f(u)^b\, du\, dt$$

	$\int$	$\sum$
+	$f(\boldsymbol{x})^{\boldsymbol{a}+\boldsymbol{b}} = \int_0^x \int_0^{x-t} \frac{\partial}{\partial t} f(\boldsymbol{t})^{\boldsymbol{a}} \cdot \frac{\partial}{\partial u} f(\boldsymbol{u})^{\boldsymbol{b}}\, du\, dt$	$f(\boldsymbol{x})^{\boldsymbol{a}+\boldsymbol{b}} = \sum_{t=1}^{x} \sum_{u=1}^{x-t} \nabla_t f(\boldsymbol{t})^{\boldsymbol{a}} \cdot \nabla_u f(\boldsymbol{u})^{\boldsymbol{b}}$
*	$f(\boldsymbol{x})^{\boldsymbol{a}+\boldsymbol{b}} = \int_1^x \int_1^{\frac{x}{t}} \frac{\partial}{\partial t} f(\boldsymbol{t})^{\boldsymbol{a}} \cdot \frac{\partial}{\partial u} f(\boldsymbol{u})^{\boldsymbol{b}}\, du\, dt$	$f(\boldsymbol{x})^{\boldsymbol{a}+\boldsymbol{b}} = \sum_{t=2}^{x} \sum_{u=2}^{\frac{x}{t}} \nabla_t f(\boldsymbol{t})^{\boldsymbol{a}} \cdot \nabla_u f(\boldsymbol{u})^{\boldsymbol{b}}$

$$\frac{\partial}{\partial x} f(x)^{a+b} = \frac{\partial}{\partial x} f(x)^a \cdot f(x)^b$$

$$\frac{\partial}{\partial x} f(\boldsymbol{x})^{a+b} = \frac{\partial}{\partial x} f(\boldsymbol{x})^{a} \cdot f(\boldsymbol{x})^{b} \,/\, \nabla_x f(\boldsymbol{x})^{a+b} = \nabla_x f(\boldsymbol{x})^{a} \cdot f(\boldsymbol{x})^{b}$$

$$\frac{\partial}{\partial x} f(x)^{a+b} = \int_0^x \frac{\partial}{\partial t} f(t)^{a} \cdot \frac{\partial}{\partial x} f(x)^{b} + \frac{\partial}{\partial x} f(x)^{a} \cdot \frac{\partial}{\partial t} f(t)^{b}\, dt$$

	$\int$	$\sum$
+	$\frac{\partial}{\partial x} f(\boldsymbol{x})^{a+b} = \int_0^x \frac{\partial}{\partial t} f(\boldsymbol{t})^{a} \cdot \frac{\partial}{\partial t} f(\boldsymbol{x}-\boldsymbol{t})^{b}\, dt$	$\nabla_x f(\boldsymbol{x})^{a+b} = \sum_{t=1}^{x-1} \nabla_t f(\boldsymbol{t})^{a} \cdot \nabla_t f(\boldsymbol{x}-\boldsymbol{t})^{b}$
*	$\frac{\partial}{\partial x} f(\boldsymbol{x})^{a+b} = \int_1^x \frac{\partial}{\partial t} f(\boldsymbol{t})^{a} \cdot \frac{\partial}{\partial t} f\left(\frac{\boldsymbol{x}}{\boldsymbol{t}}\right)^{b} dt$	$\nabla_x f(\boldsymbol{x})^{a+b} = \sum_{t \mid x,\, 1<t<x} \nabla_t f(\boldsymbol{t})^{a} \cdot \nabla_t f\left(\frac{\boldsymbol{x}}{\boldsymbol{t}}\right)^{b}$

Section 2:
$(1+\boldsymbol{x})^z$

Overview

In this section, I'll introduce the convolution equivalent of Newton's Generalized Binomial Theorem.

In the previous section, when looking at our terms of the form $\boldsymbol{x}^{\boldsymbol{k}}$, we only dealt with k as a whole number. We'll revisit this constraint later, but generally, even once we find ways to generalize k, most of the identities for $\boldsymbol{x}^{\boldsymbol{k}}$ only work if k is a whole number. You might be familiar with this from $\boldsymbol{x}^{k+\Sigma}$, the closed form of which is the binomial coefficient $\binom{x}{k}$. It is possible to generalize the binomial coefficient as $\binom{x}{z}=\frac{\Gamma(x+1)}{\Gamma(z+1)\cdot\Gamma(x-z+1)}$, where $\Gamma(z)$ is the gamma function, generally regarded as the complex extension of the factorial function. Nevertheless, many important identities for the binomial coefficients don't hold when z is complex.

Fortunately, no such problems arise when investigating $(1+\boldsymbol{x})^z$, which we will define momentarily with the binomial theorem.

Newton's Generalized Binomial Theorem

Newton's generalized binomial theorem provides a very well known way to express complex exponentiation. If $0<x<2$, then

$$(1+x)^z=\sum_{k=0}^{\infty}\binom{z}{k}x^k$$

where $\binom{z}{k}$ is defined as $\frac{z\cdot(z-1)\cdot\ldots\cdot(z-k+1)}{k!}$ and z can be any complex value. Now here's something crucially important; with this definition, we can take a derivative of x^z with respect to *z*.

It turns out we can define an analogous identity for any of our variations of $\boldsymbol{x}^{\boldsymbol{k}}$, like so.

$$(1+\boldsymbol{x})^z=\sum_{k=0}^{\infty}\binom{z}{k}\boldsymbol{x}^{\boldsymbol{k}}$$

This will converge for any *x* that is a positive whole number.

Explicit Definition of Generalized Binomial

Let me write this out more explicitly. Newton's Generalized Binomial expression, and then the four convolution equivalents, is

$$(1+x)^z \quad = \quad 1 \quad + \quad \binom{z}{1}\int_0^x dt \quad + \quad \binom{z}{2}\int_0^x\int_0^x du\,dt \quad + \quad \binom{z}{3}\int_0^x\int_0^x\int_0^x dv\,du\,dt \quad + \ldots$$

$$(1+\boldsymbol{x})^{z+\int} \quad = \quad 1 \quad + \quad \binom{z}{1}\int_0^x dt \quad + \quad \binom{z}{2}\int_0^x\int_0^{x-t} du\,dt \quad + \quad \binom{z}{3}\int_0^x\int_0^{x-t}\int_0^{x-t-u} dv\,du\,dt \quad + \ldots$$

$$(1+\boldsymbol{x})^{z+\Sigma} \quad = \quad 1 \quad + \quad \binom{z}{1}\sum_{t=1}^{x} 1 \quad + \quad \binom{z}{2}\sum_{t=1}^{x}\sum_{u=1}^{x-t} 1 \quad + \quad \binom{z}{3}\sum_{t=1}^{x}\sum_{u=1}^{x-t}\sum_{v=1}^{x-t-u} 1 \quad + \ldots$$

$$(1+\boldsymbol{x})^{z*\int} \quad = \quad 1 \quad + \quad \binom{z}{1}\int_1^x dt \quad + \quad \binom{z}{2}\int_1^x\int_1^{\frac{x}{t}} du\,dt \quad + \quad \binom{z}{3}\int_1^x\int_1^{\frac{x}{t}}\int_1^{\frac{x}{tu}} dv\,du\,dt \quad + \ldots$$

$$(1+\boldsymbol{x})^{z*\Sigma} \quad = \quad 1 \quad + \quad \binom{z}{1}\sum_{t=2}^{x} 1 \quad + \quad \binom{z}{2}\sum_{t=2}^{x}\sum_{u=2}^{\lfloor\frac{x}{t}\rfloor} 1 \quad + \quad \binom{z}{3}\sum_{t=2}^{x}\sum_{u=2}^{\lfloor\frac{x}{t}\rfloor}\sum_{v=2}^{\lfloor\frac{x}{t\cdot u}\rfloor} 1 \quad + \ldots$$

Recursive Definition for $(1+\boldsymbol{x})^z$

To highlight the symmetry between these expressions, I note they all can be written as differently parameterized versions of the following recursive expression:

$$f_k(x)=1+d\cdot\left(\frac{z+1}{k}-1\right)\sum_{t=1}^{\lfloor\frac{x-I}{d}\rfloor} f_{k+1}(\theta(x,d\cdot t+I))$$
$$f(x,d,\theta,I)=f_1(x)$$

Given this, Newton's generalized binomial theorem (for 1 < x < 2) can be written as

$$(1+x)^z=\lim_{d\to 0} f(x,d,\theta_=,1) \quad \text{with} \quad \theta_=(x,t)=x$$

If two helper functions, $\theta_+(x,t)=x-t$ and $\theta_*(x,t)=\frac{x}{t}$, are defined, we can also express our four variants of $(1+\boldsymbol{x})^z$ as

$$(1+\boldsymbol{x})^z=$$

	$\int$	Σ
$+$	$\lim_{d\to 0} f(x,d,\theta_+)$	$f(x,1,\theta_+)$
$*$	$\lim_{d\to 0} f(x,d,\theta_*)$	$f(x,1,\theta_*)$

Mathematica Implementation of $(1+\boldsymbol{x})^z$

This recursive definition, for the 4 variants of $(1+\boldsymbol{x})^z$, can be implemented in Mathematica as

```
delta=.01;
thetaAdd[x_,t_]:=x-t
thetaMul[x_,t_]:=x/t
thetaEq[x_,t_]:=x
f[x_,z_,k_,d_,fn_,I_]:=1+d ((z+1)/k-1) Sum[ f[fn[x,d t+I],z,k+1,d,fn,I],{t,1,(x-
I)/d}]
```

$(1+\boldsymbol{x})^z$ can then be computed as

	$\int$	$\sum$
+	`f[x,z,1,delta,thetaAdd,0]`	`f[x,z,1,1,thetaAdd,0]`
*	`f[x,z,1,delta,thetaMul,1]`	`f[x,z,1,1,thetaMul,1]`

with the values on the left side of the table getting closer to their actual values as delta gets closer to 0.

Note that this is a reasonable way to compute small values of $(1+\boldsymbol{x})^{z\ +\sum}$ and $(1+\boldsymbol{x})^{z\ *\sum}$, but not at all reasonable to compute $(1+\boldsymbol{x})^{z\ +\int}$ or $(1+\boldsymbol{x})^{z\ *\int}$. Fortunately, straightforward closed form values of those exist.

