$$[\boldsymbol{\infty}^z]_n = \sum_{k=0}^{n} [x^k] \left(\frac{1}{1-x}\right)^z$$

$$[\boldsymbol{\infty}^z_{\mathbf{1/k}}]_n = \sum_{k=0}^{n} [x^k] \left(\frac{1}{1-x^k}\right)^z$$

$$[\mathbf{0}^z]_n = \sum_{k=0}^{n} [x^k] (1-x)^z$$

$$[\mathbf{0}^z_{\mathbf{1/k}}]_n = \sum_{k=0}^{n} [x^k] (1-x^k)^z$$

$$[\boldsymbol{m}^z]_n = \sum_{k=0}^{n} [x^k] \left(\frac{1-x^m}{1-x}\right)^z$$

$$[\boldsymbol{m}^z_{\mathbf{1/k}}]_n = \sum_{k=0}^{n} [x^k] \left(\frac{1-x^{k\cdot m}}{1-x^k}\right)^z$$

$$[\mathbf{2}^z]_n = \sum_{k=0}^{n} [x^k] \left(\frac{1-x^2}{1-x}\right)^z$$

$$[\mathbf{2}^z]_n = \sum_{k=0}^{n} [x^k] (1+x)^z$$

$$[\mathbf{2}^z_{\mathbf{1/2}}]_n = \sum_{k=0}^{n} [x^k] \left(\frac{1-x^4}{1-x^2}\right)^z$$

$$[\mathbf{2}^z_{\mathbf{1/2}}]_n = \sum_{k=0}^{n} [x^k] (1+x^2)^z$$

$$[\mathbf{3}^z]_n = \sum_{k=0}^{n} [x^k] (1+x+x^2)^z$$

$$[\mathbf{3}^z_{\mathbf{1/2}}]_n = \sum_{k=0}^{n} [x^k] (1+x^2+x^4)^z$$

$$[\boldsymbol{\infty}^{z}]_n = [\prod_{k=0} \mathbf{2}_{\mathbf{2}^{k}}{}^{z}]_n$$

$$[(1-x)^{-1}]_n = [\prod_{k=0} (1 + x^{2^{k}})]_n$$

$$[(1-x)^{-1}]_n = [\prod_{k=0} (1 + x^{3^{k}} + x^{2\cdot 3^{k}})]_n$$

$$\nabla[\infty^z]_n = \frac{z^{(n)}}{n!}$$

$$[\infty^z]_n = \frac{(z+1)^{(n)}}{n!}$$

$$\sum_{k=0}^{n} \frac{z^{(k)}}{k!} = \frac{(z+1)^{(n)}}{n!}$$

$$\sum_{k=0}^{n} [\infty^z]_k = [\infty^{z+1}]_n$$

$$\sum_{k=0}^{n} \nabla[\infty^z]_k = \nabla[\infty^{z+1}]_n = [\infty^z]_n$$

$$\nabla[\infty^z]_n = [\infty^{z-1}]_n$$

$$\nabla[\log\infty]_k = \frac{1}{k}$$

$$[\log\infty]_n = H_n$$

$$\nabla[\infty^{a+b}]_n = \sum_{j+k=n} \nabla[\infty^a]_j \cdot \nabla[\infty^b]_k$$

$$\nabla[\infty^{a+b+1}]_n = \sum_{j+k\leq n} \nabla[\infty^a]_j \cdot \nabla[\infty^b]_k$$

$$\ldots$$

$$[\infty^{a+b}]_n = \sum_{j+k\leq n} \nabla[\infty^a]_j \cdot \nabla[\infty^b]_k$$

$$[\infty^{a+b+1}]_n = \sum_{j+k=n} [\infty^a]_j \cdot [\infty^b]_k$$

$$\nabla[2^z]_n = (-1)^n \cdot \frac{(-z)^{(n)}}{n!} = (-1)^n \cdot \nabla[\boldsymbol{\infty}^{-z}]_n = \binom{z}{n}$$

$$[\mathbf{2}^z]_k = \sum_{k=0}^{n} \nabla[\mathbf{2}^z]_k = \sum_{k=0}^{n} (-1)^k \nabla[\boldsymbol{\infty}^{-z}]_k = \nabla[\boldsymbol{\infty}^{-z+\mathbf{1}}]_n - 2\sum_{k=0}^{\frac{n}{2}} \nabla[\boldsymbol{\infty}^{-z}]_{2k} \text{ …. MEH.}$$

$$\nabla[\log \mathbf{2}]_k = \frac{(-1)^{k+1}}{k}$$

$$t_{\frac{a}{b}}(n)=\nabla[\log\frac{\boldsymbol{a}}{\boldsymbol{b}}]_k=\lim_{x\to 1}\nabla[\log(\frac{1-x^b}{1-x^a})]_n$$

$$t_{\frac{a}{b}}(n)=b(\lfloor\frac{n}{b}\rfloor-\lfloor\frac{n-1}{b}\rfloor)-a(\lfloor\frac{n}{a}\rfloor-\lfloor\frac{n-1}{a}\rfloor)$$

$$t_{\frac{a}{b}}(n)=-t_{\frac{b}{a}}(n)$$

$$t_{\frac{a}{a}}(n)=t_1(n)=0$$

$$\lim_{x\to\infty} t_x(n)=1$$

$$\lim_{x\to\infty} t_{x^{-1}}(n)=-1$$

$$t_{-a}(n)=???$$

$$\nabla[\log m]_k=\frac{t_m(k)}{k}$$

$$\nabla[\boldsymbol{m}^z]_k=\nabla[(\frac{1}{m})^{-z}]_k$$

$$\nabla[\boldsymbol{m}^{\boldsymbol{a}+\boldsymbol{b}}]_n=\sum_{j+k=n}\nabla[\boldsymbol{m}^{\boldsymbol{a}}]_j\cdot\nabla[\boldsymbol{m}^{\boldsymbol{b}}]_k$$

$$[\boldsymbol{m}^{\boldsymbol{a}+\boldsymbol{b}}]_n=\sum_{j+k\le n}\nabla[\boldsymbol{m}^{\boldsymbol{a}}]_j\cdot\nabla[\boldsymbol{m}^{\boldsymbol{b}}]_k$$

...

