
[This is the starting point of trying to generalize the eta function]

$$\eta_x(s)=\zeta(s)-x^{1-s}\zeta(s)$$

If re(s) > 0, then

$$\eta_x(s)=\lim_{n\to\infty}\sum_{1\le j\le n} j^{-s}-x^{1-s}\sum_{1\le j\le \frac{n}{x}} j^{-s}$$

Given $H_n^{(s)}$, the generalized Harmonic Number, this is

$$\eta_x(s)=\lim_{n\to\infty} H_n^{(s)}-x^{1-s}H_{\frac{n}{x}}^{(s)}$$

...

Note that $H_{n,s}=\zeta(s)-\zeta(s,n+1)$, which is important.

Thus, we are interested in cases where $\frac{\partial}{\partial x}\eta_x(s)=0$. Right? So,

$$\frac{\partial}{\partial x}\eta_x(s)=\frac{\partial}{\partial x}(\lim_{n\to\infty}\sum_{1\le j\le n} j^{-s}-x^{1-s}\sum_{1\le j\le \frac{n}{x}} j^{-s})$$

$$\frac{\partial}{\partial x}\eta_x(s)=\frac{\partial}{\partial x}(\lim_{n\to\infty} -x^{1-s}\sum_{1\le j\le \frac{n}{x}} j^{-s})$$

Mathematica claims this is

$$\frac{\partial}{\partial x}\eta_x(s)=\lim_{n\to\infty} x^{-s-1}((s-1)x\cdot H_{\frac{n}{x}}^{(s)}+n\cdot s\cdot\zeta(s+1,\frac{n+x}{x}))$$

$$\lim_{x\to 1}\frac{\partial}{\partial x}\eta_x(s)=\lim_{n\to\infty}((s-1)H_n^{(s)}+n\cdot s\cdot\zeta(s+1,n+1))$$

(Which is

$$\frac{\partial}{\partial x}\eta_x(s)=(s-1)x^{-s}\zeta(s)$$

and

$$\lim_{x\to 1}\frac{\partial}{\partial x}\eta_x(s)=(s-1)\zeta(s)$$

)

And empirical tests seem to back this up. Notably, the two terms in the addition are not zero – they just happen to be opposites at zeta zeros.

Now,

$$H_n^{(s)}=\sum_{j=1}^{n}\frac{1}{j^s}$$

and, when re(s)>1, if a is an integer

$$\zeta(s,a)=\sum_{j=a}\frac{1}{j^{s}}$$

So

$$\frac{\partial}{\partial x}\eta_x(s)=\lim_{n\to\infty}x^{-s-1}((s-1)x\cdot(\sum_{j=1}^{\frac{n}{x}}\frac{1}{j^{s}})+n\cdot s\cdot(\sum_{j=0}\frac{1}{(j+\frac{n}{x}+1)^{s+1}}))$$

$$\frac{\partial}{\partial x}\eta_x(s)=\lim_{n\to\infty}x^{-s-1}((s-1)x\cdot(\sum_{j=1}^{\frac{n}{x}}\frac{1}{j^{s}})+n\cdot s\cdot(\sum_{j=1}\frac{1}{(j+n/x)^{s+1}}))$$

Suppose x=1. Then

$$\lim_{x\to1}\frac{\partial}{\partial x}\eta_x(s)=\lim_{n\to\infty}(1-s)\cdot(\sum_{j=1}^{n}\frac{1}{j^{s}})-(s)\cdot(\sum_{j=n+1}^{\infty}\frac{n}{j^{s+1}})$$

So then we want to find these zeros:

$$\boxed{0=\lim_{n\to\infty}(1-s)\cdot(\sum_{j=1}^{n}\frac{1}{j^{s}})-(s)\cdot(\sum_{j=n+1}^{\infty}\frac{n}{j^{s+1}})}$$

which is

$$0=\lim_{n\to\infty}((s)\cdot n(\zeta(s+1)-H_n^{(s+1)})-(1-s)(H_n^{(s)}))$$

A bit of further investigation suggests subsequent derivatives also have these zeros. Check below for more!

…

I need to understand the derivation of the derivative better. Mathematica says before simplifying that

$$\frac{\partial}{\partial x}\eta_x(s)=(s-1)\cdot x^{-s}\cdot\zeta(s)=\lim_{n\to\infty}(s)\cdot(n\cdot x^{-s}\cdot(\zeta(1+s)-H_{\frac{n}{x}}^{(s+1)}))-(1-s)\cdot(x^{-s}\cdot H_{\frac{n}{x}}^{(s)})$$

$$\lim_{x\to1}\frac{\partial}{\partial x}\eta_x(s)=(s-1)\cdot\zeta(s)=\lim_{n\to\infty}(s)\cdot(n\cdot(\zeta(1+s)-H_n^{(s+1)}))-(1-s)\cdot H_n^{(s)}$$

...

So, IF re(s) > 0,

$$\zeta(s)=\lim_{n\to\infty}\frac{s}{s-1}\cdot(n\cdot(\zeta(1+s)-H_n^{(s+1)}))+H_n^{(s)}$$

which is the same thing as

$$\boxed{\zeta(s)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{j^{s}}-\frac{s}{s-1}\cdot n\cdot\zeta(1+s,1+n)}$$

…

Ah ha! Taking multiple derivatives pushes the Re(s) line left by 1 each time. WRITE THOSE DOWN.

…

Generalizes to the hurwitz zeta function, too:

$$\lim_{x\to 1} \frac{\partial}{\partial x}(1-x^{1-s})\zeta_x(s, y)=(s-1)\zeta(s, y)$$

$$\lim_{x\to 1} \frac{\partial}{\partial x}(\lim_{n\to\infty} \sum_{1\le j\le n} (j+y)^{-s}-x^{1-s} \sum_{1\le j\le \frac{n}{x}} (j+y)^{-s})=\lim_{n\to\infty} s\cdot n\cdot\zeta(1+s, 1+n+y)-(1-s)(\zeta(s, y)-\zeta(s, 1+n+y))$$

$$\zeta(s, y)=\lim_{n\to\infty} -\frac{s\cdot n}{1-s}\cdot\zeta(1+s, y+n+1)+(\zeta(s, y)-\zeta(s, y+n+1))$$

…

So, about that derivative…

$$\zeta(s)=\lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{j^s}-\sum_{k=1}^{t} (-1)^k\cdot\binom{t}{k}\cdot\frac{s-1+k}{s-1}\cdot n^k\zeta(s+k, 1+n)$$

Or something like this. For what it's worth.

…

If t=2, this is

$$\zeta(s)=\lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{j^s}-2\cdot\frac{s}{s-1}\cdot n\cdot\zeta(1+s, 1+n)+\frac{s+1}{s-1}\cdot n^2\cdot\zeta(2+s, 1+n)$$

If s=0, then this simplifies to

$$\zeta(0)=\lim_{n\to\infty} n-\sum_{j=n+1}^{\infty} \frac{n^2}{j^2}=\lim_{n\to\infty} n-n^2(\zeta(2)-\sum_{j=1}^{n} \frac{1}{j^2})=\frac{1}{2}$$

$$\lim_{n\to\infty} n-n^2(\frac{\pi^2}{6}-\sum_{j=1}^{n} \frac{1}{j^2})=\frac{1}{2}$$

$$\lim_{n\to\infty} \frac{1}{n}+\sum_{j=1}^{n} \frac{1}{j^2}-\frac{1}{2n^2}=\frac{\pi^2}{6}$$

...

$$\zeta(s)=\lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{j^s}-3\cdot\frac{s}{s-1}\cdot n\cdot\zeta(1+s, 1+n)+3\cdot\frac{s+1}{s-1}\cdot n^2\cdot\zeta(2+s, 1+n)-3\cdot\frac{s+2}{s-1}\cdot n^3\cdot\zeta(3+s, 1+n)$$

If s=-1, then this simplifies to

$$\zeta(s)=\lim_{n\to\infty} \frac{n(n+1)}{2}-\frac{3}{2}\cdot\sum_{j=n+1}^{\infty} n+\frac{3}{2}\cdot\sum_{j=n+1}^{\infty} \frac{n^3}{j^2}=-\frac{1}{12}$$

How does this connect with the known zeta identities more generally?

…..

$$\boxed{0=\lim_{n\to\infty}(1-s)\cdot(\sum_{j=1}^{n}\frac{1}{j^{s}})-(s)\cdot(\sum_{j=n+1}^{\infty}\frac{n}{j^{s+1}})}$$

$$0=\lim_{n\to\infty}(1-s)\cdot(\sum_{j=1}^{n}\frac{1}{j^{s}})-(s)\cdot(n\cdot(\zeta(s+1)-\sum_{j=1}^{n}\frac{1}{j^{s+1}}))$$

Now, because of the reflection formula, it needs to be the case that if $\zeta(s)=0$, that $\zeta(1-s)=0$. And their complex conjugates need to be 0 as well. So if we sum all of them in this formula, the answer still must equal 0.

$$0=\zeta(s+ti)-\zeta(1-s+ti)$$

…..

What about reflection formula?

$$\zeta(s)=2^{s}\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)$$

…

The hurwitz zeta fn, for re(s) > 0

$$\zeta(s,y)=\lim_{n\to\infty}\sum_{j=0}^{n}\frac{1}{(j+y)^{s}}-\frac{s}{1-s}\cdot n\cdot(\sum_{j=0}^{\infty}\frac{1}{(j+y)^{s+1}}-\sum_{j=0}^{n}\frac{1}{(j+y)^{s+1}})$$

or

$$\zeta(s,y)=\lim_{n\to\infty}\sum_{j=0}^{n}\frac{1}{(j+y)^{s}}-\frac{s}{1-s}\cdot(\sum_{j=n}^{\infty}\frac{n}{(j+y)^{s+1}})$$

or, if y is a positive integer,

or

$$\zeta(s,y)=\lim_{n\to\infty}\sum_{j=y}^{n}\frac{1}{j^{s}}-\frac{s}{1-s}\cdot n\cdot(\sum_{j=y}^{\infty}\frac{1}{j^{s+1}}-\sum_{j=y}^{n}\frac{1}{j^{s+1}})$$

$$\boxed{\zeta(s,y)=\lim_{n\to\infty}\sum_{j=y}^{n}\frac{1}{j^{s}}-\frac{s}{s-1}\cdot(\sum_{j=y+n}^{\infty}\frac{n}{j^{s+1}})}$$

…

GOOD. Now, verify that multiple derivative formula...

$$\zeta(s,y)=\zeta(s,y)+\lim_{n\to\infty}\sum_{j=0}^{k}(-1)^{j}\cdot\binom{k}{j}\frac{(s-1+j)}{s-1}\cdot n^{j}\cdot\zeta(s+j,y+n+1)$$

$$\zeta(s,y)=\lim_{n\to\infty}\sum_{j=y}^{n+y-1}j^{-s}-\sum_{j=0}^{k}(-1)^{j}\cdot\binom{k}{j}\frac{(s-1+j)}{s-1}\cdot n^{j}\cdot\zeta(s+j,y+n)$$

$$0=\lim_{n\to\infty}\sum_{j=0}^{k}(-1)^{j}\cdot\binom{k}{j}\frac{(s-1+j)}{s-1}\cdot n^{j}\cdot\zeta(s+j,y+n)$$

$$\zeta(s)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{j^{s}}-2\cdot\frac{s}{s-1}\cdot n\cdot\zeta(1+s,1+n)+\frac{s+1}{s-1}\cdot n^{2}\cdot\zeta(2+s,1+n)$$

....

