

```

binomial[z_, k_] := binomial[z, k] = Product[z - j, {j, 0, k - 1}] / k!
zetaHurwitz[n_, s_, y_, 0] := UnitStep[n - 1]
zetaHurwitz[n_, s_, y_, 1] :=
  zetaHurwitz[n, s, y, 1] = HarmonicNumber[Floor[n], s] - HarmonicNumber[y, s]
zetaHurwitz[n_, s_, y_, 2] := zetaHurwitz[n, s, y, 2] =
  Sum[(m^(-2 s)) + 2 (m^(-s) (zetaHurwitz[Floor[n / m], s, m, 1]), {m, y + 1, Floor[n^(1 / 2)]}]
zetaHurwitz[n_, s_, y_, k_] := zetaHurwitz[n, s, y, k] =
  Sum[(m^(-s k)) + k (m^(-s (k - 1))) zetaHurwitz[Floor[n / (m^(k - 1))], s, m, 1] +
    Sum[binomial[k, j] (m^(-s))^j zetaHurwitz[Floor[n / (m^j)], s, m, k - j], {j, 1, k - 2}],
    {m, y + 1, Floor[n^(1 / k)]}]
zeta[n_, s_, z_] := Expand@Sum[binomial[z, k] zetaHurwitz[n, s, 1, k], {k, 0, Log2@n}]

```

```

eta[n_, s_, z_] :=
  Expand@Sum[(-1)^j binomial[z, j] 2^(j (1 - s)) zeta[n / (2^j), s, z], {j, 0, Log2@n}]

```

```

etaZeros[n_, s_] := If[(c = Exponent[f = eta[n, s, z], z]) == 0, {},
  If[c == 1, List@NRroots[f == 0, z][[2]], List@@NRroots[f == 0, z][[All, 2]]]]
etaR[n_, s_, z_] := Chop@Expand@Product[1 - z / rho, {rho, etaZeros[n, s]}]

```

```

DetaZeros[n_, s_] := If[(c = Exponent[f = (D[eta[n, t, z], t] /. t -> s) - 1, z]) == 0,
  {}, If[c == 1, List@NRroots[f == 0, z][[2]], List@@NRroots[f == 0, z][[All, 2]]]]
DetaR[n_, s_, z_] := -1 + Chop@Expand@Product[1 - z / rho, {rho, DetaZeros[n, s]}]

```

```

eta[100 000 000, N@ZetaZero@1, z]

```

```

(1. + 0. i) + (178.811 - 111.793 i) z - (648.401 - 338.823 i) z^2 + (1006. - 434.963 i) z^3 -
(899.348 - 326.894 i) z^4 + (522.784 - 169.271 i) z^5 - (211.134 - 67.6491 i) z^6 +
(61.8198 - 22.3314 i) z^7 - (13.5292 - 6.1622 i) z^8 + (2.26506 - 1.38782 i) z^9 -
(0.294909 - 0.248205 i) z^10 + (0.0300519 - 0.03466 i) z^11 - (0.00236985 - 0.00375092 i) z^12 +
(0.000138919 - 0.000313588 i) z^13 - (5.54652 × 10^-6 - 0.0000201624 i) z^14 +
(1.14854 × 10^-7 - 9.88521 × 10^-7 i) z^15 + (1.63804 × 10^-9 + 3.65482 × 10^-8 i) z^16 -
(2.23903 × 10^-10 + 1.00481 × 10^-9 i) z^17 + (8.61981 × 10^-12 + 2.01412 × 10^-11 i) z^18 -
(1.88538 × 10^-13 + 2.86703 × 10^-13 i) z^19 + (2.58061 × 10^-15 + 2.83791 × 10^-15 i) z^20 -
(2.39143 × 10^-17 + 2.01128 × 10^-17 i) z^21 + (1.59503 × 10^-19 + 1.13567 × 10^-19 i) z^22 -
(7.70253 × 10^-22 + 5.18788 × 10^-22 i) z^23 + (1.96369 × 10^-24 + 1.83034 × 10^-24 i) z^24 -
(1.58641 × 10^-28 + 2.07171 × 10^-27 i) z^25 - (2.91933 × 10^-31 - 7.98362 × 10^-32 i) z^26

```

```

etaZeros[100 000 000, N@ZetaZero@1]