$$[(\log\boldsymbol{m})^k]_n=\sum_{j\le n}\nabla[\log\boldsymbol{m}]_j\cdot[(\log\boldsymbol{m})^{k-1}]_{n-j}$$

$$[(\log\boldsymbol{m})^a]_n=\sum_{j+k\le n}\nabla[\log\boldsymbol{m}]_j\cdot\nabla[(\log\boldsymbol{m})^{a-1}]_k$$

$$[(\log\boldsymbol{m})^{a+b}]_n=\sum_{j+k\le n}\nabla[(\log\boldsymbol{m})^a]_j\cdot\nabla[(\log\boldsymbol{m})^b]_k$$

$$[(\log\boldsymbol{m})^k]_n=\sum_{j\le n}t_m(j)\cdot[(\log\boldsymbol{m})^{k-1}]_{n-j}$$

$$\nabla[\boldsymbol{m}^z]_n=\sum_{j=0}^{m-1}\nabla[\boldsymbol{m}^{z-1}]_{n-j}$$

$$\nabla[\boldsymbol{m}^z]_n=\nabla[\boldsymbol{m}^z]_{(m-1)z-n}$$

$$[\boldsymbol{m}^z]_n=\sum_{j=0}^{m-1}[\boldsymbol{m}^{z-1}]_{n-j}$$

$$[\boldsymbol{m}^z]_n=[\boldsymbol{m}^z]_{(m-1)z}-[\boldsymbol{m}^z]_{(m-1)z-n-1}$$

$$\sum_{j=0}^{(m-1)k}\nabla[\boldsymbol{m}^{\boldsymbol{k}}]_j=m^k$$

$$\sum_{j=0}\nabla[\boldsymbol{m}^z]_j=m^z \ for \ \Re(z)>0$$

$$\sum_{j=0}^{(m-1)k} t_m(j)\cdot[\boldsymbol{m}^{\boldsymbol{k}}]_j=0$$

$$\sum_{j=0} t_m(j)\cdot[\boldsymbol{m}^z]_j=0$$

$$f(n)=(g(n)/g(n/2))$$

$$f(n)=\frac{g(n)}{g(n)/2}$$

$$lf(n)=lg(n)-lg(\frac{n}{2})$$

$$lg=\sum_{k=0} lf(\frac{n}{2^k})$$

NOW EXPRESS $[\infty^z]_n$ IN TERMS OF $[2^z]_n$ WITH THIS!!!

$$[2^z]_n=[(\frac{\boldsymbol{\infty}}{\boldsymbol{\infty}_{\mathbf{1/2}}})^z]_n$$

$$[3^z]_n=[(\frac{\boldsymbol{\infty}}{\boldsymbol{\infty}_{\mathbf{1/3}}})^z]_n$$

$$[(1+x)^z]_n=[(\frac{1-x^2}{1-x})^z]_n \qquad [(1+x)^z]_n=\sum_{a+2b\le n}\nabla[(\frac{1}{1-x})^z]_a\cdot\nabla[(\frac{1}{1-x^2})^{-z}]_b$$

$$[\boldsymbol{\infty}^z]_n=[\prod_{k=0}\mathbf{2}_{\mathbf{1/2^k}}{}^z]_n$$

$$\nabla[\boldsymbol{\infty}^z]_n=\sum_{a+2b+4c+8d+\ldots=n}\nabla[\mathbf{2}^z]_a\cdot\nabla[\mathbf{2}^z]_b\cdot\nabla[\mathbf{2}^z]_c\cdots$$

$$\nabla[\boldsymbol{\infty}^z]_n=\sum_{a+3b+9c+27d+\ldots=n}\nabla[3^z]_a\cdot\nabla[3^z]_b\cdot\nabla[3^z]_c\cdots$$

$$\nabla[\mathbf{2}^z]_n=\sum_{a+2b=n}\nabla[\boldsymbol{\infty}^z]_a\cdot\nabla[\boldsymbol{\infty}^{-z}]_b$$

$$\nabla[3^z]_n=\sum_{a+3b=n}\nabla[\boldsymbol{\infty}^z]_a\cdot\nabla[\boldsymbol{\infty}^{-z}]_b$$

$$\nabla[3^z]_n=\sum_{a+3b=n}\nabla[\boldsymbol{\infty}^z]_a\cdot\nabla[\boldsymbol{\infty}^{-z}]_b$$

$$[\log\boldsymbol{\infty}]_n=\sum_{k=0}[\log\mathbf{2}_{\mathbf{1/2^k}}]_n$$

$$[\log\boldsymbol{\infty}]_n=\sum_{k=0}[\log 3]_{\frac{n}{3^k}}$$

$$[\log 1]_n=0=[\log\boldsymbol{\infty}]_n-[\log\boldsymbol{\infty}]_n$$

$$[\log\mathbf{2}]_n=[\log\boldsymbol{\infty}]_n-[\log\boldsymbol{\infty}_{\mathbf{1/2}}]_n$$

$$[\log 3]_n=[\log\boldsymbol{\infty}]_n-[\log\boldsymbol{\infty}_{\mathbf{1/3}}]_n$$

$$[\log 3]_n=\sum_{k=0}[\log\mathbf{2}]_{\frac{n}{2^k}}-\sum_{k=0}[\log\mathbf{2}]_{\frac{n}{3\cdot 2^k}}$$

$$[\log b]_n=\sum_{k=0}[\log a-\log a_{1/b}]_{\frac{n}{a^k}}$$

$$\log b = \lim_{n \to \infty} \sum_{k=0} [\log a]_{\frac{n}{a^k}} - \sum_{k=0} [\log a]_{\frac{n}{b \cdot a^k}} \quad \text{OR} \quad \log b = \lim_{n \to \infty} \sum_{k=0} \sum_{j=\lfloor \frac{n}{b \cdot a^k} \rfloor + 1}^{\lfloor \frac{n}{a^k} \rfloor} \nabla [\log a]_j$$

$$\log b = \lim_{n \to \infty} \sum_{k=0} [\log a]_{\frac{n}{a^k}} - [\log a]_{\frac{n}{b \cdot a^k}} \quad \log b = \lim_{n \to \infty} \sum_{k=0} [\log a - \log a_{1/b}]_{\frac{n}{a^k}}$$

$$[\log \mathbf{4}]_n = [\log \mathbf{2}]_n + [\log \mathbf{2}_{\mathbf{1/2}}]_n$$

$$[\log \mathbf{8}]_n = [\log \mathbf{2}]_n + [\log \mathbf{2}]_{\frac{n}{\mathbf{2}}} + [\log \mathbf{2}]_{\frac{n}{4}}$$

$$[\log \mathbf{9}]_n = [\log \mathbf{3}]_n + [\log \mathbf{3}]_{\frac{n}{3}}$$

$$\nabla [\mathbf{4}^z]_n = \sum_{a+2b=n} \nabla [\mathbf{2}^z]_a \cdot \nabla [\mathbf{2}^z]_b$$

$$\nabla [\mathbf{8}^z]_n = \sum_{a+2b+4c=n} \nabla [\mathbf{2}^z]_a \cdot \nabla [\mathbf{2}^z]_b \dot{\nabla} [\mathbf{2}^z]_c$$

$$\nabla [\mathbf{9}^z]_n = \sum_{a+3b=n} \nabla [\mathbf{3}^z]_a \cdot \nabla [\mathbf{3}^z]_b$$

$$[\log \mathbf{2}]_n = \sum_{k=0} (-1)^k [\log \mathbf{4}]_{\frac{n}{2^k}}$$

$$[\log \mathbf{2}]_n = [\prod_{k=0} \mathbf{4}^{(-1)^k \cdot z} / (2^k)]_n$$

$$[\mathbf{2}^z]_n = \sum_{a+2b+4c+8d+\ldots \le n} \nabla [\mathbf{4}^z]_a \cdot \nabla [\mathbf{4}^{-z}]_b \cdot \nabla [\mathbf{4}^z]_c \cdot \nabla [\mathbf{4}^{-z}]_d \cdot \ldots$$

$$\nabla [(\boldsymbol{m}^2)^z]_n = \sum_{a+m \cdot b=n} \nabla [\boldsymbol{m}^z]_a \cdot \nabla [\boldsymbol{m}^z]_b$$

$$[(\boldsymbol{m}^2)^z]_n = \sum_{a+m \cdot b \le n} \nabla [\boldsymbol{m}^z]_a \cdot \nabla [\boldsymbol{m}^z]_b$$

(is this fine for m as a non-integer?)

