```
fs[n_, k_, s_] := Expand[(-1)^(k) ((1/(1-s))^(k) -
    n^(1-s)/(1-s) Sum[1/j! (-Log[n])^j (1/(1-s))^(k-j-1), {j, 0, k-1}])]
Fa4[n_, a_, s_] := (-1)^a Integrate[t^(a-1) E^(-(1-s) t), {t, 0, -Log[n]}]/Gamma[a]
```

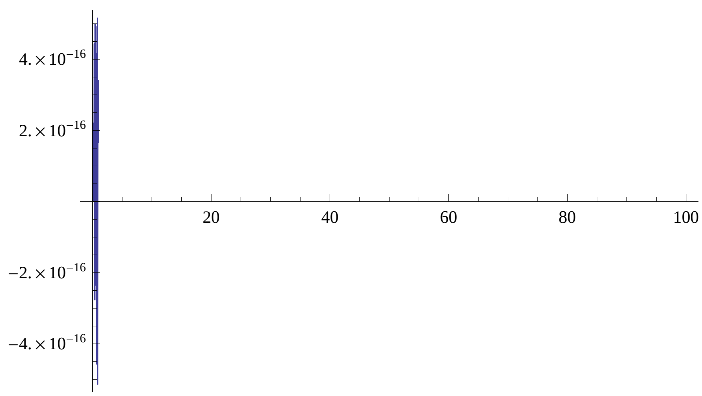
$$\text{Fa5}[n\_, a\_, s\_] := (-1)^a \frac{(\text{Integrate}[\text{Log}[t^{-1}]^{(a-1)}, \{t, 1/(n^{(s-1)}), 1\}])\ (1-s)^{-a}}{\text{Integrate}[\text{Log}[t^{-1}]^{(a-1)}, \{t, 0, 1\}]}$$

$$\text{Fa3}[n\_, a\_, s\_] := (-1)^a \frac{(\text{Gamma}[a, 0, -(1-s)\ \text{Log}[n]])\ (1-s)^{-a}}{\text{Gamma}[a]}$$

```
N[{Fa4[100, ac = 2, ca = -1], Fa5[100, ac, ca], Fa3[100, ac, ca], fs[100, ac, ca]}]
```

$\{20\,526.1,\ 20\,526.1,\ 20\,526.1 - 2.51369 \times 10^{-12}\, i,\ 20\,526.1\}$

```
Plot[fs[n, cc = 2, dc = 1/2] - Fa3[n, cc, dc], {n, 0, 100}]
```




```
Limit[(Fa3[n, a, s] - 1)/a, {a → 0}]
```

$\{i\pi - \text{Gamma}[0, (-1+s)\ \text{Log}[n]] - \text{Log}[1-s]\}$

```
cc[n_, s_] := -iπ - Gamma[0, (-1 + s) Log[n]] - Log[1 - s]
N[cc[100, -1]]
```

1245.44 + 0. i

```
cd[n_, s_] := -iπ - Gamma[0, Log[n^(-1 + s)]] - Log[1 - s]
N[cd[100, -1]]
```

1245.44 + 0. i

```
N[ExpIntegralEi[Log[100]]]
```

30.1261

```
N[ExpIntegralEi[Log[100^2]]]
```

1246.14

```
N[LogIntegral[100]]
```

30.1261

```
N[LogIntegral[10 000]] - Log[2]
```

1245.44

```
N[cc[100, 2]]
```

−0.00182974 − 6.28319 i

```
N[LogIntegral[100^-1]]
```

```
-0.00182974
```

```
N[cc[100, 3]]
```

```
-0.693157 - 6.28319 i
```

```
N[LogIntegral[100^-2]] - Log[2]
```

```
-0.693157
```

```
cc[100, -1]
```

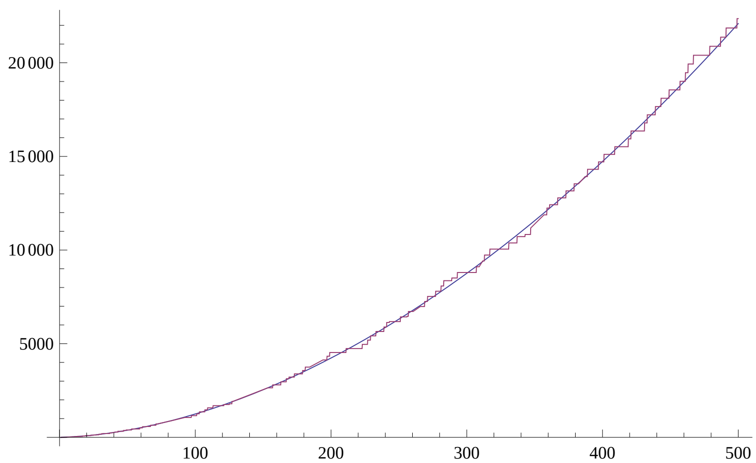
```
-i π - Gamma[0, -2 Log[100]] - Log[2]
```

```
jj[n_, s_] := Sum[ N[MangoldtLambda[j] / Log[j]] j^-s, {j, 2, n}]
```

