```
s2[n_, s_] := Sum[ j^(-1/2) (s Cosh[ s Log[n/j]] - 1/2 Sinh[s Log[n/j]]) /
    (s Cosh[s Log[n]] + s Log[Pi] + Log[Gamma[1/2 - s/2]] -
      Log[Gamma[1/4 + s/2]] - (1/2) Sinh[s Log[n]] + s Log[Pi] +
      Log[Gamma[1/2 - s/2]] - Log[Gamma[1/4 + s/2]]), {j, 1, n}]
```

s3[n_, s_] := Sum[j^(-1/2) (2 s Cosh[s Log[n/j]] - Sinh[s Log[n/j]]) /

$$\left(2\,s\,\mathrm{Cosh}\left[s\,\mathrm{Log}[n] + s\,\mathrm{Log@Pi} + \mathrm{Log@Gamma}\left[\frac{1}{4} - \frac{s}{2}\right] - \mathrm{Log@Gamma}\left[\frac{1}{4} + \frac{s}{2}\right]\right] - \mathrm{Sinh}\left[s\,\mathrm{Log}[n] + s\,\mathrm{Log@Pi} + \mathrm{Log@Gamma}\left[\frac{1}{4} - \frac{s}{2}\right] - \mathrm{Log@Gamma}\left[\frac{1}{4} + \frac{s}{2}\right]\right]\right), \{j, 1, n\}\Big]$$

s4[n_, s_] := Sum[j^(-1/2) (s Cosh[s Log[n/j]] - (1/2) Sinh[s Log[n/j]]) /

$$\left(s\,\mathrm{Cosh}\left[s\,\mathrm{Log}[n] + s\,\mathrm{Log@Pi} + \mathrm{Log@Gamma}\left[\frac{1}{4} - \frac{s}{2}\right] - \mathrm{Log@Gamma}\left[\frac{1}{4} + \frac{s}{2}\right]\right] - (1/2)\,\mathrm{Sinh}\left[s\,\mathrm{Log}[n] + s\,\mathrm{Log@Pi} + \mathrm{Log@Gamma}\left[\frac{1}{4} - \frac{s}{2}\right] - \mathrm{Log@Gamma}\left[\frac{1}{4} + \frac{s}{2}\right]\right]\right), \{j, 1, n\}\Big]$$

s5[n_, s_] := Sum[j^(-1/2) Cosh[s Log[n/j] - ArcCoth[2 s]] / Cosh[

$$s\,\mathrm{Log}[n] + s\,\mathrm{Log@Pi} + \mathrm{Log@Gamma}\left[\frac{1}{4} - \frac{s}{2}\right] - \mathrm{Log@Gamma}\left[\frac{1}{4} + \frac{s}{2}\right] - \mathrm{ArcCoth}[2\,s]\Big], \{j, 1, n\}\Big]$$

```
s5a[n_, s_] := Sum[j^(-1/2) Cosh[s Log[n/j] - ArcCoth[2 s]] / Cosh[s Log[n] - ArcCoth[2 s]],
  {j, 1, n}]
```

s5b[n_, s_] := Sum[j^(-1/2) Cosh[s Log[n/j] - ArcCoth[2 s]] /

$$\mathrm{Cosh}\left[s\,\mathrm{Log}[n] - \mathrm{ArcCoth}[2\,s] + \mathrm{Log}\left[\mathrm{Pi}\verb|^|s\,\mathrm{Gamma}\left[\frac{1}{4} - \frac{s}{2}\right] \Big/ \mathrm{Gamma}\left[\frac{1}{4} + \frac{s}{2}\right]\right]\right], \{j, 1, n\}\Big]$$

s6[n_, s_] := Sum[j^(-1/2) Cos[s Log[n/j] + ArcCot[2 s]] /

$$\mathrm{Cos}\left[\mathrm{ArcCot}[2\,s] + s\,\mathrm{Log}[n] - i\,\mathrm{Log}\left[\frac{\pi^{i s}\,\mathrm{Gamma}\left[\frac{1}{4} - \frac{i s}{2}\right]}{\mathrm{Gamma}\left[\frac{1}{4} + \frac{i s}{2}\right]}\right]\right], \{j, 1, n\}\Big]$$

```
s6a[n_, s_] := Sum[j^(-1/2) Cos[s Log[n/j] + ArcCot[2 s]] / Cos[ArcCot[2 s] + s Log[n]],
  {j, 1, n}]

s5b[10 000, -.3 + 10 I]

1.66262 - 0.112298 i

Zeta[-.2 + 10 I]

1.86083 - 0.0867377 i

Chop@s5[100 000, N@ZetaZero@1 - 1/2]

-0.00164403

s6[10 000, -.7 I + 10]

1.86081 + 0.0867579 i
```

```
FullSimplify[Cosh[(s / I) Log[n] + (s / I) Log@Pi +
    Log@Gamma[1/4 - (s / I)/2] - Log@Gamma[1/4 + (s / I)/2] - ArcCoth[2 (s / I)]]]
```

$$\mathrm{Cos}\left[\mathrm{ArcCot}[2\,s] + s\,\mathrm{Log}[n\,\pi] - i\left(\mathrm{Log}\left[\mathrm{Gamma}\left[\frac{1}{4}(1-2\,i\,s)\right]\right] - \mathrm{Log}\left[\mathrm{Gamma}\left[\frac{1}{4}(1+2\,i\,s)\right]\right]\right)\right]$$

```
s6a[10 000, .3 I - 10]
```

1.4463 − 0.114155 ⅈ

```
Zeta[.8 + 10 I]
```

```
Log[ Pi^(s / I) Gamma[1/4 - (s / I)/2] / Gamma[1/4 + (s / I)/2]]
```

$$\mathrm{Log}\left[\frac{\pi^{-i\,s}\,\mathrm{Gamma}\left[\frac{1}{4}+\frac{i\,s}{2}\right]}{\mathrm{Gamma}\left[\frac{1}{4}-\frac{i\,s}{2}\right]}\right]$$

```
Cosh[(-(s I)) Log[n] - ArcCoth[2 (-(s I))] +
   Log[Pi^(-(s I)) Gamma[1/4 - (-(s I))/2] / Gamma[1/4 + (-(s I))/2]]]
```

$$\mathrm{Cos}\left[\mathrm{ArcCot}[2\,s] + s\,\mathrm{Log}[n] + i\,\mathrm{Log}\left[\frac{\pi^{-i\,s}\,\mathrm{Gamma}\left[\frac{1}{4}+\frac{i\,s}{2}\right]}{\mathrm{Gamma}\left[\frac{1}{4}-\frac{i\,s}{2}\right]}\right]\right]$$

```
Cosh[((s / I)) Log[n / j] - ArcCoth[2 ((s / I))]]
```

$$\mathrm{Cos}\left[\mathrm{ArcCot}[2\,s] + s\,\mathrm{Log}\left[\frac{n}{j}\right]\right]$$

```
ac[n_, t_] :=
 Sum[ j^(-1 / 2) (Cos[t Log[j]] + Tan[t Log[n] + ArcCot[2 t]] Sin[t Log[j]]), {j, 1, n}]
ac2[n_, t_] := Sum[ j^(-1 / 2) (Cos[t Log[j]] - I Sin[t Log[j]]), {j, 1, n}]
aca[n_, t_] :=
 Sum[ N[j^(-1 / 2) (Cos[t Log[j]] + Tan[t Log[n] + ArcCot[2 t]] Sin[t Log[j]])], {j, 1, n}]
ac2a[n_, t_] := Sum[N[ j^(-1 / 2) (Cos[t Log[j]] - I Sin[t Log[j]])], {j, 1, n}]
```

$\texttt{ac2b[n\_, t\_] := Sum}\left[j^{-\frac{1}{2}-i\,t}, \{j, 1, n\}\right]$

$\texttt{ac2c[n\_, t\_] := HarmonicNumber}\left[n, \frac{1}{2} + i\,t\right]$

```
aca[1 000 000, -.3 I + 5]
```

0.737982 + 0.198464 ⅈ

```
Zeta[.8 + 5 I]
```

0.738 + 0.198579 ⅈ

```
Tan[t Log[n] + Pi / 2 - ArcTan[2 t]]
```

Cot[ArcTan[2 t] − t Log[n]]

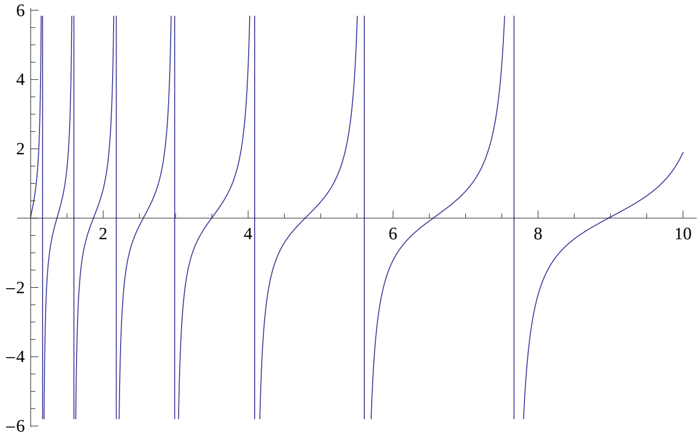
```
Limit[Tan[t Log[n] + ArcCot[2 t]] /. t → -.3 I + 5, n → Infinity]
```

0. − 1. ⅈ

```
Tan[t Log[n] + ArcCot[2 t]] /. n → 1 000 000 /. t → -.3 I + 5
```

0.0000626735 − 0.999496 ⅈ

```
Plot[Re[Tan[t Log[n] + ArcCot[2 t]]] /. t → 10, {n, 1, 10}]
```




```
ArcCot[100.]
```

0.00999967

```
FullSimplify[j^(-1/2) (Cos[t Log[j]] - I Sin[t Log[j]])]
```

$j^{-\frac{1}{2}-i t}$

Sum$\left[j^{-\frac{1}{2}-i t}, \{j, 1, n\}\right]$

HarmonicNumber$\left[n, \frac{1}{2} + i t\right]$

```
ext5[n_, x_] := (1/2) (HarmonicNumber[n, 1/2 - I x] + HarmonicNumber[n, 1/2 + I x]) +
   (Tan[x Log[n] + ArcCot[2 x]]
     ((1/(2 I)) (HarmonicNumber[n, (1/2 - I x)] - HarmonicNumber[n, (1/2 + I x)])))
```

$$\text{ext5a}[n\_, s\_] := \frac{1}{2}\left(\text{HarmonicNumber}\left[n, \frac{1}{2} - i s\right] + \text{HarmonicNumber}\left[n, \frac{1}{2} + i s\right]\right) - \frac{1}{2} i \left(\text{HarmonicNumber}\left[n, \frac{1}{2} - i s\right] - \text{HarmonicNumber}\left[n, \frac{1}{2} + i s\right]\right) \text{Tan}[\text{ArcCot}[2 s] + s \text{Log}[n]]$$

$$\text{ext5b}[n\_, s\_] := \frac{1}{2} \text{HarmonicNumber}\left[n, \frac{1}{2} - i s\right] (1 - i \text{Tan}[\text{ArcCot}[2 s] + s \text{Log}[n]]) + \frac{1}{2} \text{HarmonicNumber}\left[n, \frac{1}{2} + i s\right] (1 + i \text{Tan}[\text{ArcCot}[2 s] + s \text{Log}[n]])$$

$$\text{ext5c}[n\_, s\_] := \frac{1}{2} \text{HarmonicNumber}\left[n, \frac{1}{2} + s\right] (1 - \text{Tanh}[\text{ArcCoth}[2 s] - s \text{Log}[n]]) + \frac{1}{2} \text{HarmonicNumber}\left[n, \frac{1}{2} - s\right] (1 + \text{Tanh}[\text{ArcCoth}[2 s] - s \text{Log}[n]])$$

$$\text{ext5cx}[n\_, s\_] := \left\{\frac{1}{2} \text{HarmonicNumber}\left[n, \frac{1}{2} + s\right] (1 - \text{Tanh}[\text{ArcCoth}[2 s] - s \text{Log}[n]]), + \frac{1}{2} \text{HarmonicNumber}\left[n, \frac{1}{2} - s\right] (1 + \text{Tanh}[\text{ArcCoth}[2 s] - s \text{Log}[n]])\right\}$$

$$\text{ext5cy}[n\_, s\_] := \frac{1}{2} \text{HarmonicNumber}\left[n, \frac{1}{2} + s\right] (1 - \text{Tanh}[\text{ArcCoth}[2 s] - s \text{Log}[n]])$$

```
ext5b[100 000 000 000 000, .2 I + 5000]
```

0.545084 + 0.2518 i

```
ext5c[100 000 000 000, N@ZetaZero@1 - 1/2]
```

$1.58714 \times 10^{-6} + 0.\ i$

```
Zeta[N@ZetaZero@1 + .1 + .3 I]
```

0.0711203 + 0.235516 i

```
FullSimplify@ext5[n, s]
```

$$\text{HarmonicNumber}\left[n, \frac{1}{2} - i s\right] (1 - i \text{Tan}[\text{ArcCot}[2 s] + s \text{Log}[n]]) / 2 + \text{HarmonicNumber}\left[n, \frac{1}{2} + i s\right] (1 + i \text{Tan}[\text{ArcCot}[2 s] + s \text{Log}[n]]) / 2$$

$$\frac{1}{2} \text{HarmonicNumber}\left[n, \frac{1}{2} - i s\right] (1 - i \text{Tan}[\text{ArcCot}[2 s] + s \text{Log}[n]]) + \frac{1}{2} \text{HarmonicNumber}\left[n, \frac{1}{2} + i s\right] (1 + i \text{Tan}[\text{ArcCot}[2 s] + s \text{Log}[n]])$$

```
Zeta[.7 + 5000 I]
```