...

$$\nabla[\boldsymbol{m}^z]_n = \sum_{a+m\cdot b=n} \nabla[\boldsymbol{\infty}^z]_a \cdot \nabla[\boldsymbol{\infty}^{-z}]_b$$

$$[\boldsymbol{m}^z]_n = \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \nabla[\boldsymbol{\infty}^{z+1}]_{n-m\cdot k} \cdot \nabla[\boldsymbol{\infty}^{-z}]_k$$

$$\sum_{m=1}^{t} [\boldsymbol{m}^z]_n = \sum_{m=1}^{t} \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \nabla[\boldsymbol{\infty}^{z+1}]_{n-m\cdot k} \cdot \nabla[\boldsymbol{\infty}^{-z}]_k$$

$$\nabla[\infty^z]_n = \frac{z^{(n)}}{n!} \qquad [\boldsymbol{\infty}^z]_n = \frac{(z+1)^{(n)}}{n!}$$

$$\sum_{m=1}^{t} [\boldsymbol{m}^z]_n = \sum_{m=1}^{t} \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(z+1)^{(n-m\cdot k)}}{(n-m\cdot k)!} \cdot \frac{(-z)^{(k)}}{k!}$$

$$\sum_{m=1}^{t} [\boldsymbol{m}^z]_n = \sum_{k=0}^{t} \frac{(-z)^{(k)}}{k!} \cdot \sum_{m=1}^{\frac{t}{k}} \frac{(z+1)^{(n-m\cdot k)}}{(n-m\cdot k)!}$$

$$\sum_{m=1}^{t} [\boldsymbol{m}^3]_n = \sum_{m=1}^{\frac{t}{0}} \frac{(3+1)^{(n)}}{(n)!} + -3 \cdot \sum_{m=1}^{t} \frac{(3+1)^{(n-m)}}{(n-m)!} + 3 \cdot \sum_{m=1}^{\frac{t}{2}} \frac{(3+1)^{(n-m\cdot 2)}}{(n-m\cdot 2)!} + -\sum_{m=1}^{\frac{t}{3}} \frac{(3+1)^{(n-m\cdot 3)}}{(n-m\cdot 3)!} \ \ldots ?$$

$$a\nabla[\log\boldsymbol{\infty}]_n \rightarrow ???$$

$$\cdots$$

$$[m^z]_n = \sum_{k=0}^{\frac{n}{m}} [\boldsymbol{\infty}^z]_{n-m\cdot k} \cdot [\boldsymbol{\infty}^{-z-\mathbf{1}}]_k$$

$$\cdots$$

$$[\mathbf{2}^z]_n = \sum_{k=0}^{n} \nabla[\mathbf{2}^z]_k$$

$$[\mathbf{2}^{z-\mathbf{1}}]_n = \sum_{k=0}^{n} \nabla[\mathbf{2}^{z-\mathbf{1}}]_k$$

$$[\mathbf{2}^{z+\mathbf{1}}]_n = \sum_{k=0}^{\frac{n}{2}} [\boldsymbol{\infty}^{z+\mathbf{1}}]_{n-2\cdot k} \cdot [\boldsymbol{\infty}^{-z-\mathbf{2}}]_k$$

$$[\mathbf{2}^z]_n = \sum_{k=0}^{\frac{n}{2}} [\boldsymbol{\infty}^z]_{n-2\cdot k} \cdot [\boldsymbol{\infty}^{-z-\mathbf{1}}]_k$$

$$[\mathbf{2}^{z-\mathbf{1}}]_n = \sum_{k=0}^{\frac{n}{2}} [\boldsymbol{\infty}^{z-\mathbf{1}}]_{n-2\cdot k} \cdot [\boldsymbol{\infty}^{-z}]_k$$

$$\nabla[\mathbf{2}^{z-\mathbf{1}}]_n = \sum_{k=0}^{\frac{n}{2}} [\boldsymbol{\infty}^{z-\mathbf{2}}]_{n-2\cdot k} \cdot [\boldsymbol{\infty}^{-z}]_k$$

$$\nabla[\mathbf{2}^z]_n = \sum_{k=0}^{\frac{n}{2}} [\boldsymbol{\infty}^{z-\mathbf{1}}]_{n-2\cdot k} \cdot [\boldsymbol{\infty}^{-z-\mathbf{1}}]_k$$

$$\nabla[\mathbf{2}^{z+\mathbf{1}}]_n = \sum_{k=0}^{\frac{n}{2}} [\boldsymbol{\infty}^z]_{n-2\cdot k} \cdot [\boldsymbol{\infty}^{-z-\mathbf{2}}]_k$$

$$[\log \Sigma \boldsymbol{p}(\boldsymbol{n})]_n = \sum_{k=1}^{n} [\log \boldsymbol{\infty}_{\mathbf{1}/\boldsymbol{k}}]_n$$

$$[\log \boldsymbol{\infty}]_n = \sum_{k=1}^{n} \mu(k) [\log \Sigma \boldsymbol{p}(\boldsymbol{n})]_{\frac{n}{k}}$$

$$[\Sigma \boldsymbol{p}(\boldsymbol{n})^z]_n = [\prod_{k=1} \boldsymbol{\infty}_{\mathbf{1}/\boldsymbol{k}}^z]_n$$

$$p(n) = \nabla [\prod_{k=1} \boldsymbol{\infty}_{\mathbf{1}/\boldsymbol{k}}]_n$$

$$a(n) = b(n) + b(\frac{n}{2}) + b(\frac{n}{3}) + b(\frac{n}{4}) \ldots$$

$$a(n) - a(\frac{n}{2}) = b(n) + b(\frac{n}{3}) + b(\frac{n}{5}) + b(\frac{n}{7}) \ldots$$

$$a(n) - a(\frac{n}{2}) - a(\frac{n}{3}) = b(n) + b(\frac{n}{5}) - b(\frac{n}{6}) + b(\frac{n}{7}) \ldots$$

Investigate relationship between additive and multiplicative identities. For example,

$$[\log \Sigma \boldsymbol{p}(\boldsymbol{n})]_n = \sum_{k=1}^{n} [\log \boldsymbol{\infty}_{1/k}]_n$$

vs

$$[\log \Sigma \boldsymbol{a}(\boldsymbol{n})]_n = \sum_{k=1}^{n} [\log \boldsymbol{\zeta}_{1/k}(\boldsymbol{0})]_n$$

ALSO. Is there an additive equivalence to the s parameter in $\zeta(s)$? Probably a multiplication by s instead, with s=-1 disappearing the way s=0 does in the multiplicative case? And with s=0 being a weird nullity the way s=1 is in the multiplicative case?

