Section 8 $[\zeta_n(s)^z]_n$:

8.1 Scaling Partial Sums of the Hurwitz Zeta Function

Back in section 1, we defined the partial sum of the Hurwitz Zeta function as

$$\boxed{[\zeta(s,1+y)^k]_n = \sum_{j=1} (j+y)^{-s} \cdot [\zeta(s,1+y)^{k-1}]_{n(j+y)^{-1}}}$$

(8.1.1)

We are now going to use a version of the Hurwitz Zeta function that incorporates a scaling factor, here labeled *y*. It is defined as

$$\boxed{[(y^{1-s} \cdot \zeta(s,1+y^{-1}))^k]_n = y \cdot \sum_{j=1} (1+j\cdot y)^{-s} [(y^{1-s} \cdot \zeta(s,1+y^{-1}))^{k-1}]_{n(1+j\cdot y)^{-1}}}$$

(8.1.2)

Perhaps a bit of description will help make sense of this. We can picture, say, $[\zeta(0,2)^2]_n$ as being the total area of 1x1 squares entirely bounded by the lines $x=1, y=1,$ and $x \cdot y = n$. Likewise, we can think of $[\zeta_n(0,2)^3]_n$ as being the total volume of 1x1x1 cubes entirely bounded by $x=1, y=1, z=1,$ and $x \cdot y \cdot z = n$.

The important point here is that $[\zeta(s,2)^k]_n$ can be thought of as sampling at a scale of 1. Our new term, $[(y^{1-s} \cdot \zeta_n(s,1+y^{-1}))^k]_n$, effectively let's us choose the scale that we are sampling at. So, for example, $[(\frac{1}{2}^{1-0} \cdot \zeta(0,1+(\frac{1}{2})^{-1}))^2]_n$ is the total area of $\frac{1}{2} x \frac{1}{2}$ squares bounded by $x=1, y=1,$ and $x \cdot y = n$. Or $[(\frac{1}{3}^{1-0} \cdot \zeta(0,1+(\frac{1}{3})^{-1}))^3]_n$ is the total volume of cubes, with sides measuring $\frac{1}{3} x \frac{1}{3} x \frac{1}{3}$ entirely bounded by $x=1, y=1, z=1,$ and $x \cdot y \cdot z = n$.

Quite obviously, $[(1^{1-s} \cdot \zeta(s,1+1^{-1}))^k]_n = [\zeta(s,2)^k]_n$

In this section, we're going to look at what we can do with this generalization – in particular, we can use it without much fuss to connect the Riemann Prime counting function to the logarithmic integral in an elementary way.

Note that with a bit more algebraic manipulation, (8.1.2) can be rewritten as

$$\boxed{[(y^{1-s} \cdot \zeta(s,1+y^{-1}))^k]_n = y^{1-s} \cdot \sum_{j=1} (j+y^{-1})^{-s} [(y^{1-s} \cdot \zeta(s,1+y^{-1}))^{k-1}]_{n(1+j\cdot y)^{-1}}}$$

(8.1.3)

Some examples of this function written more explicitly include

$$[y^{1-s}\cdot\zeta(s,1+y)]_n = y\sum_{j=1}(1+j\cdot y)^{-s}$$
$$[(y^{1-s}\cdot\zeta(s,1+y))^2]_n = y^2\sum_{j=1}\sum_{k=1}((1+j\cdot y)\cdot(1+k\cdot y))^{-s}$$
$$[(y^{1-s}\cdot\zeta(s,1+y))^3]_n = y^3\sum_{j=1}\sum_{k=1}\sum_{l=1}((1+j\cdot y)\cdot(1+k\cdot y)\cdot(1+l\cdot y))^{-s}$$

(8.1.4)

The limit of (8.1.2) as n approaches infinity, if $\Re(s)>1$, is

$$(y^{1-s}\cdot\zeta_n(s,1+y^{-1}))^k$$

(8.1.5)

