## Overview

Start with the well known identity for the Dirichlet eta function:

$$(1-2^{1-s})\zeta(s)=\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j^s}, \Re(s)>0$$

Generalize this to a real valued $x$ to be

$$(1-x^{1-s})\zeta(s)=\lim_{n\to\infty}\left(\sum_{j=1}^{\infty}\left(\frac{1}{j^s}-\frac{1}{j+x\cdot n^s}\right)-x^{1-s}\cdot\sum_{1\le j\le n}\frac{1}{j^s}\right), \Re(s)>0$$

Use fractional integro-differentiation on this to show that, for some complex $z$,

$$\zeta(s)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{j^s}+n^z\cdot\left(\frac{s-1+z}{s-1}\right)\cdot\left(\zeta(s+z)-\sum_{j=1}^{n}\frac{1}{j^{s+z}}\right), \Re(s)>0, \Re(s+z)>0$$

This leads immediately to

$$\boxed{\lim_{n\to\infty} n^y\cdot(s-1+y)\left(\zeta(s+y)-\sum_{j=1}^{n}\frac{1}{j^{s+y}}\right)-n^x\cdot(s-1+x)\left(\zeta(s+x)-\sum_{j=1}^{n}\frac{1}{j^{s+x}}\right)=0}$$

for $, \Re(s)>0, \Re(s+x)>0, \Re(s+y)>0$ .

Now, suppose that $s=\frac{1}{2}$ and $y=-x$ . Then this is

$$\lim_{n\to\infty} n^{-x}\cdot\left(-\frac{1}{2}-x\right)\left(\zeta\left(\frac{1}{2}-x\right)-\sum_{j=1}^{n}\frac{1}{j^{\frac{1}{2}-x}}\right)-n^x\cdot\left(-\frac{1}{2}+x\right)\left(\zeta\left(\frac{1}{2}+x\right)-\sum_{j=1}^{n}\frac{1}{j^{\frac{1}{2}+x}}\right)=0$$

for $\Re(x)<\frac{1}{2}$ .

Now, obviously, we are most interested in cases where $\zeta(\frac{1}{2}+x)=0$ (and thus, by the reflection formula, necessarily, $\zeta(\frac{1}{2}-x)=0$ ). The above equations suggests that, for these two terms to be 0, it is a necessary but not sufficient condition for

$$\boxed{\lim_{n\to\infty} n^{-x}\cdot\left(\frac{1}{2}+x\right)\left(\sum_{j=1}^{n}\frac{1}{j^{\frac{1}{2}-x}}\right)+n^x\cdot\left(x-\frac{1}{2}\right)\left(\sum_{j=1}^{n}\frac{1}{j^{\frac{1}{2}+x}}\right)=0}$$

to be true, and the Riemann Hypothesis equivalent to the statement that this equation can never be satisfied if x has a non-zero real component.

This formula can be rewritten in a number of different ways. In particular, it can lead to the equivalent statement that

$$\boxed{\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\left(2x\cos\left(x\cdot\log\frac{j}{n}\right)+\sin\left(x\cdot\log\frac{j}{n}\right)\right)=0}$$

and

$$\boxed{\begin{aligned}&\lim_{n\to\infty}(2x\sin(x\log n)+\cos(x\log n))\cdot\left(\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\sin(x\log j)\right)+\\ &\qquad(2x\cos(x\log n)-\sin(x\log n))\cdot\left(\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\cos(x\log j)\right)=0\end{aligned}}$$

can't be true if $x$ has a non-zero imaginary component. Empirically, it appears that neither of the equations ever converge if

$x$ has a non-zero imaginary component.

## Step 1:

## Generalize

$$(1-2^{1-s})\zeta(s)=\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j^s}$$

## to

$$(1-x^{1-s})\zeta(s)=\lim_{n\to\infty}\left(\sum_{j=1}^{\infty}\left(\frac{1}{j^s}-\frac{1}{(j+x\cdot n)^s}\right)-x^{1-s}\cdot\sum_{1\le j\le n}\frac{1}{j^s}\right)$$

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## Step 2:

## Fractional Integro-Differentiation of

$$(1-x^{1-s})\zeta(s)=\lim_{n\to\infty}\left(\sum_{j=1}^{\infty}\left(\frac{1}{j^s}-\frac{1}{(j+x\cdot n)^s}\right)-x^{1-s}\cdot\sum_{1\le j\le n}\frac{1}{j^s}\right)$$

## Leads to

$$\zeta(s)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{j^s}+n^z\cdot\left(\frac{s-1+z}{s-1}\right)\cdot\left(\zeta(s+z)-\sum_{j=1}^{n}\frac{1}{j^{s+z}}\right)$$