YEP

...

$$\nabla[\log \boldsymbol{\infty}(\boldsymbol{s})]_n = \frac{1-(1+s)^n}{n}$$

$$\nabla[\boldsymbol{\infty}(\boldsymbol{-1})]_n = 1$$

$$[\boldsymbol{\infty}(\boldsymbol{-1})^z]_n = \frac{z^{(n)}}{n!}$$

$$\sum_{j=0}^{n} \nabla[\boldsymbol{\infty}(\boldsymbol{-1})^z]_j = [\boldsymbol{\infty}(\boldsymbol{-1})^{z+1}]$$

...

$$\nabla[\boldsymbol{\infty}(\boldsymbol{s})]_n = -s$$

$$[\boldsymbol{\infty}(\boldsymbol{s})]_n = -s \cdot n$$

$$\nabla[\log \boldsymbol{\infty}(\boldsymbol{s})]_n = \frac{1-(1+s)^n}{n}$$

$$[\log \boldsymbol{\infty}(\boldsymbol{s})]_n = H_n + (1+s)^{n+1} \cdot \Phi(1+s, 1, 1+n) + \log(-s)$$

Lerch transcendental here.

$$[\boldsymbol{\infty}(\boldsymbol{s})^z]_n = \sum_{k=0}^{n} \binom{z}{k} [(\boldsymbol{\infty}(\boldsymbol{s})-1)^k]_n$$

...

$$[(\boldsymbol{\infty}(\boldsymbol{s})-1)]_n = \sum_{j=1}^{n} -s$$

$$[(\boldsymbol{\infty}(\boldsymbol{s})-1)^2]_n = \sum_{j+k\le n;\ j,k>0} (-s)^2$$

...

$$\nabla[\boldsymbol{\infty}(\boldsymbol{s})-1]_n = -s$$

$$\nabla[(\boldsymbol{\infty}(\boldsymbol{s})-1)^k]_n = (-s)^k \cdot \frac{(n-k+1)^{(k-1)}}{(k-1)!}$$

$$\nabla[\boldsymbol{\infty}(\boldsymbol{s})^z]_n = -s z\ {}_2F_1(1-n, 1-z, 2, -s)$$

Good.

$$[(\boldsymbol{\infty}(\boldsymbol{s})-1)^k]_n = s^k \cdot \frac{(-n)^{(k)}}{k!}$$

$$\frac{(z+1)^{(n)}}{n!} = \sum_{k=0} \binom{z}{k} (-1)^k \cdot \frac{(-n)^{(k)}}{k!}$$

$$[\boldsymbol{\infty}(\boldsymbol{s})^z]_n = \sum_{k=0} \binom{z}{k} \cdot s^k \cdot \frac{(-n)^{(k)}}{k!}$$

$$[\boldsymbol{\infty}(\boldsymbol{s})^z]_n = {}_2F_1(-n, -z, 1, -s)$$

(Remember, $[\boldsymbol{\infty}^z]_n = \frac{(z+1)^{(n)}}{n!}$)

$$[\boldsymbol{\infty}(\boldsymbol{s})^z]_n = [\boldsymbol{\infty}(\boldsymbol{s})^{\boldsymbol{n}}]_z$$

...

What happens for $[\mathbf{2}(\boldsymbol{s})^z]_n$? For $[\boldsymbol{m}(\boldsymbol{s})^z]_n$?

$$[\infty(\boldsymbol{s})]_n = \lim_{x \to 1} [\frac{-s}{1-x}]_n$$

$$[\infty(\boldsymbol{s})^z]_n = \lim_{x \to 1} [(\frac{-s}{1-x})^z]_n$$

$$[(\infty(\boldsymbol{s})-1)^k]_n = \lim_{x \to 1} [(\frac{-s}{1-x}-1)^k]_n$$

$$[\log \infty(\boldsymbol{s})]_n = \lim_{x \to 1} [\log(\frac{-s}{1-x})]_n$$

$$[(1+x)^z]_n = [(\frac{1-x^2}{1-x})^z]_n$$

$$[(1+x)^2]_n = \sum_{a+b\le n} \nabla[1+x]_a \cdot \nabla[1+x]_b$$

$$[(\frac{1}{1-x})^2]_n = \sum_{a+b\le n} \nabla[(\frac{1}{1-x})]_a \cdot \nabla[\frac{1}{1-x}]_b$$

$$[(\frac{1}{1-x^2})^2]_n = \sum_{2a+2b\le n} \nabla[(\frac{1}{1-x^2})]_a \cdot \nabla[\frac{1}{1-x^2}]_b$$

$$[1+x]_n = \{1,1,0,0,0,0,0...\}$$

$$[1+x+x^2]_n = \{1,1,1,0,0,0,0...\}$$

$$[\frac{1}{1-x}]_n = \{1,1,1,1,1,1,1...\}$$

$$[\frac{1}{1-x^2}]_n = \{1,0,1,0,1,0,1,0...\}$$

… ?

$$[1-x]_n = \{1,-1,0,0,0,0,0...\}$$

$$[\frac{1}{1+x}]_n = \{1,-1,1,-1,1,-1,1...\}$$

Questions:

d_z(n) and pochhammer(z,n)/n! are, respectively, the multiplicative and additive convolutions of the sequence 1,1,1,1,1,1….

What other similar mappings are there? And what are their relationships?

Variant of					
$\{1,1,0,0,0,0,0...\}$	$(1+x)^z$	$\binom{z}{n}$	$\frac{(-1)^{n+1}}{n}$		$n=2^k\,?\,\frac{(-1)^{k+1}}{k}:0$
$\{1,1,1,0,0,0,0...\}$	$(1+x+x^2)^z$	$\binom{z}{n}_2$	$\frac{t_3(n)}{n}$		
$\{1,1,1,1,1,1,1...\}$	$(\frac{1}{1-x})^z$	$\frac{z^{(n)}}{n!}$	$\frac{1}{n}$	$d_z(n)$	$\frac{\kappa(n)}{n}$
$\{1,0,1,0,1,0,1,0...\}$	$(\frac{1}{1-x^2})^z$				
$\{1,-1,0,0,0,0,0...\}$	$(1-x)^z$				
$\{1,-1,1,-1,1,-1,1...\}$	$(\frac{1}{1+x})^z$		$\frac{(-1)^{n+1}\cdot(2^n-1)}{n}$		
$\{1,\frac{1}{2},\frac{1}{6},\frac{1}{24},...\}$	e^z	$\frac{z^n}{n!}$	$n=1\,?\,1:0$		$n=p\,?\,1:0$

$$\boxed{\sum_{k=1}^{n} C_k \cdot H_{n-k} = f(n) = 1}$$

$$\boxed{\sum_{k=1}^{n} \frac{\mu(j)}{j} H_{\lfloor \frac{n}{j} \rfloor} = f(n) = 1}$$

$$lh(n) = \sum_{k=1}^{n} [\frac{1}{k} \cdot lf(k)]$$

$$h(n) = [\prod_{k=1}^{n} f(k)^{1/k}]$$

$$lf(n) = \sum_{k=1}^{n} \frac{\mu(j)}{j} \cdot [lh(\frac{n}{j})]$$

$$f(n) = [\prod_{k=1}^{n} h(k)^{\frac{\mu(j)}{j}}]$$

I've connected e to 1/(1-x). Now connect it to (1+x) – which is to say, connect 2^z to e^z. Then generalize it to m^z.