Closed Form Values for $(1+\boldsymbol{x})^z$

In fact, the closed form values of $(1+\boldsymbol{x})^z$ are as follows:

$$(1+\boldsymbol{x})^z=$$

	$\int$	$\sum$
+	$L_z(-x)$	$\left(\!\binom{z+1}{x}\!\right)$
*	$x\cdot L_{z-1}(-\log x)$	$D_z(x)$

Here, $L_z(x)$ is the Laguerre polynomials $D_z(x)$ is the seldom seen generalized divisor summatory function, also defined as $D_z(x)=\sum_{n=1}^{x}\prod_{p^k|n}\left(\!\binom{z}{k}\!\right)$, where the product is over the prime factorization of n, and $\left(\!\binom{x}{k}\!\right)$ is the multiset coefficient defined as $\left(\!\binom{x}{k}\!\right)=\frac{x(x+1)(x+2)\ldots(x+k-1)}{k\,!}$

Closed Form Values for Derivatives / Finite Differences of $(1+\boldsymbol{x})^z$

We can also get closed forms for the derivatives / finite differences with respect to x. For reference, we have

$$\frac{\partial}{\partial x}(1+x)^z = z\cdot(1+x)^{z-1}$$

The convolution equivalent of this is

$$\frac{\partial}{\partial x}(1+\boldsymbol{x})^z=$$
OR
$$\nabla_x(1+\boldsymbol{x})^z=$$

	$\int$	$\sum$
+	$L_{z-1}^{1}(-x)$	$\left(\!\binom{z}{x}\!\right)$

*	$L^{1}_{z-1}(-\log x)$	$\prod_{p^k \mid x} \left(\!\binom{z}{k}\!\right)$

Here, $L^{(k)}_{z}(x)$ is the generalized Laguerre polynomials, and $d_z(x)$ is the seldom seen generalized divisor function, also defined as $d_z(x)=\prod_{p^k \mid x} \left(\!\binom{z}{k}\!\right)$, where the product is over the prime factorization of x.

The symmetries in this table should be particularly visually obvious. But let me point out just a few more.

$$(1+\boldsymbol{x})^{z\ +\Sigma}=\sum_{a=1}^{x} \nabla_a (1+\boldsymbol{a})^{z\ +\Sigma}$$

$$(1+\boldsymbol{x})^{z\ *\Sigma}=\sum_{a=1}^{x} \prod_{p^{\alpha} \mid a} \nabla_{\alpha} (1+\boldsymbol{\alpha})^{z\ +\Sigma}$$

$$\frac{(x+z)!}{x!\,z!}=\sum_{a=1}^{x} \frac{z^{(a)}}{a!}$$

$$D_z(x)=\sum_{a=1}^{x} \prod_{p^{\alpha} \mid a} \frac{z^{(\alpha)}}{\alpha!}$$

$$(1+\boldsymbol{x})^{z\ +\Sigma}=\sum_{a=1}^{x} (1+\boldsymbol{a})^{z-1\ +\Sigma}$$

$$(1+\boldsymbol{x})^{z\ *\Sigma}=\sum_{a=1}^{x} \prod_{p^{\alpha} \mid a} (1+\boldsymbol{\alpha})^{z-1\ +\Sigma}$$

$$\frac{(x+z)!}{x!\,z!}=\sum_{a=1}^{x} \frac{(a+z-1)!}{a!\,(z-1)!}$$

$$D_z(x)=\sum_{a=1}^{x} \prod_{p^{\alpha} \mid a} \frac{(\alpha+z-1)!}{\alpha!\,(z-1)!}$$

Just as $\frac{\partial}{\partial x}(1+x)^z = z\cdot(1+x)^{z-1}$ can be integrated to give $(1+x)^z = 1+\int_0^x z\cdot(1+t)^{z-1}\,dt$, we can take these derivatives / finite differences to give

	$\int$	$\sum$
+	$L_z(-x)=1+\int_0^x L^{1}_{z-1}(-t)\,dt$	$\left(\!\binom{z+1}{x}\!\right)=1+\sum_{t=1}^{x}\left(\!\binom{z}{t}\!\right)$
*	$x\cdot L_{z-1}(-\log x)=1+\int_1^x L^{1}_{z-1}(-\log t)\,dt$	$D_z(x)=1+\sum_{t=2}^{x}\prod_{p^{\tau} \mid t}\left(\!\binom{z}{\tau}\!\right)$

Closed Forms of $(1+\boldsymbol{x})^z$ In Terms of $\boldsymbol{x}^k$

Now that I have listed closed form expressions for the various forms of $(1+\boldsymbol{x})^z$, I can take the closed form expressions for $\boldsymbol{x}^k$ from the previous chapter to write rewrite the binomial theorem in more standard notation. Specifically,

$$(1+\boldsymbol{x})^{z}=\sum_{k=0}^{\infty}\binom{z}{k}\boldsymbol{x}^{k}$$

	$\int$	$\sum$
$+$	$L_z(-x)=\sum_{k=0}^{\infty}\binom{z}{k}\frac{x^k}{k!}$	$\left(\!\binom{z+1}{x}\!\right)=\sum_{k=0}^{\infty}\binom{z}{k}\binom{x}{k}$
$*$	$x\cdot L_{z-1}(-\log x)=\sum_{k=0}^{\infty}\binom{z}{k}(-1)^{-k}\cdot P(k,-\log x)$	$D_z(x)=\sum_{k=0}^{\infty}\binom{z}{k}D_k'(x)$

Closed Forms of $\boldsymbol{x}^k$ in Terms of $(1+\boldsymbol{x})^z$

The binomial expression can also be inverted, like so:

$$x^k=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(1+x)^j$$

This idea also holds for the variants of $(1+\boldsymbol{x})^z$, giving

$$\boldsymbol{x}^k=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(1+\boldsymbol{x})^j$$

	$\int$	$\sum$
$+$	$\frac{x^k}{k!}=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}L_j(-x)$	$\binom{x}{k}=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\cdot\left(\!\binom{x+1}{j}\!\right)$
$*$	$(-1)^{-k}\cdot P(k,-\log x)=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}x\cdot L_{j-1}(-\log x)$	$D_k'(x)=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}D_j(x)$

Multiplication: $(1+x)^{a+b}=(1+x)^a\cdot(1+x)^b$

The usual rule of multiplication of exponents says that

$$(1+x)^{a+b}=(1+x)^a\cdot(1+x)^b$$

This also works for our convolutions as

$$(1+\boldsymbol{x})^{a+b}=(1+\boldsymbol{x})^a\cdot(1+\boldsymbol{x})^b$$

Another, more elaborate, way to say this, is

$$(1+x)^{a+b}=1+\int_0^x\frac{\partial}{\partial t}(1+t)^a\,dt+\int_0^x\frac{\partial}{\partial u}(1+u)^b\,du+\int_0^x\int_0^x\frac{\partial}{\partial t}(1+t)^a\cdot\frac{\partial}{\partial u}(1+u)^b\,du\,dt$$

which, by analogy, is

	$\int$	$\sum$

$+$	$(1+\boldsymbol{x})^{a+b}=1+\int_0^x \frac{\partial}{\partial t}(1+\boldsymbol{t})^a\,dt+\int_0^x \frac{\partial}{\partial u}(1+\boldsymbol{u})^b\,du$ $+\int_0^x\int_0^{x-t} \frac{\partial}{\partial t}(1+\boldsymbol{t})^a\cdot\frac{\partial}{\partial u}(1+\boldsymbol{u})^b\,du\,dt$	$(1+\boldsymbol{x})^{a+b}=1+\sum_{t=1}^{x}\nabla_t(1+\boldsymbol{t})^a+\sum_{u=1}^{x}\nabla_u(1+\boldsymbol{u})^b$ $+\sum_{t=1}^{x}\sum_{u=1}^{x-t}\nabla_t(I+\boldsymbol{t})^a\cdot\nabla_u(1+\boldsymbol{u})^b$
$*$	$(1+\boldsymbol{x})^{a+b}=1+\int_1^x \frac{\partial}{\partial t}(1+\boldsymbol{t})^a\,dt+\int_1^x \frac{\partial}{\partial u}(1+\boldsymbol{u})^b\,du$ $+\int_1^x\int_1^{\frac{x}{t}} \frac{\partial}{\partial t}(1+\boldsymbol{t})^a\cdot\frac{\partial}{\partial u}(1+\boldsymbol{u})^b\,du\,dt$	$(1+\boldsymbol{x})^{a+b}=1+\sum_{t=2}^{x}\nabla_t(1+\boldsymbol{t})^a+\sum_{u=2}^{x}\nabla_u(1+\boldsymbol{u})^b$ $+\sum_{t=2}^{x}\sum_{u=1}^{\frac{x}{t}}\nabla_t(1+\boldsymbol{t})^a\cdot\nabla_u(1+\boldsymbol{u})^b$

With the closed forms applied, this is

$$(1+x)^{a+b}=1+\int_0^x a(1+t)^{a-1}\,dt+\int_0^x b(1+u)^{b-1}\,du+\int_0^x\int_0^x a(1+t)^{a-1}\cdot b(1+u)^{b-1}\,du\,dt$$

An analogous approach can be applied to the four variants of $(1+\boldsymbol{x})^z$ gives

	$\int$	$\sum$
$+$	$L_{a+b}(-x)=1+\int_0^x L_{a-1}^{(1)}(-t)\,dt+\int_0^x L_{b-1}^{(1)}(-t)\,dt$ $+\int_0^x\int_0^{x-t} L_{a-1}^{(1)}(-t)\cdot L_{b-1}^{(1)}(-u)\,du\,dt$	$\left(\!\binom{a+b+1}{x}\!\right)=\sum_{t=0}^{x}\sum_{u=0}^{x-t}\left(\!\binom{a}{t}\!\right)\cdot\left(\!\binom{b}{u}\!\right)$
$*$	$x\cdot L_{a+b-1}(-\log x)=1+\int_1^x L_{a-1}^{(1)}(-\log t)\,dt+\int_1^x L_{b-1}^{(1)}(-\log t)\,dt$ $+\int_1^x\int_1^{\frac{x}{t}} L_{a-1}^{(1)}(-\log t)\cdot L_{b-1}^{(1)}(-\log u)\,du\,dt$	$D_{a+b}(x)=\sum_{t=1}^{x}\sum_{u=1}^{\lfloor\frac{x}{t}\rfloor} d_a(t)\cdot d_b(u)$

