$$D_1(n)=\frac{1^1}{1!}\sum_{j>1}^{j\le n} p(j)+\frac{1^2}{2!}\sum_{j,k>1}^{j\cdot k\le n} p(j)\cdot p(k)+\frac{1^3}{3!}\sum_{j,k,m>1}^{j\cdot k\cdot m\le n} p(j)\cdot p(k)\cdot p(m)+\frac{1^4}{4!}\sum_{j,k,m,o>1}^{j\cdot k\cdot m\cdot o\le n} p(j)\cdot p(k)\cdot p(m)\cdot p(o)+\ldots$$

$$D_2(n)=\frac{2^1}{1!}\sum_{j>1}^{j\le n} p(j)+\frac{2^2}{2!}\sum_{j,k>1}^{j\cdot k\le n} p(j)\cdot p(k)+\frac{2^3}{3!}\sum_{j,k,m>1}^{j\cdot k\cdot m\le n} p(j)\cdot p(k)\cdot p(m)+\frac{2^4}{4!}\sum_{j,k,m,o>1}^{j\cdot k\cdot m\cdot o\le n} p(j)\cdot p(k)\cdot p(m)\cdot p(o)+\ldots$$

$$D_3(n)=\frac{3^1}{1!}\sum_{j>1}^{j\le n} p(j)+\frac{3^2}{2!}\sum_{j,k>1}^{j\cdot k\le n} p(j)\cdot p(k)+\frac{3^3}{3!}\sum_{j,k,m>1}^{j\cdot k\cdot m\le n} p(j)\cdot p(k)\cdot p(m)+\frac{3^4}{4!}\sum_{j,k,m,o>1}^{j\cdot k\cdot m\cdot o\le n} p(j)\cdot p(k)\cdot p(m)\cdot p(o)+\ldots$$

$$D_a(n)=\frac{a^1}{1!}\sum_{j>1}^{j\le n} p(j)+\frac{a^2}{2!}\sum_{j,k>1}^{j\cdot k\le n} p(j)\cdot p(k)+\frac{a^3}{3!}\sum_{j,k,m>1}^{j\cdot k\cdot m\le n} p(j)\cdot p(k)\cdot p(m)+\frac{a^4}{4!}\sum_{j,k,m,o>1}^{j\cdot k\cdot m\cdot o\le n} p(j)\cdot p(k)\cdot p(m)\cdot p(o)+\ldots$$

$$D_0(n)=1$$

$$D_1(n)=\sum_{j>1}^{j\le n} p(j)+\frac{1}{2}\sum_{j,k>1}^{j\cdot k\le n} p(j)\cdot p(k)+\frac{1}{6}\sum_{j,k,m>1}^{j\cdot k\cdot m\le n} p(j)\cdot p(k)\cdot p(m)+\frac{1}{24}\sum_{j,k,m,o>1}^{j\cdot k\cdot m\cdot o\le n} p(j)\cdot p(k)\cdot p(m)\cdot p(o)+\ldots$$

$$D_2(n)=2\sum_{j>1}^{j\le n} p(j)+2\sum_{j,k>1}^{j\cdot k\le n} p(j)\cdot p(k)+\frac{4}{3}\sum_{j,k,m>1}^{j\cdot k\cdot m\le n} p(j)\cdot p(k)\cdot p(m)+\frac{2}{3}\sum_{j,k,m,o>1}^{j\cdot k\cdot m\cdot o\le n} p(j)\cdot p(k)\cdot p(m)\cdot p(o)+\ldots$$

$$D_3(n)=3\sum_{j>1}^{j\le n} p(j)+\frac{9}{2}\sum_{j,k>1}^{j\cdot k\le n} p(j)\cdot p(k)+\frac{9}{2}\sum_{j,k,m>1}^{j\cdot k\cdot m\le n} p(j)\cdot p(k)\cdot p(m)+\frac{27}{8}\sum_{j,k,m,o>1}^{j\cdot k\cdot m\cdot o\le n} p(j)\cdot p(k)\cdot p(m)\cdot p(o)+\ldots$$

$$D_4(n)=4\sum_{j>1}^{j\le n} p(j)+8\sum_{j,k>1}^{j\cdot k\le n} p(j)\cdot p(k)+\frac{32}{3}\sum_{j,k,m>1}^{j\cdot k\cdot m\le n} p(j)\cdot p(k)\cdot p(m)+\frac{32}{3}\sum_{j,k,m,o>1}^{j\cdot k\cdot m\cdot o\le n} p(j)\cdot p(k)\cdot p(m)\cdot p(o)+\ldots$$

