4. More Exponential-Style Dirichlet Convolutions as Power Series

One goal here is to show, with mind-numbing repetition, the direct and fruitful connection between exponential-style Dirichlet convolutions and Dirichlet series. In particular, the goal is to show that the convolution operations turn into the normal multiplication and exponentiation operations of math as the limit of these partial sums goes to infinity. Once established, the next hope is that this should inspire the reverse move – taking established power series operations on Dirichlet series and finding partial sum equivalents, as the Dirichlet series versions of identities tend to be much more familiar.

This is most evident when looking at various power series equivalents. So this section will include a number of examples of power series-like identities with exponential-style Dirichlet convolutions that, in their limit, are normal power series involving Dirichlet series.

4.1 Equations for $[(\zeta(s)-1)^k]_n$

There are a few useful ways to express $[(\zeta(s)-1)^k]_n$ in terms of other functions as well. For example,

$$\boxed{[(\zeta(s)-1)^k]_n = \sum_{j=0} (-1)^{k-j} \binom{k}{j} [\zeta(s)^j]_n} \tag{4.1.1}$$

```
Dm1[ n_, 0 ] := UnitStep[n-1]
Dm1[ n_, k_ ] := Sum[ Dm1[ n(1+j)^-1, k-1 ], { j, 1, n-1 } ]
Dz[n_,z_,k_]:=1+((z+1)/k-1)Sum[Dz[n/j,z,k+1],{j,2,n}]
Dm1Alt[n_,k_]:=Sum[(-1)^(k-j)Binomial[k,j]Dz[n,j,1],{j,0,k}]
Grid[ Table[Dm1[ n, k ]-Dm1Alt[n,k], { n, 1, 50 }, { k, 1, 7 } ]]
```

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$(\zeta(s)-1)^k = \sum_{j=0}^{\infty} (-1)^{(k-j)} \binom{k}{j} \zeta(s)^j \tag{4.1.2}$$

```
{(Zeta[s]-1)^j,Sum[ (-1)^(j-k) Binomial[j,k] Zeta[s]^k,{k,0,Infinity}]}
```

$$[(f-1)^k]_n = \sum_{j=0} (-1)^{k-j} \binom{k}{j} [f^j]_n \tag{4.1.1}$$

4.1 Equations for $[(\zeta(s)-1)^k]_n$

$[(\zeta(s)-1)^k]_n$ can also be expressed in terms of $[(\log\zeta(s))^k]_n$ as

$$\boxed{[(\zeta(s)-1)^k]_n = \sum_{j=0}^{\infty} \left(\lim_{x\to 0} \frac{\partial^j}{\partial x^j} (e^x-1)^k\right) [(\log\zeta(s))^j]_n} \tag{4.1.3}$$

```
Dm1[n_,0]:=UnitStep[n-1]
```

```
Dm1[n_,k_]:=Sum[Dm1[n (1+j)^-1,k-1],{j,1,n-1}]
logD[n_,k_]:=Sum[FullSimplify[MangoldtLambda[j]/Log[j]] logD[n/j,k-1],{j,2,n}];logD[n_,0]:=UnitStep[n-1]
Dm1Alt[n_,k_]:=Sum[(Limit[D[(E^x-1)^k,{x,j}],x->0])/j!logD[n,j],{j,0,Log[2,n]}]
Grid[Table[Dm1[n,k]-Dm1Alt[n,k],{n,1,50},{k,1,7}]]
```

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$(\zeta(s)-1)^k=\sum_{j=0}^{\infty}\left(\lim_{x\to 0}\frac{\partial^j}{\partial x^j}(e^x-1)^k\right)(\log\zeta(s))^j \tag{4.1.4}$$

```
Grid[Table[Chop[(Zeta[s]-1)^k-N[Sum[Limit[D[(E^x-1)^k,{x,j}],x->0]/(j!)Log[Zeta[s]]^j,{j,0,50}]]],{s,2.,5.},{n,0.35,1.75,.2},
{k,1,5}]]
```

