```
Zeta[2]
```

$$\frac{\pi^2}{6}$$

```
Sum[j^-s, {j, 1, n}]

HarmonicNumber[n, s]

D[HarmonicNumber[n, s], n]

s (-HarmonicNumber[n, 1 + s] + Zeta[1 + s])

D[Zeta[s] - Zeta[s, n / x + 1], x] /. x → 1

-n s Zeta[1 + s, 1 + n]

D[1 / j^s - 1 / (j + n)^s - x / (x j)^s + x / (x j + n)^s, x] /. x → 1
```

$-j^{-s} + (j+n)^{-s} + j^{-s}\, s - j\,(j+n)^{-1-s}\, s$

`Sum`$\left[-j^{-s} + (j+n)^{-s} + j^{-s}\, s - j\,(j+n)^{-1-s}\, s\right.$`, {n, j, Infinity}`$\left.\right]$

$$\sum_{n=j}^{\infty}\left(-j^{-s} + (j+n)^{-s} + j^{-s}\, s - j\,(j+n)^{-1-s}\, s\right)$$

```
pa[n_, s_] := Sum[ (-1)^(j + 1) / j^s - (-1)^(j + n + 1) / (j + n)^s, {j, 1, Infinity}]
pa2[n_, s_] := Sum[ (-1)^(j + 1) / j^s, {j, 1, n}]

pa[99, 1.5]

0.765651

pa2[99, 1.5]

0.765651

Expand[Sum[ j^-s - 2 / (2 j)^s, {j, 1, Infinity}]]
```

`Zeta[s] -` 2^{1-s} `Zeta[s]`

```
Sum[ 1 / (j + n)^s - 2 / (2 j + 2 n)^s, {j, 1, Infinity}]
```

$2^{-s}\,(-2 + 2^{s})$ `HurwitzZeta[s, 1 + n]`

`FullSimplify[D[`$x^{-s}\,(-x + x^{s})$` HurwitzZeta[s, 1 + n], x]]`

$(-1 + s)\, x^{-s}$ `HurwitzZeta[s, 1 + n]`

```
D[Sum[ j^-s, {j, 1, x n}] - (1 - x^(1 - s)) Sum[ j^-s, {j, 1, n}], x] /. x → 1

(1 - s) HarmonicNumber[n, s] + n s (-HarmonicNumber[n, 1 + s] + Zeta[1 + s])

Sum[ j^-s, {j, 1, x n}]

HarmonicNumber[n x, s]

D[(-x^(1 - s)) Sum[ j^-s, {j, 1, n}], x] /. x → 1

-(1 - s) HarmonicNumber[n, s]

D[Sum[ j^-s, {j, 1, x n}], x] /. x → 1

n s (-HarmonicNumber[n, 1 + s] + Zeta[1 + s])

Sum[ j^-s, {j, 1, x n}]

HarmonicNumber[n x, s]
```

```
D[Zeta[s] - Zeta[s, n x + 1], x] /. x → 1
```

n s Zeta[1 + s, 1 + n]

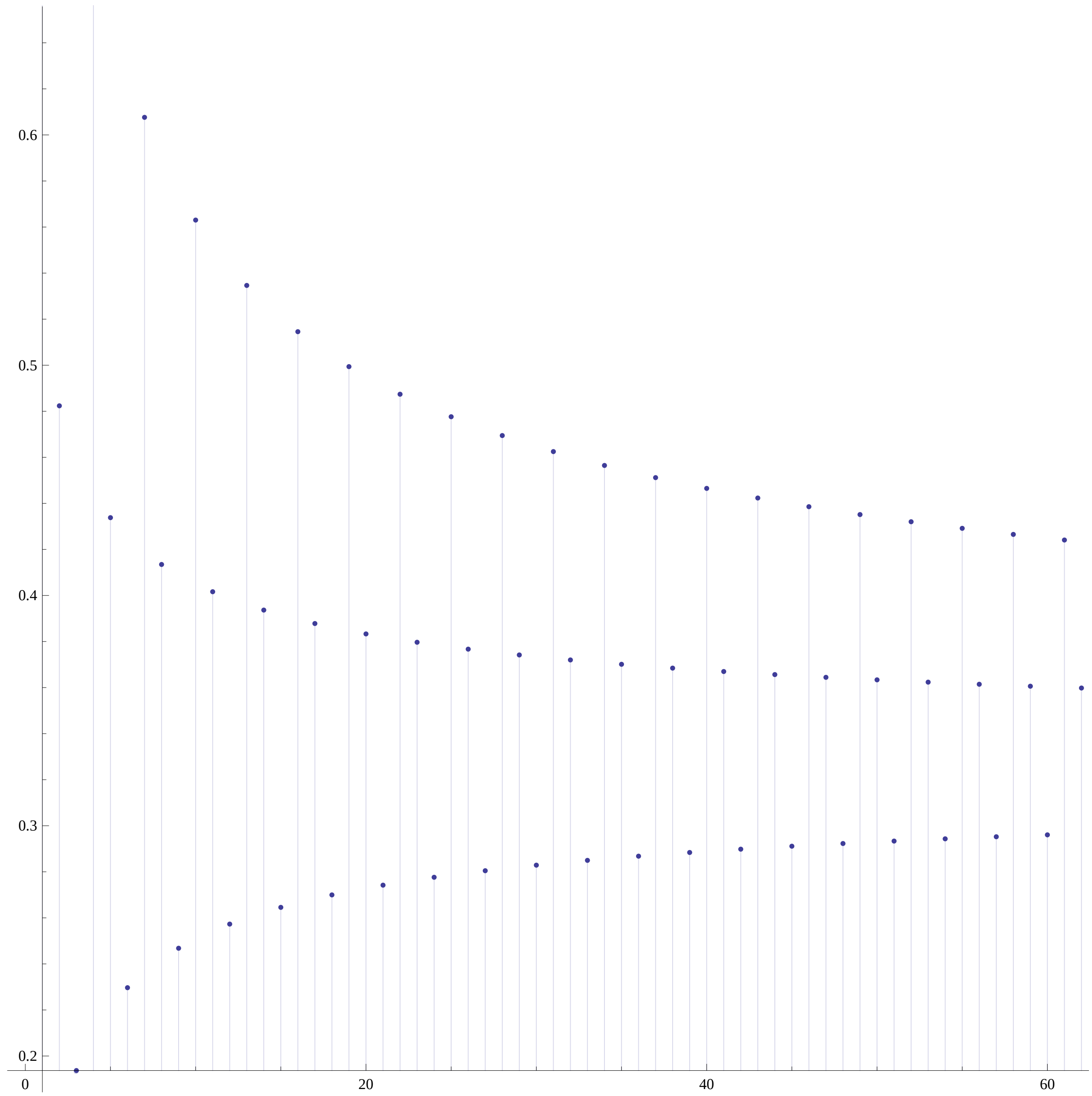
```
D[ (j + n x) ^ - s, x] /. x → 1
```

$-n\,(j+n)^{-1-s}\,s$

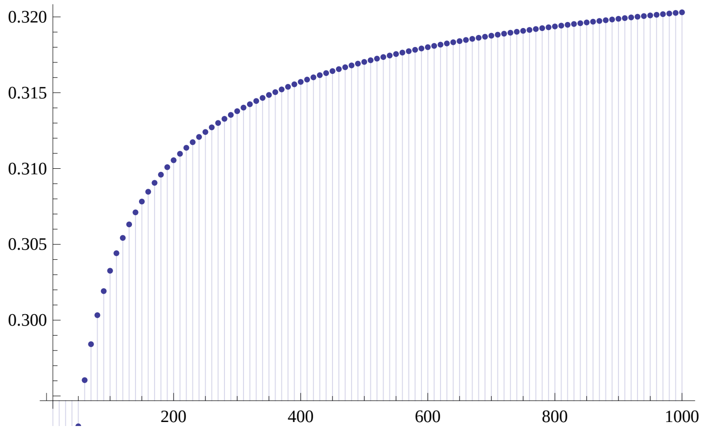
```
Sum[-n (j + n)^(-1-s) s, {n, 1, Infinity}]
```

$$\sum_{n=1}^{\infty} -n\,(j+n)^{-1-s}\,s$$

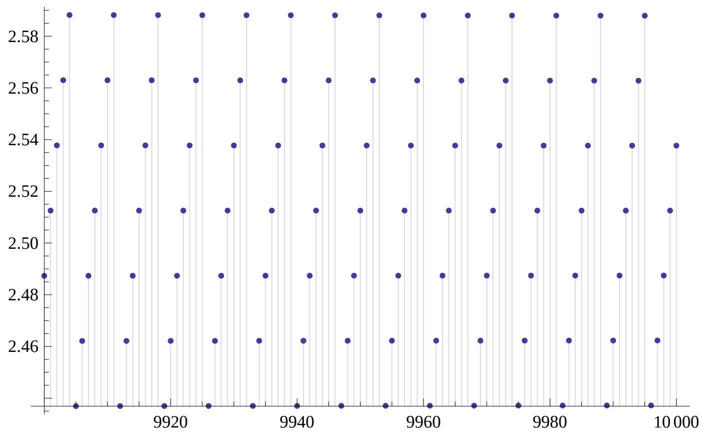
```
tx[n_, s_, k_] := Sum[ j^-s, {j, 1, n}] - k^(1 - s) Sum[ j^-s, {j, 1, Floor[n / k]}]
tx2[n_, s_, k_] := (Zeta[s] - Zeta[s, 1 + n]) - k^(1 - s) (Zeta[s] - Zeta[s, 1 + n / k])
DiscretePlot[{ tx[n, .5, 3 / 2]}, {n, 1, 100, 1}]
```



```
DiscretePlot[ tx2[n, .5, 3 / 2], {n, 10, 1000, 10}]
```




```
DiscretePlot[ {tx[n, .4, 7]}, {n, 10 000 - 100, 10 000}]
```



$$\text{HarmonicNumber}\left[\frac{n}{4}, s\right] \text{ /. } n \to 100 \text{ /. } s \to .5$$

```
8.63931

Zeta[.5] - Zeta[.5, 1 + 100 / 4]

8.63931

D[Sum[ j^-s, {j, 1, x n}], {x, 3}] /. x → 1
```

n^3 s (1 + s) (2 + s) (-HarmonicNumber[n, 3 + s] + Zeta[3 + s])

```
D[Sum[ j^-s - (j + x n)^-s, {j, 1, Infinity}], {x, 3}] /. x → 1
```

n^3 s (1 + s) (2 + s) HurwitzZeta[3 + s, 1 + n]

```
D[(-x^(1 - s)) Sum[ j^-s, {j, 1, n}], {x, 3}] /. x → 1

(-1 - s) (1 - s) s HarmonicNumber[n, s]

D[(1 - x^(1 - s)) Zeta[s], {x, 2}] /. x → 1

(1 - s) s Zeta[s]
```

```
FullSimplify[
 (D[Sum[ j^-s - (j + x n)^-s, {j, 1, Infinity}] - x^(1 - s) Sum[ j^-s,  {j, 1, n}], {x, 3}] /.
    x → 1) / ((-1 - s) (1 - s) s)]
```

$$\text{HarmonicNumber}[n, s] + \frac{n^3 (2 + s) \text{HurwitzZeta}[3 + s, 1 + n]}{-1 + s}$$

```
ff[n_, s_] := HarmonicNumber[n, s] + n^3 (2 + s) HurwitzZeta[3 + s, 1 + n] / (-1 + s)
```

```
ff[1 000 000 000, N[ZetaZero[1]]]
```