```

```

{-23.4433 + 115.492 i, -3.00749 + 7.24046 i, -0.00399172 - 0.0024432 i,
  1. + 3.6082 × 10^-6 i, 1.59804 + 38.0367 i, 1.61973 - 1.50287 i, 2.00164 - 0.0052567 i,
  2.70904 + 0.134875 i, 3.0931 - 8.44949 i, 3.1986 - 1.21641 i, 3.56358 + 1.39473 i,
  4.17529 - 3.86039 i, 4.4097 + 16.3853 i, 4.60627 + 3.92814 i, 5.18591 + 0.00864523 i,
  9.63811 - 0.0407831 i, 12.2774 - 154.546 i, 12.4618 - 21.1794 i,
  21.8501 + 3.95519 i, 24.4498 - 2.96747 i, 32.8255 - 0.284314 i, 66.7753 - 26.1226 i,
  68.0094 + 67.8245 i, 87.9623 - 5862.85 i, 133.035 - 68.5173 i, 820.076 - 843.833 i}

```

```

(1. + 0. i) + (178.81051161715766 - 111.79320482065665 i) z -
(648.4009901367455 - 338.8234288642313 i) z^2 +
(1006.000443562261 - 434.96273548637834 i) z^3 -
(899.3482292498704 - 326.89440580503936 i) z^4 +
(522.7840561419488 - 169.27097662108687 i) z^5 -
(211.1342932244821 - 67.64907060016728 i) z^6 +
(61.819813901114195 - 22.331353485676786 i) z^7 -
(13.52923759262924 - 6.162203505463005 i) z^8 +
(2.2650644834902467 - 1.3878215687228064 i) z^9 -
(0.29490859956352167 - 0.24820529273554204 i) z^10 +
(0.03005186880695021 - 0.03466004231688902 i) z^11 -
(0.002369845310801477 - 0.00375092087177209 i) z^12 +
(0.00013891936203590943 - 0.00031358812397073305 i) z^13 -
(5.546518335751427 - 0.000020162419997292805 i) z^14 +
(1.1485356710117488 - 9.885211195309346 i) z^15 +
(1.6380368547837514 - 3.6548209158648697 i) z^16 -
(2.2390306818658378 - 1.0048058537999287 i) z^17 +
(8.619808760641594 - 2.0141218476377515 i) z^18 -
(1.885383255894536 - 2.8670326570284277 i) z^19 +
(2.5806065619431393 - 2.8379118121650123 i) z^20 -
(2.3914347064086936 - 2.0112780944037475 i) z^21 +
(1.5950302820319473 - 1.13566963696527 i) z^22 -
(7.7025343089350135 - 5.187880760531311 i) z^23 +
(1.9636884264889935 - 1.830344457362511 i) z^24 -
(1.5864116063510988 - 2.07170897448175 i) z^25 -
(2.9193336374987864 - 7.98361908841194 i) z^26 /
z -> 1.0000009675281858 + 3.6081952893011924 i

```

```

-3.34801 x 10^-13 + 5.20396 x 10^-13 i

```

```

eta[100 000 000, N@ZetaZero@10, z]

```

```

(1. + 0. i) - (19.305 + 171.071 i) z + (147.583 + 543.521 i) z^2 - (358.526 + 733.154 i) z^3 +
(431.217 + 569.367 i) z^4 - (311.539 + 290.624 i) z^5 + (150.299 + 105.524 i) z^6 -
(51.7726 + 28.7404 i) z^7 + (13.3094 + 6.05808 i) z^8 - (2.62276 + 0.994781 i) z^9 +
(0.400621 + 0.124838 i) z^10 - (0.0473431 + 0.0114548 i) z^11 + (0.0042793 + 0.00070475 i) z^12 -
(0.00029105 + 0.000022248 i) z^13 + (0.0000147163 - 3.47282 x 10^-7 i) z^14 -
(5.5459 x 10^-7 - 6.96649 x 10^-8 i) z^15 + (1.57383 x 10^-8 - 3.14973 x 10^-9 i) z^16 -
(3.33521 x 10^-10 - 6.88693 x 10^-11 i) z^17 + (5.48199 x 10^-12 - 3.17276 x 10^-13 i) z^18 -
(8.83361 x 10^-14 + 5.03437 x 10^-14 i) z^19 + (1.32478 x 10^-15 + 2.28761 x 10^-15 i) z^20 -
(1.18062 x 10^-17 + 4.31953 x 10^-17 i) z^21 + (4.02455 x 10^-20 + 3.92966 x 10^-19 i) z^22 +
(2.60574 x 10^-24 - 1.70906 x 10^-21 i) z^23 - (2.34875 x 10^-26 - 3.36209 x 10^-24 i) z^24 +
(2.97818 x 10^-28 - 2.37389 x 10^-27 i) z^25 + (2.80431 x 10^-32 + 3.01375 x 10^-31 i) z^26