Multiplication: $\frac{\partial}{\partial x}(1+x)^{a+b}=\frac{\partial}{\partial x}(1+x)^a\cdot(1+x)^b$

$$\frac{\partial}{\partial x}(1+x)^{a+b}=\frac{\partial}{\partial x}\left(1+\int_0^x \frac{\partial}{\partial t}(1+t)^a\,dt+\int_0^x \frac{\partial}{\partial u}(1+u)^b\,du+\int_0^x\int_0^x \frac{\partial}{\partial t}(1+t)^a\cdot\frac{\partial}{\partial u}(1+u)^b\,du\,dt\right)$$

$$\frac{\partial}{\partial x}(1+x)^{a+b}=\frac{\partial}{\partial x}(1+x)^a+\frac{\partial}{\partial x}(1+x)^b+\int_0^x \frac{\partial}{\partial t}(1+t)^a\cdot\frac{\partial}{\partial x}(1+x)^b+\frac{\partial}{\partial x}(1+x)^a\cdot\frac{\partial}{\partial t}(1+t)^b\,dt$$

$$(a+b)(1+x)^{a+b-1}=a(1+x)^{a-1}+b(1+x)^{b-1}+\int_0^x a(1+t)^{a-1}\cdot b(1+x)^{b-1}+a(1+x)^{a-1}\cdot b(1+t)^{b-1}\,dt$$

	$\int$	$\sum$
$+$	$L_{a+b-1}^{(1)}(-x)=L_{a-1}^{(1)}(-x)+L_{b-1}^{(1)}(-x)$ $+\int_0^x L_{a-1}^{(1)}(-t)\cdot L_{b-1}^{(1)}(-(x-t))\,dt$	$\left(\!\binom{a+b}{x}\!\right)=\sum_{t=0}^{x}\left(\!\binom{a}{t}\!\right)\cdot\left(\!\binom{b}{x-t}\!\right)$

*	$L^{(1)}_{-(a+b)-1}(\log x) = \frac{1}{x} \cdot L^{(1)}_{-a-1}(\log x) + \frac{1}{x} \cdot L^{(1)}_{-b-1}(\log x)$??? $+ \int_1^x dt$	$d_{a+b}(x) = \sum_{t \mid x} d_a(t) \cdot d_b(\frac{x}{t})$

General Transformations Using $(1+x)^z$

It's worth taking a momentary detour to show how the derivatives or finite differences of $(1+x)^z$ with respect to x lead naturally to a specific kind of transform. In particular, this exercise provides a natural way to think about Moebius inversion and to see how it relates to some closely related transforms.

To begin, let's start with the following (not especially interesting) transform pair. Given some function, we have

$$\hat{f}(x) = f(x) + \int_0^x (\frac{\partial}{\partial t}(1+t)^z) \cdot f(x)\, dt$$
$$f(x) = \hat{f}(x) + \int_0^x (\frac{\partial}{\partial t}(1+t)^{-z}) \cdot \hat{f}(x)\, dt$$

	$\int$	$\sum$
+	$\hat{f}(x) = f(x) + \int_0^x \frac{\partial}{\partial t}(1+t)^z \cdot f(x-t)\, dt$ $f(x) = \hat{f}(x) + \int_0^x \frac{\partial}{\partial t}(1+t)^{-z} \cdot \hat{f}(x-t)\, dt$	$\hat{f}(x) = f(x) + \sum_{t=1}^{x} \nabla_t (1+t)^z \cdot f(x-t)$ $f(x) = \hat{f}(x) + \sum_{t=1}^{x} \nabla_t (1+t)^{-z} \cdot \hat{f}(x-t)$
*	$\hat{f}(x) = f(x) + \int_1^x \frac{\partial}{\partial t}(1+t)^z \cdot f(\frac{x}{t})\, dt$ $f(x) = \hat{f}(x) + \int_1^x \frac{\partial}{\partial t}(1+t)^{-z} \cdot \hat{f}(\frac{x}{t})\, dt$	$\hat{f}(x) = f(x) + \sum_{t=2}^{x} \nabla_t (1+t)^z \cdot f(\frac{x}{t})$ $f(x) = \hat{f}(x) + \sum_{t=2}^{x} \nabla_t (1+t)^{-z} \cdot \hat{f}(\frac{x}{t})$

Now let's fill in our closed form values for these functions. We have

$$\hat{f}(x) = f(x) + \int_0^x (z \cdot (1+t)^{z-1}) \cdot f(x)\, dt$$
$$f(x) = \hat{f}(x) + \int_0^x (-z \cdot (1+t)^{-z-1}) \cdot \hat{f}(x)\, dt$$

and in our related convolution forms, we have

	$\int$	$\sum$

+	$\hat{f}(x)=f(x)+\int_0^x L_{z-1}^{(1)}(-t)\cdot f(x-t)\,dt$ $f(x)=\hat{f}(x)+\int_0^x L_{-z-1}^{(1)}(-t)\cdot \hat{f}(x-t)\,dt$	$\hat{f}(x)=f(x)+\sum_{t=1}^{x}\left(\!\!\binom{z}{t}\!\!\right)\cdot f(x-t)$ $f(x)=\hat{f}(x)+\sum_{t=1}^{x}\left(\!\!\binom{-z}{t}\!\!\right)\cdot \hat{f}(x-t)$
*	$\hat{f}(x)=f(x)+\int_1^x L_{z-1}^{(1)}(-\log t)\cdot f\left(\frac{x}{t}\right)dt$ $f(x)=\hat{f}(x)+\int_1^x L_{-z-1}^{(1)}(-\log t)\cdot \hat{f}\left(\frac{x}{t}\right)dt$	$\hat{f}(x)=f(x)+\sum_{t=2}^{x} d_z(t)\cdot f\left(\frac{x}{t}\right)$ $f(x)=\hat{f}(x)+\sum_{t=2}^{x} d_{-z}(t)\cdot \hat{f}\left(\frac{x}{t}\right)$

Now let's take a look at the special case where z=1.

For our non-convolution version, this leaves us with

$$\hat{f}(x)=f(x)+\int_0^x f(x)\,dt$$

$$f(x)=\hat{f}(x)-\int_0^x \frac{1}{(1+t)^2}\cdot \hat{f}(x)\,dt$$

which, if you work out the integrals and simplify, amounts to the obvious and uninteresting $\hat{f}(x)=(1+x)f(x)$ and $f(x)=\frac{1}{1+x}\hat{f}(x)$.

The convolution equivalents are more interesting, I think.

	$\int$	$\sum$
+	$\hat{f}(x)=f(x)+\int_0^x f(x-t)\,dt$ $f(x)=\hat{f}(x)-\int_0^x e^{-t}\cdot \hat{f}(x-t)\,dt$	$\hat{f}(x)=f(x)+\sum_{t=1}^{x} f(x-t)$ $f(x)=\hat{f}(x)-\hat{f}(x-1)$
*	$\hat{f}(x)=f(x)+\int_1^x f\left(\frac{x}{t}\right)dt$ $f(x)=\hat{f}(x)-\int_1^x \frac{1}{t}\cdot \hat{f}\left(\frac{x}{t}\right)dt$	$\hat{f}(x)=f(x)+\sum_{t=2}^{x} f\left(\frac{x}{t}\right)$ $f(x)=\hat{f}(x)+\sum_{t=2}^{x}\mu(t)\cdot \hat{f}\left(\frac{x}{t}\right)$

Here, $\mu(t)$ is the Moebius function.

There are two things interesting about this table.

Alright. I finally have enough defined for me to talk about $\log(1+\boldsymbol{x})$.

Section 3:
$\log(1+\boldsymbol{x})$

Overview

The identities in this section are really the key motivators of this whole paper. In essence, our various convolution equivalents of $\log(1+x)$ are the exponential integral, the Harmonic number, the logarithmic integral, and the Riemann Prime Counting function. This section will show a number of different ways to express these functions in the closed forms of $\boldsymbol{x}^{\boldsymbol{k}}$ and $(1+\boldsymbol{x})^z$ that we've already explored.

The Mercator Series Equivalent

The Mercator series for the logarithm is very well known to be, for 0 < x < 2,

$$\log(1+x)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}x^k$$

It turns out that a very useful analogous identity holds for the four variants of $\boldsymbol{x}^{\boldsymbol{k}}$ as

$$\log(1+\boldsymbol{x})=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\boldsymbol{x}^{\boldsymbol{k}}$$

This Series for $\log(1+\boldsymbol{x})$ Written Out Explicitly

Written out more explicitly, what I'm saying here is

$$\begin{array}{lclclclc}
\log(1+x) & = & \int_0^x dt & - & \frac{1}{2}\int_0^x\int_0^x du\,dt & + & \frac{1}{3}\int_0^x\int_0^x\int_0^x dv\,du\,dt & -\ldots \\
\log(1+\boldsymbol{x})^{+\int} & = & \int_0^x dt & - & \frac{1}{2}\int_0^x\int_0^{x-t} du\,dt & + & \frac{1}{3}\int_0^x\int_0^{x-t}\int_0^{x-t-u} dv\,du\,dt & -\ldots \\
\log(1+\boldsymbol{x})^{+\Sigma} & = & \sum_{t=1}^{x}1 & - & \frac{1}{2}\sum_{t=1}^{x}\sum_{u=1}^{x-t}1 & + & \frac{1}{3}\sum_{t=1}^{x}\sum_{u=1}^{x-t}\sum_{v=1}^{x-t-u}1 & -\ldots \\
\log(1+\boldsymbol{x})^{*\int} & = & \int_1^x dt & - & \frac{1}{2}\int_1^x\int_1^{\frac{x}{t}} du\,dt & + & \frac{1}{3}\int_1^x\int_1^{\frac{x}{t}}\int_1^{\frac{x}{tu}} dv\,du\,dt & -\ldots \\
\log(1+\boldsymbol{x})^{*\Sigma} & = & \sum_{t=2}^{x}1 & - & \frac{1}{2}\sum_{t=2}^{x}\sum_{u=2}^{\lfloor\frac{x}{t}\rfloor}1 & + & \frac{1}{3}\sum_{t=2}^{x}\sum_{u=2}^{\lfloor\frac{x}{t}\rfloor}\sum_{v=2}^{\lfloor\frac{x}{t\cdot u}\rfloor}1 & -\ldots
\end{array}$$

It's hard not to notice the general symmetry between these different statements, I think.