$-2.3703 \times 10^{-6} - 2.37262 \times 10^{-6}\, i$

```
Zeta[.5]
```

-1.46035

```
FullSimplify[
 (D[Sum[ j^-s - (j + x n)^-s, {j, 1, Infinity}] - x^(1 - s) Sum[ j^-s,  {j, 1, n}], {x, 2}] /.
    x → 1) / ((1 - s) s)]
```

$$\text{HarmonicNumber}[n, s] + \frac{n^2 (1 + s) \text{HurwitzZeta}[2 + s, 1 + n]}{-1 + s}$$

```
al[n_, a_, b_] := b (Floor[n / b] - Floor[(n - 1) / b]) - a (Floor[n / a] - Floor[(n - 1) / a])
ala[n_, a_, b_] := -a (Floor[n / a] - Floor[(n - 1) / a])
alb[n_, a_, b_] := b (Floor[n / b] - Floor[(n - 1) / b])
```

```
N@Sum[(1 / 3) al[n, 4, 3] / (n / 3)^2, {n, 1, 12 * 1000}]
```

0.411234

```
N@Limit[(1 - x^(1 - s)) Zeta[s], s → 2] /. x → (4 / 3)
```

0.411234

```
Table[ al[n, 4, 5], {n, 1, 20}]
```

{0, 0, 0, -4, 5, 0, 0, -4, 0, 5, 0, -4, 0, 0, 5, -4, 0, 0, 0, 1}

```
Table[(n / 5)^-2, {n, 1, 40}]
```

$$\left\{25, \frac{25}{4}, \frac{25}{9}, \frac{25}{16}, 1, \frac{25}{36}, \frac{25}{49}, \frac{25}{64}, \frac{25}{81}, \frac{1}{4}, \frac{25}{121}, \frac{25}{144}, \frac{25}{169}, \frac{25}{196}, \frac{1}{9}, \frac{25}{256}, \frac{25}{289}, \frac{25}{324}, \frac{25}{361}, \frac{1}{16}, \frac{25}{441}, \frac{25}{484}, \frac{25}{529}, \frac{25}{576}, \frac{1}{25}, \frac{25}{676}, \frac{25}{729}, \frac{25}{784}, \frac{25}{841}, \frac{1}{36}, \frac{25}{961}, \frac{25}{1024}, \frac{25}{1089}, \frac{25}{1156}, \frac{1}{49}, \frac{25}{1296}, \frac{25}{1369}, \frac{25}{1444}, \frac{25}{1521}, \frac{1}{64}\right\}$$

```
Table[ala[n, 5, 6] n^-s, {n, 1, 60}]
```

$$\{0, 0, 0, 0, -5^{1-s}, 0, 0, 0, 0, -2^{-s} 5^{1-s}, 0, 0, 0, 0, -3^{-s} 5^{1-s}, 0, 0, 0, 0, -4^{-s} 5^{1-s}, 0, 0, 0, 0, -5^{1-2s}, 0, 0, 0, 0, -5^{1-s} 6^{-s}, 0, 0, 0, 0, -5^{1-s} 7^{-s}, 0, 0, 0, 0, -5^{1-s} 8^{-s}, 0, 0, 0, 0, -5^{1-s} 9^{-s}, 0, 0, 0, 0, -2^{-s} 5^{1-2s}, 0, 0, 0, 0, -5^{1-s} 11^{-s}, 0, 0, 0, 0, -5^{1-s} 12^{-s}\}$$

```
Table[alb[n, 5, 6] (n)^-s , {n, 1, 30}]
```

$$\{0, 0, 0, 0, 0, 6^{1-s}, 0, 0, 0, 0, 0, 2^{1-2s} 3^{1-s}, 0, 0, 0, 0, 0, 2^{1-s} 3^{1-2s}, 0, 0, 0, 0, 0, 2^{1-3s} 3^{1-s}, 0, 0, 0, 0, 0, 5^{-s} 6^{1-s}\}$$

```
tt[n_, s_, a_, b_] := Sum[ (a n + j) - s, {j, 1, a}] - (a / b)^(1 - s) Sum[ (b n + j)^-s, {j, 1, b}]
```

```
tt[0, s, 2, 1]
```

$3 - 2^{1-s} - 2\,s$

```
N@Sum[(1 / 1) al[n, 3, 1] (n / 1)^-s, {n, 1, 2 * 2}]
```

$1. + 2.^{-1.\,s} - 2. \times 3.^{-1.\,s} + 4.^{-1.\,s}$

```
ala[n_, a_, b_] := -a (Floor[n / a] - Floor[(n - 1) / a])
alb[n_, a_, b_] := b (Floor[n / b] - Floor[(n - 1) / b])
ala2[n_, a_, b_] := (Floor[n / a] - Floor[(n - 1) / a])
alb2[n_, a_, b_] := (Floor[n / b] - Floor[(n - 1) / b])
f1[nn_, s_, a_, b_] := Sum[(1 / b) al[n, a, b] (n / b)^-s, {n, 1, (a b) nn}]
f2[nn_, s_, a_, b_] :=
 Sum[Sum[(1 / b) al[a b n + j, a, b] ((n a b + j) / b)^-s, {j, 1, a b}], {n, 0, nn}]
f3[nn_, s_, a_, b_] := (1 / b) Sum[Sum[ al[j, a, b] ((n a b + j) / b)^-s, {j, 1, a b}], {n, 0, nn}]
f4[nn_, s_, a_, b_] := b^(s - 1) Sum[Sum[ al[j, a, b] ((n a b + j))^-s, {j, 1, a b}], {n, 0, nn}]
f5[nn_, s_, a_, b_] := b^(s - 1) Sum[Sum[ ala[j, a, b] ((n a b + j))^-s, {j, 1, a b}] +
    Sum[ alb[j, a, b] ((n a b + j))^-s, {j, 1, a b}], {n, 0, nn}]
f6[nn_, s_, a_, b_] := b^(s - 1) Sum[Sum[ ala[j a, a, b] ((n a b + j a))^-s, {j, 1, b}] +
    Sum[ alb[j b, a, b] ((n a b + j b))^-s, {j, 1, a}], {n, 0, nn}]
f7[nn_, s_, a_, b_] := b^(s - 1) Sum[Sum[ -a ala2[j a, a, b] ((n a b + j a))^-s, {j, 1, b}] +
    Sum[ b alb2[j b, a, b] ((n a b + j b))^-s, {j, 1, a}], {n, 0, nn}]
f8[nn_, s_, a_, b_] := b^(s - 1) Sum[Sum[ -a ((n a b + j a))^-s, {j, 1, b}] +
    Sum[ b ((n a b + j b))^-s, {j, 1, a}], {n, 0, nn}]
f9[nn_, s_, a_, b_] := b^(s - 1) Sum[Sum[ -a (a^-s) ((n b + j ))^-s, {j, 1, b}] +
    Sum[ b (b^-s) ((n a + j ))^-s, {j, 1, a}], {n, 0, nn}]
f10[nn_, s_, a_, b_] := b^(s - 1) Sum[-a^(1 - s) Sum[ ((n b + j ))^-s, {j, 1, b}] +
    (b^(1 - s)) Sum[ ((n a + j ))^-s, {j, 1, a}], {n, 0, nn}]
f11[nn_, s_, a_, b_] := Sum[-a^(1 - s) (1 / b^(1 - s)) Sum[ ((n b + j ))^-s, {j, 1, b}] +
   (b^(1 - s)) (1 / b^(1 - s)) Sum[ ((n a + j ))^-s, {j, 1, a}], {n, 0, nn}]
f12[nn_, s_, a_, b_] := Sum[Sum[(n a + j)^-s, {j, 1, a}] -
   (a / b)^(1 - s) Sum[ (n b + j )^-s, {j, 1, b}], {n, 0, nn}]
```

```
N@f12[1000, 1, 3, 2]
```

0.405382

```
Sum[ j^-s - (j + a n)^-s, {j, 1, Infinity}] -
 (a / b)^(1 - s) Sum[ j^-s - (j + b n)^-s, {j, 1, Infinity}]
```

$-\text{HurwitzZeta}[s, 1 + a\,n] + \text{Zeta}[s] - \left(\frac{a}{b}\right)^{1-s} (-\text{HurwitzZeta}[s, 1 + b\,n] + \text{Zeta}[s])$

$\mathbf{D}\left[-\text{HurwitzZeta}[s, 1 + a\,n] + \text{Zeta}[s] - \left(\frac{a}{b}\right)^{1-s} (-\text{HurwitzZeta}[s, 1 + b\,n] + \text{Zeta}[s]),\ b\right]\ /.\ b \to 1$

$-a^{1-s}\,n\,s\,\text{HurwitzZeta}[1 + s, 1 + n] + a^{1-s}\,(1 - s)\,(-\text{HurwitzZeta}[s, 1 + n] + \text{Zeta}[s])$

```
d1[n_, s_, x_] := Sum[j^-s - (j + n x)^-s, {j, 1, Infinity}] - x^(1 - s) Sum[ j^-s, {j, 1, n}]
d2[n_, s_, x_] := Sum[j^-s, {j, 1, n}] - x^(1 - s) Sum[ j^-s - (j + n / x)^-s, {j, 1, Infinity}]
```

```
Table[FullSimplify[ (D[d1[n, s, x], {x, k}] /. x → 1) /
    (D[(1 - x^(1 - s)) Zeta[s], {x, k}] /. x → 1) Zeta[s]], {k, 1, 5}] // TableForm
```

$$\text{HarmonicNumber}[n,s] + \frac{n\,s\,\text{HurwitzZeta}[1+s,1+n]}{-1+s}$$
$$\text{HarmonicNumber}[n,s] + \frac{n^2\,(1+s)\,\text{HurwitzZeta}[2+s,1+n]}{-1+s}$$
$$\text{HarmonicNumber}[n,s] + \frac{n^3\,(2+s)\,\text{HurwitzZeta}[3+s,1+n]}{-1+s}$$
$$\text{HarmonicNumber}[n,s] + \frac{n^4\,(3+s)\,\text{HurwitzZeta}[4+s,1+n]}{-1+s}$$
$$\text{HarmonicNumber}[n,s] + \frac{n^5\,(4+s)\,\text{HurwitzZeta}[5+s,1+n]}{-1+s}$$

```
Table[FullSimplify[ (D[d2[n, s, x], {x, k}] /. x → 1) /
    (D[(1 - x^(1 - s)) Zeta[s], {x, k}] /. x → 1) Zeta[s]], {k, 1, 5}] // TableForm
```