0.545082 - 0.251794 i

$\frac{1}{2}$ HarmonicNumber[n, $\frac{1}{2}$ - ⅈ (s / I)] (1 - ⅈ Tan[ArcCot[2 (s / I)] + (s / I) Log[n]]) +
$\frac{1}{2}$ HarmonicNumber[n, $\frac{1}{2}$ + ⅈ (s / I)] (1 + ⅈ Tan[ArcCot[2 (s / I)] + (s / I) Log[n]])

$\frac{1}{2}$ HarmonicNumber[n, $\frac{1}{2}$ + s] (1 - Tanh[ArcCoth[2 s] - s Log[n]]) +
$\frac{1}{2}$ HarmonicNumber[n, $\frac{1}{2}$ - s] (1 + Tanh[ArcCoth[2 s] - s Log[n]])

ext5cy[n_, s_] := $\frac{1}{2}$ HarmonicNumber[n, $\frac{1}{2}$ + s] (1 - Tanh[ArcCoth[2 s] - s Log[n]])

ext5cyc[n_, s_] := $\frac{1}{2}$ HarmonicNumber[n, s] (1 - Tanh[ArcCoth[2 s - 1] - (s - 1 / 2) Log[n]])

ext5cy2[n_, s_] := {$\frac{1}{2}$ HarmonicNumber[n, $\frac{1}{2}$ + s], (1 - Tanh[ArcCoth[2 s] - s Log[n]])}

ext5cy3[n_, s_] := $\frac{1}{2}$ HarmonicNumber[n, $\frac{1}{2}$ + s] (1 - Tanh[ArcCoth[2 s] - s Log[n]]) +
$\frac{1}{2}$ HarmonicNumber[n, $\frac{1}{2}$ - s] (1 + Tanh[ArcCoth[2 s] - s Log[n]])

ext5cyc[1 000 000 000 000, N@ZetaZero@1]

$4.91847 \times 10^{-7} + 69\,217.7\, i$

sb[n_, s_] := n^(s - 1 / 2) (1 - s) HarmonicNumber[n, s]
sb2[n_, s_] := n^(s - 1 / 2) ((1 - s) / s)^(1 / 2) HarmonicNumber[n, s]
sb3[n_, s_] := n^s (($\frac{1}{2}$ - s) / (1 / 2 + s))^(1 / 2) HarmonicNumber[n, $\frac{1}{2}$ + s]

sb3[10 000 000 000, N@ZetaZero@3 - 1 / 2]

$3997.46 - 4.9992 \times 10^{-6}\, i$

sb[n, s + 1 / 2]

$n^s \left(\frac{1}{2} - s\right)$ HarmonicNumber[n, $\frac{1}{2}$ + s]

ext5cyc[n_, s_] := $\frac{1}{2}$ HarmonicNumber[n, s] (1 - Tanh[ArcCoth[2 s - 1] - (s - 1 / 2) Log[n]])

ext5cyc2[n_, s_] := $\frac{1}{2}$ HarmonicNumber[n, s + 1 / 2] (1 - Tanh[ArcCoth[2 s] - s Log[n]])

ext5cyc5[1 000 000 000 000, N@ZetaZero@1 - .5]

$4.91858 \times 10^{-7} + 69\,217.7\, i$

ext5cyc[1 000 000 000 000, N@ZetaZero@1]

$4.91847 \times 10^{-7} + 69\,217.7\, i$

Tanh[ArcCoth[2 s - 1]]

$-\frac{1}{1 - 2 s}$

```
ext3[n_, x_] := Sum[(((1 / 2) (j^(-1 / 2 + I x) + j^(-1 / 2 - I x))) +
    Tan[x Log[n] + ArcCot[2 x]] ((1 / (2 I)) (j^(-1 / 2 + I x) - j^(-1 / 2 - I x)))), {j, 1, n}]
```

```
ext3[100 000, .2 I + 10]
```

```
1.47421 + 0.114901 i
```

```
Zeta[.7 + 10 I]
```

```
1.47708 - 0.114696 i
```

$$\text{ext5c}[n\_, s\_] := \frac{1}{2}\,\text{HarmonicNumber}\left[n, \frac{1}{2}+s\right](1-\text{Tanh}[\text{ArcCoth}[2 s]-s\,\text{Log}[n]]) + \frac{1}{2}\,\text{HarmonicNumber}\left[n, \frac{1}{2}-s\right](1+\text{Tanh}[\text{ArcCoth}[2 s]-s\,\text{Log}[n]])$$

$$\text{ext5d}[n\_, s\_] := \frac{1}{2}\,\text{HarmonicNumber}\left[n, \frac{1}{2}+s\right]$$

```
  (1 - ((E^(ArcCoth[2 s] - s Log[n]) - E^(-(ArcCoth[2 s] - s Log[n]))) /
      (E^(ArcCoth[2 s] - s Log[n]) + E^(-(ArcCoth[2 s] - s Log[n]))))) + 1/2 HarmonicNumber[
    n, 1/2 - s] (1 + ((E^(ArcCoth[2 s] - s Log[n]) - E^(-(ArcCoth[2 s] - s Log[n]))) /
      (E^(ArcCoth[2 s] - s Log[n]) + E^(-(ArcCoth[2 s] - s Log[n])))))
```

$$\text{ext5e}[n\_, s\_] := \frac{1}{2}\,\text{HarmonicNumber}\left[n, \frac{1}{2}+s\right]$$

```
  (1 - ((E^ArcCoth[2 s] E^(-s Log[n]) - E^(s Log[n]) E^(-ArcCoth[2 s])) / (E^(ArcCoth[2 s])
        E^(-s Log[n]) + E^(s Log[n]) E^(-ArcCoth[2 s])))) + 1/2 HarmonicNumber[n, 1/2 - s]
  (1 + ((E^ArcCoth[2 s] E^(-s Log[n]) - E^(s Log[n]) E^(-ArcCoth[2 s])) /
      (E^(ArcCoth[2 s]) E^(-s Log[n]) + E^(s Log[n]) E^(-ArcCoth[2 s]))))
```

$$\text{ext5f}[n\_, s\_] := \frac{1}{2}\,\text{HarmonicNumber}\left[n, \frac{1}{2}+s\right]\left(\frac{2\,n^{2 s}}{e^{2\,\text{ArcCoth}[2 s]}+n^{2 s}}\right) + \frac{1}{2}\,\text{HarmonicNumber}\left[n, \frac{1}{2}-s\right]\left(\frac{2}{1+e^{-2\,\text{ArcCoth}[2 s]}\,n^{2 s}}\right)$$

$$\text{ext5g}[n\_, s\_] := \frac{1}{2}\,\text{HarmonicNumber}\left[n, \frac{1}{2}+s\right]\left(\frac{2\,n^{2 s}}{\frac{1+2 s}{-1+2 s}+n^{2 s}}\right) + \frac{1}{2}\,\text{HarmonicNumber}\left[n, \frac{1}{2}-s\right]\left(\frac{2}{1+\frac{-1+2 s}{1+2 s}\,n^{2 s}}\right)$$

$$\text{ext5h}[n\_, s\_] := \text{HarmonicNumber}\left[n, \frac{1}{2}+s\right]\left(\frac{n^{2 s}}{\frac{2 s+1}{2 s-1}+n^{2 s}}\right) + \text{HarmonicNumber}\left[n, \frac{1}{2}-s\right]\left(\frac{1}{1+\frac{2 s-1}{2 s+1}\,n^{2 s}}\right)$$

$$\text{ext5i}[n\_, s\_] := \text{HarmonicNumber}\left[n, \frac{1}{2}+s\right]\left(\frac{1}{1+\frac{2 s+1}{2 s-1}\,n^{-2 s}}\right) + \text{HarmonicNumber}\left[n, \frac{1}{2}-s\right]\left(\frac{1}{1+\frac{2 s-1}{2 s+1}\,n^{2 s}}\right)$$

$$\text{ext5j}[n\_, s\_] := \text{HarmonicNumber}\left[n, \frac{1}{2}+s\right]\left(\frac{1}{1-\frac{1/2+s}{1/2-s}\,n^{-2 s}}\right) + \text{HarmonicNumber}\left[n, \frac{1}{2}-s\right]\left(\frac{1}{1-\frac{1/2-s}{1/2+s}\,n^{2 s}}\right)$$

```
ext5j[100 000 000, N@ZetaZero@1 - .5]
```

$-0.0000531084 + 0.\,i$

```
Zeta[.7 + 10 I]
```

$1.47708 - 0.114696\,i$

```
Expand[E^(ArcCoth[2 s])]
```

$e^{\text{ArcCoth}[2 s]}$

```
E^(-2 ArcCoth[ 2 s]) /. s → 1.2 + I
```

0.562982 + 0.257069 i

```
E^(-Log[(2 s + 1) / (2 s - 1)]) /. s → 1.2 + I
```

0.562982 + 0.257069 i

$\frac{-1+2 s}{1+2 s}$ /. s → 1.2 + I

0.562982 + 0.257069 i

```
E^(2 ArcCoth[ 2 s]) /. s → 1.2 + I
```

1.4698 - 0.671141 i

```
E^(Log[(2 s + 1) / (2 s - 1)]) /. s → 1.2 + I
```

1.4698 - 0.671141 i

$\frac{1+2 s}{-1+2 s}$ /. s → 1.2 + I

1.4698 - 0.671141 i

```
Integrate[ Cos[2 x] Cos[x], {x, -Pi, Pi}]
```

0

```
FullSimplify[ (1 - ((E^ArcCoth[2 s] E^(-s Log[n]) - E^(s Log[n]) E^(-ArcCoth[2 s])) /
    (E^(ArcCoth[2 s]) E^(-s Log[n]) + E^(s Log[n]) E^(-ArcCoth[2 s]))))]
```

$$\frac{2 n^{2 s}}{e^{2 \text{ArcCoth}[2 s]} + n^{2 s}}$$

```
FullSimplify[(1 + ((E^ArcCoth[2 s] E^(-s Log[n]) - E^(s Log[n]) E^(-ArcCoth[2 s])) /
    (E^(ArcCoth[2 s]) E^(-s Log[n]) + E^(s Log[n]) E^(-ArcCoth[2 s]))))]
```

$$\frac{2}{1 + e^{-2 \text{ArcCoth}[2 s]} n^{2 s}}$$

$e^{-2 \text{ArcCoth}[2 s]}$ /. s → 1.3

0.444444

$\frac{-1+2 s}{1+2 s}$ /. s → 1.3

0.444444

```
E^(-Log[(2 s + 1) / (2 s - 1)])
```

$$\frac{-1+2 s}{1+2 s}$$

FullSimplify$\left[\frac{2 s+1}{2 s-1}\right]$

$$1 + \frac{2}{-1+2 s}$$

$$\texttt{FullSimplify}\left[\texttt{HarmonicNumber}\left[n, \frac{1}{2}+s\right] \left(\frac{1}{1+\frac{2s+1}{2s-1} n^{-2s}}\right) + \texttt{HarmonicNumber}\left[n, \frac{1}{2}-s\right] \left(\frac{1}{1+\frac{2s-1}{2s+1} n^{2s}}\right) \texttt{ /. } s \to s - 1/2\right]$$

$$\frac{n\, s\, \texttt{HarmonicNumber}[n, 1-s] + n^{2s} (-1+s)\, \texttt{HarmonicNumber}[n, s]}{n^{2s} (-1+s) + n\, s}$$

$$\frac{1}{2} \texttt{HarmonicNumber}\left[n, \frac{1}{2}+s\right] (1 - \texttt{Tanh}[\texttt{ArcCoth}[2 s] - s\, \texttt{Log}[n]]) \texttt{ /. } n \to 100\,000\,000 \texttt{ /. } s \to \texttt{N@ZetaZero@1} - 1/2$$

$-0.0000265542 - 375.73\, i$

$$\texttt{HarmonicNumber}\left[n, \frac{1}{2}+s\right] \left(\frac{1}{1+\frac{2s+1}{2s-1} n^{-2s}}\right) \texttt{ /. } n \to 100\,000\,000 \texttt{ /. } s \to \texttt{N@ZetaZero@1} - 1/2$$

$-0.0000265542 - 375.73\, i$

$$\texttt{HarmonicNumber}\left[n, \frac{1}{2}+s\right] \left(\frac{1}{1+\frac{2s+1}{2s-1} n^{-2s}}\right) \texttt{ /. } n \to 100\,000\,000 \texttt{ /. } s \to \texttt{N@ZetaZero@1} - 1/2$$

$$\texttt{HarmonicNumber}\left[n, \frac{1}{2}+s\right] \left(1 - \frac{1/2+s}{1/2-s} n^{-2s}\right) \texttt{^} -1 \texttt{ /. } n \to 100\,000\,000 \texttt{ /. } s \to \texttt{N@ZetaZero@1} - 1/2$$

$-0.0000265542 - 375.73\, i$

$$\texttt{FullSimplify}\left[\texttt{HarmonicNumber}\left[n, \frac{1}{2}+s\right] \left(\frac{1}{1-\frac{1/2+s}{1/2-s} n^{-2s}}\right) + \texttt{HarmonicNumber}\left[n, \frac{1}{2}-s\right] \left(\frac{1}{1-\frac{1/2-s}{1/2+s} n^{2s}}\right) \texttt{ /. } s \to s - 1/2\right]$$

$$\frac{n\, s\, \texttt{HarmonicNumber}[n, 1-s] + n^{2s} (-1+s)\, \texttt{HarmonicNumber}[n, s]}{n^{2s} (-1+s) + n\, s}$$

```
o2[n_, s_] := n^s ((1 / 2 - s) / (1 / 2 + s)) ^ (1 / 2) HarmonicNumber[n, 1 / 2 + s]
o3[n_, s_] := n^s ((1 / 2 - s) / (1 / 2 + s)) ^ (1 / 2) HarmonicNumber[n, 1 / 2 + s]
```

```
o3[10 000 000, N@ZetaZero@1 - 1 / 2]
```