And why not trig functions, while I'm at it? For fun.

$$[(\frac{1}{1-x})^{z}]_{n} = [\prod_{k=0} (1+x^{(2^k)})^{z}]_{n}$$

$$[\frac{1}{1-x}]_{n} = [\prod_{k=0} (1+x^{(2^k)})]_{n}$$

$$e = \prod_{k=1} (\frac{1}{1-x})^{\frac{\mu(k)}{k}}$$

$$e = \prod_{j=1} (\prod_{k=0} (1+x^{(2^k)}))^{\frac{\mu(j)}{j}}$$

$$e^{z} = \prod_{k=1} (\frac{1}{1-x})^{\frac{z \cdot \mu(k)}{k}}$$

$$[e^{z}]_{n} = \sum_{a+2b+3c+4d+\ldots \le n} \nabla[(\frac{1}{1-x})^{z}]_{a} \cdot \nabla[(\frac{1}{1-x})^{-\frac{z}{2}}]_{b} \cdot \nabla[(\frac{1}{1-x})^{-\frac{z}{3}}]_{c} \cdot \nabla[(\frac{1}{1-x})^{-\frac{z}{5}}]_{d} \cdot \ldots$$

$$\nabla[(\frac{1}{1-x})^{z}]_{n} = \sum_{a+2b+4c+8d+\ldots = n} \nabla[\mathbf{2}^{z}]_{a} \cdot \nabla[\mathbf{2}^{z}]_{b} \cdot \nabla[\mathbf{2}^{z}]_{c} \cdots$$

$$[(\frac{1}{1-x})^z]_n = \prod_{a+2b+3c+4d+\ldots\le n} \nabla[e^z]_a \cdot \nabla[e^{\frac{z}{2}}]_b \cdot \nabla[e^{\frac{z}{3}}]_c \cdot \nabla[e^{\frac{z}{4}}]_d \cdot \ldots$$

$$[(1-x)^z]_n = \prod_{a+2b+3c+4d+\ldots\le n} \nabla[e^{-z}]_a \cdot \nabla[e^{-\frac{z}{2}}]_b \cdot \nabla[e^{-\frac{z}{3}}]_c \cdot \nabla[e^{-\frac{z}{4}}]_d \cdot \ldots$$

$$[e^z]_n = \prod_{a+2b+3c+4d+\ldots\le n} \nabla[(\frac{1}{1-x})^z]_a \cdot \nabla[(\frac{1}{1-x})^{-\frac{z}{2}}]_b \cdot \nabla[(\frac{1}{1-x})^{-\frac{z}{3}}]_c \cdot \nabla[(\frac{1}{1-x})^{-\frac{z}{5}}]_d \cdot \ldots$$

$$[e^z]_n = \sum_{a_1+2a_2+\ldots+k\cdot a_k\le n} \prod_k \nabla[(\frac{1}{1-x})^{\mu(k)\cdot\frac{z}{k}}]_{a_k}$$

$$[(\frac{1}{1-x})^z]_n = \sum_{a_1+2a_2+\ldots+k\cdot a_k\le n} \prod_k \nabla[e^{\frac{z}{k}}]_{a_k}$$

$$[e^z]_n = \sum_{a_1+2a_2+\ldots+k\cdot a_k\le n} \prod_k \frac{(\mu(k)/k\cdot z)^{(k)}}{k!}$$

$$[(\frac{1}{1-x})^z]_n = \sum_{a_1+2a_2+\ldots+k\cdot a_k\le n} \prod_k \nabla[e^{\frac{z}{k}}]_{a_k}$$

```
r1[n_,z_,k_]:=If[k>n,1,Sum[ (z/k)^j/j! r1[n-k j,z,k+1],{j,0,n/k}]]
r2[n_,z_,k_]:=If[k>n,1,Sum[ Pochhammer[z MoebiusMu[k]/k,j]/j! r2[n-k j,z,k+1],
{j,0,n/k}]]

(call with k=1)
```

$$[(\frac{1}{1-(1)})^z]_n = \sum_{k=0}^{n} \frac{z^{(k)}}{k!} = \frac{(z+1)^{(n)}}{n!}$$

$$[e^z]_n = \sum_{k=0}^{n} \frac{z^k}{k!} = \frac{e^z\cdot\gamma(n+1,z)}{n!}$$

$$[(\frac{1}{1-(1)})^z]_n = \sum_{a=0}^{n} \frac{z^a}{a!} \cdot \sum_{b=0}^{\lfloor \frac{1}{2}\cdot\frac{n}{a}\rfloor} \frac{(z/2)^b}{b!} \cdot \sum_{c=0}^{\lfloor \frac{1}{3}\cdot\frac{n}{a\cdot 2b}\rfloor} \frac{(z/3)^c}{c!} \cdot \sum_{d=0}^{\lfloor \frac{1}{4}\cdot\frac{n}{a\cdot 2b\cdot 3c}\rfloor} \frac{(z/4)^d}{d!} \cdot \ldots$$

$$[e^z]_n = \sum_{a=0}^{n} \frac{z^{(a)}}{a!} \cdot \sum_{b=0}^{\lfloor \frac{1}{2}\cdot\frac{n}{a}\rfloor} \frac{(-z/2)^{(b)}}{b!} \cdot \sum_{c=0}^{\lfloor \frac{1}{3}\cdot\frac{n}{a\cdot 2b}\rfloor} \frac{(-z/3)^{(c)}}{c!} \cdot \sum_{d=0}^{\lfloor \frac{1}{5}\cdot\frac{n}{a\cdot 2b\cdot 3c}\rfloor} \frac{(-z/5)^{(d)}}{d!} \cdot \ldots$$

SO. From a certain, perhaps most natural, perspective, there is a way to interpret all of the following

$$\lim_{x\to 1} \prod_{k=1} (\frac{1}{1-x})^{\frac{z\cdot\mu(k)}{k}} = e^z$$

$\lim_{x\to 1} \prod_{k=1} (1-x)^{-z\cdot\frac{\mu(k)}{k}} = e^z$ (but something like this is already known. Boo.)

$$\prod_{k=1} e^{\frac{z}{k}} = \lim_{x\to 1} \left(\frac{1}{1-x}\right)^{z}$$

$$\lim_{x\to 1} \prod_{k=1} \left(\frac{1}{1-x}\right)^{\frac{\mu(k)}{k}} = e$$

$$\prod_{k=1} e^{\frac{1}{k}} = \lim_{x\to 1} \left(\frac{1}{1-x}\right)$$

$$\lim_{x\to 1} \left(\frac{1-x^{a}}{1-x}\right)^{z} = a^{z}$$

to mean that they are true (if the limit is taken as the equations above).