This can be generalized as

$$\boxed{[(f-1)^k]_n=\sum_{j=0}^{\infty}\left(\lim_{x\to 0}\frac{\partial^j}{\partial x^j}(e^x-1)^k\right)[(\log f)^j]_n} \tag{4.1.3}$$

4.1 Equations for $[(\zeta(s)-1)^k]_n$

One more, slightly more complicated way to express $[\zeta_n(s)-1]^{*k}$ looks like this

$$\boxed{[(\zeta(s)-1)^k]_n=\sum_{m=0}\frac{1}{m!}\left(\lim_{x\to 0}\frac{\partial^m}{\partial x^m}\frac{x}{\log(1+x)}\right)\cdot[(\zeta(s)-1)^{k-1+m}\cdot\log\zeta(s)]_n} \tag{4.1.5}$$

```
RiemannPrimeCount[n_]:=Sum[PrimePi[n^(1/j)]/j,{j,1,Log[2,n]}]
Dm1[n_,0]:=UnitStep[n-1]
Dm1[n_,k_]:=Dm1[n,k]=Sum[Dm1[Floor[n (1+j)^-1],k-1],{j,1,n-1}]
dm1[n_, k_] := Dm1[n,k]-Dm1[n-1,k]; dm1[n_, 0] := If[ n>=1,1,0]
Dm1Alt[ n_, k_ ] := Sum[ Limit[D[x/Log[1+x],{x,m}],x->0]/(m!)Sum[ dm1[ j, k-1+m] RiemannPrimeCount[ n/j],{j,1,n}],
{m,0,Log[2,n]}]
Grid[Table[Dm1[n,k]-Dm1Alt[n,k],{n,1,50},{k,1,5}]]
```

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$(\zeta(s)-1)^k=\sum_{m=0}^{\infty}\frac{1}{m!}\left(\lim_{x\to 0}\frac{\partial^m}{\partial x^m}\frac{x}{\log(1+x)}\right)(\zeta(s)-1)^{k-1+m}\log\zeta(s) \tag{4.1.6}$$

```
Grid[Table[Chop[(Zeta[s]-1)^k-N[Sum[Limit[ D[ x/Log[1+x],{x,m}],x->0]/(m!)(Zeta[s]-1)^(k-1+m)Log[Zeta[s]],{m,0,50}]]],
{s,2.,5.},{n,0.35,1.75,.2},{k,-3,3}]]
```

This can be generalized as

$$[(f-1)^k]_n=\sum_{m=0}\frac{1}{m!}\left(\lim_{x\to 0}\frac{\partial^m}{\partial x^m}\frac{x}{\log(1+x)}\right)\cdot[(f-1)^{k-1+m}\cdot\log f]_n \tag{4.1.5}$$

In particular, if k=1 and s=0, this reduces to

$$n-1=\sum_{m=0}\frac{1}{m!}\left(\lim_{x\to 0}\frac{\partial^m}{\partial x^m}\frac{x}{\log(1+x)}\right)\sum_{j=1}[\nabla(\zeta(s)-1)^m]_j\cdot\Pi\left(\frac{n}{j}\right)$$

(4.1.7)

```
RiemannPrimeCount[n_]:=Sum[PrimePi[n^(1/j)]/j,{j,1,Log[2,n]}]
Dm1[n_,k_]:=Dm1[n,k]=Sum[Dm1[Floor[n (1+j)^-1],k-1],{j,1,n-1}];Dm1[n_,0]:=UnitStep[n-1]
dm1[n_, k_] := Dm1[n,k]-Dm1[n-1,k]; dm1[n_, 0] := If[ n □1,1,0]
nm1[ n_ ] := Sum[ Limit[D[x/Log[1+x],{x,m}],x->0]/(m!)Sum[ dm1[ j, m] RiemannPrimeCount[ n/j],{j,1,n}],{m,0,Log[2,n]}]
Table[ {n-1, nm1[n]},{n,1,100}]
```

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\zeta(s)-1=\sum_{m=0}^{\infty}\frac{1}{m!}\left(\lim_{x\to 0}\frac{\partial^m}{\partial x^m}\frac{x}{\log(1+x)}\right)(\zeta(s)-1)^m\log\zeta(s)$$