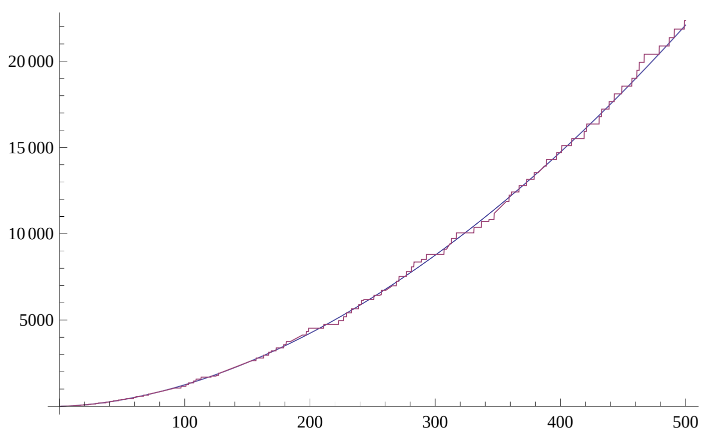
```
jj[100, -1]
```

```
1156.48
```

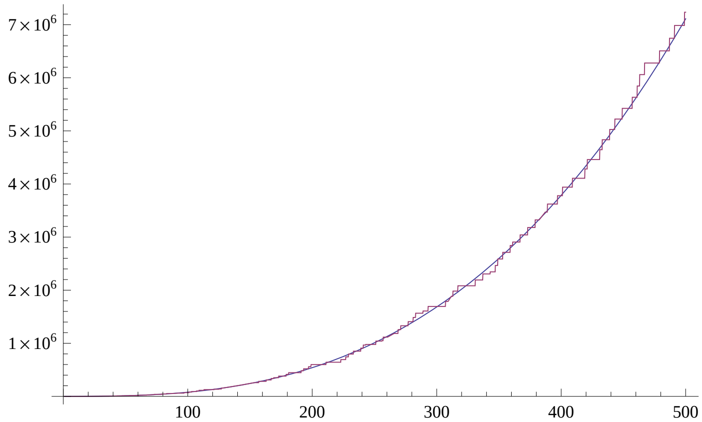
```
Plot[{ Re[ -i π - Gamma[0, -2 Log[n]] - Log[2]], jj[Floor[n], -1]}, {n, 1, 500}]
```



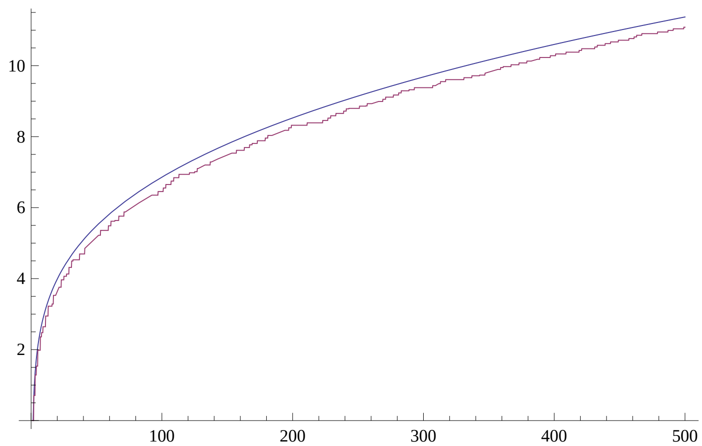

```
Plot[{ Re[ cc[n, s2 = -1]], jj[Floor[n], s2]}, {n, 1, 500}]
```



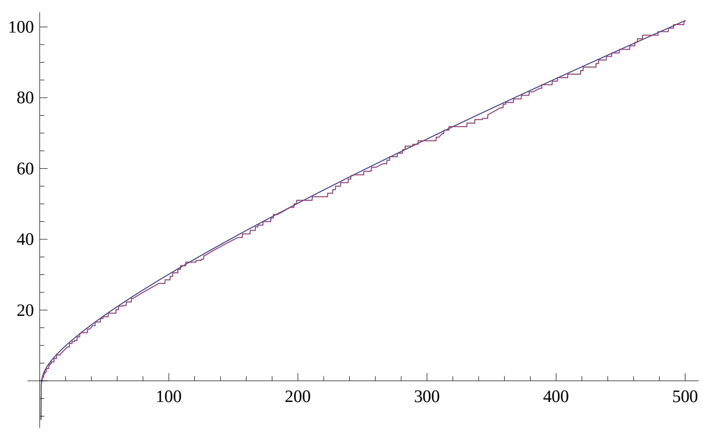
```
Plot[{ Re[ cc[n, s2 = -2]], jj[Floor[n], s2]}, {n, 1, 500}]
```



