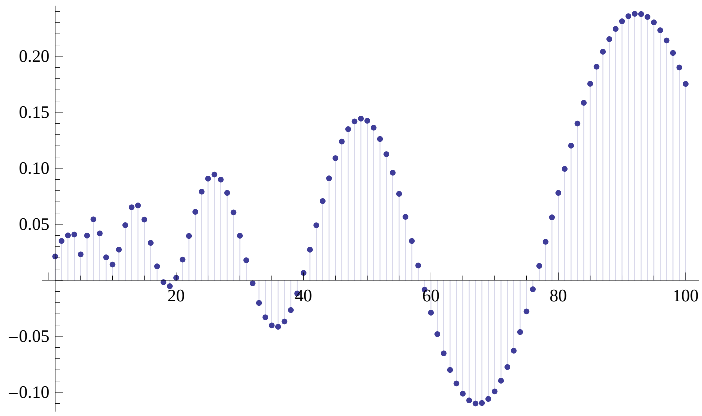
```
po[n_, s_] := 1 / 2 s (s - 1) Pi^(-s / 2) Gamma[s / 2] HarmonicNumber[n, s]
pl[n_, s_] := 1 / 2 s (s - 1) Pi^(-s / 2) Gamma[s / 2] Zeta[s]
```

```
DiscretePlot[Re@po[n, 10 I], {n, 1, 100}]
```




```
FullSimplify[Limit[po[n, s], s → 0]]
```

-n

```
p1[n_, s_] :=
 (1 / 2 + s) Sum[(n / j)^(1 / 2 - s), {j, 1, n}] - (1 / 2 - s) Sum[(n / j)^(1 / 2 + s), {j, 1, n}]
p2[n_, s_] := Sum[(n / j)^(1 / 2) (2 s Cosh[s Log[n / j]] - Sinh[s Log[n / j]]), {j, 1, n}]
```

```
p2[100, .2 + 8 I]
```

117.705 + 260.386 i

```
p1[100, .2 + 8 I]
```

117.705 + 260.386 i

```
p3[n_, s_] :=
 (1 / 2 + s) Sum[(n / j)^(1 / 2 - s), {j, 1, n}] + (1 / 2 - s) Sum[(n / j)^(1 / 2 + s), {j, 1, n}]
p4[n_, s_] := Sum[(n / j)^(1 / 2) (Cosh[s Log[n / j]] - 2 s Sinh[s Log[n / j]]), {j, 1, n}] - 2 n
```

```
N@p4[10000, -.5 + ZetaZero[1]]
```

0.496666 + 0. i

```
N@p4[10000, -2]
```

-2.01223×10^{10}

```
p3[100, .2 + 8 I]
```

45.4413 - 176.352 i

```
z1[n_, s_] := -2 n + (1 / 2 + s) (-Sum[(n / j)^(1 / 2 - s), {j, 1, n}]) +
  (1 / 2 - s) (-Sum[(n / j)^(1 / 2 + s), {j, 1, n}])
```

```
N@z1[10000, -ZetaZero@1]
```

5990.79 - 3199.43 i

```
h1[n_, s_] := -n + (1 + s) (Sum[(n / j) ^ (-s), {j, 1, n}] - n^-s Zeta[-s])
h2[n_, s_] :=
 -n + (1 + (s - 1 / 2)) (Sum[(n / j) ^ (- (s - 1 / 2)), {j, 1, n}] - n^- (s - 1 / 2) Zeta[- (s - 1 / 2)])
h3[n_, s_] := -n + (1 / 2 + s) (Sum[(n / j) ^ (1 / 2 - s), {j, 1, n}] - n^ (1 / 2 - s) Zeta[1 / 2 - s])
h4[n_, s_] := -n + (1 / 2 - s) (Sum[(n / j) ^ (1 / 2 + s), {j, 1, n}] - n^ (1 / 2 + s) Zeta[1 / 2 + s])
hadd[n_, s_] := -n + (1 / 2 + s) (Sum[(n / j) ^ (1 / 2 - s), {j, 1, n}] - n^ (1 / 2 - s) Zeta[1 / 2 - s]) +
  -n + (1 / 2 - s) (Sum[(n / j) ^ (1 / 2 + s), {j, 1, n}] - n^ (1 / 2 + s) Zeta[1 / 2 + s])
hsub[n_, s_] := -n + (1 / 2 + s) (Sum[(n / j) ^ (1 / 2 - s), {j, 1, n}] - n^ (1 / 2 - s) Zeta[1 / 2 - s]) +
  n - (1 / 2 - s) (Sum[(n / j) ^ (1 / 2 + s), {j, 1, n}] - n^ (1 / 2 + s) Zeta[1 / 2 + s])

N@(hsub[10 000, .8])

0.8

(1 + (s - 1 / 2)) / 2 /. s → .8

0.65

N[h3[n, .8 + I] + h4[n, .8 + I] /. n → 100]

$Aborted

hsubx[n_, s_] :=
 -n + (1 / 2 + s) Sum[(n / j) ^ (1 / 2 - s), {j, 1, n}] + n - (1 / 2 - s) Sum[(n / j) ^ (1 / 2 + s), {j, 1, n}]

N@hsubx[100, -.5 + ZetaZero@1]

0. + 14.1347 i

hsub[n_, s_] := -n + (1 / 2 + s) (Sum[(n / j) ^ (1 / 2 - s), {j, 1, n}] - n^ (1 / 2 - s) Zeta[1 / 2 - s]) +
  n - (1 / 2 - s) (Sum[(n / j) ^ (1 / 2 + s), {j, 1, n}] - n^ (1 / 2 + s) Zeta[1 / 2 + s])
hsub1[n_, s_] := (1 / 2 + s) (-n^ (1 / 2 - s) Zeta[1 / 2 - s]) -
  (1 / 2 - s) (-n^ (1 / 2 + s) Zeta[1 / 2 + s]) +
  (1 / 2 + s) (Sum[(n / j) ^ (1 / 2 - s), {j, 1, n}]) - (1 / 2 - s) (Sum[(n / j) ^ (1 / 2 + s), {j, 1, n}])
hsub2[n_, s_] :=
 (1 / 2 + s) (-n^ (1 / 2 - s) Zeta[1 / 2 - s]) - (1 / 2 - s) (-n^ (1 / 2 + s) Zeta[1 / 2 + s]) +
  Sum[(n / j) ^ (1 / 2) (2 s Cosh[s Log[n / j]] - Sinh[s Log[n / j]]), {j, 1, n}]

hsub2[1000, -.2 + 3 I]

-0.2 + 3. i

hsub[1000, .2 + 4 I]

0.2 + 4. i

hadd[n_, s_] := -n + (1 / 2 + s) (Sum[(n / j) ^ (1 / 2 - s), {j, 1, n}] - n^ (1 / 2 - s) Zeta[1 / 2 - s]) +
  -n + (1 / 2 - s) (Sum[(n / j) ^ (1 / 2 + s), {j, 1, n}] - n^ (1 / 2 + s) Zeta[1 / 2 + s])
hadd1[n_, s_] := -2 n + (1 / 2 + s) (-n^ (1 / 2 - s) Zeta[1 / 2 - s]) +
  (1 / 2 - s) (-n^ (1 / 2 + s) Zeta[1 / 2 + s]) +
  (1 / 2 + s) (Sum[(n / j) ^ (1 / 2 - s), {j, 1, n}]) + (1 / 2 - s) (Sum[(n / j) ^ (1 / 2 + s), {j, 1, n}])
hadd2[n_, s_] :=
 -2 n + (1 / 2 + s) (-n^ (1 / 2 - s) Zeta[1 / 2 - s]) + (1 / 2 - s) (-n^ (1 / 2 + s) Zeta[1 / 2 + s]) +
  Sum[(n / j) ^ (1 / 2) ( Cosh[s Log[n / j]] - 2 s Sinh[s Log[n / j]]), {j, 1, n}]

hadd2[10 000, .2 + 4 I]

0.49973 + 0.0000266666 i

hadd[10 000, .2 + 4 I]

0.49973 + 0.0000266667 i
```

```
hsub2x[n_, s_] := Sum[(n / j) ^ (1 / 2) (2 s Cosh[s Log[n / j]] - Sinh[s Log[n / j]]), {j, 1, n}]
hadd2x[n_, s_] :=
 -2 n + Sum[(n / j) ^ (1 / 2) ( Cosh[s Log[n / j]] - 2 s Sinh[s Log[n / j]]), {j, 1, n}]
```

```
hadd2x[10 000, N@Im@ZetaZero@1 I]
```