$$\frac{-(-1+s)\,\text{HurwitzZeta}[s,1+n]+n\,s\,\text{HurwitzZeta}[1+s,1+n]+(-1+s)\,\text{Zeta}[s]}{-1+s}$$
$$\frac{-(-1+s)\,\text{HurwitzZeta}[s,1+n]+2\,n\,s\,\text{HurwitzZeta}[1+s,1+n]-n^2\,(1+s)\,\text{HurwitzZeta}[2+s,1+n]+(-1+s)\,\text{Zeta}[s]}{-1+s}$$
$$\frac{-(-1+s)\,\text{HurwitzZeta}[s,1+n]+3\,n\,s\,\text{HurwitzZeta}[1+s,1+n]-3\,n^2\,(1+s)\,\text{HurwitzZeta}[2+s,1+n]+n^3\,(2+s)\,\text{HurwitzZeta}[3+s,1+n]+(-1+s)\,\text{Zeta}[s]}{-1+s}$$
$$\frac{-(-1+s)\,\text{HurwitzZeta}[s,1+n]+4\,n\,s\,\text{HurwitzZeta}[1+s,1+n]-6\,n^2\,(1+s)\,\text{HurwitzZeta}[2+s,1+n]+4\,n^3\,(2+s)\,\text{HurwitzZeta}[3+s,1+n]-n^4\,(3+s)\,\text{Hurwit}}{-1+s}$$
$$\frac{-(-1+s)\,\text{HurwitzZeta}[s,1+n]+5\,n\,s\,\text{HurwitzZeta}[1+s,1+n]-10\,n^2\,(1+s)\,\text{HurwitzZeta}[2+s,1+n]+10\,n^3\,(2+s)\,\text{HurwitzZeta}[3+s,1+n]-5\,n^4\,(3+s)\,\text{Hu}}{-1+s}$$

```
dif[n_, s_, m_, k_] := n^m (s - 1 + m) (Zeta[s + m] - Sum[ 1 / (j^(s + m)), {j, 1, n}]) -
  n^k (s - 1 + k) (Zeta[s + k] - Sum[ 1 / (j^(s + k)), {j, 1, n}])
dif2[n_, s_, m_, k_] := {n^m (s - 1 + m) (Zeta[s + m] - Sum[ 1 / (j^(s + m)), {j, 1, n}]),
  n^k (s - 1 + k) (Zeta[s + k] - Sum[ 1 / (j^(s + k)), {j, 1, n}])}
```

```
FullSimplify[dif[n, s, m, k]]
```

$-n^k\,(-1+k+s)\,\text{HurwitzZeta}[k+s,1+n]+n^m\,(-1+m+s)\,\text{HurwitzZeta}[m+s,1+n]$

$-n^k\,(-1+k+s)\,\text{HurwitzZeta}[k+s,1+n]+n^m\,(-1+m+s)\,\text{HurwitzZeta}[m+s,1+n]$

$-n^k\,(-1+k+s)\,\text{HurwitzZeta}[k+s,1+n]+n^m\,(-1+m+s)\,\text{HurwitzZeta}[m+s,1+n]$

```
d3[n_, s_, m_, k_] :=
 -n^k (-1 + k + s) HurwitzZeta[k + s, 1 + n] + n^m (-1 + m + s) HurwitzZeta[m + s, 1 + n]
```

```
d3[100 000 000 000., 0, .25 + 4 I, .5 + 8 I]
```

0.124969 + 1.9993 i

```
ark[n_, x_] := Sum[ j^(-1 / 2 + x) n^-x (1 / 2 + x) - j^(-1 / 2 - x) n^x (1 / 2 - x), {j, 1, n}]
ark2[n_, s_] := (1 - s) Sum[ j^-s, {j, 1, n}] + s n^(1 - 2 s) Sum[ j^(-1 + s), {j, 1, n}]
```

```
ark[1 000 000, 14.134725141734695` i]
```

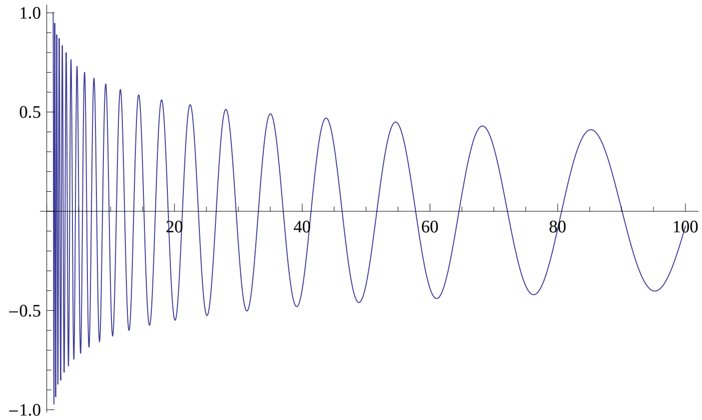
0. + 0.0141347 i

```
N[ZetaZero[1]]
```

0.5 + 14.1347 i

```
ts[n_] := n^(1 - 2 (.1 + ZetaZero[1]))
```

```
Plot[Re[ts[n]], {n, 1, 100}]
```




```
ark2[12 000, 14.134725141734695` ⅈ]
```

70 864.4 − 27 477.3 ⅈ

```
ark3[n_, s_, x_] := (1 - s) (Zeta[s] - Sum[1 / j^s, {j, 1, n}]) +
  (s - 1 + x) n^x (Zeta[s + x] - Sum[1 / j^(s + x), {j, 1, n}])
ark4[n_, s_, x_] := (Sum[1 / j^s, {j, 1, n}]) -
  (s - 1 + x) n^x / (1 - s) (Zeta[s + x] - Sum[1 / j^(s + x), {j, 1, n}])
ark5[n_, s_, x_] := {(Sum[1 / j^s, {j, 1, n}]),
  (s - 1 + x) n^x / (1 - s) (Zeta[s + x] - Sum[1 / j^(s + x), {j, 1, n}])}
ark6[n_, s_, x_] := {(1 - s) (Zeta[s] - Sum[1 / j^s, {j, 1, n}]),
  (s - 1 + x) n^x (Zeta[s + x] - Sum[1 / j^(s + x), {j, 1, n}])}
```

```
ark4[10 000 000, .5 + 3 I, 1]
```

0.532685 − 0.0789056 ⅈ

```
Zeta[.5 + 3 I]
```

0.532737 − 0.0788965 ⅈ

```
ark3[1 000 000 000, .3 + 3 I, .3]
```

−0.00023606 − 0.000183986 ⅈ

```
ark5[10 000 000, .5 + 3 I, 1]
```

{−1023.29 − 181.364 ⅈ, −1023.82 − 181.285 ⅈ}

```
ark6[10 000 000, .5 + 3 I, 1]
```

{1055.77 − 2980.83 ⅈ, −1055.77 + 2980.83 ⅈ}

```
ark3[n, s, 1 - 2 s]
```

$-n^{1-2s}\, s\, (-\text{HarmonicNumber}[n, 1-s] + \text{Zeta}[1-s]) + (1-s)\, (-\text{HarmonicNumber}[n, s] + \text{Zeta}[s])$

$\textbf{ark7}[n\_, s\_] := n^{1-2s}\, s\, \text{HarmonicNumber}[n, 1-s] + (-1+s)\, \text{HarmonicNumber}[n, s]$

$\textbf{ark7a}[n\_, s\_] := \{n^{1-2s}\, s\, \text{HarmonicNumber}[n, 1-s], (-1+s)\, \text{HarmonicNumber}[n, s]\}$

```
ark7[100 000 000 000, N[ZetaZero[1]]]
```

$-5.84131 \times 10^{-6} + 0.0000443068\, ⅈ$

$\textbf{FullSimplify}[-n^{1-2s}\, s\, (-\text{HarmonicNumber}[n, 1-s]) + (1-s)\, (-\text{HarmonicNumber}[n, s])]$

$n^{1-2s}\, s\, \text{HarmonicNumber}[n, 1-s] + (-1+s)\, \text{HarmonicNumber}[n, s]$

```
ark7a[100 000 000 000, .6 + I]
```

```
{24 639.4 - 4884.31 i, -24 638.7 + 4884.84 i}
```

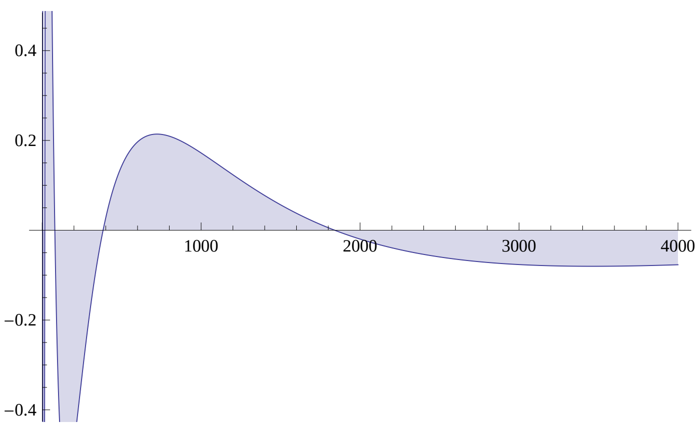
```
ark9[n_, j_, x_] := j^(-1/2+x) n^-x (1/2+x) - j^(-1/2-x) n^x (1/2-x)
ark9t[n_, j_, x_, t_] := j^(-1/2+x) n^-(x t) (1/2+x) - j^(-1/2-x) n^(x t) (1/2-x)
ark9ts[n_, x_, t_] :=
 Sum[j^(-1/2+x) n^-(x t) (1/2+x) - j^(-1/2-x) n^(x t) (1/2-x), {j, 1, n}]
```

ark10[n_, x_] := $\left(n^{-x}\left(\frac{1}{2}+x\right)\text{HarmonicNumber}\left[n,\frac{1}{2}-x\right]\right)-\left(n^{x}\left(\frac{1}{2}-x\right)\text{HarmonicNumber}\left[n,\frac{1}{2}+x\right]\right)$

ark11[n_, x_] := $\left(\left(\frac{1}{2}+x\right)\text{HarmonicNumber}\left[n,\frac{1}{2}-x\right]\right)-\left(\left(\frac{1}{2}-x\right)\text{HarmonicNumber}\left[n,\frac{1}{2}+x\right]\right)$

ark12[n_, x_] := $(n^{-x}) - (n^{x})$

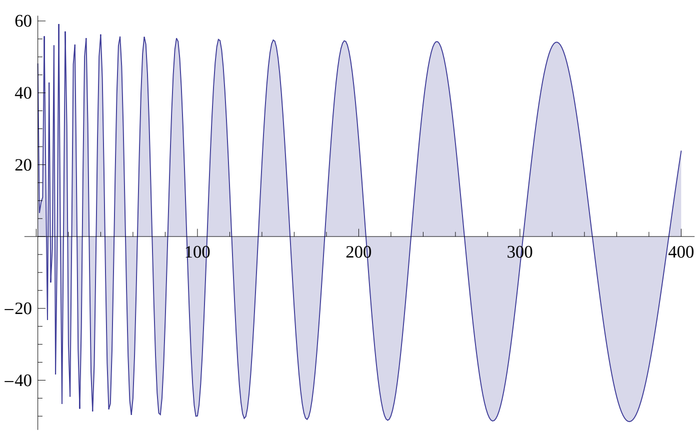
```
DiscretePlot[Re[ark9[100 000 000, j, .1 + 2 I]], {j, 1, 4000}]
```




```
n^(1 - 2 s) (n^(-1/2 + s))
```

$n^{\frac{1}{2}-s}$

```
DiscretePlot[Im[ark9ts[j, 3 I + 21.022039638771556` i, 1]], {j, 1, 400, 1}]
```




```
N[ZetaZero[2]]
```

```
0.5 + 21.022 i
```

```
Expand[ark9ts[n, x, 1]]
```

$$\frac{1}{2} n^{-x} \text{HarmonicNumber}\left[n, \frac{1}{2} - x\right] + n^{-x} x \,\text{HarmonicNumber}\left[n, \frac{1}{2} - x\right] - \frac{1}{2} n^{x} \text{HarmonicNumber}\left[n, \frac{1}{2} + x\right] + n^{x} x \,\text{HarmonicNumber}\left[n, \frac{1}{2} + x\right]$$