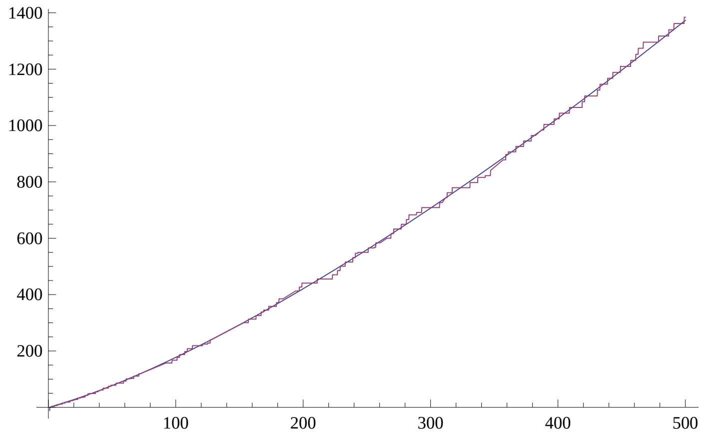

```
Plot[{ Re[ cc[n, s2 = 1 / 2]], jj[Floor[n], s2]}, {n, 1, 500}]
```



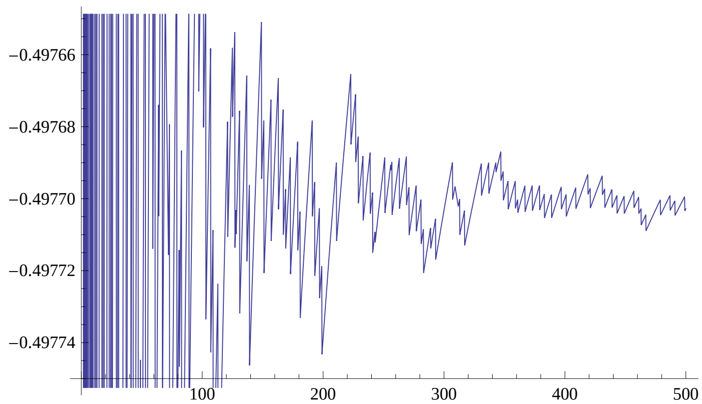

```
Plot[{ Re[ cc[n, s2 = 0]], jj[Floor[n], s2]}, {n, 1, 500}]
```



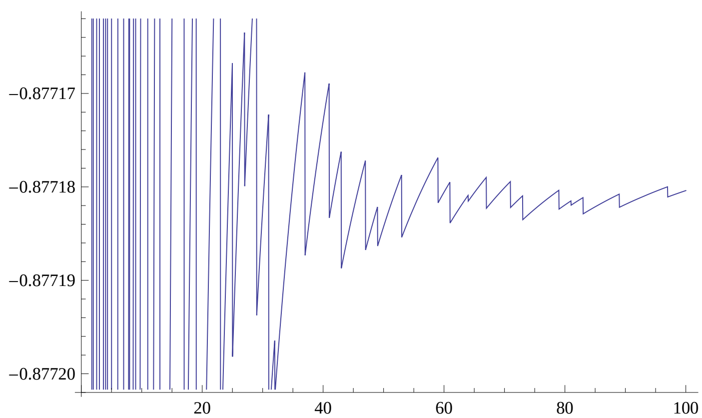
```
Plot[{ Re[ cc[n, s2 = -1 / 2]], jj[Floor[n], s2]}, {n, 1, 500}]
```



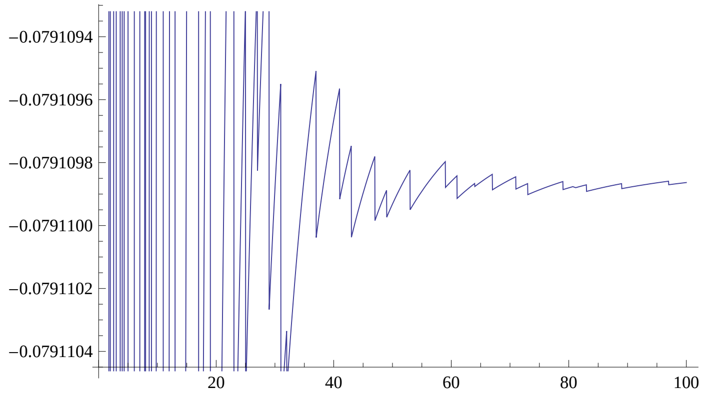

```
Plot[{ Re[ cc[n, s2 = 2]] - jj[Floor[n], s2]}, {n, 1, 500}]
```




```
Plot[{ Re[ cc[n, s2 = 3]] - jj[Floor[n], s2]}, {n, 1, 100}]
```



```
Plot[{ Re[ ce[n, s2 = 4]] - jj[Floor[n], s2]}, {n, 1, 100}]
```




```
cc[n, 4]
```