0.496666 + 0. i

```
hsub2y[n_, s_] := Sum[(n / j) ^ (1 / 2) (2 s Cos[s Log[n / j]] - Sin[s Log[n / j]]), {j, 1, n}]
hadd2y[n_, s_] := -2 n + Sum[(n / j) ^ (1 / 2) ( Cos[s Log[n / j]] + 2 s Sin[s Log[n / j]]), {j, 1, n}]
```

```
hsub2y[100 000, N@Im@ZetaZero@1 ]
```

14.1347

```
(1 / 2 + s) / 2 + I^z (1 / 2 - s) / 2
```

$$\frac{1}{2}\, i^z \left(\frac{1}{2} - s\right) + \frac{1}{2}\left(\frac{1}{2} + s\right)$$

```
Expand[(1 - i) (i + 2 s)]
```

$(1 + i) + (2 - 2\, i)\, s$

```
Abs[I^z]
```

```
FullSimplify[-n - n I^z]
```

$-(1 + i^z)\, n$

```
I^(-z / 2) (n / j)^-s + I^(z / 2) (n / j)^s
```

```
I^(z / 2)
```

$(-1)^{z/4}$

```
E^(Log[I] / 2)
```

$e^{\frac{i\pi}{4}}$

```
N[I^(-z / 2) (n / j)^-s + I^(z / 2) (n / j)^s /. n → 10 /. j → 3 /. z → 1 /. s → 2 I]
```

$-1.99732 + 3.88578 \times 10^{-16}\, i$

```
N[E^(-z / 2 Log[I]) (n / j)^-s + E^(z / 2 Log[I]) (n / j)^s /. n → 10 /. j → 3 /. z → 1 /. s → 2 I]
```

$-1.99732 + 0.\, i$

```
N[E^(-z / 2 Log[I]) E^(-s Log[n / j]) + E^(z / 2 Log[I]) E^(s Log[n / j]) /. n → 10 /. j → 3 /.
   z → 1 /. s → 2 I]
```

$-1.99732 + 0.\, i$

```
N[E^(-z / 2 Log[I] - s Log[n / j]) + E^(z / 2 Log[I] + s Log[n / j]) /. n → 10 /. j → 3 /. z → 1 /.
  s → 2 I]
```

$-1.99732 + 0.\, i$

```
ExpToTrig[E^(-z / 2 Log[I] - s Log[n / j]) + E^(z / 2 Log[I] + s Log[n / j])]
```

$$2\,\mathrm{Cos}\left[\frac{\pi z}{4} - i\, s\, \mathrm{Log}\left[\frac{n}{j}\right]\right]$$

$$\mathrm{N}\left[2\,\mathrm{Cos}\left[\frac{\pi z}{4} - i\, s\, \mathrm{Log}\left[\frac{n}{j}\right]\right] \text{ /. } n \to 10 \text{ /. } j \to 3 \text{ /. } z \to 1 \text{ /. } s \to 2\, I\right]$$

-1.99732

```
N[I^(-z / 2) (n / j)^-s + I^(z / 2) (n / j)^s /. n → 10 /. j → 3 /. z → 2 /. s → .1 + 2 I]
```

$-1.34888 - 0.17928\,i$

$$\text{N}\left[2\,\text{Cos}\left[\frac{\pi z}{4} - i\,s\,\text{Log}\left[\frac{n}{j}\right]\right] \text{ /. } n \to 10 \text{ /. } j \to 3 \text{ /. } z \to 2 \text{ /. } s \to .1 + 2\,I\right]$$

$-1.34888 - 0.17928\,i$

$I^{z/2}$

```
E^(z / 2 Log[I])
```

$e^{\frac{i \pi z}{4}}$

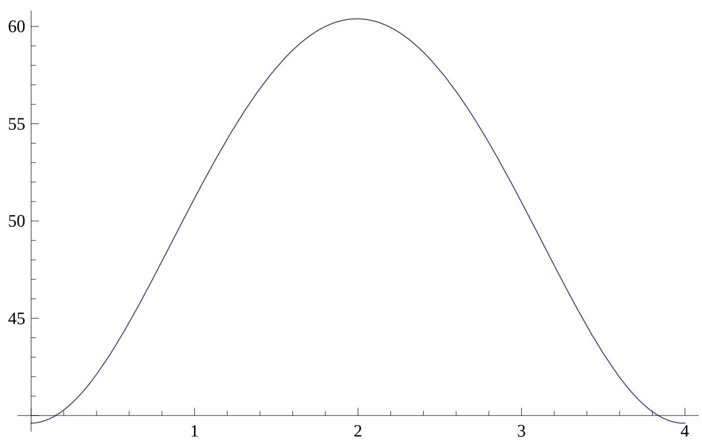
```
ExpToTrig[E^(-z / 2 Log[I] - s Log[n / j]) - E^(z / 2 Log[I] + s Log[n / j])]
```

$$-2\,i\,\text{Sin}\left[\frac{\pi z}{4} - i\,s\,\text{Log}\left[\frac{n}{j}\right]\right]$$

```
I^2
```

-1

```
rr[s_, z_] := (1 / 2 + s) / 2 + I^z (1 / 2 - s) / 2
Plot[Abs[rr[100 I, z + I]], {z, 0, 4}]
```




```
N@I^(1 + I)
```

$1.2729 \times 10^{-17} + 0.20788\,i$

```
(-2 s + 1 / (2 s))
```

$$\frac{1}{2 s} - 2 s$$

```
ar[n_, s_] := -2 n + (-2 s + 1 / (2 s)) Sum[(n / j)^(1 / 2) Sin[s Log[n / j]], {j, 1, n}]
ar[10 000, N@Im@ZetaZero@1]
```

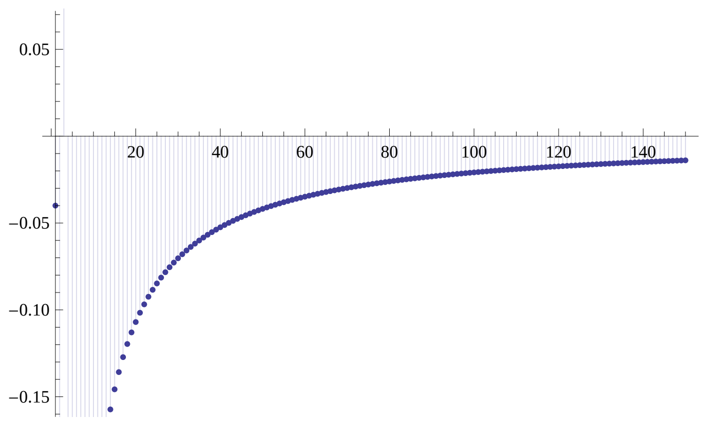
$4.89495 \times 10^{59} + 0.\,i$

```
hsub2y[n_, s_] := Sum[(n / j) ^ (1 / 2) (2 s Cos[s Log[n / j]] - Sin[s Log[n / j]]), {j, 1, n}]
hsub2ya[n_, s_] :=
 Sum[(n / j) ^ (1 / 2) (Cos[s Log[n / j]] - (1 / 2 / s) Sin[s Log[n / j]]), {j, 1, n}]
hadd2y[n_, s_] := -2 n + Sum[(n / j) ^ (1 / 2) ( Cos[s Log[n / j]] + 2 s Sin[s Log[n / j]]), {j, 1, n}]
bh[n_, s_] := -2 n + Sum[(n / j) ^ (1 / 2) ( Cos[s Log[n / j]] + 2 s Sin[s Log[n / j]]), {j, 1, n}] -
  Sum[(n / j) ^ (1 / 2) (Cos[s Log[n / j]] - (1 / 2 / s) Sin[s Log[n / j]]), {j, 1, n}]
bh2[n_, s_] := -2 n + Sum[(n / j) ^ (1 / 2) ( 2 s Sin[s Log[n / j]]), {j, 1, n}] -
  Sum[(n / j) ^ (1 / 2) (- (1 / 2 / s) Sin[s Log[n / j]]), {j, 1, n}]
bh3[n_, s_] := -2 n + (2 s + 1 / (2 s)) Sum[(n / j) ^ (1 / 2) Sin[s Log[n / j]], {j, 1, n}]
bhx[n_, s_] :=
 -2 n / (2 s) - s - 1 / (4 s) + (2 s + 1 / (2 s)) Sum[(n / j) ^ (1 / 2) Cos[s Log[n / j]], {j, 1, n}]
```

