```
Cm1[x_, y_, k_] := y^-1 Sum[ Cm1[ x y / ( j + y), y, k - 1], {j, 1, x y - y }];
Cm1[ x_, y_, 0] := UnitStep[ x - 1]

Cm1[ 100, 3, 3]
```

$$\frac{14\,602}{27}$$

```
l[x_, k_, y_] := Sum[l[x / (j + y), k - 1, y], {j, 0, Floor[x - y]}];
l[x_, 1, y_] := Sum[Log[(j + y) / (y - 1)], {j, 0, x - y}]; l[x_, 0, y_] := 1


referenceRiemanPrimeCount[n_] := Sum[FullSimplify[MangoldtLambda[j] / Log[j]], {j, 2, n}]
E2[n_, k_] := Sum[(-1)^(j + 1) E2[n / j, k - 1], {j, 2, n}]; E2[n_, 0] := UnitStep[ n - 1]
RiemanPrimeCountAlt[n_] :=
 Sum[1 / k (2^k E2[n / 2^k, 0] + (-1)^(k + 1) E2[n, k]), {k, 1, Log[2, n]}]
Table[{referenceRiemanPrimeCount[n] - RiemanPrimeCountAlt[n]}, {n, 1, 100}]

{{0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0},
 {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0},
 {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0},
 {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0},
 {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0},
 {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}}

num[c_] := Numerator[c]; den[c_] := Denominator[c]
alpha[n_, c_] := den[c] (Floor[n / den[c]] - Floor[(n - 1) / den[c]]) -
   num[c] (Floor[n / num[c]] - Floor[(n - 1) / num[c]])
E1[n_, k_, c_] := (1 / den[c]) Sum[If[alpha[j, c] == 0, 0,
     alpha[j, c] E1[(den[c] n) / j, k - 1, c]], {j, 1, den[c] n}]; E1[n_, 0, c_] := 1
E1x[n_, k_, c_] := (1 / den[c]) Sum[If[alpha[j, c] == 0, 0, alpha[j, c]
      E1x[(den[c] n) / j, k - 1, c]], {j, 1, 2 den[c] n}]; E1x[n_, 0, c_] := UnitStep[ n - 1]
Table[ E1[ n, k, 3 / 2] - E1x[n, k, 3 / 2], {n, 10, 50}, {k, 1, 4}]

{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0},
 {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0},
 {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0},
 {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0},
 {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0},
 {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}
```

```
d1[n_, z_] := Product[(-1)^p[[2]] Binomial[-z, p[[2]]], {p, FI[n]}];
FI[n_] := FactorInteger[n]; FI[1] := {}
ReferenceD1[n_, z_] := Sum[d1[j, z], {j, 1, n}]
MertensReference[n_] := Sum[MoebiusMu[j], {j, 1, n}]
num[c_] := Numerator[c]; den[c_] := Denominator[c]
alpha[n_, c_] := den[c] (Floor[n / den[c]] - Floor[(n - 1) / den[c]]) -
  num[c] (Floor[n / num[c]] - Floor[(n - 1) / num[c]])
E2[n_, k_, c_] := E2[n, k, c] = (1 / den[c])
   Sum[If[alpha[j, c] == 0, 0, alpha[j, c] E2[(den[c] n) / j, k - 1, c]],
    {j, den[c] + 1, 2 den[c] n}]; E2[n_, 0, c_] := UnitStep[n - 1]
DAlt[n_, c_] := Sum[(j + 1) c^j (E2[n / c^j, 0, c] + 2 E2[n / c^j, 1, c] + E2[n / c^j, 2, c]),
  {j, 0, 2 Log[n] / Log[c]}]
MertensAlt[n_, c_] := Sum[(-1)^k (E2[n, k, c] - c E2[n / c, k, c]),
  {k, 0, 2 Floor[Log[n] / Log[c]]}]
Grid[Table[{ReferenceD1[n, 2] - DAlt[n, (b + 1) / b]}, {n, 10, 60, 10}, {b, 1, 5}]]
Grid[Table[{MertensReference[n] - MertensAlt[n, (b + 1) / b]}, {n, 10, 60, 10}, {b, 1, 4}]]
```

```
{0} {0} {0} {0} {0}
{0} {0} {0} {0} {0}
{0} {0} {0} {0} {0}
{0} {0} {0} {0} {0}
{0} {0} {0} {0} {0}
{0} {0} {0} {0} {0}
```

```
{0} {0} {0} {0}
{0} {0} {0} {0}
{0} {0} {0} {0}
{0} {0} {0} {0}
{0} {0} {0} {0}
{0} {0} {0} {0}
```

```
referenceRiemanPrimeCount[n_] := Sum[PrimePi[n^(1/k)] / k, {k, 1, Floor[Log[2, n]]}]
num[c_] := Numerator[c]; den[c_] := Denominator[c]
alpha[n_, c_] := den[c] (Floor[n / den[c]] - Floor[(n - 1) / den[c]]) -
  num[c] (Floor[n / num[c]] - Floor[(n - 1) / num[c]])
E2[n_, k_, c_] := E2[n, k, c] = (1 / den[c]) Sum[
    If[alpha[j, c] == 0, 0, alpha[j, c] E2[(den[c] n) / j, k - 1, c]], {j, den[c] + 1, 2 den[c] n}]
E2[n_, 0, c_] := UnitStep[n - 1]
P[n_, c_] := Sum[c^j / j, {j, 1, Floor[Log[n] / Log[c]]}] +
  Sum[(-1)^(k + 1) / k E2[n, k, c], {k, 1, If[c < 2, Floor[Log[n] / Log[c]], 2 Log[2, n]]}]
Table[{n, referenceRiemanPrimeCount[n], P[n, 5/2], P[n, 3/2], P[n, 4/3]}, {n, 1, 40, 5}] //
 TableForm
```

