Generalizing Linnik's Identity: Divisor-Style Functions Where $f_1(1)=0$

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11/27/2011

Summary: This paper will generalize the core idea behind Linnik's identity $\frac{\Lambda(n)}{\log n}=d_1'(n)-\frac{1}{2}d_2'(n)+\frac{1}{3}d_3'(n)-\ldots$. *It will construct a more general theory about divisor-style functions based on arithmetic functions, and identify a much, much wider range of similar identities for a broad range of arithmetic functions. This approach will essentially reside partway between Dirichlet convolution and the mechanics of power series.*

1. Divisor-Style Functions and Power Series

1.1 Preliminary Connections

Let's suppose we have some function $f(n)$. We will now define the following combinatorial identities:

$$f_1(n)=\begin{cases} f(n): n\neq 1 \\ 0: n=1 \end{cases}$$
$$f_k(n)=\sum_{j|n} f_1(j) f_{k-1}\left(\frac{n}{j}\right)$$
$$f_0(n)=\begin{cases} 1: n=1 \\ 0: n\neq 1 \end{cases}$$
$$F_k(n)=\sum_{j=1}^{n} f_k(j)$$

(L1)

Given these identities, we will have the following subsequent identities

$$f_k(n)=\sum_{j|n} f_a(j) f_{k-a}\left(\frac{n}{j}\right)$$

(L2)

for $0\leq a\leq k$ and

$$F_k(n)=\sum_{j=1}^{n} f_a(j) F_{k-a}\left(\frac{n}{j}\right)$$

(L3)

for $0\leq a\leq k$. We can generalize *(L2)* as

$$(a\cdot b) f_{c+d}(n)=\sum_{j|n} (a f_c(j))\left(b f_d\left(\frac{n}{j}\right)\right)$$

(L4)

This last expression is surprisingly versatile and powerful. First, let's compare our functions to an elementary property of exponents:

$$(a\cdot b) x^{c+d}=(a x^c)\cdot(b x^d)$$

(L4) is perhaps a bit more complicated, of course, but the close similarities here should be clear.

We can take this a step further. If we multiply the following two polynomials together

$$\left(\sum_{j=0} a_j x^{c+j}\right)\cdot\left(\sum_{k=0} b_k x^{d+k}\right)$$

we have

$$\sum_{j=0}\sum_{k=0}^{j} a_k b_{k-j} x^{c+d+j}$$

We can do something very similar with our functions. For

$$\sum_{m|n}\left(\sum_{j=0} a_j f_{c+j}(m)\right)\cdot\left(\sum_{k=0} b_k f_{d+k}\left(\frac{n}{m}\right)\right)$$

we have

$$\sum_{j=0}\sum_{k=0}^{j} a_k b_{k-j} f_{c+d+j}(n)$$

The important point here is that, because of this deep structural similarity between power series and our series $f_k(n)$, there are patterns, properties, and ideas from power series that can be applied in close analogue here.

As one example, two series of coefficients s_k and t_k can be said to be multiplicative inverses if they satisfy

$$x^k=\left(\sum_{a=0} s_a x^{a+c}\right)\cdot\left(\sum_{b=0} t_b x^{b+k-c}\right)$$

and especially

$$1=(\sum_{a=0} s_a x^a)\cdot(\sum_{b=0} t_b x^b)$$

This relationship is likewise present with the same coefficients for our functions.

$$f_k(n)=\sum_{j|n}(\sum_{a=0} s_a f_{a+c}(j))\cdot(\sum_{b=0} t_b f_{b+k-c}(\frac{n}{j}))$$

(L5)

and

$$1=\sum_{j|n}(\sum_{a=0} s_a f_a(j))\cdot(\sum_{b=0} t_b f_b(\frac{n}{j}))$$

One important such coefficient pair is $s_k=\frac{1}{(k+1)!}$ and $t_k=\frac{B_k}{(k+1)!}$ where B_k are the Bernoulli numbers with $B_1=-\frac{1}{2}$, but there a multitude of such pairs.

1.2 Series Transformation

One very important role for power series and their coefficients is to provide a way of expressing one function in terms of another. For example, if we have a series of coefficients $a_k=\frac{-1^{k+1}}{k}$ then we have

$$\log(1+x)=\sum_{k=1} a_k x^k \, for -1< x\leq 1$$

This powerful idea also works for our functions $f_k(n)$ and $F_k(n)$, but it only works if we have convergence.

It is here that it is critical that $f_1(1)=0$. If $f_1(1)=0$, then $f_k(n)$ and $F_k(n)$ are only nonzero when $n>2^k$. This means that we can talk meaningfully about the values of sums of the form $\sum_{k=1} a_k F_k(n)$, as we only have to evaluate $\log_2 n$ terms. Essentially, $f_1(1)=0$ gives us a guaranteed form of convergence. It should just take a moments thought to recognize that this is generally not the case if $|f_1(1)|\geq 1$, which should be familiar from functions like the number of divisors function. (The case where $0<|f_1(1)|<1$ is potentially interesting but will not be discussed here).