[Here I'm trying to show that the dirichlet eta function can be written as $(1-2^{1-s})\zeta(s)=\lim_{n\to\infty}(\sum_{j=1}^{\infty}(\frac{1}{j^s}-\frac{1}{(j+2n)^s})-2^{1-s}\cdot\sum_{1\le j\le n}\frac{1}{j^s}),\Re(s)>0$ *]*

Start with the following very well known definition for the dirichlet eta function:

$$(1-2^{1-s})\zeta(s)=\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j^s}$$

Re-index the summation so we are counting pairs of terms.

$$(1-2^{1-s})\zeta(s)=\sum_{j=1}^{\infty}\frac{1}{(2j-1)^s}-\frac{1}{(2j)^s},\Re(s)>0$$

Change the infinite sum into an explicit limit.

$$(1-2^{1-s})\zeta(s)=\lim_{n\to\infty}\sum_{1\le j\le n}\frac{1}{(2j-1)^s}-\frac{1}{(2j)^s},\Re(s)>0$$

Use the fact that $-\frac{1}{(2j)^s}=\frac{1}{(2j)^s}-\frac{2}{(2j)^s}$ to rewrite our sum as

$$(1-2^{1-s})\zeta(s)=\lim_{n\to\infty}\sum_{1\le j\le n}\frac{1}{(2j-1)^s}+\frac{1}{(2j)^s}-\frac{2}{(2j)^s},\Re(s)>0$$

Hopefully it is clear that $\sum_{1\le j\le n}\frac{1}{(2j-1)^s}+\frac{1}{(2j)^s}$ is the same thing as $\sum_{1\le j\le 2n}\frac{1}{j^s}$. So split our sum into two sums as

$$(1-2^{1-s})\zeta(s)=\lim_{n\to\infty}(\sum_{1\le j\le 2n}\frac{1}{j^s}-\sum_{1\le j\le n}\frac{2}{(2j)^s}),\Re(s)>0$$

The second sum can have a factor of 2^{1-s} extracted, so do that.

$$(1-2^{1-s})\zeta(s)=\lim_{n\to\infty}(\sum_{1\le j\le 2n}\frac{1}{j^s}-2^{1-s}\sum_{1\le j\le n}\frac{1}{j^s}),\Re(s)>0$$

In general, it is the case that $\sum_{j=1}^{n}\frac{1}{j^s}=\sum_{j=1}^{\infty}\frac{1}{j^s}-\frac{1}{(j+n)^s}$, which finally leaves us with

$$\boxed{(1-2^{1-s})\zeta(s)=\lim_{n\to\infty}(\sum_{j=1}^{\infty}(\frac{1}{j^s}-\frac{1}{(j+2n)^s})-2^{1-s}\cdot\sum_{1\le j\le n}\frac{1}{j^s}),\Re(s)>0}$$

...

[Same thing for 3]

$$(1-3^{1-s})\zeta(s)=\sum_{j=1}^{\infty}\frac{1}{(3j-2)^s}+\frac{1}{(3j-1)^s}-\frac{2}{(3j)^s},\Re(s)>0$$

$$(1-3^{1-s})\zeta(s)=\lim_{n\to\infty}\left(\sum_{1\le j\le 3n}\frac{1}{j^s}-3^{1-s}\sum_{1\le j\le n}\frac{1}{j^s}\right),\Re(s)>0$$

$$(1-3^{1-s})\zeta(s)=\lim_{n\to\infty}\left(\sum_{j=1}^{\infty}(j^{-s}-(j+3n)^{-s})-3^{1-s}\cdot\sum_{1\le j\le n}j^{-s}\right),\Re(s)>0$$

Okay. This should be obvious for integers at the very least. And it lines up with the numerators for rational fractions. But make that more explicit though.

[Now I'm showing it can be generalized to some rational fraction x. I want to take its derivative.]

$$(1-x^{1-s})\zeta(s)=\lim_{n\to\infty}\left(\sum_{1\le j\le xn}\frac{1}{j^s}-x^{1-s}\sum_{1\le j\le n}\frac{1}{j^s}\right),\Re(s)>0$$

$$(1-x^{1-s})\zeta(s)=\lim_{n\to\infty}\left(\sum_{j=1}^{\infty}(j^{-s}-(j+x\cdot n)^{-s})-x^{1-s}\cdot\sum_{1\le j\le n}j^{-s}\right),\Re(s)>0$$

$$(s-1)\zeta(s)=\lim_{n\to\infty}\left(n\cdot s\cdot(\zeta(s+1)-\sum_{j=1}^{n}j^{-s-1})-(1-s)\sum_{j=1}^{n}j^{-s}\right),\Re(s)>0$$

Write down multi derivative here

…

$$(1-x^{1-s})\zeta(s)=\lim_{n\to\infty}H_{x\cdot n}^{(s)}-x^{1-s}H_n^{(s)},\Re(s)>0$$

$$(1-x^{1-s})\zeta(s)=\lim_{n\to\infty}H_n^{(s)}-x^{1-s}H_{\frac{n}{x}}^{(s)},\Re(s)>0$$

$$\frac{\partial H_z^{(s)}}{\partial z}=s(\zeta(s+1)-H_z^{(s+1)})$$

$$\frac{\partial^n H_z^{(s)}}{\partial^n z}=(-1)^n\cdot(s)_n(H_z^{(s+n)}-\zeta(s+n))$$

…

$$(1-x^{1-s})\zeta(s)=\lim_{n\to\infty}H_{x\cdot n}^{(s)}-x^{1-s}H_n^{(s)},\Re(s)>0$$

$$H_n^{(s)}=\zeta(s)-\zeta(s,n+1)$$

$$(1-x^{1-s})\zeta(s)=\lim_{n\to\infty}(\zeta(s)-\zeta(s,nx+1))-x^{1-s}\cdot(\zeta(s)-\zeta(s,n+1)),\Re(s)>0$$

$$(1-x^{1-s})\zeta(s)=\lim_{n\to\infty}(1-x^{1-s})\zeta(s)+x^{1-s}\cdot\zeta(s,n+1)-\zeta(s,nx+1),\Re(s)>0$$

So the trick here is to make clear that the sum on the right DOES converge for re(s)>0, even though it individually only converges for re(s)>1

$$(1-x^{1-s})\zeta(s)=\lim_{n\to\infty}(1-x^{1-s})\zeta(s)+\sum_{j=1}^{n}x^{1-s}\cdot(j+n)^{-s}-(j+nx)^{-s},\Re(s)>0$$

If re(s)>0,

$$\lim_{n\to\infty} x^{1-s}\cdot(j+n)^{-s}-(j+n\,x)^{-s}=0$$

because, for re(s)>0,

$$\lim_{n\to\infty} n^{-s}=0$$

…

$$\alpha_{\frac{a}{b}}(n)=b\cdot(\lfloor\frac{n}{b}\rfloor-\lfloor\frac{n-1}{b}\rfloor)-a\cdot(\lfloor\frac{n}{a}\rfloor-\lfloor\frac{n-1}{a}\rfloor)$$

$$(1-(\frac{a}{b})^{1-s})\zeta(s)=\lim_{n\to\infty}\sum_{j=1}^{b\cdot n} b^{-1}\alpha_{\frac{a}{b}}(j)\cdot(\frac{j}{b})^{-s}$$

$$(1-(\frac{a}{b})^{1-s})\zeta(s)=\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=1}^{a\cdot b} b^{-1}\alpha_{\frac{a}{b}}(j)\cdot(\frac{a\cdot b\cdot k+j}{b})^{-s}$$

$$(1-(\frac{a}{b})^{1-s})\zeta(s)=\lim_{n\to\infty} b^{-1}\cdot\sum_{k=1}^{n}(-a\sum_{j=1}^{b}(\frac{a\cdot b\cdot k+a\cdot j}{b})^{-s})+(b\sum_{j=1}^{a}(\frac{a\cdot b\cdot k+b\cdot j}{b})^{-s})$$

$$(1-(\frac{a}{b})^{1-s})\zeta(s)=\lim_{n\to\infty} b^{s-1}\cdot\sum_{k=1}^{n}(-a\sum_{j=1}^{b}(a\cdot b\cdot k+a\cdot j)^{-s})+(b\sum_{j=1}^{a}(a\cdot b\cdot k+b\cdot j)^{-s})$$

$$(1-(\frac{a}{b})^{1-s})\zeta(s)=\lim_{n\to\infty} b^{s-1}\cdot\sum_{k=1}^{n}(-a^{1-s}\sum_{j=1}^{b}(b\cdot k+j)^{-s})+(b^{1-s}\sum_{j=1}^{a}(a\cdot k+j)^{-s})$$

$$(1-(\frac{a}{b})^{1-s})\zeta(s)=\sum_{n=0}^{\infty}(\sum_{j=1}^{a}(a\cdot n+j)^{-s})-((\frac{a}{b})^{1-s}\sum_{j=1}^{b}(b\cdot n+j)^{-s})$$

$$(1-(\frac{a}{b})^{1-s})\zeta(s)=\lim_{t\to\infty}\sum_{n=0}^{t}(\sum_{j=1}^{a}(a\cdot n+j)^{-s})-((\frac{a}{b})^{1-s}\sum_{j=1}^{b}(b\cdot n+j)^{-s})$$

Now reindex:

$$(1-(\frac{a}{b})^{1-s})\zeta(s)=\lim_{n\to\infty}\sum_{1\le j\le a\cdot n} j^{-s}-(\frac{a}{b})^{1-s}\sum_{1\le j\le b\cdot n} j^{-s}$$

$$(1-(\frac{a}{b})^{1-s})\zeta(s)=\lim_{n\to\infty}(\sum_{j=1}^{\infty}(j^{-s}-(j+a\cdot n)^{-s})-(\frac{a}{b})^{1-s}\cdot(\sum_{j=1}^{\infty} j^{-s}-(j+b\cdot n)^{-s})),\ \Re(s)>0$$

But! I want show that this is equal to

$$(1-(\frac{a}{b})^{1-s})\zeta(s)=\lim_{n\to\infty}(\sum_{j=1}^{\infty}(j^{-s}-(j+\frac{a}{b}\cdot n)^{-s})-(\frac{a}{b})^{1-s}\cdot\sum_{1\le j\le n} j^{-s}),\ \Re(s)>0$$

Easily done. Take the second equation, change its limiting variable to m,

$$(1-(\frac{a}{b})^{1-s})\zeta(s)=\lim_{m\to\infty}(\sum_{j=1}^{\infty}(j^{-s}-(j+\frac{a}{b}\cdot m)^{-s})-(\frac{a}{b})^{1-s}\cdot\sum_{1\le j\le m} j^{-s}),\ \Re(s)>0$$

And now suppose that m = n * b

$$(1-(\frac{a}{b})^{1-s})\zeta(s)=\lim_{n\to\infty}(\sum_{j=1}^{\infty}(j^{-s}-(j+\frac{a}{b}\cdot n\cdot b)^{-s})-(\frac{a}{b})^{1-s}\cdot\sum_{1\le j\le n\cdot b}j^{-s}),\Re(s)>0$$

which is, as desired,

$$(1-(\frac{a}{b})^{1-s})\zeta(s)=\lim_{n\to\infty}(\sum_{j=1}^{\infty}(j^{-s}-(j+a\cdot n)^{-s})-(\frac{a}{b})^{1-s}\cdot\sum_{1\le j\le\infty}j^{-s}-(j+b\cdot n)^{-s}),\Re(s)>0$$

…

$$(1-2^{1-s})\zeta(s)=\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j^s},\Re(s)>0$$

$$(1-2^{1-s})\zeta(s)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{(-1)^{j+1}}{j^s},\Re(s)>0$$

Suppose that n is a positive even integer.

$$(1-2^{1-s})\zeta(s)=\lim_{n\to\infty}(\sum_{1\le j\le n}\frac{1}{j^s}-\sum_{1\le 2j\le n}\frac{2}{(2j)^s}),\Re(s)>0$$

$$(1-2^{1-s})\zeta(s)=\lim_{n\to\infty}(\sum_{1\le j\le n}\frac{1}{j^s}-2^{1-s}\sum_{1\le 2j\le n}\frac{1}{j^s}),\Re(s)>0$$

This will be equal to the previous term if n is a positive even integer.