$$[\zeta(0)^{z}]_{n} = \sum_{a=0}^{\frac{\log n}{\log 2}} \frac{z^{(a)}}{a!} \cdot \sum_{b=0}^{\frac{\log n - a\log 2}{\log 3}} \frac{z^{(b)}}{b!} \cdot \sum_{c=0}^{\frac{\log n - a\log 2 - b\log 3}{\log 5}} \frac{z^{(c)}}{c!} \cdot \sum_{d=0}^{\frac{\log n - a\log 2 - b\log 3 - c\log 5}{\log 7}} \frac{z^{(d)}}{d!} \cdot \ldots$$

$$[\zeta(s)^{z}]_{n} = \sum_{a=0}^{\frac{\log n}{\log 2}} \frac{z^{(a)}}{a!} \cdot 2^{-as} \cdot \sum_{b=0}^{\frac{\log n - a\log 2}{\log 3}} \frac{z^{(b)}}{b!} \cdot 3^{-bs} \cdot \sum_{c=0}^{\frac{\log n - a\log 2 - b\log 3}{\log 5}} \frac{z^{(c)}}{c!} \cdot 5^{-cs} \cdot \sum_{d=0}^{\frac{\log n - a\log 2 - b\log 3 - c\log 5}{\log 7}} \frac{z^{(d)}}{d!} \cdot 7^{-ds} \cdot \ldots$$

$$[e^{z}]^{*}_{n} = \sum_{a=0}^{\frac{\log n}{\log 2}} \frac{z^{a}}{a!} \cdot 2^{-as} \cdot \sum_{b=0}^{\frac{\log n - a\log 2}{\log 3}} \frac{z^{b}}{b!} \cdot 3^{-bs} \cdot \sum_{c=0}^{\frac{\log n - a\log 2 - b\log 3}{\log 5}} \frac{z^{c}}{c!} \cdot 5^{-cs} \cdot \sum_{d=0}^{\frac{\log n - a\log 2 - b\log 3 - c\log 5}{\log 7}} \frac{z^{d}}{d!} \cdot 7^{-ds} \cdot \ldots$$

$$\left[\left(\frac{1}{1-(1)}\right)^{z}\right]_{n} = \sum_{a=0}^{n} \frac{z^{a}}{a!} \cdot \sum_{b=0}^{\lfloor \frac{1}{2} \cdot \frac{n}{a} \rfloor} \frac{(z/2)^{b}}{b!} \cdot \sum_{c=0}^{\lfloor \frac{1}{3} \cdot \frac{n}{a\cdot 2b} \rfloor} \frac{(z/3)^{c}}{c!} \cdot \sum_{d=0}^{\lfloor \frac{1}{4} \cdot \frac{n}{a\cdot 2b\cdot 3c} \rfloor} \frac{(z/4)^{d}}{d!} \cdot \ldots$$

$$[e^{z}]_{n} = \sum_{a=0}^{n} \frac{z^{(a)}}{a!} \cdot \sum_{b=0}^{\lfloor \frac{1}{2} \cdot \frac{n}{a} \rfloor} \frac{(-z/2)^{(b)}}{b!} \cdot \sum_{c=0}^{\lfloor \frac{1}{3} \cdot \frac{n}{a\cdot 2b} \rfloor} \frac{(-z/3)^{(c)}}{c!} \cdot \sum_{d=0}^{\lfloor \frac{1}{5} \cdot \frac{n}{a\cdot 2b\cdot 3c} \rfloor} \frac{(-z/5)^{(d)}}{d!} \cdot \ldots .$$

$$\ldots$$

$$[e^{z}]^{*}_{n} = \sum_{a\cdot b^{2}\cdot c^{3}\cdot d^{5}\cdot \ldots \le n} d_{z}(a) \cdot d_{-\frac{z}{2}}(b) \cdot d_{-\frac{z}{3}}(c) \cdot d_{-\frac{z}{5}}(d) \cdot \ldots$$

$$[\zeta(s)^{z}]_{n} = \sum_{a\cdot b^{2}\cdot c^{3}\cdot d^{5}\cdot \ldots \le n} \nabla[e^{z}]^{*}_{a} \cdot \nabla[e^{\frac{z}{2}}]^{*}_{b} \cdot \nabla[e^{\frac{z}{3}}]^{*}_{c} \cdot \nabla[e^{\frac{z}{4}}]^{*}_{d} \cdot \ldots$$

$$\prod_{k=1} (\zeta(k\cdot s))^{\frac{\mu(k)}{k}} = ???$$

$$\left[\left(\prod_{k=1} \zeta_{1/k}(k\, s)^{\frac{\mu(k)}{k}}\right)^{z}\right]_{n} = [e*(s)]_{n}$$

$$[(\prod_{k=1} e*_{1/k}(k\,s)^{\frac{1}{k}})]_n = [\zeta(s)]_n$$

$$\prod_{k=1} e*(k\,s)^{\frac{1}{k}} = \zeta(s)$$

$$\prod_{k=1} \zeta(k\,s)^{\frac{\mu(k)}{k}} = e*(s)$$

…

Okay. So here is the BIG question. How many properties does $[e*(s)^z]$ share with e^z ?

This is the right way to add the s back in to make it line up with euler products.

$$\sum_{k=0}^{\infty} \frac{z^{(k)}}{k!} \cdot x^{(-s k)} = \left(\frac{1}{1-x^{-s}}\right)^{z}$$

$$\sum_{k=0}^{\infty} \binom{z}{k} \cdot x^{(-s k)} = (1+x^{-s})^{z}$$

$$\sum_{k=0}^{\infty} \frac{z^{k}}{k!} \cdot x^{(-s k)} = e^{x^{-s} \cdot z}$$

$$\ldots$$

$$[e^{z}]_{n}^{*} = \sum_{a=0}^{\frac{\log n}{\log 2}} \frac{z^{a}}{a!} \cdot 2^{-a s} \cdot \sum_{b=0}^{\frac{\log n - a \log 2}{\log 3}} \frac{z^{b}}{b!} \cdot 3^{-b s} \cdot \sum_{c=0}^{\frac{\log n - a \log 2 - b \log 3}{\log 5}} \frac{z^{c}}{c!} \cdot 5^{-c s} \cdot \sum_{d=0}^{\frac{\log n - a \log 2 - b \log 3 - c \log 5}{\log 7}} \frac{z^{d}}{d!} \cdot 7^{-d s} \cdot \ldots$$

$$e*(s) = \prod_{k=1} \zeta(k s)^{\frac{\mu(k)}{k}}$$

$$e*(s) = \sum_{j=1} \prod_{p^{k}|j} \frac{p^{-s k}}{k!} = \prod_{p} e^{p^{-s}}$$

$$e*(s)^{z} = \sum_{j=1} \prod_{p^{k}|j} \frac{p^{-s k}}{k!} = \prod_{p} e^{p^{-s} \cdot z}$$