```

```

etaZeros[100 000 000, N@ZetaZero@10]

```

```

{-49.6949 - 25.1369 i, -15.7218 + 33.7737 i, 0.000541244 - 0.0057767 i,
0.382278 + 1.90549 i, 0.976956 - 5.54277 i, 1. - 3.31977 x 10^-6 i, 1.72374 + 22.6148 i,
2.00143 + 0.00318553 i, 2.78763 - 0.114924 i, 3.15418 + 3.89716 i,
3.27105 + 6.44863 i, 3.3842 + 1.12569 i, 3.784 - 1.3535 i, 5.18499 + 0.0277175 i,
5.84783 - 4.8559 i, 7.18344 - 32.7999 i, 9.98146 - 0.585674 i, 12.4519 + 3.49082 i,
20.5116 - 9.48804 i, 37.9348 - 11.8786 i, 78.8952 - 0.225229 i, 80.6185 - 14.7305 i,
146.437 - 65.8895 i, 325.627 + 201.245 i, 875.187 - 404.029 i, 6155.18 + 2008.47 i}

```

$$\begin{aligned}
& (1. + 0. i) - (19.304967646701293 + 171.07087432715835 i) z + \\
& (147.58283657486345 + 543.521212613564 i) z^2 - \\
& (358.52601955478434 + 733.1543880954837 i) z^3 + \\
& (431.216624716372 + 569.3669970645752 i) z^4 - \\
& (311.5392316617495 + 290.6238818092178 i) z^5 + \\
& (150.29941124982744 + 105.52392268125644 i) z^6 - \\
& (51.7726108707123 + 28.740372270745798 i) z^7 + \\
& (13.309391454005784 + 6.058077642651738 i) z^8 - \\
& (2.622758685730081 + 0.994781479777612 i) z^9 + \\
& (0.4006207243016371 + 0.12483762910361951 i) z^{10} - \\
& (0.04734314037291354 + 0.011454833980272833 i) z^{11} + \\
& (0.0042792995287336925 + 0.0007047503054294357 i) z^{12} - \\
& (0.0002910503481999493 + 0.00002224803550082005 i) z^{13} + \\
& (0.000014716276182868146 - 3.472824181059795 i) z^{14} - \\
& (5.54590022969686 i - 6.966488619279004 i) z^{15} + \\
& (1.5738322559502116 i - 3.1497270341325565 i) z^{16} - \\
& (3.3352119929199406 i - 6.886927922954629 i) z^{17} + \\
& (5.4819934206654634 i - 3.172757015068706 i) z^{18} - \\
& (8.833612317424254 i + 5.034371070377254 i) z^{19} + \\
& (1.3247802017315104 i + 2.2876072747455275 i) z^{20} - \\
& (1.180622486626005 i + 4.319531249903527 i) z^{21} + \\
& (4.024554434701075 i + 3.9296644602094103 i) z^{22} + \\
& (2.605740098577079 i - 1.709062648343275 i) z^{23} - \\
& (2.3487544592050405 i - 3.362094609691875 i) z^{24} + \\
& (2.978181028802646 i - 2.3738904642440183 i) z^{25} + \\
& (2.804306343095682 i - 3.0137493437568326 i) z^{26} / . \\
& z \rightarrow 1.0000000399154443 - 3.31976516561226 i
\end{aligned}$$

$$4.28906 \times 10^{-14} - 3.37096 \times 10^{-14} i$$

eta[1 000 000 000, N@ZetaZero@1, z]