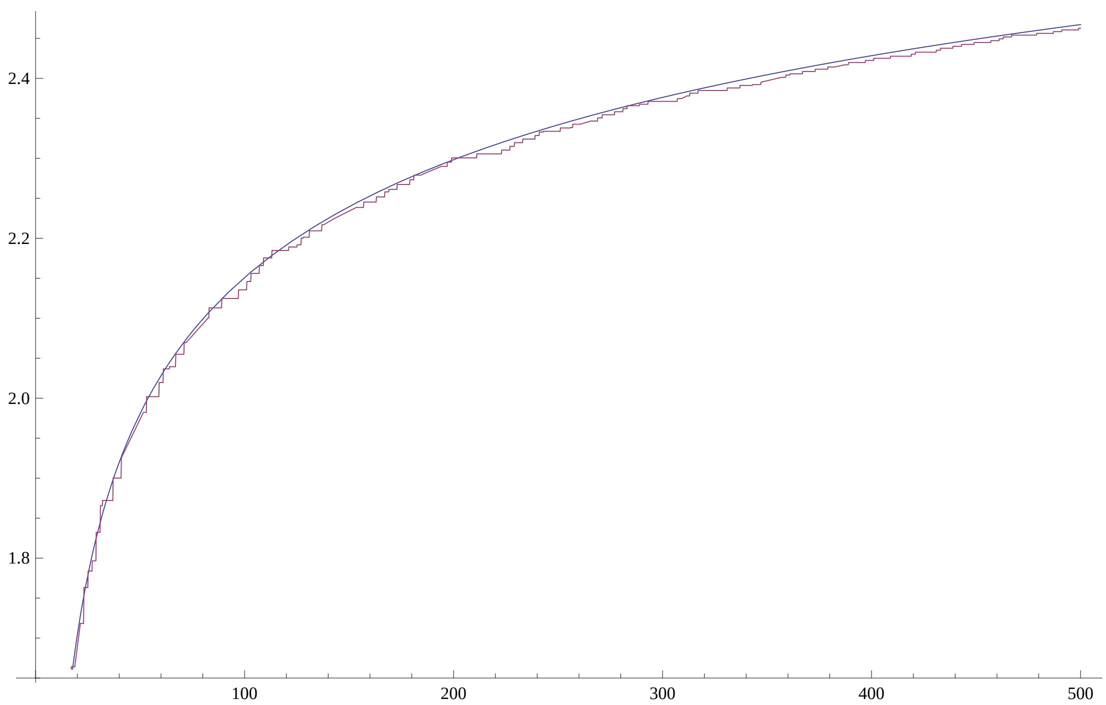
−2 ⅈ π − Gamma[0, 3 Log[n]] − Log[3]

```
N[Log[3]]
```

