More Informal Work with Linnik's Identity and Various Explicit Prime Counting Formulas

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[7] Sums of Counts of Divisors >= 2 [D']

Just to clear up some notation, through out this document, I am going to refer frequently to the following function:

$$D_k'(n)=\sum_{j=2} \tau_k'(n) \tag{7.1}$$

where

$$\tau_k'(n) = |\{n_1, \dots, n_k \geqslant 2; \;\; n_1 \dots n_k = n\}|.$$

According to Linnik's Identity, this means that

$$\pi^*(n) = \sum_{k=1} \frac{-1^{k+1}}{k} D_k'(n) \tag{7.2}$$

where π* is the prime power counting function. Thus, the problem of prime counting turns into the problem of calculating these various D'(n,k) values.

My goal is to count these values as quickly as possible. Thus, I am including here as many properties of these sums as I have been able to figure out, in hope that they might lead to productive ideas.

[8] Properties of D'

The fundamental relationships for D' are

$$D_1'(n)=n-1$$
$$D_0'(n)=1$$

and

$$D_k'(n)=\sum_{j=2} D_{k-1}'(\lfloor\frac{n}{j}\rfloor) \tag{8.1}$$

(8.1) is a special case of the following relationship

$$D_k'(n)=\sum_{j=2}(D_a'(j)-D_a'(j-1))\,D_{k-a}'(\lfloor\frac{n}{j}\rfloor)$$

and

$$D_k'(n)=\sum_{j=2} D_a'(j)(D_{k-a}'(\lfloor\frac{n}{j}\rfloor)-D_{k-a}'(\lfloor\frac{n}{j+1}\rfloor)) \tag{8.2}$$

where a is an integer >= 1 and <= k-1.

D_k'(n) is an interesting function because it is so close to the arithmetic sum of counts of divisors functions, but it appears to have different identities and properties because it begins summation at 2, rather than 1. So, for example, if, in an effort to perform a mobius inversion, we take the following

$$D_k'(n)=D_{k-1}'(\frac{n}{2})+D_{k-1}'(\frac{n}{3})+D_{k-1}'(\frac{n}{4})+D_{k-1}'(\frac{n}{5})+...$$

$$D_k'(\frac{n}{2})=D_{k-1}'(\frac{n}{4})+D_{k-1}'(\frac{n}{6})+D_{k-1}'(\frac{n}{8})+D_{k-1}'(\frac{n}{10})+...$$

and then subtract the second from the first, we have

$$D_k'(n)-D_k'(\frac{n}{2})=D_{k-1}'(\frac{n}{2})+D_{k-1}'(\frac{n}{3})+D_{k-1}'(\frac{n}{5})+D_{k-1}'(\frac{n}{7})+...$$

and if we continue by subtracting D_k'(n/3), then subtracting D_k'(n/5), then adding D_k'(n/6), and so on, eventually we are left with terms which, if D_k'(n) is on one side of the equation and all other terms are collected to the other side, look like this:

$$D_k'(n)=\sum_{j=2}-\mu(j)(D_k'(\lfloor\frac{n}{j}\rfloor)+D_{k-1}'(\lfloor\frac{n}{j}\rfloor)) \tag{8.3}$$

Connected to (8.3) is the following extremely interesting identity:

$$\sum_{j=2}-\mu(j)D_k'(\lfloor\frac{n}{j}\rfloor)=\sum_{m=1}-1^{m+1}D_{k+m}'(n) \tag{8.4}$$

Essentially, and perhaps somewhat surprisingly, this function relates one count of divisor series to all the series that come above it. If (8.4) is added together for two consecutive values of k, (8.3) is recovered. If k is 0 for (8.4), the relationship between D_k' and the Merten's function is made explicit.

Several other identities follow without comment:

$$D_k'(n) - D_{k-1}'(n) = \sum_{j=2} -\mu(j)\left(D_k'\left(\lfloor \frac{n}{j} \rfloor\right) - D_{k-2}'\left(\lfloor \frac{n}{j} \rfloor\right)\right) \tag{8.5}$$

$$D_k'(n) + D_{k-1}'(n) = \sum_{j=2} -\mu(j)\left(D_k'\left(\lfloor \frac{n}{j} \rfloor\right) + 2\, D_{k-1}'\left(\lfloor \frac{n}{j} \rfloor\right) + D_{k-2}'\left(\lfloor \frac{n}{j} \rfloor\right)\right) \tag{8.6}$$

$$\sum_{k=1} D_k'(n) = \sum_{j=2} -\mu(j)\left(1 + 2\sum_{k=1} D_k'\left(\frac{n}{j}\right)\right) \tag{8.7}$$

$$D_k'(n) = D_{k+1}'(2\text{n}) - \sum_{j=3} D_{k+1}'\left(\lfloor \frac{4\text{n}}{j} \rfloor\right) + \sum_{j,l=3} D_{k+1}'\left(\lfloor \frac{8\text{n}}{jl} \rfloor\right) - \sum_{j,l,m=3} D_{k+1}'\left(\lfloor \frac{16\text{n}}{jlm} \rfloor\right) + \ldots \tag{8.8}$$