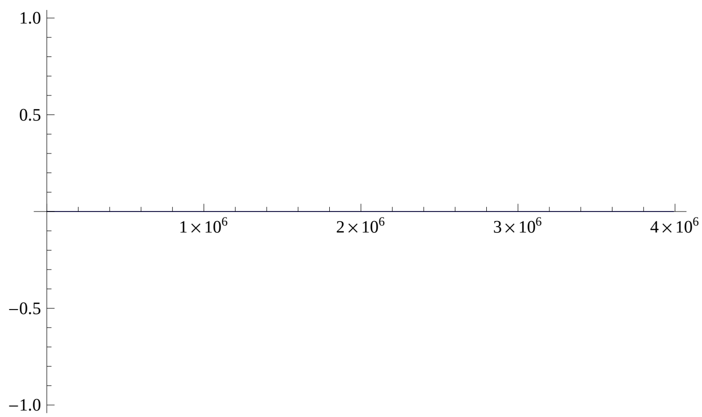
```
FullSimplify@Sum[ j^(-1/2+x) n^-(x ) (1/2+x), {j, 1, n}]
```

$$n^{-x} \left(\frac{1}{2} + x\right) \text{HarmonicNumber}\left[n, \frac{1}{2} - x\right]$$

```
FullSimplify@Sum[ j^(-1/2-x) n^(x) (1/2-x), {j, 1, n}]
```

$$n^{x} \left(\frac{1}{2} - x\right) \text{HarmonicNumber}\left[n, \frac{1}{2} + x\right]$$

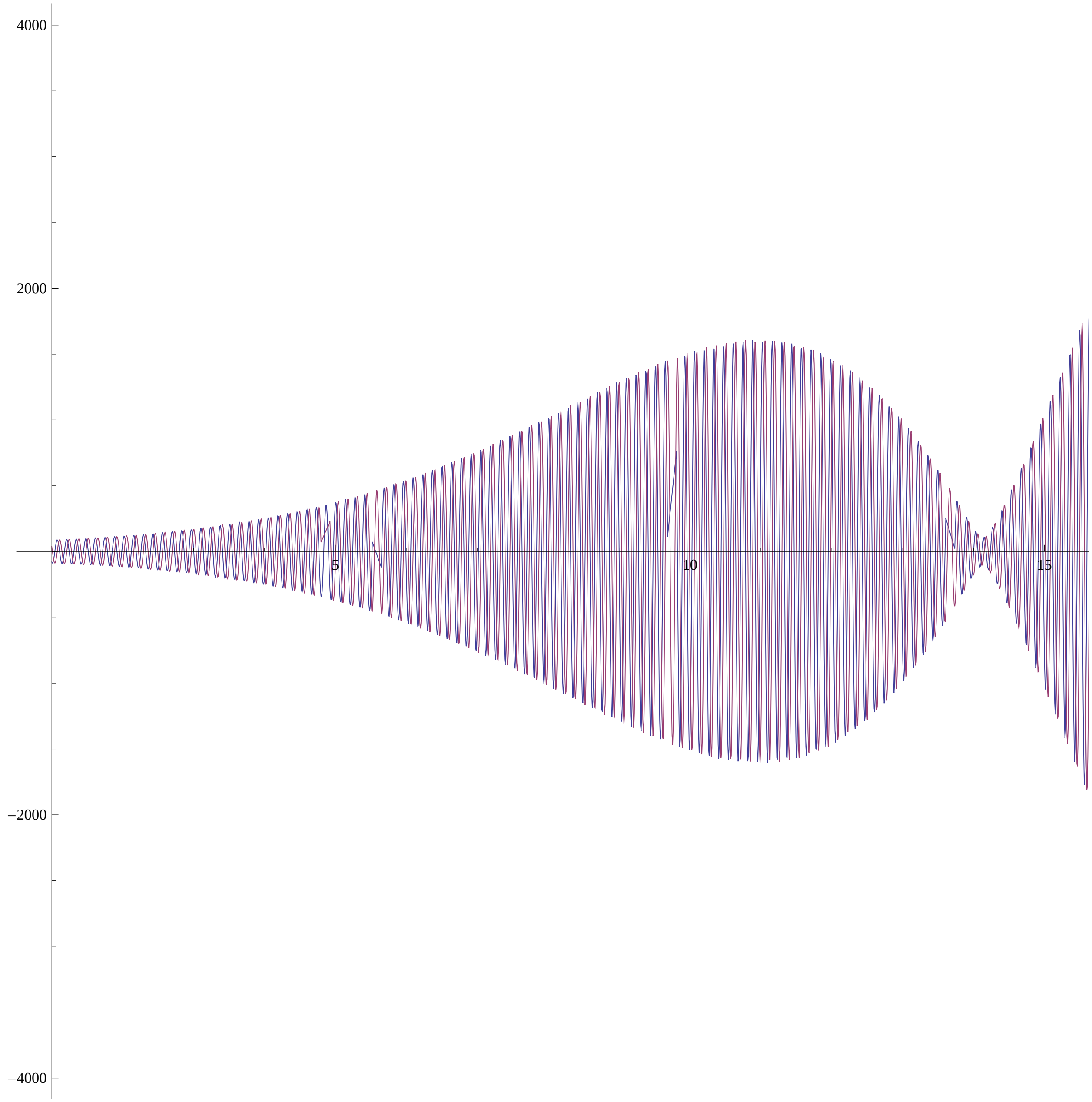
```
DiscretePlot[Re[ark10[j, 122.94682929355258` ⅈ]], {j, 1, 4 000 000, 10 000}]
```



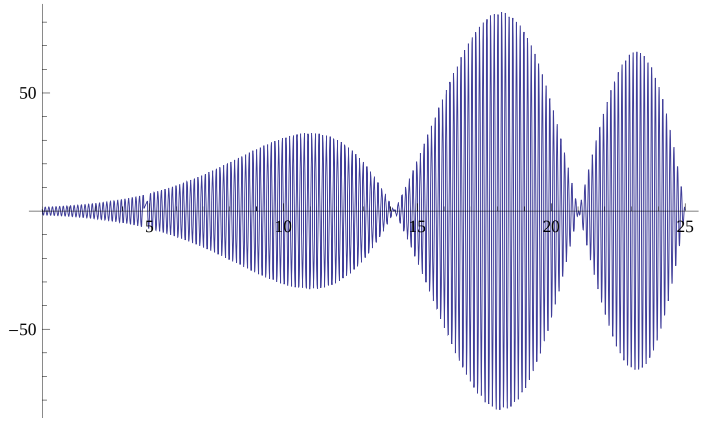

```
N[ZetaZero[40]]
```

$0.5 + 122.947\, i$

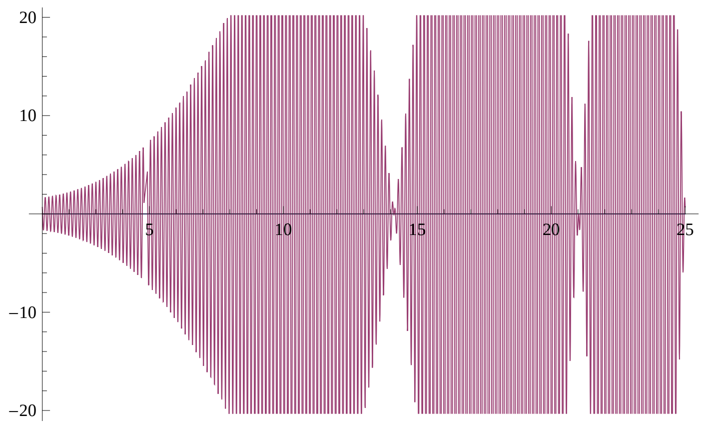
```
Plot[{Re[ark10[10^20, .1 + s I]], Im[ark10[10^20, .1 + s I]]}, {s, 1, 25}]
```



```
Plot[{Im[ark10[10^20, s I]]}, {s, 1, 25}]
```



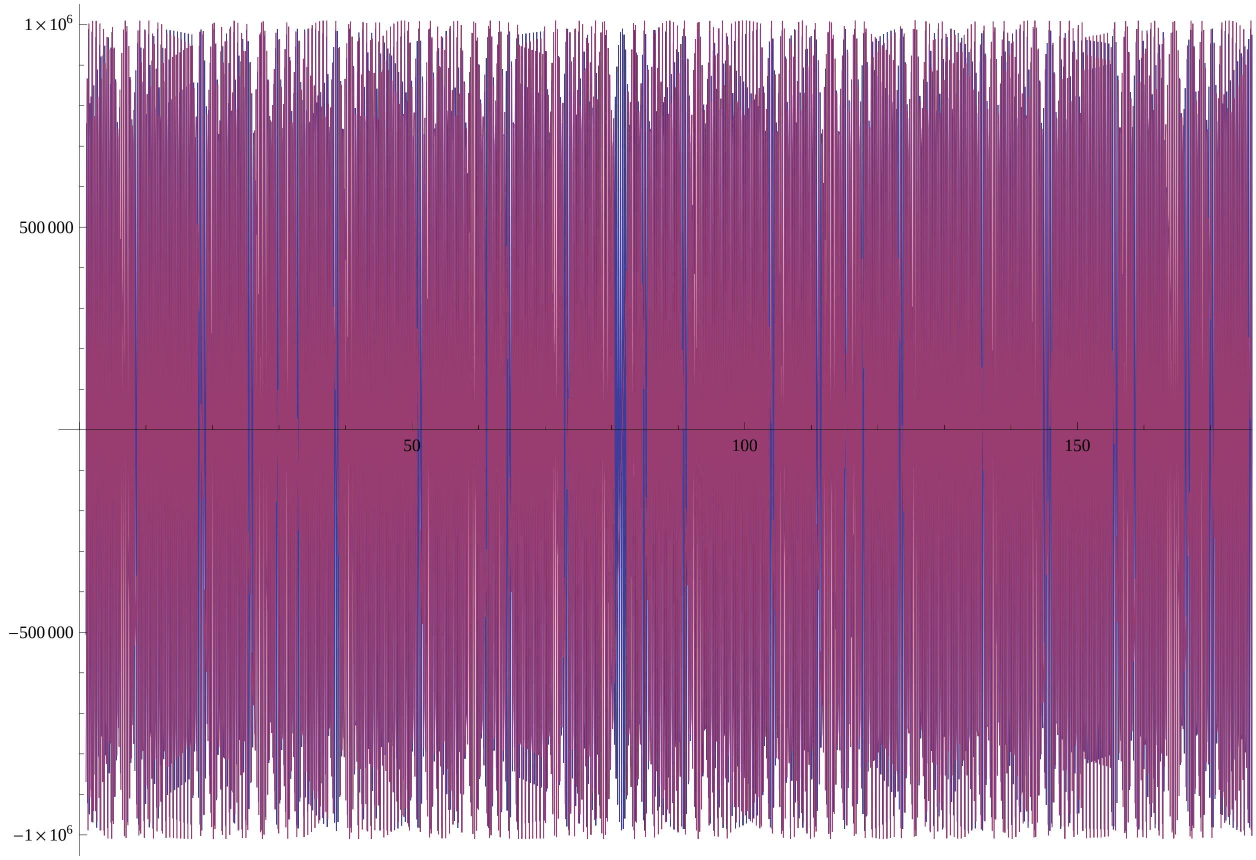

```
Plot[{Re[ark10z[10^20, s I]], Im[ark10z[10^20, s I]]}, {s, 1, 25}]
```



