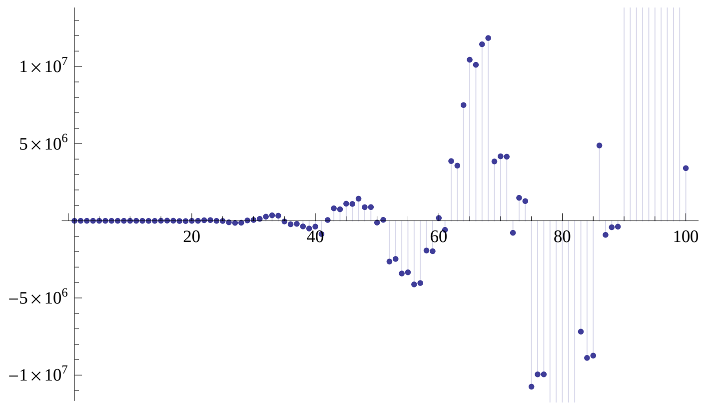
```
bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!

Clear[d2]
d2[n_, x_, k_] := d2[n, x, k] = -d2[n, x, k - 1] - x Sum[ d2[n / (j x), x, k - 1], {j, 1, n}]
d2[n_, x_, 0] := UnitStep[n - 1]
ld2[n_, x_]  := Sum[ (-1)^(k + 1) / k d2[n, x, k], {k, 1, 3 Log2@n}]
```

```
DiscretePlot[ld2[n, 1.2], {n, 1, 100}]
```




```
Table[d2[100, 1.3, k], {k, 0, 16}]
```

$\{1, -99.8, 619.41, -2620.98, 9079.25, -27745.2, 77846.1, -204231., 507862., -1.21081\times10^{6}, 2.7898\times10^{6}, -6.23944\times10^{6}, 1.35731\times10^{7}, -2.87525\times10^{7}, 5.93833\times10^{7}, -1.19772\times10^{8}, 2.36387\times10^{8}\}$

```
Sum[ ((-1)^(k + 1) / k (j / y)^k) / Log[n], {k, 1, Infinity}]
```

$$\frac{\mathrm{Log}\left[\frac{j+y}{y}\right]}{\mathrm{Log}[n]}$$

```
Sum[ ((-1)^(k + 1) / k ((j) / y)^k) / Log[n], {k, 1, Infinity}] +
 Sum[ ((-1)^(k + 1) / k ((t) / y)^k) / Log[n], {k, 1, Infinity}]
```

$$\frac{\mathrm{Log}\left[\frac{j+y}{y}\right]}{\mathrm{Log}[n]} + \frac{\mathrm{Log}\left[\frac{t+y}{y}\right]}{\mathrm{Log}[n]}$$

```
Sum[ ((-1)^(k + 1) / k (j / y)^k), {k, 1, Infinity}]
```

$$\mathrm{Log}\left[\frac{j+y}{y}\right]$$

```
Sum[ ((-1)^(k + 1) / k (n - 1)^-k), {k, 1, Infinity}]
```

$$\mathrm{Log}\left[\frac{n}{-1+n}\right]$$

```
Sum[ ((-1)^(k + 1) / k ((j) / y)^k) / Log[n] +
   ((-1)^(k + 1) / k ((t) / y)^k) / Log[n], {k, 1, Infinity}]
```

$$\frac{\mathrm{Log}\left[\frac{j+y}{y}\right] + \mathrm{Log}\left[\frac{t+y}{y}\right]}{\mathrm{Log}[n]}$$

```
Sum[ ((-1)^(k+1) / k ((j) / y)^k) + ((-1)^(k+1) / k ((t) / y)^k), {k, 1, Infinity}]
```

$$\mathrm{Log}\left[\frac{j+y}{y}\right]+\mathrm{Log}\left[\frac{t+y}{y}\right]$$

```
Sum[ ((-1)^(k+1) / k (j / y)^k) + ((-1)^(k+1) / k (t / y)^k), {k, 1, Infinity}]
```