1	0	0	0	0
6	$\frac{7}{2}$	$\frac{7}{2}$	$\frac{7}{2}$	$\frac{7}{2}$
11	$\frac{19}{3}$	$\frac{19}{3}$	$\frac{19}{3}$	$\frac{19}{3}$
16	$\frac{91}{12}$	$\frac{91}{12}$	$\frac{91}{12}$	$\frac{91}{12}$
21	$\frac{115}{12}$	$\frac{115}{12}$	$\frac{115}{12}$	$\frac{115}{12}$
26	$\frac{133}{12}$	$\frac{133}{12}$	$\frac{133}{12}$	$\frac{133}{12}$
31	$\frac{161}{12}$	$\frac{161}{12}$	$\frac{161}{12}$	$\frac{161}{12}$
36	$\frac{817}{60}$	$\frac{817}{60}$	$\frac{817}{60}$	$\frac{817}{60}$

```
Cm1[x_, y_, k_] := y^-1 Sum[ Cm1[ x y / ( j + y), y, k - 1], {j, 1, x y - y }];
Cm1[ x_, y_, 0] := UnitStep[ x - 1]
```

```
bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
Ds2[ n_, s_, k_] := Sum[ j^-s Ds2[ Floor[n / j], s, k - 1], {j, 2, n}];
Ds2[n_, s_, 0] := UnitStep[n - 1]
Ds[ n_, s_, z_] := Sum[bin[ z, k] Ds2[ n, s, k], {k, 0, Log[2, n]}]
```

```
N[D[D[Ds[ 100, s, z], z], s] /. s → 0] /. z → 0
```

-94.0453

```
D[Ds[ 100, s, z], z] /. {z → 0, s → 0}
```

$$\frac{428}{15}$$

```
N[D[D[Ds[ 100, s, z], z], s] /. {s → 0, z → 0}]
```

-94.0453

```
y / (j + y)
```

$$\frac{y}{j+y}$$

```
1 / (j / y + 1)
```

$$\frac{1}{1+\frac{j}{y}}$$

```
FullSimplify[(1 + j / y) ^ -1]
```

$$\frac{y}{j+y}$$

```
Dd[n_, 0, y_] := UnitStep[n - 1]
Dd[n_, k_, y_] := Sum[Dd[n / (j + y), k - 1, y], {j, 1, n - y}]
```

```
Dd[ 100, 2, 0]
```