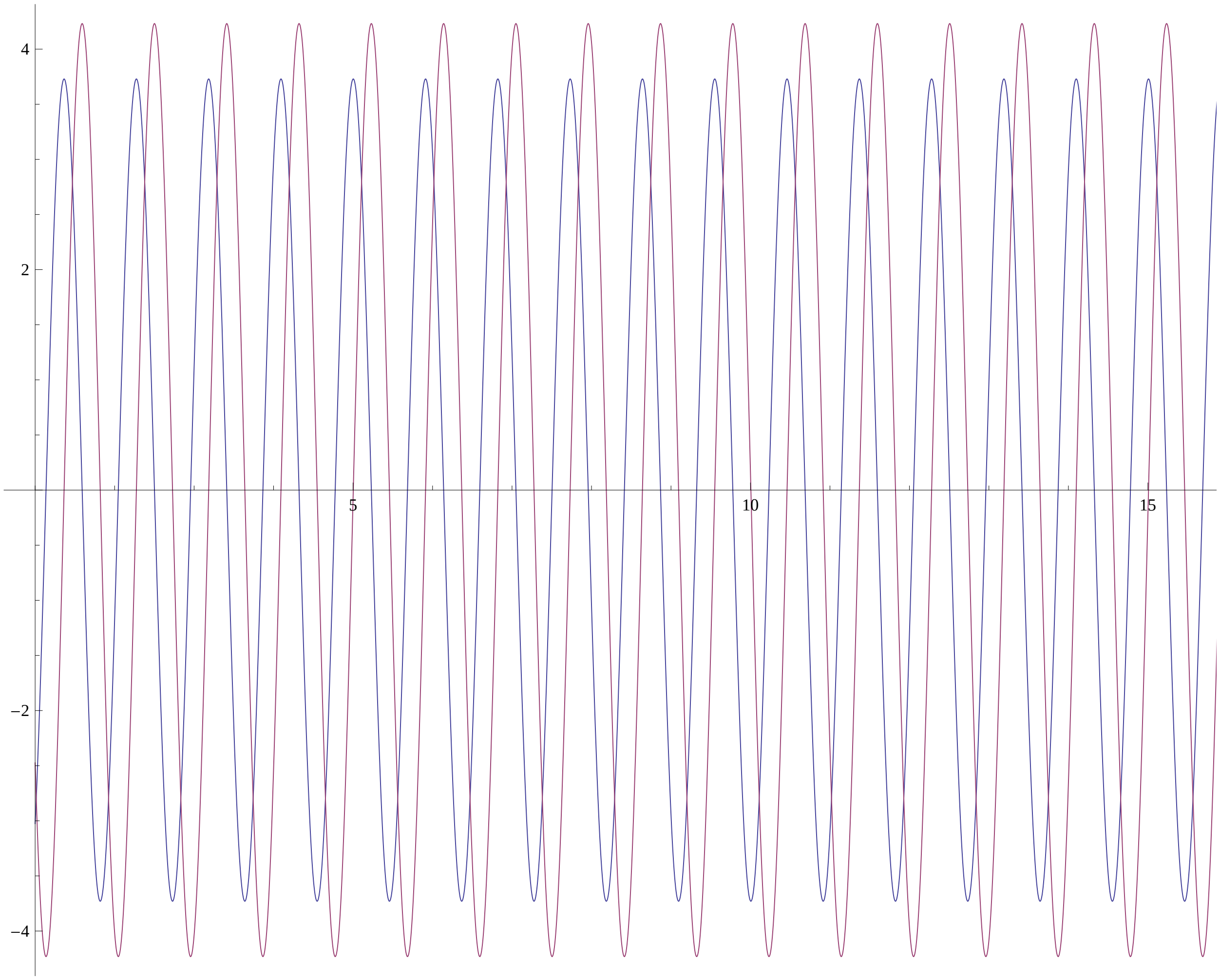

```
N[ZetaZero[2]]
```

0.5 + 21.022 ⅈ

```
Plot[{Re[ark11[10^10, .1 + s I]], Im[ark11[10^10, .1 + s I]]}, {s, 1, 200}]
```



```
Plot[{Re[ark12[10^3, .2 + s I]], Im[ark12[10^3, .2 + s I]]}, {s, 1, 20}]
```



$$\text{ark10a}[n\_, x\_] := \left(n^{-x}\left(\frac{1}{2}+x\right)\text{HarmonicNumber}\left[n, \frac{1}{2}-x\right]\right) - \left(n^{x}\left(\frac{1}{2}-x\right)\text{HarmonicNumber}\left[n, \frac{1}{2}+x\right]\right)$$

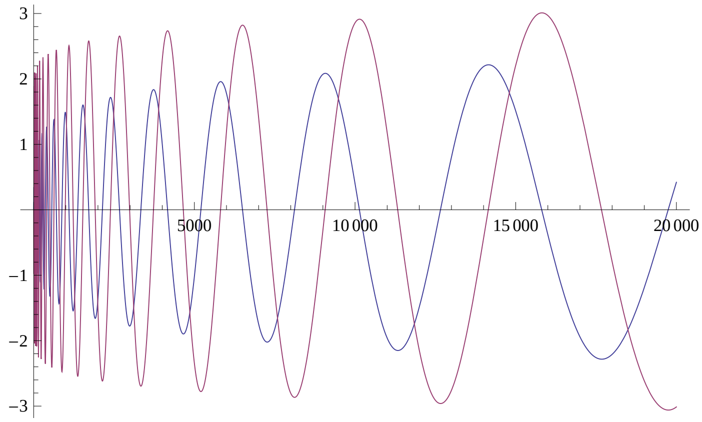
$$\text{ark11a}[n\_, x\_] := \left(\ \left(\frac{1}{2}+x\right)\text{HarmonicNumber}\left[n, \frac{1}{2}-x\right]\right) - \left(\ \left(\frac{1}{2}-x\right)\text{HarmonicNumber}\left[n, \frac{1}{2}+x\right]\right)$$

$$\text{ark12a}[n\_, x\_] := (n^{-x}\ ) - (n^{x}\ )$$

```
FullSimplify[n^-x - n^x]
```

$n^{-x} - n^{x}$

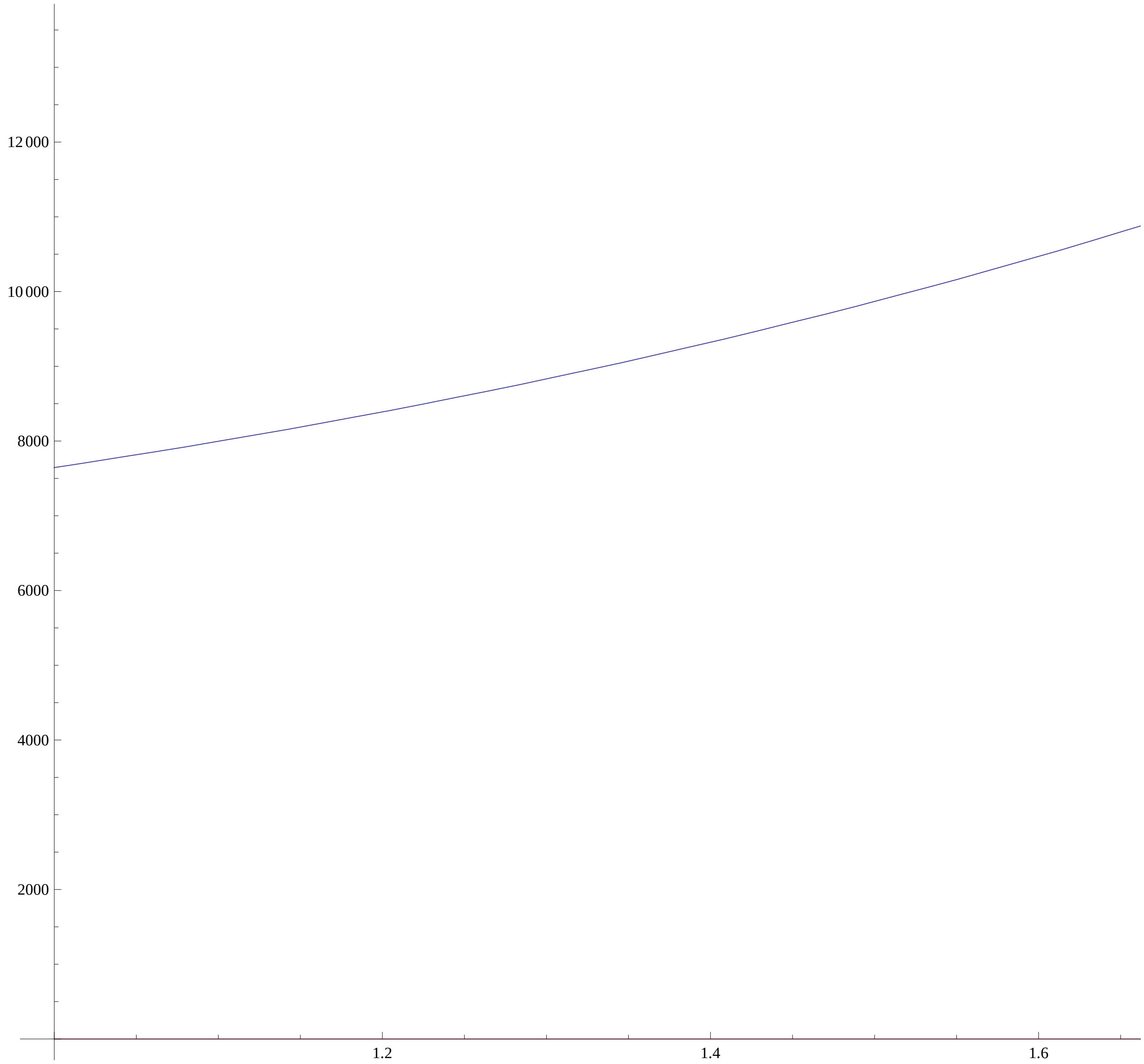
```
Plot[{Re[ark12[n, .1 + 14.134725141734695` ⅈ ]],
  Im[ark12[n, .1 + 14.134725141734695` ⅈ]]}, {n, 1, 20000}]
```




```
N[ZetaZero[1]]
```

```
0.5 + 14.1347 ⅈ
```

```
Plot[{(Re[ark10[10^20, .1 + s I]])^2 + (Im[ark10[10^20, .1 + s I]]^2), -1}, {s, 1, 2}]
```



$$\text{ark10z}[n\_, x\_] := \left(n^{-x} \left(\frac{1}{2}+x\right) \text{Zeta}\left[\frac{1}{2}-x\right]\right) - \left(n^{x} \left(\frac{1}{2}-x\right) \text{Zeta}\left[\frac{1}{2}+x\right]\right)$$

$$\text{ark10z2}[n\_, x\_] := \left(n^{-x} \left(\frac{1}{2}+x\right) \left(\text{Zeta}\left[\frac{1}{2}-x\right] - \text{HarmonicNumber}\left[n, \frac{1}{2}-x\right]\right)\right) - \left(n^{x} \left(\frac{1}{2}-x\right) \left(\text{Zeta}\left[\frac{1}{2}+x\right] - \text{HarmonicNumber}\left[n, \frac{1}{2}+x\right]\right)\right)$$

$$\text{ark10z3}[n\_, x\_] := \left(n^{-x} \left(\frac{1}{2}+x\right) \left(n\text{^}\left(1-\left(\frac{1}{2}-x\right)\right)\Big/ (1-(1/2-x))\right)\right) - \left(n^{x} \left(\frac{1}{2}-x\right) \left(n\text{^}\left(1-\left(\frac{1}{2}+x\right)\right)\Big/ (1-(1/2+x))\right)\right)$$

