

```

Clear[logpw]
t[n_, a_, b_] := b (Floor[n / b] - Floor[(n - 1) / b]) - a (Floor[n / a] - Floor[(n - 1) / a])
logpw[n_, a_, b_, k_] :=
  logpw[n, a, b, k] = Sum[t[j, a, b] / j logpw[n - j, a, b, k - 1], {j, 1, n - 1}]
logpw[n_, a_, b_, 1] := t[n, a, b] / n
logpw[n_, a_, b_, 0] := 0
pw[n_, a_, b_, z_] := Sum[z^k / k! logpw[n, a, b, k], {k, 0, n}]
pw[n_, a_, b_, 0] := 1
pw[0, a_, b_, z_] := 1
bn[z_, k_, m_] := pw[k, m, 1, z]
bna[z_, k_, m_] := Sum[bn[z - 1, k - j, m], {j, 0, m - 1}]

```

```
pw[10, 13, 1, 2 z]
```

$$\frac{z}{5} + \frac{7129 z^2}{6300} + \frac{1303 z^3}{504} + \frac{9046 z^4}{2835} + \frac{19 z^5}{8} + \frac{3013 z^6}{2700} + \frac{z^7}{3} + \frac{58 z^8}{945} + \frac{2 z^9}{315} + \frac{4 z^{10}}{14175}$$

```
Expand@Sum[pw[j, 13, 1, z] pw[10 - j, 13, 1, z], {j, 0, 10}]
```

$$\frac{z}{5} + \frac{7129 z^2}{6300} + \frac{1303 z^3}{504} + \frac{9046 z^4}{2835} + \frac{19 z^5}{8} + \frac{3013 z^6}{2700} + \frac{z^7}{3} + \frac{58 z^8}{945} + \frac{2 z^9}{315} + \frac{4 z^{10}}{14175}$$

```
Table[FullSimplify[pw[n, 3, 1, z]], {n, 1, 10}] // TableForm
```

$$\begin{array}{l} z \\ \frac{1}{2} z (1 + z) \\ \frac{1}{6} (-1 + z) z (4 + z) \\ \frac{1}{24} (-1 + z) z (-6 + z (7 + z)) \\ \frac{1}{120} (-2 + z) (-1 + z) z (1 + z) (12 + z) \\ \frac{1}{720} (-2 + z) (-1 + z) z (-120 + z (1 + z) (17 + z)) \\ \frac{(-3 + z) (-2 + z) (-1 + z) z (-120 + z (116 + z (27 + z)))}{5040} \\ \frac{(-3 + z) (-2 + z) (-1 + z) z (1 + z) (10 + z) (-84 + z (23 + z))}{40320} \\ \frac{(-4 + z) (-3 + z) (-2 + z) (-1 + z) z (-3360 + z (86 + z (467 + z (46 + z))))}{362880} \\ \frac{(-4 + z) (-3 + z) (-2 + z) (-1 + z) z (15120 + z (-16626 + z (-895 + z (665 + z (55 + z)))))}{3628800} \end{array}$$

```
bn[11, 7, 4]
```

```
16302
```

```
bn[11, ((4 - 1) 11) - 7, 4]
```

```
16302
```

```
{bn[5, 3, 2] + bn[5, 4, 2], bn[5 + 1, 4, 2]}
```

```
{15, 15}
```

```
{bn[5, 3, 3] + bn[5, 4, 3] + bn[5, 5, 3], bn[5 + 1, 5, 3]}
```

```
{126, 126}
```

```
{bn[7, 3, 4] + bn[7, 4, 4] + bn[7, 5, 4] + bn[7, 6, 4], bn[7 + 1, 6, 4]}
```

```
{1428, 1428}
```

```
{Expand@bna[z, 6, 7], Expand@bn[z, 6, 7]}
```

$$\left\{ \frac{z}{6} + \frac{137 z^2}{360} + \frac{5 z^3}{16} + \frac{17 z^4}{144} + \frac{z^5}{48} + \frac{z^6}{720}, \frac{z}{6} + \frac{137 z^2}{360} + \frac{5 z^3}{16} + \frac{17 z^4}{144} + \frac{z^5}{48} + \frac{z^6}{720} \right\}$$

```
Sum[(-1)^j bn[10, j, 2], {j, 0, 10}]
```

```
0
```

```
Sum[t[j, 3, 1] bn[10, j, 3], {j, 0, 20}]
```

```
0
```

```
Table[Sum[(-1)^j bn[n, j, 7], {j, 0, (7-1)n}], {n, 1, 10}]
```

```
{1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
```

```
Table[Table[Sum[t[j, 5, 1] bn[n, j, m], {j, 0, (m-1)n}], {n, 1, 15}], {m, 3, 9}] // TableForm
```