$$\begin{aligned}
& (1. + 0. i) - (93.5648 - 392.35 i) z + (225.017 - 1267.8 i) z^2 - \\
& (176.463 - 1743.28 i) z^3 + (5.36837 - 1374.61 i) z^4 + (87.4171 + 700.52 i) z^5 - \\
& (75.5523 + 244.251 i) z^6 + (36.2922 + 59.4331 i) z^7 - (11.9458 + 9.88393 i) z^8 + \\
& (2.89884 + 0.990177 i) z^9 - (0.536718 + 0.0136626 i) z^{10} + (0.0772313 - 0.0153263 i) z^{11} - \\
& (0.00872934 - 0.00338873 i) z^{12} + (0.000777866 - 0.000469932 i) z^{13} - \\
& (0.0000542162 - 0.0000502147 i) z^{14} + (2.854 \times 10^{-6} - 4.29003 \times 10^{-6} i) z^{15} - \\
& (1.01926 \times 10^{-7} - 2.89673 \times 10^{-7} i) z^{16} + (1.51985 \times 10^{-9} - 1.52128 \times 10^{-8} i) z^{17} + \\
& (6.92672 \times 10^{-11} + 6.17127 \times 10^{-10} i) z^{18} - (5.63363 \times 10^{-12} + 1.92407 \times 10^{-11} i) z^{19} + \\
& (2.04482 \times 10^{-13} + 4.55931 \times 10^{-13} i) z^{20} - (4.71337 \times 10^{-15} + 8.11539 \times 10^{-15} i) z^{21} + \\
& (7.23795 \times 10^{-17} + 1.08344 \times 10^{-16} i) z^{22} - (7.32319 \times 10^{-19} + 1.09077 \times 10^{-18} i) z^{23} + \\
& (4.78898 \times 10^{-21} + 8.17878 \times 10^{-21} i) z^{24} - (2.05179 \times 10^{-23} + 4.35777 \times 10^{-23} i) z^{25} + \\
& (5.91626 \times 10^{-26} + 1.4927 \times 10^{-25} i) z^{26} - (1.38071 \times 10^{-28} + 2.80186 \times 10^{-28} i) z^{27} + \\
& (1.55581 \times 10^{-31} + 2.96726 \times 10^{-31} i) z^{28} + (7.47556 \times 10^{-35} - 8.27372 \times 10^{-35} i) z^{29}
\end{aligned}$$

etaZeros[1 000 000 000, N@ZetaZero@1]

```
{-10.0127 + 12.4723 i, -4.01144 + 3.20132 i, -1.09667 - 10.4134 i,
 0.000557961 + 0.00242129 i, 1. - 2.0444 × 10-7 i, 1.66144 + 19.7489 i,
 1.99949 - 0.000125545 i, 2.36916 + 2.78868 i, 2.93515 - 0.00529677 i,
 3.2266 - 3.67003 i, 3.65815 - 0.0014028 i, 3.79497 + 1.58381 i, 3.82397 - 1.50088 i,
 5.56562 + 5.22071 i, 5.73226 - 5.678 i, 9.11472 - 156.208 i, 10.4981 + 0.15342 i,
 12.3661 - 0.109525 i, 14.4188 - 27.1005 i, 15.794 + 48.0736 i, 20.7779 + 0.540955 i,
 32.0328 - 2.57705 i, 44.8093 + 10.6115 i, 62.13 - 22.2897 i, 85.8335 + 82.5627 i,
 108.951 - 2980.4 i, 121.746 - 68.4464 i, 127.915 + 418.704 i, 352.045 - 146.525 i}
```

```
(1. + 0. i) - (93.56481491992147 - 392.35049348616826 i) z +
(225.0168916180092 - 1267.8035853745575 i) z2 -
(176.46300065210008 - 1743.2823688157448 i) z3 +
(5.368368711745703 - 1374.6112205131615 i) z4 +
(87.41706794545041 + 700.519675758609 i) z5 -
(75.552278664022 + 244.25110377019843 i) z6 +
(36.29224837739953 + 59.43313786763662 i) z7 -
(11.945820213278544 + 9.883929932840232 i) z8 +
(2.8988391975361836 + 0.9901767973373801 i) z9 -
(0.5367182439912863 + 0.013662612519050654 i) z10 +
(0.07723134739174053 - 0.015326319619118678 i) z11 -
(0.008729341832612092 - 0.0033887330747927047 i) z12 +
(0.000777865951489598 - 0.00046993169740686125 i) z13 -
(0.000054216226464051664 - 0.000050214744063834665 i) z14 +
(2.8539967697333024 i - 4.290026144459103 i) z15 -
(1.0192563367850214 i - 2.8967253396297837 i) z16 +
(1.5198509104898117 i - 1.521279231574943 i) z17 +
(6.926722719945014 i - 6.171269852637664 i) z18 -
(5.633629382748262 i - 1.9240746876646925 i) z19 +
(2.0448213994039887 i - 4.55931445315398 i) z20 -
(4.713373810906278 i - 8.115391776790271 i) z21 +
(7.237945733439867 i - 1.0834356932297339 i) z22 -
(7.323187943119842 i - 1.0907708096908202 i) z23 +
(4.788984143506904 i - 8.178778927042958 i) z24 -
(2.0517896647007632 i - 4.357773111143897 i) z25 +
(5.916264504311941 i - 1.4926985053368194 i) z26 -
(1.3807078438159591 i - 2.801859253499078 i) z27 +
(1.5558053295280045 i - 2.9672561911931094 i) z28 +
(7.475556324054337 i - 8.273716988262671 i) z29 / .
z -> 0.999999543667157 - 2.0444024033339393 i
```