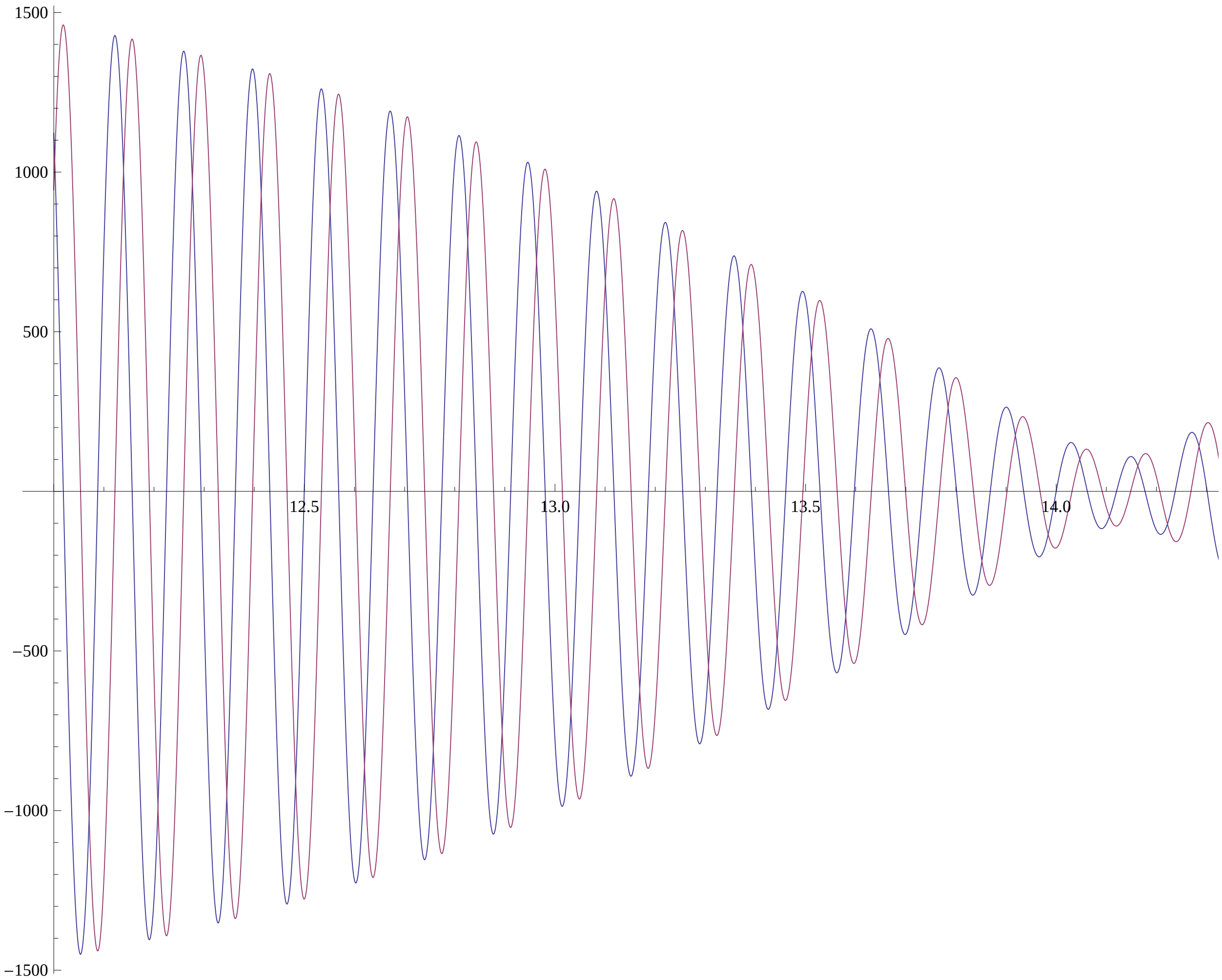
$$\text{ark10z4}[n\_, x\_] := \left(\text{Zeta}\left[\frac{1}{2}-x\right]\right) - \left(\text{Zeta}\left[\frac{1}{2}+x\right]\right)$$

$$\text{ark10z5}[n\_, x\_] := \left(\left(\frac{1}{2}+x\right) \text{Zeta}\left[\frac{1}{2}-x\right]\right) - \left(\left(\frac{1}{2}-x\right) \text{Zeta}\left[\frac{1}{2}+x\right]\right)$$

$$\text{ark10z6}[n\_, x\_] := \left(n^{-x}\, \text{Zeta}\left[\frac{1}{2}-x\right]\right) - \left(n^{x}\, \text{Zeta}\left[\frac{1}{2}+x\right]\right)$$

$$\text{ark10z7}[n\_, x\_] := \left(n^{-x}\, \text{HarmonicNumber}\left[n, \frac{1}{2}-x\right]\right) - \left(n^{x}\, \text{HarmonicNumber}\left[n, \frac{1}{2}+x\right]\right)$$

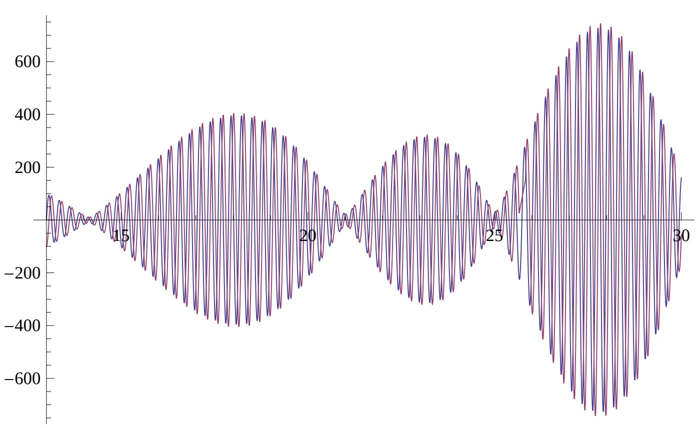
```
Plot[{Re[ark10z[10^20, .1 + s I]], Im[ark10z[10^20, .1 + s I]]}, {s, 12, 15}]
```



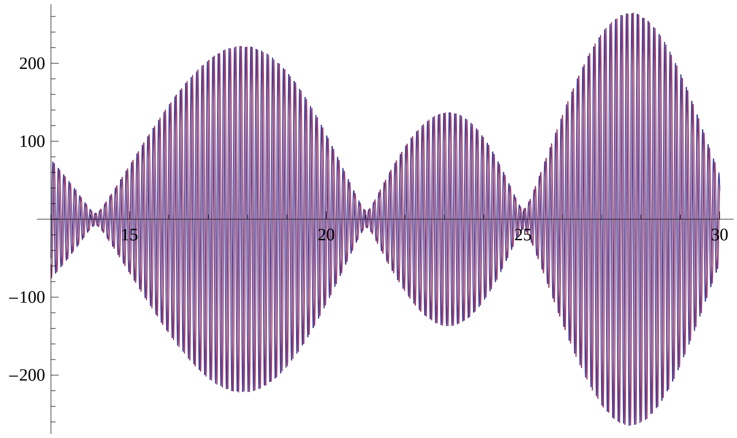
$$\left(n^{-x}\left(\frac{1}{2}+x\right)\left(n^{\wedge}\left(1-\left(\frac{1}{2}-x\right)\right)\Big/\,(1-(1/2-x))\right)\right)-\left(n^{x}\left(\frac{1}{2}-x\right)\left(n^{\wedge}\left(1-\left(\frac{1}{2}+x\right)\right)\Big/\,(1-(1/2+x))\right)\right)$$

0

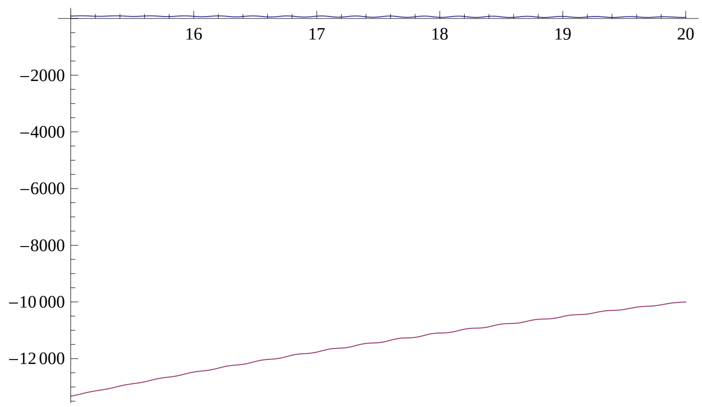
```
Plot[{Re[ark10z[10^10, .1 + s I]], Im[ark10z[10^10, .1 + s I]]}, {s, 13, 30}]
```



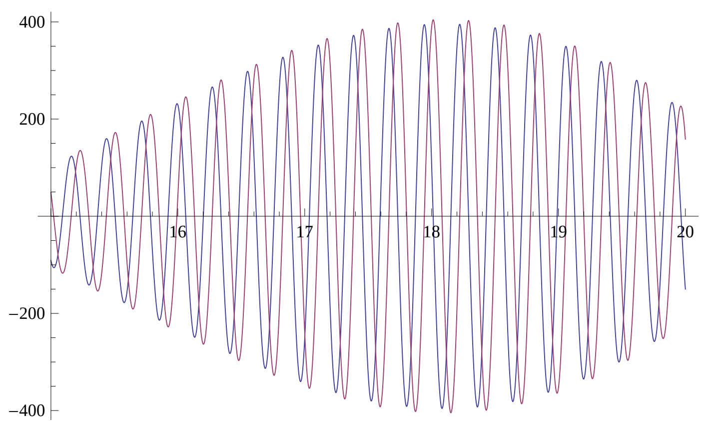
```
Plot[{Re[ark10z6[10^20, .1 + s I]], Im[ark10z6[10^20, .1 + s I]]}, {s, 13, 30}]
```




```
Plot[{Re[ark10z7[10^10, .1 + s I]], Im[ark10z7[10^10, .1 + s I]]}, {s, 15, 20}]
```