$$\mathrm{Log}\left[\frac{j+y}{y}\right]+\mathrm{Log}\left[\frac{t+y}{y}\right]$$

```
Sum[ ((-1)^(k+1) / k ( (j / y)^k) + ( (t / y)^k)), {k, 1, Infinity}]
```

$$\frac{-t+t\,\mathrm{Log}\left[\frac{j+y}{y}\right]-y\,\mathrm{Log}\left[\frac{j+y}{y}\right]}{t-y}$$

```
Sum[ ((-1)^(k+1) / k ( (j / y)^k + (t / y)^k)), {k, 1, Infinity}]
```

$$\mathrm{Log}\left[\frac{j+y}{y}\right]+\mathrm{Log}\left[\frac{t+y}{y}\right]$$

```
Sum[ ((-1)^(k+1) / k ( (j^k / y^k) + (t^k / y^k))), {k, 1, Infinity}]
```

$$\mathrm{Log}\left[\frac{j+y}{y}\right]+\mathrm{Log}\left[\frac{t+y}{y}\right]$$

```
Sum[ (-1)^(k+1) / k ( j^k / y^k + t^k / y^k), {k, 1, Infinity}]
```

$$\mathrm{Log}\left[\frac{j+y}{y}\right]+\mathrm{Log}\left[\frac{t+y}{y}\right]$$

```
Sum[ (-1)^(k+1) / k ( (j^k + t^k) / y^k), {k, 1, Infinity}]
```

$$\mathrm{Log}\left[\frac{j+y}{y}\right]+\mathrm{Log}\left[\frac{t+y}{y}\right]$$

```
Sum[ (-1)^(k+1) / k ( n - 1)^k, {k, 1, Infinity}]
```

Log[n]

```
Grid@Table[ (-1)^(k - j) (-1)^(j - m) Bin[z, k] Binomial[k, j]
    Binomial[j, m] f[ n y^(j - k), m], {j, 0, k}, {m, 0, j}] /. k → 0
```

Table::iterb : Iterator {j, 0, k} does not have appropriate bounds. ≫

Bin[z, 0] f[n, 0]

```
Grid@Table[ (-1)^(k - j) (-1)^(j - m) Bin[z, k] Binomial[k, j]
    Binomial[j, m] f[ n y^(j - k), m], {j, 0, k}, {m, 0, j}] /. k → 1
```

Table::iterb : Iterator {j, 0, k} does not have appropriate bounds. ≫

$-\mathrm{Bin}[z,1]\ f\left[\frac{n}{y},0\right]$

-Bin[z, 1] f[n, 0] Bin[z, 1] f[n, 1]

```
Grid@Table[ (-1)^(k-j) (-1)^(j-m) Bin[z, k] Binomial[k, j]
    Binomial[j, m] f[ n y^(j-k), m], {j, 0, k}, {m, 0, j}] /. k → 2
```

Table::iterb : Iterator {j, 0, k} does not have appropriate bounds. ≫

$\text{Bin}[z, 2]\, f\left[\frac{n}{y^2}, 0\right]$		
$2\,\text{Bin}[z, 2]\, f\left[\frac{n}{y}, 0\right]$	$-2\,\text{Bin}[z, 2]\, f\left[\frac{n}{y}, 1\right]$	
$\text{Bin}[z, 2]\, f[n, 0]$	$-2\,\text{Bin}[z, 2]\, f[n, 1]$	$\text{Bin}[z, 2]\, f[n, 2]$

```
Grid@Table[ (-1)^(k-j) (-1)^(j-m) Bin[z, k] Binomial[k, j]
    Binomial[j, m] f[ n y^(j-k), m], {j, 0, k}, {m, 0, j}] /. k → 3
```