Suppose we have some sequence of coefficients a_k. We can use these to define a new set of functions like so

$$g_1(n)=\sum_{k=1} a_k f_k(n)$$

$$g_k(n)=\sum_{j|n} g_1(j)\, g_{k-1}(\frac{n}{j})$$

$$g_0(n)=\begin{cases}1: n=1\\ 0: n\neq 1\end{cases}$$

$$G_k(n)=\sum_{j=1}^{n} g_k(j)$$

(L6)

We will immediately find that the following relationships are true.

$$G_1(n)=v_1(n) \text{ for } v_k(n)=\sum_{j=1}^{\lfloor n\rfloor} f_1(j)(a_k+v_{k+1}(\frac{n}{j}))$$

(L7)

and

$$G_1(n)=$$
$$\sum_{j=1}^{n}(a_1 f_0(j)+a_2 f_1(j)+a_3 f_2(j)+a_4 f_3(j)+\ldots)F_1(\frac{n}{j})$$

(L8)

If we define the following constants

$$\alpha(n,k)=\sum_{j=1}^{k}\alpha(1,j)\cdot\alpha(n-1,k-j+1), where\, \alpha(1,k)=a_k$$

(L9)

then we will further find that

$$G_k(n)=\sum_{j=0}\alpha(k,j+1)F_{k+j}(n)$$

(L10)

and

$$g_k(n)=\sum_{j=0}\alpha(k,j+1)f_{k+j}(n)$$

(L11)

and

$$\sum_{j=1}^{n} f_m(j)\,G_1(\frac{n}{j})=\sum_{k=1} a_k F_{k+m-1}(n)$$

(L12)

If c_k is the multiplicative inverse of a_k, then

$$F_1(n)=\sum_{j=1}^{n}\sum_{k=0} c_k f_k(j) G_1(\frac{n}{j})$$

(L13)

$$F_k(n)=\sum_{j=1}^{n}\sum_{m=0} c_m f_{k+m-1}(j)\,G_1(\frac{n}{j})$$

(L14)

$$F_1(n)=\sum_{j=1}^{n} g_1(j)(\sum_{k=0} c_k F_k(\frac{n}{j}))$$

(L15)

$$F_1(n)=\sum_{j=1}^{n}(G_1(\frac{n}{j})-G_1(\frac{n}{j+1}))(\sum_{k=0} c_k F_k{}'(j))$$

$$F_1(n)=v_0(n) \quad \text{for}$$
$$v_k(n)=\sum_{j=1}^{\lfloor n\rfloor} c_k g_1(j)+f_1(j)v_{k+1}(\frac{n}{j})$$

(L16)

(L17)

1.3 Chained Transformations

Multiple series transformations can also be chained together. Let's suppose we have

$$f_1(n)=\begin{cases} f(n): n\neq 1 \\ 0: n=1 \end{cases}$$
$$f_k(n)=\sum_{j|n} f_1(j) f_{k-1}(\frac{n}{j})$$
$$f_0(n)=\begin{cases} 1: n=1 \\ 0: n\neq 1 \end{cases}$$
$$F_k(n)=\sum_{j=1}^{n} f_k(j)$$

and, for some some set of coefficients a_k ,

$$g_1(n)=\sum_{k=1} a_k f_k(n)$$
$$g_k(n)=\sum_{j|n} g_1(j) g_{k-1}(\frac{n}{j})$$
$$g_0(n)=\begin{cases} 1: n=1 \\ 0: n\neq 1 \end{cases}$$
$$G_k(n)=\sum_{j=1}^{n} g_k(j)$$

and, for another set of coefficients b_k ,

$$h_1(n)=\sum_{k=1} b_k g_k(n)$$
$$h_k(n)=\sum_{j|n} h_1(j) h_{k-1}(\frac{n}{j})$$
$$h_0(n)=\begin{cases} 1: n=1 \\ 0: n\neq 1 \end{cases}$$
$$H_k(n)=\sum_{j=1}^{n} h_k(j)$$

We can then find explicit coefficients for expressing h_k in terms of f_k . So, if we have

$$\alpha(n,k)=\sum_{j=1}^{k} \alpha(1,j)\cdot\alpha(n-1,k-j+1), where\, \alpha(1,k)=a_k$$

(L18)

then we can calculate the following coefficients c_k ,

$$c_k=\sum_{j=1}^{k} \alpha(j,k-j+1)\cdot b_j$$

(L19)

such that

$$h_1(n)=\sum_{k=1} c_k f_k(n)$$

1.4 Series Reversion

When dealing with power series, there is a process for inverting the relationship between functions. So, if $a_k=\frac{-1^{k+1}}{k}$ and we have

$$\log(1+x)=\sum_{k=1} a_k x^k \; for -1< x\leq 1$$

then we have the coefficients $b_k=\frac{1}{k!}$ such that

$$x=\sum_{k=0} b_k \log(x)^k$$

We have a similar process, which uses identical coefficients, for our functions $f_k(n)$ and $g_k(n)$.