Recursive Definition for $\log(1+\boldsymbol{x})$

All five of these can be expressed with the same recursive form. If we define the following recursive function,

$$f_k(x)=d\cdot\sum_{j=1}^{\lfloor\frac{x-I}{d}\rfloor}\frac{1}{k}-f_{k+1}(\theta(x,j\cdot d+I))$$
$$f(x,d,\theta,I)=f_1(x)$$

then we can compute $\log(1+x)$, for 1 < x < 2, as

$$\log(1+x)=\lim_{d\to 0} f(x,d,\theta_{=},1) \quad \text{with} \quad \theta_{=}(x,t)=x$$

And if we define two helper functions, $\theta_{+}(x,t)=x-t$ and $\theta_{*}(x,t)=\frac{x}{t}$, we can also express our four variants of $\log(1+\boldsymbol{x})$ as

$$\log(1+\boldsymbol{x})=$$

	$\int$	$\sum$
+	$\lim_{d\to 0} f(x,d,\theta_{+},0)$	$f(x,1,\theta_{+},0)$
*	$\lim_{d\to 0} f(x,d,\theta_{*},1)$	$f(x,1,\theta_{*},1)$

Mathematica Implementation of $\log(1+\boldsymbol{x})$

Just to keep things clear, this means that in Mathematica, given the following code,

```
delta=.01;
thetaAdd[x_,t_]:=x-t
thetaMul[x_,t_]:=x/t
thetaEq[x_,t_]:=x
f[x_,k_,d_,fn_,I_]:=d Sum[1/k-f[fn[x,d t+I],k+1,d,fn,I],{t,1,(x-I)/d}]
```

we can then compute $\log(1+\boldsymbol{x})$ as

	$\int$	$\sum$
+	`f[x,1,delta,thetaAdd, 0]`	`f[x,1,1,thetaAdd, 0]`
*	`f[x,1,delta,thetaMul, 1]`	`f[x,1,1,thetaMul, 1]`

again with the values on the left side of the table getting closer as delta gets closer to 0.

Note that this is a reasonable way to compute small values of $\log(1+\boldsymbol{x})^{+\Sigma}$ and $\log(1+\boldsymbol{x})^{*\Sigma}$, but not at all reasonable to compute $\log(1+\boldsymbol{x})^{+\int}$ or $\log(1+\boldsymbol{x})^{+\int}$.

Closed form for $\log(1+\boldsymbol{x})$

$\log(1+\boldsymbol{x})$ has some *really* interesting closed forms. As a matter of fact, the closed forms of $\log(1+\boldsymbol{x})$ are the central point of this entire paper. They are, in fact,

$$\log(1+\boldsymbol{x})=$$

	$\int$	$\sum$
+	$-(Ei(-x)-\log(x)-\gamma)$	H_x
*	$li(x)-\log\log x-\gamma$	$\Pi(x)$

Here, $\Gamma(k,x)$ is the incomplete gamma function, $li(x)$ is the logarithmic integral, H_x is the Harmonic Number, $\Pi(x)$ is the Riemann Prime Counting function, and γ is Euler's constant gamma.

Closed Form Values for Derivatives / Finite Differences

The derivatives and finite differences of $\log(1+\boldsymbol{x})$ are important to note, and well as perhaps a bit illuminating. So let's list those here.

Just as we can easily see that

$$\frac{\partial}{\partial x}\log(1+x)=\frac{1}{1+x}$$

we have have the following four closely related terms for our four variants.

$$\frac{\partial}{\partial x}\{\log(I+x)\}=$$
OR
$$\nabla_x\log(1+\boldsymbol{x})=$$

	$\int$	$\sum$
+	$\frac{1}{x}-\frac{e^{-x}}{x}$	$\frac{1}{x}$
*	$\frac{1}{\log x}-\frac{1}{x\log x}$	$\kappa(x)$

Here, once again, $\kappa(x)=\frac{1}{a}$ if x is a prime to the whole number power a, and $\kappa(x)=0$ if x has two or more prime factors.

$\log(1+\boldsymbol{x})$ in Terms of $\boldsymbol{x}^k$

With these various closed forms, this Mercator series equivalent can be rewritten as

	$\int$	$\sum$
+	$-(Ei(-x)-\log(x)-\gamma)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\cdot\frac{x^k}{k!}$	$H_x=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\binom{x}{k}$
*	$li(x)-\log\log x-\gamma=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}(-1)^{-k}\cdot P(k,-\log x)$	$\Pi(x)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}D_k'(x)$

$\log(1+\boldsymbol{x})$ in Terms of $(1+\boldsymbol{x})^z$

One very simple way to define $\log(1+x)$ is as the derivative of $(1+x)^z$ with respect to z, as

$$\log(1+x)=\lim_{z\to 0}\frac{\partial}{\partial z}(1+x)^z$$

With the closed forms of our convolutions, this is

$$\log(1+\boldsymbol{x})=\lim_{z\to 0}\frac{\partial}{\partial z}(1+\boldsymbol{x})^z$$

	$\int$	$\sum$
$+$	$-(Ei(-x)-\log(x)-\gamma)=\lim_{z\to 0}\frac{\partial}{\partial z}L_z(-x)$	$H_x=\lim_{z\to 0}\frac{\partial}{\partial z}\cdot\left(\left(\binom{x+1}{z}\right)\right)$
$*$	$li(x)-\log\log x-\gamma=\lim_{z\to 0}\frac{\partial}{\partial z}x\cdot L_{z-1}(-\log x)$	$\Pi(x)=\lim_{z\to 0}\frac{\partial}{\partial z}D_z(x)$

$\log(1+\boldsymbol{x})$ in Terms of $(1+\boldsymbol{x})^z$, Version 2

Another well known way to define the logarithm is as

$$\log(1+x)=\lim_{z\to 0}\frac{(1+x)^z-1}{z}$$

This has an analog with our convolutions as

$$\log(1+\boldsymbol{x})=\lim_{z\to 0}\frac{(1+\boldsymbol{x})^z-1}{z}$$

	$\int$	$\sum$
$+$	$-(Ei(-x)-\log(x)-\gamma)=\lim_{z\to 0}\frac{L_z(-x)-1}{z}$	$H_x=\lim_{z\to 0}\frac{\left(\left(\binom{x+1}{z}\right)\right)-1}{z}$
$*$	$li(x)-\log\log x-\gamma=\lim_{z\to 0}\frac{x\cdot L_{z-1}(-\log x)-1}{z}$	$\Pi(x)=\lim_{z\to 0}\frac{D_z(x)-1}{z}$

Section 4:
$\log^k(1+\boldsymbol{x})$

Overview

This section details expressions of the form $\log^k(1+\boldsymbol{x})$, which will function similarly to $\log^k(1+x)$. The section contains a number of identities for expressing $\boldsymbol{x}^{\boldsymbol{k}}$ and $(1+\boldsymbol{x})^z$ in terms of $\log^k(1+\boldsymbol{x})$, the most noteworthy being a convolution analogue to the familiar expression $x^z=\sum_{k=0}^{\infty}\frac{z^k}{k!}\log^k x$.

$\log^k(1+\boldsymbol{x})$ Written Out Explicitly

To begin with, I'll write out, explicitly, $\log^k(1+\boldsymbol{x})$ for the first few values of k. To do this, I'll be relying on the differential / finite difference form of $\frac{\partial}{\partial x}\log(1+\boldsymbol{x})=$ / $\nabla_x\log(1+\boldsymbol{x})=$ detailed in Section 3.

So, if k=1, we can expression the log of 1+x in the very familiar form of

$$\log(1+x)=\int_0^x\frac{1}{1+t}dt$$

The convolution equivalents of this are

$$\log(1+\boldsymbol{x})=$$

	$\int$	$\sum$
+	$\int_0^x\frac{1}{t}-\frac{e^{-t}}{t}dt$	$\sum_{t=1}^{x}\frac{1}{t}$
*	$\int_1^x\frac{1}{\log t}-\frac{1}{t\log t}dt$	$\sum_{t=2}^{x}\kappa(t)$

For k=2, we can express log of 1+x squared pretty simply as

$$\log^2(1+x)=\int_0^x\int_0^x\frac{1}{1+t}\cdot\frac{1}{1+u}du\,dt$$

Here, the convolution equivalent is slight more complicated, as

$$\log^2(1+\boldsymbol{x})=$$

	$\int$	$\sum$
+	$\int_0^x\int_0^{x-t}(\frac{1}{t}-\frac{e^{-t}}{t})\cdot(\frac{1}{u}-\frac{e^{-u}}{u})du\,dt$	$\sum_{t=1}^{x}\sum_{u=1}^{x-t}\frac{1}{t}\cdot\frac{1}{u}$

*	$\int_1^x \int_1^{\frac{x}{t}} \left(\frac{1}{\log t}-\frac{1}{t\log t}\right)\cdot\left(\frac{1}{\log u}-\frac{1}{u\log u}\right) du\, dt$	$\sum_{t=2}^{x} \sum_{u=2}^{\lfloor\frac{x}{t}\rfloor} \kappa(t)\cdot\kappa(u)$

For k=3, log of 1+x cubed can be written as

$$\log^3(1+x)=\int_0^x\int_0^x\int_0^x \frac{1}{1+t}\cdot\frac{1}{1+u}\cdot\frac{1}{1+v}\, dv\, du\, dt$$

And obviously, continuing the pattern, the convolution equivalents can be written as

$$\log^3(1+\boldsymbol{x})=$$

	$\int$	$\sum$
+	$\int_0^x \int_0^{x-t} \int_0^{x-t-u} \left(\frac{1}{t}-\frac{e^{-t}}{t}\right)\cdot\left(\frac{1}{u}-\frac{e^{-u}}{u}\right)\cdot\left(\frac{1}{v}-\frac{e^{-v}}{v}\right) dv\, du\, dt$	$\sum_{t=1}^{x}\sum_{u=1}^{x-t}\sum_{v=1}^{x-t-u} \frac{1}{t}\cdot\frac{1}{u}\cdot\frac{1}{v}$
*	$\int_1^x \int_1^{\frac{x}{t}} \int_1^{\frac{x}{tu}} \left(\frac{1}{\log t}-\frac{1}{t\log t}\right)\cdot\left(\frac{1}{\log u}-\frac{1}{u\log u}\right)\cdot\left(\frac{1}{\log v}-\frac{1}{v\log v}\right) dv\, du\, dt$	$\sum_{t=2}^{x}\sum_{u=2}^{\lfloor\frac{x}{t}\rfloor}\sum_{v=2}^{\lfloor\frac{x}{t\cdot u}\rfloor} \kappa(t)\cdot\kappa(u)\cdot\kappa(v)$

And so on. Hopefully the pattern should be clearly established here.