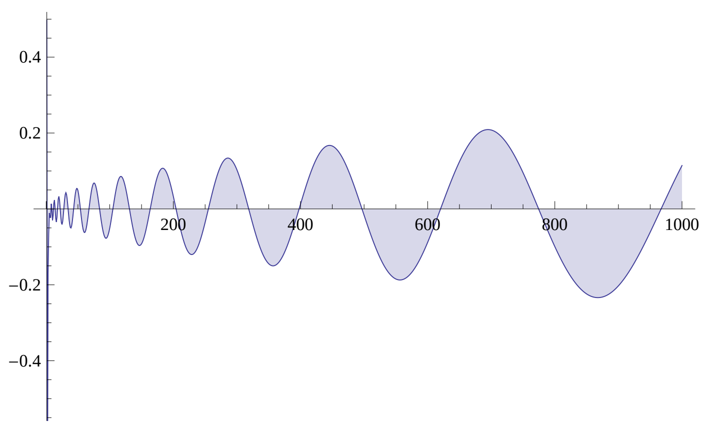
bh3a[n_, s_] := $-\frac{4\,s\,n}{1+4\,s^2}$ + Sum[(n / j) ^ (1 / 2) Sin[s Log[n / j]], {j, 1, n}]

bhxa[n_, s_] := $-\frac{2\,n}{1+4\,s^2}-\frac{1}{2}$ + Sum[(n / j) ^ (1 / 2) Cos[s Log[n / j]], {j, 1, n}]

```
DiscretePlot[Re@bh3a[n, N@Im@ZetaZero@3], {n, 1, 150}]
```




```
DiscretePlot[Re@bhxa[n, N@Im@ZetaZero@1 + .01], {n, 1, 1000}]
```




```
1 / (n^(1 / 2 - s))
```

$n^{-\frac{1}{2}+s}$

```
FullSimplify[(s + (1 / (4 s))) / (2 s + 1 / (2 s))]
```

$\frac{1}{2}$

```
FullSimplify[2 n / (2 s) / (2 s + 1 / (2 s))]
```

$$\frac{2\,n}{1+4\,s^2}$$

```
FullSimplify[(-s - 1 / (4 s)) / (2 s + 1 / (2 s))]
```

$$-\frac{1}{2}$$

```
FullSimplify[(-2 n / (2 s)) / (2 s + 1 / (2 s))]
```

$$-\frac{2\,n}{1+4\,s^2}$$

$$\text{FullSimplify}\left[-\frac{2\,n}{1+4\,s^2}-\frac{1}{2}\ /.\ s\to \text{Im@ZetaZero@3}\right]$$

$$-\frac{1}{2}-\frac{2\,n}{1+4\,\text{Im}[\text{ZetaZero}[3]]^2}$$

```
FullSimplify[-n / ( s + 1 / (4 s))]
```

$$-\frac{4\,n\,s}{1+4\,s^2}$$

```
Expand[(1 / 2 - s I) (1 / 2 + s I)]
```

$$\frac{1}{4}+s^2$$

```
gadd[n_, s_] := -2 n + (1 / 2 + s) (Sum[(n / j) ^ (1 / 2 - s), {j, 1, n}] - n^ (1 / 2 - s) Zeta[1 / 2 - s]) +
   (1 / 2 - s) (Sum[(n / j) ^ (1 / 2 + s), {j, 1, n}] - n^ (1 / 2 + s) Zeta[1 / 2 + s])
gsub[n_, s_] := (1 / 2 + s) (Sum[(n / j) ^ (1 / 2 - s), {j, 1, n}] - n^ (1 / 2 - s) Zeta[1 / 2 - s]) -
   (1 / 2 - s) (Sum[(n / j) ^ (1 / 2 + s), {j, 1, n}] - n^ (1 / 2 + s) Zeta[1 / 2 + s])
gsubi[n_, s_] := ((1 / 2 + s) (Sum[(n / j) ^ (1 / 2 - s), {j, 1, n}] - n^ (1 / 2 - s) Zeta[1 / 2 - s]) -
     (1 / 2 - s) (Sum[(n / j) ^ (1 / 2 + s), {j, 1, n}] - n^ (1 / 2 + s) Zeta[1 / 2 + s])) / (2 s)
```

$$\text{gsubi2}[n\_, s\_] := \Big((1/2 + 1/(4 s))\ (\text{Sum}[(n/j)^\wedge(1/2 - s), \{j, 1, n\}] - n^\wedge(1/2 - s)\ \text{Zeta}[1/2 - s]) + \left(\frac{1}{2}-\frac{1}{4\,s}\right)(\text{Sum}[(n/j)^\wedge(1/2 + s), \{j, 1, n\}] - n^\wedge(1/2 + s)\ \text{Zeta}[1/2 + s])\Big)$$

$$\text{gall}[n\_, s\_] := -2 n + (1/2 + s)\ (\text{Sum}[(n/j)^\wedge(1/2 - s), \{j, 1, n\}] - n^\wedge(1/2 - s)\ \text{Zeta}[1/2 - s]) + (1/2 - s)\ (\text{Sum}[(n/j)^\wedge(1/2 + s), \{j, 1, n\}] - n^\wedge(1/2 + s)\ \text{Zeta}[1/2 + s]) - \Big((1/2 + 1/(4 s))\ (\text{Sum}[(n/j)^\wedge(1/2 - s), \{j, 1, n\}] - n^\wedge(1/2 - s)\ \text{Zeta}[1/2 - s]) + \left(\frac{1}{2}-\frac{1}{4\,s}\right)(\text{Sum}[(n/j)^\wedge(1/2 + s), \{j, 1, n\}] - n^\wedge(1/2 + s)\ \text{Zeta}[1/2 + s])\Big)$$