Table::iterb : Iterator {j, 0, k} does not have appropriate bounds. ≫

$-\text{Bin}[z, 3]\, f\left[\frac{n}{y^3}, 0\right]$			
$-3\,\text{Bin}[z, 3]\, f\left[\frac{n}{y^2}, 0\right]$	$3\,\text{Bin}[z, 3]\, f\left[\frac{n}{y^2}, 1\right]$		
$-3\,\text{Bin}[z, 3]\, f\left[\frac{n}{y}, 0\right]$	$6\,\text{Bin}[z, 3]\, f\left[\frac{n}{y}, 1\right]$	$-3\,\text{Bin}[z, 3]\, f\left[\frac{n}{y}, 2\right]$	
$-\text{Bin}[z, 3]\, f[n, 0]$	$3\,\text{Bin}[z, 3]\, f[n, 1]$	$-3\,\text{Bin}[z, 3]\, f[n, 2]$	$\text{Bin}[z, 3]\, f[n, 3]$

```
Grid@Table[ (-1)^(k-j) (-1)^(j-m) Bin[z, k] Binomial[k, j]
    Binomial[j, m] f[ n y^(j-k), m], {j, 0, k}, {m, 0, j}] /. k → 4
```

Table::iterb : Iterator {j, 0, k} does not have appropriate bounds. ≫

$\text{Bin}[z, 4]\, f\left[\frac{n}{y^4}, 0\right]$				
$4\,\text{Bin}[z, 4]\, f\left[\frac{n}{y^3}, 0\right]$	$-4\,\text{Bin}[z, 4]\, f\left[\frac{n}{y^3}, 1\right]$			
$6\,\text{Bin}[z, 4]\, f\left[\frac{n}{y^2}, 0\right]$	$-12\,\text{Bin}[z, 4]\, f\left[\frac{n}{y^2}, 1\right]$	$6\,\text{Bin}[z, 4]\, f\left[\frac{n}{y^2}, 2\right]$		
$4\,\text{Bin}[z, 4]\, f\left[\frac{n}{y}, 0\right]$	$-12\,\text{Bin}[z, 4]\, f\left[\frac{n}{y}, 1\right]$	$12\,\text{Bin}[z, 4]\, f\left[\frac{n}{y}, 2\right]$	$-4\,\text{Bin}[z, 4]\, f\left[\frac{n}{y}, 3\right]$	
$\text{Bin}[z, 4]\, f[n, 0]$	$-4\,\text{Bin}[z, 4]\, f[n, 1]$	$6\,\text{Bin}[z, 4]\, f[n, 2]$	$-4\,\text{Bin}[z, 4]\, f[n, 3]$	$\text{Bin}[z, 4]\, f[n, 4]$

```
Sum[ (-1)^k Binomial[z, k] , {k, 0, Infinity}] f[n, 0]
```

f[n, 0] HypergeometricPFQ[{-z}, {}, 1]

```
Sum[ (-1)^k Binomial[k, k-1] Binomial[z, k] , {k, 0, Infinity}] f[n / y, 0]
```

$-z\, f\left[\frac{n}{y}, 0\right]$ HypergeometricPFQ[{1 - z}, {}, 1]

```
Sum[ (-1)^k Binomial[k, k-2] Binomial[z, k], {k, 0, Infinity}] f[n / y^2, 0]
```

$\frac{1}{2} f\left[\frac{n}{y^2}, 0\right]$ (z HypergeometricPFQ[{1 - z}, {}, 1] - z HypergeometricPFQ[{2, 1 - z}, {1}, 1])

```
Table[Binomial[k, k - 1], {k, 0, 10}]
```

