```
Clear[f, p]
bin[z_, k_] := bin[z, k] = Product[z - j, {j, 0, k - 1}] / k!
f[n_, y_, z_] := f[n, y, z] = If[n < y, 1, If[n < y^2, 1 + z (n - y + 1),
    Sum[bin[z, k] f[Floor[n / y^k], y + 1, z - k], {k, 0, Log[y, n]}]]]
p[n_, y_] := p[n, y] = If[n < y, 0, If[n < y^2, n - y + 1,
    p[n, y + 1] + Sum[(-1)^(k + 1) / k f[Floor[n / y^k], y + 1, -k], {k, 1, Log[y, n]}]]]
f2[n_, z_] := f[n, 2, z]
f2d[n_, y_, z_] := f[n, y, z] - f[n - 1, y, z]
```

```
Expand@f[100, 2, z]
```

$$1 + \frac{428\,z}{15} + \frac{16\,289\,z^2}{360} + \frac{331\,z^3}{16} + \frac{611\,z^4}{144} + \frac{67\,z^5}{240} + \frac{7\,z^6}{720}$$

```
Clear[dd]
dd[n_, y_, k_] := dd[n, y, k] = Sum[ dd[Floor[n / j], y, k - 1], {j, y, n}]
dd[n_, y_, 0] := UnitStep[n - 1]
dz[n_, y_, z_] := Sum[bin[z, k] dd[n, y, k], {k, 0, Log[y, n]}]
ddz[n_, y_, z_] := dz[n, y, z] - dz[n - 1, y, z]
de[n_, k_, y_, z_] :=
 bin[z, k] ddz[n, y - 1, z - k] - If[Mod[n, y - 1] == 0, de[n / (y - 1), k + 1, y, z], 0]
```

```
Expand@ddz[100, 3, z]
```

$$-\frac{z}{2} + z^2 + \frac{z^3}{2}$$

```
Expand@de[100, 0, 3, z]
```

$$-\frac{z}{2} + z^2 + \frac{z^3}{2}$$

```
Expand@f2d[100, 2, z]
```

$$\frac{z^2}{4} + \frac{z^3}{2} + \frac{z^4}{4}$$

```
Table[{y, f[40, y, z]}, {y, 2, 42}] // TableForm
```

2	$1+35\,z+(1+3\,(-1+z))\,z+(1+6\,(-1+z))\,z+(1+10\,(-1+z))\,z+\left(1+16\,(-1+z)+(1+3\,(-2+z)\right.$
3	$1+35\,z+(1+3\,(-1+z))\,z+(1+6\,(-1+z))\,z+(1+10\,(-1+z))\,z+\frac{3}{2}\,(-1+z)\,z+\frac{1}{6}\,(-2+z)\,(-1+z$
4	$1+35\,z+(1+3\,(-1+z))\,z+(1+6\,(-1+z))\,z+\frac{3}{2}\,(-1+z)\,z$
5	$1+35\,z+(1+3\,(-1+z))\,z+(-1+z)\,z$
6	$1+35\,z+\frac{1}{2}\,(-1+z)\,z$
7	$1+34\,z$
8	$1+33\,z$
9	$1+32\,z$
10	$1+31\,z$
11	$1+30\,z$
12	$1+29\,z$
13	$1+28\,z$
14	$1+27\,z$
15	$1+26\,z$
16	$1+25\,z$
17	$1+24\,z$
18	$1+23\,z$
19	$1+22\,z$
20	$1+21\,z$
21	$1+20\,z$
22	$1+19\,z$
23	$1+18\,z$
24	$1+17\,z$
25	$1+16\,z$
26	$1+15\,z$
27	$1+14\,z$
28	$1+13\,z$
29	$1+12\,z$
30	$1+11\,z$
31	$1+10\,z$
32	$1+9\,z$
33	$1+8\,z$
34	$1+7\,z$
35	$1+6\,z$
36	$1+5\,z$
37	$1+4\,z$
38	$1+3\,z$
39	$1+2\,z$
40	$1+z$
41	1
42	1

```
Sum[MoebiusMu[j], {j, 1, 20}]
```

```
-3
```

```
f[20, 3, -1]
```

```
-7
```

```
13 + 3 + (-1) (-4) (-5) / 2
```

```
6
```

```
Expand[z (1 + 7 (z - 1)) + z (z - 1) + z (z - 1) (z - 2) / 2]
```

$$-6\,z+\frac{13\,z^2}{2}+\frac{z^3}{2}$$

```
Expand@(z f[10, 3, z - 1])
```

$$-6\,z+\frac{13\,z^2}{2}+\frac{z^3}{2}$$

```
Expand[(1 + 16 z) + z (1 + (z - 1)) + z (z - 1) / 2 + z (1 + 3 (z - 1)) + z (z - 1) / 2]
```

$$1+13\,z+5\,z^2$$

```
Expand[f[20, 3, z]]
```

$$1+13\,z+5\,z^2$$

```
p[100 000, 2]
```

$$\frac{991\,892\,879}{102\,960}$$

```
Sum[ PrimePi[100 000^(1 / k)] / k, {k, 1, Log2@100 000}]
```

$$\frac{991\,892\,879}{102\,960}$$

```
$RecursionLimit = 10 000
```

$$10\,000$$

```
pp[n_, y_] := If[n < y, 0, If[n < y^2, ad[n - y + 1],
    pp[n, y + 1] + Sum[(-1)^(k + 1) / k fo[Floor[n / y^k], y + 1, -k], {k, 1, Log[y, n]}]]]
```

```
pp[1000, 2]
```

$$\text{ad}[969] + \frac{1}{9}\,\text{fo}[1, 3, -9] - \frac{1}{6}\,\text{fo}[1, 4, -6] - \frac{1}{4}\,\text{fo}[1, 6, -4] + \frac{1}{3}\,\text{fo}[1, 9, -3] + \frac{1}{3}\,\text{fo}[1, 10, -3] + \frac{1}{3}\,\text{fo}[1, 11, -3] - \frac{1}{2}\,\text{fo}[1, 24, -2] - \frac{1}{2}\,\text{fo}[1, 25, -2] - \frac{1}{2}\,\text{fo}[1, 26, -2] - \frac{1}{2}\,\text{fo}[1, 27, -2] - \frac{1}{2}\,\text{fo}[1, 28, -2] - \frac{1}{2}\,\text{fo}[1, 29, -2] - \frac{1}{2}\,\text{fo}[1, 30, -2] - \frac{1}{2}\,\text{fo}[1, 31, -2] - \frac{1}{2}\,\text{fo}[1, 32, -2] + \frac{1}{3}\,\text{fo}[2, 8, -3] - \frac{1}{2}\,\text{fo}[2, 20, -2] - \frac{1}{2}\,\text{fo}[2, 21, -2] - \frac{1}{2}\,\text{fo}[2, 22, -2] - \frac{1}{2}\,\text{fo}[2, 23, -2] - \frac{1}{8}\,\text{fo}[3, 3, -8] - \frac{1}{4}\,\text{fo}[3, 5, -4] - \frac{1}{2}\,\text{fo}[3, 17, -2] - \frac{1}{2}\,\text{fo}[3, 18, -2] - \frac{1}{2}\,\text{fo}[3, 19, -2] + \frac{1}{5}\,\text{fo}[4, 4, -5] + \frac{1}{3}\,\text{fo}[4, 7, -3] - \frac{1}{2}\,\text{fo}[4, 16, -2] - \frac{1}{2}\,\text{fo}[5, 14, -2] - \frac{1}{2}\,\text{fo}[5, 15, -2] - \frac{1}{2}\,\text{fo}[6, 13, -2] + \frac{1}{7}\,\text{fo}[7, 3, -7] + \frac{1}{3}\,\text{fo}[8, 6, -3] - \frac{1}{2}\,\text{fo}[8, 12, -2] - \frac{1}{2}\,\text{fo}[10, 11, -2] - \frac{1}{4}\,\text{fo}[12, 4, -4] - \frac{1}{2}\,\text{fo}[12, 10, -2] - \frac{1}{6}\,\text{fo}[15, 3, -6] + \frac{1}{3}\,\text{fo}[15, 5, -3] - \frac{1}{2}\,\text{fo}[15, 9, -2] - \frac{1}{2}\,\text{fo}[20, 8, -2] - \frac{1}{2}\,\text{fo}[27, 7, -2] + \frac{1}{5}\,\text{fo}[31, 3, -5] + \text{fo}[32, 32, -1] + \text{fo}[33, 31, -1] + \text{fo}[34, 30, -1] + \text{fo}[35, 29, -1] + \frac{1}{3}\,\text{fo}[37, 4, -3] + \text{fo}[37, 28, -1] + \text{fo}[38, 27, -1] - \frac{1}{2}\,\text{fo}[40, 6, -2] + \text{fo}[40, 26, -1] + \text{fo}[41, 25, -1] + \text{fo}[43, 24, -1] + \text{fo}[45, 23, -1] + \text{fo}[47, 22, -1] + \text{fo}[50, 21, -1] + \text{fo}[52, 20, -1] + \text{fo}[55, 19, -1] + \text{fo}[58, 18, -1] - \frac{1}{4}\,\text{fo}[62, 3, -4] - \frac{1}{2}\,\text{fo}[62, 5, -2] + \text{fo}[62, 17, -1] + \text{fo}[66, 16, -1] + \text{fo}[71, 15, -1] + \text{fo}[76, 14, -1] + \text{fo}[83, 13, -1] + \text{fo}[90, 12, -1] + \text{fo}[100, 11, -1] - \frac{1}{2}\,\text{fo}[111, 4, -2] + \text{fo}[111, 10, -1] + \frac{1}{3}\,\text{fo}[125, 3, -3] + \text{fo}[125, 9, -1] + \text{fo}[142, 8, -1] + \text{fo}[166, 7, -1] + \text{fo}[200, 6, -1] - \frac{1}{2}\,\text{fo}[250, 3, -2] + \text{fo}[250, 5, -1] + \text{fo}[333, 4, -1] + \text{fo}[500, 3, -1]$$

```
1000^.5
```

```
31.6228
```

```
(-1 / 2) f[25, 6, -2] + (-2) (-1 / 2) f[5, 6, -3]
```