(4.1.8)

```
Table[Chop[(Zeta[s]-1)-N[Sum[Limit[D[x/Log[1+x],{x,m}],x->0]/(m!)(Zeta[s]-1)^m Log[Zeta[s]],{m,0,50}]]],{s,2.,5.},
{n,0.35,1.75,.2}]
```

which is possibly interesting in that it shows a way to sum the Riemann prime counting function so that it equals an easy-to-calculate value, n-1. Note that this is the same as saying

$$\begin{aligned}
&n-1=\Pi(n)\\
&+\frac{1}{2}\sum_{j=2}^{\lfloor n\rfloor}\Pi\left(\frac{n}{j}\right)\\
&-\frac{1}{12}\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor \frac{n}{j}\rfloor}\Pi\left(\frac{n}{j\cdot k}\right)\\
&+\frac{1}{24}\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor \frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor \frac{n}{j\cdot k}\rfloor}\Pi\left(\frac{n}{j\cdot k\cdot l}\right)\\
&-\frac{19}{720}\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor \frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor \frac{n}{j\cdot k}\rfloor}\sum_{m=2}^{\lfloor \frac{n}{j\cdot k\cdot l}\rfloor}\Pi\left(\frac{n}{j\cdot k\cdot l\cdot m}\right)\\
&+\ldots
\end{aligned}$$

(4.1.9)

```
pi[n_]:=Sum[PrimePi[n^(1/j)]/j,{j,1,Log[2,n]}]
(*sum is trunctated and stops working after n=2^6-1*)
sum[n_]:=pi[n]+(1/2)Sum[pi[n/j],{j,2,n}]-1/12Sum[pi[n/(j k)],{j,2,n},{k,2,n/j}]+1/24Sum[pi[n/(j k l)],{j,2,n},{k,2,n/j},{l,2,n/(j k)}]-19/720Sum[pi[n/(j k l m)],{j,2,n},{k,2,n/j},{l,2,n/(j k)},{m,2,n/(j k l)}]+3/160Sum[pi[n/(j k l m o)],{j,2,n},{k,2,n/j},
{l,2,n/(j k)},{m,2,n/(j k l)},{o,2,n/(j k l m)}]
Table[{n-1, sum[n]},{n,1,63}]//TableForm
```

Combining (2.5) with (D1) gives us

$$[\zeta(0)^z]_n=1+\sum_{k=1}^{\infty}\binom{z}{k}\sum_{m=0}\frac{1}{m!}\left(\lim_{x\to 0}\frac{\partial^m}{\partial x^m}\frac{x}{\log(1+x)}\right)\cdot\sum_{j=1}[\nabla(\zeta(0)-1)^{k-1+m}]_j\,\Pi\left(\frac{n}{j}\right)$$

(4.1.10)

```
Dz[n_,z_,k_]:=1+((z+1)/k-1)Sum[Dz[n/j,z,k+1],{j,2,n}]
RiemannPrimeCount[n_]:=Sum[PrimePi[n^(1/j)]/j,{j,1,Log[2,n]}]
Dm1[n_,k_]:=Dm1[n,k]=Sum[Dm1[Floor[n (1+j)^-1],k-1],{j,1,n-1}];Dm1[n_,0]:=UnitStep[n-1]
dm1[n_, k_] := Dm1[n,k]-Dm1[n-1,k]; dm1[n_, 0] := If[ n□1,1,0]
DzAlt[ n_, z_ ] := 1+Sum[ Binomial[z,k]Sum[ Limit[D[x/Log[1+x],{x,m}],x->0]/(m!)Sum[ dm1[ j, k-1+m]
RiemannPrimeCount[ n/j],{j,1,n}],{m,0,Log[2,n]}],{k,1,Log[2,n]}]
Grid[Table[Chop[Dz[a=55,s+t I,1]-DzAlt[a,s+t I]],{s,-1.5,4,.7},{t,-1.1,4,.7}]]
```

which lets us express, for example, the Dirichlet Divisor function $D(n)$ or the Mertens function $M(n)$ as sums of the Riemann Prime Counting function $\Pi(n)$ in a fashion similar to (2.7) if you keep in mind that

$$\sum_{j=2}[\nabla(\zeta(0)-1)^2]_j \Pi(\frac{n}{j\,k})=\sum_{j=2}\sum_{k=2}\Pi(\frac{n}{j}) \tag{4.1.10}$$

$$\sum_{j=2}[\nabla(\zeta(0)-1)^3]_j \Pi(\frac{n}{j})=\sum_{j=2}\sum_{k=2}\sum_{l=2}\Pi(\frac{n}{j\,k\,l}) \tag{4.1.10}$$

and so on.