$3.13053 \times 10^{-13} - 4.3248 \times 10^{-13} i$

**eta[10 000 000 000, N@ZetaZero@1, z]**

$$\begin{aligned}
 & (1. + 0. i) + (1349.62 + 286.85 i) z - (4933.77 + 1194.84 i) z^2 + \\
 & (7938.42 + 2153.69 i) z^3 - (7599.08 + 2235.79 i) z^4 + (4913.5 + 1501.74 i) z^5 - \\
 & (2304.68 + 693.153 i) z^6 + (821.145 + 226.35 i) z^7 - (229.447 + 52.2189 i) z^8 + \\
 & (51.4356 + 7.98979 i) z^9 - (9.4004 + 0.559707 i) z^{10} + (1.41661 - 0.0801028 i) z^{11} - \\
 & (0.177508 - 0.0329653 i) z^{12} + (0.0186243 - 0.00587783 i) z^{13} - \\
 & (0.00164619 - 0.000708021 i) z^{14} + (0.000123076 - 0.0000634468 i) z^{15} - \\
 & (7.7814 \times 10^{-6} - 4.4315 \times 10^{-6} i) z^{16} + (4.13498 \times 10^{-7} - 2.50264 \times 10^{-7} i) z^{17} - \\
 & (1.82658 \times 10^{-8} - 1.17729 \times 10^{-8} i) z^{18} + (6.62068 \times 10^{-10} - 4.66793 \times 10^{-10} i) z^{19} - \\
 & (1.94382 \times 10^{-11} - 1.54118 \times 10^{-11} i) z^{20} + (4.56156 \times 10^{-13} - 4.13042 \times 10^{-13} i) z^{21} - \\
 & (8.42616 \times 10^{-15} - 8.76504 \times 10^{-15} i) z^{22} + (1.2057 \times 10^{-16} - 1.44031 \times 10^{-16} i) z^{23} - \\
 & (1.32049 \times 10^{-18} - 1.79501 \times 10^{-18} i) z^{24} + (1.1064 \times 10^{-20} - 1.67585 \times 10^{-20} i) z^{25} - \\
 & (7.21693 \times 10^{-23} - 1.18316 \times 10^{-22} i) z^{26} + (3.78195 \times 10^{-25} - 6.55948 \times 10^{-25} i) z^{27} - \\
 & (1.59062 \times 10^{-27} - 2.89897 \times 10^{-27} i) z^{28} + (5.04986 \times 10^{-30} - 9.77167 \times 10^{-30} i) z^{29} - \\
 & (1.11632 \times 10^{-32} - 2.10332 \times 10^{-32} i) z^{30} + (1.11687 \times 10^{-35} - 2.33909 \times 10^{-35} i) z^{31} - \\
 & (1.73196 \times 10^{-39} - 1.54054 \times 10^{-38} i) z^{32} + (8.07148 \times 10^{-43} + 5.85322 \times 10^{-42} i) z^{33}
 \end{aligned}$$

**etaZeros[10 000 000 000, N@ZetaZero@1]**

$$\begin{aligned}
 & \{-3864.73 - 1018.79 i, -26.2144 + 176.538 i, -23.4619 - 181.371 i, \\
 & -4.12866 + 7.34176 i, -3.72207 - 18.2745 i, -1.57817 + 30.8767 i, \\
 & -0.000707144 + 0.000149944 i, 1. - 7.40352 \times 10^{-8} i, 1.99997 - 0.0012656 i, \\
 & 2.05938 - 5.29269 i, 2.49167 + 5.68823 i, 2.494 - 1.73396 i, 2.95185 + 0.0543244 i, \\
 & 3.22142 - 0.46806 i, 3.37038 + 0.822981 i, 4.08412 - 2.42114 i, 4.29356 + 2.34112 i, \\
 & 7.72879 - 0.00991479 i, 8.12663 - 5.15547 i, 8.14801 + 5.31789 i, 16.3646 + 6.26176 i, \\
 & 16.4991 - 0.0467327 i, 17.145 - 6.17218 i, 20.2719 - 33.6102 i, 26.1258 + 44.9315 i, \\
 & 40.7551 - 3.18062 i, 49.9124 + 24.7706 i, 58.4182 - 27.7668 i, 92.6007 + 108.398 i, \\
 & 120.861 - 70.0834 i, 228.429 - 79.127 i, 311.514 - 371.285 i, 330.177 + 764.904 i\}
 \end{aligned}$$