$223.584 - 0.000158015\, i$

$$\texttt{HarmonicNumber}\left[n, \frac{1}{2}+s\right] \left(1 - \frac{1/2+s}{1/2-s} n^{-2s}\right) \texttt{^} -1 \texttt{ /. } n \to 100\,000\,000 \texttt{ /. } s \to \texttt{N@ZetaZero@1} - 1/2$$

$-0.0000265542 - 375.73\, i$

$$\frac{1}{2}\,\text{HarmonicNumber}\left[n,\frac{1}{2}+s\right]\,(1-\text{Tanh}[\text{ArcCoth}[2\,s]-s\,\text{Log}[n]])+\\ \frac{1}{2}\,\text{HarmonicNumber}\left[n,\frac{1}{2}-s\right]\,(1+\text{Tanh}[\text{ArcCoth}[2\,s]-s\,\text{Log}[n]])\ /.\ s\to s-1/2$$

$$\frac{1}{2}\,\text{HarmonicNumber}[n,s]\left(1-\text{Tanh}\left[\text{ArcCoth}\left[2\left(-\frac{1}{2}+s\right)\right]-\left(-\frac{1}{2}+s\right)\text{Log}[n]\right]\right)+\\ \frac{1}{2}\,\text{HarmonicNumber}[n,1-s]\left(1+\text{Tanh}\left[\text{ArcCoth}\left[2\left(-\frac{1}{2}+s\right)\right]-\left(-\frac{1}{2}+s\right)\text{Log}[n]\right]\right)$$

$$\text{bb}[n\_,s\_]:=\frac{1}{2}\,\text{HarmonicNumber}[n,s]\left(1-\text{Tanh}\left[\text{ArcCoth}\left[2\left(-\frac{1}{2}+s\right)\right]-\left(-\frac{1}{2}+s\right)\text{Log}[n]\right]\right)+\\ \frac{1}{2}\,\text{HarmonicNumber}[n,1-s]\left(1+\text{Tanh}\left[\text{ArcCoth}\left[2\left(-\frac{1}{2}+s\right)\right]-\left(-\frac{1}{2}+s\right)\text{Log}[n]\right]\right)$$

```
bb2[n_, s_] := HarmonicNumber[n, s] (1 / 2 - Tanh[ArcCoth[2 s - 1] + (1 / 2 - s) Log[n]] / 2) +
  HarmonicNumber[n, 1 - s] (1 / 2 + Tanh[ArcCoth[2 s - 1] + (1 / 2 - s) Log[n]] / 2)
```

$$\text{bb3}[n\_,s\_]:=\frac{1}{2}\,\text{HarmonicNumber}[n,1-s]+\frac{1}{2}\,\text{HarmonicNumber}[n,s]-\\ \frac{1}{2}\,\text{HarmonicNumber}[n,1-s]\,\text{Tanh}\left[\text{ArcCoth}[1-2\,s]-\left(\frac{1}{2}-s\right)\text{Log}[n]\right]+\\ \frac{1}{2}\,\text{HarmonicNumber}[n,s]\,\text{Tanh}\left[\text{ArcCoth}[1-2\,s]-\left(\frac{1}{2}-s\right)\text{Log}[n]\right]$$

```
bb3[1000, 2.]
```

1.64494

```
Zeta[.7 + 31 I]
```

0.539212 + 0.181647 i

$$\text{FullSimplify}\left[\left(1-\text{Tanh}\left[\text{ArcCoth}\left[2\left(-\frac{1}{2}+s\right)\right]-\left(-\frac{1}{2}+s\right)\text{Log}[n]\right]\right)\Big/\,2\right]$$

$$\frac{1}{2}\left(1+\text{Tanh}\left[\text{ArcCoth}[1-2\,s]-\frac{\text{Log}[n]}{2}+s\,\text{Log}[n]\right]\right)$$

```
Expand[HarmonicNumber[n, s] (1 / 2 - Tanh[ArcCoth[2 s - 1] + (1 / 2 - s) Log[n]] / 2) +
  HarmonicNumber[n, 1 - s] (1 / 2 + Tanh[ArcCoth[2 s - 1] + (1 / 2 - s) Log[n]] / 2)]
```

$$\frac{1}{2}\,\text{HarmonicNumber}[n,1-s]+\frac{1}{2}\,\text{HarmonicNumber}[n,s]-\\ \frac{1}{2}\,\text{HarmonicNumber}[n,1-s]\,\text{Tanh}\left[\text{ArcCoth}[1-2\,s]-\left(\frac{1}{2}-s\right)\text{Log}[n]\right]+\\ \frac{1}{2}\,\text{HarmonicNumber}[n,s]\,\text{Tanh}\left[\text{ArcCoth}[1-2\,s]-\left(\frac{1}{2}-s\right)\text{Log}[n]\right]$$

$$\text{HarmonicNumber}\left[n,\frac{1}{2}+s\right]\left(1-\frac{1/2+s}{1/2-s}\,n^{-2\,s}\right)\text{^}-1\ /.\ n\to 100\,000\,000\ /.\ s\to \text{N@ZetaZero@1}-1/2$$

-0.0000265542 - 375.73 i

```
HarmonicNumber[n, 1/2 + s] ((1/2 + s)/(1/2 - s))^(-1/2) / (((1/2 + s)/(1/2 - s))^(-1/2) - ((1/2 + s)/(1/2 - s))^(1/2) n^(-2 s)) /.
  n → 100 000 000 /. s → N@ZetaZero@1 - 1/2
```

$-0.0000265542 - 375.73\,i$

```
HarmonicNumber[n, 1/2 + s] ((1/2 + s)/(1/2 - s))^(-1/2)
  n^s / (((1/2 + s)/(1/2 - s))^(-1/2) n^s - ((1/2 + s)/(1/2 - s))^(1/2) n^-s) /.
  n → 100 000 000 000 /. s → N@ZetaZero@1 - 1/2
```

$7.93571\times10^{-7} + 11230.6\,i$

```
HarmonicNumber[n, 1/2 + s] ((1/2 - s)/(1/2 + s))^(1/2)
  n^s / (((1/2 - s)/(1/2 + s))^(1/2) n^s - ((1/2 - s)/(1/2 + s))^(-1/2) n^-s) /.
  n → 100 000 000 000 /. s → N@ZetaZero@1 - 1/2
```

$7.93571\times10^{-7} + 11230.6\,i$

```
HarmonicNumber[n, 1/2 + s] ((1/2 - s)/(1/2 + s))^(1/2) n^s /. n → 100 000 000 000 /. s → N@ZetaZero@1 - 1/2
```

$22358.4 - 1.57988\times10^{-6}\,i$

```
HarmonicNumber[n, 1/2 + s] (1/2 - s) n^s /. n → 10 000 000 000 000 000 /. s → N@ZetaZero@1 - 1/2
```

$1.\times10^{8} + 2.08616\times10^{-7}\,i$

```
HarmonicNumber[n, 1/2 + s] (1/2 - s) n^s / ((1/2 - s) n^s - (1/2 + s) n^-s) /.
  n → 100 000 000 000 000 /. s → N@ZetaZero@1 - 1/2
```

$-4.61587\times10^{-8} - 357759.\,i$

```
HarmonicNumber[n, 1/2 + s] (1/2 - s) n^s / ((1/2 - s) n^s - (1/2 + s) n^-s) /. s → s - 1/2
```

$$\frac{n^{-\frac{1}{2}+s}\,(1-s)\,\mathrm{HarmonicNumber}[n,\,s]}{n^{-\frac{1}{2}+s}\,(1-s) - n^{\frac{1}{2}-s}\,s}$$

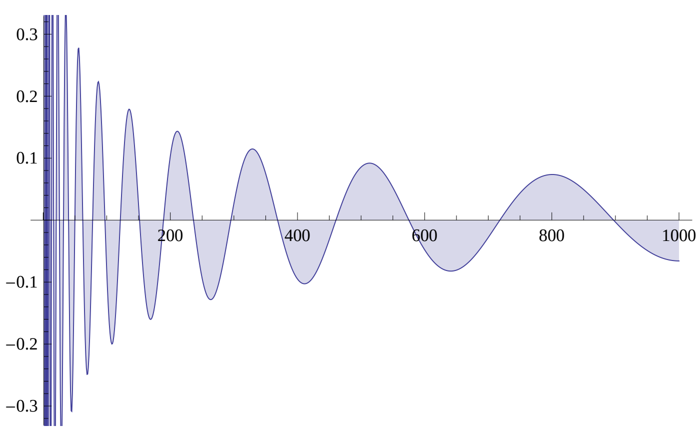
```
HarmonicNumber[n, 1/2 + s] (1 - (1/2 + s)/(1/2 - s) n^(-2 s))^-1 /. s → s - 1/2
```

$$\frac{\mathrm{HarmonicNumber}[n,\,s]}{1-\frac{n^{-2\left(-\frac{1}{2}+s\right)}\,s}{1-s}}$$

$$\mathtt{FullSimplify}\left[\frac{1}{1-\frac{n^{-2\left(-\frac{1}{2}+s\right)}\,s}{1-s}}\right]$$

$$\frac{1}{1+\frac{n^{1-2s}\,s}{-1+s}}$$

```
cc[n_, s_] := Sum[1/2 (1 + Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]]) j^-s, {j, 1, n}]
cd[n_, s_] := Sum[ (1 + Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]]) j^-s +
   (1 - Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]]) j^(s - 1), {j, 1, n}]
cd2[n_, s_] := (1 / 2) Sum[j^-s + j^(s - 1) + Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]] j^-s -
    Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]] j^(s - 1), {j, 1, n}]
cd3[n_, s_] := (1 / 2) Sum[j^-s + j^(s - 1) +
    Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]] (j^-s - j^(s - 1)), {j, 1, n}]
cd4[n_, s_] := (1 / 2) Sum[j^-s + j^(s - 1) +
    Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]] (j^-s - j^(s - 1)), {j, 1, n}]
cd4a[n_, s_] := DiscretePlot[Re[j^-s + j^(s - 1) +
    Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]] (j^-s - j^(s - 1))], {j, 1, n}]
cd4a[1000, N@ZetaZero@1]
```



```
FullSimplify[(1 / 2) Sum[
   j^-s + j^(s - 1) + Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]] (j^-s - j^(s - 1)), {j, 1, n}]]
```

$$\frac{1}{2}\left(\text{HarmonicNumber}[n, 1-s]+\text{HarmonicNumber}[n, s]+\left(-\text{HarmonicNumber}[n, 1-s]+\text{HarmonicNumber}[n, s]\right)\text{Tanh}\left[\text{ArcCoth}[1-2s]-\frac{\text{Log}[n]}{2}+s\,\text{Log}[n]\right]\right)$$

```
pd[n_, s_] := (1 / 2) Sum[
   j^-s + j^(s - 1) + Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]] (j^-s - j^(s - 1)), {j, 1, n}]
pdp[n_, s_] := (1 / 2) Sum[j^-s - j^(s - 1) +
    Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]] (j^-s + j^(s - 1)), {j, 1, n}]
pd0[n_, s_, j_] := j^-s + j^(s - 1) + Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]]
   (j^-s - j^(s - 1))
pda[n_, s_, j_] := {j^-s, +j^(s - 1), +Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]] (j^-s),
  Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]] (-j^(s - 1))}
pdb[n_, s_, j_] := {j^-s + j^(s - 1),
  +Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]] (j^-s - j^(s - 1))}
pdc[n_, s_, j_] := { j^-s + Tanh[ArcCoth[1 - 2 s] + (s - 1 / 2) Log[n]] (j^-s),
  +Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]] (-j^(s - 1)) + j^(s - 1)}
pdc2[n_, s_] := (1 / 2) Sum[j^-s + Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]] (j^-s), {j, 1, n}]
pdc3[n_, s_] := (1 / 2) (1 + Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]]) HarmonicNumber[n, s]
pdd[n_, s_, j_] := { j^-s + Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]] (-j^(s - 1)),
  +Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]] (j^-s) + j^(s - 1)}
pdt[n_, s_, t_] := (1 / 2) Sum[j^-(s + t I) + j^((s + t I) - 1) +
    Tanh[ArcCoth[1 - 2 (s + t I)] - Log[n]/2 + (s + t I) Log[n]]
     (j^-(s + t I) - j^((s + t I) - 1)), {j, 1, n}]
pdt2[n_, s_, t_] := (1 / 2) Sum[j^-(s + t I) + j^((s + t I) - 1) +
    Tanh[ArcCoth[1 - 2 (s + t I)] - Log[n]/2 + (s + t I) Log[n]]
     (j^-(s + t I) - j^((s + t I) - 1)), {j, 1, n}]
```

```
pdc3[10 000 000 000 000, .4 + 10 I]
```

12 631. − 9571.55 ⅈ

```
pdc[100 000, N@ZetaZero@1, 2]
```

{−0.484159 + 0.703598 ⅈ, −0.484159 − 0.703598 ⅈ}

```
FullSimplify[j^(1/2 + b I) + j^(1 - (1/2 + b I)) - c (j^(1/2 + b I) - j^(1 - (1/2 + b I)))]
```

$j^{\frac{1}{2}-i b}\left(1+c-(-1+c)\, j^{2 i b}\right)$

```
FullSimplify[j^(a + b I) + j^(1 - (a + b I)) - c (j^(a + b I) - j^(1 - (a + b I)))]
```