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The derivative of $(1-x^{1-s})\zeta(s)=\lim_{n\to\infty}\left(\sum_{j=1}^{\infty}\left(\frac{1}{j^s}-\frac{1}{(j+x\cdot n)^s}\right)-x^{1-s}\cdot\sum_{1\le j\le n}\frac{1}{j^s}\right)$ , with respect to x, is

$$(1-s)\,x^{-s}\cdot\zeta(s)=\lim_{n\to\infty} n\cdot s\cdot\sum_{j=1}^{\infty}\frac{1}{(j+n\,x)^{s+1}}+(1-s)\cdot x^{-s}\cdot\sum_{1\le j\le n}\frac{1}{j^s}$$

Its 2nd derivative is, in turn,

$$(1-s)(-s)\cdot x^{-s-1}\cdot\zeta(s)=\lim_{n\to\infty} -n^2\cdot s(s+1)\cdot\sum_{j=1}^{\infty}\frac{1}{(j+n\,x)^{s+2}}+(1-s)(-s)\cdot x^{-s-1}\cdot\sum_{1\le j\le n}\frac{1}{j^s}$$

and the kth derivative can clearly be calculated through standard techniques.

If we take these derivatives at x=1 and isolate $\zeta(s)$ by dividing out the other terms on the left hand side of the equation, this gives us

$$\zeta(s)=\lim_{n\to\infty} n\cdot\frac{s}{1-s}\cdot\sum_{j=1}^{\infty}\frac{1}{(j+n)^{s+1}}+\sum_{1\le j\le n}\frac{1}{j^s}$$

and

$$\zeta(s)=\lim_{n\to\infty} n^2\cdot\frac{s+1}{1-s}\cdot\sum_{j=1}^{\infty}\frac{1}{(j+n)^{s+2}}+\sum_{1\le j\le n}\frac{1}{j^s}$$

and so on.

If we turn to fractional integro-differentiation, the derivative generalizes in the standard way to

$$(1-s)\frac{\Gamma(1-s)}{\Gamma(2-s-z)}\cdot x^{1-s-z}\cdot\zeta(s)=\lim_{n\to\infty} -n^z\cdot\frac{\Gamma(1-s)}{\Gamma(1-s-z)}\cdot\sum_{j=1}^{\infty}(j+n\,x)^{-s-z}+(1-s)\frac{\Gamma(1-s)}{\Gamma(2-s-z)}\cdot x^{1-s-z}\cdot\sum_{1\le j\le n}\frac{1}{j^s}$$

which, if we take the derivative at x=1 and isolate $\zeta(s)$ on the left hand side, leaves us with

$$\boxed{\zeta(s)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{j^s}+n^z\cdot\left(\frac{s-1+z}{s-1}\right)\cdot\left(\zeta(s+z)-\sum_{j=1}^{n}\frac{1}{j^{s+z}}\right)}$$

## Step 3:

$$\zeta(s)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{j^{s}}+n^{z}\cdot(\frac{s-1+z}{s-1})\cdot(\zeta(s+z)-\sum_{j=1}^{n}\frac{1}{j^{s+z}})$$

**Implies**

$$\lim_{n\to\infty} n^{y}\cdot(s-1+y)(\zeta(s+y)-\sum_{j=1}^{n}\frac{1}{j^{s+y}})-n^{z}\cdot(s-1+z)(\zeta(s+z)-\sum_{j=1}^{n}\frac{1}{j^{s+z}})=0$$

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This part is pretty straightforward. If we have

$$\zeta(s)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{j^{s}}+n^{z}\cdot(\frac{s-1+z}{s-1})\cdot(\zeta(s+z)-\sum_{j=1}^{n}\frac{1}{j^{s+z}})$$

we can rearrange terms to get

$$\lim_{n\to\infty}\zeta(s)-\sum_{j=1}^{n}\frac{1}{j^{s}}-n^{z}\cdot(\frac{s-1+z}{s-1})\cdot(\zeta(s+z)-\sum_{j=1}^{n}\frac{1}{j^{s+z}})=0$$

Then, multiply every term by (s-1) to get

$$\lim_{n\to\infty}(s-1)(\zeta(s)-\sum_{j=1}^{n}\frac{1}{j^{s}})-n^{z}\cdot(s-1+z)\cdot(\zeta(s+z)-\sum_{j=1}^{n}\frac{1}{j^{s+z}})=0$$

Now swap z with some other variable y, to get

$$\lim_{n\to\infty}(s-1)(\zeta(s)-\sum_{j=1}^{n}\frac{1}{j^{s}})-n^{y}\cdot(s-1+y)\cdot(\zeta(s+y)-\sum_{j=1}^{n}\frac{1}{j^{s+y}})=0$$

Now subtract this from the previous equation, to get

$$\boxed{\lim_{n\to\infty} n^{y}\cdot(s-1+y)(\zeta(s+y)-\sum_{j=1}^{n}\frac{1}{j^{s+y}})-n^{x}\cdot(s-1+x)(\zeta(s+x)-\sum_{j=1}^{n}\frac{1}{j^{s+x}})=0}$$