Our goal is to produce an equation of $F_1(n)$ exclusively in terms of $G_k(n)$. The essence of series reversion works like this. We begin with our basic series transformation equation *(L10),* with the coefficients $\alpha(n,k)$ from *(L9)*

$$G_1(n)=\alpha(1,1)F_1(n)+\alpha(1,2)F_2(n)+\alpha(1,3)F_3(n)+...$$

We also know from *(L10)* that

$$G_2(n)=\alpha(2,1)F_2(n)+\alpha(2,2)F_3(n)+\alpha(2,3)F_4(n)+...$$

Multiply both sides of this second equation by $\alpha(1,2)/\alpha(2,1)$, and subtract the result from our first equation. We will then have

$$G_1(n)-\frac{\alpha(1,2)}{\alpha(2,1)}G_2(n)= \alpha(1,1)F_1(n)+ (\alpha(1,3)-\frac{\alpha(1,2)\alpha(2,2)}{\alpha(2,1)})F_3(n)+...$$

The important point here is that we have removed $F_2(n)$ from the right hand side of our equation by doing this. Once again, we know from *(L10)* that

$$G_3(n)=\alpha(3,1)F_3(n)+\alpha(3,2)F_4(n)+\alpha(3,3)F_5(n)+...$$

So, multiply both sides of this equation by $(\alpha(1,3)-\frac{\alpha(1,2)\alpha(2,2)}{\alpha(2,1)})/\alpha(3,1)$ and subtract the result from our main equation. We will have added a $G_3(n)$ term to the left hand side of the equation and eliminated an $F_3(n)$ term from the right hand side of the equation. And then we return to *(L10)* once again.

If we repeatedly apply *(L10)* to ever higher values of *k* for $G_k(n)$, we will eventually reach our goal of isolating $F_1(n)$ and expressing it exclusively in terms of $G_k(n)$, which will immediately give us $f_1(n)$ in terms of $g_1(n)$. This is enough for us to express all $F_k(n)$ and $f_k(n)$ in terms of $G_k(n)$ and $g_k(n)$.

Calculating those coefficients works as follows.

If a_k were the coefficients we used to express $g_k(n)$ in terms of $f_k(n)$, then, if $a_1 \neq 0$, a_k has a corresponding series of coefficients b_k that can be used to invert these relationships. If we have $\alpha(n,k)$ from above, then

$$b_1=\frac{1}{\alpha(1,1)}$$

and, for $k>1$,

$$b_k=-\frac{1}{\alpha(k,1)}\sum_{j=1}^{k-1} b_j \alpha(j,k-j+1)$$

(L21)

We then have

$$\sum_{k=1} b_k g_k(n)=f_1(n)$$

(L22)

and

$$\sum_{k=1} b_k G_k(n)=F_1(n)$$

(L23)

and

$$F_1(n)=\sum_{j=1}^{n}(b_1 g_0(j)+b_2 g_1(j)+b_3 g_2(j)+b_4 g_3(j)+\ldots)G_1(\frac{n}{j})$$

(L24)

We will also find the following relationships hold:

$$v_1(n)=F_1(n) \text{ for } v_k(n)=\sum_{j=1}^{\lfloor n\rfloor} g_1(j)(b_k+v_{k+1}(\frac{n}{j}))$$

(L25)

If we define the following constants

$$\beta(n,k)=\sum_{j=1}^{k}\beta(1,j)\cdot\beta(n-1,k-j+1),\, where\, \beta(1,k)=b_k$$

(L26)

then we will find that

$$F_k(n)=\sum_{j=0}\beta(k,j+1)G_{k+j}(n)$$

(L27)

and

$$f_k(n)=\sum_{j=0}\beta(k,j+1)g_{k+j}(n)$$

(L28)

Some series reversion coefficient pairs follow are listed here, though this is, of course, an already well-understood problem:

a_k	b_k	a_k	b_k	a_k	b_k
1	-1^{k+1}	$-\frac{1}{k}$	$-\frac{1}{k!}$	$-\frac{1}{c}$	$-c^k$
-1	-1	$\frac{-1^k}{k}$	$\frac{-1^k}{k!}$	$\frac{-1^k}{c}$	$(-c)^k$
-1^k	-1^k	$\frac{-1^{k+1}}{k}$	$\frac{1}{k!}$	$\frac{-1^{k+1}}{c}$	c^k
$\frac{1}{k}$	$\frac{-1^{k+1}}{k!}$	$\frac{1}{c}$	$-1^{k+1}c^k$		

1.5 Inversion

The coefficient pair $a_k=-1^k, b_k=-1^k$ generates the Dirichlet Inverse. For some function $f_k(n)$ as defined in *(L1)*, if we have

$$g_1(n)=\sum_{k=1} -1^k f_k(n)$$

$$g_k(n)=\sum_{j|n} g_1(j)g_{k-1}(\frac{n}{j})$$

$$g_0(n)=1\, if\; n=1, 0\; otherwise$$

$$G_k(n)=\sum_{j=1}^{n} g_k(j)$$

then

$$F_1(n)=\sum_{j=1}^{\lfloor n\rfloor} g_1(j)(1+F_1(\frac{n}{j}))$$

(L30)

$$F_1(n) = -G_1(n) - \sum_{j=1}^{n} f_1(j) G_1\left(\frac{n}{j}\right)$$

(L31)