$(1+c)\, j^{1-a-i b}-(-1+c)\, j^{a+i b}$

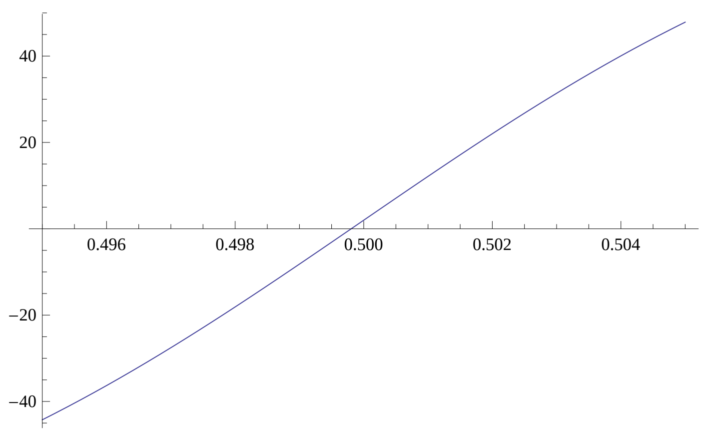
```
pdt3[100 000, 0, N@Im@ZetaZero@1]
```

0.542101 − 0.0498584 ⅈ

```
j^-(s + t I) + j^((s + t I) - 1) +
 Tanh[ArcCoth[1 - 2 (s + t I)] - Log[n]/2 + (s + t I) Log[n]] (j^-(s + t I) - j^((s + t I) - 1))
```

$j^{-s-i t}+j^{-1+s+i t}+\left(j^{-s-i t}-j^{-1+s+i t}\right) \mathrm{Tanh}\left[\mathrm{ArcCoth}[1-2(s+i t)]-\frac{\mathrm{Log}[n]}{2}+(s+i t)\, \mathrm{Log}[n]\right]$

```
Plot[Re[pdc3[1 000 000, s + N@ZetaZero@1 - .5 + 1 I]], {s, .495, .505}]
```



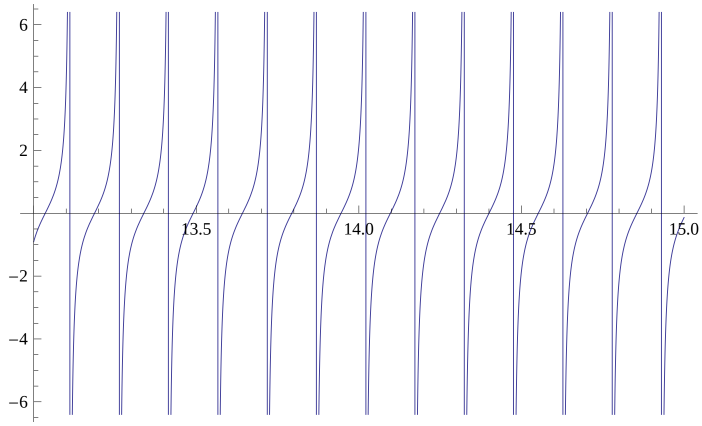

```
Re[pdc3[10 000 000 000 000, .5 + N@ZetaZero@1 - .5 + 1 I]]
```

−0.216907

```
(1 - Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]]) / 2 /. s → N@ZetaZero@1 + .1 /.
 n → 1 000 000 000 000 000 000
```

−0.000250713 − 0.0000182582 ⅈ

```
Plot[ Im[ Tanh[ArcCoth[1 - 2 (1 / 2 + s I)] - Log[n]/2 + (1 / 2 + s I) Log[n]]] /. n → 1 000 000 000,
 {s, 13, 15}]
```




```
(1 / 2) (1 + Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]]) /. s → N@ZetaZero@10 + .1 /. n → 10 000 000
```

1.02716 + 0.030594 ⅈ

```
(1 / 2) (1 + Tanh[ArcCoth[-2 s] + s Log[n]]) /. s → N@ZetaZero@10 - .5 + .1 /. n → 1 000 000 000
```

1.0077 + 0.0139921 ⅈ

```
pdc3x[n_, s_] := {(1 + Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]]) / 2, HarmonicNumber[n, s],
   (1 + Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]]) / 2 HarmonicNumber[n, s]}
```

```
pdc3x[100 000 000, N@ZetaZero@1 + .1]
```

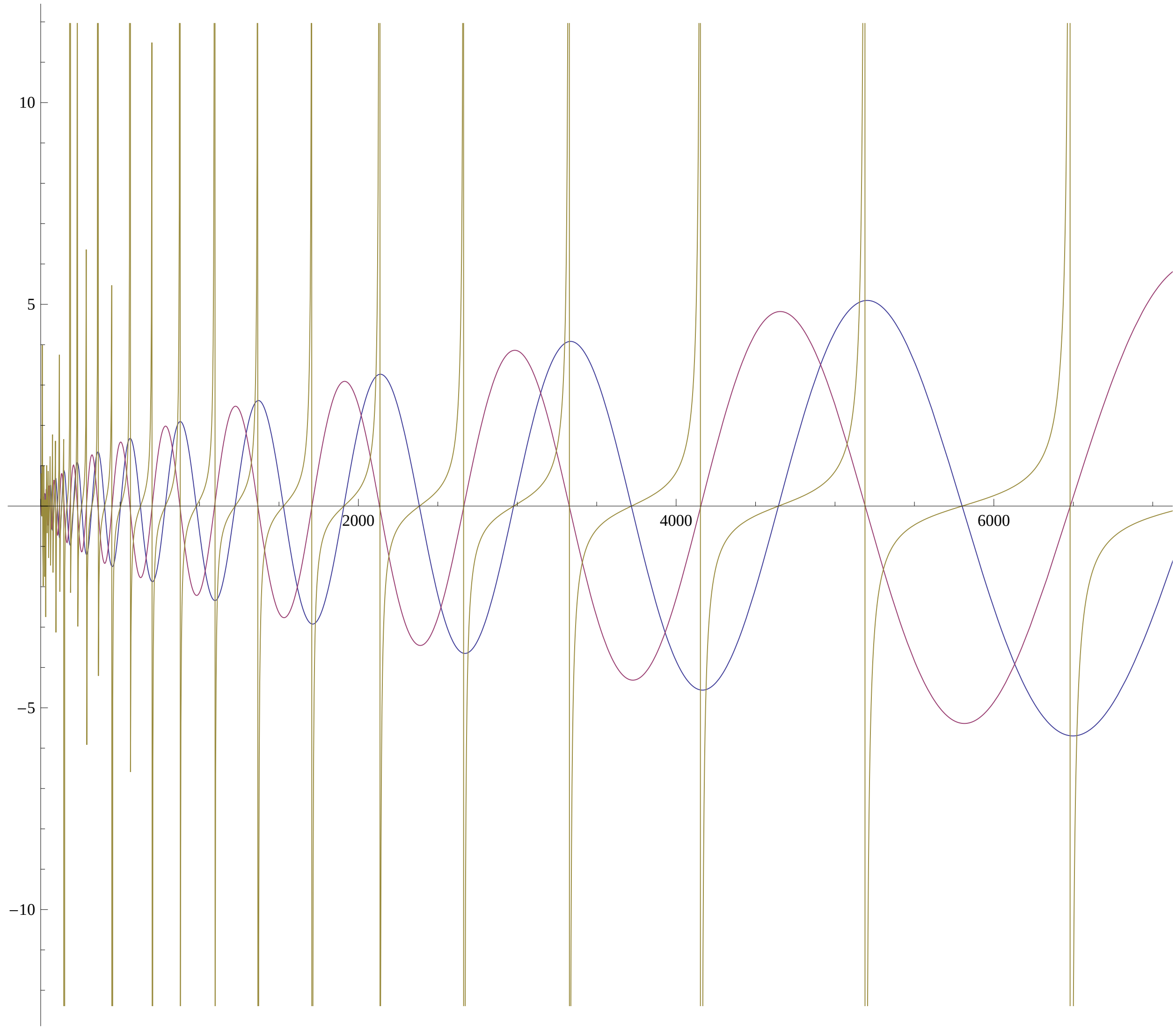
{0.980761 - 0.0154127 ⅈ, 38.786 - 105.174 ⅈ, 36.4188 - 103.749 ⅈ}

```
(0.5` - 1.3930927462873348` ⅈ) (-6654.70681368401` + 2388.4651062216653` ⅈ)
```

$7.39581 \times 10^{-6} + 10\,464.9\, i$

```
dx[n2_, s_] := Plot[{ Re[ HarmonicNumber[ n, s]], Im[ HarmonicNumber[ n, s]],
    Im[(1 + Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + N@s Log[n]]) / 2]}, {n, 1, n2}]
```

```
dx[10 000, N@ZetaZero@1]
```




```
pdc3e[100 000, 1 / 2, N@Im@ZetaZero@1]
```

−6.26931 + 4.24706 ⅈ

```
Tanh[-x]
```

−Tanh[x]

```
ook[n_, s_] := 1/2 (HarmonicNumber[n, 1 - s] +
    HarmonicNumber[n, s] + (-HarmonicNumber[n, 1 - s] + HarmonicNumber[n, s])
      Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]])
ook2[n_, s_] := {1/2 (HarmonicNumber[n, s] (1 + Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]])),
  1/2 (HarmonicNumber[n, 1 - s] (1 - Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]]))}
ook2x[n_, s_] := {{HarmonicNumber[n, s], (1 + Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]])},
  {HarmonicNumber[n, 1 - s], (1 - Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]])}}
ook2a[n_, s_] := Re[1/2 (HarmonicNumber[n, s] (1 + Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]]))] +
  Re[1/2 (HarmonicNumber[n, 1 - s] (1 - Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]]))]
ook3[n_, s_] := {1/2 (HarmonicNumber[n, 1 - s] + HarmonicNumber[n, s]),
  (1 / 2) ((HarmonicNumber[n, s] - HarmonicNumber[n, 1 - s])
      Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]])}
ook3a[n_, s_] := 1/2 (HarmonicNumber[n, s] + (HarmonicNumber[n, s])
      Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]])
ook3b[n_, s_] := (1 / 2) (HarmonicNumber[n, 1 - s] -
    (HarmonicNumber[n, 1 - s]) Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]])
ook4[n_, s_] := {1/2 (HarmonicNumber[n, 1 - s] + HarmonicNumber[n, s]),
  (1 / 2) ((HarmonicNumber[n, s] - HarmonicNumber[n, 1 - s])
      Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]])}
ook4o[n_, s_] := {{HarmonicNumber[n, s], HarmonicNumber[n, 1 - s]}, {HarmonicNumber[n, s],
   -HarmonicNumber[n, 1 - s], Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]]}}
ook4a[n_, s_] := 1/2 (HarmonicNumber[n, 1 - s] + HarmonicNumber[n, s])
ook4b[n_, s_] := (1 / 2)
  ((HarmonicNumber[n, s] - HarmonicNumber[n, 1 - s]) Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]])
ook[1 000 000 000 000 000 000 000, N@ZetaZero@1 + .1]
0.0753326 + 0.011364 i
```

```
Zeta[10 I + .6]
```

1.50992 - 0.115339 i

```
ook3[10 000 000 000, 10 I + .6]
```

{-41 919. + 27 425.3 i, 41 920.5 - 27 425.5 i}

```
ook4o[1 000 000, .9 + 20 I]
```

{{0.629021 - 0.538723 i, -1314.46 - 12 476.1 i},
 {0.629021 - 0.538723 i, 1314.46 + 12 476.1 i, 0.999969 - 7.84716 × 10^{-6} i}}

```
(0.6290209794817144` - 0.5387228886380006` i) + (-1314.456626857418` - 12476.12256231477` i)
```

-1313.83 - 12 476.7 i

```
-((0.6290209794817144` - 0.5387228886380006` i) +
    (1314.456626857418` + 12476.12256231477` i))
 (0.9999692566250131` - 7.847163490931001`*^-6 i)
```

-1315.14 - 12 475.2 i

```
-((0.6290209794817144` - 0.5387228886380006` i) + (1314.456626857418` + 12476.12256231477` i))
```

-1315.09 - 12 475.6 i

```
ook4o[1 000 000, N@ZetaZero@10 + .3]
```

{{0.407264 - 0.179835 i, 424.325 + 1194.49 i},
 {0.407264 - 0.179835 i, -424.325 - 1194.49 i, 0.999614 - 0.000321891 i}}

```
ook2[100 000 000 000, N@ZetaZero@10 + .1]
```

{-401.038 - 307.544 i, 401.154 + 307.601 i}

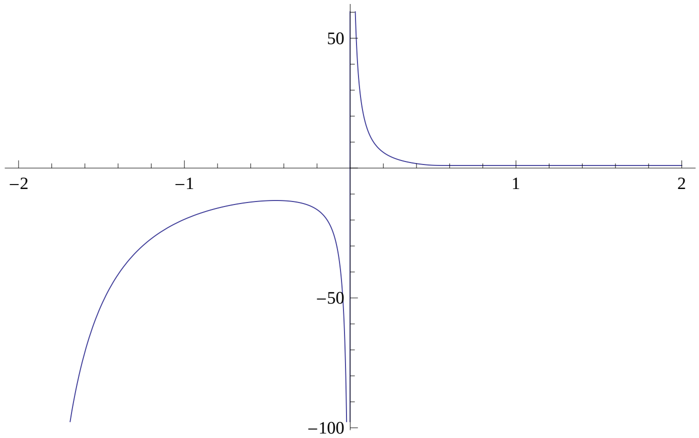
```
FullSimplify[j^-s - j^(s - 1)]
```

$-j^{-1+s} + j^{-s}$

```
tt[n_, s_] := (1 + Tanh[ArcCoth[1 - 2 s] + (s - 1 / 2) Log[n]]) / 2
tt2[n_, s_] := (1 - Tanh[ArcCoth[1 - 2 s] + (s - 1 / 2) Log[n]]) / 2
tt3[n_, s_] :=
 (1 (1 + Tanh[ArcCoth[1 - 2 s] + (s - 1 / 2) Log[n]]) + (2^s Pi^(s - 1) Sin[Pi s / 2] Gamma[1 - s])
     (1 - Tanh[ArcCoth[1 - 2 s] + (s - 1 / 2) Log[n]])) / 2
```

tt4[n_, s_] := $\Big($1 (1 + Tanh[ArcCoth[1 - 2 s] + (s - 1 / 2) Log[n]]) +
 $\left(2^{1-s}\,\pi^{-s}\right.$ Gamma[s] Sin$\left[\frac{1}{2}\,\pi\,(1-s)\right]\Big)$ (1 - Tanh[ArcCoth[1 - 2 s] + (s - 1 / 2) Log[n]])$\Big)$ / 2

```
Plot[ tt4[10 000 000 000 000 000, t], {t, -2, 2}]
```




```
(2^s Pi^(s - 1) Sin[Pi s / 2] Gamma[1 - s]) Zeta[1 - s] /. s → 2.0000000001
```