$$D_0(n)=1$$

$$D_1(n)=P_1(n)+\frac{1}{2}P_2(n)+\frac{1}{6}P_3(n)+\frac{1}{24}P_4(n)+\ldots$$

$$D_2(n)=2P_1(n)+2P_2(n)+\frac{4}{3}P_3(n)+\frac{2}{3}P_4(n)+\ldots$$

$$D_3(n)=3P_1(n)+\frac{9}{2}P_2(n)+\frac{9}{2}P_3(n)+\frac{27}{8}P_4(n)+\ldots$$

$$D_4(n)=4P_1(n)+8P_2(n)+\frac{32}{3}P_3(n)+\frac{32}{3}P_4(n)+\ldots$$

$$D_1'(n)=D_1(n)-D_0(n)$$

$$D_2'(n)=D_2(n)-2D_1(n)+D_0(n)$$

$$D_2'(n)=1+\frac{\binom{2}{2}2^1-\binom{2}{1}1^1}{1!}P_1(n)+\frac{\binom{2}{2}2^2-\binom{2}{1}1^2}{2!}P_2(n)+\frac{\binom{2}{2}2^3-\binom{2}{1}1^3}{3!}P_3(n)+\frac{\binom{2}{2}2^4-\binom{2}{1}1^4}{4!}P_4(n)+\ldots$$

$$D_3'(n)=D_3(n)-3D_2(n)+3D_1(n)-D_0(n)$$

$$D_3'(n)=-1+\frac{\binom{3}{3}3^1-\binom{3}{2}2^1+\binom{3}{1}1^1}{1!}P_1(n)+\frac{\binom{3}{3}3^2-\binom{3}{2}2^2+\binom{3}{1}2^2}{2!}P_2(n)+\frac{\binom{3}{3}3^3-\binom{3}{2}2^3+\binom{3}{1}1^3}{3!}P_3(n)+\frac{\binom{3}{3}3^4-\binom{3}{2}2^4+\binom{3}{1}1^4}{4!}P_4(n)+\ldots$$

$$D_4'(n)=D_4(n)-4D_3(n)+6D_2(n)-4D_1(n)+D_0(n)$$

$$d_a(n)=\frac{a^1}{1!}\sum_{j|n}^{1<j<n} p(j)+\frac{a^2}{2!}\sum_{j\cdot k=n}^{1<j,k<n} p(j)\cdot p(k)+\frac{a^3}{3!}\sum_{j\cdot k\cdot m=n}^{1<j,k,m<n} p(j)\cdot p(k)\cdot p(m)+\frac{a^4}{4!}\sum_{j\cdot k\cdot m\cdot o=n}^{1<j,k,m,o<n} p(j)\cdot p(k)\cdot p(m)\cdot p(o)+...$$

$$d_{a,k}(n)=\sum_{j|n}^{1<j<n} p(j)(\frac{a^k}{k!}+d_{a,k+1}(\frac{n}{j}))$$
$$p_k(n)=\sum_{j|n}^{1<j<n}\frac{a^k}{k}-p_{k+1}(\frac{n}{j})$$

$$P_k(n)=\sum_{j=2}^{n} d_a(j)(\frac{1}{k}-P_{k+1}(\frac{n}{j}))$$

$$\Pi(n)=\frac{P_1(n)}{a}$$

$$\Pi(n)=\sum_{j>1}^{j\le n} d_1(j)-\frac{1}{2}\sum_{j,k>1}^{j\cdot k\le n} d_1(j)\cdot d_1(k)+\frac{1}{3}\sum_{j,k,m>1}^{j\cdot k\cdot m\le n} d_1(j)\cdot d_1(k)\cdot d_1(m)-\frac{1}{4}\sum_{j,k,m,o>1}^{j\cdot k\cdot m\cdot o\le n} d_1(j)\cdot d_1(k)\cdot d_1(m)\cdot d_1(o)+...$$

$$\Pi(n)=\frac{1}{2}\sum_{j>1}^{j\le n} d_2(j)-\frac{1}{4}\sum_{j,k>1}^{j\cdot k\le n} d_2(j)\cdot d_2(k)+\frac{1}{6}\sum_{j,k,m>1}^{j\cdot k\cdot m\le n} d_2(j)\cdot d_2(k)\cdot d_2(m)-\frac{1}{8}\sum_{j,k,m,o>1}^{j\cdot k\cdot m\cdot o\le n} d_2(j)\cdot d_2(k)\cdot d_2(m)\cdot d_2(o)+...$$

$$\Pi(n)=\frac{1}{3}\sum_{j>1}^{j\le n} d_3(j)-\frac{1}{6}\sum_{j,k>1}^{j\cdot k\le n} d_3(j)\cdot d_3(k)+\frac{1}{9}\sum_{j,k,m>1}^{j\cdot k\cdot m\le n} d_3(j)\cdot d_3(k)\cdot d_3(m)-\frac{1}{12}\sum_{j,k,m,o>1}^{j\cdot k\cdot m\cdot o\le n} d_3(j)\cdot d_3(k)\cdot d_3(m)\cdot d_3(o)+...$$

$$d_1(n)=d_1'(n)$$
$$d_2(n)=d_2'(n)+2d_1'(n)$$
$$d_3(n)=d_3'(n)+3d_2'(n)+3d_1'(n)$$
$$d_4(n)=d_4'(n)+4d_3'(n)+6d_2'(n)+4d_1'(n)$$
$$d_5(n)=d_5'(n)+5d_4'(n)+10d_3'(n)+10d_2'(n)+5d_1'(n)$$

$$2\Pi(n)=\sum_{j>1}^{j\le n} d_2(j)-\frac{1}{2}\sum_{j,k>1}^{j\cdot k\le n} d_2(j)\cdot d_2(k)$$
$$+\frac{1}{3}\sum_{j,k,m>1}^{j\cdot k\cdot m\le n} d_2(j)\cdot d_2(k)\cdot d_2(m)-...$$