4.2 Equations for $[(\zeta(s)-1)^k]_n$

Specify symbols to be used next here:

$$[(\zeta(s)-1)^k]_n=\sum_{j=2} j^{-s}[(\zeta(s)-1)^{k-1}]_{n\,j^{-1}}$$

$$[\zeta_n(s)^{-1}-1]^{*k}=\sum_{j=1}[\zeta_{\Delta(j+1)}(s)]^{*-1}[\zeta_{n(j+1)^{-1}}(s)-1]^{*k-1}$$

$$[f^k]_n=\sum_{j=1} f(j)[f^{k-1}]_{n\,j^{-1}}$$

$$[(f-1)^k]_n=\sum_{j=2} f(j+1)[(f-1)^{k-1}]_{n(j+1)^{-1}}$$

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4.2 Equations for $[(\zeta(s)-1)^k]_n$

$$[(\zeta(s)-1)^{k+a}]_n=[(\zeta(s)-1)^k\cdot(\zeta(s)-1)^a]_n \tag{4.1.10}$$

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$(\zeta(s)-1)^{k+a}=(\zeta(s)-1)^k\cdot(\zeta(s)-1)^a \tag{4.1.10}$$

```
(Zeta[s]-1)^a (Zeta[s]-1)^k
```

This can be generalized as

$$[(f-1)^{*k+a}]_n=[(f-1)^{*k}\cdot(f-1)^{*a}]_n \tag{4.1.10}$$

4.2 Equations for $[(\zeta(s)-1)^k]_n$

$$[\zeta(s)^{-1}-1]_n=\sum_{k=1}(-1)^k[(\zeta(s)-1)^k]_n$$

(4.1.10)

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\zeta(s)^{-1}-1=\sum_{k=1}(-1)^k(\zeta(s)-1)^k$$

(4.1.10)

```
((1/(Zeta[s])-1)-FullSimplify[Sum[ (-1)^k (Zeta[s]-1)^k,{k,1,Infinity}]])
```

This can be generalized as

$$[f^{-1}-1]_n=\sum_{k=1}(-1)^k[(f-1)^k]_n$$

(4.1.10)

4.2 Equations for $[(\zeta(s)-1)^k]_n$

$$\begin{gathered}[(\zeta(s)^{-1}-1)\cdot(\zeta(s)-1)^k]_n= \\ \sum_{j=0}(-1)^j[(\zeta(s)-1)^{j+k}]_n= \\ [(\zeta(s)-1)^k\cdot\zeta(s)^{-1}]_n\end{gathered}$$

(4.1.10)

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\begin{gathered}(\zeta(s)^{-1}-1)\cdot(\zeta(s)-1)^k= \\ \sum_{j=0}(-1)^j(\zeta(s)-1)^{j+k}= \\ (\zeta(s)-1)^k\cdot\zeta(s)^{-1}\end{gathered}$$

(4.1.10)

```
Sum[ (-1)^j (Zeta[s]-1)^(j+k),{j,0,Infinity}]
```

This can be generalized as

$$\begin{gathered}[(f^{-1}-1)\cdot(f-1)^k]_n= \\ \sum_{j=0}(-1)^j[(f-1)^{j+k}]_n= \\ [(f-1)^k\cdot f^{-1}]_n\end{gathered}$$

(4.1.10)

4.2 Equations for $[(\zeta(s)-1)^k]_n$

$$-[(\zeta(s)^{-1}-1)\cdot((\zeta(s)-1)^{k-1}+(\zeta(s)-1)^k)]_n=[(\zeta(s)-1)^k]_n$$