Recursive Definition for $\log^k(1+\boldsymbol{x})$

One way to capture the pattern of the previous sections is to write log of 1+x to the kth power recursively as

$$\log^k(1+x)=\int_0^x \frac{1}{1+t}\cdot\log^{k-1}(1+x)\, dt$$

The convolution equivalents of this are

$$\log^k(1+\boldsymbol{x})=$$

	$\int$	$\sum$
+	$\int_0^x \left(\frac{1}{t}-\frac{e^{-t}}{t}\right)\cdot\log^{k-1}(1+(\boldsymbol{x-t}))\, dt$	$\sum_{t=1}^{x} \frac{1}{t}\cdot\log^{k-1}(1+(\boldsymbol{x-t}))$
*	$\int_1^x \left(\frac{1}{\log t}-\frac{1}{t\log t}\right)\cdot\log^{k-1}(1+\frac{\boldsymbol{x}}{\boldsymbol{t}})\, dt$	$\sum_{t=2}^{x} \kappa(t)\cdot\log^{k-1}(1+\frac{\boldsymbol{x}}{\boldsymbol{t}})$

Multiplication: $\log^{a+b}(1+x)=\log^a(1+x)\cdot\log^b(1+x)$

We can also express the normal rule of adding exponents here. If *a* and *b* are both whole numbers, we can express the product of *log(1+x)* raised to the powers of *a* and *b* as

$$\log^{a+b}(1+x)=\log^{a}(1+x)\cdot\log^{b}(1+x)$$

The convolution equivalent of this is, of course,

$$\log^{a+b}(1+\boldsymbol{x})=\log^{a}(1+\boldsymbol{x})\cdot\log^{b}(1+\boldsymbol{x})$$

which, when written out, is

$$\log^{a+b}(1+\boldsymbol{x})=$$

	$\int$	$\sum$
+	$\int_0^x\int_0^{x-t}\frac{\partial}{\partial t}\log^a(1+\boldsymbol{t})\cdot\frac{\partial}{\partial u}\log^b(1+\boldsymbol{u})\,du\,dt$	$\sum_{t=1}^{x}\sum_{u=1}^{x-t}\nabla_t\log^a(1+\boldsymbol{t})\cdot\nabla_u\log^b(1+\boldsymbol{u})$
*	$\int_1^x\int_1^{\frac{x}{t}}\frac{\partial}{\partial t}\log^a(1+\boldsymbol{t})\cdot\frac{\partial}{\partial u}\log^b(1+\boldsymbol{u})\,du\,dt$	$\sum_{t=2}^{x}\sum_{u=2}^{\lfloor\frac{x}{t}\rfloor}\nabla_t\log^a(1+\boldsymbol{t})\cdot\nabla_u\log^b(1+\boldsymbol{u})$

Closed Form Values for $\log^k(1+\boldsymbol{x})$

Unfortunately, to the best of my knowledge, there aren't any. Although there are closed forms for all four variants of $\log(1+\boldsymbol{x})$ **,** there are no closed form expressions for any of the more general variants of $\log^k(1+\boldsymbol{x})$.

Convergence of Sums of $\log^k(1+\boldsymbol{x})$

The recursive definitions of $\log^k(1+\boldsymbol{x})^{+\Sigma}$ and $\log^k(1+\boldsymbol{x})^{*\Sigma}$ should be enough to show that $\log^k(1+\boldsymbol{x})^{+\int}=0$ when k>x, and $\log^k(1+\boldsymbol{x})^{*\Sigma}=0$ when $k>\lfloor\frac{\log x}{\log 2}\rfloor$, just as was the case with both $\boldsymbol{x}^{k+\int}$ and $\boldsymbol{x}^{k*\Sigma}$.

$\log^k \boldsymbol{x}$ in terms of $(1+\boldsymbol{x})^z$

One of the tidier was to represent $\log^k(1+x)$ is as the derivative of $(1+x)^z$ with respect to *z*, as

$$\log^k(1+x)=\lim_{z\to 0}\frac{\partial^k}{\partial z^k}(1+x)^z$$

This translates over to our convolution expressions as

$$\log^k(1+\boldsymbol{x})=\lim_{z\to 0}\frac{\partial^k}{\partial z^k}(1+\boldsymbol{x})^z$$

Our various convolutions of the form $(1+\boldsymbol{x})^z$ all have tidy closed forms, and thus

$$\log^k(1+\boldsymbol{x})=$$

	$\int$	$\sum$

+	$\lim_{z\to 0}\frac{\partial^k}{\partial z^k}L_z(-x)$	$\lim_{z\to 0}\frac{\partial^k}{\partial z^k}(\binom{z+1}{x})$
*	$\lim_{z\to 0}\frac{\partial^k}{\partial z^k}x\cdot L_{z-1}(-\log x)$	$\lim_{z\to 0}\frac{\partial^k}{\partial z^k}D_z(x)$

$\log^k(1+\mathbf{x})$ in terms of $\mathbf{x}^k$

We can also express $\log^k(1+x)$ in terms of x^k with the series.

$$\log^k(1+x)=\sum_{j=0}^{\infty}\frac{1}{j!}\cdot(\lim_{t\to 1}\frac{\partial^j}{\partial t^j}(\log(t))^k)\,x^j$$

This can be rewritten as

$$\log^k(1+x)=\sum_{j=0}^{\infty}\frac{s(j,k)\cdot k!}{j!}x^j$$

where $s(j,k)$ are signed Stirling numbers of the first kind.

There is a straightforward convolution version of this, as

$$\log^k(1+\mathbf{x})=\sum_{j=0}^{\infty}\frac{s(j,k)\cdot k!}{j!}\mathbf{x}^j$$

Using the various closed forms for $\mathbf{x}^k$, this means that

$$\log^k(1+\mathbf{x})=$$

	$\int$	$\sum$
+	$\sum_{j=0}^{\infty}\frac{s(j,k)\cdot k!}{j!}\cdot\frac{x^j}{j!}$	$\sum_{j=0}^{\infty}\frac{s(j,k)\cdot k!}{j!}\cdot\binom{x}{j}$
*	$\sum_{j=0}^{\infty}\frac{s(j,k)\cdot k!}{j!}\cdot(-1)^{-j}\cdot P(j,-\log x)$	$\sum_{j=0}^{\infty}\frac{s(j,k)\cdot k!}{j!}\cdot D_j{}'(x)$

$(1+\mathbf{x})^z$ in terms of $\log^k(1+\mathbf{x})$

A particularly important use of $\log^k(1+x)$ is to provide another basic expression for $(1+x)^z$, in the familiar form of

$$(1+x)^z=\sum_{k=0}^{\infty}\frac{z^k}{k!}\log^k(1+x)$$

The convolution equivalent of this is

$$(1+\mathbf{x})^z=\sum_{k=0}^{\infty}\frac{z^k}{k!}\log^k(1+\mathbf{x})$$

Written out explicitly, this is

$$\begin{array}{llllll}
(1+x)^z & =1+ & z\cdot\int_0^x \frac{1}{1+t}dt & + & \frac{z^2}{2}\cdot\int_0^x\int_0^x \frac{1}{1+t}\cdot\frac{1}{1+u}du\,dt & +\frac{z^3}{6}\cdot\ldots \\
(1+\boldsymbol{x})^{z+\int} & =1+ & z\cdot\int_0^x \frac{1}{t}-\frac{e^{-t}}{t}dt & + & \frac{z^2}{2}\cdot\int_0^x\int_0^{x-t} (\frac{1}{t}-\frac{e^{-t}}{t})(\frac{1}{u}-\frac{e^{-u}}{u})du\,dt & +\frac{z^3}{6}\cdot\ldots \\
(1+\boldsymbol{x})^{z+\Sigma} & =1+ & z\cdot\sum_{t=1}^x \frac{1}{t} & + & \frac{z^2}{2}\cdot\sum_{t=1}^x\sum_{u=1}^{x-t} \frac{1}{t}\cdot\frac{1}{u} & +\frac{z^3}{6}\cdot\ldots \\
(1+\boldsymbol{x})^{z*\int} & =1+ & z\cdot\int_1^x \frac{1}{\log t}-\frac{1}{t\log t}dt & + & \frac{z^2}{2}\cdot\int_1^x\int_1^{\frac{x}{t}} (\frac{1}{\log t}-\frac{1}{t\log t})(\frac{1}{\log u}-\frac{1}{u\log u})du\,dt & +\frac{z^3}{6}\cdot\ldots \\
(1+\boldsymbol{x})^{z*\Sigma} & =1+ & z\cdot\sum_{t=2}^x \kappa(t) & + & \frac{z^2}{2}\cdot\sum_{t=2}^x\sum_{u=2}^{\lfloor\frac{x}{t}\rfloor} \kappa(t)\cdot\kappa(u) & +\frac{z^3}{6}\cdot\ldots
\end{array}$$

Here, once again, $\kappa(t)=\frac{1}{a}$ if t is a prime to the whole number power a, and $\kappa(t)=0$ if t has two or more prime factors.

$\boldsymbol{x}^k$ in terms of $\log^k(1+\boldsymbol{x})$

We can also express x^k in terms of $\log^k(1+x)$ as

$$x^k=\sum_{j=0}^{\infty}\frac{1}{j!}\cdot(\lim_{t\to 0}\frac{\partial^j}{\partial t^j}(e^t-1)^k)\log^j(1+x)$$

which we can rewrite as

$$x^k=\sum_{j=0}^{\infty}\frac{S(j,k)\cdot k!}{j!}\cdot\log^j(1+x)$$

where $S(j,k)$ are Stirling numbers of the second kind.

This also maps cleanly into the convolution context as

$$\boldsymbol{x}^k=\sum_{j=0}^{\infty}\frac{S(j,k)\cdot k!}{j!}\cdot\log^j(1+\boldsymbol{x})$$

Section 5:
x^z

Overview

In section 1, we worked with the constraint that for our various convolutions of the form $\boldsymbol{x}^{k}$, k had to be a whole number. There's a straight foward reason for this - most interesting identities involving $\boldsymbol{x}^{k}$ stop working if we extend $\boldsymbol{x}^{k}$ by replacing the k with a more general complex value z in the most obvious ways.

Nevertheless, it is still interesting to explore these forms. In particular, it is interesting, for some conceptual reasons, to compare the forms of $\boldsymbol{x}^{z}$ to the corresponding forms of $(1+\boldsymbol{x})^{z}$, and, likewise, the forms of $\log \boldsymbol{x}$ to $\log(1+\boldsymbol{x})$.

