

```

bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
f[n_, s_, k_] := Sum[(-1)^(j + 1) j^(-s) f[n / j, s, k - 1], {j, 2, Floor[n]}]
f[n_, s_, 0] := 1
g[n_, s_, z_] := Expand[Sum[bin[z, k] f[n, s, k], {k, 0, Log[2, n]}]]
zeros[n_, s_] := List @@ Roots[g[n, s, z] == 0, z][[All, 2]]

Expand[g[900, 0, z]]

1 +  $\frac{13\,753\,z}{504}$  -  $\frac{253\,619\,z^2}{3360}$  +  $\frac{6\,854\,461\,z^3}{90\,720}$  -  $\frac{70\,447\,z^4}{1920}$  +  $\frac{5161\,z^5}{540}$  -  $\frac{1271\,z^6}{960}$  +  $\frac{2983\,z^7}{30\,240}$  -  $\frac{41\,z^8}{13\,440}$  +  $\frac{z^9}{45\,360}$ 

N[zeros[10 000, N[ZetaZero[1]]]] // TableForm

-0.056997 + 0.256175 i
0.993717 - 0.00552038 i
1.22366 - 4.3064 i
1.86509 - 0.0871545 i
2.15019 + 4.33561 i
2.78845 - 0.368826 i
2.87575 + 0.156775 i
3.01969 + 15.2867 i
9.93096 - 1.25319 i
14.6253 - 2.87026 i
38.4628 + 54.5394 i
54.7341 - 21.7694 i
116.761 - 712.137 i

0

frange[n_, s_] := frange[n, s] = Sum[N[m^(-s) (-1)^(m + 1)], {m, 1, n}]
ffast[n_, 0, s_, a_] := UnitStep[n - 1]
ffast[n_, 1, s_, a_] := ffast[n, 1, s, a] = frange[Floor[n], s] - frange[a, s]
ffast[n_, k_, s_, a_] := ffast[n, k, s, a] =
  N[Sum[Binomial[k, j] (m^(-s) (-1)^(m + 1))^j ffast[Floor[n / (m^j)], k - j, s, m],
    {j, 1, k}, {m, a + 1, Floor[n^(1 / k)}]]]
gfast[n_, s_, z_] := Sum[bin[z, k] ffast[n, k, s, 1], {k, 0, Log[2, n]}]
zerosfast[n_, s_] := List @@ NRoots[gfast[n, s, z] == 0, z][[All, 2]]

zerosfast[10 000, 0] // TableForm

0.00691499
1.
1.38296 - 0.491005 i
1.38296 + 0.491005 i
2.75946 - 0.210986 i
2.75946 + 0.210986 i
5.37564 - 9.06206 i
5.37564 + 9.06206 i
10.2876
31.0016 - 28.5431 i
31.0016 + 28.5431 i
32.6661
824.

```

```
(*Table[{ FullSimplify[ (D[g[n,s,z],z]-D[g[n-1,s,z],z])/.z->0],
  (FullSimplify[MangoldtLambda[n]/Log[n]-If[ Log[2,n]==Floor[Log[2,n]],n/Log[2,n],0]])
  n^-s},{n,2,40}]}//TableForm*)
```

```
theta[n_, k_] := (1+Floor[Floor[Log[k, n]] - Log[k, n]]) n / Log[2, n]
```

```
Table[{ FullSimplify[ (D[g[n, s, z], z] - D[g[n - 1, s, z], z]) /. z -> 0],
  (FullSimplify[MangoldtLambda[n] / Log[n] - theta[n, 2])) n^-s}, {n, 2, 40}] // TableForm
```

