In this set of notes, I was trying to enumerate the ways that different divisor-style summatory functions could be expressed in terms of each other. The coefficients are the same as the equivalent situation for taylor series, which is what the notation was meant to express.

Most of these functions are not, in and of themselves, all that interesting. But I was just trying to work through the mechanics.

...

To work through an example of what I'm getting at, suppose that

$$D_0'(n)=1$$
$$D_k'(n)=\sum_{j=2}^{n} D_{k-1}'\left(\frac{n}{j}\right)$$

and

$$M_0'(n)=1$$
$$M_k'=\sum_{j=2}^{n} \mu(j)\cdot M_{k-1}'\left(\frac{n}{j}\right)$$

where the mu is the moebius mu function.

THEN with some effort we can find that

$$M_1'(n)=-D_1'(n)+D_2'(n)-D_3'(n)+D_4'(n)-..$$
$$M_2'(n)=D_2'(n)-2D_3'(n)+3D_4'(n)-4D_5'(n)$$
$$M_3'(n)=-D_3'(n)+3D_4'(n)-6D_5'(n)+10D_6'($$

and so on. M and D could be swapped here, too.

These coefficients will be identical to those found in the relationships

$$\frac{1}{1+x}-1=-x+x^2-x^3+x^4-\ldots$$
$$\left(\frac{1}{1+x}-1\right)^2=x^2-2x^3+3x^4-4x^5+\ldots$$
$$\left(\frac{1}{1+x}-1\right)^3=-x^3+3x^4-6x^5+10x^6+\ldots$$

So, essentially, from a certain perspective, $D_k'(n)$ *and* $M_k'(n)$ *here have the same relationship to each other that* $(x-1)^k$ *and* $\left(\frac{1}{x}-1\right)^k$ *hold to each other.*

...

The later notation that I adopted makes these relationships much, much more obvious – in fact, that was a major motivation for adopting it. The jumble of letters found below obscures a bunch of natural relationships.

Using my later notation,

$$D_z=[\zeta(0)^z]_n\ ,$$
$$D_k'=[(\zeta(0)-1)^k]_n$$
$$M_z=[\zeta(0)^{-z}]_n$$
$$M_k'=[(\zeta(0)^{-1}-1)^k]_n$$

$$P_k'=[(\log\zeta(0))^k]_n$$
$$P_z=[(\log\zeta(0)+1)^z]_n$$
$$iP_z=\left[\left(\frac{1}{\log(\zeta(0))+1}\right)^z\right]_n$$
$$iP_k'=\left[\left(\frac{1}{\log(\zeta(0))+1}-1\right)^k\right]_n$$

$$lP_k'=[(\log(\log(\zeta(0)+1)))^k]_n$$
$$lP_z=[(\log(\log(\zeta(0)+1))+1)^z]_n$$