-2	4	7	-4	-22	-11	47	76	-47	-246	-123	521	843	-52
-1	1	-1	1	4	1	-1	1	-1	-4	-1	1	-1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0
-4	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4
-3	-1	3	11	22	29	18	-29	-123	-246	-322	-199	322	136
-2	4	7	-4	-22	-11	47	76	-47	-246	-123	521	843	-52
-1	1	-1	1	4	1	-1	1	-1	-4	-1	1	-1	1

```
Table[Sum[logpw[n, 2, 1, j], {j, 0, n}], {n, 1, 10}]
```

$$\left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}, \frac{7}{60}, \frac{19}{360}, \frac{3}{70}, \frac{5}{336}, \frac{13}{756}, \frac{199}{75600} \right\}$$

```
logpw[3, 2, 1, 3]
```

```
1
```

```
Table[Sum[logpw[n, 2, 1, j], {j, 0, n}], {n, 1, 10}]
```

$$\left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}, \frac{7}{60}, \frac{19}{360}, \frac{3}{70}, \frac{5}{336}, \frac{13}{756}, \frac{199}{75600} \right\}$$

```
Series[1/(1-Log[1+x]), {x, 0, 10}]
```

$$1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{6} + \frac{7x^5}{60} + \frac{19x^6}{360} + \frac{3x^7}{70} + \frac{5x^8}{336} + \frac{13x^9}{756} + \frac{199x^{10}}{75600} + O[x]^{11}$$

```
Table[Sum[logpw[n, 3, 1, j], {j, 0, n}], {n, 1, 10}]
```

$$\left\{ 1, \frac{3}{2}, \frac{4}{3}, \frac{5}{3}, \frac{107}{60}, \frac{709}{360}, \frac{307}{140}, \frac{4099}{1680}, \frac{2528}{945}, \frac{225469}{75600} \right\}$$

```
Sum[bn[8, j, 2], {j, 0, 8}]
```

```
256
```

```
Sum[bn[10, j, 3], {j, 0, 20}]
```

```
59 049
```

```
3^10
```

```
59 049
```

```
Sum[bn[10, 2j, 2], {j, 0, 10(2-1)}]
```

```
512
```

```
Sum[bn[10, 3 j, 3], {j, 0, 10 (3 - 1)}]
```

```
19 683
```

```
3^9
```

```
19 683
```

```
Sum[bn[9, 4 j, 4], {j, 0, 9 (4 - 1)}]
```

```
65 536
```

```
4^8
```

```
65 536
```

```
Sum[bn[9, 4 j + 3, 4], {j, 0, 9 (4 - 1)}]
```

```
65 536
```

```
3^7
```

```
2187
```

```
bn[z, 3, 2]
```

$$\frac{z}{3} - \frac{z^2}{2} + \frac{z^3}{6}$$

```
(-1)^3 Expand[bn[3 - z - 1, 3, 2]]
```

$$\frac{z}{3} - \frac{z^2}{2} + \frac{z^3}{6}$$

```
Table[bn[z, 3, 2], {z, -10, 12}]
```

```
{-220, -165, -120, -84, -56, -35, -20, -10,  
-4, -1, 1, 0, 0, 1, 4, 10, 20, 35, 56, 84, 120, 165, 220}
```

```
Table[bn[z, 4, 3], {z, -20, 12}]
```

```
{4445, 3515, 2736, 2091, 1564, 1140, 805, 546, 351, 209, 110, 45, 6, -14, -21,  
-20, -15, -9, -4, -1, 1, 0, 1, 6, 19, 45, 90, 161, 266, 414, 615, 880, 1221}
```

```
Table[bn[z, 4, 2] - bn[1 - z, 4, 2], {z, 1, 12}]
```

```
{-1, -1, -5, -14, -30, -55, -91, -140, -204, -285, -385, -506}
```

```
Sum[bn[j, 5, 3], {j, 0, 10}]
```

```
3301
```

```
bn[10 + 1, 5 + 1, 3]
```

```
4917
```

```
Sum[bn[5, j, 3]^2, {j, 0, 5}]
```

```
5777
```

```
bn[15, 5, 3]
```

```
9828
```

```
Table[{k, FullSimplify@Expand[(bn[z, 3, k] - bn[z, 3, 12])], {k, 2, 12}} // TableForm
```

```

2      - z2
3      - z
4      0
5      0
6      0
7      0
8      0
9      0
10     0
11     0
12     0