$$2\Pi(n)=\sum_{j>1}^{j\le n}(d_2'(j)+2d_1'(j))-\frac{1}{2}\sum_{j,k>1}^{j\cdot k\le n}(d_2'(j)+2d_1'(j))\cdot(d_2'(k)+2d_1'(k))$$
$$+\frac{1}{3}\sum_{j,k,m>1}^{j\cdot k\cdot m\le n}(d_2'(j)+2d_1'(j))\cdot(d_2'(k)+2d_1'(k))\cdot(d_2'(m)+2d_1'(m))-...$$

$$2\Pi(n)=\sum_{j>1}^{j\le n}(d_2'(j)+2)-\frac{1}{2}\sum_{j,k>1}^{j\cdot k\le n}(d_2'(j)+2)\cdot(d_2'(k)+2)$$
$$+\frac{1}{3}\sum_{j,k,m>1}^{j\cdot k\cdot m\le n}(d_2'(j)+2)\cdot(d_2'(k)+2)\cdot(d_2'(m)+2)-...$$

$$P_{k,a}(n)=\sum_{j=2}^{n}\frac{a^k}{k}-P_{k+1,a}(\frac{n}{j})$$

$$\Pi(n)=\frac{P_{1,1}(n)}{1}$$

$$n-1=\frac{P_{1,0}(n)}{0}$$

$$F_k(n)=\sum_{j=2}^{n}a(j)(b(k)+F_{k+1}(\frac{n}{j}))$$

$$a(j)=1, b(k)=\frac{(-1)^{k+1}}{k}\rightarrow\Pi(n)$$

$$a(j)=\frac{\Lambda(j)}{\log j}, b(k)=\frac{1}{k!}\rightarrow n-1$$

$$a(j)=j, b(k)=\frac{(-1)^{k+1}}{k}\rightarrow\text{Sum of Primes}$$

$$d_1(n)=d_1{}'(n)$$
$$d_2(n)=d_2{}'(n)+2\,d_1{}'(n)$$
$$d_3(n)=d_3{}'(n)+3\,d_2{}'(n)+3\,d_1{}'(n)$$
$$d_4(n)=d_4{}'(n)+4\,d_3{}'(n)+6\,d_2{}'(n)+4\,d_1{}'(n)$$
$$d_5(n)=d_5{}'(n)+5\,d_4{}'(n)+10\,d_3{}'(n)+10\,d_2{}'(n)+5\,d_1{}'(n)$$

$$d_3{}'(n)=\binom{3}{3}d_3(n)-\binom{3}{2}d_2(n)+\binom{3}{1}d_1(n)-\binom{3}{0}d_0(n)+\binom{3}{-1}d_{-1}(n)-\binom{3}{-2}d_{-2}(n)+\ldots$$
$$d_k{}'(n)=\sum_{j=0}(-1)^j\binom{k}{k-j}d_{k-j}(n)$$

$$d_k(n) = \prod \binom{p_j + k - 1}{p_j}$$

$$d_k(n) = \prod \frac{(p_j + k - 1)!}{p_j!\, k - 1!}$$

$$d_1(n) = \prod \frac{p_j!}{p_j!}$$

$$d_1(n) = \prod 1$$

$$d_k(n) = \prod \frac{(p_j + k - 1)!}{p_j!\, k - 1!}$$

$$\zeta(s)=\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\frac{1}{6^s}+\ldots$$

$$\zeta(s)=\frac{d_1(1)}{1^s}+\frac{d_1(2)}{2^s}+\frac{d_1(3)}{3^s}+\frac{d_1(4)}{4^s}+\frac{d_1(5)}{5^s}+\frac{d_1(6)}{6^s}+\ldots$$

$$\zeta(s)^2=(\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\frac{1}{6^s}+\ldots)(\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\frac{1}{6^s}+\ldots)$$

$$\zeta(s)^2=\frac{1}{1^s}+\frac{2}{2^s}+\frac{2}{3^s}+\frac{3}{4^s}+\frac{2}{5^s}+\frac{4}{6^s}+\ldots$$

$$\zeta(s)^2=\frac{d_2(1)}{1^s}+\frac{d_2(2)}{2^s}+\frac{d_2(3)}{3^s}+\frac{d_2(4)}{4^s}+\frac{d_2(5)}{5^s}+\frac{d_2(6)}{6^s}+\ldots$$

$$\zeta(s)^k=\frac{d_k(1)}{1^s}+\frac{d_k(2)}{2^s}+\frac{d_k(3)}{3^s}+\frac{d_k(4)}{4^s}+\frac{d_k(5)}{5^s}+\frac{d_k(6)}{6^s}+\ldots$$

$$(\zeta(s)-1)=\frac{d_1(2)}{2^s}+\frac{d_1(3)}{3^s}+\frac{d_1(4)}{4^s}+\frac{d_1(5)}{5^s}+\frac{d_1(6)}{6^s}+\ldots$$

$$(\zeta(s)-1)^2=\frac{d_2'(2)}{2^s}+\frac{d_2'(3)}{3^s}+\frac{d_2'(4)}{4^s}+\frac{d_2'(5)}{5^s}+\frac{d_2'(6)}{6^s}+\ldots$$