$$\frac{41}{2}$$

```
f[25, 6, -2]
```

```
-39
```

```
fa[n_, y_, z_] := If[n < y, fl[1], If[n < y^2, fb[1 + z (n - y + 1)],
   Sum[bin[z, k] faa[Floor[n / y^k], y + 1, z - k], {k, 0, Log[y, n]}]]]
```

```
fa[25, 5, -2]
```

```
3 faa[1, 6, -4] - 2 faa[5, 6, -3] + faa[25, 6, -2]
```

```
(-1 / 2) bin[-2, 1]
```

```
1
```

```
Expand@de[1200, 0, 4, z]
```

$$3\,z - 4\,z^2 - \frac{z^3}{2} + \frac{5\,z^4}{2}$$

```
Expand@ddz[1200, 4, z]
```

$$3\,z - 4\,z^2 - \frac{z^3}{2} + \frac{5\,z^4}{2}$$

```
Expand@dz[100, 3, z]
```

$$1 + \frac{341\,z}{12} + \frac{1391\,z^2}{24} + \frac{139\,z^3}{12} + \frac{z^4}{24}$$

```
Expand@Sum[de[j, 0, 3, z], {j, 1, 100}]
```

$$1 + \frac{341\,z}{12} + \frac{1391\,z^2}{24} + \frac{139\,z^3}{12} + \frac{z^4}{24}$$

```
8^2
```

```
64
```

```
1 + bin[z, 1] Sum[1, {a, y, n}]
```

```
1 + (1 + n - y) z
```

```
pp[n_, y_, z_] :=
 1 + bin[z, 1] Sum[1, {a, y, n}] + bin[z, 2] Sum[1, {a, y, n}, {b, y, Floor[n / a]}]
ppa[n_, y_, z_] := bin[z, 2] Sum[1, {a, y, n}, {b, y, Floor[n / a]}]
pr[n_, y_, z_] :=
 1 + bin[z, 1] Sum[1, {a, y, n}] + If[n > y^2, bin[z, 2] Sum[Floor[n / a] - y + 1, {a, y, n}], 0]
```

```
Expand@pp[100, 5, z]
```

$$1 + 55\,z + 41\,z^2$$

```
Expand@f[100, 5, z]
```

$$1 + 55\,z + 41\,z^2$$

```
Expand@dz[100, 4, z]
```

$$1 + \frac{227\,z}{6} + \frac{115\,z^2}{2} + \frac{5\,z^3}{3}$$

```
Expand@f[100, 4, z]
```

$$1 + \frac{227\,z}{6} + \frac{115\,z^2}{2} + \frac{5\,z^3}{3}$$

```
Table[{2 (ppa[100, k, z] - ppa[100, k + 1, z]) / z / (z - 1), 1 + 2 (Floor[100 / k] - k)}, {k, 2, 6}] //
 TableForm
```

```
97    97
61    61
43    43
31    31
21    21
```

```
Expand[pp[100, 5, z] - pp[100, 6, z]]
```

$$-\frac{29\,z}{2}+\frac{31\,z^2}{2}$$

```
Expand[z + z (z - 1) / 2 (1 + 2 (Floor[100 / 5] - 5))]
```

$$-\frac{29\,z}{2}+\frac{31\,z^2}{2}$$

```
fq[n_, y_, z_, c_] :=
 fq[n, y, z, c] = If[n < y, 1, If[n < y^2, 1 + z (n - y + 1), If[n < c, ff[n, y, z],
     Sum[bin[z, k] fq[Floor[n / y^k], y + 1, z - k, c], {k, 0, Log[y, n]}]]]]
pq[n_, y_, c_] := pq[n, y, c] = If[n < y, 0, If[n < y^2, n - y + 1,
    pq[n, y + 1, c] + Sum[(-1)^(k + 1) / k fq[Floor[n / y^k], y + 1, -k, c], {k, 1, Log[y, n]}]]]
```

```
pq[10 000, 2, 100]
```