$$(1-2^{1-s})\zeta(s)=\lim_{n\to\infty}(\sum_{j=1}^{\infty}\frac{1}{j^s}-\sum_{j=1}^{\infty}\frac{1}{(j+n)^s}-2^{1-s}(\sum_{j=1}^{\infty}\frac{1}{j^s}-\sum_{j=1}^{\infty}\frac{1}{(j+n/2)^s})),\Re(s)>0$$

$$(1-2^{1-s})\zeta(s)=\lim_{n\to\infty}(\sum_{j=1}^{\infty}(\frac{1}{j^s}-\frac{1}{(j+n)^s})-2^{1-s}(\sum_{j=1}^{\infty}\frac{1}{j^s}-\frac{1}{(j+n/2)^s})),\Re(s)>0$$

Now suppose that n is divisible by some positive real x.

$$(1-x^{1-s})\zeta(s)=\lim_{n\to\infty}(\sum_{j=1}^{\infty}(\frac{1}{j^s}-\frac{1}{(j+n)^s})-x^{1-s}(\sum_{j=1}^{\infty}\frac{1}{j^s}-\frac{1}{(j+n/x)^s})),\Re(s)>0$$

$$(1-x^{1-s})\zeta(s)=\lim_{n\to\infty}(\sum_{j=1}^{\infty}(\frac{1}{j^s}-\frac{1}{(j+n)^s})-\sum_{j=1}^{\infty}\frac{x^{1-s}}{j^s}-\frac{x^{1-s}}{(j+n/x)^s}),\Re(s)>0$$

$$\frac{\partial}{\partial x}(1-x^{1-s})\zeta(s)=\lim_{n\to\infty}(\frac{\partial}{\partial x}\sum_{j=1}^{\infty}(\frac{1}{j^s}-\frac{1}{(j+n)^s})-\sum_{j=1}^{\infty}\frac{x^{1-s}}{j^s}-\frac{x^{1-s}}{(j+n/x)^s}),\Re(s)>0$$

$$(s-1)x^{-s}\zeta(s)=\lim_{n\to\infty}(\frac{\partial}{\partial x}-\sum_{j=1}^{\infty}(\frac{x^{1-s}}{j^s})-(\frac{x^{1-s}}{(j+n/x)^s})),\Re(s)>0$$

$$(s-1)x^{-s}\zeta(s)=(\lim_{n\to\infty}-\sum_{j=1}^{\infty}(\frac{1-s}{jx^s})-(\frac{ns}{(jx+n)^{s+1}}+\frac{1-s}{(jx+n)^s})),\Re(s)>0$$

$$\lim_{x\to1}(s-1)x^{-s}\zeta(s)=\lim_{n\to\infty}(\lim_{x\to1}-\sum_{j=1}^{\infty}(\frac{1-s}{jx^s})-(\frac{ns}{(jx+n)^{s+1}}+\frac{1-s}{(jx+n)^s})),\Re(s)>0$$

$$(s-1)\zeta(s)=\lim_{n\to\infty}\left(-\sum_{j=1}^{\infty}\frac{1-s}{j^s}-\frac{ns}{(j+n)^{s+1}}-\frac{1-s}{(j+n)^s}\right), \Re(s)>0$$

$$(1-s)\zeta(s)=\lim_{n\to\infty}\left(\sum_{j=1}^{\infty}\frac{1-s}{j^s}-\frac{1-s}{(j+n)^s}-\frac{ns}{(j+n)^{s+1}}\right), \Re(s)>0$$

$$(1-s)\zeta(s)=\lim_{n\to\infty}\left((1-s)\sum_{j=1}^{n}\frac{1}{j^s}-s\sum_{j=1}^{\infty}\frac{n}{(j+n)^{s+1}}\right), \Re(s)>0$$

$$(1-s)\zeta(s)=\lim_{n\to\infty}\left((1-s)\sum_{j=1}^{n}\frac{1}{j^s}-s\sum_{j=n+1}^{\infty}\frac{n}{j^{s+1}}\right), \Re(s)>0$$

[This is the formula that results from that derivative]

$$\boxed{\zeta(s)=\lim_{n\to\infty}\left(\sum_{j=1}^{n}\frac{1}{j^s}-\frac{s}{1-s}\cdot n\cdot\zeta(1+s,n+1)\right), \Re(s)>0}$$

$$\boxed{\zeta(s)=\lim_{n\to\infty}\left(\sum_{j=1}^{n}\frac{1}{j^s}-\frac{s}{1-s}\cdot n\cdot\left(\zeta(s+1)-\sum_{j=1}^{n}\frac{1}{j^{s+1}}\right)\right), \Re(s)>0}$$

[And from the other derivative]

$$\zeta(s)=\lim_{n\to\infty}\left(\sum_{j=1}^{n}\frac{1}{j^s}-\sum_{k=1}^{t}(-1)^k\cdot\binom{t}{k}\cdot\left(1+\frac{k}{s-1}\right)\cdot n^k\zeta(s+k,1+n)\right)$$

$$\boxed{\zeta(s)=\lim_{n\to\infty}\left(\sum_{j=1}^{n}\frac{1}{j^s}-\sum_{k=1}^{t}(-1)^k\cdot\binom{t}{k}\cdot\left(1+\frac{k}{s-1}\right)\cdot n^k\left(\zeta(s+k)-\sum_{j=1}^{n}\frac{1}{j^{s+k}}\right)\right), \Re(s)>-t+1}$$

$$0=\lim_{n\to\infty}\left(\sum_{k=0}^{t}(-1)^k\cdot\binom{t}{k}\cdot\left(1+\frac{k}{s-1}\right)\cdot n^k\left(\zeta(s+k)-\sum_{j=1}^{n}\frac{1}{j^{s+k}}\right)\right), \Re(s)>-t+1$$

[Here, as a note, is the euler macluarin summation formula and some variants]

$$\zeta(s)=\lim_{n\to\infty}\left(\sum_{j=1}^{n}j^{-s}-\frac{1}{1-s}\sum_{k=0}^{\lfloor 1-s\rfloor}\binom{1-s}{k}\cdot B_k\cdot n^{s-k+1}\right)\ \ldots?$$

$$\zeta(s)=\lim_{n\to\infty}\left(\sum_{j=1}^{n}j^{-s}-\frac{n^{1-s}}{1-s}-\frac{n^{-s}}{2}+\frac{n^{-s-1}\cdot s}{12}-\frac{n^{-s-3}\cdot s(s+1)(s+2)}{720}+\ldots\right)$$

$$\zeta(s)=\lim_{n\to\infty}\left(\sum_{j=1}^{n}j^{-s}-\frac{n^{1-s}}{1-s}\right), \Re(s)>0$$

$$n^{1-s}\approx s\cdot n\cdot\zeta(1+s,n+1), \Re(s)>0$$

particularly as n gets bigger.

Note: $H_x=\sum_{k=1}\frac{x}{k(x+k)}$

[So now I take the identity from the derivative of the generalized eta function and I try to generalize it.]

Okay. New claim.

$$\boxed{\lim_{n\to\infty}(1-s)\zeta(s,n)+(s-1+x)(n^x)\zeta(s+x,n)=0}$$

which is

$$\boxed{\lim_{n\to\infty}(1-s)(\zeta(s)-\sum_{j=1}^{n}\frac{1}{j^s})+(s-1+x)(n^x)(\zeta(s+x)-\sum_{j=1}^{n}\frac{1}{j^{s+x}})=0}$$

With 2 very special cases:

if x = 1, we get that original form,

$$\boxed{\lim_{n\to\infty}(1-s)(\zeta(s)-\sum_{j=1}^{n}\frac{1}{j^s})+(s)(n)(\zeta(s+1)-\sum_{j=1}^{n}\frac{1}{j^{s+1}})=0}$$

and if x = 1 – 2 s,

$$\boxed{\lim_{n\to\infty}(1-s)(\zeta(s)-\sum_{j=1}^{n}\frac{1}{j^s})+s\cdot(n^{1-2s})(\zeta(1-s)-\sum_{j=1}^{n}\frac{1}{j^{1-s}})=0}$$

which means (plus might be minus in there)

$$\boxed{\lim_{n\to\infty}(1-s)(\zeta(s)-\sum_{j=1}^{n}\frac{1}{j^s})+s\cdot(n^{1-2s})(\zeta(1-s)-\sum_{j=1}^{n}\frac{1}{j^{1-s}})=0}$$

[This is the main idea right here]

So, if zeta(s) = 0, the following must be true:

$$\boxed{\lim_{n\to\infty}(1-s)(\sum_{j=1}^{n}\frac{1}{j^s})+s\cdot n^{1-2s}\cdot(\sum_{j=1}^{n}\frac{1}{j^{1-s}})=0}$$

Also interesting if s is one zeta zero, and x is the difference between it and another zeta zero

AND ALSO

$$\boxed{\lim_{n\to\infty}\frac{\zeta(s,n)}{n^x\cdot\zeta(s+x,n)}=1+\frac{x}{s-1}}$$

$$\frac{\partial}{\partial s}\sum_{j=1}^{n}\frac{1}{j^s}=-\sum_{j=1}\frac{\log j}{j^s}$$

$$\zeta(s)=\lim_{n\to\infty}\left(\sum_{j=1}^{n}\frac{1}{j^s}-\frac{s}{1-s}\cdot n\cdot\left(\zeta(s+1)-\sum_{j=1}^{n}\frac{1}{j^{s+1}}\right)\right), \Re(s)>0$$

$$\boxed{\zeta'(s)=\lim_{n\to\infty}-\sum_{j=1}^{n}\frac{\log j}{j^s}+n\cdot\frac{s}{1-s}\cdot\sum_{j=n+1}^{\infty}\frac{\log j}{j^{s+1}}-\frac{n}{(s-1)^2}\cdot\sum_{j=n+1}^{\infty}\frac{1}{j^{s+1}}, \Re(s)>0}$$

…

$$\boxed{\lim_{n\to\infty}(1-s)\zeta(s,n)+(s-1+x)(n^x)\zeta(s+x,n)=0}$$

If this is true, then the derivative with respect to x needs to be 0.

So what is the derivative here?

$$\frac{\partial}{\partial x}\lim_{n\to\infty}(1-s)\zeta(s,n)+n^x\cdot(s-1+x)\left(\zeta(s+x)-\sum_{j=1}^{n}j^{-s-x}\right)=0$$

$$\lim_{n\to\infty}\frac{\partial}{\partial x}n^x\cdot(s-1+x)\cdot\left(\zeta(s+x)-\sum_{j=1}^{n}j^{-s-x}\right)=0$$

[And now generalize the identity from the previous section even more]

Can generalize to this? Need much better verification / proof.

$$\boxed{\lim_{n\to\infty}(s-1+y)(n^y)(\zeta(s+y)-\sum_{j=1}^{n}\frac{1}{j^{s+y}})-(s-1+x)(n^x)(\zeta(s+x)-\sum_{j=1}^{n}\frac{1}{j^{s+x}})=0}$$

so

$$\boxed{\zeta(s+y)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{j^{s+y}}-\frac{(s-1+x)(n^x)}{(s-1+y)(n^y)}(\zeta(s+x)-\sum_{j=1}^{n}\frac{1}{j^{s+x}})}$$

If y = 0,

$$\boxed{\zeta(s)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{j^s}-\frac{n^x(s-1+x)}{s-1}(\zeta(s+x)-\sum_{j=1}^{n}\frac{1}{j^{s+x}})}$$

Suppose lim x=1-s. Is this right? Is there some way to make sense of this?

$$\boxed{\zeta(s)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{j^s}-\frac{n^{1-s}}{s-1}(\zeta(1)-\sum_{j=1}^{n}\frac{1}{j})}$$

$$\boxed{\zeta(s)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{j^s}+\frac{n^{1-s}}{s-1}\cdot 0\cdot\sum_{j=n+1}^{\infty}\frac{1}{j}}$$

If x = 0

$$\boxed{\zeta(s+y)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{j^{s+y}}-\frac{(s-1)}{(s-1+y)(n^y)}(\zeta(s)-\sum_{j=1}^{n}\frac{1}{j^s})}$$

AND ALSO

$$\boxed{\lim_{n\to\infty}n^{y-x}\cdot\frac{\zeta(s+y,n)}{\zeta(s+x,n)}=\frac{s-1+x}{s-1+y}}$$

…

[And now the identity that leads to all the trig stuff gets introduced.]