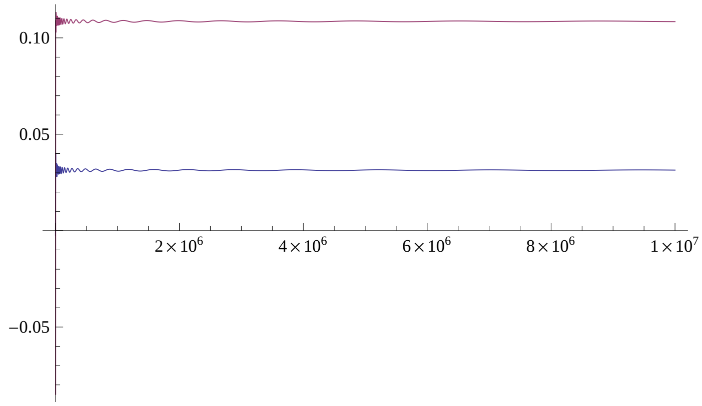

```
Plot[{Re[ark10[10^10, .1 + s I]], Im[ark10[10^10, .1 + s I]]}, {s, 15, 20}]
```



```
Plot[{Re[ark10[10^15, .1 + s I]], Im[ark10[10^15, .1 + s I]]}, {s, 15, 100}]
```


```
pa[s_] := Plot[ {Re[HarmonicNumber[n, s] - n^(1 - s) / (1 - s)],
   Im[HarmonicNumber[n, s] - n^(1 - s) / (1 - s)]}, {n, 1, 10 000 000}]
pa2[s_] := Plot[ {Re[HarmonicNumber[n, s]], Im[HarmonicNumber[n, s]]}, {n, 1, 10 000 000}]
pa3[s_] := Plot[ {Re[(Zeta[s] - HarmonicNumber[n, s]) + n^(1 - s) / (1 - s)],
   Im[(Zeta[s] - HarmonicNumber[n, s]) + n^(1 - s) / (1 - s)]}, {n, 1, 10 000 000}]
```

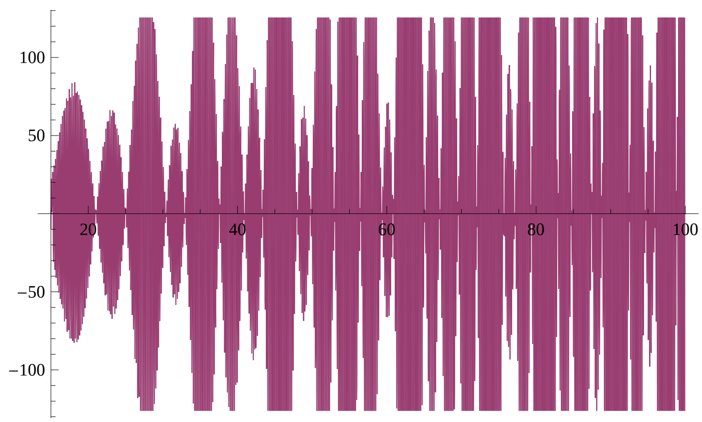
```
pa[N[ZetaZero[2] + .1 I]]
```



$$\text{ark10h1}[\text{n\_}, \text{x\_}] := \left(n^{-x}\left(\frac{1}{2}+x\right)\text{HarmonicNumber}\left[n, \frac{1}{2}-x\right]\right)$$

$$\text{ark10h2}[\text{n\_}, \text{x\_}] := \left(n^{x}\left(\frac{1}{2}-x\right)\text{HarmonicNumber}\left[n, \frac{1}{2}+x\right]\right)$$

```
Plot[{Re[ark10h1[10^15, s I] - ark10h2[10^15, s I]],
  Im[ark10h1[10^15, s I] - ark10h2[10^15, s I]]}, {s, 15, 100}]
```



$$\text{ark20}[\text{n\_}, \text{x\_}] := \left(n^{-x}\left(\frac{1}{2}+x\right)\text{HarmonicNumber}\left[n, \frac{1}{2}-x\right]\right) - \left(n^{x}\left(\frac{1}{2}-x\right)\text{HarmonicNumber}\left[n, \frac{1}{2}+x\right]\right)$$

$$\text{ark20a}[\text{n\_}, \text{x\_}] := \left\{\text{HarmonicNumber}\left[n, \frac{1}{2}-x\right], \text{HarmonicNumber}\left[n, \frac{1}{2}+x\right]\right\}$$

$$\text{ark20s}[\text{n\_}, \text{x\_}, \text{s\_}] :=$$
$$\left(n^{-x}\left(\frac{1}{2}+x\right)\text{HarmonicNumber}\left[n, \frac{1}{2}-x\right]\right) - \left(n^{x}\left(\frac{1}{2}-x\right)\text{HarmonicNumber}\left[n, \frac{1}{2}+x\right]\right)$$

$$\text{ark20z}[\text{n\_}, \text{x\_}] := \left(n^{-x}\left(\frac{1}{2}+x\right)\text{Zeta}\left[\frac{1}{2}-x\right]\right) - \left(n^{x}\left(\frac{1}{2}-x\right)\text{Zeta}\left[\frac{1}{2}+x\right]\right)$$

$$\text{ark20za}[\text{n\_}, \text{x\_}] := \left\{\text{Zeta}\left[\frac{1}{2}-x\right], \text{Zeta}\left[\frac{1}{2}+x\right]\right\}$$

$$\text{ark20f}[\text{n\_}, \text{x\_}] :=$$
$$\left\{\left\{n^{-x}, \left(\frac{1}{2}+x\right), \text{HarmonicNumber}\left[n, \frac{1}{2}-x\right]\right\}, \left\{n^{x}, \left(\frac{1}{2}-x\right), \text{HarmonicNumber}\left[n, \frac{1}{2}+x\right]\right\}\right\}$$

```
N@ark20a[10^5, 4 I]
```

{64.3166 + 45.6794 ⅈ, 64.3166 − 45.6794 ⅈ}

```
ark20za[10^20, 4.0 I]
```

{0.606784 − 0.0911121 ⅈ, 0.606784 + 0.0911121 ⅈ}

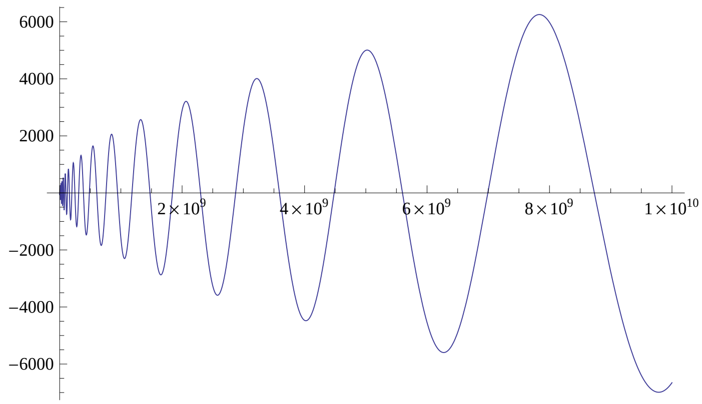
```
N@ark20f[10^5, 14.134725141734695` ⅈ] // Grid
```

0.807567 + 0.589776 ⅈ	0.5 + 14.1347 ⅈ	−12.5386 − 18.5117 ⅈ
0.807567 − 0.589776 ⅈ	0.5 − 14.1347 ⅈ	−12.5386 + 18.5117 ⅈ

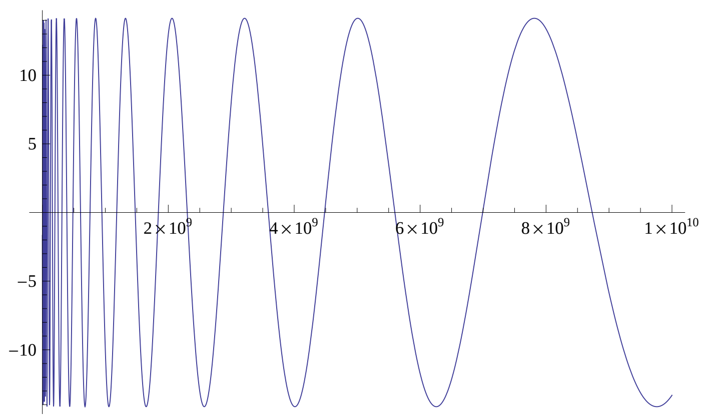
```
FullSimplify[Expand[(a + b I) (1 / 2 + x I ) (e + f I) ] - Expand[(a - b I) (1 / 2 - x I ) (e - f I) ]]
```

ⅈ (a (f + 2 e x) + b (e − 2 f x))

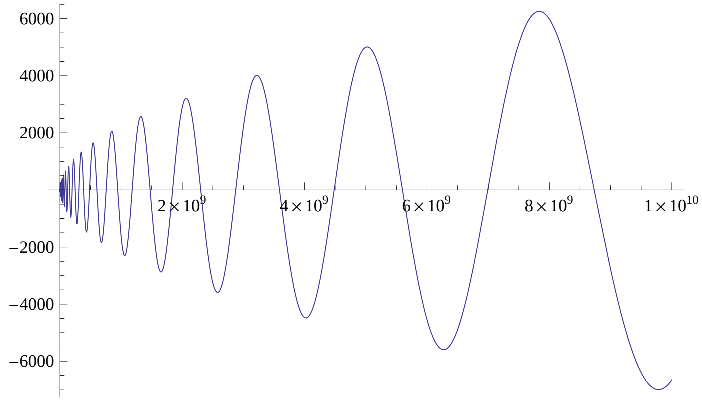
```
Plot[Re[HarmonicNumber[ n, 1 / 2 - z1]], {n, 1, 10 000 000 000}]
```




```
Plot[Re[(1 / 2 + z1) n^-z1], {n, 1, 10 000 000 000}]
```



```
Plot[Re[n^(1/2+z1)/(1/2+z1)], {n, 1, 10000000000}]
```




```
z1 = (N@ZetaZero[1] - 1/2)
```

0. + 14.1347 i

```
(n^(1/2+z)/(1/2+z)) ((1/2+z) n^-z)
```

$\sqrt{n}$

ark40[n_, x_] := $\left(n^{-x}\left(\frac{1}{2}+x\right)\text{HarmonicNumber}\left[n, \frac{1}{2}-x\right]\right)$

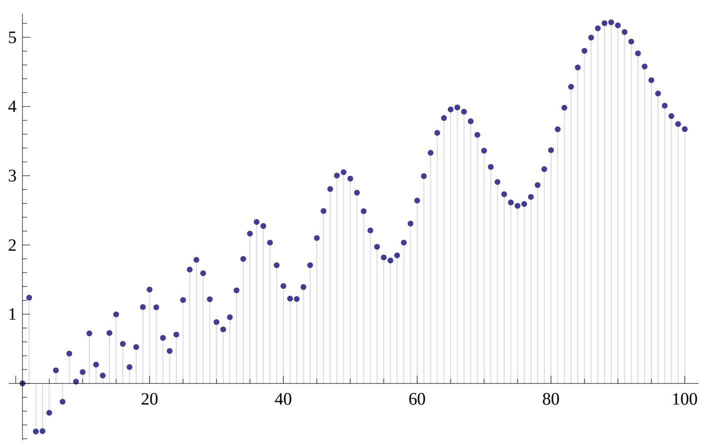
ark41[n_, x_] := $\left(n^{x}\left(\frac{1}{2}-x\right)\text{HarmonicNumber}\left[n, \frac{1}{2}+x\right]\right)$

ark41b[n_, x_] := $\left(n^{x}\ \text{HarmonicNumber}\left[n, \frac{1}{2}+x\right]\right)$

ark40a[n_, x_] := $\left(n^{-x}\left(\frac{1}{2}+x\right)\left(\text{HarmonicNumber}\left[n, \frac{1}{2}-x\right]-\text{Zeta}[1/2-x]\right)\right)$

```
ark42[n_, s_] := Sum[ (n/j)^s / (j^(1/2)), {j, 1, n}]
ark42a[n_, s_] := Sum[ (n/j)^s, {j, 1, n}]
ark42b[n_, s_] := Table[ (n/j)^s, {j, 1, n}]
ark45[n_, x_] := (n^x HarmonicNumber[n, x])
```

```
DiscretePlot[Im[ark45[n, .1 I + N[ZetaZero[2]]]], {n, 1, 100}]
```