$$\text{gall2}[n\_, s\_] := -2 n + (s)\ (\text{Sum}[(n/j)^\wedge(1/2 - s), \{j, 1, n\}] - n^\wedge(1/2 - s)\ \text{Zeta}[1/2 - s]) + (-s)\ (\text{Sum}[(n/j)^\wedge(1/2 + s), \{j, 1, n\}] - n^\wedge(1/2 + s)\ \text{Zeta}[1/2 + s]) - \Big((1/(4 s))\ (\text{Sum}[(n/j)^\wedge(1/2 - s), \{j, 1, n\}] - n^\wedge(1/2 - s)\ \text{Zeta}[1/2 - s]) + \left(-\frac{1}{4\,s}\right)(\text{Sum}[(n/j)^\wedge(1/2 + s), \{j, 1, n\}] - n^\wedge(1/2 + s)\ \text{Zeta}[1/2 + s])\Big)$$

```
gall3[n_, s_] := -2 n + s (Sum[(n / j) ^ (1 / 2 - s), {j, 1, n}] - n^ (1 / 2 - s) Zeta[1 / 2 - s]) -
  s (Sum[(n / j) ^ (1 / 2 + s), {j, 1, n}] - n^ (1 / 2 + s) Zeta[1 / 2 + s]) -
  ((1 / (4 s)) (Sum[(n / j) ^ (1 / 2 - s), {j, 1, n}] - n^ (1 / 2 - s) Zeta[1 / 2 - s]) -
    1/(4 s) (Sum[(n / j) ^ (1 / 2 + s), {j, 1, n}] - n^ (1 / 2 + s) Zeta[1 / 2 + s]))
gall4[n_, s_] := -2 n + (s - (1 / (4 s)))
    (Sum[(n / j) ^ (1 / 2 - s), {j, 1, n}] - n^ (1 / 2 - s) Zeta[1 / 2 - s]) +
  (1/(4 s) - s) (Sum[(n / j) ^ (1 / 2 + s), {j, 1, n}] - n^ (1 / 2 + s) Zeta[1 / 2 + s])
gall5[n_, s_] := -2 n + (s - (1 / (4 s)))
    (Sum[(n / j) ^ (1 / 2 - s), {j, 1, n}] - n^ (1 / 2 - s) Zeta[1 / 2 - s]) -
  (s - 1/(4 s)) (Sum[(n / j) ^ (1 / 2 + s), {j, 1, n}] - n^ (1 / 2 + s) Zeta[1 / 2 + s])
gall6[n_, s_] := -2 n + (s - 1/(4 s))
    ((Sum[(n / j) ^ (1 / 2 - s), {j, 1, n}] - n^ (1 / 2 - s) Zeta[1 / 2 - s]) -
      (Sum[(n / j) ^ (1 / 2 + s), {j, 1, n}] - n^ (1 / 2 + s) Zeta[1 / 2 + s]))
gall7[n_, s_] := (-2 n) / (s - 1/(4 s)) + Sum[(n / j) ^ (1 / 2 - s), {j, 1, n}] -
  n^ (1 / 2 - s) Zeta[1 / 2 - s] - Sum[(n / j) ^ (1 / 2 + s), {j, 1, n}] + n^ (1 / 2 + s) Zeta[1 / 2 + s]
gall8[n_, s_] := (8 s)/(1 - 4 s^2) n + Sum[(n / j) ^ (1 / 2 - s), {j, 1, n}] - n^ (1 / 2 - s) Zeta[1 / 2 - s] -
  Sum[(n / j) ^ (1 / 2 + s), {j, 1, n}] + n^ (1 / 2 + s) Zeta[1 / 2 + s]
gall9[n_, s_] := (8 s)/(1 - 4 s^2) n + Sum[(n / j) ^ (1 / 2 - s), {j, 1, n}] -
  Sum[(n / j) ^ (1 / 2 + s), {j, 1, n}] - n^ (1 / 2 - s) Zeta[1 / 2 - s] + n^ (1 / 2 + s) Zeta[1 / 2 + s]
gall10[n_, s_] := -(8 s)/(1 - 4 s^2) n + 2 Sum[(n / j) ^ (1 / 2) Sinh[s Log[n / j]], {j, 1, n}] +
  n^ (1 / 2 - s) Zeta[1 / 2 - s] - n^ (1 / 2 + s) Zeta[1 / 2 + s]
gall10[13 000, -1.3 + 3 I]
```

0.0000167228 − 0.0000384691 i

```
Expand@FullSimplify[(1 / 2 + s) / (2 s)]
```

$\frac{1}{2} + \frac{1}{4 s}$

$\left(\frac{1}{2} + \frac{1}{4 s}\right)$

$\frac{1}{2} + \frac{1}{4 s}$

```
Expand[(1 / 2 - s) / (2 s)]
```

$\left(-\frac{1}{2} + \frac{1}{4 s}\right)$

```
FullSimplify[(-2 n) / (s - 1/(4 s))]
```

$$\frac{8\,n\,s}{1-4\,s^2}$$

```
Expand[(1 - 2 s) (1 + 2 s)]
```

$$1-4\,s^2$$

```
Sum[(n / j) ^ (1 / 2 - s), {j, 1, n}] - Sum[(n / j) ^ (1 / 2 + s), {j, 1, n}] /. n → 100 /. s → .2 + 4 I
```

$$-10.6959-43.6213\,i$$

```
-2 Sum[(n / j) ^ (1 / 2) Sinh[s Log[n / j]], {j, 1, n}] /. n → 100 /. s → .2 + 4 I
```

$$-10.6959-43.6213\,i$$

```
call10[n_, s_] := -4/(1 - 4 s^2) n - 1 + 2 Sum[(n / j) ^ (1 / 2) Cosh[s Log[n / j]], {j, 1, n}] -
  n^(1 / 2 - s) Zeta[1 / 2 - s] - n^(1 / 2 + s) Zeta[1 / 2 + s]
```

```
call10[30 000, -.75 + 40 I]
```

$$-2.77383\times10^{-6}+7.567\times10^{-10}\,i$$

```
FullSimplify[-4/(1 - 4 s^2) n - 1 + 2 Sum[(n / j) ^ (1 / 2) Cosh[s Log[n / j]], {j, 1, n}] -
   n^(1 / 2 - s) Zeta[1 / 2 - s] - n^(1 / 2 + s) Zeta[1 / 2 + s] /. s → 1 / 2 - s]
```

$$-1+\frac{n}{(-1+s)\,s}+2\sum_{j=1}^{n}\sqrt{\frac{n}{j}}\;\text{Cosh}\left[\left(\frac{1}{2}-s\right)\text{Log}\left[\frac{n}{j}\right]\right]-n^{1-s}\,\text{Zeta}[1-s]-n^{s}\,\text{Zeta}[s]$$

```
FullSimplify[-8 s/(1 - 4 s^2) n + 2 Sum[(n / j) ^ (1 / 2) Sinh[s Log[n / j]], {j, 1, n}] +
   n^(1 / 2 - s) Zeta[1 / 2 - s] - n^(1 / 2 + s) Zeta[1 / 2 + s] /. s → 1 / 2 - s]
```

$$\frac{n-2\,n\,s}{(-1+s)\,s}+2\sum_{j=1}^{n}\sqrt{\frac{n}{j}}\;\text{Sinh}\left[\left(\frac{1}{2}-s\right)\text{Log}\left[\frac{n}{j}\right]\right]-n^{1-s}\,\text{Zeta}[1-s]+n^{s}\,\text{Zeta}[s]$$

```
-4/(1 - 4 s^2) n - 1 + 2 Sum[(n / j) ^ (1 / 2) Cosh[s Log[n / j]], {j, 1, n}] -
  n^(1 / 2 - s) Zeta[1 / 2 - s] - n^(1 / 2 + s) Zeta[1 / 2 + s] /. s → s I
```

$$-1-\frac{4\,n}{1+4\,s^2}+2\sum_{j=1}^{n}\sqrt{\frac{n}{j}}\;\text{Cos}\left[s\,\text{Log}\left[\frac{n}{j}\right]\right]-n^{\frac{1}{2}-i\,s}\,\text{Zeta}\left[\frac{1}{2}-i\,s\right]-n^{\frac{1}{2}+i\,s}\,\text{Zeta}\left[\frac{1}{2}+i\,s\right]$$

```
Expand[(-8 s/(1 - 4 s^2) n + 2 Sum[(n / j) ^ (1 / 2) Sinh[s Log[n / j]], {j, 1, n}] +
     n^(1 / 2 - s) Zeta[1 / 2 - s] - n^(1 / 2 + s) Zeta[1 / 2 + s]) I^3] /. s → s I
```

$$-\frac{8\,n\,s}{1+4\,s^2}-2\,i\sum_{j=1}^{n}i\,\sqrt{\frac{n}{j}}\;\text{Sin}\left[s\,\text{Log}\left[\frac{n}{j}\right]\right]-i\,n^{\frac{1}{2}-i\,s}\,\text{Zeta}\left[\frac{1}{2}-i\,s\right]+i\,n^{\frac{1}{2}+i\,s}\,\text{Zeta}\left[\frac{1}{2}+i\,s\right]$$