(4.1.10)

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$-(\zeta(s)^{-1}-1)^1\cdot((\zeta(s)-1)^{k-1}+(\zeta(s)-1)^k)=(\zeta(s)-1)^k$$

(4.1.10)

```
FullSimplify[-(Zeta[s]^-1 - 1 )( (Zeta[s]-1)^k + (Zeta[s]-1)^(k-1) )]
```

This can be generalized as

$$-[(f^{-1}-1)\cdot((f-1)^{k-1}+(f-1)^{k})]_n=[(f-1)^{k}]_n \tag{4.1.10}$$

4.2 Equations for $[(\zeta(s)-1)^k]_n$

$$\sum_{k=1}\frac{(-1)^{k+1}}{k}[(\zeta(s)-1)^{k+a}]_n=[(\zeta(s)-1)^{a}\cdot\log\zeta(s)]_n \tag{4.1.10}$$

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\sum_{k=1}\frac{(-1)^{k+1}}{k}(\zeta(s)-1)^{k+a}=(\zeta(s)-1)^{a}\cdot\log\zeta(s) \tag{4.1.10}$$

```
Sum[ (-1)^(k+1)/k (Zeta[s]-1)^(k+a),{k,1,Infinity}]
```

This can be generalized as

$$\sum_{k=1}\frac{(-1)^{k+1}}{k}[(f-1)^{k+a}]_n=[(f-1)^{a}\cdot\log f]_n \tag{4.1.10}$$

4.2 Equations for $[(\zeta(s)-1)^k]_n$

$$[(\zeta(s)-1)^{k}]_n=\sum_{m=0}\frac{1}{m!}(\lim_{x\to 0}\frac{\partial^m}{\partial x^m}\frac{x}{\log(1+x)})[(\zeta(s)-1)^{k-1+m}\cdot\log\zeta(s)]_n \tag{4.1.10}$$

$$(\zeta(s)-1)^{k}=\sum_{m=0}\frac{1}{m!}(\lim_{x\to 0}\frac{\partial^m}{\partial x^m}\frac{x}{\log(1+x)})(\zeta(s)-1)^{k-1+m}\cdot\log\zeta(s) \tag{4.1.10}$$

This can be generalized as

$$[(f-1)^{k}]_n=\sum_{m=0}\frac{1}{m!}(\lim_{x\to 0}\frac{\partial^m}{\partial x^m}\frac{x}{\log(1+x)})[(f-1)^{k-1+m}\cdot\log f]_n \tag{4.1.10}$$

4.2 Equations for $[(\zeta(s)-1)^k]_n$

(There are some other ideas from 9-G that are worth keeping)

4.2 Equations for $[(\zeta(s)-1)^k]_n$

$$[(\zeta(s)^{-1}-1)^k]_n=[(\zeta(s)-1)\cdot(-(\zeta(s)^{-1}-1)^{k-1}-(\zeta(s)^{-1}-1)^k)]_n \tag{4.1.10}$$

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$(\zeta(s)^{-1}-1)^k=(\zeta(s)-1)^1\cdot(-(\zeta(s)^{-1}-1)^{k-1}-(\zeta(s)^{-1}-1)^k) \tag{4.1.10}$$

```
FullSimplify[(Zeta[s]-1)( -(Zeta[s]^-1-1)^(k-1)-(Zeta[s]^-1-1)^k)]
```

This can be generalized as

$$[(f^{-1}-1)^k]_n=[(f-1)\cdot(-(f^{-1}-1)^{k-1}-(f^{-1}-1)^k)]_n \tag{4.1.10}$$

4.2 Equations for $[(\zeta(s)-1)^k]_n$

$$[(\zeta(s)^{-1}-1)^k]_n=\sum_{j=0}(-1)^{k+j}\binom{k+j-1}{k-1}[(\zeta(s)-1)^{k+j}]_n \tag{4.1.10}$$

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$(\zeta(s)^{-1}-1)^k=\sum_{j=0}(-1)^{k+j}\binom{k+j-1}{k-1}(\zeta(s)-1)^{k+j} \tag{4.1.10}$$

```
FullSimplify[Sum[ (-1)^(k+j) Binomial[ k+j-1, k-1] (Zeta[s]-1)^(k+j),{j,0, Infinity}]]
```

