$$[\log(\sum_{j=1}^{n}\chi_k(j))]_n=\sum_{j=1}^{n}\chi_k(j)\cdot[\nabla\log\zeta(0)]_j$$

$$\log(\sum_{j=1}^{n}\chi_k(j))=\sum_{s=0}^{t}\chi_k(s)\cdot\sum_{j=0}^{\lfloor\frac{n}{t}\rfloor}\nabla*\log(j\cdot t+s)$$

$$[(((1-x^{1-s})\zeta(s))^z-1)^k]_n=\sum_{j=2}^{n}[\nabla((1-x^{1-s})\zeta(s))^z]_j\cdot[(((1-x^{1-s})\zeta(s))^z-1)^{k-1}]_{n\cdot j^{-1}}$$

$$[\log((1-x^{1-s})\zeta(s)^z)]_n=-z\cdot\sum_{k=1}^{\lfloor\frac{\log n}{\log x}\rfloor}\frac{x^{(1-s)k}}{k}+z\cdot[\log\zeta(s)]_n$$

```
bin[z_, k_] := Product[z - j, {j, 0, k - 1}]/k!
E2[n_, k_] :=  E2[n, k] = Sum[(-1)^(j + 1)  E2[Floor[n/j], k - 1], {j, 2, n}]
E2[n_, 0] := UnitStep[n - 1]
Etz[n_, z_] := Sum[ bin[ z, k] E2[n, k], {k, 0, Log[2, n]}]
etz[n_, z_] := Etz[n, z] - Etz[n - 1, z]
D1xD[n_, k_, z2_] := D1xD[n, k, z2] = Sum[etz[j, z2]  D1xD[n/j, k - 1, z2], {j, 2, n}]
D1xD[n_, 0, z2_] := UnitStep[n - 1]
E1[n_, z_] := Sum[ (-1)^(k + 1)/k D1xD[n, k, z], {k, 1, Log2@n}]
fo[n_] := -Sum[ 2^k/k, {k, 1, Log2@n}]
pr[n_] := Sum[ PrimePi[ n^(1/k)]/k, {k, 1, Log2@n}]
DiscretePlot[ E1[n, 2] - (2 pr[n] + 2 fo[n]), {n, 1, 100}]
```

$$\log((1-x^{1-s})\zeta(s))=\log(1-x^{1-s})+\log\zeta(s)$$

$$\log((1-x^{1-s})\zeta(s)^2)=\log(1-x^{1-s})+\log(\zeta(s)^2)$$

$$\boxed{\lim_{x \to 1^+} \sum_{k=1}^{\lfloor \frac{\log n}{\log x} \rfloor} \frac{x^{k(1-s)}-1}{k} = li\left(n^{1-s}\right) - \log \log n^{1-s} - \gamma}$$

$$\boxed{\sum_{k=1}^{\infty} \frac{x^{k(1-s)}}{k} = -\log\left(1 - x^{(1-s)}\right)}$$

AND...

$$\boxed{\lim_{x \to 1^+} \sum_{k=1}^{\lfloor \frac{\log n}{\log x} \rfloor} \frac{x^{k}-1}{k} = li(n) - \log \log n - \gamma}$$

$$\boxed{\sum_{k=1}^{\infty} \frac{x^{k}}{k} = -\log(1 - x)}$$

$$\boxed{\lim_{x \to 1^+} \sum_{k=1}^{\lfloor \frac{\log n}{\log x} \rfloor} x^{k} \cdot \log x = n - 1}$$

$$\boxed{\sum_{k=1}^{\infty} x^{k} \cdot \log x = -\frac{x \log x}{x - 1}}$$

$$\boxed{\pi(n)=\sum_{k=1}^{\lfloor \log_2 n \rfloor} \frac{1}{k}\cdot\mu(k)[\log\zeta_{1/k}(0)]_n}$$

$$\boxed{[\log(\prod_{k=1} \zeta_{1/k}(0)^{\frac{1}{k}\cdot[\nabla\zeta(0)^{-1}]_k})]_n=\sum_{k=1} \frac{1}{k}\cdot[\nabla\zeta(0)^{-1}]_k\cdot[\log\zeta_{1/k}(0)]_n}$$