482

```
F2[f_, n_, k_] := F2[f, n, k] = Sum[f[j] F2[f, n / j, k - 1], {j, 2, Floor[n]}];
F2[f_, n_, 0] := UnitStep[n - 1]
bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
F1[f_, n_, z_] := Sum[bin[z, k] F2[f, n, k], {k, 0, Log[2, n]}]
f2[f_, n_, k_] := F2[f, n, k] - F2[f, n - 1, k]
f2[f_, n_, 0] := If[ n == 1, 1, 0]
f1[f_, n_, z_] := F1[f, n, z] - F1[f, n - 1, z]
LF[f_, n_, k_] := D[F1[f, n, z], {z, k}] /. z → 0
lf[f_, n_, k_] := LF[f, n, k] - LF[f, n - 1, k]
bern[ f_, n_, k_] :=
 Sum[BernoulliB[ b] / b! f2[f, j, 1] LF[ f, n / j, b + k - 1], {j, 1, n}, {b, 0, Log[2, n]}]
bern2[ f_, n_, k_] :=
 Sum[BernoulliB[ b] / b! lf[ f, j, b + k - 1] F2[f, n / j, 1], {j, 1, n}, {b, 0, Log[2, n]}]
t1[ f_, n_, k_] := Sum[ f[j] LF[ f, n / j, k], {j, 1, n}] -
  Sum[ 1 / j! LF[ f, n, k + j], {j, 0, Log[2, n]}]
F2A[ f_, n_, k_ ] := Sum[ f2[f, j, k - 1 + s]
   (D[ Series[ z / Log[1 + z], {z, 0, 20}], {z, s}] / (s) ! /. z → 0)
   LF[f, n / j, 1], {j, 1, n}, {s, 0, Log[2, n]}]
id[n_] := n
id0[n_] := 1
```

```
LF[id0, 100, 1]
```

$$\frac{428}{15}$$

```
bern2[id0, 100, 1]
```

$$\frac{428}{15}$$

```
t1[ f_, n_, k_] :=
 Sum[ f[j] LF[ f, n / j, k], {j, 1, n}] - Sum[ 1 / j! LF[ f, n, k + j], {j, 0, Log[2, n]}]
t1a[ f_, n_, k_] := Sum[ f[j] LF[ f, n / j, k], {j, 2, n}]
```

```
t1[LiouvilleLambda, 120, 4]
```

0

```
D[ Series[ z / Log[1 - z], {z, 0, 20}], {z, 1}] / 1! /. z → 0
```

$$\frac{1}{2}$$

```
F2A[id0, 100, 1]
```

99

```
Sum[BernoulliB[b] / b! (x - 1) Log[x]^(b + k - 1), {b, 0, Infinity}]
```

$\text{Log}[x]^k$

```
{x Log[x]^k, Sum[1 / (j!) Log[x]^(k + j), {j, 0, Infinity}]}
```

$\{x\,\text{Log}[x]^k,\ x\,\text{Log}[x]^k\}$

```
Sum[ Limit[ D[ x / Log[1 + x], {x, s}], x → 0] (x - 1)^{k - 1 + s} Log[x], {s, 0, Infinity}]
```

$$\left\{\sum_{s=0}^{\infty}(-1+x)^{-1+k+s}\,\text{Limit}\left[\partial_{\{x,s\}}\frac{x}{\text{Log}[1+x]},\ x\to 0\right]\text{Log}[x]\right\}$$

SeriesCoefficient::argmu : SeriesCoefficient called with 1 argument; 2 or more arguments are expected. ≫

```
ff[n_] := SeriesCoefficient[Series[ x / Log[1 + x], {x, 0, n}], n]
ff[3]
```

$\frac{1}{24}$

```
Grid[Table[Chop[(n - 1)^z -
    N[Sum[Limit[ D[ x / Log[1 + x], {x, k}], x → 0] / (k!) (n - 1)^(z + k - 1) Log[n], {k, 0, 50}]]],
  {n, 0.35, 1.75, .2}, {z, -3, 3}]]
```

0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
3.1367×10^{-10}	2.35251×10^{-10}	1.76439×10^{-10}	1.32329×10^{-10}	0	0	0

```
Limit[ D[ x / Log[1 + x], {x, 2}], x → 0]
```

$-\frac{1}{6}$

```
SeriesCoefficient[Series[ x / Log[1 + x], {x, 0, 40}], 2]
```

$-\frac{1}{12}$