$$-\frac{614\,990\,093}{120\,120}-\frac{1}{10}\,\text{ff}[9,3,-10]+\frac{1}{9}\,\text{ff}[19,3,-9]-\text{ff}[20,4,-6]+4\,\text{ff}[26,5,-5]+\frac{1}{4}\,(-525+4\,\text{ff}[30,5,-5])+6\,\text{ff}[34,5,-5]+4\,\text{ff}[37,6,-5]-\frac{1}{8}\,\text{ff}[39,3,-8]-\frac{1}{4}\,\text{ff}[39,5,-4]-\text{ff}[40,6,-4]+\frac{1}{5}\,\text{ff}[41,4,-5]-3\,\text{ff}[41,6,-4]+4\,\text{ff}[46,5,-5]-3\,\text{ff}[50,6,-4]+\frac{1}{6}\,(-2974-21\,\text{ff}[17,4,-8]+6\,\text{ff}[39,5,-7]+6\,\text{ff}[52,4,-7])-\text{ff}[52,5,-4]-3\,\text{ff}[52,7,-4]+2\,(1302-3\,\text{ff}[55,7,-4])+\text{ff}[55,7,-3]+\text{ff}[61,4,-5]-3\,\text{ff}[62,6,-4]-3\,\text{ff}[66,6,-4]+\frac{1}{2}\,(-774+2\,\text{ff}[66,7,-3])+\text{ff}[68,8,-3]+\frac{1}{2}\,(-1871-3\,\text{ff}[44,6,-4]+2\,(1163-3\,\text{ff}[55,6,-4])-3\,\text{ff}[69,5,-4])+\text{ff}[69,7,-3]+2\,\text{ff}[71,8,-3]+\frac{1}{7}\,\text{ff}[78,3,-7]+\frac{1}{5}\,(3465-5\,(836-6\,\text{ff}[26,5,-7])+15\,\text{ff}[34,4,-7]-5\,\text{ff}[52,7,-6]-5\,\text{ff}[62,6,-6]-5\,\text{ff}[78,5,-6])-\text{ff}[78,5,-4]-3\,\text{ff}[79,8,-4]+2\,\text{ff}[79,8,-3]+\frac{1}{3}\,\text{ff}[80,6,-3]+\frac{1}{3}\,(64+6\,(1061-5\,\text{ff}[34,5,-6])-10\,\text{ff}[46,4,-6]+6\,\text{ff}[50,6,-5]-3\,(1787-4\,\text{ff}[52,7,-5]-4\,\text{ff}[62,6,-5])+6\,\text{ff}[78,5,-5]-3\,(212+10\,\text{ff}[26,5,-6]-4\,\text{ff}[69,7,-5]-4\,\text{ff}[83,6,-5]))+2\,(1711-3\,\text{ff}[69,7,-4]-3\,\text{ff}[83,6,-4])+2\,\text{ff}[83,7,-3]+2\,\text{ff}[83,9,-3]+\frac{1}{4}\,(5684+20\,\text{ff}[23,4,-7]-10\,\text{ff}[39,5,-6]+4\,(1031-5\,\text{ff}[41,6,-6]-5\,\text{ff}[52,5,-6])-10\,\text{ff}[69,4,-6]+4\,\text{ff}[89,8,-5])+\frac{1}{2}\,(-1093+2\,\text{ff}[89,8,-3])+2\,\text{ff}[89,9,-3]+\frac{1}{2}\,(-7174-5\,\text{ff}[30,4,-6]+4\,\text{ff}[39,5,-5]+2\,(2656-3\,(1229-4\,\text{ff}[41,6,-5])+6\,\text{ff}[52,5,-5])-3\,(1378-4\,\text{ff}[55,6,-5]-4\,\text{ff}[69,5,-5])-3\,\text{ff}[69,7,-4]+2\,(2108-3\,\text{ff}[71,8,-4]-3\,\text{ff}[83,7,-4])+2\,(1908-3\,\text{ff}[89,8,-4])+4\,\text{ff}[92,4,-5])+\frac{1}{3}\,(852-3\,\text{ff}[61,7,-4]-3\,\text{ff}[74,6,-4]-3\,\text{ff}[92,5,-4])-3\,\text{ff}[92,7,-4]+\text{ff}[92,7,-3]+2\,\text{ff}[95,8,-3]$$

```
bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
FI[n_] := FactorInteger[n]; FI[1] := {}
odz[n_, z_] := Product[(-1)^p[[2]] Binomial[-z, p[[2]]], {p, FI[n]}]
```

```
Table[{n, FullSimplify@ddz[n, 3, z]}, {n, 1, 20}] // TableForm
```

1	1
2	0
3	z
4	z
5	z
6	z
7	z
8	z
9	$\frac{1}{2} z (1 + z)$
10	z
11	z
12	z^2
13	z
14	z
15	z^2
16	$\frac{1}{2} z (1 + z)$
17	z
18	z^2
19	z
20	z^2