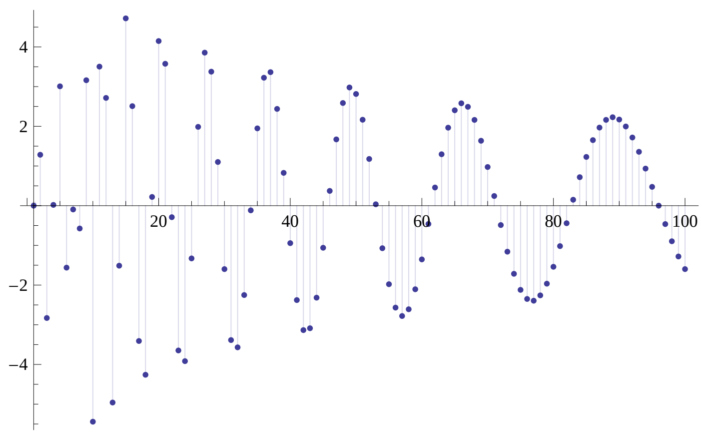
ark42b[**30, s**]

$\left\{30^s, 15^s, 10^s, \left(\frac{15}{2}\right)^s, 6^s, 5^s, \left(\frac{30}{7}\right)^s, \left(\frac{15}{4}\right)^s, \left(\frac{10}{3}\right)^s, 3^s, \left(\frac{30}{11}\right)^s, \left(\frac{5}{2}\right)^s, \left(\frac{30}{13}\right)^s, \left(\frac{15}{7}\right)^s, 2^s, \left(\frac{15}{8}\right)^s, \left(\frac{30}{17}\right)^s, \left(\frac{5}{3}\right)^s, \left(\frac{30}{19}\right)^s, \left(\frac{3}{2}\right)^s, \left(\frac{10}{7}\right)^s, \left(\frac{15}{11}\right)^s, \left(\frac{30}{23}\right)^s, \left(\frac{5}{4}\right)^s, \left(\frac{6}{5}\right)^s, \left(\frac{15}{13}\right)^s, \left(\frac{10}{9}\right)^s, \left(\frac{15}{14}\right)^s, \left(\frac{30}{29}\right)^s, 1\right\}$

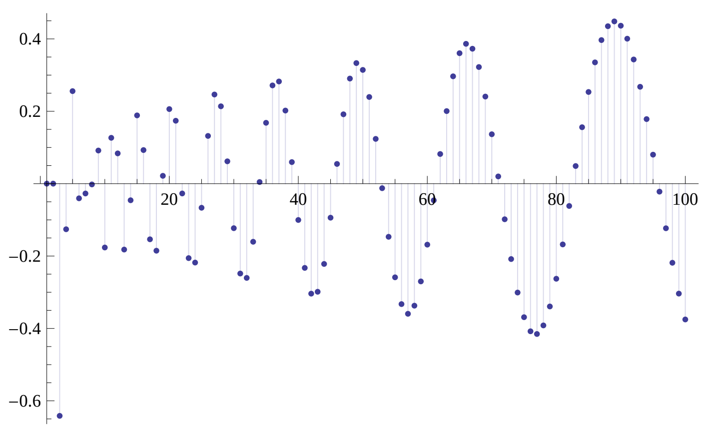
ark42b[**31, s**]

$\left\{31^s, \left(\frac{31}{2}\right)^s, \left(\frac{31}{3}\right)^s, \left(\frac{31}{4}\right)^s, \left(\frac{31}{5}\right)^s, \left(\frac{31}{6}\right)^s, \left(\frac{31}{7}\right)^s, \left(\frac{31}{8}\right)^s, \left(\frac{31}{9}\right)^s, \left(\frac{31}{10}\right)^s, \left(\frac{31}{11}\right)^s, \left(\frac{31}{12}\right)^s, \left(\frac{31}{13}\right)^s, \left(\frac{31}{14}\right)^s, \left(\frac{31}{15}\right)^s, \left(\frac{31}{16}\right)^s, \left(\frac{31}{17}\right)^s, \left(\frac{31}{18}\right)^s, \left(\frac{31}{19}\right)^s, \left(\frac{31}{20}\right)^s, \left(\frac{31}{21}\right)^s, \left(\frac{31}{22}\right)^s, \left(\frac{31}{23}\right)^s, \left(\frac{31}{24}\right)^s, \left(\frac{31}{25}\right)^s, \left(\frac{31}{26}\right)^s, \left(\frac{31}{27}\right)^s, \left(\frac{31}{28}\right)^s, \left(\frac{31}{29}\right)^s, \left(\frac{31}{30}\right)^s, 1\right\}$

DiscretePlot[**Im**[**n^**(**N**[**ZetaZero**[**2**]]) - (**n - 1**) **^** (**N**[**ZetaZero**[**2**]])], {**n, 1, 100**}]



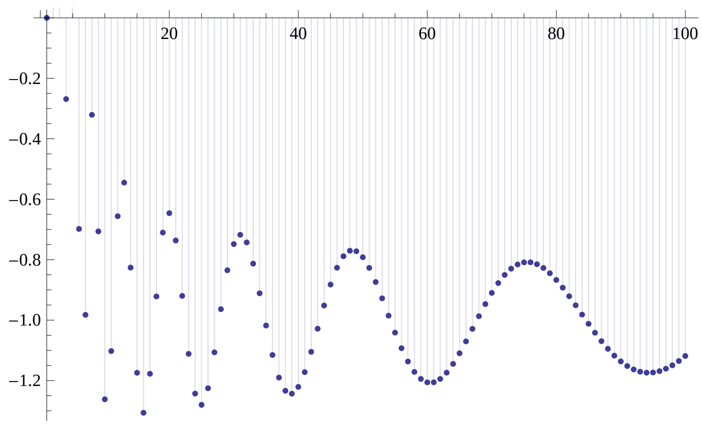
DiscretePlot[**Im**[**HarmonicNumber**[**n - 1, N**[**ZetaZero**[**2**]]]], {**n, 1, 100**}]



```
teo[z_] := DiscretePlot[ Im[ ( n^z) (HarmonicNumber[n, z]) ], {n, 1, 100}]
te[z_] := DiscretePlot[ Im[ ( n^z - (n - 1)^z) (HarmonicNumber[n - 1, z]) ], {n, 1, 100}]
te2[z_] := DiscretePlot[ Re[ (n^(-1+z) z ) (HarmonicNumber[n - 1, z]) ], {n, 1, 100}]
te3[s_] := DiscretePlot[Re[Sum[ (n / j)^s , {j, 1, n}]], {n, 1, 100}]
te3x[m_] := Animate[DiscretePlot[
   Re[ m Sum[ (n / j)^(.5 + t I) , {j, m, n, m}]], {n, 1, 100}, PlotRange → 8], {t, 14, 15}]
te3y[m_] := Animate[DiscretePlot[Re[ n / (-.5 + t I)], {n, 1, 100}, PlotRange → 8], {t, 14, 15}]
```

```
te3x[1 / 2]
```

```
te2[N[ZetaZero[1] + .1]]
```

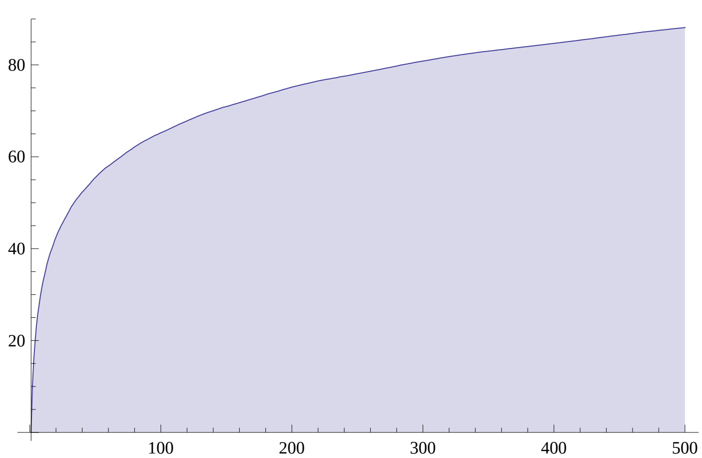



```
D[n^z, n]
```

$n^{-1+z} z$

```
Clear[da]
bin[z_, k_] := bin[z, k] = Product[z - j, {j, 0, k - 1}] / k!
da[n_, s_, k_] := da[n, s, k] = Sum[ j^-s da[Floor[n / j], s, k - 1], {j, 2, n}]
da[n_, s_, 0] := UnitStep[n - 1]
dz[n_, s_, z_] := Sum[ bin[z, k] da[n, s, k], {k, 0, Log2@n}]
te4[z_, x_] := DiscretePlot[ Re[ n^(x z) (dz[n, z, x]) ], {n, 1, 300}]
te5[z_, x_] := DiscretePlot[ Re[ (dz[n, z, x]) ], {n, 1, 300}]
te6[s_] := DiscretePlot[ Im[D[ n^(z s) (dz[n, s, z]), z] /. z → 0 ], {n, 1, 500}]
```

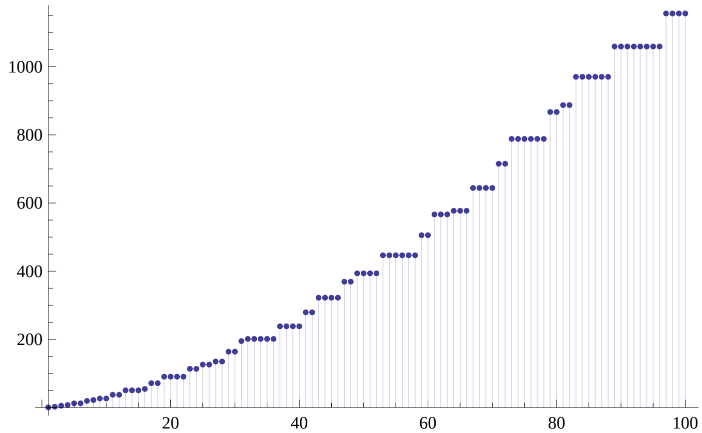
```
te6[N[ZetaZero[1]]]
```



D[100^(z s) (dz[100, s, z]), z] /. z → 0 /. s → 1

$$\frac{292\,149\,953\,504\,274\,361\,788\,974\,787\,095\,433\,526\,022\,627}{139\,440\,750\,459\,424\,954\,329\,067\,617\,870\,624\,607\,113\,600} + \text{Log}[100]$$

DiscretePlot[D[dz[n, -1, z], z] /. z → 0, {n, 1, 100}]



D[(dz[100, s, z]), z] /. z → 0 /. s → 1

$$\frac{292\,149\,953\,504\,274\,361\,788\,974\,787\,095\,433\,526\,022\,627}{139\,440\,750\,459\,424\,954\,329\,067\,617\,870\,624\,607\,113\,600}$$

N[ZetaZero[1]]

0.5 + 14.1347 ⅈ

Limit[m Sum[(n / j)^z , {j, m, n, m}], m → 0]

$$\text{Limit}\left[m\,\text{Sum}\left[\left(\frac{n}{j}\right)^z, \{j, m, n, m\}\right], m \to 0\right]$$

Integrate[(n / j)^z, {j, 0, n}]

$$\text{ConditionalExpression}\left[-\frac{n}{-1+z}, \text{Re}[z] < 1 \,\&\&\, n > 0\right]$$