$$\boxed{\lim_{n\to\infty}(-\frac{1}{2}-x)(n^{-x})(\zeta(\frac{1}{2}-x)-\sum_{j=1}^{n}\frac{1}{j^{\frac{1}{2}-x}})-(-\frac{1}{2}+x)(n^x)(\zeta(\frac{1}{2}+x)-\sum_{j=1}^{n}\frac{1}{j^{\frac{1}{2}+x}})=0}$$

…

For zeta(1/2-x) (and thus zeta(1/2+x)) to be 0, the following must be true:

$$\boxed{\lim_{n\to\infty}(-\frac{1}{2}-x)(n^{-x})(-\sum_{j=1}^{n}\frac{1}{j^{\frac{1}{2}-x}})-(-\frac{1}{2}+x)(n^x)(-\sum_{j=1}^{n}\frac{1}{j^{\frac{1}{2}+x}})=0}$$

...

$$\lim_{n\to\infty}\left(\sum_{j=1}^{n}\frac{n^{-x}(\frac{1}{2}+x)}{j^{\frac{1}{2}-x}}\right)-\left(\sum_{j=1}^{n}\frac{(\frac{1}{2}-x)n^{x}}{j^{\frac{1}{2}+x}}\right)=0$$

$$\lim_{n\to\infty}\left(\sum_{j=1}^{n}\frac{n^{-x}(\frac{1}{2}+x)}{j^{\frac{1}{2}-x}}-\frac{(\frac{1}{2}-x)n^{x}}{j^{\frac{1}{2}+x}}\right)=0$$

$$\lim_{n\to\infty}\left(\sum_{j=1}^{n}\frac{j^{x}n^{-x}(\frac{1}{2}+x)}{j^{\frac{1}{2}}}-\frac{j^{-x}n^{x}(\frac{1}{2}-x)}{j^{\frac{1}{2}}}\right)=0$$

$$\boxed{\lim_{n\to\infty}\sum_{j=1}^{n}j^{-\frac{1}{2}+x}n^{-x}(\frac{1}{2}+x)-j^{-\frac{1}{2}-x}n^{x}(\frac{1}{2}-x)=0}$$

...

if lim s → 1 and x = (zeta zero - 1)

$$\boxed{\zeta(s)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{j^{s}}-\frac{n^{x}(s-1+x)}{s-1}(\zeta(s+x)-\sum_{j=1}^{n}\frac{1}{j^{s+x}})}$$

$$\boxed{(s-1)\zeta(s)=\lim_{n\to\infty}(s-1)\sum_{j=1}^{n}\frac{1}{j^{s}}-n^{x-1}\cdot(x-1)\cdot(-\sum_{j=1}^{n}\frac{1}{j^{x}})}$$

So, for x to be a zeta zero, the following must be true…?

$$\boxed{1=\lim_{n\to\infty}n^{x-1}\cdot(1-x)\cdot\sum_{j=1}^{n}\frac{1}{j^{x}}}$$

which is this again… hmm.

$$\boxed{\lim_{n\to\infty}\frac{n^{1-x}}{1-x}=\sum_{j=1}^{n}\frac{1}{j^{x}}}$$

…

If y = 0,

$$\boxed{\zeta(s)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{j^{s}}+\frac{n^{x}(s-1+x)}{s-1}(\zeta(s+x)-\sum_{j=1}^{n}\frac{1}{j^{s+x}})}$$

$$(s-1)\zeta(s)=\lim_{n\to\infty}(s-1)\sum_{j=1}^{n}\frac{1}{j^{s}}-n^{x}(s-1+x)(\zeta(s+x)-\sum_{j=1}^{n}\frac{1}{j^{s+x}})$$

Suppose lim x=1-s. Is this right? Is there some way to make sense of this?

$$\zeta(s)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{j^{s}}+\frac{n^{x}(s-1+x)}{s-1}\left(\zeta(s+x)-\sum_{j=1}^{n}\frac{1}{j^{s+x}}\right)$$

$$\dots$$

$$\boxed{\zeta(s)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{j^{s}}-\frac{n^{1-s}}{1-s}}$$

[Here I rederive the euler maclaurin summation formula through brute force]

Here is what I think I'm seeing:

$$(s-1)\left(n^s\cdot\zeta(s)-\sum_{j=1}^{n}\left(\frac{n}{j}\right)^s\right)\approx n-\frac{1-s}{2}$$

$$(s-1)\left(n^s\cdot\zeta(s,n+1)\right)\approx n-\frac{1-s}{2}$$

$$\sum_{j=1}^{n}\frac{1}{j^s}\approx\zeta(s)-\frac{n-(1-s)/2}{(s-1)n^s}$$

Like, very, very close… up to a point. But why? Can it be exact?

…

$$(s-1)n^s\cdot\zeta(s,n+1)$$

$$(s-1)\sum_{j=1}^{\infty}\left(\frac{n}{j+n}\right)^s$$

$$(s-1)\left(n^s\zeta(s)-\sum_{j=1}^{n}\left(\frac{n}{j}\right)^s\right)$$

...

$$\sum_{j=1}^{n}\left(\frac{n}{j}\right)^{-1}=\frac{n+1}{2}$$

$$\sum_{j=1}^{n}\left(\frac{n}{j}\right)^{-2}=\frac{(1+n)(1+2n)}{6n}$$

$$\sum_{j=1}^{n}\left(\frac{n}{j}\right)^{-3}=\frac{(n+1)^2}{4n}$$

etc...

…

Bah.

$$\lim_{n\to\infty}(s-1)n^{s-1}\cdot\zeta(s,n+1)=1$$

$$\boxed{\lim_{n\to\infty}(s-1)n^{s-1}\cdot\left(\zeta(s)-\sum_{j=1}^{n}j^{-s}\right)=1}$$

(for re(s)<1,

$$\lim_{n\to\infty}(s-1)n^{s-1}\cdot\left(C-\sum_{j=1}^{n}j^{-s}\right)=1$$

but not so for re(s)>1)

and for re(s)>0,

$$\boxed{\zeta(s)=\lim_{n\to\infty}\sum_{j=1}^{n} j^{-s}-\frac{n^{1-s}}{1-s}}$$

…

$$\zeta(s)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{j^{s}}-\sum_{k=1}^{t}(-1)^{k}\cdot\binom{t}{k}\cdot\frac{s-1+k}{s-1}\cdot n^{k}\zeta(s+k,1+n)$$

$$\zeta(s)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{j^{s}}-\frac{s}{s-1}\cdot n\cdot(\zeta(1+s)-\sum_{j=1}^{n}\frac{1}{j^{1+s}})$$

$$\boxed{\zeta(s)=\lim_{n\to\infty}\sum_{j=1}^{n} j^{-s}-\frac{n^{1-s}}{1-s}}$$

...

$$\zeta(s)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{j^{s}}-\sum_{k=1}^{t}(-1)^{k}\cdot\binom{t}{k}\cdot\frac{n^{k}}{s-1}\cdot(s-1+k)(\zeta(s+k,n+1))$$

…

Look at this again:

$$\boxed{\lim_{n\to\infty}(s-1)n^{s-1}\cdot(\zeta(s)-\sum_{j=1}^{n} j^{-s})=1}$$

It was trivial to see that in the cases of s integers less than 0, why the limit was 1. But the bernoulli polynomials fail horribly for non integers (to say nothing of s>1 too, or complex s).

This is REALLY REALLY similar to the issues with $x^{s}=\sum_{j=0}^{s}\binom{s}{j}(x-1)^{j}$ only converging when s is an integer < 0. But there was that alternative identity, $x^{z}=\sum_{j=0}^{\infty}\binom{z}{j}(x-1)^{z-j}$, which is valid for any s.

...

$$\zeta(s)=1+\sum_{k=0}^{\infty}\binom{-s}{k}\zeta(s+k)$$

(for what range? There is some fixed limit here, anyway. And yet it seems to work sometimes.)

$$\zeta(s)=1+\sum_{k=0}^{\infty}\binom{-s}{k}\zeta(s+k)$$

…

$$H_{n,s}=1+\sum_{k=0}^{\infty}\binom{-s}{k}H_{n-1,s+k}$$

(Again, for some range.)

$$n^{-s}-1=\sum_{k=1}^{\infty}\binom{-s}{k}H_{n-1,s+k}$$

...

Alright. Let's give this a go. We are looking at something in the ballpark of

$$\sum_{j=1}^{n} j^{-s}=\zeta(s)+\frac{1}{1-s}\sum_{k=0}^{\infty}\binom{1-s}{k}\cdot B_k\cdot n^{s-k+1}$$

$$\zeta(s)=\lim_{n\to\infty}\sum_{j=1}^{n} j^{-s}-\frac{1}{1-s}\sum_{k=0}^{\lfloor 1-s\rfloor}\binom{1-s}{k}\cdot B_k\cdot n^{s-k+1}$$

$$\zeta(s)=\lim_{n\to\infty}\sum_{j=1}^{n} j^{-s}-\frac{n^{1-s}}{1-s}-\frac{n^{-s}}{2}+\frac{n^{-s-1}\cdot s}{12}-\frac{n^{-s-3}\cdot s(s+1)(s+2)}{720}+\ldots$$

But what is the relationship between this and the Euler Maclaurin Summation formula?

…

I MUST be missing something here.

Sigh. And indeed I am. This is known.

[Here I draw out some of the conclusions of that one symmetric identity]

Drawing from

$$\boxed{\lim_{n\to\infty}(s-1+y)(n^y)(\zeta(s+y)-\sum_{j=1}^{n}\frac{1}{j^{s+y}})-(s-1+x)(n^x)(\zeta(s+x)-\sum_{j=1}^{n}\frac{1}{j^{s+x}})=0}$$

...

$$\boxed{\lim_{n\to\infty}(1-s)(\sum_{j=1}^{n}\frac{1}{j^s})+s\cdot n^{1-2s}\cdot(\sum_{j=1}^{n}\frac{1}{j^{1-s}})=0}$$

$$\boxed{\lim_{n\to\infty}(1-s)n^{-\frac{1}{2}+s}(\sum_{j=1}^{n}\frac{1}{j^s})+s\cdot n^{\frac{1}{2}-s}\cdot(\sum_{j=1}^{n}\frac{1}{j^{1-s}})=0}$$

$$\boxed{\lim_{n\to\infty}\sum_{j=1}^{n}j^{-\frac{1}{2}+x}n^{-x}(\frac{1}{2}+x)-j^{-\frac{1}{2}-x}n^{x}(\frac{1}{2}-x)=0}$$

Are these actually interesting? Are they in any sense more interesting than

$$\lim_{n\to\infty}\sum_{j=1}^{n}-1^{j+1}\cdot(\frac{1}{j^s}-\frac{1}{j^{1-s}})=0$$

Maybe yes, in that the alternating series is more difficult to reason about?

…

$$\boxed{\lim_{n\to\infty}\sum_{j=1}^{n}j^{-\frac{1}{2}+x}n^{-x}(\frac{1}{2}+x)-j^{-\frac{1}{2}-x}n^{x}(\frac{1}{2}-x)=0}$$

For this one:

IF re(x)=0, then the real part of this limit is 0. Either the imaginary part converges to 0, or it doesn't converge, and the curve bounding it on either side grows.

If re(x) <> 0, then real part doesn't converge. I don't think the imaginary part converges either, but it's hard to tell.

This can be rewritten as

$$\boxed{\lim_{n\to\infty}(\frac{1}{2}+x)n^{-x}H_n^{(\frac{1}{2}-x)}-(\frac{1}{2}-x)n^{x}H_n^{(\frac{1}{2}+x)}}$$

or

$$\boxed{\lim_{n\to\infty}(1-s)n^{s-\frac{1}{2}}H_n^{(s)}-s\cdot n^{\frac{1}{2}-s}H_n^{(1-s)}}$$

Here is the chain of thought.