$$(\zeta(s)-1)^k=\frac{d_k'(2)}{2^s}+\frac{d_k'(3)}{3^s}+\frac{d_k'(4)}{4^s}+\frac{d_k'(5)}{5^s}+\frac{d_k'(6)}{6^s}+\ldots$$

$$(2^0+2^1+2^2+2^3+\ldots)(3^0+3^1+3^2+3^3+\ldots)(5^0+5^1+5^2+5^3+\ldots)(7^0+7^1+7^2+7^3+\ldots)(\ldots)=1+2+3+4+5+6+7+8+\ldots$$

$$(2^{0s}+2^{1s}+2^{2s}+2^{3s}+\ldots)(3^{0s}+3^{1s}+3^{2s}+3^{3s}+\ldots)(5^{0s}+5^{1s}+5^{2s}+5^{3s}+\ldots)(7^{0s}+7^{1s}+7^{2s}+7^{3s}+\ldots)(\ldots)=1^s+2^s+3^s+4^s+5^s+6^s+7^s+8^s+\ldots$$

$$1+\frac{1}{n^s}+\frac{1}{n^{2s}}+\frac{1}{n^{3s}}+\frac{1}{n^{4s}}+\ldots=\frac{1}{(1-\frac{1}{n^s})}$$

$$(\frac{1}{1-\frac{1}{2^s}})(\frac{1}{1-\frac{1}{3^s}})(\frac{1}{1-\frac{1}{5^s}})(\frac{1}{1-\frac{1}{7^s}})(\ldots)=\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\frac{1}{6^s}+\frac{1}{7^s}+\frac{1}{8^s}+\ldots$$

$$\log((\frac{1}{1-\frac{1}{2^s}})(\frac{1}{1-\frac{1}{3^s}})(\frac{1}{1-\frac{1}{5^s}})(\frac{1}{1-\frac{1}{7^s}})(\ldots))=\log(\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\frac{1}{6^s}+\frac{1}{7^s}+\frac{1}{8^s}+\ldots)$$

$$\log(\frac{1}{1-\frac{1}{2^s}})+\log(\frac{1}{1-\frac{1}{3^s}})+\log(\frac{1}{1-\frac{1}{5^s}})+\log(\frac{1}{1-\frac{1}{7^s}})+\log(\ldots)=\log(\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\frac{1}{6^s}+\frac{1}{7^s}+\frac{1}{8^s}+\ldots)$$

$$-\log(1-\frac{1}{2^s})-\log(1-\frac{1}{3^s})-\log(1-\frac{1}{5^s})-\log(1-\frac{1}{7^s})-\log(\ldots)=\log(\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\frac{1}{6^s}+\frac{1}{7^s}+\frac{1}{8^s}+\ldots)$$

$$-\log(1-n)=n+\frac{1}{2}n^2+\frac{1}{3}n^3+\frac{1}{4}n^4+\ldots$$

$$\frac{1}{2^s}+\frac{1}{2}\frac{1}{2^{2s}}+\frac{1}{3}\frac{1}{2^{3s}}+\ldots+\frac{1}{3^s}+\frac{1}{2}\frac{1}{3^{2s}}+\frac{1}{3}\frac{1}{3^{3s}}+\ldots+\frac{1}{5^s}+\frac{1}{2}\frac{1}{5^{2s}}+\frac{1}{3}\frac{1}{5^{3s}}+\ldots=\log(\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\frac{1}{6^s}+\frac{1}{7^s}+\frac{1}{8^s}+\ldots)$$

$$\Pi_1(n)=\sum_{j=2}^{n} 1-\Pi_2\left(\frac{n}{j}\right)$$

$$\Pi_2(n)=\sum_{j=2}^{n} \frac{1}{2}-\Pi_3\left(\frac{n}{j}\right)$$

$$\Pi_3(n)=\sum_{j=2}^{n} \frac{1}{3}-\Pi_4\left(\frac{n}{j}\right)$$

$$\Pi_4(n)=\sum_{j=2}^{n} \frac{1}{4}-\Pi_5\left(\frac{n}{j}\right)$$

$$\Pi_5(n)=\sum_{j=2}^{n} \frac{1}{5}-\Pi_6\left(\frac{n}{j}\right)$$

$$\sum_{j=2}^{n} \frac{\Lambda(j)}{\log j}+\frac{1}{2}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor} \frac{\Lambda(j)}{\log j}\cdot\frac{\Lambda(k)}{\log k}+\frac{1}{6}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\sum_{m=2}^{\lfloor\frac{n}{j\cdot k}\rfloor} \frac{\Lambda(j)}{\log j}\cdot\frac{\Lambda(k)}{\log k}\cdot\frac{\Lambda(m)}{\log m}+\ldots=n-1$$

$$\sum_{j=2}^{n} \frac{\Lambda(j)}{\log j}\left(1+\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor} \frac{\Lambda(k)}{\log k}\left(\frac{1}{2}+\sum_{m=2}^{\lfloor\frac{n}{j\cdot k}\rfloor} \frac{\Lambda(m)}{\log m}\left(\frac{1}{6}+\sum_{o=2}^{\lfloor\frac{n}{j\cdot k\cdot m}\rfloor} \frac{\Lambda(o)}{\log o}\left(\frac{1}{24}+\ldots\right)\right)\right)\right)=n-1$$