Additionally,

$$\sum_{j=1} g_1(j) F_k\left(\frac{n}{j}\right) = \sum_{j=1} -1^j F_{k+j}(n)$$

(L32)

and thus

$$\sum_{j=1} -g_1(j)\left(F_k\left(\frac{n}{j}\right) + F_{k-1}\left(\frac{n}{j}\right)\right) = F_k(n)$$

(L33)

Also,

$$F_k(n) = \sum_{j=0} -1^{k+j} \binom{k+j-1}{k-1} G_{k+j}(n)$$

(L34)

With these functions, the essence of Dirichlet Inversion looks like this, going forward:

$$\sum_{j=1}^{n} (f_0(j) + f_1(j)) \sum_{m=0}^{k} \binom{k}{m} F_m\left(\frac{n}{j}\right) = \sum_{m=0}^{k+1} \binom{k+1}{m} F_m(n)$$

(the coefficients here mirror those found in the similar $(1+x)\cdot(1+x)^k = (1+x)^{k+1}$)

and going backward,

$$\sum_{j=1}^{n} (g_0(j) + g_1(j)) \sum_{m=0}^{k} \binom{k}{m} F_m\left(\frac{n}{j}\right) = \sum_{m=0}^{k-1} \binom{k-1}{m} F_m(n)$$

(the coefficients here mirror $\frac{(1+x)^k}{(1+x)} = (1+x)^{k-1}$)

with

$$\sum_{j=1}^{n} (g_0(j) + g_1(j))\left(F_1\left(\frac{n}{j}\right) + F_0\left(\frac{n}{j}\right)\right) = 1$$

f's and g's can be swapped here too.

$$\sum_{j=1}^{n} (g_0(j) + g_1(j))\left(F_{k+1}\left(\frac{n}{j}\right) + F_k\left(\frac{n}{j}\right)\right) = F_k(n)$$

1.6 Other Properties

As an aside, a generalization of Dirichlet's hyperbola method also works

$$F_k'(n,a) = \sum_{m=a}^{\lfloor n^{\frac{1}{k}} \rfloor} \sum_{j=0}^{k-1} f(m)^{k-j} \binom{k}{j} F_j'\left(\frac{n}{m^{k-j}}, m+1\right)$$
$$F_1'(n,a) = F_1(n) - F_1(a-1) \qquad F_0'(n,a) = 1$$

$$F_k(n) = F_k'(n,2)$$

(L35)

2. Transformation Pairs

What follows is a few tables of some interesting function transformation pairs.

2.1 Table A:
Transformations connected to $f(n) = n^a$

#	$f(n)$	a_k	$g(n)$
A1	1	$\frac{-1^{k+1}}{k}$	$\frac{\Lambda(n)}{\log n}$
A2	1	-1^k	$\mu(n)$
A3	$\frac{\Lambda(n)}{\log n}$	$\frac{1}{k!}$	1
A4	$\mu(n)$	-1^k	1
A5	$\mu(n)$	$\frac{-1^k}{k}$	$\frac{\Lambda(n)}{\log n}$
A6	$\frac{\Lambda(n)}{\log n}$	$\frac{-1^k}{k!}$	$\mu(n)$
A7	n	$\frac{-1^{k+1}}{k}$	$n \cdot \frac{\Lambda(n)}{\log n}$
A8	$n \cdot \frac{\Lambda(n)}{\log n}$	$\frac{1}{k!}$	n
A9	$\frac{1}{n}$	$\frac{-1^{k+1}}{k}$	$\frac{1}{n} \cdot \frac{\Lambda(n)}{\log n}$
A10	$\frac{1}{n} \cdot \frac{\Lambda(n)}{\log n}$	$\frac{1}{k!}$	$\frac{1}{n}$
A11	n^a	$\frac{-1^{k+1}}{k}$	$n^a \cdot \frac{\Lambda(n)}{\log n}$
A12	$n^a \cdot \frac{\Lambda(n)}{\log n}$	$\frac{1}{k!}$	n^a
A13	$\frac{\Lambda(n)}{\log n}$	-1^k	r(n)
A14	r(n)	-1^k	$\frac{\Lambda(n)}{\log n}$

(*) Derivation of #A11

We start, by definition, with

$$G_1(n)=\sum_{j=1}^{n} f(j)-\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{\lfloor\frac{n}{j}\rfloor} f(j)\cdot f(k)$$
$$+\frac{1}{3}\sum_{j=1}^{n}\sum_{k=1}^{\lfloor\frac{n}{j}\rfloor}\sum_{l=1}^{\lfloor\frac{n}{j\cdot k}\rfloor} f(j)\cdot f(k)\cdot f(l)-\frac{1}{4}\ldots$$

Now, our $f(n)=0\, if\ n=1, n^a\, if\ n\neq 1$, so

$$G_1(n)=\sum_{j=2}^{n} j^a-\frac{1}{2}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor} j^a\cdot k^a$$
$$+\frac{1}{3}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor\frac{n}{j\cdot k}\rfloor} j^a\cdot k^a\cdot l^a-\frac{1}{4}\ldots$$

By the rules of exponents, we can rewrite this as

$$G_1(n)=\sum_{j=2}^{n} j^a-\frac{1}{2}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor} (jk)^a$$
$$+\frac{1}{3}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor\frac{n}{j\cdot k}\rfloor} (jkl)^a-\frac{1}{4}\ldots$$