{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

```
bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
Clear[dd]
dd[n_, y_, k_] := dd[n, y, k] = Sum[ dd[Floor[n / j], y, k - 1], {j, y, n}]
dd[n_, y_, 0] := UnitStep[n - 1]
dz[n_, y_, z_] := Sum[ bin[z, k] dd[n, y, k], {k, 0, Log[y, n]}]
da[n_, y_, k_] := Sum[bin[k, j] dd[n (y^(j - k)), y + 1, j], {j, 0, k}]
db[n_, y_, j_] := Sum[(-1)^(j - m) bin[j, m] dz[n, y, m], {m, 0, j}]
dzp1[n_, y_, z_] :=
 Sum[ bin[z, k] Sum[bin[k, j] dd[n (y^(j - k)), y + 1, j], {j, 0, k}], {k, 0, Log[y, n]}]
dzm1[n_, y_, z_] := Sum[ bin[z, k]
   Sum[(-1)^(k - j) bin[k, j] dd[n ((y - 1)^(j - k)), y - 1, j], {j, 0, k}], {k, 0, Log[y, n]}]
dzp1a[n_, y_, z_] := Sum[ bin[z, k]
   Sum[bin[k, j] Sum[(-1)^(j - m) bin[j, m] dz[n (y^(j - k)), y + 1, m], {m, 0, j}],
    {j, 0, k}], {k, 0, Log[y, n]}]
dzm1a[n_, y_, z_] := Sum[ bin[z, k] Sum[(-1)^(k - j) bin[k, j] Sum[(-1)^(j - m) bin[j, m]
       dz[n ((y - 1)^(j - k)), y - 1, m], {m, 0, j}], {j, 0, k}], {k, 0, Log[y, n]}]
dzp1b[n_, y_, z_] := Sum[bin[z, k] bin[k, j] (-1)^(j - m) bin[j, m] dz[n (y^(j - k)), y + 1, m],
  {k, 0, Log[y, n]}, {j, 0, k}, {m, 0, j}]
dzm1b[n_, y_, z_] := Sum[ bin[z, k] bin[k, j] bin[j, m] (-1)^(k - m)
   dz[n ((y - 1)^(j - k)), y - 1, m], {k, 0, Log[y, n]}, {j, 0, k}, {m, 0, j}]
```

```
Expand@dz[100, 2, z]
```

$$1+\frac{428\,z}{15}+\frac{16\,289\,z^2}{360}+\frac{331\,z^3}{16}+\frac{611\,z^4}{144}+\frac{67\,z^5}{240}+\frac{7\,z^6}{720}$$

```
Expand@dz[100, 3, z]
```

$$1+\frac{341\,z}{12}+\frac{1391\,z^2}{24}+\frac{139\,z^3}{12}+\frac{z^4}{24}$$

```
Expand@dzp1[100, 2, z]
```

$$1+\frac{428\,z}{15}+\frac{16\,289\,z^2}{360}+\frac{331\,z^3}{16}+\frac{611\,z^4}{144}+\frac{67\,z^5}{240}+\frac{7\,z^6}{720}$$

```
dd[100, 2, 2]
```

283

```
da[100, 2, 2]
```

283

```
Expand@dzp1b[100, 3, z]
```

$$1+\frac{341\,z}{12}+\frac{1391\,z^2}{24}+\frac{139\,z^3}{12}+\frac{z^4}{24}$$

```
Expand@dz[100, 3, z]
```

$$1+\frac{341\,z}{12}+\frac{1391\,z^2}{24}+\frac{139\,z^3}{12}+\frac{z^4}{24}$$

```
db[100, 2, 2]
```

283

```
Binomial[k, k - 2]
```

$$\frac{1}{2}(-1+k)k$$

```
Sum[ Binomial[z, k] x^(k), {k, 0, Infinity}]
```

$$(1+x)^z$$

```
Sum[ Binomial[z, k] k x^(k - 1), {k, 0, Infinity}]
```

$$(1+x)^{-1+z}z$$

```
Sum[ Binomial[z, k] (k) (k - 1) / 2 x^(k - 2), {k, 0, Infinity}]
```