-2^{-s}	-2^{-s}
3^{-s}	3^{-s}
$-3 \cdot 2^{-1-2s}$	$-3 \cdot 2^{-1-2s}$
5^{-s}	5^{-s}
0	0
7^{-s}	7^{-s}
$-\frac{7 \times 8^{-s}}{3}$	$-\frac{7 \times 8^{-s}}{3}$
$\frac{9^{-s}}{2}$	$\frac{9^{-s}}{2}$
0	0
11^{-s}	11^{-s}
0	0
13^{-s}	13^{-s}
0	0
0	0
$-15 \cdot 4^{-1-2s}$	$-15 \cdot 4^{-1-2s}$
17^{-s}	17^{-s}
0	0
19^{-s}	19^{-s}
0	0
0	0
0	0
23^{-s}	23^{-s}
0	0
$\frac{5^{-2s}}{2}$	$\frac{25^{-s}}{2}$
0	0
3^{-1-3s}	3^{-1-3s}
0	0
29^{-s}	29^{-s}
0	0
31^{-s}	31^{-s}
$-\frac{31}{5} \cdot 2^{-5s}$	$-\frac{31 \times 32^{-s}}{5}$
0	0
0	0
0	0
0	0
37^{-s}	37^{-s}
0	0
0	0
0	0

```
theta[n_, k_] := (1+Floor[Floor[Log[k, n]] - Log[k, n]]) n / Log[k, n]
```



```

FI[n_] := FactorInteger[n]; FI[1] := {}
dz[n_, z_] := Product[(-1)^p[[2]] Binomial[-z, p[[2]]], {p, FI[n]}]
dz2[n_, z_] := If[Log[2, n] == Floor[Log[2, n]], (-1)^(Log[2, n]) n bin[-z, Log[2, n]], 0]
Table[{n, FullSimplify[g[n, 0, z] - g[n - 1, 0, z]], dz[n, z], dz2[n, z]}, {n, 2, 40}] //
  TableForm

```

2	$-z$	$-z$
3	z	z
4	$\frac{1}{2}(-3+z)z$	$\frac{3}{2}(-1-z)z$
5	z	z
6	$-z^2$	z^2
7	z	z
8	$-\frac{1}{6}(-7+z)(-2+z)z$	$-\frac{7}{6}(-2-z)(-1-z)z$
9	$\frac{1}{2}z(1+z)$	$-\frac{1}{2}(-1-z)z$
10	$-z^2$	z^2
11	z	z
12	$\frac{1}{2}(-3+z)z^2$	$-\frac{1}{2}(-1-z)z^2$
13	z	z
14	$-z^2$	z^2
15	z^2	z^2
16	$\frac{1}{24}(-5+z)z(18+(-13+z)z)$	$\frac{5}{8}(-3-z)(-2-z)(-1-z)z$
17	z	z
18	$-\frac{1}{2}z^2(1+z)$	$-\frac{1}{2}(-1-z)z^2$
19	z	z
20	$\frac{1}{2}(-3+z)z^2$	$-\frac{1}{2}(-1-z)z^2$
21	z^2	z^2
22	$-z^2$	z^2
23	z	z
24	$-\frac{1}{6}(-7+z)(-2+z)z^2$	$\frac{1}{6}(-2-z)(-1-z)z^2$
25	$\frac{1}{2}z(1+z)$	$-\frac{1}{2}(-1-z)z$
26	$-z^2$	z^2
27	$\frac{1}{6}z(1+z)(2+z)$	$\frac{1}{6}(-2-z)(-1-z)z$
28	$\frac{1}{2}(-3+z)z^2$	$-\frac{1}{2}(-1-z)z^2$
29	z	z
30	$-z^3$	z^3
31	z	z
32	$-\frac{1}{120}(-4+z)z(-186+z(171+(-26+z)z))$	$-\frac{31}{120}(-4-z)(-3-z)(-2-z)(-1-z)z$
33	z^2	z^2
34	$-z^2$	z^2
35	z^2	z^2
36	$\frac{1}{4}(-3+z)z^2(1+z)$	$\frac{1}{4}(-1-z)^2z^2$
37	z	z
38	$-z^2$	z^2
39	z^2	z^2
40	$-\frac{1}{6}(-7+z)(-2+z)z^2$	$\frac{1}{6}(-2-z)(-1-z)z^2$

```

fk[n_, s_, 0] := If[n == 1, 1, 0]
fk[n_, s_, 1] := If[n == 1, 0, (-1)^(n+1) n^-s]
fk[n_, s_, k_] := Sum[fk[j, s, 1] fk[n/j, s, k-1], {j, Divisors[n]}]
fz[n_, s_, z_] := Sum[bin[z, k] fk[n, s, k], {k, 0, Log[2, n]}]

FullSimplify[fk[100, s, 4]]

 $3 \times 2^{1-2s} 5^{-2s}$ 

FullSimplify[f[100, s, 4] - f[99, s, 4]]

 $3 \times 2^{1-2s} 5^{-2s}$ 

FullSimplify[fz[100, s, z]]

 $4^{-1-s} 25^{-s} (-3+z) z^2 (1+z)$ 

FullSimplify[g[100, s, z] - g[99, s, z]]

 $4^{-1-s} 25^{-s} (-3+z) z^2 (1+z)$ 

FullSimplify[(a^j / b^j) / j]


$$\frac{a^j b^{-j}}{j}$$


a[n_, a_, b_] := b (Floor[n/b] - Floor[(n-1)/b]) - a (Floor[n/a] - Floor[(n-1)/a])
Grid[Table[{Sum[N[a[n, a, b] / n], {n, 1, 100 000}}, N[Log[a/b]]], {a, 1, 10}, {b, 1, 6}]]