Let's turn for a second to the strict divisor, given as

$$d_1'(n)=0\, if\ n=1, 1\, if\ n\neq 1 \qquad d_k'(n)=\sum_{j|n} d_1'(j)\cdot d_{k-1}'(\frac{n}{j})$$

This is the divisor function used in Linnik's identity. One core property of this function is

$$\sum_{j=2}^{n} f(j)=\sum_{j=2}^{n} d_1'(j) f(j)$$
$$\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor} f(jk)=\sum_{j=2}^{n} d_2'(j) f(j)$$
$$\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor\frac{n}{jk}\rfloor} f(jkl)=\sum_{j=2}^{n} d_3'(j) f(j)$$

and so on. Turning back to our sum $G_1(n)$, this means we can rewrite our sum as

$$G_1(n)=\sum_{j=2}^{n} d_1'(j)\, j^a-\frac{1}{2}\sum_{j=2}^{n} d_2'(j)\, j^a+\frac{1}{3}\sum_{j=2}^{n} d_3'(j)\, j^a-\frac{1}{4}\ldots$$

which is

$$G_1(n)=\sum_{j=2}^{n}(d_1'(j)-\frac{1}{2}d_2'(j)+-\frac{1}{3}d_3'(j)-\frac{1}{4}\ldots)\, j^a$$

The sum of divisors here is just Linnik's identity, and so we're left with

$$G_1(n)=\sum_{j=2}^{n}\frac{\Lambda(j)}{\log j} j^a$$

By definition, $G_1(n-1)+g_1(n)=G_1(n)$, so it must be the case that $g_1(n)=\frac{\Lambda(n)}{\log n} n^a$.

It turns out that, by a similar process, many of the identities in Tables 1-5 can have a factor of n^a multiplied in for both *f(n)* and *g(n)* without consequence.

2.2 Table B: Transformations connected to $f(n)=\lambda(n)\cdot n^a$

In the following table, we will use these two functions

$$\lambda(n)=-1^{\Omega(n)}$$

which is the liouville function, and

$$p'(n)=\frac{-1^{a+1}}{a} \text{ if } n=p^a \text{ where } p \text{ is prime}, 0 \text{ otherwise}$$

which is very similar to $\frac{\Lambda(n)}{\log n}$ but negative when *a* is even.

#	$f(n)$	a_k	$g(n)$
B1	$\lambda(n)$	$\frac{-1^k}{k}$	$p'(n)$
B2	$\lambda(n)$	-1^k	$\mu(n)^2$
B3	$p'(n)$	$\frac{-1^k}{k!}$	$\lambda(n)$
B4	$\mu(n)^2$	-1^k	$\lambda(n)$
B5	$\mu(n)^2$	$\frac{-1^{k+1}}{k}$	$p'(n)$
B6	$p'(n)$	$\frac{1}{k!}$	$\mu(n)^2$
B7	$n\cdot\lambda(n)$	$\frac{-1^k}{k}$	$n\cdot p'(n)$
B8	$n\cdot p'(n)$	$\frac{-1^k}{k!}$	$n\cdot\lambda(n)$
B9	$\frac{1}{n}\cdot\lambda(n)$	$\frac{-1^k}{k}$	$\frac{1}{n}\cdot p'(n)$
B10	$\frac{1}{n}\cdot p'(n)$	$\frac{-1^k}{k!}$	$\frac{1}{n}\cdot\lambda(n)$

#	$f(n)$	a_k	$g(n)$
B11	$n^a \cdot \lambda(n)$	$\frac{-1^k}{k}$	$n^a \cdot p'(n)$
B12	$n^a \cdot p'(n)$	$\frac{-1^k}{k!}$	$n^a \cdot \lambda(n)$

Similar properties and relationships can be found for k-free numbers more generally.

2.3 Table C: Transformations connected to $\rho(n)=1$ if n is prime, 0 otherwise

In this following table, we will use these two functions:

$$\rho(n)=1 \text{ if } n \textit{ is prime}, 0 \textit{ otherwise}$$

$$\theta(n)=\prod_{j=1}^{r} \frac{1}{a_j!} \quad \text{where} \quad n=\prod_{j=1}^{r} p_j^{a_j}$$

#	$f(n)$	a_k	$g(n)$
C1	$\theta(n)$	$\frac{-1^{k+1}}{k}$	$\rho(n)$
C2	$\theta(n)$	-1^k	$\theta(n) \cdot -1^{\omega(n)}$
C3	$\rho(n)$	$\frac{1}{k!}$	$\theta(n)$
C4	$\theta(n) \cdot -1^{\omega(n)}$	-1^k	$\theta(n)$
C5	$\theta(n) \cdot -1^{\omega(n)}$	$\frac{-1^k}{k}$	$\rho(n)$
C6	$\rho(n)$	$\frac{-1^k}{k!}$	$\theta(n) \cdot -1^{\omega(n)}$
C7	$n^a \cdot \theta(n)$	$\frac{-1^{k+1}}{k}$	$n^a \cdot \rho(n)$
C8	$n^a \cdot \rho(n)$	$\frac{1}{k!}$	$n^a \cdot \theta(n)$