$$\frac{1}{2}(1+x)^{-2+z}(-1+z)z$$

```
Sum[ Binomial[z, k] (k) (k - 1) (k - 2) / 6 x^(k - 3), {k, 0, Infinity}]
```

$$\frac{1}{6}(1+x)^{-3+z}z\left(2-3z+z^2\right)$$

```
Sum[ Binomial[z, k] x^(k), {k, 0, Infinity}]
```

$$(1+x)^z$$

```
Sum[ Binomial[z, k] (1 + Zeta[s, a]) ^ (z - k), {k, 0, Infinity}]
```

$$(1+\mathrm{Zeta}[s,a])^z\left(\frac{2+\mathrm{Zeta}[s,a]}{1+\mathrm{Zeta}[s,a]}\right)^z$$

```
N@(2 + Zeta[2, 5]) ^2
```

4.93428

```
Binomial[k, k - 1]
```

k

```
Binomial[k, k - 2]
```

$$\frac{1}{2}(-1+k)k$$

```
Sum[ Binomial[z, k] Binomial[k, k - 2] x^(k - 2), {k, 0, Infinity}]
```

$$\frac{1}{2}(1+x)^{-2+z}(-1+z)z$$

```
Sum[ Binomial[z, k] Binomial[k, k - 3] x^(k - 3), {k, 0, Infinity}]
```

$$\frac{1}{6}\,(1+x)^{-3+z}\,z\,\left(2-3\,z+z^2\right)$$

```
Binomial[z, 1]
```

z

```
bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
Clear[dd]
dd[n_, y_, k_] := dd[n, y, k] = Sum[ dd[Floor[n / j], y, k - 1], {j, y, n}]
dd[n_, y_, 0] := UnitStep[n - 1]
dz[n_, y_, z_] := Sum[ bin[z, k] dd[n, y, k], {k, 0, Log[y, n]}]
da[n_, y_, k_] := Sum[bin[k, j] dd[n (y^(j - k)), y + 1, j], {j, 0, k}]
dza[n_, y_, z_] :=
 Sum[ bin[z, k] Sum[bin[k, j] dd[n (y^(j - k)), y + 1, j], {j, 0, k}], {k, 0, Log[y, n]}]
dzb[n_, y_, z_] :=
 Sum[ bin[z, k] (dd[n , y + 1, k] + Sum[bin[k, j] dd[n (y^(j - k)), y + 1, j], {j, 0, k - 1}]),
  {k, 0, Log[y, n]}]
dzc[n_, y_, z_] := Sum[ bin[z, k] dd[n , y + 1, k], {k, 0, Log[y, n]}] +
  Sum[ bin[z, k] Sum[bin[k, j] dd[n (y^(j - k)), y + 1, j], {j, 0, k - 1}], {k, 0, Log[y, n]}]
dzd[n_, y_, z_] := dz[n , y + 1, z] +
  Sum[ bin[z, k] Sum[bin[k, j] dd[n (y^(j - k)), y + 1, j], {j, 0, k - 1}], {k, 0, Log[y, n]}]
dze[n_, y_, z_] := dz[n , y + 1, z] +
  Sum[ bin[z, k] bin[k, k - 1] dd[n (y^(-1)), y + 1, k - 1], {k, 0, Log[y, n]}] +
  Sum[ bin[z, k] Sum[bin[k, j] dd[n (y^(j - k)), y + 1, j], {j, 0, k - 2}], {k, 0, Log[y, n]}]
dzf[n_, y_, z_] := dz[n , y + 1, z] + z dz[n / y , y + 1, z - 1] +
  Sum[ bin[z, k] Sum[bin[k, j] dd[n (y^(j - k)), y + 1, j], {j, 0, k - 2}], {k, 0, Log[y, n]}]
dzg[n_, y_, z_] := dz[n , y + 1, z] + Binomial[z, 1] dz[n / y , y + 1, z - 1] +
  Binomial[z, 2] dz[n / y^2 , y + 1, z - 2] +
  Sum[ bin[z, k] Sum[bin[k, j] dd[n (y^(j - k)), y + 1, j], {j, 0, k - 3}], {k, 0, Log[y, n]}]
dzh[n_, y_, z_] := Sum[ bin[z, k] dz[n / y^k , y + 1, z - k], {k, 0, Log[y, n]}]
dzi[n_, y_, z_] := If[ n < y, 1, Sum[ bin[z, k] dzi[n / y^k , y + 1, z - k], {k, 0, Log[y, n]}]]
dzj[n_, y_, z_] := Sum[(-1)^k bin[z, k] dz[n / (y - 1)^k , y - 1, z - k], {k, 0, Log[y - 1, n]}]
d2z[n_, y_, z_, t_] := Sum[ (-1)^k bin[z, k] dz[n , y, z - k], {k, 0, t}]
```

```
dd[100, 3, 3]
```

