I written elsewhere, pretty extensively, about the two techniques I've implemented for counting primes.

One of those techniques, first implement by me in 2005, relied on a combination of, essentially, Linnik's identity summed, the hyperbola method, and huge wheels to count primes in more or less O(n^4/5) time and O(epsilon) space. I'd never found a good way to utilize memory to speed that up, aside from the wheel.

The second technique, which dates from 2011, was heavily inspired by an existing algorithm for computing the Mertens function. It's also combinatorial in nature, and relies on Linnik's identity summed, some combinatorial techniques, and an interleaved sieving process. It runs in something like O(n^2/3 log n) time and O(n^1/3 log n) space, making it competitive with the fastest combinatorial prime counting algorithms, on paper.

I'd never found a way to combine these two approaches, though.

In mid 2015, I

$$\boxed{[(1+\zeta(0,y))^z]_n = \sum_{k=0} \binom{z}{k} \cdot [1+\zeta(0,y+1)^{z-k}]_{n/y^k}}$$

...

$$\begin{aligned}
&[(1+\zeta(0,2))^z]_{100} = \\
&\quad [(1+\zeta(0,3))^z]_{100} + \\
&\quad z[(1+\zeta(0,3))^{z-1}]_{50} + \\
&\quad \frac{z(z-1)}{2}[(1+\zeta(0,3))^{z-2}]_{25} + \\
&\quad \frac{z(z-1)(z-2)}{6}[(1+\zeta(0,3))^{z-3}]_{12} + \\
&\quad \frac{z(z-1)(z-2)(z-3)}{24}[(1+\zeta(0,3))^{z-4}]_{6} + \\
&\quad \frac{z(z-1)(z-2)(z-3)(z-4)}{120}[(1+\zeta(0,3))^{z-5}]_{3} + \\
&\quad \frac{z(z-1)(z-2)(z-3)(z-4)(z-5)}{720}[(1+\zeta(0,3))^{z-6}]_{1}
\end{aligned}$$

$$\begin{aligned}
&[(1+\zeta(0,2))^z]_{100} = \\
&\quad [(1+\zeta(0,3))^z]_{100} + \\
&\quad z[(1+\zeta(0,3))^{z-1}]_{50} + \\
&\quad \frac{z(z-1)}{2}([(1+\zeta(0,4))^{z-2}]_{25} + (z-2)[(1+\zeta(0,4))^{z-3}]_{8} + \frac{(z-2)(z-3)}{2}[(1+\zeta(0,4))^{z-4}]_{2}) + \\
&\quad \frac{z(z-1)(z-2)}{6}([(1+\zeta(0,4))^{z-3}]_{12} + (z-3)[(1+\zeta(0,4))^{z-4}]_{4} + \frac{(z-3)(z-4)}{2}[(1+\zeta(0,4))^{z-5}]_{1}) + \\
&\quad \frac{z(z-1)(z-2)(z-3)}{24}([(1+\zeta(0,4))^{z-4}]_{4} + (z-4)[(1+\zeta(0,4))^{z-5}]_{2}) + \\
&\quad \frac{z(z-1)(z-2)(z-3)(z-4)}{120}([(1+\zeta(0,4))^{z-5}]_{3} + (z-5)[(1+\zeta(0,4))^{z-6}]_{1}) + \\
&\quad \frac{z(z-1)(z-2)(z-3)(z-4)(z-5)}{720}[(1+\zeta(0,3))^{z-6}]_{1}
\end{aligned}$$

$$
\begin{aligned}
&[(1+\zeta(0,2))^{z}]_{100}= \\
&\quad [(1+\zeta(0,3))^{z}]_{100}+ \\
&\quad z[(1+\zeta(0,3))^{z-1}]_{50}+ \\
&\quad \frac{z(z-1)}{2}([(1+\zeta(0,4))^{z-2}]_{25}+(z-2)[(1+\zeta(0,4))^{z-3}]_{8}+\frac{(z-2)(z-3)}{2}(1))+ \\
&\quad \frac{z(z-1)(z-2)}{6}([(1+\zeta(0,4))^{z-3}]_{12}+(z-3)(1+(z-4))+\frac{(z-3)(z-4)}{2}(1))+ \\
&\quad \frac{z(z-1)(z-2)(z-3)}{24}((1+(z-4)1)+(z-4)(1))+ \\
&\quad \frac{z(z-1)(z-2)(z-3)(z-4)}{120}(1+(z-5)(1))+ \\
&\quad \frac{z(z-1)(z-2)(z-3)(z-4)(z-5)}{720}(1)
\end{aligned}
$$

$$
\begin{aligned}
&[(1+\zeta(0,2))^{z}]_{100} = \\
&\quad [(1+\zeta(0,3))^{z}]_{100} + \\
&\quad [(1+\zeta(0,3))^{-1}]_{50} + \\
&\quad -\frac{1}{2}[(1+\zeta(0,3))^{-2}]_{25} + \\
&\quad \frac{1}{3}[(1+\zeta(0,3))^{-3}]_{12} + \\
&\quad -\frac{1}{4}[(1+\zeta(0,3))^{-4}]_{6} + \\
&\quad -\frac{1}{6}[(1+\zeta(0,3))^{-5}]_{3} + \\
&\quad \frac{1}{7}[(1+\zeta(0,3))^{-6}]_{1}
\end{aligned}
$$

$$
\begin{aligned}
&[(1+\zeta(0,2))^{z}]_{100} = \\
&\quad [(1+\zeta(0,3))^{z}]_{100} + \\
&\quad [(1+\zeta(0,3))^{-1}]_{50} + \\
&\quad -\frac{1}{2}[(1+\zeta(0,3))^{-2}]_{25} + \\
&\quad \frac{1}{3}[(1+\zeta(0,3))^{-3}]_{12} + \\
&\quad -\frac{1}{4}([(1+\zeta(0,4))^{-4}]_{6} + (-4)(1)) + \\
&\quad -\frac{1}{6}(1 + (-5)(1)) \\
&\quad \frac{1}{7}(1)
\end{aligned}
$$