Closed Form Values for $\boldsymbol{x}^{z}$

The two continuous convolutions $\boldsymbol{x}^{k+\int}$ and $\boldsymbol{x}^{k*\int}$ can be extended to complex values of z simply by treating the gamma function as the natural extension of the factorial function. The discrete additive convolution $\boldsymbol{x}^{k+\Sigma}$, which we saw previously has the closed form of binomial coefficients, can be generalized through the use of the gamma function as well. But our last convolution, $\boldsymbol{x}^{z*\Sigma}$, the one connected to the Riemann Prime Counting function, is much trickier to extend. So let's list the obvious extensions first.

$$\boldsymbol{x}^{z}=$$

	$\int$	Σ
+	$\frac{x^z}{\Gamma(z+1)}$	$\frac{\Gamma(x+1)}{\Gamma(z+1)\cdot\Gamma(x-z+1)}$
*	$(-1)^{-z}\cdot P(z,-\log x)$	

Ultimately there is a different way of extending the binomial coefficient $\binom{x}{z}$ to complex values of z that also works our discrete multiplicative case of $\boldsymbol{x}^{z*\Sigma}$. That extension is

$$\boldsymbol{x}^{z}=$$

	$\int$	Σ
+	...	$\frac{\sin(\pi z)}{\pi}\cdot\sum_{k=0}^{\infty}\frac{(-1)^k}{z-k}\cdot\binom{x}{k}$
*	...	$\frac{\sin(\pi z)}{\pi}\cdot\sum_{k=0}^{\infty}\frac{(-1)^k}{z-k}\cdot D_k'(x)$

Interestingly, all four convolution variations of $\boldsymbol{x}^{z}$ produce binomial-style curves.

Closed Form Values for Derivatives / Finite Differences

As usual, we can start with the well known derivative

$$\frac{\partial}{\partial x} x^z = z \cdot x^{z-1}$$

The convolution equivalent of this derivative / finite difference is

$$\frac{\partial}{\partial x} \boldsymbol{x}^z = / \nabla_x \boldsymbol{x}^z =$$

	$\int$	$\sum$
$+$	$\frac{x^{z-1}}{\Gamma(z)}$	$\frac{\Gamma(x)}{\Gamma(z) \cdot \Gamma(x-z+1)}$
$*$	$\frac{\log^{z-1} x}{\Gamma(z)}$	$\frac{\sin(\pi z)}{\pi} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{z-k} \cdot d'_k(x)$

Closed Form Values for $\log \boldsymbol{x}$

Two tidy ways to express $\log x$, given x^z are as

$$\log x = \lim_{z \to 0} \frac{\partial}{\partial z} x^z$$

and

$$\log x = \lim_{z \to 0} \frac{x^z - 1}{z}$$

With the closed forms of our convolutions, we can equivalently do either, as

$$\log \boldsymbol{x} = \lim_{z \to 0} \frac{\partial}{\partial z} \boldsymbol{x}^z$$

and

$$\log \boldsymbol{x} = \lim_{z \to 0} \frac{\boldsymbol{x}^z - 1}{z}$$

which result in the closed forms

$$\log \boldsymbol{x} =$$

	$\int$	$\sum$
$+$	$\log x + \gamma$	H_x
$*$	$li(x)$	$\Pi(x)$

Closed Form Values for Derivatives / Finite Differences

The derivative of *log x* is, of course,

$$\frac{\partial}{\partial x}\log x=\frac{1}{x}$$

Likewise, if we take the derivatives or finite differences of our closed forms for $\log \boldsymbol{x}$, we have

$$\frac{\partial}{\partial x}\log \boldsymbol{x}= \;/\; \nabla_x \log \boldsymbol{x}=$$

	$\int$	$\sum$
$+$	$\frac{1}{x}$	$\frac{1}{x}$
$*$	$\frac{1}{\log x}$	$\kappa(x)$

Section 6:
Various Binomial Forms

Overview

Recursive Definition for $(1+\boldsymbol{x})^z$

$$C_j = \left(\lim_{t \to 0} \frac{\partial^j}{\partial t^j} \frac{t}{\log(1+t)}\right)$$

$$x^k = \sum_{j=0}^{\infty} \frac{C_j}{j!} \cdot x^{k+j-1} \cdot \log(1+x)$$

$$\boldsymbol{x}^{\boldsymbol{k}} = \sum_{j=0}^{\infty} \frac{C_j}{j!} \cdot \boldsymbol{x}^{\boldsymbol{k+j-1}} \cdot \log(1+\boldsymbol{x})$$

$$\boldsymbol{x}^{\boldsymbol{k}} =$$

	$\int$	$\sum$
+	$\sum_{j=0}^{\infty} \frac{C_j}{j!} \cdot \int_0^x \int_0^{x-t} \frac{\partial}{\partial t} \boldsymbol{t}^{\boldsymbol{k+j-1}} \cdot \frac{\partial}{\partial u} \log(1+\boldsymbol{u})\, du\, dt$	$\sum_{j=0}^{\infty} \frac{C_j}{j!} \cdot \sum_{t=1}^{x} \sum_{u=1}^{x-t} \nabla_t \boldsymbol{t}^{\boldsymbol{k+j-1}} \cdot \nabla_u \log(1+\boldsymbol{u})$
*	$\sum_{j=0}^{\infty} \frac{C_j}{j!} \cdot \int_1^x \int_1^{\frac{x}{t}} \frac{\partial}{\partial t} \boldsymbol{t}^{\boldsymbol{k+j-1}} \cdot \frac{\partial}{\partial u} \log(1+\boldsymbol{u})\, du\, dt$	$\sum_{j=0}^{\infty} \frac{C_j}{j!} \cdot \sum_{t=2}^{x} \sum_{u=2}^{\lfloor \frac{x}{t} \rfloor} \nabla_t \boldsymbol{t}^{\boldsymbol{k+j-1}} \cdot \nabla_u \log(1+\boldsymbol{u})$

$$\boldsymbol{x}^{\boldsymbol{k}} =$$

	$\int$	$\sum$
+	$\frac{x^k}{k!} = \sum_{j=0}^{\infty} \frac{C_j}{j!} \cdot \int_0^x \int_0^{x-t} \frac{t^{k+j-2}}{(k+j-2)!} \cdot \left(\frac{1}{u} - \frac{e^{-u}}{u}\right) du\, dt$	$\binom{x}{k} = \sum_{j=0}^{\infty} \frac{C_j}{j!} \cdot \sum_{t=1}^{x} \sum_{u=1}^{x-t} \binom{t-1}{k+j-2} \cdot \frac{1}{u}$
*	$(-1)^{-k} P(k, -\log x) = \sum_{j=0}^{\infty} \frac{C_j}{j!} \cdot \int_1^x \int_1^{\frac{x}{t}} \frac{\log^{k+j-2} t}{(k+j-2)} \cdot \left(\frac{1}{\log u} - \frac{1}{u \log u}\right) du\, dt$	$D_k'(x) = \sum_{j=0}^{\infty} \frac{C_j}{j!} \cdot \sum_{t=2}^{x} \sum_{u=2}^{\lfloor \frac{x}{t} \rfloor} d_{k+j-1}'(t) \cdot \kappa(u)$

$$\sum_{k=1} \frac{(-1)^{k+1}}{k} x^{k+a} = x^a \cdot \log(I + x)$$

$$\sum_{k=1} \frac{(-1)^{k+1}}{k} \boldsymbol{x}^{\boldsymbol{k+a}} = \boldsymbol{x}^{\boldsymbol{a}} \cdot \log(1 + \boldsymbol{x})$$

	$\int$	$\sum$
+	$\sum_{k=1} \frac{(-1)^{k+1}}{k} \boldsymbol{x}^{\boldsymbol{k+a}} = \int_0^x \int_0^{x-t} \frac{\partial}{\partial t} \boldsymbol{t}^{\boldsymbol{a}} \cdot \frac{\partial}{\partial u} \log(1 + \boldsymbol{u})\, du\, dt$	$\sum_{k=1} \frac{(-1)^{k+1}}{k} \boldsymbol{x}^{\boldsymbol{k+a}} = \sum_{t=1}^{x} \sum_{u=1}^{x-t} \nabla_t \boldsymbol{t}^{\boldsymbol{a}} \cdot \nabla_u \log(1 + \boldsymbol{u})$
*	$\sum_{k=1} \frac{(-1)^{k+1}}{k} \boldsymbol{x}^{\boldsymbol{k+a}} = \int_1^x \int_1^{\frac{x}{t}} \frac{\partial}{\partial t} \boldsymbol{t}^{\boldsymbol{a}} \cdot \frac{\partial}{\partial u} \log(1 + \boldsymbol{u})\, du\, dt$	$\sum_{k=1} \frac{(-1)^{k+1}}{k} \boldsymbol{x}^{\boldsymbol{k+a}} = \sum_{t=2}^{x} \sum_{u=2}^{\lfloor \frac{x}{t} \rfloor} \nabla_t \boldsymbol{t}^{\boldsymbol{a}} \cdot \nabla_u \log(1 + \boldsymbol{u})$

	$\int$	$\sum$
+	$\sum_{k=1} \frac{(-1)^{k+1}}{k} \cdot \frac{x^{k+a}}{(k+a)!} = \int_0^x \int_0^{x-t} \frac{t^{a-1}}{(a-1)!} \cdot \left(\frac{1}{u} - \frac{e^{-u}}{u}\right) du\, dt$	$\sum_{k=1} \frac{(-1)^{k+1}}{k} \binom{x}{k+a} = \sum_{t=1}^{x} \sum_{u=1}^{x-t} \binom{t-1}{a-1} \cdot \frac{1}{u}$
*	$\sum_{k=1} \frac{(-1)^{k+1}}{k} (-1)^{-(k+a)} P(k+a, -\log x) =$ $\int_1^x \int_1^{\frac{x}{t}} \frac{\log^{a-1} t}{(a-1)!} \cdot \left(\frac{1}{\log u} - \frac{1}{u \log u}\right) du\, dt$	$\sum_{k=1} \frac{(-1)^{k+1}}{k} D_{k+a}'(x) = \sum_{t=2}^{x} \sum_{u=2}^{\lfloor \frac{x}{t} \rfloor} d_a'(t) \cdot \kappa(u)$

$$\log^{a}(1+x)=\sum_{k=1}\frac{(-1)^{k+1}}{k}x^{k}\cdot\log^{a-1}(1+x)$$

$$\log^{a}(1+x)=\sum_{k=0}\frac{B_{k}}{k!}x\cdot\log^{k+a-1}(1+x)$$

$$\log(1+x)=\frac{x}{1+x}+\sum_{k=2}\frac{(-1)^{k}}{k(k-1)}\cdot\frac{x^{k}}{1+x}$$

$$\log(1+x)=\sum_{k=1}(-1)^{k+1}\cdot H_{k}\cdot x^{k}\cdot(1+x)$$

$$\log(1+x)=\sum_{k=1}(-1)^{k+1}\cdot H_k\cdot x^k\cdot(1+x)$$

$$\log(1+\boldsymbol{x})=\sum_{k=1}(-1)^{k+1}\cdot H_k\cdot \boldsymbol{x}^{\boldsymbol{k}}\cdot(1+\boldsymbol{x})$$

$$\log(1+\boldsymbol{x})=\sum_{k=1}(-1)^{k+1}\cdot H_k\cdot \boldsymbol{x}^{\boldsymbol{k}}+\boldsymbol{x}^{\boldsymbol{k+1}}$$