Start with this identity which, if it is true, is just true for any zeta(s), for re(s)>0 (?), and s+x > 0, and s+y > 0.

$$\boxed{\lim_{n\to\infty}(s-1+y)(n^{y})(\zeta(s+y)-H_n^{(s+y)})-(s-1+x)(n^{x})(\zeta(s+x)-H_n^{(s+x)})=0}$$

Because of the reflection formula, nontrivial zeta zeros in the critical strip have to come in pairs of (½-x) and (1/2-x) So let's let *s* be ½, and *y* be *-x*. The following is still a true statement for any 0 < re(x) < ½.

$$\boxed{\lim_{n\to\infty}(-\frac{1}{2}-x)(n^{-x})(\zeta(\frac{1}{2}-x)-H_n^{(\frac{1}{2}-x)})-(-\frac{1}{2}+x)(n^{x})(\zeta(\frac{1}{2}+x)-H_n^{(\frac{1}{2}+x)})=0}$$

Now, IF $\zeta(\frac{1}{2}-x)=0$ (and thus $\zeta(\frac{1}{2}+x)=0$), the following will thus HAVE to be true

$$\lim_{n\to\infty}(-\frac{1}{2}-x)(n^{-x})(-H_n^{(\frac{1}{2}-x)})-(-\frac{1}{2}+x)(n^{x})(-H_n^{(\frac{1}{2}+x)})=0$$

which is to say

$$\boxed{\lim_{n\to\infty}(-\frac{1}{2}-x)\cdot n^{-x}\cdot H_n^{(\frac{1}{2}-x)}-(-\frac{1}{2}+x)\cdot n^{x}\cdot H_n^{(\frac{1}{2}+x)}=0}$$

in order for the original identity to remain satisfied. And conversely, if there is no way for this equation to be satisfied with certain values of x, then there is no way for the zeta function to have zeros for those values of x either. It is not the case that if this identity is satisfied for some value of x, the zeta function will have a corresponding zero. It is necessary but not sufficient.

Some observations:

If re(s)=0, then the real part of the limit will always be 0. This makes satisfying the equation much simpler.

This thing is worth graphing.

My intuition is that there is some way these waves are fundamentally in the same phase with each other such that they can never equal 0 at the same time, similar to sin(x)+cos(x) = 0.

So what causes the shapes?

This equation is actually really interesting because the contours bounding the waves are clearly visible regardless of the value of x – and the shapes are visible for both real and imaginary graphs.

IF re(s)=0, then the values of of the imaginary part of the limit seem to be bound by fixed sized curves, no matter how big n gets. It's the same curves.

If re(s) <> 0, the the values of both the imaginary and real parts of the limit seem to be bound by the same curves, but the curves scale out bigger and bigger as n gets bigger.

As re(s) gets bigger, the pinched parts get looser and looser, as well as the bounding curve getting much bigger more generally.

$$\boxed{\lim_{n\to\infty}(-\frac{1}{2}-x)\cdot e^{-x\log n}\cdot H_n^{(\frac{1}{2}-x)}-(-\frac{1}{2}+x)\cdot e^{x\log n}\cdot H_n^{(,\frac{1}{2}+x)}}$$

$$\boxed{\lim_{n\to\infty}(-\frac{1}{2}-x)\cdot e^{-x\log n}\cdot H_n^{(\frac{1}{2}-x)}-(-\frac{1}{2}+x)\cdot e^{x\log n}\cdot H_n^{(\frac{1}{2}+x)}=0}$$

$$\cdots$$

$$\boxed{\lim_{n\to\infty}\sum_{j=1}^{n} j^{-\frac{1}{2}+x}e^{-x\log n}(\frac{1}{2}+x)-j^{-\frac{1}{2}-x}e^{x\log n}(\frac{1}{2}-x)=0}$$

[Now I take a leap and finally express that symmetric identity in terms of trig functions]

So here's an interesting question. For re(x)=0, in this equation,

$$\boxed{\lim_{n\to\infty}(-\frac{1}{2}-x)\cdot n^{-x}\cdot H_n^{(\frac{1}{2}-x)}-(-\frac{1}{2}+x)\cdot n^{x}\cdot H_n^{(\frac{1}{2}+x)}}$$

the harmonic number terms are complex conjugates. They have the same magnitudes and their real parts are identical. Their complex parts have opposite signs.

So what, exactly, makes them = 0?

At the zeroes,

$$(-\frac{1}{2}-x)\cdot e^{-x\log n}\cdot H_n^{(\frac{1}{2}-x)}\approx\sqrt{n}$$

and

$$(-\frac{1}{2}+x)\cdot e^{x\log n}\cdot H_n^{(\frac{1}{2}+x)}\approx\sqrt{n}$$

…

$$H_n^{(s)}=\sum_{j=1}^{n} j^{-s}=\sum_{j=1}^{n} e^{-s\log j}$$

$$H_n^{(\frac{1}{2}+x)}=\sum_{j=1}^{n} j^{-\frac{1}{2}+x}=\sum_{j=1}^{n}\frac{1}{\sqrt{j}}e^{-x\log j}$$

$$H_n^{(\frac{1}{2}-x)}=\sum_{j=1}^{n} j^{-\frac{1}{2}-x}=\sum_{j=1}^{n}\frac{1}{\sqrt{j}}e^{x\log j}$$

$$\frac{1}{2}(H_n^{(\frac{1}{2}+x)}+H_n^{(\frac{1}{2}-x)})=\frac{1}{2}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}(e^{-x\log j}+e^{x\log j})=\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cosh(x\log j)$$

$$\frac{1}{2}(H_n^{(\frac{1}{2}+x)}-H_n^{(\frac{1}{2}-x)})=-\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\sinh(x\log j)$$

$$(s+\frac{1}{2})n^{-s}+(s-\frac{1}{2})n^{s}=2s\cosh(s\cdot\log n)-\sinh(s\cdot\log n)$$

$$\boxed{\lim_{n\to\infty}(-\frac{1}{2}-x)\cdot e^{-x\log n}\cdot(\sum_{j=1}^{n}e^{-(\frac{1}{2}-x)\log j})-(-\frac{1}{2}+x)\cdot e^{x\log n}\cdot(\sum_{j=1}^{n}e^{-(\frac{1}{2}+x)\log j})}$$

$$\boxed{\lim_{n\to\infty}(-\frac{1}{2}-x)\cdot(\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot e^{x(\log j-\log n)})-(-\frac{1}{2}+x)\cdot(\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot e^{x(\log n-\log j)})}$$

$$\boxed{\lim_{n\to\infty}(-\frac{1}{2}-x)\cdot(\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot(\frac{j}{n})^{x})-(-\frac{1}{2}+x)\cdot(\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot(\frac{j}{n})^{-x})}$$

$$\boxed{\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot((-\frac{1}{2}-x)\cdot(\frac{j}{n})^{x}-(-\frac{1}{2}+x)\cdot(\frac{j}{n})^{-x})}$$

$$\boxed{\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot((x+\frac{1}{2})\cdot(\frac{j}{n})^{x}+(x-\frac{1}{2})\cdot(\frac{j}{n})^{-x})}$$

$$\boxed{\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\left(x\left(\left(\frac{j}{n}\right)^{x}+\left(\frac{j}{n}\right)^{-x}\right)+\frac{1}{2}\left(\left(\frac{j}{n}\right)^{x}-\left(\frac{j}{n}\right)^{-x}\right)\right)}$$

…

$$\lim_{n\to\infty}\sum_{j=1}^{n} j^{-\frac{1}{2}}\left(\left(\frac{j}{n}\right)^{x}\left(x+\frac{1}{2}\right)+\left(\frac{j}{x}\right)^{-x}\left(x-\frac{1}{2}\right)\right)=0$$

$$\lim_{n\to\infty}\sum_{j=1}^{n} j^{-\frac{1}{2}}\left(x\left(\left(\frac{j}{n}\right)^{x}+\left(\frac{j}{n}\right)^{-x}\right)+\frac{1}{2}\left(\left(\frac{j}{n}\right)^{x}-\left(\frac{j}{n}\right)^{-x}\right)\right)=0$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\left(x\left(e^{x\cdot\log j/n}+e^{-x\cdot\log j/n}\right)+\frac{1}{2}\left(e^{x\cdot\log j/n}-e^{-x\cdot\log j/n}\right)\right)$$

$$\boxed{\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\left(2x\cosh\left(x\cdot\log\frac{j}{n}\right)+\sinh\left(x\cdot\log\frac{j}{n}\right)\right)}$$

Can swap. If x is a real value of the imaginary part of zeta zeros, then the following converges to 0

$$\boxed{\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\left(2x\cos\left(x\cdot\log\frac{j}{n}\right)+\sin\left(x\cdot\log\frac{j}{n}\right)\right)}$$

…

Okay. Here is an analysis kind of question. Take

$$\sum_{j=1}^{k}\frac{1}{\sqrt{j}}\cdot\sin\left(x\log\frac{j}{n}\right)$$

If you graph it over the range k = 1 to n, it's basically the same a growing wave, and it's essentially the same wave as n gets bigger (other than, kind of, eliminating discretizing artifacts in the very early terms). The main thing that changes is the line that it oscillates around, which drifts around UNLESS x is the imaginary part of zeta zeros.

Actually, it even stays centered on zero if we add an offset into the sine wave like this:

$$\sum_{j=1}^{k}\frac{1}{\sqrt{j}}\cdot\sin\left(x\log\frac{j}{n}+C\right)$$

…

Another one:

Can rewrite earlier identity as

$$\boxed{\begin{aligned}&\lim_{n\to\infty}\left(2x\sin(x\log n)+\cos(x\log n)\right)\cdot\left(\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\sin(x\log j)\right)+\\&\qquad\left(2x\cos(x\log n)-\sin(x\log n)\right)\cdot\left(\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\cos(x\log j)\right)=0\end{aligned}}$$

The key part here where the zeta zeros show up is

$$\sum_{j=1}^{n} \frac{1}{\sqrt{j}} \cdot \sin(x \log j)$$

and

$$\sum_{j=1}^{n} \frac{1}{\sqrt{j}} \cdot \cos(x \log j)$$

These waves are centered on 0 at zeta zeros.

Actually, it's much more interesting than that. For any real C,

$$\boxed{\sum_{j=1}^{n} \frac{1}{\sqrt{j}} \cdot \sin(x \log j + C)}$$

is centered on 0. (Which the cos version is just one special version of)

OR

$$\boxed{\sum_{j=1}^{n} \frac{1}{\sqrt{j}} \cdot \sin(x \log(j \cdot C))}$$

Now. This is REALLY similar to being a fourier series.

…

$$\boxed{\begin{aligned} &\lim_{n\to\infty} (2x\sin(x\log n + c) + \cos(x \log n + c)) \cdot (\sum_{j=1}^{n} \frac{1}{\sqrt{j}} \cdot \sin(x \log j + c)) + \\ &\quad (2x\cos(x \log n + c) - \sin(x \log n + c)) \cdot (\sum_{j=1}^{n} \frac{1}{\sqrt{j}} \cdot \cos(x \log j + c)) = 0 \end{aligned}}$$

at zeta zeros.

…

Actually, looks like this property is shared by

$$\boxed{\sum_{j=1}^{n} \frac{1}{\sqrt{j}} \cdot (-1)^{j} \sin(x \log j + C)}$$

Is this right? And

$\boxed{\sum_{j=1}^{n} \frac{1}{\sqrt{j}} \cdot (-1)^{j} \sin(x \log j + C)}$ where j is only odd values. Or even. And if j is rescaled by a real both inside the sine and outside it.