A useful extra property is

$$[(y^{1-s}\cdot\zeta(s,1+y^{-1}))^k]_n = y^{k(1-s)}\cdot[\zeta(s,1+y^{-1})^k]_{n\cdot y^{-k}}$$

and if s=1,

$$\lim_{s\to 1}[(y^{1-s}\cdot\zeta(s,1+y^{-1}))^k]_n = [\zeta(1,1+y^{-1})^k]_{n\cdot y^{-k}}$$

8.2 Scaling the Partial Sum of the Hurwitz Zeta Function for k=1

We're going to be using our scaling factor to, essentially, smooth $[\zeta(s,2)^k]_n$ - and then, use that smoothed $[\zeta(s,2)^k]_n$ to smooth $[\zeta(s)^z]_n$ and $[\log\zeta(s)]_n$, since those functions can be expressed in terms of $[\zeta(s,2)^k]_n$.

So let's start with the simplest example Suppose we take (8.1.2) and take k as 1. If y=1, we evidently have

$$[1^{1-s}\cdot\zeta(s,1+1^{-1})]_n = [\zeta(s)-1]_n = \sum_{y=2}^{n} y^{-s}$$

(8.2.1)

```
Dd[x_,1,y_]:=Sum[(j+y)^0,{j,0,Floor[x-y-1]}]
Cc[x_,1,y_]:=y^(-1) Dd[x y,1,y]
{Floor[x]-1,Cc[x,1,1]}
```

Meanwhile, if we take the limit of y as it approaches 0, we have

$$\lim_{y\to 0}[y^{1-s}\cdot\zeta(s,1+y^{-1})]_n = \int_1^n x^{-s}\,dx = \frac{1}{s-1}\cdot(1-n^{1-s})$$

(8.2.2)

unless s=1, in which case we have

$$\lim_{y\to 0}[y^{1-(1)}\cdot\zeta(1,1+y^{-1})]_n = \int_1^n \frac{1}{x}\,dx = \log n$$

```
Dd[x_,1,y_]:=Sum[(j+y)^0,{j,0,Floor[x-y-1]}]
Cc[x_,1,y_]:=y^(-1) Dd[x y,1,y]
Table[ {n/7-1, Limit[ Cc[n/7,1,z], z->Infinity]}, {n,1,20}]//TableForm
```

which means we can relate (8.2.1) to (8.2.2) as

$$\boxed{[\zeta(s)-1]_n=\frac{1}{s-1}(1-n^{1-s})+\int_0^1 \frac{\partial}{\partial y}[y^{1-s}\cdot\zeta(s,1+y^{-1})]_n dy}$$

(8.2.3)

unless s=1, which gives

$$[\zeta(1)-1]_n=\log n+\int_0^1 \frac{\partial}{\partial y}[\zeta(1,1+y^{-1})]_n dy$$

which, because $[\zeta(1)-1]_n=\sum_{j=2}^{n}\frac{1}{j}=H_n-1$, gives a relationship between log n and harmonic numbers.

The limit of (8.2.1) as n approaches infinity, if $\Re(s)>1$, is

$$1^{1-s}\cdot\zeta(s,1+\frac{1}{1})=\sum_{j=2}^{\infty} j^{-s}=\zeta(s)-1$$

(8.2.4)

and the limit of (8.2.2) as n approaches infinity, if $\Re(s)>1$, is

$$\lim_{y\to 0} y^{1-s}\zeta(s,1+y^{-1})=\int_1^{\infty} x^{-s}dx=\frac{1}{s-1}$$

(8.2.5)

```
{Limit[y^(s-1) HurwitzZeta[s,y+1],y->Infinity],1/(s-1)}
```

and so the limit of (8.2.3), if $\Re(s)>1$, is

$$\zeta(s)-1=\frac{1}{s-1}+\int_0^1 \frac{\partial}{\partial y} y^{1-s}\zeta(s,1+y^{-1})dy$$

(8.2.6)

```
Table[{Zeta[s]-1,1/(s-1)-Integrate[D[y^(s-1) Zeta[s,y+1],y],{y,1,Infinity}]},{s,2,6}]
```

Simple.