## Step 4:

## Using

$$\lim_{n\to\infty} n^{y}\cdot(s-1+y)(\zeta(s+y)-\sum_{j=1}^{n}\frac{1}{j^{s+y}})-n^{z}\cdot(s-1+z)(\zeta(s+z)-\sum_{j=1}^{n}\frac{1}{j^{s+z}})=0$$

## to Establish the New Function

$$f(x)=\lim_{n\to\infty}(n^{-x}\cdot(\frac{1}{2}+x)\sum_{j=1}^{n}\frac{1}{j^{\frac{1}{2}-x}})+(n^{x}\cdot(x-\frac{1}{2})\sum_{j=1}^{n}\frac{1}{j^{\frac{1}{2}+x}})$$

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In the previous step, we arrived at the following identity

$$\lim_{n\to\infty} n^{y}\cdot(s-1+y)(\zeta(s+y)-\sum_{j=1}^{n}\frac{1}{j^{s+y}})-n^{z}\cdot(s-1+z)(\zeta(s+z)-\sum_{j=1}^{n}\frac{1}{j^{s+z}})=0$$

We are, of course, primarily interested in the question of under what circumstances $\zeta(s)=0$ . We know from the reflection formula $\zeta(s)=2^{s}\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)$ that for non-trivial zeroes of $\zeta(s)$ , if $\zeta(\frac{1}{2}+x)=0$ , then it must be that $\zeta(\frac{1}{2}-x)=0$ as well.

Let's make our identity reflect this. Say that $s=\frac{1}{2}$ , $y=-x$ , and $z=x$ . Then we have

$$\lim_{n\to\infty}(-\frac{1}{2}-x)(n^{-x})(\zeta(\frac{1}{2}-x)-\sum_{j=1}^{n}\frac{1}{j^{\frac{1}{2}-x}})-(-\frac{1}{2}+x)(n^{x})(\zeta(\frac{1}{2}+x)-\sum_{j=1}^{n}\frac{1}{j^{\frac{1}{2}+x}})=0$$

But of course, our entire

$$\lim_{n\to\infty} n^{-x}\cdot(\frac{1}{2}+x)(\sum_{j=1}^{n}\frac{1}{j^{\frac{1}{2}-x}})+n^{x}\cdot(x-\frac{1}{2})(\sum_{j=1}^{n}\frac{1}{j^{\frac{1}{2}+x}})$$

## Step 5:

$$f(x)=\lim_{n\to\infty} n^{-x}\cdot\left(\frac{1}{2}+x\right)\left(\sum_{j=1}^{n}\frac{1}{j^{\frac{1}{2}-x}}\right)+n^{x}\cdot\left(x-\frac{1}{2}\right)\left(\sum_{j=1}^{n}\frac{1}{j^{\frac{1}{2}+x}}\right)$$

## Can be Rewritten as

$$f(x)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\left(2x\cosh\left(x\cdot\log\frac{j}{n}\right)+\sinh\left(x\cdot\log\frac{j}{n}\right)\right)$$

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$$\lim_{n\to\infty} n^{-x}\cdot\left(\frac{1}{2}+x\right)\left(\sum_{j=1}^{n}\frac{1}{j^{\frac{1}{2}-x}}\right)+n^{x}\cdot\left(x-\frac{1}{2}\right)\left(\sum_{j=1}^{n}\frac{1}{j^{\frac{1}{2}+x}}\right)$$

$$\lim_{n\to\infty}\sum_{j=1}^{n} n^{-x}\cdot\left(\frac{1}{2}+x\right)\frac{1}{j^{\frac{1}{2}-x}}+n^{x}\cdot\left(x-\frac{1}{2}\right)\frac{1}{j^{\frac{1}{2}+x}}$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\left(n^{-x}\cdot\left(\frac{1}{2}+x\right)\frac{1}{j^{-x}}+n^{x}\cdot\left(x-\frac{1}{2}\right)\frac{1}{j^{x}}\right)$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\left(\left(\frac{1}{2}+x\right)\left(\frac{j}{n}\right)^{x}+\left(x-\frac{1}{2}\right)\left(\frac{j}{n}\right)^{-x}\right)$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\left(x\left(\left(\frac{j}{n}\right)^{x}+\left(\frac{j}{n}\right)^{-x}\right)+\frac{1}{2}\left(\left(\frac{j}{n}\right)^{-x}-\left(\frac{j}{n}\right)^{-x}\right)\right)$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\left(x\left(e^{x\cdot\log\frac{j}{n}}+e^{-x\cdot\log\frac{j}{n}}\right)+\frac{1}{2}\left(e^{x\cdot\log\frac{j}{n}}-e^{-x\cdot\log\frac{j}{n}}\right)\right)$$

$$\boxed{\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\left(2x\cosh\left(x\cdot\log\frac{j}{n}\right)+\sinh\left(x\cdot\log\frac{j}{n}\right)\right)}$$

**Step 6:**