…

$$\boxed{\lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{\sqrt{j}} \cdot (2x\cos(x \cdot \log \frac{j}{n}) + \sin(x \cdot \log \frac{j}{n}))}$$

$$\boxed{\begin{aligned}&\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\left(2s\cos\left(s\cdot\log\frac{j}{n}\right)\cosh\left(t\cdot\log\frac{j}{n}\right)+2t\sin\left(s\cdot\log\frac{j}{n}\right)\sinh\left(t\log\frac{j}{n}\right)+\sin\left(s\log\frac{j}{n}\right)\cosh\left(t\log\frac{j}{n}\right)\right)\\&\quad+i\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\left(2t\cos\left(s\cdot\log\frac{j}{n}\right)\cosh\left(t\cdot\log\frac{j}{n}\right)-2s\sin\left(s\cdot\log\frac{j}{n}\right)\sinh\left(t\log\frac{j}{n}\right)+\cos\left(s\log\frac{j}{n}\right)\sinh\left(t\log\frac{j}{n}\right)\right)\end{aligned}}$$

…

[trying to see how eta fits in with what I was just doing]

$$\zeta(s)=(1-2^{1-s})^{-1}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j^{s}}$$

$$(1-2^{\frac{1}{2}-ix})\zeta(\frac{1}{2}+ix)=\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j^{\frac{1}{2}+ix}}$$

$$(1-2^{\frac{1}{2}-ix})\cdot\zeta(\frac{1}{2}+ix)=\sum_{j=1}^{\infty}\frac{1}{\sqrt{j}}\frac{(-1)^{j+1}}{j^{ix}}$$

$$(1-2^{\frac{1}{2}+ix})\cdot\zeta(\frac{1}{2}-ix)=\sum_{j=1}^{\infty}\frac{1}{\sqrt{j}}\frac{(-1)^{j+1}}{j^{-ix}}$$

$$(1-2^{\frac{1}{2}-ix})\cdot\zeta(\frac{1}{2}+ix)-(1-2^{\frac{1}{2}+ix})\cdot\zeta(\frac{1}{2}-ix)=\sum_{j=1}^{\infty}\frac{1}{\sqrt{j}}(\frac{(-1)^{j+1}}{j^{ix}}-\frac{(-1)^{j+1}}{j^{-ix}})$$

$$(1-2^{\frac{1}{2}-ix})\cdot\zeta(\frac{1}{2}+ix)-(1-2^{\frac{1}{2}+ix})\cdot\zeta(\frac{1}{2}-ix)=\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{\sqrt{j}}(j^{-ix}-j^{ix})$$

$$(1-2^{\frac{1}{2}-ix})\cdot\zeta(\frac{1}{2}+ix)-(1-2^{\frac{1}{2}+ix})\cdot\zeta(\frac{1}{2}-ix)=\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{\sqrt{j}}(e^{-ix\log j}-e^{ix\log j})$$

$$(1-2^{\frac{1}{2}-ix})\cdot\zeta(\frac{1}{2}+ix)-(1-2^{\frac{1}{2}+ix})\cdot\zeta(\frac{1}{2}-ix)=-2i\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{\sqrt{j}}\sin(x\log j)$$

$$(1-2^{\frac{1}{2}-ix})\cdot\zeta(\frac{1}{2}+ix)+(1-2^{\frac{1}{2}+ix})\cdot\zeta(\frac{1}{2}-ix)=2\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{\sqrt{j}}\cos(x\log j)$$

…

$$(1-2^{\frac{1}{2}-ix})\cdot\zeta(\frac{1}{2}+ix)=\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{\sqrt{j}}\cos(x\log j)-i\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{\sqrt{j}}\sin(x\log j)$$

$$(1-2^{\frac{1}{2}-ix})\cdot\zeta(\frac{1}{2}+ix)=\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{\sqrt{j}}(\cos(x\log j)-i\sin(x\log j))$$

$$\zeta(\frac{1}{2}+ix)=(1-2^{\frac{1}{2}-ix})^{-1}\cdot\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{\sqrt{j}}(\cos(x\log j)-i\sin(x\log j))$$

...

$$\boxed{\lim_{n\to\infty}\left(-\frac{1}{2}-x\right)\cdot n^{-x}\cdot H_n^{\left(\frac{1}{2}-x\right)}-\left(-\frac{1}{2}+x\right)\cdot n^{x}\cdot H_n^{\left(\frac{1}{2}+x\right)}=0}$$

$$\lim_{n\to\infty}\left(-\frac{1}{2}-x\right)\cdot n^{-x}\cdot H_n^{\left(\frac{1}{2}-x\right)}-\left(-\frac{1}{2}+x\right)\cdot n^{x}\cdot H_n^{\left(\frac{1}{2}+x\right)}=0$$

$$\lim_{n\to\infty}\left(x+\frac{1}{2}\right)\cdot n^{-x}\cdot H_n^{\left(\frac{1}{2}-x\right)}+\left(x-\frac{1}{2}\right)\cdot n^{x}\cdot H_n^{\left(\frac{1}{2}+x\right)}=0$$

$$\lim_{n\to\infty}x\cdot\left(n^{-x}\cdot H_n^{\left(\frac{1}{2}-x\right)}+n^{x}\cdot H_n^{\left(\frac{1}{2}+x\right)}\right)+\frac{1}{2}\cdot\left(n^{-x}\cdot H_n^{\left(\frac{1}{2}-x\right)}-n^{x}\cdot H_n^{\left(\frac{1}{2}+x\right)}\right)=0$$

$$\boxed{\lim_{n\to\infty} (-\frac{1}{2}-x)\cdot n^{-x}\cdot H_n^{(\frac{1}{2}-x)} - (-\frac{1}{2}+x)\cdot n^{x}\cdot H_n^{(\frac{1}{2}+x)} = 0}$$

$$\lim_{n\to\infty} (-\frac{1}{2}-x)\cdot n^{-x}\cdot H_n^{(\frac{1}{2}-x)} - (-\frac{1}{2}+x)\cdot n^{x}\cdot H_n^{(\frac{1}{2}+x)} = 0$$

$$\lim_{n\to\infty} (-\frac{1}{2}-x)\cdot n^{-x}\cdot (s+t\,i) - (-\frac{1}{2}+x)\cdot n^{x}\cdot (s-t\,i) = 0$$

$$\dots$$

$$\lim_{n\to\infty} n^{-x}\cdot (x+\frac{1}{2}) H_n^{(\frac{1}{2}-x)} + n^{x}\cdot (x-\frac{1}{2}) H_n^{(\frac{1}{2}+x)} = 0$$

$$\lim_{n\to\infty} n^{-ix}\cdot (\frac{1}{2}+i\,x) H_n^{(\frac{1}{2}-i\,x)} - n^{ix}\cdot (\frac{1}{2}-i\,x) H_n^{(\frac{1}{2}+i\,x)} = 0$$

$$\dots$$

$$\lim_{n\to\infty} \Im(n^{-ix}\cdot (\frac{1}{2}+i\,x) H_n^{(\frac{1}{2}-i\,x)}) = 0$$

$$\lim_{n\to\infty} \Im(\sum_{j=1}^{n} \frac{1}{\sqrt{j}}\cdot (\frac{j}{n})^{ix}) + \Im(2\,x\,i \sum_{j=1}^{n} \frac{1}{\sqrt{j}}\cdot (\frac{j}{n})^{ix}) = 0$$

$$\dots$$

$$\boxed{\lim_{n\to\infty} s\cdot n^{\frac{1}{2}-s}\cdot H_n^{(1-s)} - (1-s)\cdot n^{-\frac{1}{2}+s}\cdot H_n^{(s)} = 0}$$

$$\boxed{\lim_{n\to\infty} s\cdot n^{\frac{1}{2}-s}\cdot \sum_{j=1}^{n} \frac{1}{j^{1-s}} - (1-s)\cdot n^{-\frac{1}{2}+s}\cdot \sum_{j=1}^{n} \frac{1}{j^{s}} = 0}$$

$$\lim_{n\to\infty} \Im((1-s)\cdot n^{-\frac{1}{2}+s}\cdot \sum_{j=1}^{n} \frac{1}{j^{s}}) = 0 \text{ at zeta zeros.}$$

$$\dots$$

$$\sum_{j=1}^{n} \frac{1}{j^{s}}$$

has both real and imaginary parts centered on 0 at the zeta zeros. It looks pretty much like the sine case. Actually, zeta is pretty much the center of the wave.

Ah ha! Here we go!

$$\boxed{\zeta(s)=\lim_{n\to\infty}\frac{s\cdot n^{1-s}H_n^{(1-s)}-(1-s)\cdot n^s\cdot H_n^{(s)}}{s\cdot n^{1-s}\cdot 2^{1-s}\cdot\pi^{-s}\cos(\frac{\pi s}{2})\cdot\Gamma(s)-(1-s)\cdot n^s}}$$

Valid for all s except s=1, (but limits need to be taken at certain special values because of the denominator, namely, ½, 0, and negative odd integers). Also, convergence on the critical line is sort of conditional convergence, because it's broken up intermittently by asymptotes. But it does converge, by some useful metric.

…

$$\lim_{n\to\infty}(-\frac{1}{2}-x)(n^{-x})(\zeta(\frac{1}{2}-x)-H_n^{(\frac{1}{2}-x)})-(-\frac{1}{2}+x)(n^x)(\zeta(\frac{1}{2}+x)-H_n^{(\frac{1}{2}+x)})$$

$$\lim_{n\to\infty}(\zeta(\frac{1}{2}-x)-H_n^{(\frac{1}{2}-x)})-\frac{(-\frac{1}{2}+x)(n^x)}{(-\frac{1}{2}-x)(n^{-x})}(\zeta(\frac{1}{2}+x)-H_n^{(\frac{1}{2}+x)})$$

$$\lim_{n\to\infty}\zeta(\frac{1}{2}-x)-H_n^{(\frac{1}{2}-x)}-n^{2x}(\frac{2}{2x+1}-1)(\zeta(\frac{1}{2}+x)-H_n^{(\frac{1}{2}+x)})$$

...

$$\zeta(s)=2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)$$

…

$$\boxed{\zeta(s)=\lim_{n\to\infty}\frac{1}{1-2^{1-s}\cdot n^{1-2s}\cdot\frac{s}{1-s}\cdot\pi^{-s}\cos(\frac{\pi s}{2})\cdot\Gamma(s)}\cdot H_n^{(s)}-\frac{1}{n^{2s-1}\cdot\frac{1-s}{s}-2^{1-s}\cdot\pi^{-s}\cos(\frac{\pi s}{2})\cdot\Gamma(s)}\cdot H_n^{(1-s)}}$$

…

$$\zeta(s)=\lim_{n\to\infty}\frac{1}{1-2^{1-s}\cdot n^{1-2s}\cdot\frac{s}{1-s}\cdot\pi^{-s}\cos(\frac{\pi s}{2})\cdot\Gamma(s)}\cdot H_n^{(s)}-\frac{1}{n^{2s-1}\cdot\frac{1-s}{s}-2^{1-s}\cdot\pi^{-s}\cos(\frac{\pi s}{2})\cdot\Gamma(s)}\cdot H_n^{(1-s)}$$

$$a=2^{1-s}\cdot\pi^{-s}\cos(\frac{\pi s}{2})\cdot\Gamma(s)$$
$$b=\frac{n^{2s}}{n}\frac{1-s}{s}$$
$$H_n^{(s)}=c+d\,i$$
$$H_n^{(1-s)}=c-d\,i$$

$$\zeta(s)=\lim_{n\to\infty}\frac{1}{1-n^{1-2s}\cdot\frac{s}{1-s}\cdot a}\cdot H_n^{(s)}-\frac{1}{n^{2s-1}\cdot\frac{1-s}{s}-a}\cdot H_n^{(1-s)}$$

$$\zeta(s)=\lim_{n\to\infty}\frac{c+d\,i}{1-\frac{a}{b}}-\frac{c-d\,i}{b-a}=0$$

$$\zeta(s)=\lim_{n\to\infty}\frac{b\cdot c+b\,d\,i}{b-a}-\frac{c-d\,i}{b-a}=0$$

$$\zeta(s)=\lim_{n\to\infty}\frac{b\,c+b\,d\,i-c+d\,i}{b-a}=0$$

$$\zeta(s)=\lim_{n\to\infty}\frac{(b-1)c+(b+1)d\,i}{b-a}=0$$

…

$$\sum_{j=1}^{n} j^{-(s+ti)}=\sum_{j=1}^{n} j^{-s}\cdot j^{-ti}=\sum_{j=1}^{n} j^{-s}\cdot e^{-it\log j}=\sum_{j=1}^{n} j^{-s}\cdot(\cos(t\log j)-i\sin(t\log j))=\sum_{j=1}^{n}\frac{\cos(t\log j)}{j^s}-i\sum_{j=1}^{n}\frac{\sin(t\log j)}{j^s}$$

…

$\boxed{\lim_{n\to\infty} s\,n^{1-s}H_n^{(1-s)}-(1-s)n^s H_n^{(s)}=\Im(s)}$ if at zeta zero. Doesn't converge otherwise… Right?