8.3 Scaling the Partial Sum of the Hurwitz Zeta Function for k as a Positive Integer

Now let's look at the case when, for $[(y^{1-s}\cdot\zeta(s,1+y^{-1}))^k]_n$, k is some positive integer.

Obviously, if $y=1$, we have, for any k,

$$[(1^{1-s}\cdot\zeta(s,1+1^{-1}))^k]_n=[(\zeta(s)-1)^k]_n$$

(8.3.1)

```
D2[ n_, k_ ] := Sum[ D2[ n/j, k-1 ], { j, 2, Floor[ n ] } ]
D2[ n_, 0 ] := UnitStep[n-1]
Dd[x_, k_, y_] := Sum[ Dd[x/(j+y),k-1,y],{j,0,Floor[x-y]}];Dd[x_, 0, y_] := UnitStep[n-1]
Cc[x_, k_, y_] := y^-k Dd[ x y^k, k, y+1]
Grid[Table[ {D2[ n, k ],Cc[n,k,1]}, { n, 1, 50 }, { k, 1, 7 } ]]
```

Now let's see what happens if we take the limit as *y* approaches infinity, for the first few values of *k*.

If k=2, we have

$$\boxed{\lim_{y\to 0}[(y^{1-s}\cdot\zeta(s,1+y^{-1}))^2]_n=\int_1^n\int_1^{\frac{n}{z}} w^{-s}\cdot z^{-s}\,dw\,dz=\frac{1}{(s-1)^2}\cdot\frac{\gamma(2,(s-1)\log n)}{\Gamma(2)}} \tag{8.3.2}$$

unless s=1, in which case

$$\lim_{y\to 0}[(y^{1-(1)}\cdot\zeta(1,1+y^{-1}))^2]_n=\int_1^n\int_1^{\frac{n}{z}} w^{-1}\cdot z^{-1}\,dw\,dz=\frac{\log(n)^2}{2}$$

```
Dd[x_,0,y_]:=UnitStep[n-1]; Dd[x_,1,y_]:=Floor[x]-y+1
Dd[x_,k_,y_]:=Sum[Binomial[k,j] Dd[x/(m^(k-j)),j,m+1],{m,y,x^(1/k)},{j,0,k-1}]
Cc[x_,k_,y_]:=y^-k Dd[x y^k,k,y+1]
Table[{Cc[x,2,3000.],N[x Log[x]-x+1],1-Gamma[2.,-Log[x]]/Gamma[2]},{x,2,40}]//TableForm
```

Here, $\gamma(n)$ is the lower incomplete gamma function.

The limit of (8.3.2) as *n* approaches infinity, if $\Re(s)>1$, is

$$\lim_{y\to 0}(y^{1-s}\cdot\zeta(s,1+y^{-1}))^2=\int_1^\infty\int_1^\infty w^{-s}\cdot z^{-s}\,dz\,dw=\frac{1}{(s-1)^2} \tag{8.3.3}$$

If k=3, we have

$$\boxed{\lim_{y\to 0}[(y^{1-s}\cdot\zeta(s,1+y^{-1}))^3]_n=\int_1^n\int_1^{\frac{n}{u}}\int_1^{\frac{n}{u\cdot z}} w^{-s}z^{-s}u^{-s}\,dw\,dz\,du=\frac{1}{(s-1)^3}\cdot\frac{\gamma(3,(s-1)\log n)}{\Gamma(3)}} \tag{8.3.4}$$

unless s=1, in which case

$$\lim_{y\to 0}[(y^{1-(1)}\cdot\zeta(1,1+y^{-1}))^3]_n=\int_1^n\int_1^{\frac{n}{u}}\int_1^{\frac{n}{u\cdot z}} w^{-1}z^{-1}u^{-1}\,dw\,dz\,du=\frac{\log(n)^3}{3!}$$

```
Dd[x_,0,y_]:=UnitStep[n-1]; Dd[x_,1,y_]:=Floor[x]-y+1
Dd[x_,k_,y_]:=Sum[Binomial[k,j] Dd[x/(m^(k-j)),j,m+1],{m,y,x^(1/k)},{j,0,k-1}]
Cc[x_,k_,y_]:=y^-k Dd[x y^k,k,y+1]
Table[{Cc[x,3,600.],N[x/2 Log[x]^2-x Log[x]+x-1],-(1-Gamma[3.,-Log[x]]/Gamma[3])},{x,2,10}]//TableForm
```