	$\int$	$\sum$
+		$\sum_{k=0}^{\infty}(-1)^{k+1}\cdot H_k\cdot\sum_{t=0}^{x}\sum_{u=1}^{x-t}\nabla_t \boldsymbol{t}^{\boldsymbol{k}}\cdot\nabla_u(1+\boldsymbol{u})$
*		$\sum_{k=0}^{\infty}(-1)^{k+1}\cdot H_k\cdot\sum_{t=1}^{x}\sum_{u=2}^{\lfloor\frac{x}{t}\rfloor}\nabla_t \boldsymbol{t}^{\boldsymbol{k}}\cdot\nabla_u(1+\boldsymbol{u})$

	$\int$	$\sum$
+		$\sum_{k=0}^{\infty}(-1)^{k+1}\cdot H_k\cdot\sum_{t=0}^{x}\sum_{u=1}^{x-t}\binom{t-1}{k-1}\cdot u$
*		$\sum_{k=0}^{\infty}(-1)^{k+1}\cdot H_k\cdot\sum_{t=2}^{x}\sum_{u=1}^{\lfloor\frac{x}{t}\rfloor}d_k{}'(t)$

	$\int$	$\sum$
+		$\sum_{k=0}^{\infty}(-1)^{k+1}\cdot H_k\cdot\sum_{t=0}^{x}\binom{t-1}{k-1}\cdot\binom{t-x}{2}$
*		$\sum_{k=0}^{\infty}(-1)^{k+1}\cdot H_k\cdot\sum_{t=2}^{x}\lfloor\frac{x}{t}\rfloor\cdot d_k{}'(t)$

	$\int$	$\sum$
+		$H_x=\sum_{k=0}^{\infty}(-1)^{k+1}\cdot H_k\cdot\left(\binom{x}{k}+\binom{x}{k+1}\right)$
*		$\Pi(x)=\sum_{k=0}^{\infty}(-1)^{k+1}\cdot H_k\cdot(D_k{}'(x)+D_{k+1}{}'(x))$

Section 7: $\log(1+\boldsymbol{x})$ Turns Convolution into Addition

Overview

Recursive Definition for $\boldsymbol{x}^z$

$$x \cdot y$$

$x \cdot y$: Rectangle

Real-valued area bounded by $t>0$, $u>0$, $\frac{t}{x} \le 1$, and $\frac{u}{y} \le 1$.

$\boldsymbol{x} \cdot \boldsymbol{y}^{+\int}$: Triangle

Real-valued area bounded by $t>0$, $u>0$, and $\frac{t}{x} + \frac{u}{y} \le 1$.

$\boldsymbol{x} \cdot \boldsymbol{y}^{+\Sigma}$ Discrete Triangle

Count of whole number pairs (t,u) satisfying $t>0$, $u>0$, and $\frac{t}{x} + \frac{u}{y} \le 1$.

$\boldsymbol{x} \cdot \boldsymbol{y}^{*\int}$: Hyperbola

Real-valued area bounded by $t>0$, $u>0$, and $\frac{\log(1+t)}{\log x} + \frac{\log(1+u)}{\log y} \le 1$.

$\boldsymbol{x} \cdot \boldsymbol{y}^{*\Sigma}$: Discrete Hyperbola

Count of ordered whole number pairs (t,u) satisfying $t>0$, $u>0$, and $\frac{\log(1+t)}{\log x} + \frac{\log(1+u)}{\log y} \le 1$.

$$x \cdot y \cdot z$$

$x \cdot y \cdot z$: Rectangular Prism

Real-valued volume bounded by $t>0$, $u>0$, $v>0$, $t \le x$, $\frac{t}{x} \le 1$, $\frac{u}{y} \le 1$,and $\frac{v}{z} \le 1$.

$\boldsymbol{x} \cdot \boldsymbol{y} \cdot \boldsymbol{z}^{+\int}$: Triangular Pyramid

Real-valued volume bounded by $t>0$, $u>0$, $v>0$, and $\frac{t}{x} + \frac{u}{y} + \frac{v}{z} \le 1$.

$\boldsymbol{x} \cdot \boldsymbol{y} \cdot \boldsymbol{z}^{+\Sigma}$: Discrete Triangular Pyramid

Count of whole number pairs (t,u,v) satisfying $t>0$, $u>0$, $v>0$, and $\frac{t}{x} + \frac{u}{y} + \frac{v}{z} \le 1$.

$\boldsymbol{x} \cdot \boldsymbol{y} \cdot \boldsymbol{z}^{*\int}$: 3-dimensional Hyperbola

Real-valued volume bounded by $t>0$, $u>0$, $v>0$, and $\frac{\log(1+t)}{\log x} + \frac{\log(1+u)}{\log y} + \frac{\log(1+v)}{\log z} \le 1$.

$\boldsymbol{x} \cdot \boldsymbol{y} \cdot \boldsymbol{z}^{*\Sigma}$: Discrete 3-dimensional hyperbola.

Count of ordered whole number pairs (t,u,v) satisfying $t>0$, $u>0$, $v>0$, and $\frac{\log(1+t)}{\log x}+\frac{\log(1+u)}{\log y}+\frac{\log(1+v)}{\log z}\le 1$

.

Recursive Definition for $\boldsymbol{x}^z$

$$\boldsymbol{x}\cdot\boldsymbol{y}^{+\Sigma}=\sum_{\frac{t_1}{x}+\frac{t_2}{y}\le 1} 1$$

$$(\boldsymbol{x}\cdot\boldsymbol{y})^{2\ +\Sigma}=\sum_{\frac{t_1}{x}+\frac{t_2}{x}+\frac{t_3}{y}+\frac{t_4}{y}\le 1} 1$$

$$(\boldsymbol{x}\cdot\boldsymbol{y})^{3\ +\Sigma}=\sum_{\frac{t_1}{x}+\frac{t_2}{x}+\frac{t_3}{x}+\frac{t_4}{y}+\frac{t_5}{y}+\frac{t_6}{y}\le 1} 1$$

$$(1+\boldsymbol{x})\cdot(1+\boldsymbol{y})^{+\Sigma}=\sum_{\frac{t_1}{x}+\frac{t_2}{y}\le 1} 1$$

$$((1+\boldsymbol{x})\cdot(1+\boldsymbol{y}))^{2\ +\Sigma}=\sum_{\frac{t_1}{x}+\frac{t_2}{x}+\frac{t_3}{y}+\frac{t_4}{y}\le 1} 1$$

$$((1+\boldsymbol{x})\cdot(1+\boldsymbol{y}))^{3\ +\Sigma}=\sum_{\frac{t_1}{x}+\frac{t_2}{x}+\frac{t_3}{x}+\frac{t_4}{y}+\frac{t_5}{y}+\frac{t_6}{y}\le 1} 1$$

$$\boldsymbol{x}\cdot\boldsymbol{y}^{*\Sigma}=\sum_{\frac{\log t_1}{\log x}+\frac{\log t_2}{\log y}\le 1} 1$$

$$(\boldsymbol{x}\cdot\boldsymbol{y})^{2\ *\Sigma}=\sum_{\frac{\log t_1}{\log x}+\frac{\log t_2}{\log x}+\frac{\log t_3}{\log y}+\frac{\log t_4}{\log y}\le 1} 1$$

$$(\boldsymbol{x}\cdot\boldsymbol{y})^{3\ *\Sigma}=\sum_{\frac{\log t_1}{\log x}+\frac{\log t_2}{\log x}+\frac{\log t_3}{\log x}+\frac{\log t_4}{\log y}+\frac{\log t_5}{\log y}+\frac{\log t_6}{\log y}\le 1} 1$$

$$(1+\boldsymbol{x})\cdot(1+\boldsymbol{y})^{*\Sigma}=\sum_{\frac{\log t_1}{\log x}+\frac{\log t_2}{\log y}\le 1} 1$$

$$((1+\boldsymbol{x})\cdot(1+\boldsymbol{y}))^{2\ *\Sigma}=\sum_{\frac{\log t_1}{\log x}+\frac{\log t_2}{\log x}+\frac{\log t_3}{\log y}+\frac{\log t_4}{\log y}\le 1} 1$$

$$((1+\boldsymbol{x})\cdot(1+\boldsymbol{y}))^{3\ *\Sigma}=\sum_{\frac{\log t_1}{\log x}+\frac{\log t_2}{\log x}+\frac{\log t_3}{\log x}+\frac{\log t_4}{\log y}+\frac{\log t_5}{\log y}+\frac{\log t_6}{\log y}\le 1} 1$$

Recursive Definition for $\boldsymbol{x}^z$

$$((1+\boldsymbol{x})\cdot(1+\boldsymbol{y})-1)^k=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}((1+\boldsymbol{x})\cdot(1+\boldsymbol{y}))^j$$

$$((1+\boldsymbol{x})\cdot(1+\boldsymbol{y}))^z=\sum_{k=0}^{\infty}\binom{z}{k}((1+\boldsymbol{x})\cdot(1+\boldsymbol{y})-1)^k$$

$$\log((1+\boldsymbol{x})\cdot(1+\boldsymbol{y}))=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}((1+\boldsymbol{x})\cdot(1+\boldsymbol{y})-1)^{k}$$

$$\log((1+\boldsymbol{x})\cdot(1+\boldsymbol{y}))=\lim_{z\to 0}\frac{\partial}{\partial z}((1+\boldsymbol{x})\cdot(1+\boldsymbol{y}))^{z}$$

$$\log((1+\boldsymbol{x})\cdot(1+\boldsymbol{y}))=\log(1+\boldsymbol{x})+\log(1+\boldsymbol{y})$$