$$d_z(n)=\prod_{p^\alpha|n}\frac{z\cdot(z+1)\cdot(z+2)\cdot\ldots(z+\alpha-1)}{\alpha!}$$

$$d_1(n)=\prod_{p^\alpha|n}\frac{1\cdot(1+1)\cdot(1+2)\cdot\ldots(1+\alpha-1)}{\alpha!}=\prod_{p^\alpha|n}\frac{\alpha!}{\alpha!}=\prod_{p^\alpha|n}1=1$$

$$d_2(n)=\prod_{p^\alpha|n}\frac{2\cdot(2+1)\cdot(2+2)\cdot\ldots(2+\alpha-1)}{\alpha!}=\prod_{p^\alpha|n}\frac{(\alpha+1)!}{\alpha!}=\prod_{p^\alpha|n}\alpha+1$$

$$d_2(n)=\prod_{p^\alpha|n}\frac{3\cdot(3+1)\cdot(3+2)\cdot\ldots(3+\alpha-1)}{\alpha!}=\prod_{p^\alpha|n}\frac{(\alpha+2)!}{2\,\alpha!}=\prod_{p^\alpha|n}\frac{(\alpha+1)(\alpha+2)}{2}$$

$$d_0(n)=\prod_{p^\alpha|n}\frac{0\cdot(0+1)\cdot(0+2)\cdot\ldots(0+\alpha-1)}{\alpha!}=\prod_{p^\alpha|n}\frac{0}{\alpha!}=0, \mathit{unless}\, n=1, \mathit{then}\ 1$$

$$\lim_{k\to 0}\frac{d_k(n)}{k}=\frac{1}{k}\prod_{p^\alpha|n}\frac{k\cdot(k+1)\cdot(k+2)\cdot\ldots(k+\alpha-1)}{\alpha!}$$

$$\mathit{if}\ n=p^\alpha$$

$$\begin{aligned}
&\lim_{k\to 0}\frac{d_k(n)}{k}\\
&=\lim_{k\to 0}\frac{1}{k}\cdot\frac{k\cdot(k+1)\cdot(k+2)\cdot\ldots(k+\alpha-1)}{\alpha!}\\
&=\lim_{k\to 0}\frac{k}{k}\cdot\frac{(k+1)\cdot(k+2)\cdot\ldots(k+\alpha-1)}{\alpha!}\\
&=\lim_{k\to 0}\frac{k}{k}\cdot\frac{(0+1)\cdot(0+2)\cdot\ldots(0+\alpha-1)}{\alpha!}\\
&=\lim_{k\to 0}\frac{k}{k}\cdot\frac{(\alpha-1)!}{\alpha!}\\
&=\frac{1}{\alpha}
\end{aligned}$$

$$\mathit{if}\ n=p_1^{\alpha_1}\cdot p_2^{\alpha_2}$$

$$\begin{aligned}
&\lim_{k\to 0}\frac{d_k(n)}{k}\\
&=\lim_{k\to 0}\frac{1}{k}\cdot\frac{k\cdot(k+1)\cdot(k+2)\cdot\ldots(k+\alpha_1-1)}{\alpha_1!}\cdot\frac{k\cdot(k+1)\cdot(k+2)\cdot\ldots(k+\alpha_2-1)}{\alpha_2!}\\
&=\lim_{k\to 0}\frac{k^2}{k}\cdot\frac{(k+1)\cdot(k+2)\cdot\ldots(k+\alpha_1-1)}{\alpha_1!}\cdot\frac{(k+1)\cdot(k+2)\cdot\ldots(k+\alpha_2-1)}{\alpha_2!}\\
&=\lim_{k\to 0}\frac{k^2}{k}\cdot\frac{(\alpha_1-1)!}{\alpha_1!}\cdot\frac{(\alpha_2-1)!}{\alpha_2!}\\
&=\lim_{k\to 0}\frac{k}{\alpha_1\cdot\alpha_2}\\
&=0
\end{aligned}$$

$$\mathit{if}\ n=p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdot\ldots p_j^{\alpha_j}$$

$$\begin{aligned}
&d_0(n)\\
&=\prod_{p^\alpha|n}\frac{0\cdot(0+1)\cdot(0+2)\cdot\ldots(0+\alpha-1)}{\alpha!}\\
&=\lim_{k\to 0}\frac{k^{j-1}}{\alpha_1\cdot\alpha_2\cdot\ldots\alpha_j}
\end{aligned}$$

$$\lim_{k\to 0}\frac{D_k(n)-1}{k}=\Pi(n)$$

$$\lim_{k\to 0}\frac{n^k-1}{k}=\log n$$

$$d_{k,a}(n)=\sum_{j|n}\frac{k\,\Lambda(j)}{a\log j}(1+d_{k,a+1}(\frac{n}{j}))$$

$$d_k(n)=\lim_{z\to 0} d_{k,1}(n) \qquad d_{k,a}(n)=\frac{k}{z\,a}\sum_{j|n} d_z(j)(1+d_{k,a+1}(\frac{n}{j}))$$