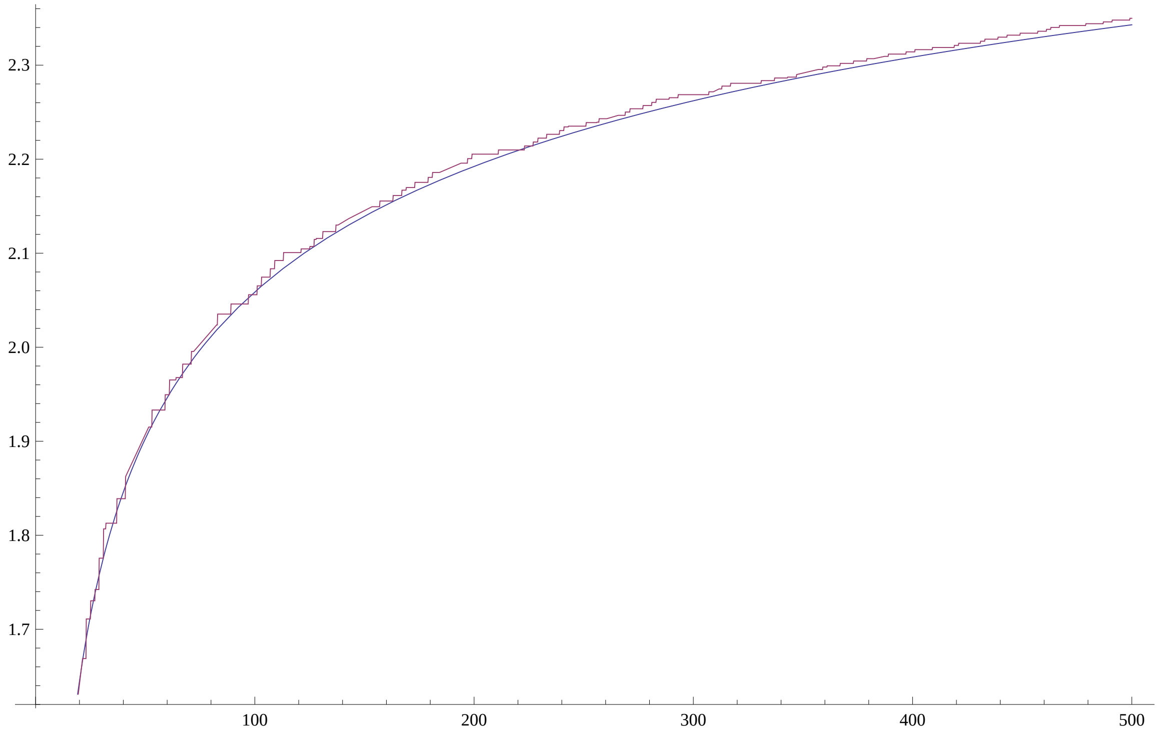
1.09861

```
ce[n_, s_] := -Gamma[0, (s - 1) Log[n]]
```

```
Plot[{ Re[ cc[n, s2 = .99]], jj[Floor[n], s2]}, {n, 1, 500}]
```



```
Plot[{ Re[ cc[n, s2 = 1.01]], jj[Floor[n], s2]}, {n, 1, 500}]
```




```
dif[n_, s_, t_] := ((cc[n + t, s]) - (cc[n - t, s])) / (2 t)
```

```
se[n_, a_, s_] := Sum[ ((a^(1 - s))^k - 1) / k, {k, 1, Log[a, n]}]
se[100, 1.00001, 0]
28.0218
N[LogIntegral[100]] - EulerGamma - Log[Log[100]]
28.0217
se[100, 1.00001, -1]
1243.34
N[LogIntegral[10 000]] - EulerGamma - Log[Log[10 000]]
1243.34
se[100, 1.000001, -2]
```

$78\,624.5 + 1.27973 \times 10^{-9}\, i$

```
N[LogIntegral[1 000 000]] - EulerGamma - Log[Log[1 000 000]]
78 624.3

4.75435
```

```
s = 1 / 2; {se[100, 1.00001, (1 - s)],
 N[LogIntegral[100^(1 - s)]] - EulerGamma - Log[Log[100^(1 - s)]]}
```

```
{4.75435, 4.75435}
```

```
s = 1; {se[100, 1.00001, s], N[LogIntegral[100^(1 - s)]] - EulerGamma - Log[Log[100^(1 - s)]]}
```

Infinity::indet : Indeterminate expression −EulerGamma + −∞ + ∞ encountered. ≫

```
{0., Indeterminate}
```

```
s = 0; {se[100, 1.00001, s], N[LogIntegral[100^(1 - s)]] - EulerGamma - Log[Log[100^(1 - s)]]}
```

```
{28.0218, 28.0217}
```

```
s = -1; {se[100, 1.00001, s], N[LogIntegral[100^(1 - s)]] - EulerGamma - Log[Log[100^(1 - s)]]}
```

```
{1243.34, 1243.34}
```

```
s = 2; {se[100, 1.00001, s], N[LogIntegral[100^(1 - s)]] - EulerGamma - Log[Log[100^(1 - s)]]}
```

```
{-2.10622, -2.10623 - 3.14159 i}
```

```
s = 3; {se[100, 1.00001, s], N[LogIntegral[100^(1 - s)]] - EulerGamma - Log[Log[100^(1 - s)]]}
```

```
{-2.79754, -2.79755 - 3.14159 i}
```

```
s = 2.5; {se[100, 1.00001, s], N[LogIntegral[100^(1 - s)]] - EulerGamma - Log[Log[100^(1 - s)]]}
```

```
{-2.50998, -2.50999 - 3.14159 i}
```

```
s = 1 / 2 + I; {se[100, 1.00001, s] + EulerGamma + Log[(1 - s) Log[100]],
 N[LogIntegral[100^(1 - s)]] - EulerGamma - Log[Log[100^(1 - s)]],
 N[ExpIntegralEi[(1 - s) Log[100]]]}
```

```
{-1.98461 - 2.81852 i, 1.54732 + 5.09858 i, -1.98461 - 2.81853 i}
```

```
s = 2; {se[100, 1.00001, s], N[LogIntegral[100^(1 - s)]],
 N[ExpIntegralEi[(1 - s) Log[100]]], N[ExpIntegralEi[Log[100^(1 - s)]]]}
```

```
{-2.10622, -0.00182974, -0.00182974, -0.00182974}
```

```
s = N[ZetaZero[4]]; {se[nn = 1200, 1.00001, s] + EulerGamma + Log[(1 - s) Log[nn]],
 N[ExpIntegralEi[(1 - s) Log[nn]]], -Gamma[0, -(1 - s) Log[nn]]}
```

```
{0.138743 - 3.22227 i, 0.138753 - 3.22241 i, 0.138753 - 0.0808157 i}
```

```
se2[a_, s_, k_] :=  ((a^(1 - s))^k - 1) / k
```

```
Sum[ N[se2[1.0001, ZetaZero[a], 1] + se2[1.0001, ZetaZero[-a], 1]], {a, 1, 5000}]
```

```
-543.018 + 0. i
```

```
Sum[ N[se2[1.0001, ZetaZero[a], 2] + se2[1.0001, ZetaZero[-a], 2]], {a, 1, 5000}]
```

```
-1038.19 + 0. i
```

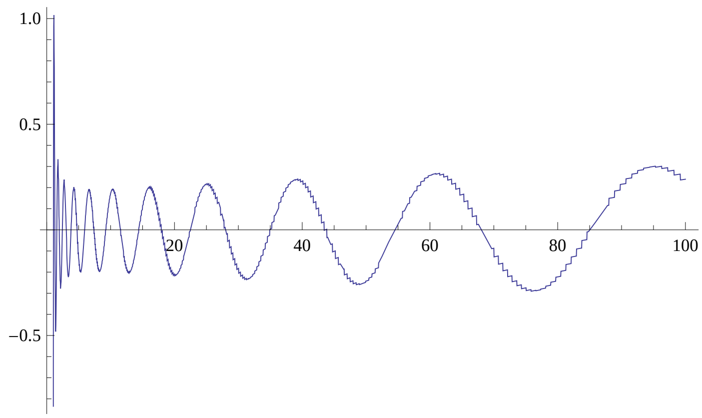
```
Sum[ N[se2[1.0001, ZetaZero[a], 3] + se2[1.0001, ZetaZero[-a], 3]], {a, 1, 5000}]
```

```
-1443.36 + 0. i
```

```
Sum[ N[se2[1.0001, ZetaZero[a], 4] + se2[1.0001, ZetaZero[-a], 4]], {a, 1, 5000}]
```

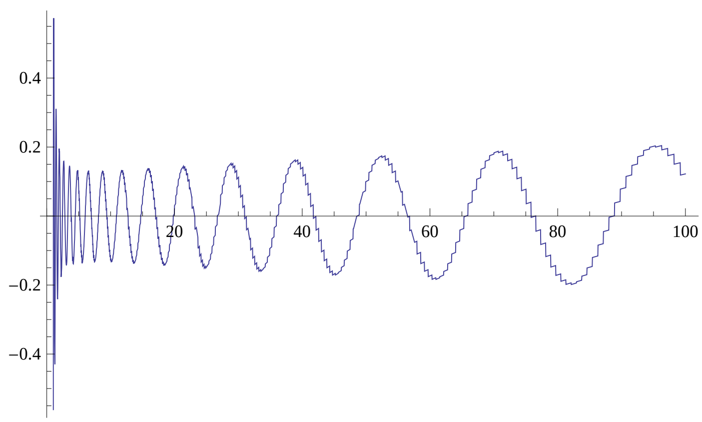
```
-1729.2 + 0. i
```

```
sse[n_, a_, s_] := se[n, a, s] + EulerGamma + Log[(1 - s) Log[n]]
Plot[Re[sse[n, aa = 1.01, N[ZetaZero[1]]] + sse[n, aa, N[ZetaZero[-1]]]], {n, 1, 100}]
```

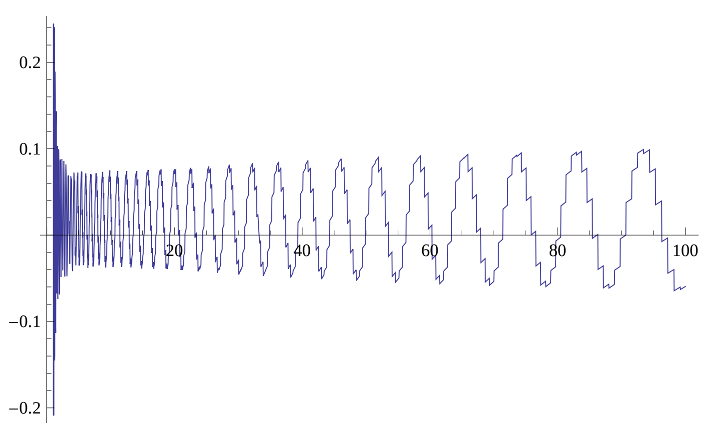



```
Re[sse[n, 2, N[ZetaZero[1]]] + sse[n, 2, N[ZetaZero[-1]]]] /. {n → 6}
Re[sse[6, 2, 0.5 - 14.1347 i] + sse[6, 2, 0.5 + 14.1347 i]]
```