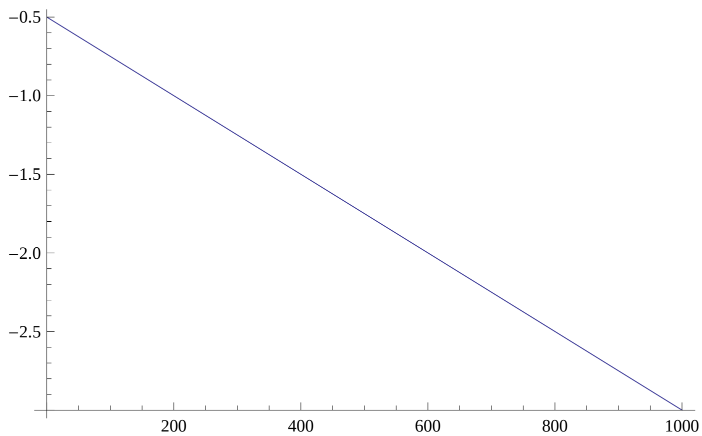
ComplexExpand[Re[n^(1 + A + f I) / (1 + A + f I)]] /. Arg[n] → 0

$$\frac{\left(n^2\right)^{\frac{1+A}{2}} \text{Cos}\left[\frac{1}{2} \text{f Log}\left[n^2\right]\right]}{(1+A)^2+f^2}+\frac{A \left(n^2\right)^{\frac{1+A}{2}} \text{Cos}\left[\frac{1}{2} \text{f Log}\left[n^2\right]\right]}{(1+A)^2+f^2}+\frac{f \left(n^2\right)^{\frac{1+A}{2}} \text{Sin}\left[\frac{1}{2} \text{f Log}\left[n^2\right]\right]}{(1+A)^2+f^2}$$

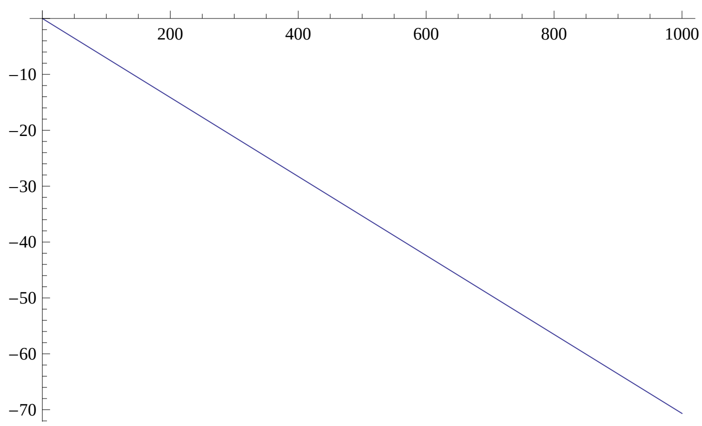
-2 / (1 + 4 (N@Im@ZetaZero@1)^2)

-0.00249949

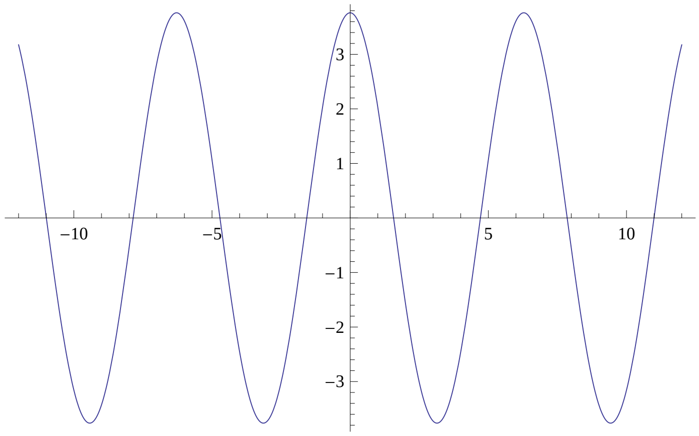
Plot[-2 / (1 + 4 (N@Im@ZetaZero@1)^2) n - 1 / 2, {n, 1, 1000}]



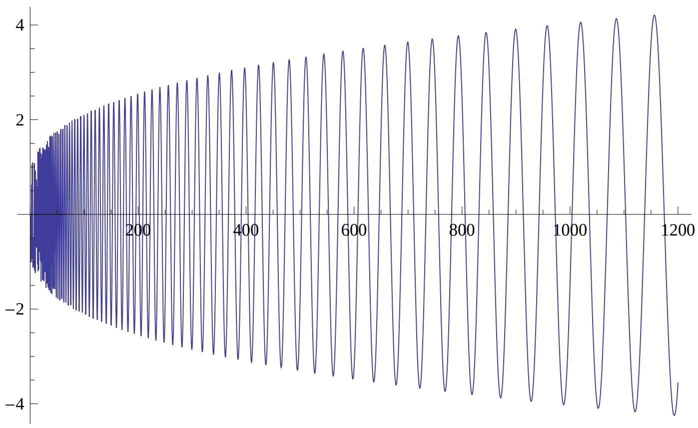
Plot[-(4 N@Im@ZetaZero@1) / (1 + 4 (N@Im@ZetaZero@1)^2) n, {n, 1, 1000}]



```
Plot[Re[Cos[s + 2 I]], {s, -12, 12}]
```




```
Plot[Re[Sin[Log[s] (100 + .3 I)]], {s, 1, 1200}]
```