$$d_{k,a}(n)=\sum_{p^\beta|n}\frac{k}{a\beta}(1+d_{k,a+1}(\frac{n}{p^\beta}))$$

$$d_k(n)=k\sum_{j|n}\frac{\Lambda(j)}{\log j}+\frac{k^2}{2}\sum_{j|n}\sum_{k|\frac{n}{j}}\frac{\Lambda(j)}{\log j}\cdot\frac{\Lambda(k)}{\log k}+\frac{k^3}{6}\sum_{j|n}\sum_{k|\frac{n}{j}}\sum_{m|\frac{n}{j\,k}}\frac{\Lambda(j)}{\log j}\cdot\frac{\Lambda(k)}{\log k}\cdot\frac{\Lambda(m)}{\log m}+\frac{k^4}{24}\ldots$$

$$d_a(n)=a\sum_{j|n}\lim_{z\to 0}\frac{d_z(j)}{z}+\frac{a^2}{2}\sum_{j|n}\sum_{k|\frac{n}{j}}\lim_{z\to 0}\frac{d_z(j)\,d_z(k)}{z^2}+\frac{a^3}{6}\sum_{j|n}\sum_{k|\frac{n}{j}}\sum_{m|\frac{n}{j\,k}}\lim_{z\to 0}\frac{d_z(j)\,d_z(k)\,d_z(m)}{z^3}+\frac{a^4}{24}\ldots$$

$$d_a(n)=\lim_{z\to 0} a\sum_{j|n}\frac{d_z(j)}{z}(1+\frac{a}{2}\sum_{k|\frac{n}{j}}\frac{d_z(k)}{z}(1+\frac{a}{3}\sum_{m|\frac{n}{j\,k}}\frac{d_z(m)}{z}(1+\frac{a}{4}\ldots)))$$

$$D_{k,a}(n)=\sum_{j=2}^{\lfloor n\rfloor}\frac{\Lambda(j)}{\log j}(\frac{k^a}{a!}+D_{k,a+1}(\frac{n}{j}))$$

$$D_{k,a}(n)=\sum_{j=2}^{\lfloor n\rfloor}\frac{k\,\Lambda(j)}{a\log j}(1+D_{k,a+1}(\frac{n}{j}))$$

$$D_{k,a}(n)=\sum_{p^\beta\le n}\frac{k}{a\beta}(1+D_{k,a+1}(\frac{n}{p^\beta}))$$

$$d_k(200)=d_k(a^3\cdot b^2)=\binom{3+k-1}{3}\cdot\binom{2+k-1}{2}=$$
$$\left(\frac{(3+k-1)!}{3!(3+k-1-3)!}\right)\cdot\left(\frac{(2+k-1)!}{2!(2+k-1-2)!}\right)=\left(\frac{(k+2)!}{3!(k-1)!}\right)\cdot\left(\frac{(k+1)!}{2!(k-1)!}\right)\left(\frac{k(k+1)(k+2)}{6}\right)\cdot\left(\frac{k(k+1)}{2}\right)$$
$$d_k(200)=\frac{k^5+4k^4+5k^3+2k^2}{12}$$

$$d_k(n)=\prod_{p^\alpha|n}\binom{\alpha+k-1}{\alpha}$$

$$d_1'(n)=d_1(n)=1$$
$$d_2'(n)=d_2(n)-2d_1(n)$$
$$d_3'(n)=d_3(n)-3d_2(n)+3d_1(n)$$
$$d_4'(n)=d_4(n)-4d_3(n)+6d_2(n)-4d_1(n)$$
$$d_5'(n)=d_5(n)-5d_4(n)+10d_3(n)-10d_2(n)+5d_1(n)$$

$$n=p^\alpha \quad -> \quad d_k(n)=\binom{\alpha+k-1}{\alpha}$$

$$d_1(n)=1$$
$$d_2(n)=\binom{\alpha+2-1}{\alpha}=\binom{\alpha+1}{\alpha}=\frac{(\alpha+1)!}{\alpha!(\alpha+1-\alpha)!}=\frac{(\alpha+1)!}{\alpha!}=\alpha+1$$
$$d_3(n)=\binom{\alpha+3-1}{\alpha}=\binom{\alpha+2}{\alpha}=\frac{(\alpha+2)!}{\alpha!(\alpha+2-\alpha)!}=\frac{(\alpha+2)!}{2\alpha!}=\frac{(\alpha+1)(\alpha+2)}{2}$$
$$d_4(n)=\binom{\alpha+4-1}{\alpha}=\binom{\alpha+3}{\alpha}=\frac{(\alpha+3)!}{\alpha!(\alpha+3-\alpha)!}=\frac{(\alpha+3)!}{6\alpha!}=\frac{(\alpha+1)(\alpha+2)(\alpha+3)}{6}$$
$$d_k(n)=\frac{(\alpha+1)(\alpha+2)...(\alpha+k-1)}{(k-1)!}$$