$$ilP_z = [(\frac{1}{\log(\log(\zeta(0)+1))+1})^z]_n$$

$$ilP_k{}' = [(\frac{1}{\log(\log(\zeta(0)+1))+1} - 1)^k]_n$$

For $D'=x$,

$D_z=(x+1)^z$

$M_z=(\frac{1}{(x+1)})^z$

$M_k'=(\frac{1}{(x+1)}-1)^k$

$P_k'=(\log(x+1))^k$

$P_z=(\log(x+1)+1)^z$

$iP_z=(\frac{1}{\log(x+1)+1})^z$

$iP_k'=(\frac{1}{\log(x+1)+1}-1)^k$

$lP_k'=(\log(1+\log(x+1)))^k$

$lP_z=(\log(1+\log(x+1))+1)^z$

$ilP_z=(\frac{1}{\log(1+\log(x+1))+1})^z$

$ilP_k'=(\frac{1}{\log(1+\log(x+1))+1}-1)^k$

For $D=x$,

$D_k'=(x-1)^k$

$M_z=x^{-z}$

$M_k'=(\frac{1}{x}-1)^k$

$P_k'=(\log(x))^k$

$P_z=(\log(x)+1)^z$

$iP_z=(\frac{1}{\log(x)+1})^z$

$iP_k'=(\frac{1}{\log(x)+1}-1)^k$

$lP_k'=(\log(\log(x+1)))^k$

$lP_z=(\log(\log(x+1))+1)^z$

$ilP_z=(\frac{1}{\log(\log(x+1))+1})^z$

$ilP_k'=(\frac{1}{\log(\log(x+1))+1}-1)^k$

For $M=x$,

$M_k'=(x-1)^k$

$D_z=x^{-z}$

$D_k'=(\frac{1}{x}-1)^k$

$P_k'=(-\log(x))^k$

$P_z=(-\log(x)+1)^z$

$iP_z=(\frac{1}{-\log(x)+1})^z$

$iP_z'=(\frac{1}{-\log(x)+1}-1)^z$

For $M'=x$,

$M_z=(x+1)^z$

$D_z=(\frac{1}{(x+1)})^z$

$D_k'=(\frac{1}{(x+1)}-1)^k$

$P_k'=(-\log(x+1))^k$

$P_z=(-\log(x+1)+1)^z$

$iP_z=(\frac{1}{-\log(x+1)+1})^z$

$iP_k'=(\frac{1}{-\log(x+1)+1}-1)^k$

$lP_k'=(\log(1-\log(x+1)))^k$

$P_z=(\log(1-\log(x+1))+1)^z$

$iP_z=(\frac{1}{\log(-\log(x+1))+1})^z$

$iP_k'=(\frac{1}{\log(1-\log(x+1))+1}-1)^k$

For $P'=x$,

$D_z=e^{zx}$

$D_k'=(e^x-1)^k$

$M_z=e^{-zx}$

$M_k'=(e^{-x}-1)^k$

$P_z=(x+1)^z$

$iP_z=(\frac{1}{x+1})^z$

$iP_k'=(\frac{1}{x+1}-1)^k$

$lP_k'=(\log(x+1))^k$

$lP_z=(\log(x+1)+1)^z$

$ilP_z=(\frac{1}{\log(x+1)+1})^z$

$ilP_k'=(\frac{1}{\log(x+1)+1}-1)^k$

For $P=x$,

$P_z=(x-1)^z$

$D_z=e^{z(x-1)}$

$D_z=(e^{(x-1)}-1)^k$

For $C'=x$,where $C'\,is\,\frac{x}{\log(x+1)}$ to D'

$D_k'=(x-1)^k$

$D_z=(x+1)^z$

$M_z=(\frac{1}{(x+1)})^z$

$M_k'=(\frac{1}{(x+1)}-1)^k$

$P_k'=(\log(x+1))^k$

$P_z=(\log(x+1)+1)^z$

$iP_z=(\frac{1}{\log(x+1)+1})^z$

$iP_k'=(\frac{1}{\log(x+1)+1}-1)^k$

Multiplicative inverses:

$$(\log(x+1))\cdot\frac{x}{(\log(x+1))}=x$$

$$\frac{1}{x^z}\cdot x^{z+1}=x$$

$$(e^x-1)\cdot(\frac{x}{e^x-1})=x$$

$$D_1{}'(n)=\sum_{j=2}^{n}\kappa(j)\sum_{k=0}^{\log_2 n} C_k\, D_k{}'(\lfloor\frac{n}{j}\rfloor) \text{ where}$$

$$C_k = \text{series expansion for } \frac{x}{\log(1+x)}$$

$$D_a{}'(n)=\sum_{j=2}^{n}\kappa(j)\sum_{k=0}^{\log_2 n} C_k\, D_k{}'(\lfloor\frac{n}{j}\rfloor) \text{ where}$$

$$C_k = \text{series expansion for } \frac{x^a}{\log(1+x)}$$