The limit of this as *n* approaches infinity, if $\Re(s)>1$, is

$$\lim_{y\to 0}(y^{1-s}\cdot\zeta(s,1+y^{-1}))^3=\int_1^\infty\int_1^\infty\int_1^\infty w^{-s}\cdot y^{-s}\cdot z^{-s}\,dz\,dy\,dw=\frac{1}{(s-1)^3} \tag{8.3.5}$$

More generally, for any k, we have

$$\boxed{\lim_{y\to 0}[(y^{1-s}\cdot\zeta(s,1+y^{-1}))^k]_n=\frac{1}{(s-1)^k}\cdot\frac{\gamma(k,(s-1)\log n)}{\Gamma(k)}} \tag{8.3.6}$$

unless s=1, in which case

$$\lim_{y\to 0}[(y^{1-(1)}\cdot\zeta(1,1+y^{-1}))^k]_n=\frac{\log(n)^k}{k!}$$

```
Dd[x_,0,y_]:=UnitStep[n-1]; Dd[x_,1,y_]:=Floor[x]-y+1
Dd[x_,k_,y_]:=Sum[Binomial[k,j] Dd[x/(m^(k-j)),j,m+1],{m,y,x^(1/k)},{j,0,k-1}]
Cc[x_,k_,y_]:=y^-k Dd[x y^k,k,y+1]
Table[{Cc[x,k,200.],N[(-1)^k(1-Gamma[k,-Log[x]]/Gamma[k])]},{x,2,7},{k,1,4}]//TableForm
```

where $\gamma(k,-\log n)$ is the lower incomplete gamma function.

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\lim_{y\to 0}(y^{1-s}\cdot\zeta(s,1+y^{-1}))^k=\frac{1}{(s-1)^k}$$

(8.3.7)

```
{Limit[(y^(1-s) HurwitzZeta[s,y^-1+1])^k,y->0],1/(s-1)^k}
```

8.4 Our Smoothed Expression for $[(\zeta(s)-1)^k]_n$

Using (8.3.1) and (8.3.6), we can thus say that

$$\boxed{[(\zeta(s)-1)^k]_n=\frac{1}{(s-1)^k}\cdot\frac{\gamma(k,(s-1)\log n)}{\Gamma(k)}+\int_0^1\frac{\partial}{\partial y}[(y^{1-s}\cdot\zeta(s,1+y^{-1}))^k]_n\,dy}$$

(8.4.1)

unless s=1, in which case

$$[(\zeta(s)-1)^k]_n=\frac{\log(n)^k}{k!}+\int_0^1\frac{\partial}{\partial y}[\zeta(1,1+y^{-1})^k]_n\,dy$$

The limit of this as n approaches 0, if $\Re(s)>1$, is

$$(\zeta(s)-1)^k=\frac{1}{(s-1)^k}+\int_0^1\frac{\partial}{\partial y}(y^{1-s}\zeta(s,1+y^{-1}))^k\,dy$$

(8.4.2)

```
Grid[Table[Chop[N[1/((s-1)^k)-Integrate[D[y^(k(s-1)) Zeta[s,y+1]^k,y],{y,1,Infinity}]]-N[(Zeta[s,2])^k]],{s,2,4},{k,1,4}]]
```

And there we are. (8.4.1) is the identity for $[\zeta(s,2)^k]_n$ that we'll use to approximate the $[\zeta(s)^z]_n$ and the Riemann Prime Counting function.

8.5 Defining $[(1+y^{1-s}\cdot\zeta(s,1+y^{-1}))^z]_n$ and $[\log(1+y^{1-s}\cdot\zeta(s,1+y^{-1}))]_n$

Let's recap what we've just done. ///??????

Now let's define a counterpart to the function we've been working with, $[(y^{1-s}\cdot\zeta(s,1+y^{-1}))^z]_n$ from (8.1.2), that includes the multiplicative identity 1.

It can be defined like this,

$$[(1+y^{1-s}\cdot\zeta(s,1+y^{-1}))^k]_n=[(1+y^{1-s}\cdot\zeta(s,1+y^{-1}))^{k-1}]_n+y^{1-s}\cdot\sum_{j=1}(j+y^{-1})^{-s}[(1+y^{1-s}\cdot\zeta(s,1+y^{-1}))^{k-1}]_{n(1+j\cdot y)^{-1}} \tag{8.5.1}$$

with examples including

$$///?????? \tag{8.5.2}$$

```
D1y1[ x_, s_, k_, y_ ]:= D1y1[x,s,k,y]=D1y1[ x, s,k-1, y] + y Sum[ (1+j y)^-s  D1y1[ x ( 1+ j y)^-1, s,k-1, y],{j,1,(x-1)/y}]
D1y1[x_,s_,0,y_] := UnitStep[ x-1]
```

But we can generalize it to complex convolution exponents by expressing it more tidily as

$$[(1+y^{1-s}\cdot\zeta(s,1+y^{-1}))^z]_n=\sum_{k=0}^{\infty}\binom{z}{k}[(y^{1-s}\cdot\zeta(s,1+y^{-1}))^k]_n \tag{8.5.3}$$

The limit of this as n approaches 0, if $\Re(s)>1$, is

$$(1+y^{1-s}\cdot\zeta(s,1+y^{-1}))^z=\sum_{k=0}^{\infty}\binom{z}{k}(y^{1-s}\cdot\zeta(s,1+y^{-1}))^k \tag{8.5.4}$$

This can also be expressed as

$$[(1+y^{1-s}\cdot\zeta(s,1+y^{-1}))^z]_n=\sum_{k=0}^{\infty}\binom{z}{k}y^{k(1-s)}\cdot[(\zeta(s,1+y^{-1}))^k]_{n\cdot y^k}$$

```
(Zeta[s,y^-1+1]y^(1-s)+1)^z-FullSimplify[Sum[ Binomial[z,k](Zeta[s,y^-1+1]y^(1-s))^k,{k,0,Infinity}]]
```

We'll also want to define a log function.

$$[\log(1+y^{1-s}\cdot\zeta(s,1+y^{-1}))]_n=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}[(y^{1-s}\cdot\zeta(s,1+y^{-1}))^k]_n \tag{8.5.5}$$

with examples including

$$///??????$$

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\log(1+y^{1-s}\cdot\zeta(s,1+y^{-1}))=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}(y^{1-s}\cdot\zeta(s,1+y^{-1}))^k \tag{8.5.6}$$

```
Log[Zeta[s,y^-1+1]y^(1-s)+1]-FullSimplify[Sum[(-1)^(k+1)/k(Zeta[s,y^-1+1]y^(1-s))^k,{k,1,Infinity}]]
```

It can also be expressed as

$$[\log(1+y^{1-s}\cdot\zeta(s,1+y^{-1}))]_n=\lim_{z\to 0}\frac{\partial}{\partial z}[(1+y^{1-s}\cdot\zeta(s,1+y^{-1}))^z]_n \tag{8.5.7}$$

or

$$[\log(1+y^{1-s}\cdot\zeta(s,1+y^{-1}))]_n=\lim_{z\to 0}\frac{1}{z}([(1+y^{1-s}\cdot\zeta(s,1+y^{-1}))^z]_n-1)$$

(8.5.8)

If

$$[(1+y^{1-s}\cdot\zeta(s,1+y^{-1}))^\rho]_n=0$$

(8.5.8)

then

$$[\log(1+y^{1-s}\cdot\zeta(s,1+y^{-1}))]_n=-\sum_\rho \rho^{-1}$$

(8.5.8)

8.6 An Expression for $[\zeta(s)^z]_n$

If we apply (8.4.1) to (D1), $[\zeta(s)^z]_n=\sum_{k=0}^{\infty}\binom{z}{k}[(\zeta(s)-1)^k]_n$,//??????????? we have the following identity for $[\zeta(s)^z]_n$

$$\boxed{[\zeta(s)^z]_n=\sum_{k=0}^{\infty}\binom{z}{k}(\frac{1}{(s-1)^k}\cdot\frac{\gamma(k,(s-1)\log n)}{\Gamma(k)}+\int_0^1\frac{\partial}{\partial y}[(y^{1-s}\cdot\zeta(s,1+y^{-1}))^k]_n\,dy)}$$

(8.6.1)

unless s=1, in which case

$$[\zeta(1)^z]_n=\sum_{k=0}^{\infty}\binom{z}{k}(\frac{(\log n)^k}{k\,!}+\int_0^1\frac{\partial}{\partial y}[\zeta(1,1+y^{-1})^k]_n\,dy)$$

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\zeta(s)^z=\sum_{k=0}^{\infty}\binom{z}{k}(\frac{1}{(s-1)^k}+\int_0^1\frac{\partial}{\partial y}(y^{1-s}\cdot\zeta(s,1+y^{-1}))^k\,dy)$$

(8.6.2)

$$\boxed{[\zeta(s)^z]_n=1+z\cdot\int_{-\log n}^{0}e^{t(s-1)}\cdot{}_1F_1(1-z;2;t)dt+\int_0^1\frac{\partial}{\partial y}[(1+y^{1-s}\cdot\zeta(s,1+y^{-1}))^k]_n\,dy}$$

unless s=1, in which case,

$$[\zeta(1)^z]_n={}_1F_1(-z;1;-\log n)+\int_0^1\frac{\partial}{\partial y}[(1+\zeta(1,1+y^{-1}))^k]_n\,dy$$

where ${}_1F_1(a;b;c)$ is the Confluent Hypergeometric Function of the First Kind.

Now, in particular, it can be shown, if s=0 and with $L_z(n)$ a Laguerre polynomial, that

$$\sum_{k=0}^{\infty}\binom{z}{k}\frac{1}{(0-1)^k}\cdot\frac{\gamma(k,-\log n)}{\Gamma(k)}=L_{-z}(\log n)$$

(8.6.3)

```
TestSum[n_,z_,t_]:=1+Sum[N[Binomial[z,k] (-1)^k (Gamma[k,0,-Log[n]]/Gamma[k])],{k,1,t}]
Grid[Table[Chop[Re[TestSum[n,k,80]]-N[LaguerreL[-k,Log[n]]]],{n,10,100,10},{k,-5,5}]]
```

So we can rewrite this, swapping the sum and integral for good measure, as

$$[\zeta(0)^z]_n = L_{-z}(\log n) + \int_0^1 \frac{\partial}{\partial y} \sum_{k=0}^{\infty} \binom{z}{k} [(y^{1-s} \cdot \zeta(0, 1+y^{-1}))^k]_n \, dy \tag{8.6.4}$$

and finally, taking advantage of (8.5.3),

$$\boxed{[\zeta(0)^z]_n = L_{-z}(\log n) + \int_0^1 \frac{\partial}{\partial y} [(1+y^{1-s} \cdot \zeta(0, 1+y^{-1}))^z]_n \, dy} \tag{8.6.5}$$

This is similar in spirit to the process that could lead one to determine that

$$\zeta(s)^z = \left(\frac{s}{s-1}\right)^z + \int_0^1 \frac{\partial}{\partial y} (1+y^{1-s} \cdot \zeta(s, 1+y^{-1}))^z \, dy \tag{8.6.6}$$

8.7 An Expression for $[\log \zeta(s)]_n$

In like fashion, if we apply (7.29) to (P3), $\Pi(n) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} [(\zeta(0)-1)^k]_n$, we have this identity for the Riemann Prime counting function

$$\boxed{[\log \zeta(s)]_n = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{1}{(s-1)^k} \frac{\gamma(k, (s-1)\log n)}{\Gamma(k)} + \int_0^1 \frac{\partial}{\partial y} [(y^{1-s} \cdot \zeta(s, 1+y^{-1}))^k]_n \, dy \right)} \tag{8.7.1}$$

unless s=1, in which case

$$[\log \zeta(1)]_n = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{(\log n)^k}{k!} + \int_0^1 \frac{\partial}{\partial y} [(\zeta(1, 1+y^{-1}))^k]_n \, dy \right)$$