**1 - (0.9999999662682961` - 7.403523811955868`\*^-8 i)**

$$3.37317 \times 10^{-8} + 7.40352 \times 10^{-8} i$$

$$\begin{aligned}
& (1. + 0. i) + (1349.62100658792 + 286.8498080505834 i) z - \\
& (4933.767046855022 + 1194.8380079941549 i) z^2 + \\
& (7938.419402922736 + 2153.6869074308997 i) z^3 - \\
& (7599.078329205213 + 2235.79395754807 i) z^4 + \\
& (4913.4973898049075 + 1501.7397359818626 i) z^5 - \\
& (2304.6819900456385 + 693.1534636360454 i) z^6 + \\
& (821.1453803767897 + 226.350174587896 i) z^7 - \\
& (229.44721333805904 + 52.21892235943409 i) z^8 + \\
& (51.435605030068615 + 7.989794094186166 i) z^9 - \\
& (9.400404740033807 + 0.5597068774652687 i) z^{10} + \\
& (1.4166113819011805 - 0.08010284922221 i) z^{11} - \\
& (0.1775076598568166 - 0.03296531689764337 i) z^{12} + \\
& (0.01862425722739902 - 0.005877834137370738 i) z^{13} - \\
& (0.0016461853950728023 - 0.0007080207152759359 i) z^{14} + \\
& (0.00012307633055110706 - 0.00006344681279928446 i) z^{15} - \\
& (7.78139931694767 \times 10^{-6} - 4.431497723468613 \times 10^{-6} i) z^{16} + \\
& (4.134977156509175 \times 10^{-7} - 2.502639441025558 \times 10^{-7} i) z^{17} - \\
& (1.826578605342965 \times 10^{-8} - 1.1772850887173012 \times 10^{-8} i) z^{18} + \\
& (6.620677133647833 \times 10^{-10} - 4.667926818641298 \times 10^{-10} i) z^{19} - \\
& (1.943819834853632 \times 10^{-11} - 1.541175678603278 \times 10^{-11} i) z^{20} + \\
& (4.561558255872053 \times 10^{-13} - 4.130415185401319 \times 10^{-13} i) z^{21} - \\
& (8.426156210930456 \times 10^{-15} - 8.765038522692641 \times 10^{-15} i) z^{22} + \\
& (1.2057046641948994 \times 10^{-16} - 1.440308240380748 \times 10^{-16} i) z^{23} - \\
& (1.3204925393185037 \times 10^{-18} - 1.7950086833630073 \times 10^{-18} i) z^{24} + \\
& (1.1063972297236223 \times 10^{-20} - 1.6758494023649118 \times 10^{-20} i) z^{25} - \\
& (7.216928070205532 \times 10^{-23} - 1.1831642881169047 \times 10^{-22} i) z^{26} + \\
& (3.7819549796583117 \times 10^{-25} - 6.559484430096852 \times 10^{-25} i) z^{27} - \\
& (1.590619378093596 \times 10^{-27} - 2.89897061167791 \times 10^{-27} i) z^{28} + \\
& (5.049860536556247 \times 10^{-30} - 9.771668999741509 \times 10^{-30} i) z^{29} - \\
& (1.1163246671792968 \times 10^{-32} - 2.1033167439037223 \times 10^{-32} i) z^{30} + \\
& (1.1168665700750259 \times 10^{-35} - 2.3390868213424994 \times 10^{-35} i) z^{31} - \\
& (1.7319600603546155 \times 10^{-39} - 1.5405404480503026 \times 10^{-38} i) z^{32} + \\
& (8.071479154510946 \times 10^{-43} + 5.853223685484761 \times 10^{-42} i) z^{33} / . \\
& z \rightarrow 0.9999999662682961 - 7.403523811955868 \times 10^{-8} i \\
& -1.15745 \times 10^{-12} - 1.09283 \times 10^{-12} i
\end{aligned}$$

```
{-3864.7328617315134` - 1018.7917480697087` i, -26.214359820164013` + 176.53807196589221` i,
-23.461887522014894` - 181.37118071120756` i, -4.128663593333958` + 7.341759497684997` i,
-3.722068766150872` - 18.274499858225308` i, -1.5781706973673029` + 30.876724102637322` i,
-0.0007071438120573267` + 0.00014994380621850703` i,
0.9999999662682961` - 7.403523811955868`*^-8 i,
1.999968084113937` - 0.0012655996405297725` i, 2.0593753716546725` - 5.2926857234895754` i,
2.4916683803732025` + 5.688233071723109` i, 2.4940037466425227` - 1.7339648164919819` i,
2.9518531627411053` + 0.05432438493847853` i, 3.2214153680680755` - 0.4680597776087114` i,
3.3703750162010793` + 0.8229808697214802` i, 4.08412254033088` - 2.421137807383802` i,
4.293557238621552` + 2.3411248064398382` i, 7.728792429357549` - 0.009914790345078455` i,
8.126633535872168` - 5.155467876717536` i, 8.14800854999952` + 5.317894734174557` i,
16.364570250248597` + 6.261763579714102` i, 16.499092415260684` - 0.04673265169444403` i,
17.144990365604517` - 6.172178946996089` i, 20.271871489778263` - 33.61023087751103` i,
26.125753600509086` + 44.931463841569126` i, 40.75508735140905` - 3.180624294259425` i,
49.912442709036654` + 24.770554951654745` i, 58.41824306936651` - 27.76677628672389` i,
92.60070895220439` + 108.39807582397253` i, 120.86075540684915` - 70.08340961443614` i,
228.42915163935317` - 79.12703146061696` i, 311.5141992460766` - 371.28504351015414` i,
330.1772109180471` + 764.9039797867878` i} // TableForm
```

```
-3864.73 - 1018.79 i
-26.2144 + 176.538 i
-23.4619 - 181.371 i
-4.12866 + 7.34176 i
-3.72207 - 18.2745 i
-1.57817 + 30.8767 i
-0.000707144 + 0.000149944 i
1. - 7.40352 × 10-8 i
1.99997 - 0.0012656 i
2.05938 - 5.29269 i
2.49167 + 5.68823 i
2.494 - 1.73396 i
2.95185 + 0.0543244 i
3.22142 - 0.46806 i
3.37038 + 0.822981 i
4.08412 - 2.42114 i
4.29356 + 2.34112 i
7.72879 - 0.00991479 i
8.12663 - 5.15547 i
8.14801 + 5.31789 i
16.3646 + 6.26176 i
16.4991 - 0.0467327 i
17.145 - 6.17218 i
20.2719 - 33.6102 i
26.1258 + 44.9315 i
40.7551 - 3.18062 i
49.9124 + 24.7706 i
58.4182 - 27.7668 i
92.6007 + 108.398 i
120.861 - 70.0834 i
228.429 - 79.127 i
311.514 - 371.285 i
330.177 + 764.904 i
```

```
et[s_] := (1 - 2^(1 - s)) Zeta[s]
```

