For some fixed value of z

$$F(n,i,k)=1 \; if \; n\leq 1 \text{ or } p_i>n$$
$$F(n,i,k)=\left(1+\frac{z-1}{k}\right)F\left(\frac{n}{p_i},i,k+1\right)+F(n,i+1,1)$$
$$D_z(n)=F(n,1,1)$$

$$F(n,i)=1 \; if \; p_i>n$$
$$F(n,i)=\sum_{a=0}^{\log_{p_i} n}(-1)^a\binom{-z}{a}F\left(\frac{n}{p_i^a},i+1\right)$$
$$D_z(n)=F(n,1)$$

$$F(n,i)=1 \; if \; p_i>n$$
$$F(n,i)=\sum_{a=0}^{\lfloor\frac{\log n}{\log p_i}\rfloor}d_z(p_i^a)F\left(\frac{n}{p_i^a},i+1\right)$$
$$D_z(n)=F(n,1)$$

$$D_z(n)=\sum_{a=0}^{\frac{\log n}{\log 2}}d_z(2^a)\sum_{b=0}^{\frac{\log n-a\log 2}{\log 3}}d_z(3^b)\sum_{c=0}^{\frac{\log n-a\log 2-b\log 3}{\log 5}}d_z(5^c)\sum_{d=0}^{\frac{\log n-a\log 2-b\log 3-c\log 5}{\log 7}}d_z(7^d)\ldots$$

$$(f)^z(n)=\binom{z}{1} f(n)+\binom{z}{2}\sum_{a\cdot b=n,1<a,b} f(a)\cdot f(b)+\binom{z}{3}\sum_{a\cdot b\cdot c=n,1<a,b,c} f(a)\cdot f(b)\cdot f(c)+\ldots$$

If f(n) is completely multiplicative

$$(f)^z(n)=\binom{z}{1} f(n)+\binom{z}{2}\sum_{a\cdot b=n,1<a,b} f(n)+\binom{z}{3}\sum_{a\cdot b\cdot c=n,1<a,b,c} f(n)+\ldots$$

$$(f)^z(n)=f(n)(\binom{z}{1}1+\binom{z}{2}\sum_{a\cdot b=n,1<a,b} 1+\binom{z}{3}\sum_{a\cdot b\cdot c=n,1<a,b,c} 1+\ldots)$$

$$(f)^z(n)=f(n)\,d_z(n)$$

$$(f)^z=f(n)\cdot\prod_{p^\alpha|n}(-1)^\alpha\binom{-z}{\alpha}$$

Expansion.

$$(f)^z(2)=z\,f(2)$$

$$(f)^z(4)=z\,f(4)+\frac{z(z-1)}{2}f(2)^2$$

$$(f)^z(4)=\frac{(z)f(4)}{1}+\frac{(z)f(2)}{1}\cdot\frac{(z-1)f(2)}{2}$$

$$(f)^z(8)=z\,f(8)+\frac{z(z-1)}{2}(2f(2)f(4))+\frac{z(z-1)(z-2)}{6}f(2)^3$$

$$(f)^z(8)=z\,f(8)+\frac{(z)f(2)}{1}\cdot\frac{(z-1)f(4)}{2}+\frac{(z)f(4)}{1}\cdot\frac{(z-1)f(2)}{2}+\frac{(z)f(2)}{1}\cdot\frac{(z-1)f(2)}{2}\cdot\frac{(z-2)f(2)}{3}$$

$$(f)^z(8)=\frac{z}{1}(f(8)+f(4)\cdot\frac{(z-1)f(2)}{2}+f(2)\cdot\frac{(z-1)f(4)}{2}+f(2)\cdot\frac{(z-1)f(2)}{2}\cdot\frac{(z-2)f(2)}{3})$$

$$(f)^z(p)=\binom{z}{1}f(p)$$

$$(f)^z(p^2)=\binom{z}{1}f(p^2)+\binom{z}{2}f(p)^2$$

$$(f)^z(p^3)=\binom{z}{1}f(p^3)+\binom{z}{2}2f(p)f(p^2)+\binom{z}{3}f(p)^3$$

$$(f)^z(p^4)=\binom{z}{1}f(p^4)+\binom{z}{2}(2f(p)\cdot f(p^3)+f(p^2)^2)+\binom{z}{3}(3f(p)^2\cdot f(p^2))+\binom{z}{4}f(p)^4$$