The limit of this as n approaches infinity, if $\Re(s) > 1$, is

$$\log \zeta(s) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{1}{(s-1)^k} + \int_0^1 \frac{\partial}{\partial y} (y^{1-s} \cdot \zeta(s, 1+y^{-1}))^k \, dy \right) \tag{8.7.2}$$

$$\boxed{\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \cdot \frac{1}{(s-1)^k} \cdot \frac{\gamma(k, (s-1)\log n)}{\Gamma(k)} = -\Gamma(0, (s-1)\log n) + \Gamma(0, s \log n) + \log\left(\frac{s}{s-1}\right)} \tag{8.7.3}$$

The limit of this as n approaches infinity, if $\Re(s) > 1$, is

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \frac{1}{(s-1)^k} = \log\left(\frac{s}{s-1}\right) \tag{7.31}$$

$$\lim_{y \to \infty} [\log(1+y^{1-s} \cdot \zeta(s, 1+y^{-1}))]_n = (\Gamma(0, s \log n) - \Gamma(0, (s-1)\log n)) + \log\left(\frac{s}{s-1}\right)$$

$$[\log\zeta(s)]_n=(\Gamma(0,s\log n)-\Gamma(0,(s-1)\log n))+\log(\frac{s}{s-1})+\int_0^1 \frac{\partial}{\partial y}[\log(1+y^{1-s}\cdot\zeta(s,1+y^{-1}))]_n\,dy$$

(8.7.4)

unless s=1, in which case

$$[\log\zeta(1)]_n=\gamma(0,\log n)+\log\log n+\gamma+\int_0^1 \frac{\partial}{\partial y}[\log(1+\zeta(1,1+y^{-1}))]_n\,dy$$

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\log\zeta(s)=\log\frac{s}{s-1}+\int_0^1 \frac{\partial}{\partial y}(\log(1+y^{1-s}\cdot\zeta(s,1+y^{-1})))\,dy$$

(8.7.5)

8.8 An Expression for $\Pi(n)$

In particular, if s=0, we have, for the Riemann Prime counting function,

$$\Pi(n)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}(\frac{1}{(0-1)^k}\frac{\gamma(k,-\log n)}{\Gamma(k)}+\int_0^1 \frac{\partial}{\partial y}[(y^{1-s}\cdot\zeta(0,1+y^{-1}))^k]_n\,dy)$$

(8.8.1)

Finally, if we take advantage of the following identity for the logarithmic integral,

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\cdot\frac{1}{(0-1)^k}\cdot\frac{\gamma(k,-\log n)}{\Gamma(k)}=\lim_{z\to 0}\frac{L_{-z}(\log n)-1}{z}=li(n)-\log\log n-\gamma$$

(8.8.2)

```
Table[{N[Sum[(-1)^(k+1)/k ((-1)^k(1-Gamma[k,-Log[n]]/Gamma[k])),{k,1,30}]],N[ Limit[ (LaguerreL[-z,Log[n]]-1)/z,z->0]],N[LogIntegral[n]-Log[Log[n]]-EulerGamma]},{n,100,600,100}]//TableForm
```

then, after swapping the sum and integral, we are at last left with an expression connecting Riemann's Prime Counting function, $\Pi(n)$, with the logarithmic integral, $li(n)$.

$$\boxed{\Pi(n)=li(n)-\log\log n-\gamma+\int_0^1 \frac{\partial}{\partial y}[\log(1+y\cdot\zeta(0,1+y^{-1}))]_n\,dy}$$

(8.8.3)

