

```

tk[n_, k_, a_] :=
  tk[n, k, a] = Sum[ tk[n / j, k - 1, a], {j, 2, n}] - a Sum[ tk[n / (a j), k - 1, a], {j, 1, n / a}];
tk[n_, 0, a_] := 1
Lina[n_, a_] := Sum[ ((-1)^(k + 1) tk[n, k, a] + 1) / k + (a^k - 1) / k, {k, 1, Log[a, n]}]
Linb[n_, a_] :=
  Sum[ (-1)^(k + 1) / k tk[n, k, a], {k, 1, Log[2, n]}] + Sum[ a^k / k, {k, 1, Log[a, n]}]
Lin2[n_, a_] := If[a >= 2, Linb[n, a], Lina[n, a]]

```

```
FullSimplify[Lin2[100, 2^(1 / 2)]]
```

$$\frac{428}{15}$$

```
FullSimplify[Lin2[100, 2^(1 / 3)]]
```

$$\frac{428}{15}$$

```
FullSimplify[Lin2[100, 2^(1 / 4)]]
```

$$\frac{428}{15}$$

```
Lin2[100, 3.27]
```

```
28.5333
```

```
f[n_, a_] := Sum[ (a^k - 1) / k, {k, 1, Log[a, n]}]
```

```
f[100, 1.00001] + f[1.45, 1.00001]
```

```
28.4309
```

```
N[LogIntegral[100]]
```

```
30.1261
```

```
g[n_, a_] := Sum[ (E^(k Log[a]) - 1) / k, {k, 1, Log[n] / Log[a]}]
```

```
g[100, 1.00001]
```

```
28.0218
```

```
Limit[g[100, s], {s -> 1}]
```

$$\left\{ \text{Limit} \left[-\text{HarmonicNumber} \left[\frac{\text{Log}[100]}{\text{Log}[s]} \right] - 100 s \text{LerchPhi} \left[s, 1, 1 + \frac{\text{Log}[100]}{\text{Log}[s]} \right] - \text{Log}[1 - s], s \rightarrow 1 \right] \right\}$$

$$\text{Limit} \left[-\text{HarmonicNumber} \left[\frac{\text{Log}[100]}{\text{Log}[s]} \right] - \text{Log}[1 - s], s \rightarrow 1 \right]$$

$$-\text{EulerGamma} - i \pi - \text{Log}[\text{Log}[100]]$$

$$\text{Limit} \left[-100 s \text{LerchPhi} \left[s, 1, 1 + \frac{\text{Log}[100]}{\text{Log}[s]} \right], s \rightarrow 1 \right]$$

$$\text{Limit} \left[-100 s \text{LerchPhi} \left[s, 1, 1 + \frac{\text{Log}[100]}{\text{Log}[s]} \right], s \rightarrow 1 \right]$$

$$\text{pp}[s_]:= \left(-100 s \operatorname{LerchPhi}\left[s, 1, 1 + \frac{\operatorname{Log}[100]}{\operatorname{Log}[s]} \right] \right) - \operatorname{EulerGamma} - i \pi - \operatorname{Log}[\operatorname{Log}[100]]$$

$$\text{pp2}[s_]:= \left(-100 s \operatorname{LerchPhi}\left[s, 1, 1 + \frac{\operatorname{Log}[100]}{\operatorname{Log}[s]} \right] \right)$$

$$\text{pp3}[s_]:= \left(-100 \operatorname{LerchPhi}\left[s, 1, 1 + \frac{\operatorname{Log}[100]}{\operatorname{Log}[s]} \right] \right)$$

N[pp[1.00001]]

28.0219 + 3.90545 × 10⁻¹¹ i

N[pp2[1.00001]]

30.1263 + 3.14159 i

N[LogIntegral[100]]

30.1261

N[pp3[1.00001]]

30.1259 + 3.14156 i

f3[n_, s_] := -n Sum[(s^k) / (Log[n] / Log[s] + 1), {k, 0, Infinity}]

$$\operatorname{Limit}\left[-\frac{n}{(1-s)\left(1+\frac{\operatorname{Log}[n]}{\operatorname{Log}[s]}\right)}, \{s \rightarrow 1\}\right]$$

$$\left\{\frac{n}{\operatorname{Log}[n]}\right\}$$

f3[100, 1.00000001]

Sum::div: Sum does not converge. >>

lerch[n_, s_] := n Integrate[E^(- (Log[n] / Log[s]) t) / (1 - (s E^(-t))), {t, 0, Infinity}]

lerch[100, .9]

Integrate::idiv: Integral of $\frac{e^{44.7087 t}}{-0.9 + e^t}$ does not converge on {0, ∞}. >>

$$100 \int_0^{\infty} \frac{e^{43.7087 t}}{1 - 0.9 e^{-t}} dt$$

lch[z_, s_, a_] :=

1 / (2 a^s) + Log[1 / z] ^ (s - 1) / z^a Gamma[1 - s, a Log[1 / z]] + 2 / (a^ (s - 1)) Integrate[
Sin[s ArcTan[t] - t a Log[z]] / ((1 + t^2) ^ (s / 2) (E^ (2 Pi a t) - 1)), {t, 0, Infinity}]

lch[1.00001, 1, Log[100] / Log[1.00001]]

$$(-0.30126 - 0.0314159 i) + 2. \int_0^{\infty} -\frac{\operatorname{Sin}[4.60517 t - \operatorname{ArcTan}[t]]}{(-1 + e^{2.89353 \times 10^6 t}) \sqrt{1 + t^2}} dt$$

$N\left[(-0.30126033010911385 - 0.03141592653589791 i) + \right.$

$$\left. 2. \int_0^{\infty} \frac{\sin[4.605170185988092 t - \text{ArcTan}[t]]}{(-1 + e^{2.8935282324919426 t}) \sqrt{1+t^2}} dt \right]$$

NIntegrate::ncvb:

NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in t near {t} = {8.99367×10⁻⁶}.

NIntegrate obtained -7.08303×10⁻¹³ and 4.4015461258204655⁻¹⁸ for the integral and error estimates. >>

-100 (-0.30126033011053044 - 0.03141592653589791 i)

30.126 + 3.14159 i

lch[r, 1, p]

$$\frac{1}{2p} + r^{-p} \text{Gamma}\left[0, p \text{Log}\left[\frac{1}{r}\right]\right] + 2 \int_0^{\infty} \frac{\sin[\text{ArcTan}[t] - p t \text{Log}[r]]}{(-1 + e^{2p\pi t}) \sqrt{1+t^2}} dt$$

$$\frac{1}{2p} + r^{-p} \text{Gamma}\left[0, p \text{Log}\left[\frac{1}{r}\right]\right] + 2 \int_0^{\infty} \frac{\sin[\text{ArcTan}[t] - p t \text{Log}[r]]}{(-1 + e^{2p\pi t}) \sqrt{1+t^2}} dt /. \{r \rightarrow s, p \rightarrow (\text{Log}[n] / \text{Log}[s])\}$$

$$\frac{\text{Gamma}\left[0, \frac{\text{Log}[n] \text{Log}\left[\frac{1}{s}\right]}{\text{Log}[s]}\right]}{n} + 2 \int_0^{\infty} \frac{\sin[\text{ArcTan}[t] - t \text{Log}[n]]}{\left(-1 + n^{\frac{2\pi t}{\text{Log}[s]}}\right) \sqrt{1+t^2}} dt + \frac{\text{Log}[s]}{2 \text{Log}[n]}$$

lchx[z_, s_, a_] :=

1 / (2 a^s) + Log[1 / z] ^ (s - 1) / z^a Gamma[1 - s, a Log[1 / z]] + 2 / (a^(s - 1)) Integrate[
Sin[s ArcTan[t] - t a Log[z]] / ((1 + t²)^(s / 2) (E^(2 Pi a t) - 1)), {t, 0, Infinity}]

N[lchx[1.00001, 1, Log[100] / Log[1.00001]]]

NIntegrate::ncvb:

NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in t near {t} = {8.99367×10⁻⁶}.

NIntegrate obtained -7.08303×10⁻¹³ and 4.4015461258204655⁻¹⁸ for the integral and error estimates. >>

-0.30126 - 0.0314159 i

lchxa[z_, s_, a_] := 1 / (2 a^s)

lchxb[z_, s_, a_] := Log[1 / z] ^ (s - 1) / z^a Gamma[1 - s, a Log[1 / z]]

lchxc[z_, s_, a_] := 2 / (a^(s - 1)) Integrate[
Sin[s ArcTan[t] - t a Log[z]] / ((1 + t²)^(s / 2) (E^(2 Pi a t) - 1)), {t, 0, Infinity}]

N[lchxa[1.00001, 1, Log[100] / Log[1.00001]] +

lchxb[1.00001, 1, Log[100] / Log[1.00001]] + lchxc[1.00001, 1, Log[100] / Log[1.00001]]]

NIntegrate::ncvb:

NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in t near {t} = {8.99367×10⁻⁶}.