2.4 Table D: Transformations connected to $f(n)=-1^{\omega(n)}$

Suppose we have $\omega(n)$ is the number of distinct prime factors of *n*. Then let's define the following functions:

$$\xi(n)=-1^{\omega(n)}$$

$$\Xi(n)=\prod_{j=1}^{r} 2^{a_j-1} \quad \text{where} \quad n=\prod_{j=1}^{r} p_j^{a_j}$$

$$\Phi(n)=-\frac{2^a-1}{a} \quad \text{, if} \quad n=p^a \quad \text{, p a prime, 0 otherwise.}$$

#	$f(n)$	a_k	$g(n)$
D1	$\xi(n)$	$\frac{-1^{k+1}}{k}$	$\Phi(n)$
D2	$\xi(n)$	-1^k	$\Xi(n)$
D3	$\Phi(n)$	$\frac{1}{k!}$	$\xi(n)$
D4	$\Xi(n)$	-1^k	$\xi(n)$
D5	$\Xi(n)$	$\frac{-1^k}{k}$	$\Phi(n)$
D6	$\Phi(n)$	$\frac{-1^k}{k!}$	$\Xi(n)$
D7	$n^a \cdot \xi(n)$	$\frac{-1^{k+1}}{k}$	$n^a \cdot \Phi(n)$
D8	$n^a \cdot \Phi(n)$	$\frac{1}{k!}$	$n^a \cdot \xi(n)$

2.5 Table E: Transformations connected to the Totient and Divisor Functions

In the following table, we will use these functions:

$\varphi(n)$ is the Euler Totient Function defined by

$$\varphi(n)=\sum_{d|n} d \cdot \mu\left(\frac{n}{d}\right)$$

$\sigma_a(n)$ is the divisor function defined by

$$\sigma_a(n)=\sum_{d|n} d^a$$

$J_a(n)$ is the Jordan totient function defined by

$$J_a(n)=\sum_{d|n} d^a \cdot \mu\left(\frac{n}{d}\right)$$

#	$f(n)$	a_k	$g(n)$
E1	$\varphi(n)$	$\frac{-1^{k+1}}{k}$	$\frac{\Lambda(n)}{\log n}(n-1)$
E2	$\frac{\Lambda(n)}{\log n}(n-1)$	$\frac{1}{k!}$	$\varphi(n)$
E3	$\sigma_1(n)$	$\frac{-1^{k+1}}{k}$	$\frac{\Lambda(n)}{\log n}(n+1)$
E4	$\frac{\Lambda(n)}{\log n}(n+1)$	$\frac{1}{k!}$	$\sigma_1(n)$
E5	$J_a(n)$	$\frac{-1^{k+1}}{k}$	$\frac{\Lambda(n)}{\log n}(n^a-1)$
E6	$\frac{\Lambda(n)}{\log n}(n^a-1)$	$\frac{1}{k!}$	$J_a(n)$
E7	$\sigma_a(n)$	$\frac{-1^{k+1}}{k}$	$\frac{\Lambda(n)}{\log n}(n^a+1)$
E8	$\frac{\Lambda(n)}{\log n}(n^a+1)$	$\frac{1}{k!}$	$\sigma_a(n)$

3. Specific Identities

All of the properties listed above as L6 through L34 apply to each entry in Table A, Table B, Table C, Table D, and Table E.

Nevertheless, certain specific instances of these properties are likely more interesting than others. Some of those more interesting identities will be listed below.

This section should be regarded as incomplete and in progress.

3.1 Relationships connected to Table A

The prime power counting function,

$$\pi(n)+\frac{1}{2}\pi(n^{\frac{1}{2}})+\frac{1}{3}\pi(n^{\frac{1}{3}})+\ldots$$

is equal to the following statements:

$$\sum_{j=2}^{n} 1-\frac{1}{2}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor} 1+\frac{1}{3}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor\frac{n}{j\cdot k}\rfloor} 1-\frac{1}{4}\ldots \quad ;$$

$$v_1(n) \text{ for } v_k(n)=\sum_{j=2}^{\lfloor n\rfloor}\frac{1}{k}-v_{k+1}\left(\frac{n}{j}\right) \quad ;$$

$$\sum_{j=2}^{n}\mu(j)\left(-1+\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\mu(k)\left(\frac{1}{2}+\sum_{l=2}^{\lfloor\frac{n}{j\cdot k}\rfloor}\mu(l)\left(-\frac{1}{3}+\sum\ldots\right)\right)\right) \quad ;$$

$$v_1(n) \text{ for } v_k(n)=\sum_{j=2}^{\lfloor n\rfloor}\mu(j)\left(\frac{-1}{k}-v_{k+1}\left(\frac{n}{j}\right)\right) \quad ;$$

$$v_0(n) \text{ for } v_k(n)=\frac{B_k}{k!}\lfloor n-1\rfloor+\sum_{j=2}^{\lfloor n\rfloor}\frac{\Lambda(j)}{\log j}v_{k+1}\left(\frac{n}{j}\right) \quad ;$$

$$v_0(n) \text{ for } v_k(n)=\frac{-B_k}{k!}(M(n)-1)-\sum_{j=2}^{\lfloor n\rfloor}\frac{\Lambda(j)}{\log j}v_{k+1}\left(\frac{n}{j}\right)$$

where $M(n)$ is the Mertens function;

$$v_1(n) \text{ for } v_k(n)=c_k(M(n)-1)-\sum_{j=2}^{\lfloor n\rfloor}v_{k+1}\left(\frac{n}{j}\right)$$

where c_k is $0,-1,\frac{1}{2},\frac{1}{6},\frac{1}{12},\frac{1}{20},\ldots$ from the Taylor expansion of $(1-z)\log(1-z)$