```

{0., 0.}	{-0.693142, -0.693147}	{-1.09861, -1.09861}	{-1.38628, -1.38629}	{-1.60942, -1.60944}	{-1.79177, -1.79176}
{0.693142, 0.693147}	{0., 0.}	{-0.40547, -0.405465}	{-0.693137, -0.693147}	{-0.916276, -0.916291}	{-1.09863, -1.09861}
{1.09861, 1.09861}	{0.40547, 0.405465}	{0., 0.}	{-0.287667, -0.287682}	{-0.510806, -0.510826}	{-0.693162, -0.693147}
{1.38628, 1.38629}	{0.693137, 0.693147}	{0.287667, 0.287682}	{0., 0.}	{-0.223139, -0.223144}	{-0.405495, -0.405465}
{1.60942, 1.60944}	{0.916276, 0.916291}	{0.510806, 0.510826}	{0.223139, 0.223144}	{0., 0.}	{-0.182357, -0.182322}
{1.79177, 1.79176}	{1.09863, 1.09861}	{0.693162, 0.693147}	{0.405495, 0.405465}	{0.182357, 0.182322}	{0., 0.}
{1.94593, 1.94591}	{1.25279, 1.25276}	{0.847318, 0.847298}	{0.559651, 0.559616}	{0.336512, 0.336472}	{0.154156, 0.154151}
{2.07941, 2.07944}	{1.38626, 1.38629}	{0.980794, 0.980829}	{0.693127, 0.693147}	{0.469989, 0.470004}	{0.287632, 0.287682}
{2.19719, 2.19722}	{1.50405, 1.50408}	{1.09858, 1.09861}	{0.810915, 0.81093}	{0.587777, 0.587787}	{0.40542, 0.405465}
{2.30254, 2.30259}	{1.6094, 1.60944}	{1.20393, 1.20397}	{0.916261, 0.916291}	{0.693122, 0.693147}	{0.510766, 0.510826}

```

 $\alpha[n_, a_, b_] := b (\text{Floor}[n / b] - \text{Floor}[(n - 1) / b]) - a (\text{Floor}[n / a] - \text{Floor}[(n - 1) / a])$ 
Table[{n,  $\alpha[n, 2, 1]$ ,  $(-1)^{(n+1)}$ }, {n, 1, 50}] // TableForm