NIntegrate obtained -7.08303×10⁻¹³ and 4.4015461258204655⁻¹⁸ for the integral and error estimates. >>

-0.30126 - 0.0314159 i

lchxa[z_, s_, a_] := 1 / (2 a¹)

lchxb[z_, s_, a_] := Log[1 / z] ^ (1 - 1) / z^a Gamma[1 - 1, a Log[1 / z]]

lchxc[z_, s_, a_] := 2 / (a^(1 - 1)) Integrate[
Sin[1 ArcTan[t] - t a Log[z]] / ((1 + t²)^(1 / 2) (E^(2 Pi a t) - 1)), {t, 0, Infinity}]

```
N[lchxa[1.00001, 1, Log[100] / Log[1.00001]] +
  lchxb[1.00001, 1, Log[100] / Log[1.00001]] + lchxc[1.00001, 1, Log[100] / Log[1.00001]]]
```

NIntegrate::ncvb :

NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in t near {t} = $\{8.99367 \times 10^{-6}\}$.

NIntegrate obtained -7.08303×10^{-13} and $4.4015461258204655 \times 10^{-18}$ for the integral and error estimates. >>

-0.30126 - 0.0314159 i

```
lchxa[z_, s_, a_] := 1 / (2 a)
```

```
lchxb[z_, s_, a_] := Log[1 / z] ^ (1 - 1) / z ^ a Gamma[1 - 1, a Log[1 / z]]
```

```
lchxc[z_, s_, a_] := 2 / (a ^ (1 - 1)) Integrate[
  Sin[1 ArcTan[t] - t a Log[z]] / ((1 + t ^ 2) ^ (1 / 2) (E ^ (2 Pi a t) - 1)), {t, 0, Infinity}]
```

```
lchxa[1.00001, 1, Log[100] / Log[1.00001]]
```

1.08573×10^{-6}

```
lchxb[1.00001, 1, Log[100] / Log[1.00001]]
```

-0.301261 - 0.0314159 i

```
N[lchxc[1.00001, 1, Log[100] / Log[1.00001]]]
```

NIntegrate::ncvb :

NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in t near {t} = $\{8.99367 \times 10^{-6}\}$.

NIntegrate obtained -7.08303×10^{-13} and $4.4015461258204655 \times 10^{-18}$ for the integral and error estimates. >>

-1.41661×10^{-12}

```
lchxa[z_, s_, a_] := 0
```

```
lchxb[z_, s_, a_] := 1 / z ^ a Gamma[0, a Log[1 / z]]
```

```
lchxc[z_, s_, a_] := 2 Integrate[
  Sin[ArcTan[t] - t a Log[z]] / ((1 + t ^ 2) ^ (1 / 2) (E ^ (2 Pi a t) - 1)), {t, 0, Infinity}]
```

```
lchxb[1.00001, 1, Log[100] / Log[1.00001]]
```

-0.301261 - 0.0314159 i

```
N[lchxc[1.00001, 1, Log[100] / Log[1.00001]]]
```

NIntegrate::ncvb :

NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in t near {t} = $\{8.99367 \times 10^{-6}\}$.

NIntegrate obtained -7.08303×10^{-13} and $4.4015461258204655 \times 10^{-18}$ for the integral and error estimates. >>

-1.41661×10^{-12}

```
FullSimplify[Sin[ArcTan[t] - t a Log[z]] / ((1 + t ^ 2) ^ (1 / 2) (E ^ (2 Pi a t) - 1))]
```

$$\frac{(-1 + \text{Coth}[a \pi t]) \sin[\text{ArcTan}[t] - a t \log[z]]}{2 \sqrt{1 + t^2}}$$

2

```
Integrate[Sin[ArcTan[t] - t a Log[z]] / ((1 + t ^ 2) ^ (1 / 2) (E ^ (2 Pi a t) - 1)), {t, 0, Infinity}]
```

$$2 \int_0^\infty \frac{\sin[\text{ArcTan}[t] - a t \log[z]]}{(-1 + e^{2 a \pi t}) \sqrt{1 + t^2}} dt$$

```
2 Integrate[
$$\frac{(-1 + \text{Coth}[a \pi t]) \text{Sin}[\text{ArcTan}[t] - a t \text{Log}[z]]}{2 \sqrt{1 + t^2}}, \{t, 0, \text{Infinity}\}]$$

```

```
$Aborted
```

```
ff[z_, a_, t_] := 
$$\frac{(-1 + \text{Coth}[a \pi t]) \text{Sin}[\text{ArcTan}[t] - a t \text{Log}[z]]}{2 \sqrt{1 + t^2}}$$