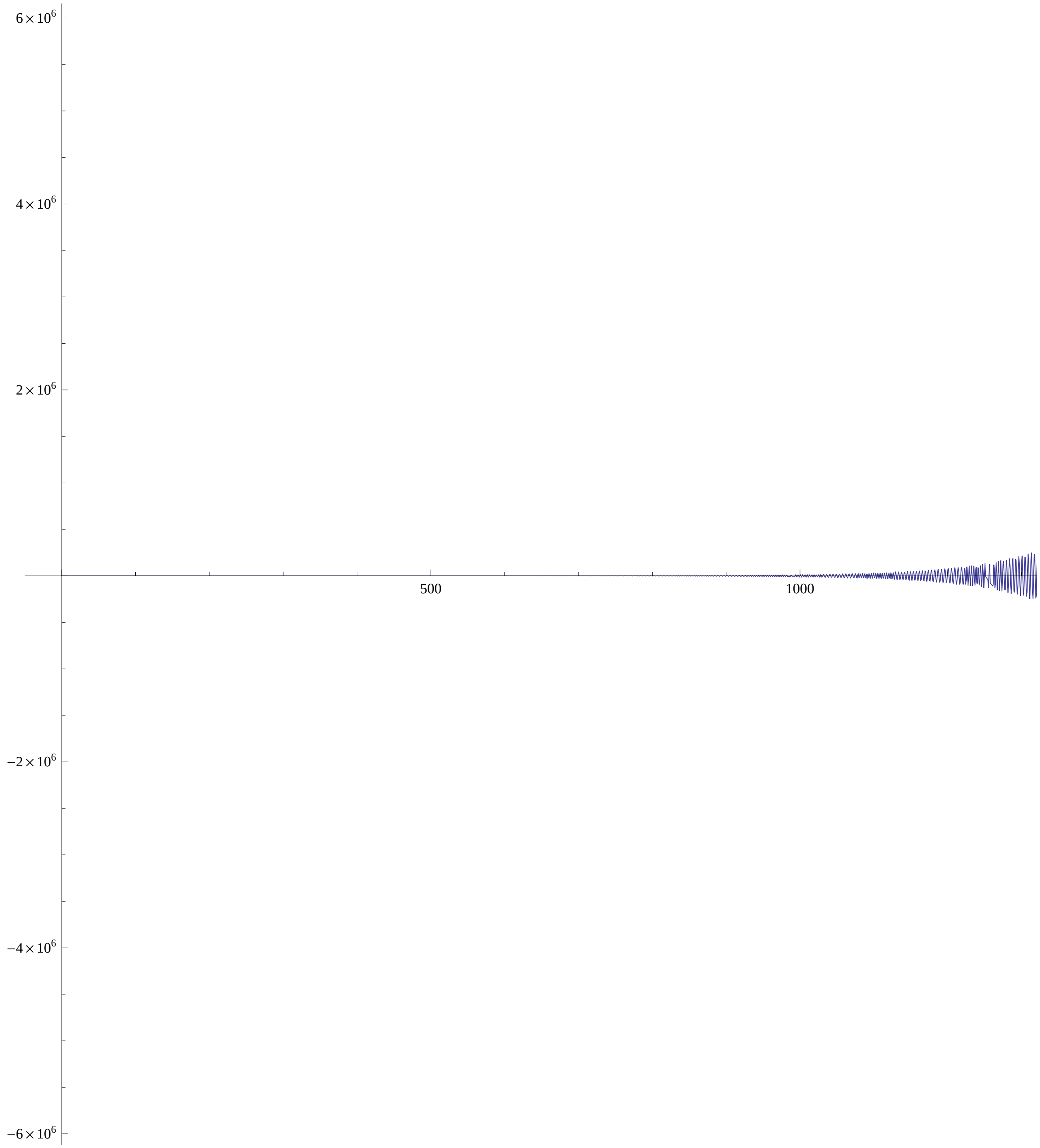

```
Sin[3. + 2 I]
```

0.530921 − 3.59056 i

```
Sin[3. + 2 I + 2 Pi]
```

0.530921 − 3.59056 i

```
Plot[Im[Cos[(10 - .01 I) s]], {s, 0, 2400}]
```




```
bebo[n_, s_] :=
 {-4 s / (1 + 4 s^2) n, Table[(n / j) ^ (1 / 2) Sin[s Log[n / j]], {j, n - 40, n}] // TableForm}
```

```
N@bebo[10^3, Im@ZetaZero@1 + .001 I]
```

```
{-70.6593 + 0.00498649 i, 0.556768 + 0.0000349184 i
                          0.543837 + 0.0000343322 i
                          0.530815 + 0.0000337231 i
                          0.517706 + 0.0000330919 i
                          0.504512 + 0.0000324391 i
                          0.491238 + 0.0000317653 i
                          0.477885 + 0.0000310712 i
                          0.464458 + 0.0000303574 i
                          0.450959 + 0.0000296246 i
                          0.437392 + 0.0000288734 i
                          0.42376 + 0.0000281044 i
                          0.410065 + 0.0000273183 i
                          0.396312 + 0.0000265158 i
                          0.382504 + 0.0000256975 i
                          0.368643 + 0.000024864 i
                          0.354733 + 0.000024016 i
                          0.340777 + 0.0000231542 i
                          0.326778 + 0.0000222791 i
                          0.312739 + 0.0000213915 i
                          0.298664 + 0.0000204921 i
                          0.284554 + 0.0000195814 i
                          0.270415 + 0.0000186601 i
                          0.256248 + 0.0000177289 i
                          0.242057 + 0.0000167884 i
                          0.227845 + 0.0000158392 i
                          0.213614 + 0.0000148821 i
                          0.199368 + 0.0000139177 i
                          0.185111 + 0.0000129465 i
                          0.170844 + 0.0000119693 i
                          0.156571 + 0.0000109866 i
                          0.142295 + 9.99922 × 10^-6 i
                          0.128018 + 9.00765 × 10^-6 i
                          0.113745 + 8.01258 × 10^-6 i
                          0.0994767 + 7.01461 × 10^-6 i
                          0.0852173 + 6.01438 × 10^-6 i
                          0.0709693 + 5.01251 × 10^-6 i
                          0.0567356 + 4.00962 × 10^-6 i
                          0.042519 + 3.00631 × 10^-6 i
                          0.0283223 + 2.00321 × 10^-6 i
                          0.0141484 + 1.0009 × 10^-6 i
                          0. + 0. i                        }
```

```
peo[n_, z_] := Sum[(n / j) ^ (1 / 2) Cosh[z Log[n / j] - ArcCoth[2 z]], {j, 1, n}]

peo[100 000, -.5 + N@ZetaZero@1 + .2 I]
```

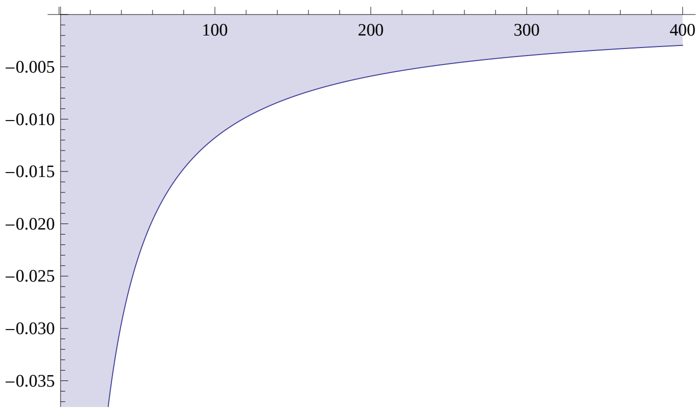
```
$Aborted
```

```
bbo[n_, s_] := -4 s / (1 + 4 s^2) n + Sum[(n / j) ^ (1 / 2) Sin[s Log[n / j]], {j, 1, n}]
bba[n_, s_] := Abs[-4 s / (1 + 4 s^2) n] - Abs[Sum[(n / j) ^ (1 / 2) Sin[s Log[n / j]], {j, 1, n}]]
```

```
bbo[10 000, N@Im@ZetaZero@1]
```

-0.000117789

```
DiscretePlot[Re@bbo[n, N@Im@ZetaZero@1], {n, 1, 400}]
```




```
-Integrate[Sin[z Log[x]] / x^(1 / 2), {x, 0, 1}]
```

ConditionalExpression$\left[\frac{4\,z}{1+4\,z^2},\ -\frac{1}{2}<\text{Im}[z]<\frac{1}{2}\right]$

```
Limit[1 / n Sum[(n / j)^(1 / 2) Sin[z Log[n / j]], {j, 1, n}], n → Infinity]
```

Limit$\left[\frac{\sum_{j=1}^{n}\sqrt{\frac{n}{j}}\ \text{Sin}\left[z\,\text{Log}\left[\frac{n}{j}\right]\right]}{n},\ n\to\infty\right]$

```
-Integrate[Cos[z Log[x]] / x^(1 / 2), {x, 0, 1}]
```

ConditionalExpression$\left[-\frac{2}{1+4\,z^2},\ z\in\text{Reals}\right]$

```
-Integrate[Tan[z Log[x]] / x^(1 / 2), {x, 0, 1}]
```

ConditionalExpression$\left[-\frac{-4\,i\,z+\text{PolyGamma}\left[0,\frac{1}{2}-\frac{i}{8\,z}\right]-\text{PolyGamma}\left[0,-\frac{i}{8\,z}\right]}{2\,z},\ \text{Im}[z]<0\right]$

```
-n^(1 / 2) - n^(-1 / 2) /. n → 10.3
```