The statement

$$n-1$$

is equal to the following statements:

$$\sum_{j=2}^{n}\frac{\Lambda(j)}{\log j}+\frac{1}{2}\sum_{j=2}^{n}\sum_{k=2}^{\frac{n}{j}}\frac{\Lambda(j)}{\log j}\cdot\frac{\Lambda(k)}{\log k}+\frac{1}{6}\sum_{j=2}^{n}\sum_{k=2}^{\frac{n}{j}}\sum_{l=2}^{\frac{n}{j\cdot k}}\frac{\Lambda(j)}{\log j}\cdot\frac{\Lambda(k)}{\log k}\cdot\frac{\Lambda(l)}{\log l}+\frac{1}{24}\ldots \quad ;$$

$$v_1(n) \text{ for } v_k(n)=\sum_{j=2}^{\lfloor n\rfloor}\frac{\Lambda(j)}{\log j}\left(\frac{1}{k!}+v_{k+1}\left(\frac{n}{j}\right)\right) \quad ;$$

$$-\sum_{j=2}^{n}\mu(j)+\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\mu(j)\cdot\mu(k)-\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor\frac{n}{j\cdot k}\rfloor}\mu(j)\cdot\mu(k)\cdot\mu(l)+\ldots \quad ;$$

$$v_1(n) \text{ for } v_k(n)=\sum_{j=2}^{\lfloor n\rfloor}\mu(j)\left(\frac{-1}{k}-v_{k+1}\left(\frac{n}{j}\right)\right) \quad ;$$

$$v_0(n) \quad \text{for} \quad v_k(n) = -C_k \Pi(n) - \sum_{j=2}^{\lfloor n \rfloor} v_{k+1}\left(\frac{n}{j}\right)$$

where $\Pi(n) = \pi(n) + \frac{1}{2}\pi(n^{\frac{1}{2}}) + \frac{1}{3}\pi(n^{\frac{1}{3}}) + \ldots$

and C_k are the Gregory coefficients,

$-1, \frac{1}{2,} \frac{1}{12,} \frac{1}{24,} \frac{19}{720,} \ldots$ the coefficients from the Taylor expansion of $\frac{z}{\log(1-z)}$

One less than the Mertens function,

$$M(n) - 1$$

is equal to the following statements:

$$-\sum_{j=2}^{n} 1 + \sum_{j=2}^{n} \sum_{k=2}^{\lfloor \frac{n}{j} \rfloor} 1 - \sum_{j=2}^{n} \sum_{k=2}^{\lfloor \frac{n}{j} \rfloor} \sum_{l=2}^{\lfloor \frac{n}{j \cdot k} \rfloor} 1 + \ldots \quad ;$$

$$v(n) \quad \text{for} \quad v(n) = \sum_{j=2}^{\lfloor n \rfloor} 1 - v\left(\frac{n}{j}\right) \quad ;$$

$$-\sum_{j=2}^{n} \frac{\Lambda(j)}{\log j} + \frac{1}{2} \sum_{j=2}^{n} \sum_{k=2}^{\lfloor \frac{n}{j} \rfloor} \frac{\Lambda(j)}{\log j} \cdot \frac{\Lambda(k)}{\log k} - \frac{1}{6} \sum_{j=2}^{n} \sum_{k=2}^{\lfloor \frac{n}{j} \rfloor} \sum_{l=2}^{\lfloor \frac{n}{j \cdot k} \rfloor} \frac{\Lambda(j)}{\log j} \cdot \frac{\Lambda(k)}{\log k} \cdot \frac{\Lambda(l)}{\log l} + \frac{1}{24} \ldots \quad ;$$

$$v_1(n) \quad \text{with} \quad v_k(n) = \sum_{j=2}^{\lfloor n \rfloor} \frac{\Lambda(j)}{\log j}\left(\frac{-1}{k!} - v_{k+1}\left(\frac{n}{j}\right)\right) \quad ;$$

$$v_0(n) \quad \text{for} \quad v_k(n) = C_k \Pi(n) - \sum_{j=2}^{\lfloor n \rfloor} \mu(j) v_{k+1}\left(\frac{n}{j}\right)$$

where $\Pi(n) = \pi(n) + \frac{1}{2}\pi(n^{\frac{1}{2}}) + \frac{1}{3}\pi(n^{\frac{1}{3}}) + \ldots$

and C_k are the Gregory coefficients, described above ;

$$v_0(n) \quad \text{for} \quad v_k(n) = c_k \Pi(n) - \sum_{j=2}^{\lfloor n \rfloor} v_{k+1}\left(\frac{n}{j}\right)$$

where c_k is $-1, -\frac{1}{2,} - \frac{5}{12,} - \frac{3}{8,} - \frac{251}{720,} - \frac{95}{288,} \ldots$ from the Taylor expansion of $\frac{z}{(1-z)\log(1-z)}$