$$\boxed{[\log(\prod_{k=1} \zeta_{1/k}(s\cdot k)^{\frac{1}{k}\cdot[\nabla\zeta(0)^{-1}]_k})]_n=\sum_{k=1} \frac{1}{k}\cdot[\nabla\zeta(0)^{-1}]_k\cdot[\log\zeta_{1/k}(s\cdot k)]_n}$$

$$\boxed{[\log\zeta_{1/k}(s\cdot k)]_n=\sum_{k=1} \frac{1}{k}\cdot[\nabla\zeta(0)]_k\cdot[\log(\prod_{k=1} \zeta_{1/k}(s\cdot k)^{\frac{1}{k}\cdot[\nabla\zeta(0)^{-1}]_k})]_n}$$

$$\boxed{[f]_n=[\prod_{k=1} \zeta_{\frac{1}{k}}(0)^{\frac{1}{k}\cdot[\nabla\zeta(0)^{-1}]_k}]_n}$$

$$\boxed{[\zeta(0)]_n=[\prod_{k=1} f_{\frac{1}{k}}(0)^{\frac{1}{k}}]_n}$$

$$\sum_{a\cdot b^2\cdot c^3\cdot d^4\cdot\ldots\le n} f_1(a)\cdot f_{\frac{1}{2}}(b)\cdot f_{\frac{1}{3}}(c)\cdot f_{\frac{1}{4}}(d)\cdot\ldots=n$$

$$\log\left(\left(1-x^{1-s}\right)\zeta(s)\right)=-\sum_{k=1}^{\infty}\frac{x^{k(1-s)}}{k}+\log\zeta_n(s)$$

$$\left[\log\left(\left(1-x^{1-s}\right)\zeta(s)\right)\right]_n=-\sum_{k=1}^{\lfloor\frac{\log n}{\log x}\rfloor}\frac{x^{k(1-s)}}{k}+\left[\log\zeta(s)\right]_n$$

$$\lim_{s\to 1}\left(1-x^{1-s}\right)\zeta(s)=\log x$$

$$\lim_{x\to 1}\lim_{s\to 1}\left[\left(1-x^{1-s}\right)\zeta(s)\right]_n=???$$

$$\lim_{x\to 1}\lim_{s\to 1}\left[\left(1-x^{1-s}\right)\zeta(s)-1\right]_n=???$$

$$\left[\log\left(\left(1-x^{1-s}\right)f(s)\right)\right]_n=???+\left[f(s)\right]_n$$

Table 2

Variant of $[\zeta(s)^z]_n$	*Value*	*Via Roots, in Mathematica*	*Value in Mathematica*
$[\log(\zeta(s)^z)]_n$	$z\cdot[\log\zeta(s)]_n$		
$[\log(t\cdot\zeta(s))]_n$	$\log t+[\log\zeta(s)]_n$		
$[\log(\zeta_{\log n}(s)\cdot\zeta_{\log m}(s))]_e$	$[\log\zeta(s)]_n+[\log\zeta(s)]_m$		
$[\log(\frac{\zeta_{\log n}(s)}{\zeta_{\log m}(s)})]_e$	$[\log\zeta(s)]_n-[\log\zeta(s)]_m$		
$[\log(\zeta(s)\cdot\zeta_{\frac{\log m}{\log n}}(s))]_n$	$[\log\zeta(s)]_n+[\log\zeta(s)]_m$		
$[\log(\frac{\zeta(s)}{\zeta_{\frac{\log m}{\log n}}(s)})]_n$	$[\log\zeta(s)]_n-[\log\zeta(s)]_m$		
$[\log((1-x^{1-s})\zeta(s))]_n$	$-\sum_{k=1}^{\lfloor\frac{\log n}{\log x}\rfloor}\frac{x^{(1-s)k}}{k}+[\log\zeta(s)]_n$		
$[\log((1-x^{1-s})\zeta(s)^z)]_n$	$-z\cdot\sum_{k=1}^{\lfloor\frac{\log n}{\log x}\rfloor}\frac{x^{(1-s)k}}{k}+z\cdot[\log\zeta(s)]_n$		
$[\log(\frac{\zeta_{1/2}(2s)}{\zeta(s)})]_n$	$[\log\zeta_{1/2}(2s)]_n-[\log\zeta(s)]_n$		
$[\log(\frac{\zeta(s)}{\zeta_{1/2}(2s)})]_n$	$[\log\zeta(s)]_n-[\log(\zeta_{1/2}(2s))]_n$		
$[\log(\zeta(s-a)\cdot\zeta(s))]_n$	$[\log\zeta(s-a)]_n+[\log\zeta(s)]_n$		
$[\log(\frac{\zeta(s-a)}{\zeta(s)})]_n$	$[\log\zeta(s-a)]_n-[\log\zeta(s)]_n$		
$[\log(\prod_{k=1}\zeta_{1/k}(0))]_n$	$\sum_{k=1}[\log\zeta_{1/k}(0)]_n$		
$[\log(\prod_{k=1}\zeta_{1/k}(0)^{\frac{\mu(k)}{k}})]_n$	$\pi(n)$		