$$d_2'(n)=d_2(n)-2d_1(n)=\alpha+1-2=\alpha-1$$
$$d_3'(n)=d_3(n)-3d_2(n)+3d_1(n)=\frac{(\alpha+1)(\alpha+2)}{2}-3(\alpha+1)+3=\frac{(\alpha+1)(\alpha+2)}{2}-\frac{6(\alpha+1)}{2}+\frac{6}{2}=\frac{(\alpha+1)(\alpha+2)-6(\alpha+1)+6}{2}$$
$$=\frac{\alpha^2}{2}-\frac{3\alpha}{2}+1$$
$$d_4'(n)=d_4(n)-4d_3(n)+6d_2(n)-4d_1(n)=\frac{(\alpha+1)(\alpha+2)(\alpha+3)}{6}-2(\alpha+1)(\alpha+2)+6(\alpha+1)-4$$
$$=\frac{\alpha^3}{6}-\alpha^2+\frac{11\alpha}{6}-1$$
$$d_1'(n)=1$$
$$d_2'(n)=\alpha-1$$
$$d_3'(n)=\frac{\alpha^2}{2}-\frac{3\alpha}{2}+1$$
$$d_3'(n)=\frac{(\alpha-1)(\alpha-2)}{2}$$
$$d_4'(n)=\frac{\alpha^3}{6}-\alpha^2+\frac{11\alpha}{6}-1$$
$$d_4'(n)=\frac{(\alpha-1)(\alpha-2)(\alpha-3)}{6}$$
$$d_5'(n)=\frac{\alpha^4}{24}-\frac{5\alpha^3}{12}+\frac{35\alpha^2}{24}-\frac{25\alpha}{12}+1$$
$$d_5'(n)=\frac{(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{24}$$
$$d_6'(n)=\frac{\alpha^5}{120}-\frac{\alpha^4}{8}+\frac{17\alpha^3}{24}-\frac{15\alpha^2}{8}+\frac{137\alpha}{60}-1$$

$$d_6{}'(n)=\frac{(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(\alpha-5)}{120}$$
$$d_k{}'(n)=\frac{(\alpha-1)!}{(k-1)!(\alpha-k)!}$$
$$d_k{}'(n)=\frac{\Gamma(\alpha)}{\Gamma(k)\Gamma(\alpha-k+1)!}$$

vs

$$d_k(n)=\frac{(\alpha-1+k)!}{(k-1)!\alpha!}$$

$$n=p^{\alpha}\cdot q^{\beta} \quad -> \quad d_k(n)=\binom{\alpha+k-1}{\alpha}\cdot\binom{\beta+k-1}{\beta}$$

$$d_1'(n)=d_1(n)=1$$
$$d_2'(n)=d_2(n)-2\,d_1(n)$$
$$d_3'(n)=d_3(n)-3\,d_2(n)+3\,d_1(n)$$
$$d_4'(n)=d_4(n)-4\,d_3(n)+6\,d_2(n)-4\,d_1(n)$$
$$d_5'(n)=d_5(n)-5\,d_4(n)+10\,d_3(n)-10\,d_2(n)+5\,d_1(n)$$

$$n=p^{\alpha} \quad -> \quad d_k(n)=\binom{\alpha+k-1}{\alpha}$$

$$d_1(n)=1$$
$$d_2(n)=\binom{\alpha+2-1}{\alpha}\binom{\beta+2-1}{\beta}=\binom{\alpha+1}{\alpha}\cdot\binom{\beta+1}{\beta}=\frac{(\alpha+1)!}{\alpha!(\alpha+1-\alpha)!}\frac{(\beta+1)!}{\beta!(\beta+1-\beta)!}=\frac{(\alpha+1)!}{\alpha!}\frac{(\beta+1)!}{\beta!}=(\alpha+1)(\beta+1)$$
$$d_3(n)=\frac{(\alpha+1)(\alpha+2)}{2}\cdot\frac{(\beta+1)(\beta+2)}{2}$$
$$d_4(n)=\frac{(\alpha+1)(\alpha+2)(\alpha+3)}{6}\cdot\frac{(\beta+1)(\beta+2)(\beta+3)}{6}$$
$$d_k(n)=\frac{(\alpha+1)(\alpha+2)\ldots(\alpha+k-1)}{(k-1)!}\cdot\frac{(\beta+1)(\beta+2)\ldots(\beta+k-1)}{(k-1)!}$$

$$d_1'(n)=d_1(n)=1$$
$$d_2'(n)=(\alpha+1)(\beta+1)-2$$
$$d_3'(n)=\frac{(\alpha+1)(\alpha+2)}{2}\cdot\frac{(\beta+1)(\beta+2)}{2}-3(\alpha+1)(\beta+1)+3$$
$$d_4'(n)=\binom{4}{4}\frac{(\alpha+1)(\alpha+2)(\alpha+3)(\beta+1)(\beta+2)(\beta+3)}{(3!)^2}-\binom{4}{3}\frac{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}{(2!)^2}+\binom{4}{2}(\alpha+1)(\beta+1)-\binom{4}{1}$$
$$d_k'(n)=\binom{k}{k}\frac{(\alpha)_{k-1}(\beta)_{k-1}}{((k-1)!)^2}-\binom{k}{k-1}\frac{(\alpha)_{k-2}(\beta)_{k-2}}{((k-2)!)^2}+\binom{k}{k-2}\frac{(\alpha)_{k-3}(\beta)_{k-3}}{((k-3)!)^2}-\binom{k}{k-3}\frac{(\alpha)_{k-4}(\beta)_{k-4}}{((k-4)!)^2}+\ldots$$