**Log@et[ZetaZero@1]**

$-\infty$

**0.9999999662682961` - 7.403523811955868`\*^-8 i**

**1. - 7.40352  $\times 10^{-8}$  i**

**etaZeros[1 000 000 000, N@ZetaZero@10] // TableForm**

**-165.584 - 104.483 i**

**-40.416 - 13.4713 i**

**-4.35653 - 0.0234938 i**

**0.0000910723 + 0.00210089 i**

**1. + 3.08594  $\times 10^{-7}$  i**

**1.67284 - 2.78239 i**

**2.00027 + 0.000484864 i**

**2.35747 + 40.3844 i**

**2.77589 + 5.44281 i**

**2.98254 + 0.0903273 i**

**3.31655 - 0.948376 i**

**3.60694 + 0.112764 i**

**3.9408 - 2.676 i**

**4.11219 + 1.77793 i**

**5.23945 + 22.9477 i**

**5.83163 + 5.65705 i**

**6.86781 - 5.55331 i**

**7.69546 - 40.0071 i**

**10.1459 + 0.0397372 i**

**11.7321 + 2.29403 i**

**16.9909 - 3.64455 i**

**29.9316 - 11.6287 i**

**38.1151 - 18.0407 i**

**77.7212 - 17.0822 i**

**104.343 + 22.8276 i**

**116.565 - 49.439 i**

**284.922 + 198.786 i**

**316.216 - 189.403 i**

**2706. + 1538.66 i**



```
etaZeros[1 000 000 000, N@ZetaZero@100] // TableForm
```

```
-1868.47 - 1086.67 i
-9.69191 + 10.8858 i
-9.44857 - 22.3523 i
-5.86101 - 2.10302 i
-0.00064706 - 0.000271471 i
0.577337 - 1.5008 i
1. - 6.61072 × 10-8 i
1.99996 - 0.00011076 i
3.00795 - 0.0169791 i
3.65143 - 1.46032 i
3.74378 - 0.364712 i
3.98086 + 0.379192 i
5.00325 - 183.726 i
5.88219 + 1.16875 i
6.67803 - 1.56609 i
8.28549 + 9.73198 i
9.75274 - 4.27455 i
10.6232 + 3.62859 i
11.3613 - 18.3927 i
11.4107 + 239.349 i
20.1913 - 63.4682 i
20.2705 - 10.3805 i
21.3182 + 17.8175 i
29.2344 + 34.4772 i
45.2309 + 82.7454 i
48.4178 - 27.6785 i
79.9948 - 25.2554 i
174.143 - 10.9472 i
356.018 + 105.371 i
```

```
etaZeros[1 000 000 000, N@ZetaZero@1000] // TableForm
```