Several other important functions emerge as n approaches

Table 2

Variant of $[\zeta(s)^z]_n$	*Value*	*Via Roots, in Mathematica*	*Value in Mathematica*
$\lim_{z\to 0}\frac{\partial}{\partial z}([\zeta(s)^z]_n\cdot[\zeta(s)^z]_m)$	$[\log\zeta(s)]_n+[\log\zeta(s)]_m$		
$\lim_{z\to 0}\frac{\partial}{\partial z}(D_z(n)\cdot D_z(m))$	$\Pi(n)+\Pi(m)$		
$\lim_{s\to 0}\frac{\partial}{\partial s}\lim_{z\to 0}\frac{\partial}{\partial z}([\zeta(s)^z]_n\cdot[\zeta(s)^z]_m)$	$\psi(n)+\psi(m)$		
$\lim_{z\to 0}\frac{\partial}{\partial z}(\frac{[\zeta(s)^z]_n}{[\zeta(s)^z]_m})$	$[\log\zeta(s)]_n-[\log\zeta(s)]_m$		
$\lim_{z\to 0}\frac{\partial}{\partial z}(\frac{D_z(n)}{D_z(m)})$	$\Pi(n)-\Pi(m)$		
$\lim_{s\to 0}\frac{\partial}{\partial s}\lim_{z\to 0}\frac{\partial}{\partial z}(\frac{[\zeta(s)^z]_n}{[\zeta(s)^z]_m})$	$\psi(n)-\psi(m)$		
$\lim_{z\to 0}\frac{\partial}{\partial z}(\frac{[\zeta(s)^z]_n}{[\zeta(s)^z]_{n-1}})$	$\kappa(n)\cdot n^{-s}$		
$\lim_{z\to 0}\frac{\partial}{\partial z}(\frac{D_z(n)}{D_z(n-1)})$	$\kappa(n)$		
$\lim_{s\to 0}\frac{\partial}{\partial s}\lim_{z\to 0}\frac{\partial}{\partial z}(\frac{[\zeta(s)^z]_n}{[\zeta(s)^z]_{n-1}})$	$\Lambda(n)$		

Several other important functions emerge as n approaches

$$\lim_{z\to 0}\frac{\partial}{\partial z}([\zeta(s)^{a\cdot z}]_n\cdot[\zeta(t)^{b\cdot z}]_m)=a[\log\zeta(s)]_n+b[\log\zeta(t)]_m$$

$$\lim_{z\to 0}\frac{\partial}{\partial z}(\frac{L_{-z}(\log n)}{L_{-z}(\log m)})=(li(n)-\log\log n-\gamma)-(li(m)-\log\log m-\gamma)$$

$$\lim_{z\to 0}\frac{\partial}{\partial z}(L_{-a\cdot z}(\log n)\cdot L_{-b\cdot z}(\log m))=a(li(n)-\log\log n-\gamma)+b(li(m)-\log\log m-\gamma)$$

$$\lim_{z\to 0}\frac{\partial}{\partial z}(\frac{[(1-2^{1-s})\zeta(s)^z]_n}{[(1-2^{1-s})\zeta(s)^z]_m})=[\log((1-2^{1-s}))\zeta(s)]_n-[\log((1-2^{1-s})\zeta(s))]_m$$

$$\lim_{z\to 0}\frac{\partial}{\partial z}(\frac{[(1+y^{s-1}\cdot\zeta(s,1+y))^z]_n}{[(1+y^{s-1}\cdot\zeta(s,1+y))^z]_m})=[\log(1+y^{s-1}\cdot\zeta(s,1+y))^z]_n-[\log(1+y^{s-1}\cdot\zeta(s,1+y))^z]_m$$