Suppose we have some function $C_k(n,x)$. Define it as the count of solutions to $(1+a_1 \cdot x)\cdot(1+a_2 \cdot x)\cdot ...(1+a_k \cdot x) \le n$ where $a_{1,} a_{2,} ... a_k$ are all integers greater than 0, and then that count multiplied by x^k.

Here's a geometric interpretation of $C_k(n,x)$:

As an example, let's look at $C_3(20,\frac{1}{2})$. You could think of $C_3(20,\frac{1}{2})$ as representing the volume of cubes with sides of length ½ (so each cube has a volume of 1/8th) that is entirely bounded by the curves $x \cdot y \cdot z \le 20, x>1, y>1, z>1$. So the parameter x represents a kind of discrete sampling factor of the continuous curve. If x is 1, $C_3(20,1)=\sum_{j=2}^{20} \sum_{k=2}^{\lfloor \frac{20}{j} \rfloor} \sum_{l=2}^{\lfloor \frac{20}{j \cdot k} \rfloor} 1$. As x approaches 0,

$$\lim_{x \to 0} C_3(20,x)=\int_1^{20} \int_1^{\frac{20}{x}} \int_1^{\frac{20}{x \cdot y}} dz\, dy\, dx$$

Now, $C_k(n,x)$ is interesting, and the parameter x is particularly interesting, because the riemann prime counting function can be expressed as

$$\Pi(n)=\sum_{k=1} \frac{(-1)^{k+1}}{k} \cdot C_k(n,1)$$

and the logarithmic integral can be expressed as

$$li(n)-\log\log n-\gamma=\lim_{x \to 0} \sum_{k=1} \frac{(-1)^{k+1}}{k} \cdot C_k(n,x)$$

And so the difference between the two (which is to say, the difference at the heart of the prime number theory) is contingent on what happens as that x value changes.

So it might seem interesting to ask what

$$\frac{\partial}{\partial x} C_k(n,x)$$

looks like…

...but probably not. Or at least, I never had any success with that line of exploration.

That's what I was trying to do here.

Incidentally, the function I'm writing as $C_k(n,x)$ here I eventually notate as $[(x^{1-(0)} \cdot \zeta((0),1+x^{-1}))^k]_n$ in my later writings.

$$\boxed{D_{k,x}(n)=\sum_{j=0}^{\lfloor n-x\rfloor} D_{k-1,x}\left(\frac{n}{j+x}\right)} \text{ with } \boxed{D_{0,x}(n)=1}$$

$$\boxed{C_k(n,x)=x^{-k} D_{k,x+1}(n x^k)}$$

$$C_k(n,x)-C_k(n,x-\epsilon)$$

$$x^{-k} D_{k,x+1}(n x^k)-(x-\epsilon)^{-k} D_{k,x-\epsilon+1}(n(x-\epsilon)^k)$$

$$x^{-k}\cdot D_{k,x+1}(n\cdot x^k)=x^{-k}\cdot \sum_{j=0}^{\lfloor n\cdot x^k-x-1\rfloor} \sum_{k=0}^{\lfloor \frac{n\cdot x^k}{j+x+1}-x-1\rfloor} \sum_{m=0}^{\lfloor \frac{n\cdot x^k}{(j+x+1)(k+x+1)}-x-1\rfloor} \sum \ldots 1$$

$$(x-\epsilon)^{-k}\cdot D_{k,x-\epsilon+1}(n\cdot(x-\epsilon)^k)=(x-\epsilon)^{-k}\cdot \sum_{j=0}^{\lfloor n\cdot(x-\epsilon)^k-x+\epsilon-1\rfloor} \sum_{k=0}^{\lfloor \frac{n\cdot(x-\epsilon)^k}{j+x-\epsilon+1}-x+\epsilon-1\rfloor} \sum_{m=0}^{\lfloor \frac{n\cdot(x-\epsilon)^k}{(j+x-\epsilon+1)(k+x-\epsilon+1)}-x+\epsilon-1\rfloor} \sum \ldots 1$$