-3.52095

```
-E^(1 / 2 Log[n]) - E^(-1 / 2 Log[n]) /. n → 10.3
```

-3.52095

```
-2 Cosh[(1 / 2) Log[n]] /. n → 10.3
```

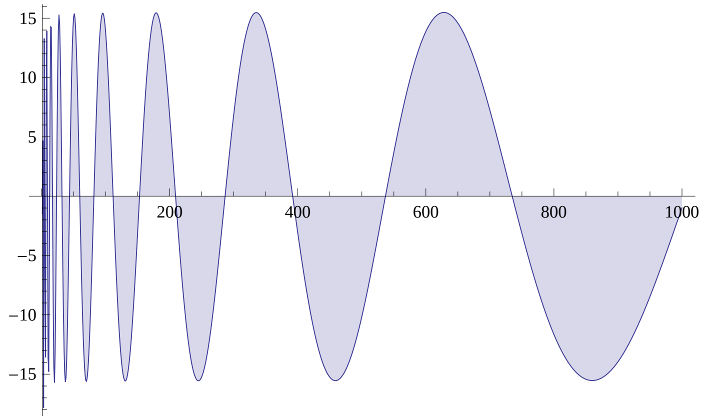
-3.52095

```
feh[n_, s_] :=
 -2 Cosh[1 / 2 Log[n]] + Sum[j^(-1 / 2) (1 / 2 Cos[s Log[n / j]] + s Sin[s Log[n / j]]), {j, 1, n}]
feha[n_, s_] := -2 Cosh[1 / 2 Log[n]]
```

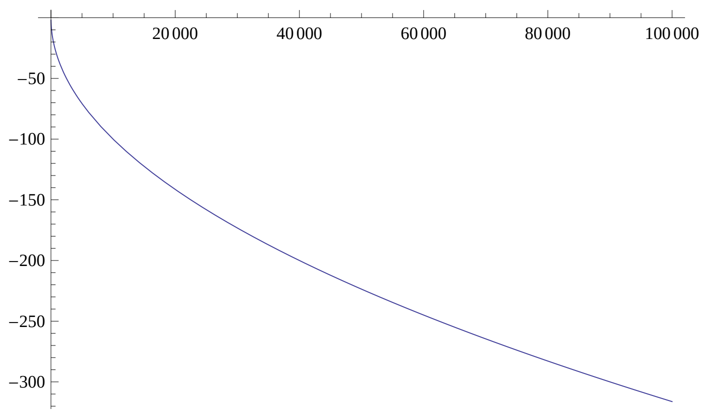
```
feh[100 000, N@Im@ZetaZero@1]
```

-0.00237224

```
DiscretePlot[Re@feh[n, 10], {n, 1, 1000}]
```




```
Plot[feha[n, 1], {n, 1, 100 000}]
```




```
pla[n_, s_] := Sum[(n j)^(-1/2) Sin[s Log[n/j]], {j, 1, n}]
```

```
po[n_] := 1/n Sum[Sin[j/n], {j, 1, n}]

N@Integrate[Sin[x], {x, 0, 1}]

0.459698

po[1 000 000.]

0.459698

pla[n_, s_] := (1/n) Sum[(j/n)^s, {j, 1, n}]

n Integrate[x^s, {x, 0, 1}]
```

ConditionalExpression$\left[\frac{n}{1+s}, \text{Re}[s] > -1\right]$

```
pla[1000, .5]

0.66716
```

```
Integrate[ (2 s Cos[s Log[x]] - Sin[ s Log[x]]) / (x^(1 / 2)), {x, 0, 1}]
```

ConditionalExpression$\left[\frac{8\,s}{1+4\,s^2},\ s\in\text{Reals}\right]$

```
pn[n_, s_] := n (1 / n) Sum[(j / n)^(-1 / 2) (2 s Cos[s Log[j / n]] + Sin[s Log[j / n]]), {j, 1, n}]

pn[100, N@Im@ZetaZero@1 ]

14.1347

Integrate[x^(-1 / 2) (2 s Cos[s Log[x]] + Sin[s Log[x]]), {x, 0, 1}]

ConditionalExpression[0, s ∈ Reals]

Integrate[x^(-1 / 2) (2 s Cos[s Log[x]]), {x, 0, 1}]
```

ConditionalExpression$\left[\frac{4\,s}{1+4\,s^2},\ s\in\text{Reals}\right]$

```
Integrate[x^(-1 / 2) (Sin[s Log[x]]), {x, 0, 1}]
```

ConditionalExpression$\left[-\frac{4\,s}{1+4\,s^2},\ -\frac{1}{2}<\text{Im}[s]<\frac{1}{2}\right]$

```
pr[n_, s_] := n (1 / n Sum[(j / n)^(-1 / 2 + s), {j, 1, n}])

pr[100, .5 + I]

50.0839 - 50.0406 i

Integrate[x^(-1 / 2 + s), {x, 0, 1}]
```

ConditionalExpression$\left[\frac{2}{1+2\,s},\ \text{Re}[s]>-\frac{1}{2}\right]$

```
FullSimplify[(1 / 2 + s) (2 / (1 + 2 s))]

1

pb[n_, s_] := Sum[j^-s, {j, 1, n}] - Integrate[j^-s, {j, 0, n}]
pbc[n_, s_] := Sum[E^-(s Log[j]), {j, 1, n}] - Integrate[E^-(s Log[j]), {j, 0, n}]
pbd[n_, s_] := Sum[E^-(s Log[j]), {j, 1, n}] - Integrate[E^-(s Log[j]), {j, 0, n}]
pbe[n_, A_, f_] :=
 Sum[j^-A (Cos[f Log[j]]), {j, 1, n}] - Integrate[j^-A (Cos[f Log[j]]), {j, 0, n}] +
  I (Sum[j^-A ( Sin[f Log[j]]), {j, 1, n}] - Integrate[j^-A (Sin[f Log[j]]), {j, 0, n}])
pbf[n_, A_, f_] := Sum[j^-A (Cos[f Log[j]]), {j, 1, n}] -
  Integrate[j^-A (Cos[f Log[j]]), {j, 0, n}] +
  I (Sum[j^-A ( Sin[f Log[j]]), {j, 1, n}] - Integrate[j^-A (Sin[f Log[j]]), {j, 0, n}])

pbe[10 000, .5, 10]

1.54189 + 0.111262 i

pb[10 000, .5 + 10 I]

1.54189 - 0.111262 i


Zeta[.5 + 10 I]

1.5449 - 0.115336 i
```

```
Integrate[j^-A (Cos[f Log[j]]), {j, 0, n}]
```

ConditionalExpression$\left[\dfrac{n^{-A}\,(-(-1+A)\,n\,\mathrm{Cos}[f\,\mathrm{Log}[n]] + 0\,\mathrm{Sin}[f\,(-\infty)] + f\,n\,\mathrm{Sin}[f\,\mathrm{Log}[n]])}{(-1+A)^2+f^2},\ \mathrm{Re}[A]<1\right]$

```
Integrate[j^-A (Sin[f Log[j]]), {j, 0, n}]
```

ConditionalExpression$\left[\dfrac{n^{-A}\,(-f\,n\,\mathrm{Cos}[f\,\mathrm{Log}[n]] + 0\,\mathrm{Sin}[f\,(-\infty)] - (-1+A)\,n\,\mathrm{Sin}[f\,\mathrm{Log}[n]])}{(-1+A)^2+f^2},\ \mathrm{Re}[A]<1\right]$

```
Integrate[j^(-A - f I), {j, 0, n}]
```

ConditionalExpression$\left[-\dfrac{n^{1-A-i f}}{-1+A+i f},\ \mathrm{Re}[A]<1+\mathrm{Im}[f]\right]$

```
ab[n_, s_] := (1 / n) Sum[(n / j) ^ (1 / 2) Sin[s Log[n / j]], {j, 1, n}]
abb[s_] := 4 s / (1 + 4 s^2)
ab[100 000, 20. + .1 I]
```