1.64493

```
tt4[10 000 000 000, 1 - s] Zeta[1 - s] /. s → 2.0000000001
```

1.64493

```
(2^s Pi^(s - 1) Sin[Pi s / 2] Gamma[1 - s]) /. s → 1 - s
```

$2^{1-s}\,\pi^{-s}\,\text{Gamma}[s]\ \text{Sin}\left[\frac{1}{2}\,\pi\,(1-s)\right]$

```
FullSimplify[(1 (1 + Tanh[ArcCoth[1 - 2 s] + (s - 1 / 2) Log[n]]) +
    f[s] (1 - Tanh[ArcCoth[1 - 2 s] + (s - 1 / 2) Log[n]])) / 2]
```

$\frac{1}{2}\left(1+f[s]-(-1+f[s])\ \text{Tanh}\left[\text{ArcCoth}[1-2\,s]+\left(-\frac{1}{2}+s\right)\text{Log}[n]\right]\right)$

```
FullSimplify[(1/2 (HarmonicNumber[n, 1 - s] +
      HarmonicNumber[n, s] + (-HarmonicNumber[n, 1 - s] + HarmonicNumber[n, s])
       Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]]))
  ((1 (1 + Tanh[ArcCoth[1 - 2 s] + (s - 1 / 2) Log[n]]) +
      f[s] (1 - Tanh[ArcCoth[1 - 2 s] + (s - 1 / 2) Log[n]])) / 2)]
```

$\left(\left(n^{2\,s}\,(-1+s)+n\,s\,f[s]\right)\left(n\,s\,\text{HarmonicNumber}[n,\,1-s]+n^{2\,s}\,(-1+s)\,\text{HarmonicNumber}[n,\,s]\right)\right)\Big/\left(n^{2\,s}\,(-1+s)+n\,s\right)^{2}$

```
ff[s_] := (2^s Pi^(s - 1) Sin[Pi s / 2] Gamma[1 - s])
ach[n_, s_] :=
 ((n^(2 s) (-1 + s) + n s ff[s]) (n s HarmonicNumber[n, 1 - s] + n^(2 s) (-1 + s) HarmonicNumber[n, s])) /
  (n^(2 s) (-1 + s) + n s)^2
ach2[n_, s_] := ((n^s (-1 + s) + n^(1 - s) s ff[s]) (n^(1 - s) s HarmonicNumber[n, 1 - s] +
      n^s (-1 + s) HarmonicNumber[n, s])) / (n^s (-1 + s) + n^(1 - s) s)^2
ach3[n_, s_] := ((n^s (-1 + s) + n^(1 - s) s ff[s]) (n^(1 - s) s HarmonicNumber[n, 1 - s])) /
   (n^s (-1 + s) + n^(1 - s) s)^2 +
  ((n^s (-1 + s) + n^(1 - s) s ff[s]) (n^s (-1 + s) HarmonicNumber[n, s])) /
   (n^s (-1 + s) + n^(1 - s) s)^2
```

```
ach3[10 000 000 000, .3]
```

-0.904264

```
Zeta[.3]
```

-0.904559

```
Expand[
 ((n^(2 s) (-1 + s) + n s ff[s]) (n s HarmonicNumber[n, 1 - s] + n^(2 s) (-1 + s) HarmonicNumber[n, s])) /
  (n^(2 s) (-1 + s) + n s)^2]
```

$$-\frac{n^{1+2s}\, s\, \mathrm{HarmonicNumber}[n, 1-s]}{\left(n^{2s}(-1+s)+n s\right)^2}+\frac{n^{1+2s}\, s^2\, \mathrm{HarmonicNumber}[n, 1-s]}{\left(n^{2s}(-1+s)+n s\right)^2}+$$
$$\frac{n^{4s}\, \mathrm{HarmonicNumber}[n, s]}{\left(n^{2s}(-1+s)+n s\right)^2}-\frac{2\, n^{4s}\, s\, \mathrm{HarmonicNumber}[n, s]}{\left(n^{2s}(-1+s)+n s\right)^2}+\frac{n^{4s}\, s^2\, \mathrm{HarmonicNumber}[n, s]}{\left(n^{2s}(-1+s)+n s\right)^2}+$$
$$\frac{2^s\, n^2\, \pi^{-1+s}\, s^2\, \mathrm{Gamma}[1-s]\, \mathrm{HarmonicNumber}[n, 1-s]\, \mathrm{Sin}\left[\frac{\pi s}{2}\right]}{\left(n^{2s}(-1+s)+n s\right)^2}-$$
$$\frac{2^s\, n^{1+2s}\, \pi^{-1+s}\, s\, \mathrm{Gamma}[1-s]\, \mathrm{HarmonicNumber}[n, s]\, \mathrm{Sin}\left[\frac{\pi s}{2}\right]}{\left(n^{2s}(-1+s)+n s\right)^2}+$$
$$\frac{2^s\, n^{1+2s}\, \pi^{-1+s}\, s^2\, \mathrm{Gamma}[1-s]\, \mathrm{HarmonicNumber}[n, s]\, \mathrm{Sin}\left[\frac{\pi s}{2}\right]}{\left(n^{2s}(-1+s)+n s\right)^2}$$

$$\mathbf{acha}[n\_, s\_] := \frac{n^{4s}\, \mathrm{HarmonicNumber}[n, s]}{\left(n^{2s}(-1+s)+n s\right)^2}-\frac{2\, n^{4s}\, s\, \mathrm{HarmonicNumber}[n, s]}{\left(n^{2s}(-1+s)+n s\right)^2}+$$
$$\frac{n^{4s}\, s^2\, \mathrm{HarmonicNumber}[n, s]}{\left(n^{2s}(-1+s)+n s\right)^2}-\frac{2^s\, n^{1+2s}\, \pi^{-1+s}\, s\, \mathrm{Gamma}[1-s]\, \mathrm{HarmonicNumber}[n, s]\, \mathrm{Sin}\left[\frac{\pi s}{2}\right]}{\left(n^{2s}(-1+s)+n s\right)^2}$$

$$\mathbf{achb}[n\_, s\_] := -\frac{n^{1+2s}\, s\, \mathrm{HarmonicNumber}[n, 1-s]}{\left(n^{2s}(-1+s)+n s\right)^2}+\frac{n^{1+2s}\, s^2\, \mathrm{HarmonicNumber}[n, 1-s]}{\left(n^{2s}(-1+s)+n s\right)^2}+$$
$$\frac{2^s\, n^2\, \pi^{-1+s}\, s^2\, \mathrm{Gamma}[1-s]\, \mathrm{HarmonicNumber}[n, 1-s]\, \mathrm{Sin}\left[\frac{\pi s}{2}\right]}{\left(n^{2s}(-1+s)+n s\right)^2}+$$
$$\frac{2^s\, n^{1+2s}\, \pi^{-1+s}\, s^2\, \mathrm{Gamma}[1-s]\, \mathrm{HarmonicNumber}[n, s]\, \mathrm{Sin}\left[\frac{\pi s}{2}\right]}{\left(n^{2s}(-1+s)+n s\right)^2}$$

```
FullSimplify[acha[n, s]]
```

$$\left(n^{2s}\, \mathrm{HarmonicNumber}[n, s] \left(n^{2s}\, \pi\, (-1+s)^2 - n\, (2\pi)^s\, s\, \mathrm{Gamma}[1-s]\, \mathrm{Sin}\left[\frac{\pi s}{2}\right]\right)\right) \Big/ \left(\pi \left(n^{2s}(-1+s)+n s\right)^2\right)$$

```
(1/2 (HarmonicNumber[n, 1 - s] + HarmonicNumber[n, s] + (-HarmonicNumber[n, 1 - s] +
        HarmonicNumber[n, s]) Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]]))
  ((1 (1 + Tanh[ArcCoth[1 - 2 s] + (s - 1 / 2) Log[n]]) + (2^s Pi^(s - 1) Sin[Pi s / 2]
        Gamma[1 - s]) (1 - Tanh[ArcCoth[1 - 2 s] + (s - 1 / 2) Log[n]])) /
    2) /. n → 1 000 000 000 000 /. s → N@ZetaZero@1
```

$-2.32786\times10^{-7}+1.46223\times10^{-6}\,i$

```
ook2s[n_, s_] := 1/2 (HarmonicNumber[n, s] (1 + Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]]) +
    (ff[1 - s] HarmonicNumber[n, 1 - s]) (1 - Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]]))
ook2t[n_, s_] := (1/2 (HarmonicNumber[n, 1 - s] + HarmonicNumber[n, s] +
       (-HarmonicNumber[n, 1 - s] + HarmonicNumber[n, s])
        Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]]))
   ((1 + Tanh[ArcCoth[1 - 2 s] + (s - 1 / 2) Log[n]] + (2^s Pi^(s - 1) Sin[Pi s / 2] Gamma[1 - s])
       (1 - Tanh[ArcCoth[1 - 2 s] + (s - 1 / 2) Log[n]])) / 2)

ach3[n_, s_] := ((n^s (-1 + s) + n^(1 - s) s ff[s]) (n^(1 - s) s HarmonicNumber[n, 1 - s])) /
    (n^s (-1 + s) + n^(1 - s) s)^2 +
   ((n^s (-1 + s) + n^(1 - s) s ff[s]) (n^s (-1 + s) HarmonicNumber[n, s])) /
    (n^s (-1 + s) + n^(1 - s) s)^2
ach4[n_, s_] := (n^(2 s-1) (-1 + s) / s + ff[s]) / (n^(2 s-1) (-1 + s) / s + 1)^2 HarmonicNumber[n, 1 - s] +
   ((n^s (-1 + s) + n^(1 - s) s ff[s]) (n^s (-1 + s) HarmonicNumber[n, s])) /
    (n^s (-1 + s) + n^(1 - s) s)^2
ach5[n_, s_] := (n^(2 s-1) (-1 + s) / s + ff[s]) / (n^(2 s-1) (-1 + s) / s + 1)^2 HarmonicNumber[n, 1 - s] +
   (1 + n^(1 - 2 s) s / (s - 1) ff[s]) / (1 + n^(1 - 2 s) s / (s - 1))^2 HarmonicNumber[n, s]
ach6[n_, s_] := (n^(2 s-1) (-1 + s) / s + ff[s]) / (n^(2 s-1) (-1 + s) / s + 1)^2 HarmonicNumber[n, 1 - s] +
   (1 + n^(1 - 2 s) s / (s - 1) ff[s]) / (1 + n^(1 - 2 s) s / (s - 1))^2 HarmonicNumber[n, s]

ach6[1 000 000 000, .3 + 3 I]
```

$0.494697-0.063197\,i$

```
Zeta[.3 + 3 I]
```

0.49469 - 0.0632084 i

```
FullSimplify[(n^(2 s-1) (-1 + s) / s + ff[s]) / (n^(2 s-1) (-1 + s) / s + 1)^2 HarmonicNumber[n, 1 - s] +
  (1 + n^(1 - 2 s) s / (s - 1) ff[s]) / (1 + n^(1 - 2 s) s / (s - 1))^2 HarmonicNumber[n, s]]
```

$$\left(\left(n\, s\, \text{HarmonicNumber}[n, 1-s] + n^{2 s} (-1+s)\, \text{HarmonicNumber}[n, s]\right) \left(n^{2 s} \pi (-1+s) + n (2 \pi)^s s\, \text{Gamma}[1-s] \,\text{Sin}\left[\frac{\pi s}{2}\right]\right)\right) \Big/ \left(\pi \left(n^{2 s} (-1+s) + n s\right)^2\right)$$

```
ook2t[n_, s_] :=
 (1/2 (HarmonicNumber[n, 1 - s] + HarmonicNumber[n, s] + (-HarmonicNumber[n, 1 - s] +
        HarmonicNumber[n, s]) Tanh[ArcCoth[1 - 2 s] - Log[n]/2 + s Log[n]]))
  ((1 + Tanh[ArcCoth[1 - 2 s] + (s - 1 / 2) Log[n]] + (2^s Pi^(s - 1) Sin[Pi s / 2] Gamma[1 - s])
      (1 - Tanh[ArcCoth[1 - 2 s] + (s - 1 / 2) Log[n]])) / 2)
ook2tr[n_, s_] := (1/2 (HarmonicNumber[n, 1 - s] + HarmonicNumber[n, s] +
      (-HarmonicNumber[n, 1 - s] + HarmonicNumber[n, s]) Tanh[-Log[n]/2 + s Log[n]]))
  ((1 + Tanh[(s - 1 / 2) Log[n]] + (2^s Pi^(s - 1) Sin[Pi s / 2] Gamma[1 - s])
      (1 - Tanh[(s - 1 / 2) Log[n]])) / 2)
ook2t[10 000 000, .2 + 10 I]
```

1.66396 - 0.110343 i

```
Zeta[.2 + 10 I]
```

1.66396 - 0.110342 i

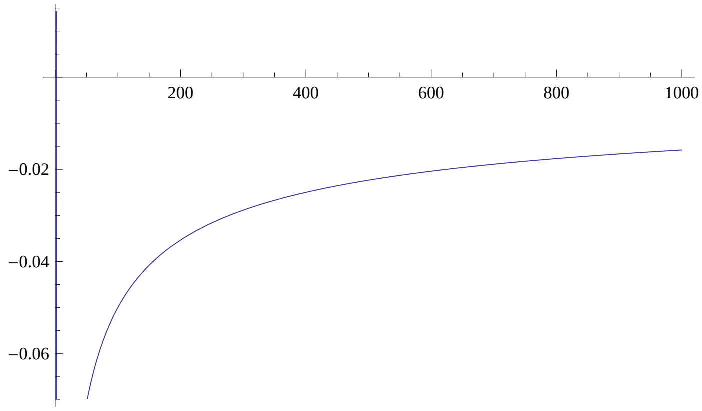
```
x Sin[x Log[n]] + 1 / 2 Cos[x Log[n]] /. x → .3 + 2 I /. n → 20
```