This can be generalized as

$$[(f^{-1}-1)^k]_n=\sum_{j=0}(-1)^{k+j}\binom{k+j-1}{k-1}[(f-1)^{k+j}]_n \tag{4.1.10}$$

4.2 Equations for $[(\zeta(s)-1)^k]_n$

$$[(\zeta(s)-1)^k]_n=\sum_{j=0}(-1)^{k+j}\binom{k+j-1}{k-1}[(\zeta(s)^{-1}-1)^{k+j}]_n \tag{4.1.10}$$

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$(\zeta(s)-1)^k=\sum_{j=0}(-1)^{k+j}\binom{k+j-1}{k-1}(\zeta(s)^{-1}-1)^{k+j} \tag{4.1.10}$$

```
FullSimplify[Sum[ (-1)^(k+j) Binomial[ k+j-1, k-1] (Zeta[s]^-1-1)^(k+j),{j,0, Infinity}]]
```

This can be generalized as

$$[(f-1)^k]_n=\sum_{j=0}(-1)^{k+j}\binom{k+j-1}{k-1}[(f^{-1}-1)^{k+j}]_n$$

(4.1.10)

4.2 Equations for $[(\zeta(s)-1)^k]_n$

$$[\log\zeta(s)]_n=\sum_{k=1}\frac{(-1)^k}{k}[(\zeta(s)^{-1}-1)^k]_n$$

(4.1.10)

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\log\zeta(s)=\sum_{k=1}\frac{(-1)^k}{k}(\zeta(s)^{-1}-1)^k$$

(4.1.10)

```
FullSimplify[Sum[(-1)^(k)/k (Zeta[s]^-1-1)^(k),{k,1,Infinity}]]
```

This can be generalized as

$$[\log f]_n=\sum_{k=1}\frac{(-1)^k}{k}[(f^{-1}-1)^k]_n$$

(4.1.10)

4.2 Equations for $[(\zeta(s)-1)^k]_n$

(There are some other ideas from 10-G that are worthy keeping)

4.2 Equations for $[(\zeta(s)-1)^k]_n$

$$[(\zeta(s)-1)^a]_n=\sum_{j=0}^{k}(-1)^k\binom{k}{j}[(\zeta(s)^{-1}-1)^k\cdot(\zeta(s)-1)^{a-j}]_n$$

(4.1.10)

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$(\zeta(s)-1)^a=\sum_{j=0}^{k}(-1)^k\binom{k}{j}(\zeta(s)^{-1}-1)^k\cdot(\zeta(s)-1)^{a-j}$$

(4.1.10)

```
Sum[ (-1)^a Binomial[a,b] (Zeta[s]^-1 -1)^a (Zeta[s]-1)^(k-b),{b,0,a}]
```

This can be generalized as

$$[(f-1)^a]_n=\sum_{j=0}^{k}(-1)^k\binom{k}{j}[(f^{-1}-1)^k\cdot(f-1)^{a-j}]_n$$

(4.1.10)

4.2 Equations for $[(\zeta(s)-1)^k]_n$

$$[(\zeta(s)^{-1}-1)^a]_n = \sum_{j=0}^{k} (-1)^k \binom{k}{j} [(\zeta(s)-1)^k \cdot (\zeta(s)^{-1}-1)^{a-j}]_n \tag{4.1.10}$$

The limit of this as n approaches infinity, if $\Re(s) > 1$, is

$$(\zeta(s)^{-1}-1)^a = \sum_{j=0}^{k} (-1)^k \binom{k}{j} (\zeta(s)-1)^k \cdot (\zeta(s)^{-1}-1)^{a-j} \tag{4.1.10}$$

```
Sum[ (-1)^a Binomial[a,b] (Zeta[s]^-1 -1)^a (Zeta[s]-1)^(k-b),{b,0,a}]
```

This can be generalized as

$$[(f^{-1}-1)^a]_n = \sum_{j=0}^{k} (-1)^k \binom{k}{j} [(f-1)^k \cdot (f^{-1}-1)^{a-j}]_n \tag{4.1.10}$$