The sum

$$\sum_{j=2}^{n} \frac{\Lambda(j)}{\log j} j^a$$

is equal to the following statements:

$$\sum_{j=2}^{n} j^a - \frac{1}{2} \sum_{j=2}^{n} \sum_{k=2}^{\lfloor \frac{n}{j} \rfloor} j^a \cdot k^a + \frac{1}{3} \sum_{j=2}^{n} \sum_{k=2}^{\lfloor \frac{n}{j} \rfloor} \sum_{l=2}^{\lfloor \frac{n}{j \cdot k} \rfloor} j^a \cdot k^a \cdot l^a - \frac{1}{4} \ldots \quad ;$$

$$v_1(n) \quad \text{for} \quad v_k(n) = \sum_{j=2}^{\lfloor n \rfloor} j^a \left(\frac{1}{k} - v_{k+1}\left(\frac{n}{j}\right)\right)$$

The sum

$$\sum_{j=2}^{n} j^a$$

is equal to the following statements:

$$\sum_{j=2}^{n} \frac{\Lambda(j)}{\log j} j^a + \frac{1}{2} \sum_{j=2}^{n} \sum_{k=2}^{\frac{n}{j}} \frac{\Lambda(j)}{\log j} \cdot \frac{\Lambda(k)}{\log k} j^a \cdot k^a + \frac{1}{6} \sum_{j=2}^{n} \sum_{k=2}^{\frac{n}{j}} \sum_{l=2}^{\frac{n}{j \cdot k}} \frac{\Lambda(j)}{\log j} \cdot \frac{\Lambda(k)}{\log k} \cdot \frac{\Lambda(l)}{\log l} j^a \cdot k^a \cdot l^a + \frac{1}{24} \ldots \quad ;$$

$$v_1(n) \quad \text{for} \quad v_k(n) = \sum_{j=2}^{\lfloor n \rfloor} j^a \cdot \frac{\Lambda(j)}{\log j}\left(\frac{1}{k!} + v_{k+1}\left(\frac{n}{j}\right)\right)$$

The prime power function

$$\frac{\Lambda(n)}{\log n}$$

is equal to the following statements:

$$1 - \sum_{\substack{j \cdot k = n \\ 1 < j, k < n}} \frac{1}{2} + \sum_{\substack{j \cdot k \cdot l = n \\ 1 < j, k, l < n}} \frac{1}{3} - \sum_{\substack{j \cdot k \cdot l \cdot m = n \\ 1 < j, k, l, m < n}} \frac{1}{4} + \ldots \quad ;$$

$$-\mu(n) + \frac{1}{2} \sum_{\substack{j \cdot k = n \\ 1 < j, k < n}} \mu(j)\mu(k) - \frac{1}{3} \sum_{\substack{j \cdot k \cdot l = n \\ 1 < j, k, l < n}} \mu(j)\mu(k)\mu(l) + \ldots$$

The moebius function

$$\mu(n)$$

is equal to the following statements:

$$-1+\sum_{\substack{j\cdot k=n\\ 1<j,k<n}}1-\sum_{\substack{j\cdot k\cdot l=n\\ 1<j,k,l<n}}1+\sum_{\substack{j\cdot k\cdot l\cdot m=n\\ 1<j,k,l,m<n}}1-\ldots \quad ;$$

$$-\frac{\Lambda(n)}{\log n}+\frac{1}{2}\sum_{\substack{j\cdot k=n\\ 1<j,k<n}}\frac{\Lambda(j)}{\log j}\cdot\frac{\Lambda(k)}{\log k}$$
$$-\frac{1}{6}\sum_{\substack{j\cdot k\cdot l=n\\ 1<j,k,l<n}}\frac{\Lambda(j)}{\log j}\cdot\frac{\Lambda(k)}{\log k}\cdot\frac{\Lambda(l)}{\log l}+\frac{1}{24}\ldots$$

$$1$$

is equal to the following statements:

$$\frac{\Lambda(n)}{\log n}+\frac{1}{2}\sum_{\substack{j\cdot k=n\\ 1<j,k<n}}\frac{\Lambda(j)}{\log j}\cdot\frac{\Lambda(k)}{\log k}$$
$$+\frac{1}{6}\sum_{\substack{j\cdot k\cdot l=n\\ 1<j,k,l<n}}\frac{\Lambda(j)}{\log j}\cdot\frac{\Lambda(k)}{\log k}\cdot\frac{\Lambda(l)}{\log l}+\frac{1}{24}\ldots \quad ;$$

$$-\mu(n)+\sum_{\substack{j\cdot k=n\\ 1<j,k<n}}\mu(j)\mu(k)-\sum_{\substack{j\cdot k\cdot l=n\\ 1<j,k,l<n}}\mu(j)\mu(k)\mu(l)+\ldots$$