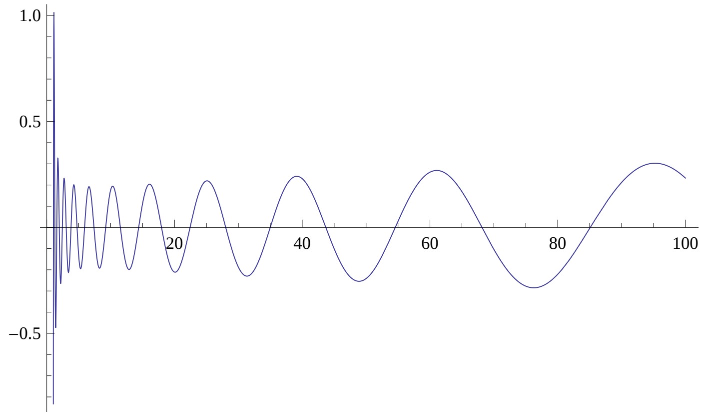
```
Plot[Re[sse[n, aa = 1.01, N[ZetaZero[2]]] + sse[n, aa, N[ZetaZero[-2]]]], {n, 1, 100}]
```



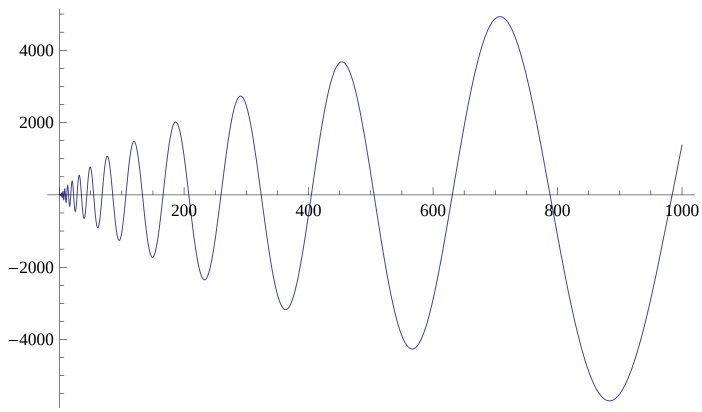

```
Plot[Re[sse[n, aa = 1.01, N[ZetaZero[11]]] + sse[n, aa, N[ZetaZero[-11]]]], {n, 1, 100}]
```



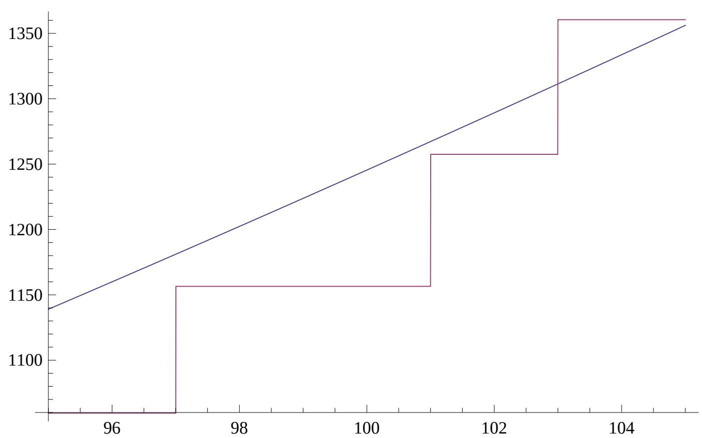
```
Plot[Re[-Gamma[0, -(1 - ZetaZero[tt = 1]) Log[n]] - Gamma[0, -(1 - ZetaZero[-tt]) Log[n]]],
 {n, 1, 100}]
```




```
Plot[
 Re[-Gamma[pa = 2, -(1 - ZetaZero[tt = 1]) Log[n]] - Gamma[pa, -(1 - ZetaZero[-tt]) Log[n]]],
 {n, 1, 1000}]
```




```
Plot[{ Re[ cc[n, s2 = -1]], jj[Floor[n], s2]}, {n, 95, 105}]
```



```
Integrate[ x^(s - 1) / (E^x - 1), {x, 0, Infinity}]
```