4.2 Equations for $[(\zeta(s)-1)^k]_n$

$$[(\log\zeta(s))^a]_n = \sum_{k=1} \frac{(-1)^{k+1}}{k} [(\zeta(s)-1)^k \cdot (\log\zeta(s))^{a-1}]_n \tag{4.1.10}$$

The limit of this as n approaches infinity, if $\Re(s) > 1$, is

$$(\log\zeta(s))^a = \sum_{k=1} \frac{(-1)^{k+1}}{k} (\zeta(s)-1)^k (\log\zeta(s))^{a-1} \tag{4.1.10}$$

```
Sum[ (-1)^(k+1)/k (Zeta[s]-1)^k Log[ Zeta[s]]^(a-1),{k,1,Infinity}]
```

This can be generalized as

$$[(\log f)^a]_n = \sum_{k=1} \frac{(-1)^{k+1}}{k} [(f-1)^k \cdot (\log f)^{a-1}]_n \tag{4.1.10}$$

4.2 Equations for $[(\zeta(s)-1)^k]_n$

$$[\log\zeta(s)]_n = \sum_{k=0} \frac{B_k}{k!} \left[(\zeta(s)-1) \cdot \lim_{z\to 0} \frac{\partial^k}{\partial z^k} \zeta(s)^z \right]_n \tag{4.1.10}$$

The limit of this as n approaches infinity, if $\Re(s) > 1$, is

$$\log\zeta(s)=\sum_{k=0}\frac{B_k}{k!}(\zeta(s)-1)\cdot\lim_{z\to 0}\frac{\partial^k}{\partial z^k}\zeta(s)^z \tag{4.1.10}$$

This can be generalized as

$$[\log f]_n=\sum_{k=0}\frac{B_k}{k!}[(f-1)\cdot\lim_{z\to 0}\frac{\partial^k}{\partial z^k}f^z]_n \tag{4.1.10}$$

4.2 Equations for $[(\zeta(s)-1)^k]_n$

$$\lim_{z\to 0}\frac{\partial^a}{\partial z^a}[\zeta(s)^z]_n=\sum_{k=0}\frac{B_k}{k!}[(\zeta(s)-1)\cdot\lim_{z\to 0}\frac{\partial^{k+a-1}}{\partial z^{k+a-1}}\zeta(s)^z]_n \tag{4.1.10}$$

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\lim_{z\to 0}\frac{\partial^a}{\partial z^a}\zeta(s)^z=\sum_{k=0}\frac{B_k}{k!}(\zeta(s)-1)\cdot\lim_{z\to 0}\frac{\partial^{k+a-1}}{\partial z^{k+a-1}}\zeta(s)^z \tag{4.1.10}$$

This can be generalized as

$$\lim_{z\to 0}\frac{\partial^a}{\partial z^a}[f^z]_n=\sum_{k=0}\frac{B_k}{k!}[(\zeta(s)-1)\cdot\lim_{z\to 0}\frac{\partial^{k+a-1}}{\partial z^{k+a-1}}f^z]_n \tag{4.1.10}$$

4.2 Equations for $[(\zeta(s)-1)^k]_n$

$$[((\log(1+\zeta(s)))^{-1}-1)^a]_n=\sum_{k=0}[(\lim_{x\to 0}\frac{\partial^k}{\partial x^k}((\log(1+x)+1)^{-1}-1)^a)(\zeta(s)-1)^k]_n \tag{4.1.10}$$

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$((\log(1+\zeta(s)))^{-1}-1)^a=\sum_{k=0}(\lim_{x\to 0}\frac{\partial^k}{\partial x^k}((\log(1+x)+1)^{-1}-1)^a)(\zeta(s)-1)^k \tag{4.1.10}$$

This can be generalized as

$$[((\log(1+f))^{-1}-1)^a]_n=\sum_{k=0}[(\lim_{x\to 0}\frac{\partial^k}{\partial x^k}((\log(1+x)+1)^{-1}-1)^a)(f-1)^k]_n \tag{4.1.10}$$