
So, I'm going to introduce a deceptively simple-seeming function. Here it is:

$$f(n,x)=\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\sin(x\cdot\log j)$$

Its looks rather similar to a Fourier series of the form $\sum_{j=1}^{\infty} a_j\cdot\sin(x\cdot j)$. Thus, you might suspect it has some properties in common.

Well, enough pondering. We have mighty workhorses of computation at our disposal. Let's take a look! So let's graph it for 4 values of x.

Here it is for x=10.

And here it is for x=100.

And for x=1000.

And, finally, let's look at one last value, a seemingly arbitrary choice, x = 1419.4224809459956.

I want to look at a little more closely at these last two values. Let's extend our sum just a bit to help out.

$$f(n,x,\theta)=\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\sin(x\cdot\log j+\theta)$$

asdf

I want to describe here what I'm seeing.

Ok. So hopefully you've played around enough to be convinced that, in a certain sense, $\sum_{j=1}^{\infty}\frac{1}{\sqrt{j}}\cdot\sin(x\cdot\log j+\theta)$ is centered on the origin at certain special frequencies for x. And those special frequencies appear to be equivalent to the zeros of the Riemann zeta function. And what's maybe a little more interesting is that it's centered on those frequencies for any value, real or complex, of θ .

But why should that be? I mean, visually, empirically, it seems to be true. But what is going on in that sum to cause it to happen? Why those frequencies?

Well, one way to look more closely at what's going on would be to approximate the sum with an integral and see how they differ. So, taking a page from Euler-Maclaurin summation, you might guess that our sum ought to be in the ballpark of

$$f(n,x,\theta)=\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\sin(x\log j+\theta)\approx\int_{1}^{n}\frac{1}{\sqrt{j}}\cdot\sin(x\log j+\theta)\,dj-\frac{1}{2}\left(\frac{\sin(x\log n+\theta)}{\sqrt{n}}\right)$$

Working through the math of that integral, we end up with

$$f(n,x,\theta)\approx-\frac{2\sqrt{n}}{1+4x^{2}}\cdot(2x\cos(x\log n+\theta)-\sin(x\log n+\theta))+\frac{2}{1+4x^{2}}(2x\cos(\theta)-\sin\theta)+\frac{1}{2}\left(\frac{\sin(x\log n+\theta)}{\sqrt{n}}\right)$$

$$\boxed{\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\left(2x\cos\left(x\cdot\log\frac{j}{n}\right)+\sin\left(x\cdot\log\frac{j}{n}\right)\right)}$$