And $\boxed{\lim_{n\to\infty}\frac{n^{1-s}}{1-s}H_n^{(1-s)}-\frac{n^s}{s}H_n^{(s)}=\Im(\frac{1}{s})}$

And $\boxed{\lim_{n\to\infty}\frac{\sqrt{s}}{\sqrt{1-s}}\cdot n^{1-s}H_n^{(1-s)}-\frac{\sqrt{1-s}}{\sqrt{s}}\cdot n^s H_n^{(s)}\approx i}$ (converges to values EXTREMELY close. How close?)

compare to

$\boxed{s\,n^{1-s}\int_0^n x^{s-1}\,dx-(1-s)n^s\int_0^n x^{-s}\,dx=0}$ (if re(s) is between 0 and 1)

…

If x is the imaginary part of zeta zeros, then

$$\boxed{\lim_{n\to\infty} n^{\frac{1}{2}-ix}(x-\frac{i}{2})H_n^{(\frac{1}{2}-ix)}+n^{\frac{1}{2}+ix}(x+\frac{i}{2})H_n^{(\frac{1}{2}+ix)}=x}$$

$$\boxed{\lim_{n\to\infty}\sum_{j=1}^{n}\sqrt{\frac{n}{j}}\cdot(2\,x\cos(x\cdot\log\frac{n}{j})-\sin(x\cdot\log\frac{n}{j}))=x}$$

$$\boxed{\begin{aligned}\lim_{n\to\infty}\ &\sqrt{n}(2\,x\sin(x\log n)+\cos(x\log n))\cdot(\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\sin(x\log j))+\\ &\sqrt{n}(2\,x\cos(x\log n)-\sin(x\log n))\cdot(\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\cos(x\log j))=x\end{aligned}}$$

…

At zeta zeros,

$$\boxed{\lim_{n\to\infty}(1-s)n^s H_n^{(s)}=n+\frac{1}{2}\cdot(1-s)}$$

$$\boxed{\lim_{n\to\infty}\sqrt{\frac{1-s}{s}}\,n^s H_n^{(s)}=(s(1-s))^{-\frac{1}{2}}(n+\frac{1-s}{2})=\frac{n}{\sqrt{s(1-s)}}+\frac{1}{2}\sqrt{\frac{s}{1-s}}}$$

$$\boxed{\lim_{n\to\infty}\frac{n^s}{s}H_n^{(s)}=\frac{n}{s(1-s)}+\frac{1}{2}\cdot\frac{1}{s}}$$

$$\boxed{\lim_{n\to\infty}H_n^{(s)}=\frac{n^{1-s}}{1-s}+\frac{n^{-s}}{2}}$$

Hrm. Right back to the Euler Maclaurin Summation Formula it would seem.

$$\sum_{j=1}^{n} j^{-s}=\zeta(s)+\frac{1}{1-s}\sum_{k=0}^{\infty}\binom{1-s}{k}\cdot B_k\cdot n^{1-s-k}$$

$$\sum_{j=1}^{n} j^{-s}=\lim_{n\to\infty}\zeta(s)+\frac{n^{1-s}}{1-s}+\frac{n^{-s}}{2}-\frac{n^{-s-1}\cdot s}{12}+\frac{n^{-s-3}\cdot s(s+1)(s+2)}{720}+\ldots$$

$$\boxed{\lim_{n\to\infty}(1-s)n^s H_n^{(s)}=n+\frac{1}{2}\cdot(1-s)}$$

$$\lim_{n\to\infty}H_n^{(s)}=\frac{n^{1-s}}{1-s}+\frac{n^{-s}}{2}-\frac{n^{-s-1}\cdot s}{12}+\frac{n^{-s-3}\cdot s(s+1)(s+2)}{720}+\ldots$$

...

$$\alpha(s)=(-\frac{1}{2}\cdot\pi^{-\frac{s}{2}}\cdot\Gamma(\frac{s}{2})\cdot s\cdot(1-s))^{-1} \qquad f(s)=\frac{n^s}{s}$$

$$t(s)=(\alpha(s)-\alpha(1-s)\cdot\frac{f(1-s)}{f(s)})^{-1}\cdot\sum_{j=1}^{n} j^{-s}$$

$$\xi(s)=\lim_{n\to\infty}t(s)+t(1-s)$$

$$\zeta(s)=\lim_{n\to\infty}\alpha(s)\cdot(t(s)+t(1-s))$$

$$\xi(s)=\frac{f(s)\cdot H_n^{(s)}-f(1-s)\cdot H_n^{(1-s)}}{\alpha(s)f(s)-\alpha(1-s)\cdot f(1-s)}$$

$$\zeta(s)=\frac{f(s)\cdot H_n^{(s)}-f(1-s)\cdot H_n^{(1-s)}}{f(s)-\frac{\alpha(1-s)}{\alpha(s)}\cdot f(1-s)}$$

$$\zeta(s)=\lim_{n\to\infty}\frac{(1-s)\cdot n^s\cdot H_n^{(s)}-s\cdot n^{1-s}H_n^{(1-s)}}{(1-s)\cdot n^s-s\cdot n^{1-s}\cdot 2^{1-s}\cdot\pi^{-s}\cos(\frac{\pi s}{2})\cdot\Gamma(s)}$$

$$\Gamma(s)\Gamma(s+\frac{1}{2})=2^{1-2s}\cdot\pi^{\frac{1}{2}}\cdot\Gamma(2s)$$

$$\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})=2^{1-s}\cdot\pi^{\frac{1}{2}}\cdot\Gamma(s)$$

$$\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})=2^{1-s}\cdot\pi^{\frac{1}{2}}\cdot\Gamma(s)$$

$$\frac{1}{2}\pi^{-s/2}s(s-1)\Gamma(\frac{s}{2})\zeta(s)$$

$$\zeta(s)=\lim_{n\to\infty}\frac{s\cdot n^{1-s}H_n^{(1-s)}-(1-s)\cdot n^s\cdot H_n^{(s)}}{s\cdot n^{1-s}\cdot 2^{1-s}\cdot\pi^{-s}\cos(\frac{\pi s}{2})\cdot\Gamma(s)-(1-s)\cdot n^s}$$

$$\zeta(s)=\lim_{n\to\infty}\frac{s\cdot n^{1-s}H_n^{(1-s)}-(1-s)\cdot n^s\cdot H_n^{(s)}}{s\cdot n^{1-s}\cdot 2^{1-s}\cdot\pi^{-s}\cos(\frac{\pi s}{2})\cdot\Gamma(s)-(1-s)\cdot n^s}$$

$$\Gamma(\frac{s}{2})=\frac{2^{1-s}\cdot\pi^{\frac{1}{2}}\cdot\Gamma(s)}{\Gamma(\frac{s}{2}+\frac{1}{2})}$$

$$\frac{\Gamma(\frac{s}{2}+\frac{1}{2})}{2^{1-s}\cdot\pi^{\frac{1}{2}}\cdot\Gamma(s)}=\frac{1}{\Gamma(\frac{s}{2})}$$

$$\Gamma(-\frac{s}{2})\Gamma(-\frac{s}{2}+\frac{1}{2})=2^{1+s}\cdot\pi^{\frac{1}{2}}\cdot\Gamma(-s)$$

$$\frac{\Gamma(-\frac{s}{2})}{2^{1+s}\cdot\pi^{\frac{1}{2}}\Gamma(-s)}=\frac{1}{\Gamma(-\frac{s}{2}+\frac{1}{2})}$$

$$\boxed{\lim_{n\to\infty}(1-s)(\zeta(s)-\sum_{j=1}^{n}\frac{1}{j^{s}})+(s-1+x)(n^{x})(\zeta(s+x)-\sum_{j=1}^{n}\frac{1}{j^{s+x}})=0}$$

$$\zeta(s)=\sum_{j=1}^{n} j^{-s}+\frac{n^{1-s}}{s-1}-s\int_{n^+}^{\infty}\frac{\{t\}}{t^{s+1}}dt$$

$$\zeta(s)-\sum_{j=1}^{n} j^{-s}=\frac{n^{1-s}}{s-1}-s\int_{n^+}^{\infty}\frac{\{t\}}{t^{s+1}}dt$$

$$\lim_{n\to\infty}(1-s)(\frac{n^{1-s}}{s-1}-s\int_{n^+}^{\infty}\frac{\{t\}}{t^{s+1}}dt)+(s-1+x)(n^{x})(\frac{n^{1-s-x}}{s+x-1}-(s+x)\int_{n^+}^{\infty}\frac{\{t\}}{t^{s+x+1}}dt)=0$$

$$\lim_{n\to\infty}-n^{1-s}-s(1-s)\int_{n^+}^{\infty}\frac{\{t\}}{t^{s+1}}dt+n^{1-s}-(s-1+x)n^{x}(s+x)\int_{n^+}^{\infty}\frac{\{t\}}{t^{s+x+1}}dt=0$$

$$\lim_{n\to\infty}-s(1-s)\int_{n^+}^{\infty}\frac{\{t\}}{t^{s+1}}dt+-(s-1+x)n^{x}(s+x)\int_{n^+}^{\infty}\frac{\{t\}}{t^{s+x+1}}dt=0$$

$$\boxed{\lim_{n\to\infty}s(1-s)\int_{n^+}^{\infty}\frac{(1-(1+\frac{x}{s})(1+\frac{x}{s-1})(\frac{n}{t})^{x})\{t\}}{t^{s+1}}dt=0}$$

…….

$$\lim_{n\to\infty}\zeta(s,n+1)\cdot(\frac{n^{1-s}}{s-1})^{-1}=\zeta(s,n+1)\cdot(1-s)\cdot n^{s-1}=1$$

But not that interesting, because everything converges to 0 at n^(s-1). The interesting range is n^(s-1/2) through n^(s).

$$\lim_{n\to\infty}(s-1+y)(n^{y})(\zeta(s+y)-\sum_{j=1}^{n}\frac{1}{j^{s+y}})-(s-1+x)(n^{x})(\zeta(s+x)-\sum_{j=1}^{n}\frac{1}{j^{s+x}})=0$$

$$\lim_{n\to\infty}(s-1)(\zeta(s)-\sum_{j=1}^{n}\frac{1}{j^{s}})-n^{x}(s-1+x)(\zeta(s+x)-\sum_{j=1}^{n}\frac{1}{j^{s+x}})=0$$

$$\lim_{n\to\infty}(s-1)(\zeta(s)-\sum_{j=1}^{n}\frac{1}{j^{s}})+n^{x}(s-1+x)\sum_{j=1}^{n}\frac{1}{j^{s+x}}-n^{x}(s-1+x)\zeta(s+x)=0$$

$$\zeta(s+x)=\lim_{n\to\infty}\frac{1}{n^{x}(s-1+x)}((s-1)(\zeta(s)-\sum_{j=1}^{n}\frac{1}{j^{s}})+n^{x}(s-1+x)\sum_{j=1}^{n}\frac{1}{j^{s+x}})$$

$$\zeta(s+x)=\lim_{n\to\infty}\frac{1}{n^{x}(s-1+x)}(1+n^{x}(s-1+x)\sum_{j=1}^{n}\frac{1}{j^{s+x}})$$

$$\zeta(1+x)=\lim_{n\to\infty}\frac{1}{n^{x}\cdot x}(1+n^{x}\cdot x\cdot\sum_{j=1}^{n}\frac{1}{j^{1+x}})$$

$$\zeta(1+(s-1))=\lim_{n\to\infty}\frac{1}{n^{(s-1)}\cdot(s-1)}(1+n^{(s-1)}\cdot(s-1)\cdot\sum_{j=1}^{n}\frac{1}{j^{1+(s-1)}})$$

$$\zeta(s)=\lim_{n\to\infty}\frac{n^{1-s}}{s-1}(1+n^{s-1}\cdot(s-1)\cdot\sum_{j=1}^{n}\frac{1}{j^{s}})$$

$$\zeta(s)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{j^{s}}-\frac{n^{1-s}}{1-s}$$