```

1	1	1
2	-1	-1
3	1	1
4	-1	-1
5	1	1
6	-1	-1
7	1	1
8	-1	-1
9	1	1
10	-1	-1
11	1	1
12	-1	-1
13	1	1
14	-1	-1
15	1	1
16	-1	-1
17	1	1
18	-1	-1
19	1	1
20	-1	-1
21	1	1
22	-1	-1
23	1	1
24	-1	-1
25	1	1
26	-1	-1
27	1	1
28	-1	-1
29	1	1
30	-1	-1
31	1	1
32	-1	-1
33	1	1
34	-1	-1
35	1	1
36	-1	-1
37	1	1
38	-1	-1
39	1	1
40	-1	-1
41	1	1
42	-1	-1
43	1	1
44	-1	-1
45	1	1
46	-1	-1
47	1	1
48	-1	-1
49	1	1
50	-1	-1

```

bin[z_, k_] := bin[z, k] = Product[z - j, {j, 0, k - 1}] / k!
FI[n_] := Fi[n] = FactorInteger[n]; FI[1] := {}
dz[n_, z_] := dz[n, z] = Product[(-1)^p[[2]] Binomial[-z, p[[2]]], {p, FI[n]}]
Dz[n_, s_, z_, k_] := 1 + ((z + 1) / k - 1) Sum[j^-s Dz[n / j, s, z, k + 1], {j, 2, n}]
DlxD[n_, s_, k_, x_] :=
  Sum[(j + 1)^-s DlxD[n / (j + 1), s, k - 1, x] - x (j x)^-s DlxD[n / (x j), s, k - 1, x], {j, 1, n}]
DlxD[n_, s_, 0, x_] := UnitStep[n - 1]
DxD[n_, s_, z_, x_] :=
  Sum[bin[z, k] DlxD[n, s, k, x], {k, 0, If[x < 2, Log[x, n], Log[2, n]]}]
DxDAlt[n_, s_, z_, x_] :=
  Sum[(-1)^j bin[z, j] x^(j (1 - s)) Dz[n / x^j, s, z, 1], {j, 0, Log[x, n]}]
DxDAlt2[n_, s_, z_, x_] := Sum[(-1)^j bin[z, j] x^(j (1 - s))
  ((Dz[n / x^j, s, z, 1] - Dz[(n - 1) / x^j, s, z, 1])), {j, 0, Log[x, n]}]
DxDAlt3[n_, s_, z_, x_] := Sum[If[(Floor[n / x^j] - Floor[(n - 1) / x^j]) == 0,
  0, (-1)^j bin[z, j] x^(j (1 - s)) ((Floor[n / x^j] - Floor[(n - 1) / x^j])
  (Floor[n / x^j]^s dz[Floor[n / x^j], z]))], {j, 0, Log[x, n]}]
K[n_] := K[n] = FullSimplify[MangoldtLambda[n] / Log[n]]
DxDAlt4[n_, z_, x_] :=
  Sum[If[(Floor[n / x^j] - Floor[(n - 1) / x^j]) == 0, 0, (-1)^j bin[z, j] x^j
  ((Floor[n / x^j] - Floor[(n - 1) / x^j]) (dz[Floor[n / x^j], z]))], {j, 0, Log[x, n]}]
f[n_, s_, k_] := Sum[(-1)^(j + 1) j^-s f[n / j, s, k - 1], {j, 2, Floor[n]}]
f[n_, s_, 0] := 1
g[n_, s_, z_] := Expand[Sum[bin[z, k] f[n, s, k], {k, 0, Log[2, n]}]]
g2[n_, s_, z_] := g[n, s, z] - g[n - 1, s, z]

D[FullSimplify[DxD[120, 0, z, 2] - DxD[119, 0, z, 2]], z] /. z -> 0
0

D[FullSimplify[g2[120, 0, z]], z] /. z -> 0
0

Table[D[Expand[DxDAlt3[n, 0, z, 1.01]], z] /. z -> 0, {n, 2, 20}] // TableForm
-4.64572
-0.12549
-0.306666
0.345764
-0.6096
0.481733
-0.14713
0.0270478
-0.461343
0.600747
-0.420617
0.606777
-0.412778
-0.378304
-0.087755
0.649108
-0.364706
0.685016
-0.391897

```



```
Table[D[Expand[DxDAlt4[n, z, 1.01]], z] /. z -> 0, {n, 2, 20}] // TableForm
```

```
-4.64572
-0.12549
-0.306666
0.345764
-0.6096
0.481733
-0.14713
0.0270478
-0.461343
0.600747
-0.420617
0.606777
-0.412778
-0.378304
-0.087755
0.649108
-0.364706
0.685016
-0.391897
```

```
D[bin[z, 0], z] /. z -> 0
```

```
0
```

```
7 / (1.01) ^ 2
```

```
6.86207
```

```
6 / (1.01) ^ 2
```

```
5.88178
```