-123.448 + 867.567 i  
-116.477 - 250.229 i  
-18.2552 - 4.10706 i  
-10.4484 - 15.445 i  
-5.32004 + 5.48851 i  
0.00186217 + 0.000964855 i  
 $0.999998 + 1.12776 \times 10^{-6} i$   
1.15844 + 0.0895617 i  
1.51321 + 3.27753 i  
2.00033 - 0.000099579 i  
3.01964 - 0.00918331 i  
3.50135 - 0.102747 i  
3.78369 + 1.87634 i  
4.14839 - 9.17109 i  
5.50192 - 0.860993 i  
5.92703 - 2.74934 i  
6.51589 + 5.85744 i  
8.24552 + 15.7857 i  
10.0418 - 5.08653 i  
13.0918 - 13.1054 i  
16.4033 + 30.3256 i  
19.5781 + 240.97 i  
22.21 + 65.0318 i  
27.9596 - 17.2299 i  
35.6098 - 71.9722 i  
44.1351 - 9.5827 i  
87.3871 - 9.82372 i  
162.747 + 16.3084 i  
389.126 - 122.321 i

**etaZeros[1 000 000 000, 2.] // TableForm**

```
-41.0585 - 211.586 i
-41.0585 + 211.586 i
-18.6392
-16.2345 - 9.65532 i
-16.2345 + 9.65532 i
-10.2071 - 16.835 i
-10.2071 + 16.835 i
-2.54153 - 21.1105 i
-2.54153 + 21.1105 i
3.57543 - 23.6483 i
3.57543 + 23.6483 i
8.76756 - 20.2914 i
8.76756 + 20.2914 i
14.7291 - 16.1728 i
14.7291 + 16.1728 i
17.841 - 44.2583 i
17.841 + 44.2583 i
19.3766 - 11.339 i
19.3766 + 11.339 i
22.8774 - 5.87528 i
22.8774 + 5.87528 i
24.7881
30.4516
56.9046 - 15.7 i
56.9046 + 15.7 i
117.75 - 74.2733 i
117.75 + 74.2733 i
346.871
2611.82
```

**eta[1 000 000 000, 2., z]**

$$1. - 0.195447 z + 0.0190997 z^2 - 0.00124433 z^3 + 0.0000607999 z^4 - 2.37659 \times 10^{-6} z^5 +$$

$$7.73956 \times 10^{-8} z^6 - 2.15377 \times 10^{-9} z^7 + 5.06526 \times 10^{-11} z^8 - 6.73798 \times 10^{-13} z^9 - 6.09754 \times 10^{-14} z^{10} +$$

$$1.15117 \times 10^{-14} z^{11} - 1.37601 \times 10^{-15} z^{12} + 1.30259 \times 10^{-16} z^{13} - 9.95823 \times 10^{-18} z^{14} + 6.15558 \times 10^{-19} z^{15} -$$

$$3.08075 \times 10^{-20} z^{16} + 1.24913 \times 10^{-21} z^{17} - 4.07852 \times 10^{-23} z^{18} + 1.05825 \times 10^{-24} z^{19} - 2.1503 \times 10^{-26} z^{20} +$$

$$3.36632 \times 10^{-28} z^{21} - 3.95496 \times 10^{-30} z^{22} + 3.37224 \times 10^{-32} z^{23} - 2.05858 \times 10^{-34} z^{24} +$$

$$9.50012 \times 10^{-37} z^{25} - 3.5933 \times 10^{-39} z^{26} + 1.06037 \times 10^{-41} z^{27} - 1.57803 \times 10^{-44} z^{28} + 4.67032 \times 10^{-48} z^{29}$$

**et[2]**

$$\frac{\pi^2}{12}$$

12

**Product[1 - 1/r, {r, etaZeros[1 000 000 000, 2.]}**

**]**  
0.822467 - 3.8707  $\times 10^{-19}$  i

**N@ZetaZero@1**

0.5 + 14.1347 i