$$d_3'(p^{\alpha}\cdot q^{\beta})=\sum_{1\le a\le\alpha,\,1\le b\le\alpha,\,1<a+b<k-1} d_2'(p^{a}\cdot q^{b})$$

$$n=p^{\alpha}\cdot q^{\beta}r^{\gamma} \quad -> \quad d_k(n)=\binom{\alpha+k-1}{\alpha}\cdot\binom{\beta+k-1}{\beta}\cdot\binom{\gamma+k-1}{\gamma}$$

$$d_k(n)=\sum_{j=1}^{k}(-1)^{k-j}\frac{k!}{j!(k-j)!}\frac{(\alpha+1)(\alpha+2)\dots(\alpha+j-1)}{(j-1)!}\cdot\frac{(\beta+1)(\beta+2)\dots(\beta+j-1)}{(j-1)!}$$

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\dots$$

$$-\log(1-x)=x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+\dots$$

$$\log(x)=(1-\frac{1}{x})+\frac{(1-\frac{1}{x})^2}{2}+\frac{(1-\frac{1}{x})^3}{3}+\frac{(1-\frac{1}{x})^4}{4}+\frac{(1-\frac{1}{x})^5}{5}+\dots$$

$$\log(x)=(1-x)+\frac{(1-x)^2}{2}+\frac{(1-x)^3}{3}+\frac{(1-x)^4}{4}+\frac{(1-x)^5}{5}+\dots(-1<x<1)$$

$$\log(x)=(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\frac{(x-1)^5}{5}-\dots(-1<x<1)$$

$$\log x=-\log\frac{1}{x}$$

$$\frac{\Lambda(n)}{\log n}=d_1{}'(n)-\frac{d_2{}'(n)}{2}+\frac{d_3{}'(n)}{3}-\frac{d_4{}'(n)}{4}+\frac{d_5{}'(n)}{5}-\dots$$

$$\log x=\lim n\rightarrow 0\,\frac{x^n-1}{n}$$

$$\frac{\Lambda(n)}{\log n}=\lim k\rightarrow 0\,\frac{d_k(n)}{k}$$

$$\log a\cdot b=\log a+\log b$$

$$\log a\cdot b=\log a+\log b$$

$$e^{\log a\cdot b}=a\,b$$

$$e^{\log a+\log b}=a\,b$$

$$e^x=\frac{1}{0!}+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots$$

$$\log(1+x)=\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\dots$$

$$\frac{\Lambda(n)}{\log n} n^{a}$$

$\rho(n) n^{a}$

$\frac{\Lambda(n)}{\log n} n^{a}$ with * -1 for even powers

$$\frac{\Lambda(n)}{\log n}\left(n^{a}-1\right)$$

$$d_5'(n)=\sum_{j|n} d_4'(j)\, d_1'(\frac{n}{j})$$

$$d_5'(n)=\sum_{j|n} d_3'(j)\, d_2'(\frac{n}{j})$$

$$d_5'(n)=\sum_{j|n} d_5'(j)\, d_0'(\frac{n}{j})$$

$$d_5'(n)=\sum_{j|n} d_k'(j)\, d_{5-k}'(\frac{n}{j})$$

$$d_5'(n)=\sum_{j|n,\, j\le n^{\frac{1}{5}}} \sum_{k|\frac{n}{j},\, j\le k\le \frac{n}{j}^{\frac{1}{4}}} \sum_{m|\frac{n}{jk},\, k\le m\le \frac{n}{jk}^{\frac{1}{3}}} \ldots$$

$$d_5'(n)=d_5(n)-\binom{5}{4}d_4(n)+\binom{5}{3}d_3(n)-\binom{5}{2}d_2(n)+\binom{5}{1}d_1(n)-\binom{5}{0}d_0(n)$$

$$\left(\quad d_k(n)=\frac{(\alpha+1)(\alpha+2)\ldots(\alpha+k-1)}{(k-1)!} \quad \right)$$

$$d_5(n)=\sum_{j|n} p(j)+\frac{5}{2}\sum_{j|n}\sum_{k|\frac{n}{j}} p(j)\,p(k)+\frac{25}{6}\sum_{j|n}\sum_{k|\frac{n}{j}}\sum_{m|\frac{n}{jk}} p(j)\,p(k)\,p(m)+\frac{125}{24.}..$$

$$d_5(n)=\sum_{j|n} d_4(\frac{n}{j})$$

$$d_1(n)=1$$

$$d_2(n)=\sum_{j|n} 1$$

$$d_3(n)=\sum_{j|n}\sum_{k|\frac{n}{j}} 1$$

$$d_4(n)=\sum_{j|n}\sum_{k|\frac{n}{j}}\sum_{m|\frac{n}{jk}} 1$$

$$d_5(n)=\sum_{j|n}\sum_{k|\frac{n}{j}}\sum_{m|\frac{n}{jk}}\sum_{s|\frac{n}{jkm}} 1$$

$$d_5'(n)=5+\sum_{j|n} -10+\sum_{k|\frac{n}{j}} +10+\sum_{m|\frac{n}{jk}} -5+\sum_{s|\frac{n}{jkm}} 1$$