0.0511243 − 0.0050391 i

```
abb[20. + .1 I]
```

0.0499675 − 0.000249526 i

```
s1[n_, s_] := -8 s / (1 + 4 s^2) n + 2 Sum[(n / j) ^ (1 / 2) Sin[s Log[n / j]], {j, 1, n}] +
  I ( n^ (1 / 2 + s I) Zeta[1 / 2 + s I] - n^ (1 / 2 - s I) Zeta[1 / 2 - s I])
```

$$\texttt{xi[s\_] := } \frac{1}{2}\pi^{-s/2}(-1+s)\, s\, \texttt{Gamma}\left[\frac{s}{2}\right] \texttt{Zeta[s]}$$

$$\texttt{zet[s\_] := xsi[s]} \Big/ \left(\frac{1}{2}\pi^{-s/2}(-1+s)\, s\, \texttt{Gamma}\left[\frac{s}{2}\right]\right)$$

$$\texttt{s2[n\_, s\_] := -8 s / (1 + 4 s\^{}2) n + 2 Sum[(n / j) \^{} (1 / 2) Sin[s Log[n / j]], \{j, 1, n\}] +}$$
$$\texttt{I}\left(\texttt{n\^{}(1/2+s I)}\left(\frac{2\pi^{\frac{1}{2}\left(\frac{1}{2}+i s\right)}\texttt{xi}\left[\frac{1}{2}+i s\right]}{\left(-\frac{1}{2}+i s\right)\left(\frac{1}{2}+i s\right)\texttt{Gamma}\left[\frac{1}{2}\left(\frac{1}{2}+i s\right)\right]}\right) - \texttt{n\^{}(1/2-s I)}\left(\frac{2\pi^{\frac{1}{2}\left(\frac{1}{2}-i s\right)}\texttt{xi}\left[\frac{1}{2}-i s\right]}{\left(-\frac{1}{2}-i s\right)\left(\frac{1}{2}-i s\right)\texttt{Gamma}\left[\frac{1}{2}\left(\frac{1}{2}-i s\right)\right]}\right)\right)$$

$$\texttt{s3[n\_, s\_] := -8 s / (1 + 4 s\^{}2) n + 2 Sum[(n / j) \^{} (1 / 2) Sin[s Log[n / j]], \{j, 1, n\}] +}$$
$$\texttt{I}\left(\left(\frac{\texttt{n\^{}(1/2+s I)}\; 2\pi^{\frac{1}{2}\left(\frac{1}{2}+i s\right)}\texttt{xi}\left[\frac{1}{2}+i s\right]}{\left(-\frac{1}{2}+i s\right)\left(\frac{1}{2}+i s\right)\texttt{Gamma}\left[\frac{1}{2}\left(\frac{1}{2}+i s\right)\right]}\right) - \left(\frac{\texttt{n\^{}(1/2-s I)}\; 2\pi^{\frac{1}{2}\left(\frac{1}{2}-i s\right)}\texttt{xi}\left[\frac{1}{2}-i s\right]}{\left(-\frac{1}{2}-i s\right)\left(\frac{1}{2}-i s\right)\texttt{Gamma}\left[\frac{1}{2}\left(\frac{1}{2}-i s\right)\right]}\right)\right)$$

$$\texttt{s4[n\_, s\_] := -8 s / (1 + 4 s\^{}2) n + 2 Sum[(n / j) \^{} (1 / 2) Sin[s Log[n / j]], \{j, 1, n\}] +}$$
$$\texttt{I}\left(\texttt{n\^{}(1/2) xi}\left[\frac{1}{2}+i s\right]\left(\left(\frac{\texttt{n\^{}(s I)}\; 2\pi^{\frac{1}{2}\left(\frac{1}{2}+i s\right)}}{\left(-\frac{1}{4}-s^2\right)\texttt{Gamma}\left[\frac{1}{2}\left(\frac{1}{2}+i s\right)\right]}\right) - \left(\frac{\texttt{n\^{}(-s I)}\; 2\pi^{\frac{1}{2}\left(\frac{1}{2}-i s\right)}}{\left(-\frac{1}{4}-s^2\right)\texttt{Gamma}\left[\frac{1}{2}\left(\frac{1}{2}-i s\right)\right]}\right)\right)\right)$$

$$\texttt{s5[n\_, s\_] := -8 s / (1 + 4 s\^{}2) n + 2 Sum[(n / j) \^{} (1 / 2) Sin[s Log[n / j]], \{j, 1, n\}] +}$$
$$\texttt{I}\left(\texttt{2 n\^{}(1/2) xi}\left[\frac{1}{2}+i s\right] \Big/ \left(-\frac{1}{4}-s^2\right)\left(\left(\frac{\texttt{n\^{}(s I)}\;\pi^{\left(\frac{1}{4}+i s/2\right)}}{\texttt{Gamma}\left[\frac{1}{2}\left(\frac{1}{2}+i s\right)\right]}\right) - \left(\frac{\texttt{n\^{}(-s I)}\;\pi^{\left(\frac{1}{4}-i s/2\right)}}{\texttt{Gamma}\left[\frac{1}{2}\left(\frac{1}{2}-i s\right)\right]}\right)\right)\right)$$

$$\texttt{s6[n\_, s\_] := -8 s / (1 + 4 s\^{}2) n + 2 Sum[(n / j) \^{} (1 / 2) Sin[s Log[n / j]], \{j, 1, n\}] +}$$
$$\texttt{I}\left(\texttt{2 n\^{}(1/2)}\;\pi^{\left(\frac{1}{4}\right)}\texttt{xi}\left[\frac{1}{2}+i s\right] \Big/ \left(-\frac{1}{4}-s^2\right)\left(\left(\frac{\texttt{n\^{}(s I)}\;\pi^{(i s/2)}}{\texttt{Gamma}\left[\frac{1}{2}\left(\frac{1}{2}+i s\right)\right]}\right) - \left(\frac{\texttt{n\^{}(-s I)}\;\pi^{(-i s/2)}}{\texttt{Gamma}\left[\frac{1}{2}\left(\frac{1}{2}-i s\right)\right]}\right)\right)\right)$$

```
s6[10 000, -.5 + .1 I]
```

$8.33329\times 10^{-6} - 1.66666\times 10^{-6}\, i$

```
1 / 2 s (s - 1) Pi^ (-s / 2) Gamma[s / 2] Zeta[s]
```

$$\frac{1}{2}\pi^{-s/2}(-1+s)\, s\, \texttt{Gamma}\left[\frac{s}{2}\right]\texttt{Zeta[s]}$$

```
zet[1 / 2 - s I]
```

$$\frac{2\pi^{\frac{1}{2}\left(\frac{1}{2}-i s\right)}\texttt{xi}\left[\frac{1}{2}-i s\right]}{\left(-\frac{1}{2}-i s\right)\left(\frac{1}{2}-i s\right)\texttt{Gamma}\left[\frac{1}{2}\left(\frac{1}{2}-i s\right)\right]}$$

```
xi[(2 + .3 I)]
```

$0.52241 + 0.0108274\, i$

```
zet[1 / 2 + s I]
```

$$\frac{2\,\pi^{\frac{1}{2}\left(\frac{1}{2}+i\,s\right)}\,\mathrm{xi}\left[\frac{1}{2}+i\,s\right]}{\left(-\frac{1}{2}+i\,s\right)\left(\frac{1}{2}+i\,s\right)\mathrm{Gamma}\left[\frac{1}{2}\left(\frac{1}{2}+i\,s\right)\right]}$$

$$\mathrm{Gamma}\left[\frac{1}{2}\left(\frac{1}{2}+i\,s\right)\right]\ /.\ s \to (-.5+.1\,I)$$

$1.57144 + 2.26644\,i$

```
Expand[(-1 / 2 + I s) (1 / 2 + I s)]
```

$$-\frac{1}{4} - s^2$$

```
Expand[(-1 / 2 - I s) (1 / 2 - I s)]
```

$$-\frac{1}{4} - s^2$$