-139.83 + 272.118 i

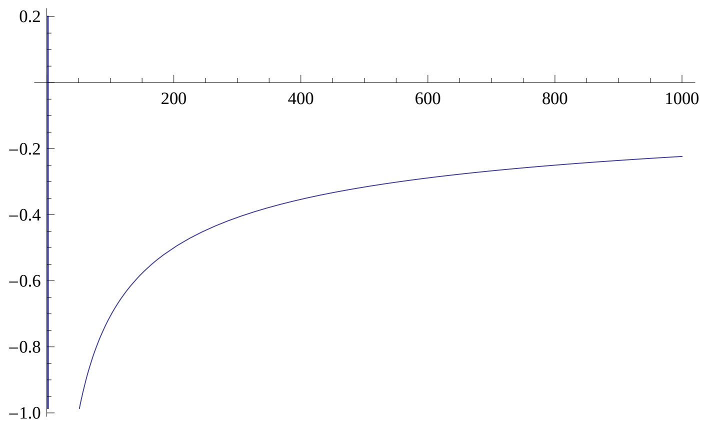
```
Cos[x Log[n]] (x Tan[x Log[n]] + 1 / 2) /. x → .3 + 2 I /. n → 20
```

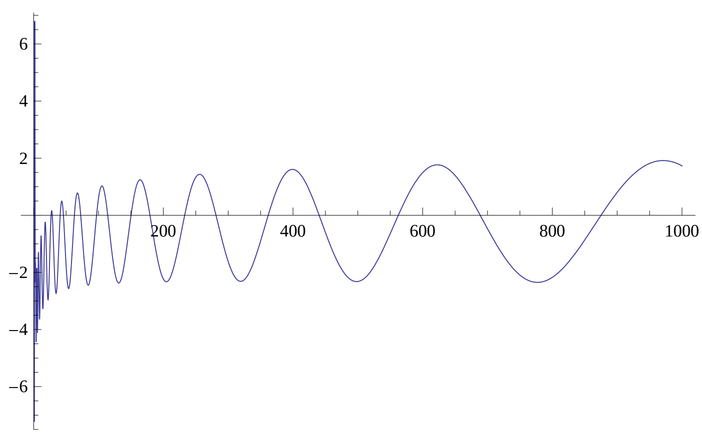
-139.83 + 272.118 i

```
Plot[Im[((1 / 2 - s) / (1 / 2 + s)) ^ (1 / 2) n^s HarmonicNumber[n, 1 / 2 + s]] /.
  s → N@ZetaZero@1 - .5, {n, 1, 1000}]
```




```
Plot[Im[((1 / 2 - s)) n^s HarmonicNumber[n, 1 / 2 + s]] /. s → N@ZetaZero@1 - .5, {n, 1, 1000}]
```






```
FullSimplify[((1 / 2 + t I) (1 / 2 - t I)) ^ (1 / 2)]
```

$$\frac{1}{2}\sqrt{1+4\,t^2}$$

$$\text{FullSimplify}\left[\frac{1}{2}\sqrt{1+4\,t^2} \Big/ (1/2 + t\,I)\right]$$

$$\frac{1-2\,i\,t}{\sqrt{1+4\,t^2}}$$

$$\text{FullSimplify}\left[\frac{1}{2}\sqrt{1+4\,t^2} \Big/ (1/2 - t\,I)\right]$$

$$\frac{1+2\,i\,t}{\sqrt{1+4\,t^2}}$$

```
FullSimplify[((1 / 2 + t I) (1 / 2 - t I)) ^ (1 / 2) / (1 / 2 + t I)]
```

$$\frac{1-2\,i\,t}{\sqrt{1+4\,t^2}}$$

```
ts[n_, t_] := n^(1 / 2 + t I) / (1 / 2 + t I) + n^(1 / 2 - t I) / (1 / 2 - t I)
```

$$\text{ts2}[n\_, t\_] := n^{\wedge}(1/2)\left(2\,n^{-i\,t} + 2\,n^{i\,t} + 4\,i\,n^{-i\,t}\,t - 4\,i\,n^{i\,t}\,t\right) \Big/ \left(1+4\,t^2\right)$$

$$\text{ts3}[n\_, t\_] := n^{\wedge}(1/2)\left(2\left(n^{i\,t} + n^{-i\,t}\right) + (4\,t\,I)\left(n^{-i\,t} - n^{i\,t}\right)\right) \Big/ \left(1+4\,t^2\right)$$

$$\text{ts4}[n\_, t\_] := n^{\wedge}(1/2)\,(4\,\text{Cos}[t\,\text{Log}[n]] + (4\,t\,I)\,(2\,I\,\text{Sin}[-t\,\text{Log}[n]])) \Big/ \left(1+4\,t^2\right)$$

$$\text{ts5}[n\_, t\_] := n^{\wedge}(1/2)\;8\,(t\,\text{Sin}[t\,\text{Log}[n]] + (1/2)\,\text{Cos}[t\,\text{Log}[n]]) \Big/ \left(1+(2\,t)^2\right)$$

```
ts6[n_, t_] := n^(1 / 2) 8 (t Sin[t Log[n]] + (1 / 2) Cos[t Log[n]]) Cos[ArcTan[2 t]]^2
```

$$\text{ts7}[n\_, t\_] := n^{\wedge}(1/2)\,(2\,t\,\text{Sin}[t\,\text{Log}[n]] + \text{Cos}[t\,\text{Log}[n]]) \Big/ \left(1/4+t^2\right)$$

```
N[ts7[10, 14]]
```

0.340949

```
FullSimplify[ts[n, t]]
```

$$\frac{2\,n^{\frac{1}{2}-i\,t}\left(1+n^{2\,i\,t}\,(1-2\,i\,t)+2\,i\,t\right)}{1+4\,t^2}$$

$$\text{Expand}\left[2\,n^{\frac{1}{2}-i\,t}\left(1+n^{2\,i\,t}\,(1-2\,i\,t)+2\,i\,t\right)\right]$$

$$2\,n^{\frac{1}{2}-i\,t} + 2\,n^{\frac{1}{2}+i\,t} + 4\,i\,n^{\frac{1}{2}-i\,t}\,t - 4\,i\,n^{\frac{1}{2}+i\,t}\,t$$

```
FullSimplify[n^(1 / 2 + t I) / (1 / 2 + t I) + n^(1 / 2 - t I) / (1 / 2 - t I)]
```

$$\frac{2\,n^{\frac{1}{2}-i\,t}\left(1+n^{2\,i\,t}\,(1-2\,i\,t)+2\,i\,t\right)}{1+4\,t^2}$$

```
(4 t I) (2 I)
```

$-8\,t$

$$n^{\wedge}(1/2)\;8\,\text{Cos}[t\,\text{Log}[n]]\,(t\,\text{Tan}[t\,\text{Log}[n]] + (1/2)) \Big/ \left(\left(1+4\,t^2\right)\right)$$

$$\frac{8\,\sqrt{n}\,\text{Cos}[t\,\text{Log}[n]]\left(\frac{1}{2}+t\,\text{Tan}[t\,\text{Log}[n]]\right)}{1+4\,t^2}$$

```
Cos[ArcTan[2 t]]
```

$$\frac{1}{\sqrt{1+4\,t^2}}$$

```
n^(1/2) 8 (t Sin[t Log[n]] + (1/2) Cos[t Log[n]]) (Cos[ArcTan[2 t]])
```

$$\frac{8\sqrt{n}\,\left(\frac{1}{2}\,\mathrm{Cos}[t\,\mathrm{Log}[n]]+t\,\mathrm{Sin}[t\,\mathrm{Log}[n]]\right)}{\sqrt{1+4\,t^2}}$$

```
FullSimplify[
 n^(1/2) 8 (t Sin[t Log[n]] + (1/2) Cos[t Log[n]]) (Cos[ArcTan[2 t]] Cos[ArcTan[2 t]])]
```

$$\frac{4\sqrt{n}\,(\mathrm{Cos}[t\,\mathrm{Log}[n]]+2\,t\,\mathrm{Sin}[t\,\mathrm{Log}[n]])}{1+4\,t^2}$$

```
FullSimplify[(1/2 + t I) (1/2 - t I)]
```

$$\frac{1}{4}+t^2$$

```
FullSimplify[n^(1/2) (2 t Sin[t Log[n]] + Cos[t Log[n]]) / (1/4 + t^2)]
```

$$\frac{\sqrt{n}\,(\mathrm{Cos}[t\,\mathrm{Log}[n]]+2\,t\,\mathrm{Sin}[t\,\mathrm{Log}[n]])}{\frac{1}{4}+t^2}$$

```
(2 t Sin[t Log[n]] + Cos[t Log[n]]) / (2 t Cos[t Log[n]] - Sin[t Log[n]]) /. t → .3 /. n → 20
```

-2.67049

```
Sin[t Log[n] + ArcCot[2 t]] / Cos[t Log[n] + ArcCot[2 t]] /. t → .3 /. n → 20
```

-2.67049

```
(Tan[t Log[n]] + 1 / (2 t)) / (1 - Tan[t Log[n]] / (2 t)) /. t → .3 /. n → 20
```

-2.67049

```
(Tan[t Log[n]] + 1 / (2 t)) / (2 t Cos[t Log[n]]) /. t → .3 /. n → 20
```

7.82594

```
FullSimplify[(Tan[t Log[n]] + 1 / (2 t)) (2 t Cos[t Log[n]])]
```

Cos[t Log[n]] + 2 t Sin[t Log[n]]

```
FullSimplify[(Tan[x] + a) / (1 - a Tan[x])]
```

$$\frac{a+\mathrm{Tan}[x]}{1-a\,\mathrm{Tan}[x]}$$

```
Tan[ArcTan[1 / (2 x)]]
```

$$\frac{1}{2\,x}$$

```
Tan[ArcCot[2 x]]
```

$$\frac{1}{2\,x}$$

```
FullSimplify[(Tan[a] + Tan[b]) / (1 - Tan[a] Tan[b])]
```

Tan[a + b]

```
ArcTan[0]
```

0

```
FullSimplify[(Tan[a]) / (1 - Tan[a])]
```

$$\frac{1}{-1+\text{Cot}[a]}$$

```
Tanh[ArcCoth[2 s]]
```

$$\frac{1}{2\,s}$$

```
FullSimplify[-1 / ((I t + 1 / 2) (I t - 1 / 2))]
```

$$\frac{4}{1+4\,t^2}$$

```
pl[t_] := 1 / (t^2 + 1 / 4)
```

```
pl[N@Im@ZetaZero@1]
```

0.00499899

```
ts7[n_, t_] := n^(1 / 2) (2 t Sin[t Log[n]] + Cos[t Log[n]]) / (1 / 4 + t^2)
ts8[n_, t_] := n^(1 / 2) (2 t Sin[t Log[n]] + Cos[t Log[n]]) / (1 / 4 + t^2)
```

```
ts7[20, 13.3]
```

0.550476

```
FullSimplify[(1 / 2 - t I) (1 / 2 + t I)]
```

$$\frac{1}{4}+t^2$$

```
os[n_, t_] := n^(1/2+tI)/((1/2+tI)) + n^(1/2-tI)/(1/2-tI)
os2[n_, t_] := ((1/2+tI)/(1/2-tI))^(-1/2) n^(1/2+tI)/
    (((1/2+tI)/(1/2-tI))^(-1/2) (1/2+tI)) + ((1/2-tI)/(1/2+tI))^(-1/2)
   n^(1/2-tI)/(((1/2-tI)/(1/2+tI))^(-1/2) (1/2-tI))
os3[n_, t_] := ((1/2+tI)/(1/2-tI))^(-1/2) n^(1/2+tI)/
    (((1/2+tI)^(1/2)/(1/2-tI)^(-1/2))) +
   ((1/2-tI)/(1/2+tI))^(-1/2) n^(1/2-tI)/
    (((1/2-tI)^(1/2)/(1/2+tI)^(-1/2)))
os4[n_, t_] := ((1/2+tI)/(1/2-tI))^(-1/2) n^(1/2+tI)/
    (((1/2+tI)^(1/2)/(1/2-tI)^(-1/2))) +
   ((1/2-tI)/(1/2+tI))^(-1/2) n^(1/2-tI)/
    (((1/2-tI)^(1/2)/(1/2+tI)^(-1/2)))
os5[n_, t_] := n^(1/2) (((1/2+tI)/(1/2-tI))^(-1/2) n^(tI)
     + ((1/2-tI)/(1/2+tI))^(-1/2) n^(-tI))/
   ((1/2-tI) (1/2+tI))^(1/2)
os6[n_, t_] := n^(1/2)
  (((1/2-tI)/(1/2+tI))^(1/2) n^(tI) + ((1/2-tI)/(1/2+tI))^(-1/2) n^(-tI))/
   ((1/2-tI) (1/2+tI))^(1/2)
os7[n_, t_] := n^(1/2) (E^Log[((1/2-tI)/(1/2+tI))^(1/2)] E^(tI Log[n]) +
     ((1/2-tI)/(1/2+tI))^(-1/2) n^(-tI))/((1/2-tI) (1/2+tI))^(1/2)
os8[n_, t_] := n^(1/2) (E^(-I ArcTan[2t]) E^(tI Log[n]) + E^(I ArcTan[2t]) n^(-tI))/
   ((1/2-tI) (1/2+tI))^(1/2)
os9[n_, t_] := n^(1/2) (E^(-I ArcTan[2t]) E^( I (t Log[n])) +
     E^(I ArcTan[2t]) E^(- I (t Log[n])))/((1/2-tI) (1/2+tI))^(1/2)
os10[n_, t_] := n^(1/2) (E^( I (t Log[n] - ArcTan[2t])) + E^(-I (t Log[n] - ArcTan[2t])))/
   ((1/2-tI) (1/2+tI))^(1/2)
os11[n_, t_] := n^(1/2) 2 Cos[t Log[n] - ArcTan[2t]]/((1/2-tI) (1/2+tI))^(1/2)
os12[n_, t_] := 4 n^(1/2) Cos[t Log[n] - ArcTan[2t]] Cos[ArcTan[2t]]
os13[n_, t_] := 2 n^(1/2) (Cos[t Log[n]] + Cos[t Log[n] - 2 ArcTan[2t]])
```

```
os13[12, 13.3]
```

0.517943

```
((1/2+tI)/(1/2-tI))^(-1/2)
```

$$\frac{1}{\sqrt{\frac{\frac{1}{2}+i\,t}{\frac{1}{2}-i\,t}}}$$

```
((1/2-tI)/(1/2+tI))^(-1/2)
```

$$\frac{1}{\sqrt{\frac{\frac{1}{2}-i\,t}{\frac{1}{2}+i\,t}}}$$

```
FullSimplify[(((1 / 2 + t I) / (1 / 2 - t I)) ^ (-1 / 2) (1 / 2 + t I))]
```

$$\frac{\sqrt{1+2\,i\,t}}{2\sqrt{\frac{1}{1-2\,i\,t}}}$$

```
(((1 / 2 - t I) ^ (1 / 2) / (1 / 2 + t I) ^ (-1 / 2)))
```

$$\sqrt{\frac{1}{2}-i\,t}\;\sqrt{\frac{1}{2}+i\,t}$$

```
(((1 / 2 + t I) ^ (1 / 2) / (1 / 2 - t I) ^ (-1 / 2)))
```

$$\sqrt{\frac{1}{2}-i\,t}\;\sqrt{\frac{1}{2}+i\,t}$$

```
FullSimplify[Log[((1 / 2 - t I) / (1 / 2 + t I)) ^ (1 / 2)]]
```

$$\mathrm{Log}\left[\sqrt{\frac{i+2\,t}{i-2\,t}}\right]$$

$$\mathbf{Log}\left[\sqrt{\frac{i+2\,t}{i-2\,t}}\right] \;/.\; t \to .7$$

$-1.11022\times10^{-16}-0.950547\,i$

```
ArcTan[2 (1.3 + I)] / I
```

$0.177123-1.32603\,i$

```
FullSimplify[ Log[((1 / 2 - t I) / (1 / 2 + t I)) ^ (1 / 2)]] /. t → (1.3 + I)
```

$0.177123-1.32603\,i$

```
ArcTanh[-2 I (1.3 + I)]
```

$0.177123-1.32603\,i$

```
ArcTanh[-2 I t]
```

$-i\,\mathrm{ArcTan}[2\,t]$

```
ArcTan[2 t] / I
```

$-i\,\mathrm{ArcTan}[2\,t]$

```
FullSimplify[2 / ((1 / 2 - t I) (1 / 2 + t I)) ^ (1 / 2)]
```

$$\frac{4}{\sqrt{1+4\,t^2}}$$

```
Cos[t Log[n] - ArcTan[2 t]] Cos[ArcTan[2 t]]
```

```
FullSimplify[Cos[t Log[n] - 2 ArcTan[2 t]]]
```

```
Cos[2 ArcTan[2 t]]
```

```
Cos[ ArcTan[2 t] + ArcTan[2 t]]
```

Cos[2 ArcTan[2 t]]

```
FullSimplify[ Log[((1 / 2 - t I) / (1 / 2 + t I)) ^ (1 / 2)]]
```

$\text{Log}\left[\sqrt{\frac{i+2t}{i-2t}}\right]$

```
FullSimplify[ Log[((1 / 2 - s) / (1 / 2 + s)) ^ (1 / 2)]] /. s → (1.3 + I)
```

-0.237467 - 1.37632 i

```
ArcTanh[-2 (1.3 + I)]
```

-0.237467 - 1.37632 i

```
ArcTanh[-2 s]
```

-ArcTanh[2 s]

```
zt[n_, s_] :=
 Sum[ j ^ (-1 / 2) Sinh[s Log[n / j] - ArcTanh[2 s]] / Sinh[s Log[n] - ArcTanh[2 s]], {j, 1, n}]
zta[n_, s_] := Sum[ j ^ (-1 / 2) (Cosh[s Log[j]] -
     (Cosh[s Log[n] - ArcTanh[2 s]] Sinh[s Log[j]]) / Sinh[s Log[n] - ArcTanh[2 s]]), {j, 1, n}]
ztb[n_, s_] := Sum[ j ^ (-1 / 2) (Cosh[s Log[j]] -
     (Cosh[s Log[n] - ArcTanh[2 s]] Sinh[s Log[j]]) / Sinh[s Log[n] - ArcTanh[2 s]]), {j, 1, n}]
zt2[n_, s_] := Sum[ j ^ (-1 / 2) (Cosh[s Log[j]] -
     Sinh[s Log[j]] Tanh[s Log[n] - ArcTanh[1 / (2 s)]]), {j, 1, n}]
zt2a[n_, s_] := Sum[ j ^ (-1 / 2) (Cos[s Log[j]] + Sin[s Log[j]] Tan[s Log[n] + ArcTan[1 / (2 s)]]),
  {j, 1, n}]
zt2b[n_, s_] := Sum[ j ^ (-1 / 2) Cos[s Log[j]], {j, 1, n}] +
  Tan[s Log[n] + ArcTan[1 / (2 s)]] Sum[ j ^ (-1 / 2) Sin[s Log[j]], {j, 1, n}]
zt2b2[n_, s_] := {Sum[ j ^ (-1 / 2) Cos[s Log[j]], {j, 1, n}],
  Tan[s Log[n] + ArcTan[1 / (2 s)]], Sum[ j ^ (-1 / 2) Sin[s Log[j]], {j, 1, n}]}
```

```
zt2b2[10 000, N@Im@ZetaZero@1]
```

{-6.98642, 6.3954, 1.08736}

```
Zeta[.7 + 10 I]
```

1.47708 - 0.114696 i

```
ArcTanh[1 / (2 s)] /. s → .3
```

0.693147 - 1.5708 i

```
ArcCoth[2 s] /. s → .3
```

0.693147 - 1.5708 i

```
ztx[n_, s_] := Sum[ j ^ (-1 / 2) Cos[s Log[n / j] + ArcTan[1 / (2 s)]], {j, 1, n}]
ztx2[n_, s_] := Sum[ j ^ (-1 / 2) Cos[s Log[j]], {j, 1, n}] +
  Tan[s Log[n] + ArcTan[1 / (2 s)]] Sum[ j ^ (-1 / 2) Sin[s Log[j]], {j, 1, n}]
ztx3[n_, s_, j_] := j ^ (-1 / 2) Cos[s Log[j]] - Tan[s Log[n] - ArcTan[1 / (2 s)]] Sin[s Log[j]]
```

```
ztx2[10 000, N@Im@ZetaZero@2]
```

0.0118368

```
FullSimplify[
 Cosh[(s / I) Log[j]] - Sinh[(s / I) Log[j]] Tanh[(s / I) Log[n] - ArcTanh[1 / (2 (s / I))]]]
```

Cos[s Log[j]] + Sin[s Log[j]] Tan[ArcCot[2 s] + s Log[n]]

```
ArcTanh[1 / (2 s)]
```

$\text{ArcTanh}\left[\frac{1}{2\,s}\right]$

```
nn = ArcTanh[1 / (2 s)]
```

$\text{ArcTanh}\left[\frac{1}{2\,s}\right]$

```
1 / (2 Tanh[nn])
```

s

```
FullSimplify[Sinh[Log[j] / (2 Tanh[t])]]
```

$\text{Sinh}\left[\frac{1}{2}\,\text{Coth}[t]\,\text{Log}[j]\right]$

```
Limit[Tan[s Log[n] + ArcCot[(2 s)]] /. s → 2 + 1 / 10 I, n → Infinity]
```

ⅈ

```
Sum[ j ^ (-1 / 2) Cos[s Log[j]], {j, 1, n}] +
 Tan[s Log[n] + ArcTan[1 / (2 s)]] Sum[ j ^ (-1 / 2) Sin[s Log[j]], {j, 1, n}]
```

$$\sum_{j=1}^{n}\frac{\text{Cos}[s\,\text{Log}[j]]}{\sqrt{j}}+\left(\sum_{j=1}^{n}\frac{\text{Sin}[s\,\text{Log}[j]]}{\sqrt{j}}\right)\text{Tan}\left[\text{ArcTan}\left[\frac{1}{2\,s}\right]+s\,\text{Log}[n]\right]$$

```
ztx2x[n_, s_] := {ztx2[n, s], Sum[ j ^ (-1 / 2) Cos[s Log[j]], {j, 1, n}],
  Tan[s Log[n] + ArcTan[1 / (2 s)]], Sum[ j ^ (-1 / 2) Sin[s Log[j]], {j, 1, n}]}
```

```
Chop@ztx2x[10 000, .4 I]
```

{-9.48154, 2219.35, 0. + 1.01142 ⅈ, 0. + 2203.66 ⅈ}

```
ztx24a[n_, s_, t_] := Sum[ j ^ (-1 / 2) Cos[(s + t I) Log[j]], {j, 1, n}] +
  Tan[(s + t I) Log[n] + ArcTan[1 / (2 (s + t I))]] Sum[ j ^ (-1 / 2) Sin[(s + t I) Log[j]], {j, 1, n}]
ztx24b[n_, s_, t_] :=
 Sum[ j ^ (-1 / 2) (Cos[s Log[j]] Cos[t I Log[j]] - Sin[s Log[j]] Sin[t I Log[j]]), {j, 1, n}] +
  Tan[(s + t I) Log[n] + ArcTan[1 / (2 (s + t I))]]
   Sum[ j ^ (-1 / 2) (Sin[s Log[j]] Cos[t I Log[j]] + Cos[s Log[j]] Sin[t I Log[j]]), {j, 1, n}]
```

```
ztx24c[n_, s_, t_] :=
 Sum[ j^(-1/2) (Cos[s Log[j]] Cos[t I Log[j]] - Sin[s Log[j]] Sin[t I Log[j]]), {j, 1, n}] +
  Tan[(s + t I) Log[n] + ArcTan[1/(2 (s + t I))]]
   Sum[ j^(-1/2) (Sin[s Log[j]] Cos[t I Log[j]] + Cos[s Log[j]] Sin[t I Log[j]]), {j, 1, n}]
```

```
ztx24c[10000, N@Im@ZetaZero@1, .3]
```

0.211083 − 0.0295144 ⅈ

```
FullSimplify[ArcCot[2 (s + t I)]]
```

ArcCot[2 (s + ⅈ t)]

```
Tan[(s + t I) Log[n] + ArcTan[1/(2 (s + t I))]] /. n → 30 /. s → 30 /. t → .1
```

0.400496 + 2.99293 ⅈ

```
(Tan[(s + t I) Log[n]] + Tan[ArcTan[1/(2 (s + t I))]]) /
   (1 - Tan[(s + t I) Log[n]] Tan[ArcTan[1/(2 (s + t I))]]) /. n → 30 /. s → 30 /. t → .1
```

0.400496 + 2.99293 ⅈ

$$\left(\text{Tan}[(s+t\,I)\,\text{Log}[n]]+\frac{1}{2\,(s+i\,t)}\right)\Big/\left(1-\text{Tan}[(s+t\,I)\,\text{Log}[n]]\,\frac{1}{2\,(s+i\,t)}\right)\ /.\ n\to 30\ /.\ s\to 30\ /.\ t\to .1$$

0.400496 + 2.99293 ⅈ

```
Tan[(s + t I) Log[n]] /. n → 30 /. s → 30 /. t → .1
```

0.527137 + 2.94655 ⅈ

```
(Tan[s Log[n]] + Tan[(t I) Log[n]]) / (1 - Tan[s Log[n]] Tan[(t I) Log[n]]) /. n → 30 /. s → 30 /.
 t → .1
```

0.527137 + 2.94655 ⅈ

```
Tan[z Log[n] + ArcTan[1/(2 z)]] /. n → 30 /. z → 30 + .1 I
```

0.400496 + 2.99293 ⅈ

```
(E^(I (z Log[n] + ArcTan[1/(2 z)])) - E^(-I (z Log[n] + ArcTan[1/(2 z)]))) /
   (I (E^(I (z Log[n] + ArcTan[1/(2 z)])) + E^(-I (z Log[n] + ArcTan[1/(2 z)])))) /. n →
   30 /. z → 30 + .1 I
```

0.400496 + 2.99293 ⅈ

```
(E^(I (z Log[n])) E^(I (ArcTan[1/(2 z)])) - E^(-I (z Log[n])) E^(-I (ArcTan[1/(2 z)]))) /
   (I (E^(I (z Log[n])) E^(I (ArcTan[1/(2 z)])) +
       E^(-I (z Log[n])) E^(-I (ArcTan[1/(2 z)])))) /. n → 30 /. z → 30 + .1 I
```

0.400496 + 2.99293 ⅈ

```
(n^(I z) E^(I (ArcTan[1/(2 z)])) - n^(-I z) E^(-I (ArcTan[1/(2 z)]))) /
   (I (n^(I z) E^(I (ArcTan[1/(2 z)])) + n^(-I z) E^(-I (ArcTan[1/(2 z)])))) /. n →
   30 /. z → 30 + .1 I
```