ConditionalExpression[Gamma[s] PolyLog[s, 1], Re[s] > 1]

```
ts[n_, s_] := Integrate[ t^(s - 1) / (E^t - 1), {t, 0, Infinity}] / Gamma[s]
ts4[n_, k_, s_] := Integrate[ t^(s - 1) / ((E^((1 - k) t) - 1)), {t, 0, Infinity}] / Gamma[s]
eta[n_, k_, s_] := Integrate[ t^(s - 1) / ((E^((1 - k) t) + 1)), {t, 0, Infinity}] / Gamma[s]
Fa4[n_, k_, s_] := (-1)^k Integrate[t^(k - 1) E^(-(1 - s) t), {t, 0, -Log[n]}] / Gamma[k]
```

$$\texttt{Fa3[n\_, k\_, s\_] := (-1)\^{}k}\ \frac{(\texttt{Gamma[k, 0, -(1-s) Log[n]]})\ (1-s)^{-k}}{\texttt{Gamma[k]}}$$

$$\texttt{Fa32[n\_, k\_, s\_] := (-1)\^{}k}\ \frac{(\texttt{Gamma[k] - Gamma[k, -(1-s) Log[n]]})\ (1-s)^{-k}}{\texttt{Gamma[k]}}$$

```
ts[100, 2]
```

$$\frac{\pi^2}{6}$$

```
ts4[100, -1, 2]
```

$$\frac{\pi^2}{24}$$

```
eta[100, -1, 2]
```

$$\frac{\pi^2}{48}$$

```
Gamma[1 - z] Gamma[z]
```

Gamma[1 - z] Gamma[z]

```
N[Fa3[100, 2, 2]]
```

0.943948

```
Fa32[100, 2, -1]
```

$$\frac{1}{4}\ (1 - \text{Gamma}[2, -2\ \text{Log}[100]])$$

```
Integrate[ 1, {j, 1, n}, {k, 1, n / j}, {m, 1, n / (j k)}]
```

$$\text{ConditionalExpression}\left[-1 + n + \frac{1}{2}\ n\ (-2 + \text{Log}[n])\ \text{Log}[n],\ \text{Re}[n] \geq 0\ ||\ n \notin \text{Reals}\right]$$

```
Integrate[ j^-s k^-s m^-s, {j, 1, n}, {k, 1, n / j}, {m, 1, n / (j k)}]
```

$$\texttt{Expand}\left[\texttt{ConditionalExpression}\left[\frac{n^{-s}\ (2\ n^{s} + n\ (-2 + (-1+s)\ \text{Log}[n]\ (-2 + \text{Log}[n] - s\ \text{Log}[n])))}{2\ (-1+s)^3},\ \text{Re}[n] \geq 0\ ||\ n \notin \text{Reals}\right]\right]$$

$$\text{ConditionalExpression}\left[\frac{1}{(-1+s)^3} - \frac{n^{1-s}}{(-1+s)^3} + \frac{n^{1-s}\ \text{Log}[n]}{(-1+s)^3} - \frac{n^{1-s}\ s\ \text{Log}[n]}{(-1+s)^3} - \frac{n^{1-s}\ \text{Log}[n]^2}{2\ (-1+s)^3} + \frac{n^{1-s}\ s\ \text{Log}[n]^2}{(-1+s)^3} - \frac{n^{1-s}\ s^2\ \text{Log}[n]^2}{2\ (-1+s)^3},\ \text{Re}[n] \geq 0\ ||\ n \notin \text{Reals}\right]$$

$$\text{fo}[\text{n\_}] := -1 + n + \frac{1}{2}\, n\, (-2 + \text{Log}[n])\, \text{Log}[n]$$

```
fo2[n_] := Fa3[n, 3, 0]
N[fo[100]]
```

698.863

```
N[fo2[100]]
```

$698.863 - 1.71417 \times 10^{-13}\, i$

$$\text{Fa32}[\text{n\_}, \text{a\_}, \text{s\_}] := (-1)^{\wedge}a\, \frac{(\text{Gamma}[a, -(1-s)\, \text{Log}[n]])\, (1-s)^{-a}}{\text{Gamma}[a]}$$

$$\text{fo3}[\text{n\_}] := \frac{1}{2}\, n\, (2\, (-1+n)\, n - \text{Log}[n]\, (2\, n + \text{Log}[n]))$$

```
fo4[n_] := Fa32[n, 3, 0]
N[fo3[100]]
```

942 888.

```
N[fo4[100]]
```

$-699.863 + 1.71417 \times 10^{-13}\, i$

```
Gamma[1, 0, -Log[n]]
```

1 - n

```
Gamma[1, -Log[n]]
```

n

```
Gamma[2, 0, -Log[n]]
```

Gamma[2, 0, -Log[n]]

```
Gamma[2, -Log[n]]
```

Gamma[2, -Log[n]]

```
Limit[(Gamma[a, 0, -Log[n]] / Gamma[a] - 1) / a, {a → 0}]
```

{-Gamma[0, -Log[n]]}

```
Limit[(Gamma[a, 0, -Log[n]] / Gamma[a] - 1) / a, {a → 4}]
```

$$\left\{-\frac{1}{24}\, \text{Gamma}[4, -\text{Log}[n]]\right\}$$