$$\ldots$$

$$\lim_{x \to 1} \prod_{k=1} \left(\frac{1}{1-x}\right)^{\frac{\mu(k)}{k}} = e$$

$$\prod_{k=1} e^{\frac{1}{k}} = \lim_{x \to 1} \left(\frac{1}{1-x}\right)$$

$$\prod_{k=1} e*(k s)^{\frac{1}{k}} = \zeta(s)$$

$$e*(s) = \frac{1}{1^{s}} + \frac{1}{2^{s}} + \frac{1}{3^{s}} + \frac{1}{2 \cdot 4^{s}} + \frac{1}{5^{s}} + \frac{1}{6^{s}} + \frac{1}{7^{s}} + \frac{1}{6 \cdot 8^{s}} + \frac{1}{2 \cdot 9^{s}} + \frac{1}{10^{s}} + \frac{1}{11^{s}} + \frac{1}{2 \cdot 12^{s}} + \frac{1}{13^{s}} + \frac{1}{14^{s}} + \frac{1}{15^{s}} + \frac{1}{24 \cdot 16^{s}} + \ldots$$

$$e^{2^{-s}\cdot z}\cdot e^{3^{-s}\cdot z}\cdot e^{5^{-s}\cdot z}\cdot e^{7^{-s}\cdot z}\cdot e^{11^{-s}\cdot z}\cdot\ldots$$

$$e^{2^{-s}}\cdot e^{3^{-s}}\cdot e^{5^{-s}}\cdot e^{7^{-s}}\cdot e^{11^{-s}}\cdot\ldots$$

$$e^{2^{-2}}\cdot e^{3^{-2}}\cdot e^{5^{-2}}\cdot e^{7^{-2}}\cdot e^{11^{-2}}\cdot\ldots$$

$$e^{\frac{1}{4}}\cdot e^{\frac{1}{9}}\cdot e^{\frac{1}{25}}\cdot e^{\frac{1}{49}}\cdot e^{\frac{1}{121}}\cdot\ldots$$

$$e^{z}=1+z+\frac{z^2}{2}+\frac{z^3}{6}+\frac{z^4}{24}+\ldots$$

$$\ldots$$

$$[e^{z\cdot x^{-s}}]^{*}=\frac{1}{1^s}+\frac{z}{2^s}+\frac{z}{3^s}+\frac{z^2}{2\cdot 4^s}+\frac{z}{5^s}+\frac{z^2}{6^s}+\frac{z}{7^s}+\frac{z^3}{6\cdot 8^s}+\frac{z^2}{2\cdot 9^s}+\frac{z^2}{10^s}+\ldots$$

$$[e^{z}]^{*}=1+z+z+\frac{z^2}{2}+z+z^2+z+\frac{z^3}{6}+\frac{z^2}{2}+z^2+\ldots$$

$$\ldots$$

$$\sum_{k=0}^{n}[\infty^{z}]_{k}=[\infty^{z+1}]_{n}$$

$$\sum_{k=1}^{n}[\zeta(0)^{z}]_{n/k}=[\zeta(0)^{z+1}]_{n}$$

$$\int_{0}^{z}e^{x}\,dx=e^{z}-1$$

$$\eta(s)=\lim_{n\to\infty}\left(\sum_{a=0}^{0}1-\sum_{a=1}^{\lfloor\frac{\log n}{\log 2}\rfloor}2^{-as}\right)\cdot\sum_{b=0}^{\lfloor\frac{\log n-a\log 2}{\log 3}\rfloor}3^{-bs}\cdot\sum_{c=0}^{\lfloor\frac{\log n-a\log 2-b\log 3}{\log 5}\rfloor}5^{-cs}\cdot\sum_{d=0}^{\lfloor\frac{\log n-a\log 2-b\log 3-c\log 5}{\log 7}\rfloor}7^{-ds}\cdot\ldots$$

$$\zeta(s)=\lim_{n\to\infty}\left(\sum_{a=0}^{0}1+\sum_{a=1}^{\lfloor\frac{\log n}{\log 2}\rfloor}2^{-as}\right)\cdot\sum_{b=0}^{\lfloor\frac{\log n-a\log 2}{\log 3}\rfloor}3^{-bs}\cdot\sum_{c=0}^{\lfloor\frac{\log n-a\log 2-b\log 3}{\log 5}\rfloor}5^{-cs}\cdot\sum_{d=0}^{\lfloor\frac{\log n-a\log 2-b\log 3-c\log 5}{\log 7}\rfloor}7^{-ds}\cdot\ldots$$

...

$$f(n,s)=\sum_{j=1}^{\lfloor(n+1)/2\rfloor}(2j-1)^{-s}$$

$$\eta(s)=\lim_{n\to\infty}f(n,s)-\sum_{a=1}^{\lfloor\frac{\log n}{\log 2}\rfloor}2^{-as}\cdot f\left(\frac{n}{2^a},s\right)$$

$$\zeta(s)=\lim_{n\to\infty}f(n,s)+\sum_{a=1}^{\lfloor\frac{\log n}{\log 2}\rfloor}2^{-as}\cdot f\left(\frac{n}{2^a},s\right)$$

...

OKAY.

$$f(n,s)=\sum_{j=1}^{\lfloor(n+1)/2\rfloor}(2j-1)^{-s}$$

$$\lim_{n\to\infty}f(n,s)=2^{1-s}\cdot f\left(\frac{n}{2,}s\right)$$

$$\lim_{n\to\infty}f(n,s)=4^{1-s}\cdot f\left(\frac{n}{4,}s\right)$$

...

$$\lim_{n\to\infty}\sum_{j=1}^{\lfloor\frac{(n+1)}{2}\rfloor}(2j-1)^{-s}=2^{1-s}\cdot\sum_{j=1}^{\lfloor\frac{(n+1)}{4}\rfloor}(2j-1)^{-s}$$

$$\lim_{n\to\infty}\sum_{j=1}^{\lfloor\frac{(n+1)}{2}\rfloor}(2j-1)^{-s}=\sum_{j=1}^{\lfloor\frac{(n+1)}{4}\rfloor}2^{1-s}(2j-1)^{-s}$$

$$\lim_{n\to\infty}\sum_{j=1}^{\lfloor\frac{(n+1)}{2}\rfloor}(2j-1)^{-s}=2\sum_{j=1}^{\lfloor\frac{(n+1)}{4}\rfloor}(4j-2)^{-s}$$

…

$$\eta(s)=\lim_{n\to\infty} f(n,s)-\sum_{a=1}^{\lfloor\frac{\log n}{\log 2}\rfloor} 2^{-as}\cdot f\left(\frac{n}{2^a},s\right)$$

// ? Is this valid? Not sure about splitting this sum. Worth checking out.

$$\eta(s)=\lim_{n\to\infty} \sum_{a=\lfloor\frac{\log n}{\log 2}\rfloor+1}^{\infty} 2^{-a} f(n,s)+\sum_{a=1}^{\lfloor\frac{\log n}{\log 2}\rfloor} 2^{-a} f(n,s)-\sum_{a=1}^{\lfloor\frac{\log n}{\log 2}\rfloor} 2^{-as}\cdot f\left(\frac{n}{2^a},s\right)$$

$$\eta(s)=\lim_{n\to\infty} \sum_{a=\lfloor\frac{\log n}{\log 2}\rfloor+1}^{\infty} 2^{-a} f(n,s)+\sum_{a=1}^{\lfloor\frac{\log n}{\log 2}\rfloor} 2^{-a} f(n,s)-2^{-as}\cdot f\left(\frac{n}{2^a},s\right)$$

$$\eta(s)=\lim_{n\to\infty} \sum_{a=\lfloor\frac{\log n}{\log 2}\rfloor+1}^{\infty} 2^{-a} f(n,s)+\sum_{a=1}^{\lfloor\frac{\log n}{\log 2}\rfloor} 2^{-a}\left(f(n,s)-2^{s}\cdot f\left(\frac{n}{2^a},s\right)\right)$$

…

Okay. Here is where I'm trying to go.