0.400496 + 2.99293 ⅈ

```
(n^(I z) ((z - I / 2) / (z + I / 2))^(-1 / 2) - n^(-I z) ((z - I / 2) / (z + I / 2))^(1 / 2)) /
   (I (n^(I z) ((z - I / 2) / (z + I / 2))^(-1 / 2) + n^(-I z) ((z - I / 2) / (z + I / 2))^(1 / 2))) /. n →
   30 /. z → 30 + .1 I
```

0.400496 + 2.99293 i

```
(n^(I z) (z + I / 2) - n^(-I z) (z - I / 2)) / (I (n^(I z) (z + I / 2) + n^(-I z) (z - I / 2))) /. n → 30 /.
 z → 30 + .1 I
```

0.400496 + 2.99293 i

```
E^(I (ArcTan[1 / (2 z)])) /. z → .3
```

0.514496 + 0.857493 i

```
E^(I (I / 2 (Log[1 - I (1 / (2 z))] - Log[1 + I (1 / (2 z))]))) /. z → .3
```

0.5144957554275265` + 0.8574929257125442` i

```
FullSimplify[E^(I (I / 2 (Log[1 - I (1 / (2 z))] - Log[1 + I (1 / (2 z))])))]
```

$$\frac{\sqrt{4+\frac{1}{z^2}}\; z}{-i+2z}$$

$$\frac{\sqrt{4+\frac{1}{z^2}}\; z}{-i+2z} \text{ /. } z \to .3$$

0.514496 + 0.857493 i

```
E^( (-1 / 2 (Log[(1 - I (1 / (2 z))) / (1 + I (1 / (2 z)))]))) /. z → .3
```

0.514496 + 0.857493 i

```
E^( ((Log[((1 - I (1 / (2 z))) / (1 + I (1 / (2 z))))^(-1 / 2)]))) /. z → .3
```

0.514496 + 0.857493 i

```
E^Log[((1 - I (1 / (2 z))) / (1 + I (1 / (2 z))))^(-1 / 2)] /. z → .3
```

0.514496 + 0.857493 i

```
((1 - I (1 / (2 z))) / (1 + I (1 / (2 z))))^(-1 / 2) /. z → .3
```

0.514496 + 0.857493 i

```
((z - I / 2) / (z + I / 2))^(-1 / 2) /. z → .3
```

0.514496 + 0.857493 i

```
ztx2[n_, s_] := Sum[ j^(-1/2) Cos[s Log[j]], {j, 1, n}] +
  Tan[s Log[n] + ArcTan[1/(2 s)]] Sum[ j^(-1/2) Sin[s Log[j]], {j, 1, n}]
ztx2c[n_, s_, c_] := Sum[ j^(-1/2) Cos[s Log[j] + c], {j, 1, n}] +
  Tan[s Log[n] + ArcTan[1/(2 s)] - c] Sum[ j^(-1/2) Sin[s Log[j] + c], {j, 1, n}]
ztx2c2[n_, s_, c_] := (2 s Cos[s Log[n] + c] - Sin[ s Log[n] + c])
   Sum[ j^(-1/2) Cos[s Log[j] + c], {j, 1, n}] +
  (2 s Sin[s Log[n] + c] + Cos[ s Log[n] + c]) Sum[ j^(-1/2) Sin[s Log[j] + c], {j, 1, n}]
ztx2c2a[n_, s_, c_] := Sum[ j^(-1/2) Cos[s Log[j] + c], {j, 1, n}] +
  (2 s Sin[s Log[n] + c] + Cos[ s Log[n] + c]) / (2 s Cos[s Log[n] + c] - Sin[ s Log[n] + c])
   Sum[ j^(-1/2) Sin[s Log[j] + c], {j, 1, n}]
ztx2c3[n_, s_, c_] := (2 s Cos[s Log[n] + c] - Sin[ s Log[n] + c]) /
    (2 s Cos[s Log[n]] - Sin[ s Log[n]]) Sum[ j^(-1/2) Cos[s Log[j] + c], {j, 1, n}] +
  (2 s Sin[s Log[n] + c] + Cos[ s Log[n] + c]) / (2 s Cos[s Log[n]] - Sin[ s Log[n]])
   Sum[ j^(-1/2) Sin[s Log[j] + c], {j, 1, n}]
```

```
ztx2c3[100 000, .3 + .2 I, 1]
```

$-1.09664 + 1.68063\, i$

```
Zeta[.7 + .3 I]
```

$-1.11132 - 1.64397\, i$

```
TrigToExp[(2 s Cos[s Log[n] + c] - Sin[ s Log[n] + c]) / (2 s Cos[s Log[n]] - Sin[ s Log[n]])
   Sum[ j^(-1/2) Cos[s Log[j] + c], {j, 1, n}] +
  (2 s Sin[s Log[n] + c] + Cos[ s Log[n] + c]) / (2 s Cos[s Log[n]] - Sin[ s Log[n]])
   Sum[ j^(-1/2) Sin[s Log[j] + c], {j, 1, n}]]
```

$$\frac{\left(-\frac{1}{2}\, i\left(e^{-i c} n^{-i s} - e^{i c} n^{i s}\right) + \left(e^{-i c} n^{-i s} + e^{i c} n^{i s}\right) s\right) \sum_{j=1}^{n} \frac{\mathrm{Cos}[c+s\,\mathrm{Log}[j]]}{\sqrt{j}}}{-\frac{1}{2}\, i\left(n^{-i s} - n^{i s}\right) + \left(n^{-i s} + n^{i s}\right) s} + \frac{\left(\frac{1}{2}\left(e^{-i c} n^{-i s} + e^{i c} n^{i s}\right) + i\left(e^{-i c} n^{-i s} - e^{i c} n^{i s}\right) s\right) \sum_{j=1}^{n} \frac{\mathrm{Sin}[c+s\,\mathrm{Log}[j]]}{\sqrt{j}}}{-\frac{1}{2}\, i\left(n^{-i s} - n^{i s}\right) + \left(n^{-i s} + n^{i s}\right) s}$$

```
ztx2z[n_, s_] := {Sum[ j^(-1/2) Cos[s Log[j]], {j, 1, n}],
  Tan[s Log[n] + ArcTan[1/(2 s)]] Sum[ j^(-1/2) Sin[s Log[j]], {j, 1, n}]}
ztx2zj[n_, s_, j_] := { j^(-1/2) Cos[s Log[j]],
  Tan[s Log[n] + ArcTan[1/(2 s)]] j^(-1/2) Sin[s Log[j]]}
ztx2zj2[n_, s_, j_] := { j^(-1/2) Cos[s Log[j]],
  Tan[s Log[n] + ArcTan[1/(2 s)]], j^(-1/2) Sin[s Log[j]]}
ztx2zj3[n_, s_, j_] := { j^(-1/2) Cos[s Log[j]],
  Tan[s Log[n] + ArcTan[1/(2 s)]], j^(-1/2) Sin[s Log[j]]}
ztx2za[n_, s_] := {Sum[ j^(-1/2) Cos[s Log[j]], {j, 1, n}],
  Tan[s Log[n] + ArcTan[1/(2 s)]], Sum[ j^(-1/2) Sin[s Log[j]], {j, 1, n}]}
ztx2zb[n_, s_] := {Sum[ j^(-1/2) Cos[s Log[j]], {j, 1, n}] /
   Sum[ j^(-1/2) Sin[s Log[j]], {j, 1, n}], Tan[s Log[n] + ArcTan[1/(2 s)]]}
ztx2zc[n_, s_] := Sum[ j^(-1/2) Cos[s Log[j]], {j, 1, n}] /
  Sum[ j^(-1/2) Sin[s Log[j]], {j, 1, n}]
ztx2zd[n_, s_] := Tan[s Log[n] + ArcTan[1/(2 s)]]
```

```
ztx2c3[n_, s_, c_] :=
 (2 s Cos[s Log[n] + c] - Sin[ s Log[n] + c]) / (2 s Cos[s Log[n]] - Sin[ s Log[n]])
   Sum[ j^(-1/2) Cos[s Log[j] + c], {j, 1, n}] +
  (2 s Sin[s Log[n] + c] + Cos[ s Log[n] + c]) / (2 s Cos[s Log[n]] - Sin[ s Log[n]])
   Sum[ j^(-1/2) Sin[s Log[j] + c], {j, 1, n}]
ztx2c3b[n_, s_] := (2 s Cos[s Log[n] + Pi/4] - Sin[ s Log[n] + Pi/4]) /
    (2 s Cos[s Log[n]] - Sin[ s Log[n]]) Sum[ j^(-1/2) Cos[s Log[j] + Pi/4], {j, 1, n}] +
  (2 s Sin[s Log[n] + Pi/4] + Cos[ s Log[n] + Pi/4]) / (2 s Cos[s Log[n]] - Sin[ s Log[n]])
   Sum[ j^(-1/2) Sin[s Log[j] + Pi/4], {j, 1, n}]
ztx2c3c[n_, s_] := (2 s Cos[s Log[n] + Pi/4] - Cos[ s Log[n] - Pi/4]) /
    (2 s Cos[s Log[n]] - Sin[ s Log[n]]) Sum[ j^(-1/2) Cos[s Log[j] + Pi/4], {j, 1, n}] +
  (2 s Cos[s Log[n] - Pi/4] + Cos[ s Log[n] + Pi/4]) / (2 s Cos[s Log[n]] - Sin[ s Log[n]])
   Sum[ j^(-1/2) Cos[s Log[j] - Pi/4], {j, 1, n}]
```

```
ztx2c3c[100 000, .3 + .2 I]
```

-1.09664 + 1.68063 i

```
Chop[Tan[ s Log[n] + ArcTan[1/(2 s)]] /. n → 1 000 000 /. s → .3 I]
```

0. + 1.00201 i

```
Chop@ztx2zj2[1 000 000, .3 I, 2]
```

{0.72245, 0. + 1.00201 i, 0. + 0.148101 i}

```
(Cos[(.3 I + 100) Log[1000]] + I Sin[(.3 I + 100) Log[1000]]) / 1000^(1/2)
```

0.00370463 - 0.00145762 i

```
(Cos[(.3 I + 100) Log[1000]] + I Sin[(.3 I + 100) Log[1000]]) / 1000^(1/2)
```

0.00370463 - 0.00145762 i

```
1000^(-1/2 - (.3 + 100 I))
```

0.00370463 + 0.00145762 i

```
N@Im@ZetaZero@5 / Pi
```

10.4836

```
(Zeta[s] - Sum[ j^(-1/2) Cos[s Log[j]], {j, 1, n}]) / Sum[ j^(-1/2) Sin[s Log[j]], {j, 1, n}] =
 Tan[s Log[n] + ArcTan[1/(2 s)]]
```

```
(1 / 2) (Tan[ArcTan[(Zeta[s] - Sum[ j^(-1 / 2) Cos[s Log[j]], {j, 1, n}]) /
        Sum[ j^(-1 / 2) Sin[s Log[j]], {j, 1, n}]] - s Log[n]])^-1 = s
```

```
ap[n_, s_] :=
 (1 / 2) (Tan[ArcTan[(Zeta[1 / 2 + s / I] - Sum[ j^(-1 / 2) Cos[s Log[j]], {j, 1, n}]) / Sum[
        j^(-1 / 2) Sin[s Log[j]], {j, 1, n}]] - s Log[n]])^-1
ap2[n_, s_] := (1 / 2) (Tan[ArcTan[(-Sum[ j^(-1 / 2) Cos[s Log[j]], {j, 1, n}]) /
        Sum[ j^(-1 / 2) Sin[s Log[j]], {j, 1, n}]] - s Log[n]])^-1 - s
```

```
ap2[100 000, N@Im@ZetaZero@1 + .1 I]
```

$-1.60955 - 0.833972\, i$

```
a1[n_, s_] := Sum[ j^(-1 / 2) Cos[s Log[j]], {j, 1, n}] +
  Tan[s Log[n] + ArcTan[1 / (2 s)]] Sum[ j^(-1 / 2) Sin[s Log[j]], {j, 1, n}]
a2[n_, s_] := Zeta[1 / 2 + s / I]
```

```
a1[100 000, .4 I + 12]
```

$0.980796 + 0.56123\, i$

```
a2[100 000, .4 I + 12]
```

$0.980765 + 0.561386\, i$

```
FullSimplify[
 (1 / 2) (Tan[ArcTan[(-Sum[ j^(-1 / 2) Cos[s Log[j]], {j, 1, n}]) / Sum[ j^(-1 / 2) Sin[s Log[j]],
        {j, 1, n}]] - s Log[n]])^-1 - s]
```

$$-s - \frac{1}{2} \mathrm{Cot}\left[\mathrm{ArcTan}\left[\frac{\sum_{j=1}^{n} \frac{\mathrm{Cos}[s\, \mathrm{Log}[j]]}{\sqrt{j}}}{\sum_{j=1}^{n} \frac{\mathrm{Sin}[s\, \mathrm{Log}[j]]}{\sqrt{j}}}\right] + s\, \mathrm{Log}[n]\right]$$

```
N@Im@ZetaZero@2 + .1 I
```

$21.022 + 0.1\, i$

$$\mathrm{Limit}\left[-s - \frac{1}{2} \mathrm{Cot}\left[\mathrm{ArcTan}\left[\frac{\sum_{j=1}^{n} \frac{\mathrm{Cos}[s\, \mathrm{Log}[j]]}{\sqrt{j}}}{\sum_{j=1}^{n} \frac{\mathrm{Sin}[s\, \mathrm{Log}[j]]}{\sqrt{j}}}\right] + s\, \mathrm{Log}[n]\right], n \to \mathrm{Infinity}\right]$$

$Aborted

```
dl[n_, s_, j_] :=
 j^(-1 / 2) Cos[s Log[j]] + Tan[s Log[n] + ArcTan[1 / (2 s)]] j^(-1 / 2) Sin[s Log[j]]
```

```
DiscretePlot[Re@dl[n, N@Im@ZetaZero@101 + .2 I, 2], {n, 1300, 1500}]
```