IF f(n) is multiplicative

$$(f)^z(6)=(z\,f(2))(z\,f(3))$$

$$(f)^z(30)=(z\,f(2))(z\,f(3))(z\,f(5))$$

$$(f)^z(12)=(z\,f(4)+\frac{z(z-1)}{2}f(2)^2)(z\,f(3))\ \ldots\ \text{ok.}$$

For some fixed power z and some fixed p for n=p^a

$$f_k(0)=1$$
$$f_k(a)=\frac{z-k+1}{k}\sum_{j=1}^{a}f(p^j)\,f_{k+1}(a-j)$$

if f(n) is multiplicative

$$(f)^z(n)=\prod_{p^a|n}f_1(a)$$

Euler Totient

$$\varphi(n)=n\prod_{p|n}(1-\frac{1}{p})$$

For a prime power,

$$\varphi(p^a)=p^{a-1}(p-1)$$

$$(\varphi)^z(p)=z(p-1)$$

$$(\varphi)^z(p^2)=\binom{z}{1}\cdot p(p-1)+\binom{z}{2}(p-1)^2$$

$$(\varphi)^z(p^3)=\binom{z}{1}p^2(p-1)+\binom{z}{2}2p(p-1)^2+\binom{z}{3}(p-1)^3$$

$$(\varphi)^z(p^4)=\binom{3}{0}\binom{z}{1}p^3(p-1)+\binom{3}{1}\binom{z}{2}((p-1)^2\cdot p^2)+\binom{3}{2}\binom{z}{3}((p-1)^3\cdot p)+\binom{3}{3}\binom{z}{4}(p-1)^4$$

$$(\varphi)^z(p^a)=\sum_{j=0}^{a-1}\binom{a-1}{j}\binom{z}{j+1}(p-1)^{j+1}p^{a-1-j}$$

$$(\varphi)^z(n)=\prod_{p^a|n}\sum_{j=0}^{a-1}\binom{a-1}{j}\binom{z}{j+1}(p-1)^{j+1}p^{a-1-j}$$

Hypergeometric 2 F 1

$$(\varphi)^z(n)=\prod_{p^a|n}(p^a-p^{a-1})z\cdot {}_2F_1(1-a,1-z;2,\frac{p-1}{p})$$

Jordan Totient

$$J_k(n)=n^k\prod_{p|n}(1-\frac{1}{p^k})$$

$$J_k(p^a)=p^{(a-1)k}(p^k-1)$$

$$(f)^z(p)=\binom{z}{1}(p^k-1)$$

$$(J_k)^z(p^2)=\binom{z}{1}p^k(p^k-1)+\binom{z}{2}(p^k-1)^2$$

$$(J_k)^z(p^3)=\binom{z}{1}f(p^3)+\binom{z}{2}2p^k(p^k-1)^2+\binom{z}{3}(p^k-1)^3$$

$$(J_k)^z(p^4)=\binom{z}{1}(p^{3k}(p^k-1))+\binom{z}{2}(3(p^{2k}(p^k-1)^2))+\binom{z}{3}(3p^k(p^k-1)^3)+\binom{z}{4}(p^k-1)^4$$

$$(J_k)^z(n)=\prod_{p^a|n}\sum_{j=0}^{a-1}\binom{a-1}{j}\binom{z}{j+1}(p^k-1)^{j+1}p^{(a-1-j)}k$$

Hypergeometric 2 F 1

$$(J_k)^z(n)=\prod_{p^a|n}(p^{ak}-p^{(a-1)k})z\cdot {}_2F_1(1-a,1-z;2,1-p^{-k})$$

Liouville Lambda

$$\lambda(n)=\prod_{p^a|n}(-1)^a$$

$$\lambda(p^a)=(-1)^a$$

$$(\lambda)^z(n)=f(n)\cdot\prod_{p^\alpha|n}(-1)^\alpha\binom{-z}{\alpha}=\prod_{p^\alpha|n}\binom{-z}{\alpha}$$

GCD(n,k)

$$gcd(n,k)=\prod_{p^a|n}\left(1+\sum_{j=1}^{a}(p^j-p^{j-1})\left(\lfloor\frac{k}{p^j}\rfloor-\lfloor\frac{k-1}{p^j}\rfloor\right)\right)$$

$$gcd(p^a,k)=1+\sum_{j=1}^{a}(p^j-p^{j-1})\left(\lfloor\frac{k}{p^j}\rfloor-\lfloor\frac{k-1}{p^j}\rfloor\right)$$

$$gcd(p^a,k)=1+(p-1)\left(\lfloor\frac{k}{p}\rfloor-\lfloor\frac{k-1}{p}\rfloor\right)+(p^2-p)\left(\lfloor\frac{k}{p^2}\rfloor-\lfloor\frac{k-1}{p^2}\rfloor\right)+(p^3-p^2)\left(\lfloor\frac{k}{p^3}\rfloor-\lfloor\frac{k-1}{p^3}\rfloor\right)+\ldots$$