71

```
Expand@dz[100, 3, 3]
```

924

```
dza[100, 3, 3]
```

924

```
dzb[100, 3, 3]
```

924

```
dzc[100, 3, 3]
```

924

```
dzd[100, 3, 3]
```

924

```
dzf[100, 3, 3]
```

924

```
Expand@dzg[100, 3, 3]
```

924

```
Expand@dzh[100, 3, z]
```

$$1+\frac{341\,z}{12}+\frac{1391\,z^2}{24}+\frac{139\,z^3}{12}+\frac{z^4}{24}$$

```
Expand@dzj[100, 3, z]
```

$$1+\frac{341\,z}{12}+\frac{1391\,z^2}{24}+\frac{139\,z^3}{12}+\frac{z^4}{24}$$

```
Expand@dz[1000, 5, z]
```

$$1-\frac{49\,z}{4}+\frac{18\,821\,z^2}{24}+\frac{893\,z^3}{4}+\frac{19\,z^4}{24}$$

```
Expand@dzh[1000, 32, z]
```

$1+969\,z$

```
N[1000^(1/2)]
```

31.6228

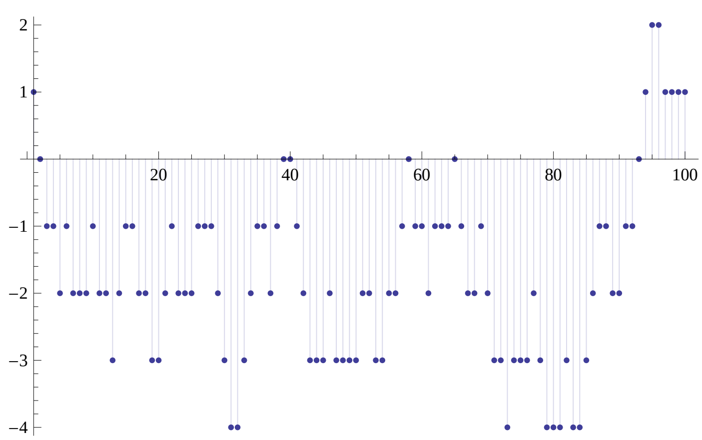
```
Expand[dzi[100, 2, z]]
```

$$1+\frac{428\,z}{15}+\frac{16\,289\,z^2}{360}+\frac{331\,z^3}{16}+\frac{611\,z^4}{144}+\frac{67\,z^5}{240}+\frac{7\,z^6}{720}$$

```
1000 - 32 + 1
```

969

```
DiscretePlot[dzi[n, 2, -1], {n, 1, 100}]
```



```
Grid@Table[Binomial[-k, j], {k, 0, 6}, {j, 0, 6}]
```

1	0	0	0	0	0	0
1	-1	1	-1	1	-1	1
1	-2	3	-4	5	-6	7
1	-3	6	-10	15	-21	28
1	-4	10	-20	35	-56	84
1	-5	15	-35	70	-126	210
1	-6	21	-56	126	-252	462