```

```
N[ff[s, Log[n] / Log[s], t] /. {s -> 1.0001, n -> 100, t -> 80}]
```

```
0.
```

```
fff[z_, a_, t_] := Sin[ArcTan[t] - t a Log[z]] / ((1 + t^2)^(1/2) (E^(2 Pi a t) - 1))
```

```
N[fff[s, Log[n] / Log[s], t] /. {s -> 3, n -> 10, t -> 10}]
```

```
-2.70307 × 10-59
```

```
ffff[z_, a_, t_] := Sin[ArcTan[t] - t (Log[n] / Log[s]) Log[s]] /  
((1 + t^2)^(1/2) (E^(2 Pi (Log[n] / Log[s]) t) - 1))
```

```
N[ffff[s, Log[n] / Log[s], t] /. {s -> 3, n -> 10, t -> 10}]
```

```
-2.70307 × 10-59
```

```
FullSimplify[Sin[ArcTan[t] - t (Log[n] / Log[s]) Log[s]] /  
((1 + t^2)^(1/2) (E^(2 Pi (Log[n] / Log[s]) t) - 1))]
```

```

$$\frac{\text{Sin}[\text{ArcTan}[t] - t \text{Log}[n]]}{\left(-1 + n^{\frac{2 \pi t}{\text{Log}[s]}}\right) \sqrt{1 + t^2}}$$

```

```
ffffff[z_, a_, t_] :=  
Sin[ArcTan[t] - t (Log[n])] / ((1 + t^2)^(1/2) (E^(2 Pi (Log[n] / Log[s]) t) - 1))
```

```
N[ffffff[s, Log[n] / Log[s], t] /. {s -> 3, n -> 10, t -> 10}]
```

```
-2.70307 × 10-59
```

```
fp[z_, a_, t_] :=  
Sin[ArcTan[t] - t (Log[n])] / ((1 + t^2)^(1/2) (E^(2 Pi (Log[n] / Log[s]) t) - 1))
```

```
lchxb[z_, s_, a_] := 1 / z^a Gamma[0, a Log[1 / z]]
```

```
lchxb[1.00001, 1, Log[100] / Log[1.00001]]
```

```
-0.301261 - 0.0314159 i
```

```
lchxb2[z_, a_] := 1 / z^a Gamma[0, a Log[1 / z]]
```

```
lchxb2[1.00001, Log[100] / Log[1.00001]]
```

```
-0.301261 - 0.0314159 i
```

```
lchxb3[n_, s_, a_] := 1 / s^a Gamma[0, a Log[1 / s]]
```

```
lchxb3[100, 1.00001, Log[100] / Log[1.00001]]
```

```
-0.301261 - 0.0314159 i
```

```
lchxb4[n_, s_] := 1 / s ^ (Log[n] / Log[s]) Gamma[0, (Log[n] / Log[s]) Log[1 / s]]
```

```
lchxb4[100, 1.00001]
```

```
-0.301261 - 0.0314159 i
```

```
lchxb5[n_, s_] := 1 / s ^ (Log[n] / Log[s]) Gamma[0, (Log[n] / Log[s]) Log[1 / s]]
```

```
lchxb5[100, 1.00001]
```

```
-0.301261 - 0.0314159 i
```

```
FullSimplify[1 / s ^ (Log[n] / Log[s]) Gamma[0, (Log[n] / Log[s]) Log[1 / s]]]
```

```
chx1[n_, s_] := 
$$\frac{\text{Gamma}\left[0, \frac{\text{Log}[n] \text{Log}\left[\frac{1}{s}\right]}{\text{Log}[s]}\right]}{n}$$

```

```
chx1[100, 1.00001]
```

```
-0.301261 - 0.0314159 i
```

```
chx2[n_, s_] := 
$$\frac{\text{Gamma}\left[0, \frac{\text{Log}[n] (-\text{Log}[s])}{\text{Log}[s]}\right]}{n}$$

```

```
chx2[100, 1.00001]
```

```
-0.301261 - 0.0314159 i
```

```
chx3[n_, s_] := (1 / n) Gamma[0, 
$$\frac{\text{Log}[n] (-\text{Log}[s])}{\text{Log}[s]}$$
]
```

```
chx3[100, 1.00001]
```

```
-0.301261 - 0.0314159 i
```

```
chx4[n_, s_] := (1 / n) Gamma[0, 
$$-\frac{\text{Log}[n] \text{Log}[s]}{\text{Log}[s]}$$
]
```

```
chx4[100, 1.00001]
```

```
-0.301261 - 0.0314159 i
```

```
chx5[n_, s_] := (1 / n) Gamma[0, -Log[n]]
```

```
chx5[100, 1.00001]
```

```

$$\frac{1}{100} \text{Gamma}[0, -\text{Log}[100]]$$

```