[9] Calculating D' Values, Method 1

[This will be a description of what my source code from section [6] in the first document does]

[10] Calculating D' Values, Method 2

[This will a description of another approach that I'm experimenting with – I'm on the fence about how promising it is, but it might be productive]

[11] Sums of Counts of Moebius Divisors >= 2 [M']

There are another set of sums with extremely similar properties to D_k' that seem to be worthy of examination.

If we start with the following functions, which are close analogs to the strict divisor function in Linnik's identity,

$$m_1'(n) = -\mu(n) \tag{11.1}$$

$$m_2'(n) = \sum_{ab=n,a,b>1} -\mu(a) \cdot -\mu(b) \tag{11.2}$$

$$m_3'(n) = \sum_{abc=n,a,b,c>1} -\mu(a) \cdot -\mu(b) \cdot -\mu(c)$$

$$(11.3)$$

...and so on, then we have the following useful sums to work with:

$$M_1'(n)=\sum_{j=2} m_1'(j) \tag{11.4}$$

$$M_2'(n)=\sum_{j=2} m_2'(j) \tag{11.5}$$

$$M_3'(n)=\sum_{j=2} m_3'(j) \tag{11.6}$$

...and so on.

[12] Moebius Analog to Linnik's Identity, and Another Recursive Prime Counting Function

The following two identities apply to m_k':

$$\sum_{k=1} m_k'(n)=1 \tag{12.1}$$

$$\sum_{k=1} \frac{1}{k}(m_k'(n))=\frac{1}{k} \mathit{if}\ n=p^k, a\ prime\ power, 0\ otherwise \tag{12.2}$$

This appears to be directly analogous to Linnik's identity. Thus,

$$\sum_{k=1} M_k'(n)=n-1 \tag{12.3}$$

and

$$\sum_{k=1} \frac{1}{k} M_k'(n)=\pi*(n) \tag{12.4}$$

Given that M_1'(n) is equal to 1-Merten(n), and given that the other M_k' values are as erratic, (12.3) in particular seems pretty surprising.

As was the case with D_k', these identities can be used to build recursive formulas for π*(n) . Those formulas are

$$v(n,k)=\frac{1-Mertens(n)}{k}+\sum_{j=2} -\mu(j)\, v(\frac{n}{j},k+1)$$

$$\pi*(n)=v(n,1)$$

(12.5)

or, which is the same thing,

$$v(n,k)=\sum_{j=2} -\mu(j)\left(\frac{1}{k}+v\left(\frac{n}{j},k+1\right)\right)$$

$$\pi*(n)=v(n,1)$$

(12.6)

Similarly,

$$v(n)=\sum_{j=2} -\mu(j)\left(1+v\left(\frac{n}{j},k+1\right)\right)$$

$$v(n)=n-1$$

(12.7)

[13] Properties of M'
[14] Connections between M' and D'
[15] Properties of Recursive Prime Counting Function *π*(n,k)*

[16] Recursive Exact Formulas for Chebyshev's ψ Function

After a bit more experimentation, if

$$v(n)=\lfloor n\rfloor\log(n)-\sum_{j=2} v\left(\frac{n}{j}\right)$$

(16.1)

then

$$v(n)-\log n=\psi(n)$$

(16.2)

where $\psi(n)$ is Chebyshev's second function, the sum of the log of all prime powers less than n.

[Note that my function ts(n) in section [4] of my first paper is essentially this identity, but with an undesirable floor function inside of the log adding error to the result]

Because, in general, recursive functions of the form

$$v(n)=f(n)-\sum_{j=2} v\left(\frac{n}{j}\right)$$

(16.3)

can be replaced by sums of the form

$$\sum_{j=1} \mu(j) f\left(\frac{n}{j}\right)$$

(16.4)

This means we can rewrite (16.1) and (16.2) as

$$-\log n+\sum_{j=1} \mu(j)\left\lfloor\frac{n}{j}\right\rfloor\log\frac{n}{j}=\psi(n)$$

(16.5)

A very similar identity to (16.5) is given by

$$\sum_{j=2} -\mu(j)\lfloor\frac{n}{j}\rfloor \log j = \psi(n) \tag{16.6}$$

Note that equations (16.1), (16.5), and (16.6) are exact formulas – there are no error terms.