```

```

FullSimplify[bn[z, 2, 2] - (FullSimplify[bn[z, 2, 12] - bn[z, 1, 12]])]
0
FullSimplify[bn[z, 3, 2] - (FullSimplify[bn[z, 3, 12] - 2 bn[z, 2, 12] + bn[z, 1, 12]])]
0
FullSimplify[
  bn[z, 4, 2] - (FullSimplify[bn[z, 4, 12] - 3 bn[z, 3, 12] + 3 bn[z, 2, 12] - bn[z, 1, 12]])]
0
FullSimplify[bn[z, 3, 3] - (FullSimplify[bn[z, 3, 12] - bn[z, 1, 12]])]
0
FullSimplify[bn[z, 4, 3] - (FullSimplify[bn[z, 4, 12] - 2 bn[z, 2, 12] + bn[z, 1, 12]])]
0
FullSimplify[bn[z, 5, 3] - (bn[z, 5, 12] - 3 bn[z, 3, 12] + 2 bn[z, 2, 12])]
0
FullSimplify[
  bn[z, 6, 3] - (bn[z, 6, 12] - 4 bn[z, 4, 12] + 3 bn[z, 3, 12] + bn[z, 2, 12] - bn[z, 1, 12])]
0
FullSimplify[bn[z, 7, 3] -
  (bn[z, 7, 12] - 5 bn[z, 5, 12] + 4 bn[z, 4, 12] + 3 bn[z, 3, 12] - 4 bn[z, 2, 12] + bn[z, 1, 12])]
0
FullSimplify[bn[z, 3, 4] - (FullSimplify[bn[z, 3, 12]])]
0
FullSimplify[bn[z, 4, 4] - (FullSimplify[bn[z, 4, 12] - bn[z, 1, 12]])]
0
FullSimplify[bn[z, 5, 4] - (FullSimplify[bn[z, 5, 12] - 2 bn[z, 2, 12] + bn[z, 1, 12]])]
0
FullSimplify[bn[z, 6, 4] - (FullSimplify[bn[z, 6, 12] - 3 bn[z, 3, 12] + 2 bn[z, 2, 12]])]
0

```

```

FullSimplify[bn[z, 7, 4] - (FullSimplify[bn[z, 7, 12] - 4 bn[z, 4, 12] + 3 bn[z, 3, 12]])]
0
FullSimplify[bn[z, 8, 4] -
  (FullSimplify[bn[z, 8, 12] - 5 bn[z, 5, 12] + 4 bn[z, 4, 12] + bn[z, 2, 12] - bn[z, 1, 12]])]
0
FullSimplify[bn[z, 4, 5] - (FullSimplify[bn[z, 4, 12]])]
0
FullSimplify[bn[z, 5, 5] - (FullSimplify[bn[z, 5, 12] - bn[z, 1, 12]])]
0
FullSimplify[bn[z, 6, 5] - (FullSimplify[bn[z, 6, 12] - 2 bn[z, 2, 12] + bn[z, 1, 12]])]
0
FullSimplify[bn[z, 7, 5] - (FullSimplify[bn[z, 7, 12] - 3 bn[z, 3, 12] + 2 bn[z, 2, 12]])]
0
FullSimplify[bn[z, 8, 5] - (FullSimplify[bn[z, 8, 12] - 4 bn[z, 4, 12] + 3 bn[z, 3, 12]])]
0
FullSimplify[bn[z, 9, 5] - (FullSimplify[bn[z, 9, 12] - 5 bn[z, 5, 12] + 4 bn[z, 4, 12]])]
0
FullSimplify[bn[z, 10, 5] -
  (FullSimplify[bn[z, 10, 12] - 6 bn[z, 6, 12] + 5 bn[z, 5, 12] + bn[z, 2, 12] - bn[z, 1, 12]])]
0
FullSimplify[bn[z, 11, 5] - (FullSimplify[bn[z, 11, 16] -
  7 bn[z, 7, 16] + 6 bn[z, 6, 16] + 3 bn[z, 3, 16] - 4 bn[z, 2, 16] + bn[z, 1, 16]])]
0

```