Recursive Definition for $\boldsymbol{x}^{z}$

$$\frac{1+\boldsymbol{x}}{1+\boldsymbol{y}}=$$

	$\int$	$\sum$
+	$1+\int_0^x \frac{\partial}{\partial t}(1+\boldsymbol{t})dt+\int_0^y \frac{\partial}{\partial u}(1+\boldsymbol{u})^{-1}du+\int_0^x \sum_0^{y(1-\frac{t}{x})} \frac{\partial}{\partial t}(1+\boldsymbol{t})\cdot\frac{\partial}{\partial u}(1+\boldsymbol{u})^{-1}du\,dt$	$\sum_{t=0}^{x}\sum_{u=0}^{\lfloor y(1-\frac{t}{x})\rfloor}\nabla_t(1+\boldsymbol{t})\cdot\nabla_u(1+\boldsymbol{u})^{-1}$
*	$1+\int_1^x \frac{\partial}{\partial t}(1+\boldsymbol{t})dt+\int_1^y \frac{\partial}{\partial u}(1+\boldsymbol{u})^{-1}du+\int_1^x \sum_1^{y^{1-\frac{\log t}{\log x}}} \frac{\partial}{\partial t}(1+\boldsymbol{t})\cdot\frac{\partial}{\partial u}(1+\boldsymbol{u})^{-1}du\,dt$	$\sum_{t=1}^{x}\sum_{u=1}^{\lfloor y^{(1-\frac{\log t}{\log x})}\rfloor}\nabla_t(1+\boldsymbol{t})\cdot\nabla_u(1+\boldsymbol{u})^{-1}$

	$\int$	$\sum$
+	$1+\int_0^x dt+\int_0^y e^{-u}du+\int_0^x \sum_0^{y(1-\frac{t}{x})} e^{-u}du\,dt$	$\sum_{t=0}^{x}\sum_{u=0}^{\lfloor y(1-\frac{t}{x})\rfloor}(\binom{-1}{u})$
*	$1+\int_1^x dt+\int_1^y \frac{1}{u}du+\int_1^x \sum_1^{y^{1-\frac{\log t}{\log x}}} \frac{1}{u}du\,dt$	$\sum_{t=1}^{x}\sum_{u=1}^{\lfloor y^{(1-\frac{\log t}{\log x})}\rfloor}\mu(u)$

$$\left(\frac{1+\boldsymbol{x}}{1+\boldsymbol{y}}\right)^{z}=$$

	$\int$	$\sum$
+	$1+\int_0^x \frac{\partial}{\partial t}(1+\boldsymbol{t})^{z}dt+\int_0^y \frac{\partial}{\partial u}(1+\boldsymbol{u})^{-z}du+\int_0^x \sum_0^{y(1-\frac{t}{x})} \frac{\partial}{\partial t}(1+\boldsymbol{t})^{z}\cdot\frac{\partial}{\partial u}(1+\boldsymbol{u})^{-z}du\,dt$	$\sum_{t=0}^{x}\sum_{u=0}^{\lfloor y(1-\frac{t}{x})\rfloor}\nabla_t(1+\boldsymbol{t})^{z}\cdot\nabla_u(1+\boldsymbol{u})^{-z}$
*	$1+\int_1^x \frac{\partial}{\partial t}(1+\boldsymbol{t})^{z}dt+\int_1^y \frac{\partial}{\partial u}(1+\boldsymbol{u})^{-z}du+\int_1^x \sum_1^{y^{1-\frac{\log t}{\log x}}} \frac{\partial}{\partial t}(1+\boldsymbol{t})^{z}\cdot\frac{\partial}{\partial u}(1+\boldsymbol{u})^{-z}du\,dt$	$\sum_{t=0}^{x}\sum_{u=0}^{\lfloor y(1-\frac{t}{x})\rfloor}\nabla_t(1+\boldsymbol{t})^{z}\cdot\nabla_u(1+\boldsymbol{u})^{-z}$

	$\int$	$\sum$
+	$1+L_z(-x)+L_{-z}(-y)+\int_0^x \sum_0^{y(1-\frac{t}{x})} L_{z-1}^{(1)}(-t)\cdot L_{-z-1}^{(1)}(-u)du\,dt$	$\sum_{t=0}^{x}\sum_{u=0}^{\lfloor y(1-\frac{t}{x})\rfloor}(\binom{z}{t})(\binom{-z}{u})$

*	$1+\int_1^x L_{z-1}^{(1)}(-\log t)\,dt+\int_1^y L_{-z-1}^{(1)}(-\log u)\,du+\int_1^x \sum_1^{y^{1-\frac{\log t}{\log x}}} L_{z-1}^{(1)}(-\log t)\cdot L_{-z-1}^{(1)}(-\log u)\,du\,dt$	$\sum_{t=1}^{x}\sum_{u=1}^{\lfloor y^{\left(1-\frac{\log t}{\log x}\right)}\rfloor} d_z(t)\cdot d_{-z}(u)$

$$\lim_{z\to 0}\frac{\partial}{\partial z}\left(\frac{1+\boldsymbol{x}}{1+\boldsymbol{y}}\right)^z=\log\frac{1+\boldsymbol{x}}{1+\boldsymbol{y}}=\log(1+\boldsymbol{x})-\log(1+\boldsymbol{y})$$

Factorial

$$\boldsymbol{x}\,!=$$

$$(1+\boldsymbol{x})!=$$

Section 8: Other Convolutions

$$\left(\frac{1+\boldsymbol{x}}{1+\frac{\boldsymbol{x}}{\boldsymbol{2}}}\right)^{z}$$

W

$$\left(\frac{1+x}{1+\frac{x}{2}}\right)^{z}=1+\int_{0}^{x}\frac{\partial}{\partial t}(1+t)^{z}\,dt+\int_{0}^{\frac{x}{2}}\frac{\partial}{\partial u}(1+u)^{-z}\,du+\int_{0}^{x}\int_{0}^{\frac{x}{2}}\frac{\partial}{\partial t}(1+t)^{z}\cdot\frac{\partial}{\partial u}(1+u)^{-z}\,du\,dt$$

$$\left(\frac{1+\boldsymbol{x}}{1+\frac{\boldsymbol{x}}{\boldsymbol{2}}}\right)^{z}$$

	$\int$	$\sum$
+	$1+\int_{0}^{x}\frac{\partial}{\partial t}(1+\boldsymbol{t})^{z}\,dt+\int_{0}^{\frac{x}{2}}\frac{\partial}{\partial u}(1+\boldsymbol{u})^{-z}\,du+$ $\int_{0}^{x}\int_{0}^{\frac{x-t}{2}}\frac{\partial}{\partial t}(1+\boldsymbol{t})^{z}\cdot\frac{\partial}{\partial u}(1+\boldsymbol{u})^{-z}\,du\,dt$	$1+\sum_{t=1}^{x}\nabla_{t}(1+\boldsymbol{t})^{z}+\sum_{u=1}^{\lfloor\frac{x}{2}\rfloor}\nabla_{u}(1+\boldsymbol{u})^{-z}+$ $\sum_{t=1}^{x}\sum_{u=1}^{\lfloor\frac{x-t}{2}\rfloor}\nabla_{t}(1+\boldsymbol{t})^{z}\cdot\nabla_{u}(1+\boldsymbol{u})^{-z}$
*	$1+\int_{1}^{x}\frac{\partial}{\partial t}(1+\boldsymbol{t})^{z}\,dt+\int_{1}^{x^{\frac{1}{2}}}\frac{\partial}{\partial u}(1+\boldsymbol{u})^{-z}\,du+$ $\int_{1}^{x}\int_{1}^{(\frac{x}{t})^{\frac{1}{2}}}\frac{\partial}{\partial t}(1+\boldsymbol{t})^{z}\cdot\frac{\partial}{\partial u}(1+\boldsymbol{u})^{-z}\,du\,dt$	$1+\sum_{t=2}^{x}\nabla_{t}(1+\boldsymbol{t})^{z}+\sum_{u=2}^{\lfloor\frac{x}{2}\rfloor}\nabla_{u}(1+\boldsymbol{u})^{-z}+$ $\sum_{t=2}^{x}\sum_{u=2}^{\lfloor(\frac{x}{t})^{\frac{1}{2}}\rfloor}\nabla_{t}(1+\boldsymbol{t})^{z}\cdot\nabla_{u}(1+\boldsymbol{u})^{-z}$

H

$$\left(\frac{1+x}{1+\frac{x}{2}}\right)^{z}=1+\int_{0}^{x}z\,(1+t)^{z-1}\,dt+\int_{0}^{\frac{x}{2}-z}(1+u)^{-z-1}\,du+\int_{0}^{x}\int_{0}^{\frac{x}{2}}z\,(1+t)^{z-1}\cdot(-z\,(1+u)^{-z-1})\,du\,dt$$

	$\int$	$\sum$

+	$1+\int_0^x L_{z-1}^1(-t)\,dt+\int_0^{\frac{x}{2}} L_{-z-1}^1(-u)\,du+\int_0^x\int_0^{\frac{x-t}{2}} L_{z-1}^1(-t)\cdot L_{-z-1}^1(-u)\,du\,dt$	$\sum_{t=0}^{x}\sum_{u=0}^{\lfloor\frac{x-t}{2}\rfloor}\left(\binom{z}{t}\right)\cdot\left(\binom{-z}{u}\right)$
*	$1+\int_1^x L_{z-1}^1(-\log t)\,dt+\int_1^{x^{\frac{1}{2}}} L_{-z-1}^1(-\log u)\,du+\int_1^x\int_1^{(\frac{x}{t})^{\frac{1}{2}}} L_{z-1}^1(-\log t)\cdot L_{-z-1}^1(-\log u)\,du\,dt$	$\sum_{t=1}^{x}\sum_{u=1}^{\lfloor(\frac{x}{t})^{\frac{1}{2}}\rfloor} d_z(t)\cdot d_{-z}(u)$

$$\left(\frac{1+\boldsymbol{x}}{1+\frac{\boldsymbol{x}}{\boldsymbol{k}}}\right)^z$$

W

$$\prod_{k=1}\left(1+\frac{\boldsymbol{x}}{\boldsymbol{k}}\right)^z$$

W

$$\left(\prod_{k=1}\left(1+\frac{\boldsymbol{x}}{\boldsymbol{k}}\right)^{z\cdot\frac{\mu(k)}{k}}\right)$$

W