8.9 Recapping

for our generalized Divisor Summatory function $[\zeta(0)^z]_n$, more generally

$$[\zeta(s)^z]_n= 1+\binom{z}{1}\sum_{j=2}^{\lfloor n\rfloor} j^{-s}+\binom{z}{2}\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor \frac{n}{j}\rfloor}(j\cdot k)^{-s}+\binom{z}{3}\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor \frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor \frac{n}{j\cdot k}\rfloor}(j\cdot k\cdot l)^{-s}+\ldots$$

$$1+z\int_{-\log n}^{0} e^{t(s-1)}{}_1F_1(1-z;2;t)\,dt=1+\binom{z}{1}\int_1^n x^{-s}dx+\binom{z}{2}\int_1^n\int_1^{\frac{n}{x}}(x\cdot y)^{-s}dy\,dx+\binom{z}{3}\int_1^n\int_1^{\frac{n}{x}}\int_1^{\frac{n}{xy}}(x\cdot y\cdot z)^{-s}dz\,dy\,dx+\ldots$$

(8.9.1)

w

$$[\zeta(0)^z]_n = \binom{z}{0}1 + \binom{z}{1}\sum_{j=2}^{\lfloor n\rfloor} 1 + \binom{z}{2}\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor \frac{n}{j}\rfloor} 1 + \binom{z}{3}\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor \frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor \frac{n}{j\cdot k}\rfloor} 1 + \ldots$$

$$L_{-z}(\log n) = \binom{z}{0}1 + \binom{z}{1}\int_1^n dx + \binom{z}{2}\int_1^n\int_1^{\frac{n}{x}} dy\,dx + \binom{z}{3}\int_1^n\int_1^{\frac{n}{x}}\int_1^{\frac{n}{xy}} dz\,dy\,dx + \ldots$$

(8.9.2)

where $L_z(n)$ is the Laguerre polynomials.

As another way of stating the general idea here, using $[((\zeta(0)-1)\cdot y)^k]_n$ from (7.24), looking at the difference between $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}[(1^{s-1}\cdot\zeta(0,1+1))^k]_n$ and $\lim_{y\to\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}[(y^{s-1}\cdot\zeta(0,1+y))^k]_n$ amounts to comparing

$$[\log\zeta(s)]_n = \sum_{j=2}^{\lfloor n\rfloor} j^{-s} - \frac{1}{2}\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor \frac{n}{j}\rfloor}(j\cdot k)^{-s} + \frac{1}{3}\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor \frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor \frac{n}{j\cdot k}\rfloor}(j\cdot k\cdot l)^{-s} - \frac{1}{4}\ldots$$

$$-\Gamma(0,(s-1)\log n) + \Gamma(0, s\log n) + \log\left(\frac{s}{s-1}\right) = \int_1^n x^{-s}\,dx - \frac{1}{2}\int_1^n\int_1^{\frac{n}{x}}(x\cdot y)^{-s}\,dy\,dx + \frac{1}{3}\int_1^n\int_1^{\frac{n}{x}}\int_1^{\frac{n}{xy}}(x\cdot y\cdot z)^{-s}\,dz\,dy\,dx - \frac{1}{4}\ldots$$

(8.9.3)

and, in particular, for Riemann's Prime counting function, which is $[\log\zeta(0)]_n$

$$\Pi(n) = \sum_{j=2}^{\lfloor n\rfloor} 1 - \frac{1}{2}\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor \frac{n}{j}\rfloor} 1 + \frac{1}{3}\sum_{j=2}^{\lfloor n\rfloor}\sum_{k=2}^{\lfloor \frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor \frac{n}{j\cdot k}\rfloor} 1 - \frac{1}{4}\ldots$$

$$li(n) - \log\log n - \gamma = \int_1^n dx - \frac{1}{2}\int_1^n\int_1^{\frac{n}{x}} dy\,dx + \frac{1}{3}\int_1^n\int_1^{\frac{n}{x}}\int_1^{\frac{n}{xy}} dz\,dy\,dx - \frac{1}{4}\ldots$$

(8.9.4)

Some of the identities glossed over in this section are covered in more detail in
http://www.icecreambreakfast.com/primecount/ApproximingThePrimeCountingFunctionWithLinniksIdentity_NathanMcKenzie.pdf