...

$$\zeta(s)=\lim_{n\to\infty}\sum_{a=0}^{\lfloor\frac{\log n}{\log 2}\rfloor}2^{-as}\cdot\sum_{b=0}^{\lfloor\frac{\log n-a\log 2}{\log 3}\rfloor}3^{-bs}\cdot\sum_{c=0}^{\lfloor\frac{\log n-a\log 2-b\log 3}{\log 5}\rfloor}5^{-cs}\cdot\sum_{d=0}^{\lfloor\frac{\log n-a\log 2-b\log 3-c\log 5}{\log 7}\rfloor}7^{-ds}\cdot\ldots$$

$$e^*(s)=\lim_{n\to\infty}\sum_{a=0}^{\frac{\log n}{\log 2}}\frac{2^{-as}}{a!}\cdot\sum_{b=0}^{\frac{\log n-a\log 2}{\log 3}}\frac{3^{-bs}}{b!}\cdot\sum_{c=0}^{\frac{\log n-a\log 2-b\log 3}{\log 5}}\frac{5^{-cs}}{c!}\cdot\sum_{d=0}^{\frac{\log n-a\log 2-b\log 3-c\log 5}{\log 7}}\frac{7^{-ds}}{d!}\cdot\ldots$$

...

$$\sum_{k=0}^{\infty}x^{-sk}=\frac{1}{1-x^{-s}}$$

$$\sum_{k=0}^{\infty}\frac{x^{-sk}}{k!}=e^{x^{-s}}$$

$$[\prod_{k=1}e^{\frac{x^{-s}}{k}}]_n=[\frac{1}{1-x^{-s}}]_n$$

$$[\prod_{k=1}(\frac{1}{1-x^{-ks}})^{\frac{\mu(k)}{k}}]_n=[e^{x^{-s}}]_n$$

...

$$\zeta(s)=\sum_{j=1}\prod_{p^k|j}p^{-sk}=\prod_p\frac{1}{1-p^{-s}}$$

$$e^*(s)=\sum_{j=1}\prod_{p^k|j}\frac{p^{-sk}}{k!}=\prod_p e^{p^{-s}}$$

$$e^*(s)=\prod_{k=1}\zeta(ks)^{\frac{\mu(k)}{k}}$$

$$\prod_{k=1}e^*(ks)^{\frac{1}{k}}=\zeta(s)$$

...

$$\eta(s)=\lim_{n\to\infty}\left(\sum_{a=0}^{0}1-\sum_{a=1}^{\lfloor\frac{\log n}{\log 2}\rfloor}2^{-as}\right)\cdot\sum_{b=0}^{\frac{\lfloor\log n-a\log 2\rfloor}{\log 3}}3^{-bs}\cdot\sum_{c=0}^{\lfloor\frac{\log n-a\log 2-b\log 3}{\log 5}\rfloor}5^{-cs}\cdot\sum_{d=0}^{\lfloor\frac{\log n-a\log 2-b\log 3-c\log 5}{\log 7}\rfloor}7^{-ds}\cdot\ldots$$

$$e^{\eta}(s)=\lim_{n\to\infty}\sum_{a=0}^{\frac{\log n}{\log 2}}(-1)^{a}\cdot\frac{2^{-as}}{a!}\cdot\sum_{b=0}^{\frac{\log n-a\log 2}{\log 3}}\frac{3^{-bs}}{b!}\cdot\sum_{c=0}^{\frac{\log n-a\log 2-b\log 3}{\log 5}}\frac{5^{-cs}}{c!}\cdot\sum_{d=0}^{\frac{\log n-a\log 2-b\log 3-c\log 5}{\log 7}}\frac{7^{-ds}}{d!}\cdot\ldots$$

...

$$\eta(s)=(1-2^{1-s})\prod_{p}\frac{1}{1-p^{-s}}$$

$$e^{\eta}(s)=e^{-2^{1-s}}\cdot\prod_{p}e^{p^{-s}}=e^{-2^{-s}}\cdot\prod_{p,\,p>2}e^{p^{-s}}$$

$$e^{\eta}(s)=\prod_{k=1}\eta(ks)^{\frac{\mu(k)}{k}}$$

$$\eta(s)=\prod_{k=1}e^{\eta}(ks)^{\frac{1}{k}}$$

$$[e^{\eta}(s)]_n=[\prod_{k=1}\eta(ks)^{\frac{\mu(k)}{k}}]_n$$

....

Ahem.

$$[\boldsymbol{\infty}^{z}]_{n}=\frac{(z+1)^{(n)}}{n!}$$

$$\nabla[\infty^{z}]_{n}=\frac{z^{(n)}}{n!}$$

$$[\boldsymbol{m}^{z}]_{n}=\sum_{k=0}^{\lfloor\frac{n}{m}\rfloor}\nabla[\boldsymbol{\infty}^{\mathbf{z+1}}]_{n-m\cdot k}\cdot\nabla[\boldsymbol{\infty}^{-z}]_{k}$$

$$[\boldsymbol{m}^{z}]_{n}=\sum_{k=0}\nabla[\boldsymbol{\infty}^{\mathbf{z+1}}]_{n-m\cdot k}\cdot\nabla[\boldsymbol{\infty}^{-z}]_{k}$$

$$[\boldsymbol{m}^{-s}]_{n}=\sum_{k=0}\nabla[\boldsymbol{\infty}^{\mathbf{1-s}}]_{n-m\cdot k}\cdot\nabla[\boldsymbol{\infty}^{s}]_{k}$$

$$[\boldsymbol{j}^{-s}]_{n}=\sum_{k=0}\nabla[\boldsymbol{\infty}^{\mathbf{1-s}}]_{n-j\cdot k}\cdot\nabla[\boldsymbol{\infty}^{s}]_{k}$$

$$[\boldsymbol{j}^{-s}]_{n}=\sum_{k=0}\frac{(1-s)^{(n-j\cdot k)}}{(n-j\cdot k)!}\cdot\frac{s^{(k)}}{k!}$$

$$\zeta(s)=\lim_{n\to\infty}\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\frac{(1-s)^{(n-j\cdot k)}}{(n-j\cdot k)!}\cdot\frac{s^{(k)}}{k!}$$

$$\zeta(s)=\lim_{n\to\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\frac{(1-s)^{(n-j\cdot k)}}{(n-j\cdot k)!}\cdot\frac{s^{(k)}}{k!}$$

…

The trig stuff: