```
RiePrimeCnt[n_] := Sum[PrimePi[n^(1 / j)] / j, {j, 1, Log[2, n]}]
LinnikApprox[n_] :=
 Sum[(-1)^(j + 1) / j (-1)^j (1 - n Sum[(-Log[n])^k / k!, {k, 0, j-1}]), {j, 1, Infinity}]
Table[{N[LinnikApprox[10^n]], N[LogIntegral[10^n] - Log[Log[10^n]] - EulerGamma]},
  {n, 1, 4}] // TableForm

$Aborted

LogIntegral[ 100.] - Log[Log[100.]] - EulerGamma

28.0217

-Gamma[ 0,  -Log[100.]] - Pi I - Log[Log[100]]  - EulerGamma

28.0217 + 0. i

TestSum[n_, z_, t_, s_] := 1 +
  Sum[N[Binomial[z, k] (-1)^k (Gamma[k, 0, (s - 1) Log[n]] / (Gamma[k] (1 - s)^k))], {k, 1, t}]
ff[ n_, z_, t_, s_]  := (Sum[ Binomial[z, k] (-1)^k
      ( 1 - Gamma[ k,  (s - 1) Log[n]] / (Gamma[k] (1 - s)^k)), {k, 0,  t}] - 1) / z
ff2[ n_, z_, t_, s_]  := 1 + Sum[ Binomial[z, k] (-1)^k
    ( Gamma[ k,  0,  (s - 1) Log[n]] / (Gamma[k] (1 - s)^k)), {k, 1,  t}]

N[Limit[ ff[ 120,  z,  30, -1],  z → 0]]
```

$1675.72 + 3.76482 \times 10^{-24}\, i$

```
1 - Log[1000.]

-5.90776

N[Gamma[2,  -Log[200]]] / 200
```

$-4.29832 + 5.26392 \times 10^{-16}\, i$

```
ff2[120., 3, 2000, 0]
```

$2644.2 - 3.92488 \times 10^{-13}\, i$

```
N[Gamma[3,  -Log[100]]]
```

$1399.73 - 3.42834 \times 10^{-13}\, i$

Integrate::idiv : Integral of $\frac{e^{-t}}{t}$ does not converge on {-Log[100], 0}. >>

```
ff4[ n_,  z_, tt_, s_]  :=
 1 + Sum[Integrate[ Binomial[z, k] (-1)^(k + 1) / ((k - 1)!) t^(k - 1) E^((s - 1) t),
    {t, -Log[n], 0}], {k, 0,  tt}]
```

```
ff2[120., 3, 1000, 2]
N[ff4[ 120, 3, 30, 2]]
```

```
7.68658
```

```
7.68658
```

```
ff5[ n_, z_, s_] :=
 Integrate[Sum[ Binomial[z, k] (-1) ^ (k + 1) / ((k - 1) !) t^ (k - 1) E^ ((s - 1) t),
   {k, 0, Infinity}], {t, -Log[n], 0}]
```

```
Limit[ (ff5[ n, z, s]) / z, z → 0]
```

$$\mathrm{Limit}\left[\frac{\int_{-\mathrm{Log}[n]}^{0} e^{(-1+s)\,t}\, z\, \mathrm{Hypergeometric1F1}[1-z, 2, t]\, dt}{z}, z \to 0\right]$$

```
Log[10.]
```

```
2.30259
```

```
-1.` + LaguerreL[-1.` z, Log[10]] /. z → 2
```

```
32.0259
```

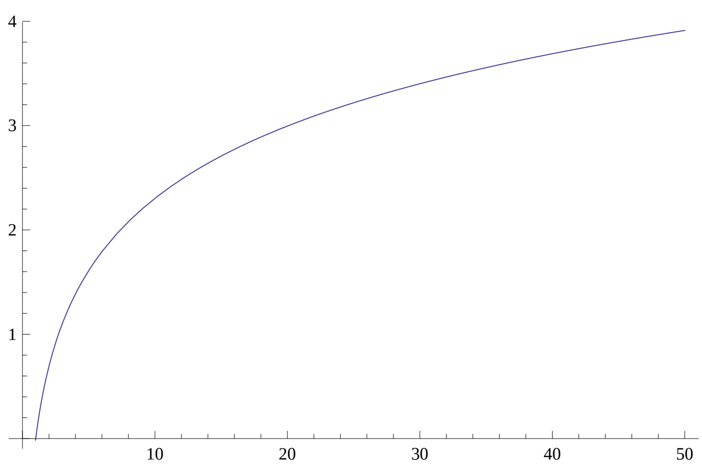
```
N[Limit[ (LaguerreL[-z, Log[100]] - 1) / z, z → 0]]
```

```
28.0217
```

```
-1. + LaguerreL[-1, Log[10]]
```

```
9.
```

```
Plot[ Re[-1 + LaguerreL[1, -Log[n]]], {n, 0, 50}]
```




```
N[ff2[ 1000, -2 + I, 1000]]
```

```
27.1002 + 19.121 i
```

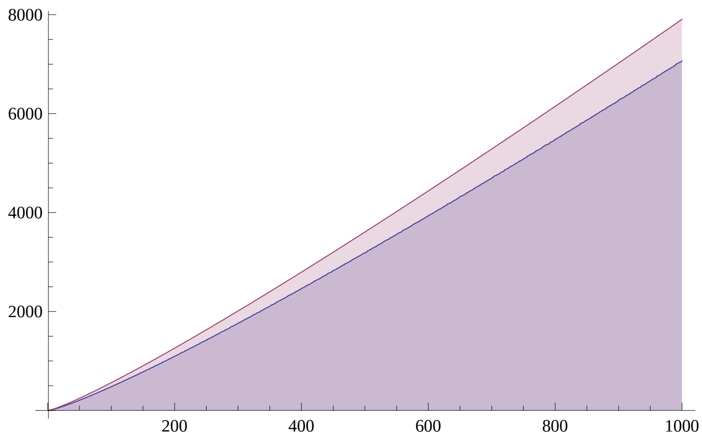
```
N[LaguerreL[ 2 + I, Log[1000]]]
```

```
27.1002 - 19.121 i
```

```
N[Limit[ (LaguerreL[-z, Log[20]] - 1) / z, z → 0]]

8.2309

LogIntegral[ 20.] - Log[Log[20.]] - EulerGamma

8.2309

ExpIntegralEi[ZetaZero[1] Log[1000.]]

-0.0879017 + 3.45317 i

Limit[ N[(LaguerreL[ -z, ZetaZero[1] Log[1000]] - 1) / z], z → 0] +
 Log[ZetaZero[1] Log[1000]] + EulerGamma

-0.0879017 + 3.45317 i

N[E^EulerGamma]

1.78107

N[Limit[( LaguerreL[-z, Log[100]] - 1) / z, z → 0]]

28.0217

N[LogIntegral[100] - Log[Log[100]] - EulerGamma]

28.0217


Sum[ N[D[ LaguerreL[ -z, Log[10]], {z, k}] /. z → 0] / k!, {k, 0, 5}]

9.99914

Sum[ N[D[ LaguerreL[ -z, Log[10]], {z, k}] /. z → 0] / k!, {k, 0, 6}]

9.99996

N[D[( LaguerreL[-z, Log[100]] - 1) / z, z]]

28.0217

N[-Pi I - Gamma[ 0, -Log[100]]]

30.1261 + 0. i

LogIntegral[100.]

30.1261

 Limit[(LaguerreL[-ee, Log[100.]] - 1) / ee, ee → 0]

28.0217

N[Limit[( LaguerreL[-z, Log[100]] - 1) / z, z → 0]]

28.0217

N[D[LaguerreL[-z, Log[100]], z]] /. z → 0

28.0217

D[ LaguerreL[-z, Log[100.]], {z, 2}] /. z → 0

80.5038
```

```
P2[n_, 0] := 1
P2[n_, k_] :=
 P2[n, k] = Sum[FullSimplify[MangoldtLambda[j] / Log[j]] P2[n / j, k - 1], {j, 2, Floor[n]}]
D2[n_, k_] := D2[n, k] = Sum[D2[Floor[n / j], k - 1], {j, 2, Floor[n]}]; D2[n_, 0] := 1
D1[n_, z_] := Sum[Binomial[z, k] D2[n, k], {k, 0, Log[2, n]}]
```

```
DiscretePlot[{ D1[n, 2], LaguerreL[-2, Log[n]]}, {n, 1, 1000}]
```




```
Integrate[ 1 / Log[x], {x, 1.1, n}, PrincipalValue → True]
```

```
ConditionalExpression[1.67577 + LogIntegral[n], Im[n] ≠ 0 || Re[n] ≥ 1.]
```

```
TestSum[ n_, z_, t_] :=
 Sum[ N[Binomial[z, k] (-1)^k ( 1 - Gamma[ k, -Log[n]] / Gamma[k])], {k, 0, t}]
Grid[ Table[ { Re[TestSum[n, k, 23]], N[LaguerreL[-k, Log[n]]]},
  {n, 10, 100, 10}, {k, -5, 5}]]
```

{0.966672, 0.966672}	{0.727895, 0.727895}	{0.0104133, 0.0104133}	{-0.954221, -0.954221}	{-1.30259, -1.30259}	{1., 1.}	{10., 10.}	{33.0259, 33.0259}	{82.5612, 82.5612}	{178.953, 178.953}	{354.26, 354.26}

{0.853716, 0.853716}	{1.37285, 1.37285}	{0.993598, 0.993598}	{-0.504259, -0.504259}	{-1.99573, -1.99573}	{1., 1.}	{20., 20.}	{79.9146, 79.9146}	{229.573, 229.573}	{558.593, 558.593}	{1223.71, 1223.71}
{0.345442, 0.345447}	{1.4452, 1.4452}	{1.59103, 1.59103}	{-0.018323, -0.0183229}	{-2.4012, -2.4012}	{1., 1.}	{30., 30.}	{132.036, 132.036}	{407.594, 407.594}	{1053.4, 1053.4}	{2433.46, 2433.46}
{-0.182647, -0.182608}	{1.31842, 1.31842}	{1.97883, 1.97883}	{0.426157, 0.426157}	{-2.68888, -2.68888}	{1., 1.}	{40., 40.}	{187.555, 187.555}	{607.267, 607.267}	{1633.79, 1633.79}	{3910.39, 3910.39}
{-0.66439, -0.664191}	{1.10954, 1.10957}	{2.2416, 2.2416}	{0.827916, 0.827916}	{-2.91202, -2.91202}	{1., 1.}	{50., 50.}	{245.601, 245.601}	{823.8, 823.8}	{2283.51, 2283.51}	{5611.57, 5611.57}
{-1.09019, -1.08949}	{0.865209, 0.86531}	{2.42308, 2.42309}	{1.19314, 1.19314}	{-3.09434, -3.09434}	{1., 1.}	{60., 60.}	{305.661, 305.661}	{1054.23, 1054.23}	{2992.07, 2992.07}	{7508.1, 7508.1}
{-1.46377, -1.46181}	{0.606799, 0.607082}	{2.54836, 2.5484}	{1.52786, 1.52787}	{-3.2485, -3.2485}	{1., 1.}	{70., 70.}	{367.395, 367.395}	{1296.53, 1296.53}	{3752.05, 3752.05}	{9578.84, 9578.84}
{-1.79198, -1.78731}	{0.344902, 0.345574}	{2.63302, 2.6331}	{1.83702, 1.83703}	{-3.38203, -3.38203}	{1., 1.}	{80., 80.}	{430.562, 430.562}	{1549.21, 1549.21}	{4557.87, 4557.87}	{11807.5, 11807.5}
{-2.08206, -2.0722}	{0.0849608, 0.0863776}	{2.68727, 2.68743}	{2.12451, 2.12452}	{-3.49981, -3.49981}	{1., 1.}	{90., 90.}	{494.983, 494.983}	{1811.14, 1811.14}	{5405.17, 5405.17}	{14181.2, 14181.2}

```
{-2.3⋱  {-0.1⋱  {2.71⋱  {2.39⋱  {-3.6⋱   {1.,   {100.,  {560.⋱  {2081⋱  {6290⋱  {1668⋱
  40⋱     70⋱    814,    343,     05⋱    1.}    100.}    517,    .41,    .43,    9.3,
  96,     25⋱    2.71⋱   2.39⋱    17,                    560.⋱   2081⋱   6290⋱   1668⋱
 -2.3⋱    9,     845}    346}    -3.6⋱                   517}    .41}    .43}    9.3}
  22⋱    -0.1⋱                    05⋱
  02}     67⋱                     17}
          53⋱
          6}
```

```
Table[{N[Sum[(-1)^(k+1)/k ((-1)^k (1 - Gamma[k, -Log[n]]/Gamma[k])), {k, 1, 30}]],
   N[ Limit[ (LaguerreL[-z, Log[n]] - 1)/z, z → 0]],
   N[LogIntegral[n] - Log[Log[n]] - EulerGamma]}, {n, 100, 600, 100}] // TableForm
```

$28.0217 - 2.09386\times 10^{-14}\, i$	28.0217	28.0217
$47.9476 - 4.32162\times 10^{-14}\, i$	47.9476	47.9476
$66.0153 - 1.89547\times 10^{-13}\, i$	66.0153	66.0153
$83.0503 - 4.70318\times 10^{-13}\, i$	83.0503	83.0503
$99.3898 - 1.32718\times 10^{-12}\, i$	99.3898	99.3898
$115.213 - 2.10999\times 10^{-12}\, i$	115.213	115.213

```
N[ Limit[ (LaguerreL[-z, Log[600]] - 1)/z, z → 0]]
```

115.213

```
N[LogIntegral[600] - Log[Log[600]] - EulerGamma]
```

115.213

```
Table[{N[Sum[(-1)^(k+1)/k ((-1)^k (1 - Gamma[k, -Log[n]]/Gamma[k])), {k, 1, 30}]],
   N[Limit[(LaguerreL[-z, Log[n]] - 1)/z, z → 0]],
   N[LogIntegral[n] - Log[Log[n]] - EulerGamma]}, {n, 100, 600, 100}] // TableForm
```

$28.0217 - 2.09386\times 10^{-14}\, i$	28.0217	28.0217
$47.9476 - 4.32162\times 10^{-14}\, i$	47.9476	47.9476
$66.0153 - 1.89547\times 10^{-13}\, i$	66.0153	66.0153
$83.0503 - 4.70318\times 10^{-13}\, i$	83.0503	83.0503
$99.3898 - 1.32718\times 10^{-12}\, i$	99.3898	99.3898
$115.213 - 2.10999\times 10^{-12}\, i$	115.213	115.213

```
Limit[ Sum[ c^j/j, {j, 2, Log[c, 100]}], c → 1]
```

$$\text{Limit}\left[-c - 100\, c\, \text{LerchPhi}\left[c, 1, 1 + \frac{\text{Log}[100]}{\text{Log}[c]}\right] - \text{Log}[1-c],\ c \to 1\right]$$

```
Binomial[z, 0]
```

1

```
Binomial[z, 1]
```

z

```
Binomial[z, 2]
```

$$\frac{1}{2}(-1+z)\, z$$

```
Binomial[z, 3]
```

$$\frac{1}{6}(-2+z)(-1+z)z$$

```
Binomial[z, 4]
```

$$\frac{1}{24}(-3+z)(-2+z)(-1+z)z$$

```
Sum[1, {j, 0, Floor[x - y]}]
```

1 + Floor[x - y]

```
Sum[1, {j, 0, Floor[x - y]}, {k, 0, Floor[x / (j + y) - y]}]
```

$$\sum_{j=0}^{\mathrm{Floor}[x-y]}\ \sum_{k=0}^{\mathrm{Floor}\left[-y+\frac{x}{j+y}\right]} 1$$

```
Dd[x_, 0, y_] := 1
Dd[x_, k_, y_] := Sum[Dd[x / (j + y), k - 1, y], {j, 0, Floor[x - y]}]
Cc[x_, k_, y_] := y^-k Dd[x y^k, k, y + 1]
CSum[ x_, y_] := Sum[ (-1)^(k + 1) / k Cc[ x, k, y], {k, 1, 10}]
```

```
Cd[x_, k_, y_] := (1 / y) Sum[ Cd[ y x / (j + y + 1), k - 1, y], {j, 0, Floor[y x - y - 1]}];
Cd[ x_, 0, y_] := 1
Cz[x_, z_, k_, y_] := (z - k + 1) / k (1 / y)
  Sum[ Cz[ y x / (j + y + 1), z, k - 1, y], {j, 0, Floor[y x - y - 1]}]; Cz[ x_, z_, 0, y_] := 1
```

```
Cc[ 100, 2, 3]
```

$$\frac{995}{3}$$

```
Cd[ 100, 4, 1]
```

184

```
Cz[ 100, 1, 1, 1]
```

99

```
pp[ x_, k_, y_] := (1 / y) Sum[ 1 / k - pp[ x y / ( j + y), k + 1, y], {j, 1, Floor[ (x - 1) y]}]
```

```
pp[100, 1, 1]
```

$$\frac{428}{15}$$

```
D[pp[10, 1, y], y]
```

$RecursionLimit::reclim : Recursion depth of 256 exceeded. ≫

$RecursionLimit::reclim : Recursion depth of 256 exceeded. ≫

$RecursionLimit::reclim : Recursion depth of 256 exceeded. ≫

General::stop : Further output of $RecursionLimit::reclim will be suppressed during this calculation. ≫

$IterationLimit::itlim : Iteration limit of 4096 exceeded. ≫

$IterationLimit::itlim : Iteration limit of 4096 exceeded. ≫

$IterationLimit::itlim : Iteration limit of 4096 exceeded. ≫

General::stop : Further output of $IterationLimit::itlim will be suppressed during this calculation. ≫

```
Expand[ (x y / ( j + y) - 1 ) y]
```

$$-y + \frac{x\, y^2}{j + y}$$

```
CSum[ 100, 13 / 10]
```

$$\frac{454\,930\,622\,410}{17\,130\,345\,141}$$

```
Expand[((x y) / (j + y) - 1 ) y]
```

$$-y + \frac{x\, y^2}{j + y}$$

```
Limit[(1 / y) Sum[ 1, {j, 1, Floor[ x y - y]}], y → Infinity]
```

$$\text{Limit}\left[\frac{\text{Floor}[-y + x\, y]}{y}, y \to \infty\right]$$

```
D[ 1 / y^2 Ccc, y]
```

$$\text{Integrate}\left[-\frac{2\,\text{Ccc}}{y^3}, \{y, 11/10, 12/10\}\right]$$

$$-\frac{575\,\text{Ccc}}{4356}$$

```
1 / ((12 / 10)^2) - 1 / ((11 / 10)^2)
```

$$-\frac{575}{4356}$$

$$\text{Integrate}\left[-\frac{2\,\text{Ccc}}{y^3}, \{y, \text{bbb}, \text{Infinity}\}\right]$$

$$\text{ConditionalExpression}\left[-\frac{\text{Ccc}}{\text{bbb}^2}, \text{Im}[\text{bbb}] \neq 0 \,||\, \text{bbb} > 0\right]$$

```
Czz[x_, z_, k_, y_] :=
 (z - k + 1) / (y k) Sum[ 1 + Czz[ x y / (j + y), z, k + 1, y], {j, 1, Floor[ x y - y]}]
1 + Czz[ 100, -3, 1, 1]
```

47

```
d1[n_, z_] := Product[(-1)^p[[2]] Binomial[-z, p[[2]]], {p, FI[n]}];
FI[n_] := FactorInteger[n]; FI[1] := {}
D1[n_, z_] := Sum[d1[j, z], {j, 1, n}]

D1[100, -3]
```

47

```
F[n_, j_, k_, z_] :=
 If[n < j, 0, (z - k + 1) / k (1 + F[n / j, 2, k + 1, z]) + F[n, j + 1, k, z]]
D1Alt[n_, z_] := 1 + F[n, 2, 1, z]
```

```
1 + F[100, 2, 1, 3]
```

1471

```
d1[n_, z_] := Product[(-1)^p[[2]] Binomial[-z, p[[2]]], {p, FI[n]}];
FI[n_] := FactorInteger[n]; FI[1] := {}
D1[n_, z_] := Sum[d1[j, z], {j, 1, n}]
F[n_, j_, k_, z_] := If[n < j, 0, (z - k + 1) / k (1 + F[n / j, 2, k + 1, z]) + F[n, j + 1, k, z]]
D1Alt[n_, z_] := 1 + F[n, 2, 1, z]
Grid[Table[{D1[a = 100, s + t I], D1Alt[a, s + t I]}, {s, -1.3, 4, .7}, {t, -1.3, 4, .7}]]
```

{10.4793 + 28.7121 i, 10.4793 + 28.7121 i}	{5.72468 + 11.2587 i, 5.72468 + 11.2587 i}	{6.03456 - 1.77709 i, 6.03456 - 1.77709 i}	{5.94691 - 15.6189 i, 5.94691 - 15.6189 i}	{15.2681 - 34.9044 i, 15.2681 - 34.9044 i}	{58.5435 - 59.5846 i, 58.5435 - 59.5846 i}	{173.704 - 80.8227 i, 173.704 - 80.8227 i}	{409.891 - 76.8935 i, 409.891 - 76.8935 i}
{-21.9794 + 33.2704 i, -21.9794 + 33.2704 i}	{-9.29577 + 6.6042 i, -9.29577 + 6.6042 i}	{-3.93641 - 0.702512 i, -3.93641 - 0.702512 i}	{-12.975 - 11.3196 i, -12.975 - 11.3196 i}	{-23.7041 - 47.2133 i, -23.7041 - 47.2133 i}	{-4.16474 - 124.722 i, -4.16474 - 124.722 i}	{95.5007 - 249.632 i, 95.5007 - 249.632 i}	{340.872 - 412.252 i, 340.872 - 412.252 i}
{-70.5899 - 1.50386 i, -70.5899 - 1.50386 i}	{-13.7213 - 18.1133 i, -13.7213 - 18.1133 i}	{3.81025 + 3.79964 i, 3.81025 + 3.79964 i}	{-26.9749 + 19.1944 i, -26.9749 + 19.1944 i}	{-89.9388 - 16.1483 i, -89.9388 - 16.1483 i}	{-144.356 - 139.879 i, -144.356 - 139.879 i}	{-126.266 - 377.33 i, -126.266 - 377.33 i}	{49.351 - 735.771 i, 49.351 - 735.771 i}
{-109.692 - 116.693 i, -109.692 - 116.693 i}	{25.3505 - 82.1743 i, 25.3505 - 82.1743 i}	{64.0826 + 14.9568 i, 64.0826 + 14.9568 i}	{-4.67506 + 101.541 i, -4.67506 + 101.541 i}	{-160.825 + 105.357 i, -160.825 + 105.357 i}	{-353.522 - 38.304 i, -353.522 - 38.304 i}	{-502.525 - 380.071 i, -502.525 - 380.071 i}	{-500.378 - 949.919 i, -500.378 - 949.919 i}
{-89.6457 - 364.055 i, -89.6457 - 364.055 i}	{165.919 - 209.786 i, 165.919 - 209.786 i}	{237.081 + 36.8175 i, 237.081 + 36.8175 i}	{110.164 + 267.906 i, 110.164 + 267.906 i}	{-190.242 + 376.688 i, -190.242 + 376.687 i}	{-601.821 + 264.819 i, -601.821 + 264.819 i}	{-1025.88 - 150.194 i, -1025.88 - 150.194 i}	{-1329.51 - 927.604 i, -1329.51 - 927.604 i}

{69.3293 − 807.552 i, 69.3293 − 807.552 i}	{497.243 − 430.789 i, 497.243 − 430.789 i}	{614.555 + 74.3692 i, 614.555 + 74.3692 i}	{404.806 + 557.989 i, 404.806 + 557.989 i}	{−102.401 + 871.308 i, −102.401 + 871.308 i}	{−831.921 + 874.96 i, −831.921 + 874.96 i}	{−1664.43 + 447.187 i, −1664.43 + 447.187 i}	{−2438.58 − 507.936 i, −2438.58 − 507.936 i}
{484.136 − 1524.88 i, 484.136 − 1524.88 i}	{1146.82 − 781.362 i, 1146.82 − 781.362 i}	{1326.79 + 133.656 i, 1326.79 + 133.656 i}	{1004.52 + 1019.94 i, 1004.52 + 1019.94 i}	{215.141 + 1678.53 i, 215.141 + 1678.53 i}	{−952.011 + 1920.83 i, −952.011 + 1920.83 i}	{−2354.8 + 1577.59 i, −2354.8 + 1577.59 i}	{−3800.46 + 508.005 i, −3800.46 + 508.005 i}
{1316.61 − 2608.99 i, 1316.61 − 2608.99 i}	{2288.19 − 1304.73 i, 2288.19 − 1304.73 i}	{2550.42 + 221.895 i, 2550.42 + 221.895 i}	{2080.38 + 1711.29 i, 2080.38 + 1711.29 i}	{919.284 + 2905.28 i, 919.284 + 2905.28 i}	{−828. + 3556.97 i, −828. + 3556.97 i}	{−2994.35 + 3440.6 i, −2994.35 + 3440.6 i}	{−5352.56 + 2361.41 i, −5352.56 + 2361.41 i}

```
zz1[n_, k_] := (-1)^k (1 - Gamma[k, -Log[n]] / Gamma[k])
zz2[n_, k_] := Integrate[(-1)^(k+1) / ((k-1)!) t^(k-1) E^(-t), {t, -Log[n], 0}]
zz2a[n_, k_] := Integrate[t^(k-1) E^(-t), {t, -Log[n], 0}]
```

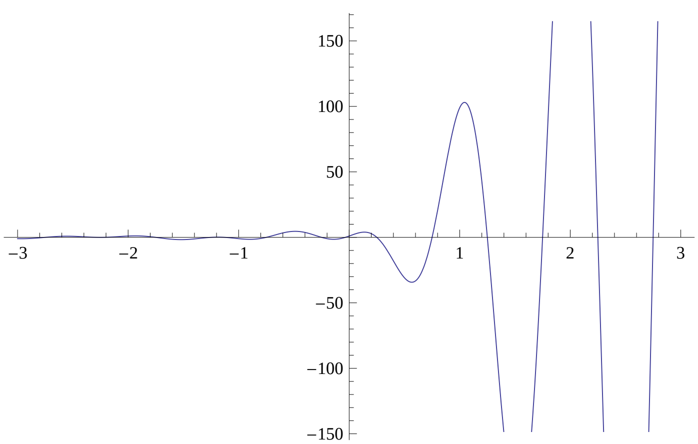
```
zz1[100, 0.01]
```

1.30358 + 0.0820144 i

```
zz2[100, 0.01]
```

1.30358 + 0.0820144 i

```
Plot[Re[zz1[100, z]], {z, -3, 3}]
```




```
Plot[Re[zz2[100, z]], {z, -3, 3}]
```

$Aborted

```
ff4a[n_, z_] :=
 Integrate[Sum[ Binomial[z, k] (-1)^(k+1) / ((k-1)!) t^(k-1) E^(-t), {k, 0, Infinity}],
  {t, -Log[n], 0}]

ff4a[100, z]
```

```
-1 + LaguerreL[-z, Log[100]]
```

```
TestSum[n_, z_, t_] :=
 Sum[N[Binomial[z, k] (-1)^k (1 - Gamma[k, -Log[n]] / Gamma[k])], {k, 0, t}]
Grid[Table[{Re[TestSum[n, k, 23]], N[LaguerreL[-k, Log[n]]]}, {n, 10, 100, 10}, {k, -5, 5}]]
```

{0.966672, 0.966672}	{0.727895, 0.727895}	{0.0104133, 0.0104133}	{-0.954221, -0.954221}	{-1.30259, -1.30259}	{1., 1.}	{10., 10.}	{33.0259, 33.0259}	{82.5612, 82.5612}	{178.953, 178.953}	{354.26, 354.26}
{0.853716, 0.853716}	{1.37285, 1.37285}	{0.993598, 0.993598}	{-0.504259, -0.504259}	{-1.99573, -1.99573}	{1., 1.}	{20., 20.}	{79.9146, 79.9146}	{229.573, 229.573}	{558.593, 558.593}	{1223.71, 1223.71}
{0.345442, 0.345447}	{1.4452, 1.4452}	{1.59103, 1.59103}	{-0.018323, -0.0183229}	{-2.4012, -2.4012}	{1., 1.}	{30., 30.}	{132.036, 132.036}	{407.594, 407.594}	{1053.4, 1053.4}	{2433.46, 2433.46}
{-0.182647, -0.182608}	{1.31842, 1.31842}	{1.97883, 1.97883}	{0.426157, 0.426157}	{-2.68888, -2.68888}	{1., 1.}	{40., 40.}	{187.555, 187.555}	{607.267, 607.267}	{1633.79, 1633.79}	{3910.39, 3910.39}
{-0.66439, -0.664191}	{1.10954, 1.10957}	{2.2416, 2.2416}	{0.827916, 0.827916}	{-2.91202, -2.91202}	{1., 1.}	{50., 50.}	{245.601, 245.601}	{823.8, 823.8}	{2283.51, 2283.51}	{5611.57, 5611.57}

{-1.09019, -1.08949}	{0.865209, 0.86531}	{2.42308, 2.42309}	{1.19314, 1.19314}	{-3.09434, -3.09434}	{1., 1.}	{60., 60.}	{305.661, 305.661}	{1054.23, 1054.23}	{2992.07, 2992.07}	{7508.1, 7508.1}
{-1.46377, -1.46181}	{0.606799, 0.607082}	{2.54836, 2.5484}	{1.52786, 1.52787}	{-3.2485, -3.2485}	{1., 1.}	{70., 70.}	{367.395, 367.395}	{1296.53, 1296.53}	{3752.05, 3752.05}	{9578.84, 9578.84}
{-1.79198, -1.78731}	{0.344902, 0.345574}	{2.63302, 2.6331}	{1.83702, 1.83703}	{-3.38203, -3.38203}	{1., 1.}	{80., 80.}	{430.562, 430.562}	{1549.21, 1549.21}	{4557.87, 4557.87}	{11807.5, 11807.5}
{-2.08206, -2.0722}	{0.0849608, 0.0863776}	{2.68727, 2.68743}	{2.12451, 2.12452}	{-3.49981, -3.49981}	{1., 1.}	{90., 90.}	{494.983, 494.983}	{1811.14, 1811.14}	{5405.17, 5405.17}	{14181.2, 14181.2}
{-2.34096, -2.32202}	{-0.170259, -0.167536}	{2.71814, 2.71845}	{2.39343, 2.39346}	{-3.60517, -3.60517}	{1., 1.}	{100., 100.}	{560.517, 560.517}	{2081.41, 2081.41}	{6290.43, 6290.43}	{16689.3, 16689.3}

```
TestSum[n_, z_, t_] :=
 Sum[N[Binomial[z, k] (-1)^k (1 - Gamma[k, -Log[n]] / Gamma[k])], {k, 0, t}]
```

```
TestSum[ 100, 2, 30]
```

$560.517 - 4.41506\times 10^{-14}\, i$

```
zz1[n_, k_] := (-1)^k ( 1 - Gamma[ k, -Log[n]] / Gamma[k])
zz2[n_, k_] := Integrate[(-1)^(k+1) / ((k-1)!) t^(k-1) E^(-t), {t, -Log[n], 0}]
```

```
N[zz2[100, 2]]
```

361.517

```
aff2[ n_, z_] :=
 Sum[ Binomial[z, k] (-1)^k ( 1 - Gamma[ k, -Log[n]] / Gamma[k]), {k, 0, Infinity}]
```

```
aff2[100, 2, 40]
```

200 - Gamma[2, -Log[100]]

Limit[Sum[(j + 1) c^j / j , {j, 1, Log[c, n]}], c → 1]

$$\text{Limit}\left[\frac{1}{-1+c}\left(-c+c\,n+c\,n\,\text{LerchPhi}\left[c,1,1+\frac{\text{Log}[n]}{\text{Log}[c]}\right]-c^2\,n\,\text{LerchPhi}\left[c,1,1+\frac{\text{Log}[n]}{\text{Log}[c]}\right]+\text{Log}[1-c]-c\,\text{Log}[1-c]\right),c\to 1\right]$$

Limit[Sum[(j + 1) (c - 1) ^2 c^j , {j, 1, Log[c, n]}], c → 1]

$1-n+n\,\text{Log}[n]$

Limit[Sum[(j + 1) (j + 2) (c - 1) ^3 c^j , {j, 1, Log[c, n]}], c → 1]

$-2+2\,n-2\,n\,\text{Log}[n]+n\,\text{Log}[n]^2$

N[Limit[Sum[(j + 1) (j + 2) (j + 3) (c - 1) ^4 c^j , {j, 1, Log[c, n]}], c → 1] /. n → 100]

5573.28

Expand[LaguerreL[4, Log[n]]]

$$1-4\,\text{Log}[n]+3\,\text{Log}[n]^2-\frac{2\,\text{Log}[n]^3}{3}+\frac{\text{Log}[n]^4}{24}$$

Gamma[4, -Log[100.]]

$-5567.28+2.04539\times 10^{-12}\,i$

Limit[Sum[j^z (c - 1) ^ (z + 1) c^j , {j, 1, Log[c, n]}], c → 1] /. {z → 4}

$24\,(-1+n)-24\,n\,\text{Log}[n]+12\,n\,\text{Log}[n]^2-4\,n\,\text{Log}[n]^3+n\,\text{Log}[n]^4$

N[Gamma[4, -Log[n]] /. n → 100]

$-5567.28+2.04539\times 10^{-12}\,i$

Limit[Sum[c^-j (c - 1) , {j, 1, Log[c, n]}], c → 1]

$$\frac{-1+n}{n}$$

Limit[Sum[c^j (c - 1) , {j, 1, Log[c, n]}], c → 1]

$-1+n$

Limit[Sum[c^j / j (c - 1) , {j, 1, Log[c, n]}], c → 1]

$$\text{Limit}\left[c\,n\,\text{LerchPhi}\left[c,1,1+\frac{\text{Log}[n]}{\text{Log}[c]}\right]-c^2\,n\,\text{LerchPhi}\left[c,1,1+\frac{\text{Log}[n]}{\text{Log}[c]}\right]+\text{Log}[1-c]-c\,\text{Log}[1-c],c\to 1\right]$$

```
TestSum[n_, z_, t_] :=
 1 + Sum[N[Binomial[z, k] (-1)^k (Gamma[k, 0, -Log[n]] / Gamma[k])], {k, 1, t}]
Grid[Table[Chop[Re[TestSum[n, k, 80]] - N[LaguerreL[-k, Log[n]]]],
  {n, 10, 100, 10}, {k, -5, 5}]]
```

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
1.47048×10^{-10}	0	0	0	0	0	0	0	0	0	0
5.51938×10^{-10}	2.2593×10^{-10}	0	0	0	0	0	0	0	0	0

```
D1y1[ x_, s_, k_, y_ ] := D1y1[x, s, k, y] = D1y1[ x, s, k - 1, y] +
   y Sum[ (1 + j y)^-s D1y1[ x ( 1 + j y)^-1, s, k - 1, y], {j, 1, (x - 1) / y}];
D1y1[x_, s_, 0, y_] := UnitStep[ x - 1]
```

```
N[D1y1[ 10, -1, 1, .0001]]
```

50.5004

```
LaguerreL[-3, Log[100.]]
```

2081.41

```
FullSimplify[
 (y (Zeta[s] - 1) + 1)^z - Sum[ Binomial[z, k] (y (Zeta[s] - 1))^k, {k, 0, Infinity}]]
```

0

```
(1 / 2) (1 / 2)^-s
```

2^{-1+s}

```
HurwitzZeta[2, 2]
```

$-1+\frac{\pi^2}{6}$

```
{Limit[x^(s - 1) HurwitzZeta[s, x + 1], x → Infinity], 1 / (s - 1)}
```

$\left\{\frac{1}{-1+s}, \frac{1}{-1+s}\right\}$

```
Integrate[ y^-s, {y, 1, x}]
```

$\text{ConditionalExpression}\left[\frac{x^{-s}\,(-x+x^s)}{-1+s}, \text{Re}[x] \geq 0 \,||\, x \notin \text{Reals}\right]$

$\textbf{Expand}\left[\frac{x^{-s}\,(-x+x^s)}{-1+s}\right]$

$$\frac{1}{-1+s}-\frac{x^{1-s}}{-1+s}\ \text{/. } s\to -3$$

$$-\frac{1}{4}+\frac{x^4}{4}$$

FullSimplify[$x^{-s}\,(-x+x^s)$]

$$1-x^{1-s}$$

Expand[(n^(1 - s) - 1) / (1 - s) /. s → -4]

$$-\frac{1}{5}+\frac{n^5}{5}$$

Sum[j^-s, {j, 2, Floor[n]}] /. s → -4

$$-1-\frac{\text{Floor}[n]}{30}+\frac{\text{Floor}[n]^3}{3}+\frac{\text{Floor}[n]^4}{2}+\frac{\text{Floor}[n]^5}{5}$$

Integrate[x^-s, {x, 1, Infinity}]

$$\text{ConditionalExpression}\left[\frac{1}{-1+s},\ \text{Re}[s]>1\right]$$

1 / (s - 1) + Integrate[D[y^(1 - s) Zeta[s, y^-1 + 1], y], {y, 0, 1}]

$$\frac{1}{-1+s}+\int_0^1\left((1-s)\,y^{-s}\,\text{Zeta}\left[s,\,1+\frac{1}{y}\right]+s\,y^{-1-s}\,\text{Zeta}\left[1+s,\,1+\frac{1}{y}\right]\right)dy$$

Table[{Zeta[s] - 1,
 1 / (s - 1) - Integrate[D[y^(s - 1) Zeta[s, y + 1], y], {y, 1, Infinity}]}, {s, 2, 6}]

$$\left\{\left\{-1+\frac{\pi^2}{6},\,-1+\frac{\pi^2}{6}\right\},\,\{-1+\text{Zeta}[3],\,-1+\text{Zeta}[3]\},\right.$$
$$\left.\left\{-1+\frac{\pi^4}{90},\,-1+\frac{\pi^4}{90}\right\},\,\{-1+\text{Zeta}[5],\,-1+\text{Zeta}[5]\},\,\left\{-1+\frac{\pi^6}{945},\,-1+\frac{\pi^6}{945}\right\}\right\}$$

```
f1[f_] := Integrate[ f[x], {x, 1, Infinity}]
f2[f_] := Integrate[ f[1 / x], {x, 0, 1}]
```

```
nn[x_] := 1 / x^2
```

f1[nn]

1

f2[nn]

$$\frac{1}{3}$$

Integrate[Sin[x] / x, {x, 1, Infinity}]

$$\frac{1}{2}\,(\pi-2\,\text{SinIntegral}[1])$$

Integrate[Sin[1 / r] / (1 / r), {r, 0, 1}]

$$\frac{1}{4}\,(-\pi+2\,(\text{Cos}[1]+\text{Sin}[1]+\text{SinIntegral}[1]))$$

```
Dy[x_, y_, k_] := Sum[y Dy[x (j y^-1 + 1) ^-1, y, k - 1], {j, 1, (x - 1) / y}];
Dy[x_, y_, 0] := UnitStep[x - 1]
```

```
Dy[100, 2, 1]
```

98

```
{Limit[(y^(s - 1) HurwitzZeta[s, y + 1]) ^k, y → Infinity], 1 / (s - 1) ^k}
```

$\left\{\left(\frac{1}{-1+s}\right)^k, (-1+s)^{-k}\right\}$

```
Expand[(1 - s) ^2]
```

$1 - 2 s + s^2$

```
Expand[(s - 1) ^2]
```

$1 - 2 s + s^2$

```
Grid[Table[
  Chop[N[1 / ((s - 1) ^k) - Integrate[D[(Zeta[s, y + 1] y^(s - 1)) ^k, y], {y, 1, Infinity}]] -
    N[(Zeta[s] - 1) ^k]], {s, 2, 4}, {k, 1, 4}]]
```

```
0 0 0 0
0 0 0 0
0 0 0 0
```

```
Log[Zeta[s, y + 1] y^(s - 1) + 1] -
 FullSimplify[Sum[ (-1) ^(k + 1) / k (Zeta[s, y + 1] y^(s - 1)) ^k, {k, 1, Infinity}]]
```

0

```
Limit[ Gamma[k, 0, -Log[100]] / Gamma[k], k → 0]
```

1

```
FullSimplify[Sum[
   (1 / k) ( 2^((1 - s) k) + (-1) ^(k + 1) ((1 - 2^(1 - s)) Zeta[s]) ^k ), {k, 1, Infinity}]] /. s → 0
```

$-i\pi + \text{Log}\left[\frac{3}{2}\right]$

```
Log[Zeta[0]]
```

$i\pi - \text{Log}[2]$

```
Table[{Log[Zeta[s]], Limit[Sum[(x^(1 - s)) ^j / j, {j, 1, Infinity}] +
    Sum[(-1) ^(k - 1) / k ((1 - x^(1 - s)) Zeta[s] - 1) ^k, {k, 1, Infinity}], x → 2]}, {s, 0, 0}]
```

$\{\{i\pi - \text{Log}[2], -i\pi - \text{Log}[2]\}\}$

```
{Log[Zeta[s]], Sum[(2^(1 - s)) ^j / j, {j, 1, Infinity}] +
   Sum[(-1) ^(k - 1) / k ((1 - 2^(1 - s)) Zeta[s] - 1) ^k, {k, 1, Infinity}]} /. s → 0
```

$\{i\pi - \text{Log}[2], -i\pi - \text{Log}[2]\}$

```
ff5[n_, z_, s_] :=
 Integrate[Sum[Binomial[z, k] (-1)^(k+1) / ((k-1)!) t^(k-1) E^((s-1) t),
   {k, 0, Infinity}], {t, -Log[n], 0}]
```

```
ff5[n, z, s] /. {s → 0}
```

ConditionalExpression[-1 + LaguerreL[-z, Log[n]], -1 ≤ Re[Log[n]] ≤ 1 || Log[n] ∉ Reals]

```
ff5[n, z, s] /. {s → -1}
```

$$\int_{-\mathrm{Log}[n]}^{0} e^{-2t}\, z\, \mathrm{Hypergeometric1F1}[1-z, 2, t]\, dt$$

```
ff5[n, z, s]
```

$$\int_{-\mathrm{Log}[n]}^{0} e^{(-1+s)\,t}\, z\, \mathrm{Hypergeometric1F1}[1-z, 2, t]\, dt$$

```
Limit[((-1 + ff5[n, z, s] /. s → 2) / z), z → 0]
```

$Aborted

```
N[-Gamma[0, -Log[100]]]
```

30.1261 + 3.14159 i

```
FullSimplify[
  Limit[Gamma[0, s Log[n]] - Gamma[0, (s-1) Log[n]] + Log[s / (s-1)], s → 3]] /. n → 100
```

$$\frac{1}{2}\left(-2\,\mathrm{ExpIntegralEi}[-3\,\mathrm{Log}[100]] + 2\,\mathrm{ExpIntegralEi}[-2\,\mathrm{Log}[100]] - \mathrm{Log}\left[\frac{1}{3\,\mathrm{Log}[100]}\right] + \mathrm{Log}\left[\frac{3}{2\,\mathrm{Log}[100]}\right] + \mathrm{Log}[\mathrm{Log}[100]] - \mathrm{Log}[2\,\mathrm{Log}[100]]\right)$$

```
Limit[ff5[n, z, s], z → 0]
```

$$\mathrm{Limit}\left[\int_{-\mathrm{Log}[n]}^{0} e^{(-1+s)\,t}\, z\, \mathrm{Hypergeometric1F1}[1-z, 2, t]\, dt,\ z \to 0\right]$$

```
D[Sum[Binomial[z, k] (-1)^(k+1) / ((k-1)!) t^(k-1) E^((s-1) t), {k, 0, Infinity}], z] /.
 z → 0
```

$$\frac{e^{(-1+s)\,t}\left(-1+e^{t}\right)}{t}$$

```
ff5[n, z, 0]
```

ConditionalExpression[-1 + LaguerreL[-z, Log[n]], -1 ≤ Re[Log[n]] ≤ 1 || Log[n] ∉ Reals]

$$\mathrm{N}\left[\frac{1}{2}\left(-2\,\mathrm{ExpIntegralEi}[-3\,\mathrm{Log}[100]] + 2\,\mathrm{ExpIntegralEi}[-2\,\mathrm{Log}[100]] - \mathrm{Log}\left[\frac{1}{3\,\mathrm{Log}[100]}\right] + \mathrm{Log}\left[\frac{3}{2\,\mathrm{Log}[100]}\right] + \mathrm{Log}[\mathrm{Log}[100]] - \mathrm{Log}[2\,\mathrm{Log}[100]]\right)\right]$$

0.405455

```
N[Gamma[0, s Log[100]] - Gamma[0, (s - 1) Log[100]] + Log[s / (s - 1)]] /. s → -2
```

77 381. + 0. i

```
N[D[ff5[ 100, z, -2], z] /. z → 0]
```

77 381.

```
N[Gamma[0, 2 Log[10]]]
```

0.00182974

```
N[Gamma[0, Log[10]]]
```

0.0323898

```
N[Gamma[0, 2]]
```

0.0489005

```
NSum[ -1 / k Gamma[k, 0, -Log[100]] / Gamma[k], {k, 1, Infinity}]
```

28.0217

```
-N[Gamma[0, -Log[100]]]
```

30.1261 + 3.14159 i

```
N[D[LaguerreL[ -z, Log[100]], z] /. z → 0]
```

28.0217

```
N[LogIntegral[100] - Log[Log[100]] - EulerGamma]
```

28.0217

```
N[D[LaguerreL[ -z, Log[100]], {z, 6}] /. z → 0]
```

$Aborted

```
ff5[ n, z, s]
```

$\int_{-\text{Log}[n]}^{0} e^{(-1+s)\,t}\, z\, \text{Hypergeometric1F1}[1 - z, 2, t]\, dt$

```
Binomial[z, 0]
```

1

```
Binomial[z, 2]
```

$\frac{1}{2}\,(-1 + z)\, z$

```
Integrate[ y^-s, {y, 1, n}]
```

$\text{ConditionalExpression}\left[\frac{n^{-s}\,(-n + n^{s})}{-1 + s}, \text{Re}[n] \geq 0 \,||\, n \notin \text{Reals}\right]$

$$\textbf{Expand}\left[\frac{n^{-s}\,(-n+n^{s})}{-1+s}\right]$$

$$\frac{1}{-1+s}-\frac{n^{1-s}}{-1+s}$$

```
Integrate[ (x y) ^-s, {x, 1, n}, {y, 1, n / x}]
```

$$\text{ConditionalExpression}\left[\frac{1}{(-1+s)^2}-\frac{n^{1-s}\,(1+(-1+s)\,\text{Log}[n])}{(-1+s)^2},\ \text{Re}[n]\geq 0\ ||\ n\notin\text{Reals}\right]$$

$$\textbf{Expand}\left[\frac{1}{(-1+s)^2}-\frac{n^{1-s}\,(1+(-1+s)\,\text{Log}[n])}{(-1+s)^2}\ \text{/. } s\to -1\right]$$

$$\frac{1}{4}-\frac{n^2}{4}+\frac{1}{2}n^2\,\text{Log}[n]$$

$$\textbf{FullSimplify}\left[\frac{1}{(-1+s)^2}-\frac{n^{1-s}\,(1+(-1+s)\,\text{Log}[n])}{(-1+s)^2}\right]$$

$$\frac{n^{-s}\,(-n+n^{s}+(n-n\,s)\,\text{Log}[n])}{(-1+s)^2}$$

$$\int_{-\text{Infinity}}^{0} e^{(-1+s)\,t}\,z\,\text{Hypergeometric1F1}[1-z,\,2,\,t]\,dt$$

$$\text{ConditionalExpression}\left[-1+\left(\frac{s}{-1+s}\right)^{z},\ \text{Re}[s]>1\right]$$

```
ff2a[ n_, z_, t_, s_] :=
 1 + Sum[ Binomial[z, k] (-1) ^k ( GammaRegularized[ k, 0, (s - 1) Log[n]] (1 - s) ^-k), {k, 1, t}]
ff2a[100, 2, 200, 2]
```

$$\frac{149}{50}+\text{GammaRegularized}[2,\,0,\,\text{Log}[100]]$$

```
N[Sum[ Binomial[z, k] (-1) ^k ( GammaRegularized[ k, 0, (s - 1) Log[n]] (1 - s) ^-k),
   {k, 1, Infinity}] /. {n → 10, s → 2, z → 3}]
```

5.11387

```
N[ff5[ 10, 3, 2]]
```

5.11387

```
E^{I Log[Zeta[s]]}
```

$$\{\text{Zeta}[s]^{i}\}$$

```
FI[n_] := FI[n] = FactorInteger[n]; FI[1] := {}
dzeta[j_, z_] := dzeta[j, z] = Product[(-1)^p[[2]] Binomial[-z, p[[2]]], {p, FI[j]}]
zeta[n_, s_, z_] := Sum[j^-s dzeta[j, z], {j, 1, n}]
cosz[ n_, s_, t_] := Sum[ (-1)^k zeta[n, s, 2 k] / ((2 k) !), {k, 0, t}]
coszp[ n_, s_, t_] :=
 Sum[ (-1)^k / ((2 k) !) Sum[ (2 Pi )^(2 k - j) zeta[n, s, j], {j, 0, 2 k}], {k, 0, t}]
coszp2[ n_, s_, t_] :=
 Sum[ (-1)^k / ((2 k) !) Sum[ (2 Pi )^(2 k - j) zet[n, s, j], {j, 0, 2 k}], {k, 0, t}]
```

```
N[cosz[10 000, 2, 30]]
```

-0.0743883

```
N[zeta[100, s, 0] - zeta[100, s, 2] / 2 + zeta[100, s, 4] / 24 - zeta[100, s, 6] / 720 +
   zeta[100, s, 8] / (8 !) - zeta[100, s, 10] / (10 !) + zeta[100, s, 12] / (12 !)] /. s → 2
```

-0.0685789

```
N[Cos[Zeta[2]]]
```

-0.0740698

```
N[coszp[1000, 2, 20]]
```

1.3782

```
Sum[ (2 Pi )^(2 r - j) zzz[n, j], {j, 0, 2 r}] /. r → 2
```

$16 \pi^4 \operatorname{zzz}[n, 0] + 8 \pi^3 \operatorname{zzz}[n, 1] + 4 \pi^2 \operatorname{zzz}[n, 2] + 2 \pi \operatorname{zzz}[n, 3] + \operatorname{zzz}[n, 4]$

```
coszp2[100, 2, 3]
```

$$\operatorname{zet}[100, 2, 0] + \frac{1}{2} \left(-4 \pi^2 \operatorname{zet}[100, 2, 0] - 2 \pi \operatorname{zet}[100, 2, 1] - \operatorname{zet}[100, 2, 2]\right) + \frac{1}{24} \left(16 \pi^4 \operatorname{zet}[100, 2, 0] + 8 \pi^3 \operatorname{zet}[100, 2, 1] + 4 \pi^2 \operatorname{zet}[100, 2, 2] + 2 \pi \operatorname{zet}[100, 2, 3] + \operatorname{zet}[100, 2, 4]\right) + \frac{1}{720} \left(-64 \pi^6 \operatorname{zet}[100, 2, 0] - 32 \pi^5 \operatorname{zet}[100, 2, 1] - 16 \pi^4 \operatorname{zet}[100, 2, 2] - 8 \pi^3 \operatorname{zet}[100, 2, 3] - 4 \pi^2 \operatorname{zet}[100, 2, 4] - 2 \pi \operatorname{zet}[100, 2, 5] - \operatorname{zet}[100, 2, 6]\right)$$

```
-Limit[D[Gamma[0, s Log[n]] - Gamma[0, (s - 1) Log[n]] + Log[s / (s - 1)], s], s → 0]
```

$-1 + n - \operatorname{Log}[n]$

```
zeta[n_, s_, z_, k_] := 1 + ((z + 1) / k - 1) Sum[j^-s zeta[n / j, s, z, k + 1], {j, 2, n}]
```

```
Limit[D[N[Expand[D[Expand[zeta[100, s, z, 1]], s] /. s → 0]], z], z → 0]
```

-94.0453

```
N[Expand[D[Expand[zeta[100, s, z, 1]], s] /. s → 0]]
```

$-94.0453 z - 169.15 z^2 - 81.6195 z^3 - 17.6846 z^4 - 1.19616 z^5 - 0.0438125 z^6$

```
tt[n_, z_] := N[Expand[D[Expand[zeta[n, s, z, 1]], s] /. s → 0]]
tt2[n_, z_] := D[zeta[n, s, z, 1], s] /. s → 0
lzeros[n_] := List @@ NRoots[tt[n, z] - 1 == 0, z][[All, 2]]
lzeros2[n_] := List @@ NRoots[tt2[n, z] - 1 == 0, z][[All, 2]]
```

```
tt[100, z]
```

$-94.0453\,z - 169.15\,z^2 - 81.6195\,z^3 - 17.6846\,z^4 - 1.19616\,z^5 - 0.0438125\,z^6$

```
lzeros[100]
```

{-10.6971 - 12.1993 i, -10.6971 + 12.1993 i,
 -2.54005 - 1.8272 i, -2.54005 + 1.8272 i, -0.816685, -0.0108436}

```
-Sum[ j^-1, {j, lzeros[100]}]
```

94.0453 + 0. i

```
-1 + Product[ 1 - 1 / j, {j, lzeros[1000]}]
```

$Aborted

```
Sum[N[Log[j]], {j, 1, 1000}]
```

363.739

```
lzeros2[100]
```

{-10.6971 - 12.1993 i, -10.6971 + 12.1993 i,
 -2.54005 - 1.8272 i, -2.54005 + 1.8272 i, -0.816685, -0.0108436}

```
-1 + Product[ 1 + 1 / j, {j, lzeros[100]}]
```

$10.0174 - 8.88178 \times 10^{-16}\,i$

```
tt[100, -1]
```

-10.0174

```
tt[100, 2]
```

-1841.68

```
D[tt[100, z], z] /. z → 0
```

-94.0453

```
D[tt[100, z], {z, 2}] /. z → 0
```

-338.3

```
-N[Sum[ Log[j] + Log[k], {j, 1, 100}, {k, 1, Floor[100 / j]}]]
```

-1841.68

```
-N[Sum[ Log[j], {j, 1, 100}, {k, 1, Floor[100 / j]}]]
```

-920.841

```
tt[100, 2] - 2 tt[100, 1] + tt[100, 0]
```

-1114.2

```
d2[n_, s_, k_] := Sum[ j^-s d2[n / j, s, k - 1], {j, 2, n}]
d2[n_, s_, 0] := UnitStep[n - 1]
```

```
N[D[Expand[d2[100, s, 2]], s] /. s → 0]
```

-1114.2

```
N[D[Sum[ (j k)^-s, {j, 2, 100}, {k, 2, 100/j}], s] /. s → 0]
```

```
-1114.2
```

```
N[Sum[ D[(j k)^-s, s] /. s → 0, {j, 2, 100}, {k, 2, 100/j}]]
```

```
-1114.2
```

```
Expand[D[(j k)^-s Kap[j] Kap[k], s] /. s → 0]
```

```
-Kap[j] Kap[k] Log[j k]
```

```
2 N[Sum[ -Log[j], {j, 2, 100}, {k, 2, 100/j}]]
```

```
-1114.2
```

```
N[D[Sum[ (j k)^-s, {j, 1, 100}, {k, 1, 100/j}], s] /. s → 0]
```

```
-1841.68
```

```
tt[100, 2]
```

```
-1841.68
```

```
N[Limit[D[Expand[Limit[D[zeta[100, s, z, 1], z], z → 0]], s], s → 0]]
```

```
-94.0453
```

```
N[Limit[D[Expand[Limit[D[zeta[100, s, z, 1], {z, 2}], z → 0]], s], s → 0]]
```

```
-338.3
```

```
N[Limit[D[Expand[Limit[D[zeta[100, s, z, 1], {z, 3}], z → 0]], s], s → 0]]
```

```
-489.717
```

```
K[n_] := K[n] = FullSimplify[MangoldtLambda[n] / Log[n]]
K[1] := 0
-2 N[Sum[ K[j] K[k] Log[k], {j, 2, 100}, {k, 2, 100/j}]]
-3 N[Sum[ K[j] K[k] K[l] Log[l], {j, 2, 100}, {k, 2, 100/j}, {l, 2, 100/(j k)}]]
```

```
-338.3
```

```
-489.717
```

```
3/2 N[ Sum[
   K[j] N[Limit[D[Expand[Limit[D[zeta[Floor[100/j], s, z, 1], {z, 2}], z → 0]], s], s → 0]],
   {j, 2, 100}]]
```

```
-489.717
```

```
D[(j + y)^-s, s] /. s → 0
```

```
-Log[j + y]
```

```
D[ (1 + j / y)^-s, s] /. s → 0
```

$$-\text{Log}\left[1+\frac{j}{y}\right]$$

```
D[ ((1 + j / y) (1 + k / y))^-s, s] /. s → 0
```

$$-\text{Log}\left[\left(1+\frac{j}{y}\right)\left(1+\frac{k}{y}\right)\right]$$

```
Sum[ (-1)^(k+1) / k x^k, {k, 1, Infinity}]
```

```
Log[1 + x]
```

```
tt[n_, z_] := N[Expand[D[Expand[zeta[n, s, z, 1]], s] /. s → 0]]
tt2[n_, z_] := D[zeta[n, s, z, 1], s] /. s → 0
lzeros[n_] := List @@ NRoots[tt[n, z] - 1 == 0, z][[All, 2]]
lzeros2[n_] := List @@ NRoots[tt2[n, z] - 1 == 0, z][[All, 2]]
pr[n_] := E^(Product[ 1 - 1 / j, {j, lzeros[n]}]) / E
```

```
pr[5]
```

```
120.
```

```
FullSimplify[ E^((1 - 1 / a) (1 - 1 / b) (1 - 1 / c) (1 - 1 / d) ) / E]
```

```
$Aborted
```

```
E^(3 + 1)
```

e^4

```
E^3 E
```

e^4

```
E^(a b)
```

$e^{a b}$

```
lzeros[5]
```

```
{-5.65162, -0.255271}
```

```
E^((1 - 1 / -5.6516195112740135`) (1 - 1 / -0.2552710843345051`)) / E
```

```
120.
```

```
Table[lzeros2[n], {n, 3, 40}] // TableForm
```

Part::partd : Part specification (z == −0.558111)[[All, 2]] is longer than depth of object. ≫

z == −0.558111	All	2	
−3.12301	−0.461957		
−5.65162	−0.255271		
−1.34947	−0.298212		
−2.25209	−0.178692		
−7.6933	−2.31459	−0.162038	
−11.3756	−1.82531	−0.138961	
−18.7891	−1.04804	−0.146528	
−18.3869	−1.4916	−0.105207	
−3.17547	−1.85831	−0.106645	
−2.529 − 1.11322 ⅈ	−2.529 + 1.11322 ⅈ	−0.082424	
−5.30607	−1.41111	−0.0840496	
−7.43365	−0.985929	−0.0858659	
−9.34366 − 4.54503 ⅈ	−9.34366 + 4.54503 ⅈ	−0.986275	−0.0812942
−9.19869 − 4.27982 ⅈ	−9.19869 + 4.27982 ⅈ	−1.29244	−0.0650671
−27.3088	−3.51173	−1.3787	−0.0654685
−27.35	−2.7197	−2.14057	−0.0543648
−41.641	−1.76741 − 0.826514 ⅈ	−1.76741 + 0.826514 ⅈ	−0.0546057
−40.9372	−2.92864	−1.30944	−0.0551387
−40.194	−4.01863	−0.962081	−0.0557025
−40.2132	−3.74722	−1.2231	−0.0469664
−4.64026 − 2.26986 ⅈ	−4.64026 + 2.26986 ⅈ	−1.23385	−0.0470746
−4.65745 − 2.78721 ⅈ	−4.65745 + 2.78721 ⅈ	−1.20282	−0.0437387
−4.77647 − 3.69903 ⅈ	−4.77647 + 3.69903 ⅈ	−0.964492	−0.0440295
−5.2026 − 3.33709 ⅈ	−5.2026 + 3.33709 ⅈ	−0.965617	−0.0420146
−8.48722	−4.50353	−0.962248	−0.0421406
−8.65164	−4.12121	−1.18562	−0.0366638
−16.2813	−1.47436 − 0.650513 ⅈ	−1.47436 + 0.650513 ⅈ	−0.0366574
−16.3055	−1.46438 − 0.88773 ⅈ	−1.46438 + 0.88773 ⅈ	−0.0324145
−14.6954 − 12.6407 ⅈ	−14.6954 + 12.6407 ⅈ	−1.45868 − 0.881441 ⅈ	−1.45868 + 0.881441 ⅈ
−14.4871 − 12.4473 ⅈ	−14.4871 + 12.4473 ⅈ	−1.66687 − 0.449788 ⅈ	−1.66687 + 0.449788 ⅈ
−14.262 − 12.2467 ⅈ	−14.262 + 12.2467 ⅈ	−2.60863	−1.1752
−14.0171 − 12.0388 ⅈ	−14.0171 + 12.0388 ⅈ	−3.32189	−0.951594
−54.1561	−4.10982 − 2.0086 ⅈ	−4.10982 + 2.0086 ⅈ	−0.951498
−54.1552	−4.01762 − 1.85249 ⅈ	−4.01762 + 1.85249 ⅈ	−1.1403
−54.2062	−4.08339 − 2.55695 ⅈ	−4.08339 + 2.55695 ⅈ	−0.95764
−54.2574	−4.11807 − 3.08061 ⅈ	−4.11807 + 3.08061 ⅈ	−0.837002
−77.312	−3.23628 − 2.85347 ⅈ	−3.23628 + 2.85347 ⅈ	−0.833711

```
Expand[(1 - 1 / r) ^4]
```

$$1+\frac{1}{r^4}-\frac{4}{r^3}+\frac{6}{r^2}-\frac{4}{r}$$

```
Log[1 - 1 / r]
```

$$\mathrm{Log}\left[1-\frac{1}{r}\right]$$

```
-Sum[ r^-k k^-1, {k, 1, Infinity}]
```

$$\mathrm{Log}\left[\frac{-1+r}{r}\right]$$

```
E^(a + b)
```

e^{a+b}

```
E^a E^b
```

e^{a+b}

```
bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
pp[n_, s_, 0] := UnitStep[n - 1]
pp[n_, s_, k_] := pp[n, s, k] =
  Sum[ (-1)^(j + 1) 1 / (2 j - 1) (2 j - 1)^-s pp[Floor[n / (2 j - 1)], s, k - 1], {j, 2, n}]
pz[n_, s_, z_] := Sum[ bin[z, k] pp[n, s, k], {k, 0, Log[2, n]}]
pzeros[n_, s_] := List @@ NRoots[pz[n, s, z] == 0, z][[All, 2]]
```

```
N[pz[1141, N[ZetaZero[1]], 1]]
```

1.20656 + 0.182392 ⅈ

```
N[Expand[pz[400, z]]]
```

$1. - 0.239591\, z + 0.0250854\, z^2 - 0.000295886\, z^3 - 0.00101557\, z^4 - 0.0000342936\, z^5$

```
Table[pzeros[n, N[ZetaZero[1]]], {n, 420, 440}]
```

{{-16.6813 + 12.2121 ⅈ, -9.25821 - 12.8105 ⅈ,
 -9.01788 - 1.39455 ⅈ, -4.33498 + 6.68041 ⅈ, 5.37644 + 8.45781 ⅈ},
 {-16.684 + 12.2159 ⅈ, -9.25778 - 12.8078 ⅈ, -9.02221 - 1.40204 ⅈ, -4.32628 + 6.67959 ⅈ,
 5.37429 + 8.45957 ⅈ}, {-16.684 + 12.2159 ⅈ, -9.25778 - 12.8078 ⅈ,
 -9.02221 - 1.40204 ⅈ, -4.32628 + 6.67959 ⅈ, 5.37429 + 8.45957 ⅈ},
 {-17.3587 + 11.5569 ⅈ, -8.9324 - 12.7647 ⅈ, -8.91464 - 1.10664 ⅈ, -4.14445 + 6.86345 ⅈ,
 5.43429 + 8.5963 ⅈ}, {-17.3587 + 11.5569 ⅈ, -8.9324 - 12.7647 ⅈ,
 -8.91464 - 1.10664 ⅈ, -4.14445 + 6.86345 ⅈ, 5.43429 + 8.5963 ⅈ},
 {-16.644 + 12.1671 ⅈ, -9.25798 - 12.7852 ⅈ, -9.04194 - 1.3936 ⅈ, -4.3379 + 6.69241 ⅈ,
 5.36587 + 8.46456 ⅈ}, {-16.644 + 12.1671 ⅈ, -9.25798 - 12.7852 ⅈ,
 -9.04194 - 1.3936 ⅈ, -4.3379 + 6.69241 ⅈ, 5.36587 + 8.46456 ⅈ},
 {-16.6245 + 12.2589 ⅈ, -9.28234 - 12.7486 ⅈ, -9.08343 - 1.4599 ⅈ, -4.31857 + 6.62727 ⅈ,
 5.39294 + 8.46768 ⅈ}, {-16.6245 + 12.2589 ⅈ, -9.28234 - 12.7486 ⅈ,
 -9.08343 - 1.4599 ⅈ, -4.31857 + 6.62727 ⅈ, 5.39294 + 8.46768 ⅈ},
 {-15.2804 + 13.5795 ⅈ, -10.0185 - 12.6962 ⅈ, -9.28282 - 2.17251 ⅈ, -4.6047 + 6.19788 ⅈ,
 5.27045 + 8.2366 ⅈ}, {-15.2804 + 13.5795 ⅈ, -10.0185 - 12.6962 ⅈ,
 -9.28282 - 2.17251 ⅈ, -4.6047 + 6.19788 ⅈ, 5.27045 + 8.2366 ⅈ},
 {-15.2776 + 13.5762 ⅈ, -10.0194 - 12.6991 ⅈ, -9.27799 - 2.16574 ⅈ, -4.61289 + 6.19942 ⅈ,
 5.27197 + 8.23447 ⅈ}, {-15.2776 + 13.5762 ⅈ, -10.0194 - 12.6991 ⅈ,
 -9.27799 - 2.16574 ⅈ, -4.61289 + 6.19942 ⅈ, 5.27197 + 8.23447 ⅈ},
 {-15.2802 + 13.5797 ⅈ, -10.0183 - 12.6963 ⅈ, -9.28322 - 2.17213 ⅈ, -4.60488 + 6.19737 ⅈ,
 5.2706 + 8.23668 ⅈ}, {-15.2802 + 13.5797 ⅈ, -10.0183 - 12.6963 ⅈ,
 -9.28322 - 2.17213 ⅈ, -4.60488 + 6.19737 ⅈ, 5.2706 + 8.23668 ⅈ},
 {-16.7897 + 12.585 ⅈ, -9.32167 - 12.8773 ⅈ, -8.97514 - 1.54043 ⅈ, -4.26161 + 6.54524 ⅈ,
 5.43217 + 8.43278 ⅈ}, {-16.7897 + 12.585 ⅈ, -9.32167 - 12.8773 ⅈ,
 -8.97514 - 1.54043 ⅈ, -4.26161 + 6.54524 ⅈ, 5.43217 + 8.43278 ⅈ},
 {-16.8321 + 12.5109 ⅈ, -9.30991 - 12.9175 ⅈ, -8.92084 - 1.49385 ⅈ, -4.25993 + 6.60726 ⅈ,
 5.40685 + 8.43841 ⅈ}, {-16.8321 + 12.5109 ⅈ, -9.30991 - 12.9175 ⅈ,
 -8.92084 - 1.49385 ⅈ, -4.25993 + 6.60726 ⅈ, 5.40685 + 8.43841 ⅈ},
 {-16.8319 + 12.5069 ⅈ, -9.31163 - 12.9194 ⅈ, -8.91387 - 1.49033 ⅈ, -4.2661 + 6.6122 ⅈ,
 5.40759 + 8.43593 ⅈ}, {-16.8319 + 12.5069 ⅈ, -9.31163 - 12.9194 ⅈ,
 -8.91387 - 1.49033 ⅈ, -4.2661 + 6.6122 ⅈ, 5.40759 + 8.43593 ⅈ}}

```
4 / N[pz[2500, -1]]
```

3.14866