...

$$f(s)=\frac{(1-s)n^s}{\left(\frac{1}{2}(1-s)\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\right)^{-1}\cdot n^{1-s}-\left(\frac{1}{2}s\cdot\pi^{-\frac{s}{2}}\cdot\Gamma\left(\frac{s}{2}\right)\right)^{-1}\cdot n^s}\cdot H_n^{(s)}$$

$$\xi(s)=f(s)+f(1-s)$$

$$\zeta(s)=\left(\frac{1}{2}\cdot s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\right)^{-1}\cdot(f(s)+f(1-s))$$

………..

$$\alpha(s)=\left(-\frac{1}{2}\cdot\pi^{-\frac{s}{2}}\cdot\Gamma\left(\frac{s}{2}\right)\cdot s\cdot(1-s)\right)^{-1}$$

$$f(s)=\frac{\frac{n^s}{s}}{\frac{\alpha(s)\cdot n^s}{s}-\frac{\alpha(1-s)\cdot n^{1-s}}{1-s}}\cdot H_n^{(s)}$$

$$\xi(s)=f(s)+f(1-s)$$

$$\zeta(s)=\alpha(s)\cdot(f(s)+f(1-s))=\alpha(s)\cdot\xi(s)$$

$$\zeta(1-s)=\alpha(1-s)\cdot(f(s)+f(1-s))=\alpha(1-s)\cdot\xi(s)$$

$$\zeta(1-s)=\frac{\alpha(1-s)}{\alpha(s)}\cdot\zeta(s)$$

$$\xi(s)=\lim_{n\to\infty}\frac{\frac{n^s\cdot H_n^{(s)}}{s}-\frac{n^{1-s}\cdot H_n^{(1-s)}}{1-s}}{\alpha(s)\frac{n^s}{s}-\alpha(1-s)\cdot\frac{n^{1-s}}{1-s}}$$

$$\zeta(s)=\lim_{n\to\infty}\frac{\frac{n^s\cdot H_n^{(s)}}{s}-\frac{n^{1-s}\cdot H_n^{(1-s)}}{1-s}}{\frac{n^s}{s}-\frac{\alpha(1-s)}{\alpha(s)}\cdot\frac{n^{1-s}}{1-s}}$$

$$\zeta(s)=\lim_{n\to\infty}\frac{\frac{n^s}{s}\cdot H_n^{(s)}-\frac{n^{1-s}}{1-s}\cdot H_n^{(1-s)}}{\frac{n^s}{s}-\pi^{\frac{1}{2}-s}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\cdot\frac{n^{1-s}}{1-s}}$$

$$\zeta(s)=\lim_{n\to\infty}\left(1-\frac{\alpha(1-s)}{\alpha(s)}\cdot n^{1-2s}\cdot\frac{s}{1-s}\right)^{-1}H_n^{(s)}-\left(n^{2s-1}\cdot\frac{1-s}{s}-\frac{\alpha(1-s)}{\alpha(s)}\right)^{-1}\cdot H_n^{(1-s)}$$

$$\zeta(s)=\lim_{n\to\infty} \left(1-\pi^{\frac{1}{2}-s}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}\cdot n^{1-2s}\cdot\frac{s}{1-s}\right)^{-1} H_n^{(s)}-\left(n^{2s-1}\cdot\frac{1-s}{s}-\pi^{s-\frac{1}{2}s}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\right)^{-1}\cdot H_n^{(1-s)}$$

...

$$\alpha(s)=(-\frac{1}{2}\cdot\pi^{-\frac{s}{2}}\cdot\Gamma(\frac{s}{2})\cdot s\cdot(1-s))^{-1} \qquad f(s)=\frac{n^s}{s}$$

$$t(s)=(\alpha(s)-\alpha(1-s)\cdot\frac{f(1-s)}{f(s)})^{-1}\cdot\sum_{j=1}^{n} j^{-s}$$

$$\xi(s)=\lim_{n\to\infty} t(s)+t(1-s)$$

$$\zeta(s)=\lim_{n\to\infty} \alpha(s)\cdot(t(s)+t(1-s))$$

$$\xi(s)=\frac{f(s)\cdot H_n^{(s)}-f(1-s)\cdot H_n^{(1-s)}}{\alpha(s)f(s)-\alpha(1-s)\cdot f(1-s)}$$

$$\zeta(s)=\frac{f(s)\cdot H_n^{(s)}-f(1-s)\cdot H_n^{(1-s)}}{f(s)-\frac{\alpha(1-s)}{\alpha(s)}\cdot f(1-s)}$$

...

$$\zeta(s)=\lim_{n\to\infty}\frac{\frac{n^s}{s}\cdot H_n^{(s)}+\frac{n^{1-s}}{1-s}\cdot H_n^{(1-s)}}{\frac{n^s}{s}-\pi^{\frac{1}{2}-s}\frac{\frac{1-s}{2}(\frac{s}{2})!}{\frac{s}{2}(\frac{1-s}{2})!}\cdot\frac{n^{1-s}}{1-s}}$$

$$\zeta(s)=\lim_{n\to\infty}\frac{\frac{n^s}{s}\cdot H_n^{(s)}+\frac{n^{1-s}}{1-s}\cdot H_n^{(1-s)}}{\frac{n^s}{s}-\pi^{\frac{1}{2}-s}\frac{(1-s)(\frac{s}{2})!}{s(\frac{1-s}{2})!}\cdot\frac{n^{1-s}}{1-s}}$$

$$\zeta(s)=\lim_{n\to\infty}\frac{n^s\cdot H_n^{(s)}+n^{1-s}\cdot\frac{s}{1-s}\cdot H_n^{(1-s)}}{n^s-\pi^{\frac{1}{2}-s}\frac{(\frac{s}{2})!}{(\frac{1-s}{2})!}\cdot n^{1-s}}$$

...

$$\zeta(s)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\left(1-\pi^{\frac{1}{2}-s}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}\cdot n^{1-2s}\cdot\frac{s}{1-s}\right)\cdot j^{s}}-\frac{1}{\left(n^{2s-1}\cdot\frac{1-s}{s}-\pi^{s-\frac{1}{2}s}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\right)j^{1-s}}$$

$$\ldots$$

$$\sum_{j=1}^{n} j^{-s}=\zeta(s)+\frac{1}{1-s}\sum_{k=0}^{\infty}\binom{1-s}{k}\cdot B_k\cdot n^{1-s-k}$$

$$\lim_{n\to\infty} H_n^{(s)}=\frac{n^{1-s}}{1-s}+\frac{n^{-s}}{2}-\frac{n^{-s-1}\cdot s}{12}+\frac{n^{-s-3}\cdot s(s+1)(s+2)}{720}+\ldots$$

$$\ldots$$

$$\zeta(s)=\lim_{n\to\infty}\frac{\frac{n^{s}}{s}\cdot H_n^{(s)}-\frac{n^{1-s}}{1-s}\cdot H_n^{(1-s)}}{\frac{n^{s}}{s}-\pi^{\frac{1}{2}-s}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}\cdot\frac{n^{1-s}}{1-s}}$$

$$f(s)=\left(\frac{1}{1-s}\sum_{k=0}^{\infty}\binom{1-s}{k}\cdot B_k\cdot n^{1-s-k}\right)^{-1}\ !!!$$

$$f(s)=\frac{1}{\frac{1}{1-s}\sum_{k=0}^{\infty}\binom{1-s}{k}\cdot B_k\cdot n^{1-s-k}}$$

$$\zeta(s)=\lim_{n\to\infty}\frac{f(s)H_n^{(s)}-f(1-s)\cdot H_n^{(1-s)}}{f(s)-\pi^{\frac{1}{2}-s}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}\cdot f(1-s)}$$

$$\boxed{\zeta(s)=\lim_{n\to\infty}\left(1-\pi^{\frac{1}{2}-s}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}\cdot\frac{f(1-s)}{f(s)}\right)^{-1}H_n^{(s)}-\left(\frac{f(s)}{f(1-s)}-\pi^{\frac{1}{2}-s}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}\right)^{-1}H_n^{(1-s)}}$$

$$g(s)=\frac{f(s)}{f(1-s)}=\frac{\frac{1}{s}\sum_{k=0}^{\infty}\binom{s}{k}\cdot B_k\cdot n^{s-k}}{\frac{1}{1-s}\sum_{k=0}^{\infty}\binom{1-s}{k}\cdot B_k\cdot n^{1-s-k}}$$

$$g(s)=\frac{1-s}{s}\cdot\frac{\sum_{k=0}^{\infty}\binom{s}{k}\cdot B_k\cdot n^{s-k}}{\sum_{k=0}^{\infty}\binom{1-s}{k}\cdot B_k\cdot n^{1-s-k}}$$

$$\boxed{\zeta(s)=\lim_{n\to\infty}\left(1-\pi^{\frac{1}{2}-s}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}\cdot g(1-s)\right)^{-1}H_n^{(s)}-\left(g(s)-\pi^{\frac{1}{2}-s}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}\right)^{-1}H_n^{(1-s)}}$$

$$\xi(s)=\lim_{n\to\infty}\frac{f(s)\cdot H_n^{(s)}-f(1-s)\cdot H_n^{(1-s)}}{\alpha(s)\,f(s)-\alpha(1-s)\cdot f(1-s)}\qquad \alpha(s)=-\frac{2\cdot\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})}$$

$$\boxed{\xi(s)=\lim_{n\to\infty}s\cdot(1-s)\cdot\left(\left(\frac{2\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})}-\frac{2\pi^{\frac{1-s}{2}}}{\Gamma(\frac{1-s}{2})}\cdot g(1-s)\right)^{-1}H_n^{(s)}-\left(\frac{2\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})}g(s)-\frac{2\pi^{\frac{1-s}{2}}}{\Gamma(\frac{1-s}{2})}\right)^{-1}H_n^{(1-s)}\right)}$$

...

$$g(s)=\frac{1-s}{s}\cdot\frac{\sum_{k=0}^{\infty}\binom{s}{k}\cdot B_k\cdot n^{s-k}}{\sum_{k=0}^{\infty}\binom{1-s}{k}\cdot B_k\cdot n^{1-s-k}}$$

For any n:

$$\boxed{\zeta(s)=\left(1-\pi^{\frac{1}{2}-s}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}\cdot g(1-s)\right)^{-1}\zeta(s)-\left(g(s)-\pi^{\frac{1}{2}-s}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}\right)^{-1}\zeta(1-s)}$$

$$\zeta(s)=\lim_{n\to\infty}\left(1-\pi^{\frac{1}{2}-s}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}\cdot\frac{f(1-s)}{f(s)}\right)^{-1}\zeta(s)-\left(\frac{f(s)}{f(1-s)}-\pi^{\frac{1}{2}-s}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}\right)^{-1}\zeta(1-s)$$

$$\zeta(s)=\lim_{n\to\infty}\left(1-\frac{b(s)}{a(s)}\right)^{-1}\zeta(s)-(a(s)-b(s))^{-1}\zeta(1-s)$$

$$(a(s)-b(s))^{-1}\zeta(1-s)=\lim_{n\to\infty}\left(1-\frac{b(s)}{a(s)}\right)^{-1}\zeta(s)-\zeta(s)$$

$$(a(s)-b(s))^{-1}\zeta(1-s)=\lim_{n\to\infty}\left(\left(1-\frac{b(s)}{a(s)}\right)^{-1}-1\right)\zeta(s)$$

$$\frac{(a(s)-b(s))^{-1}}{\left(\left(1-\frac{b(s)}{a(s)}\right)^{-1}-1\right)}\zeta(1-s)=\lim_{n\to\infty}\zeta(s)$$

Meanwhile, if n exceeds a threshold proportional to Im(s),

$$\boxed{0=\left(1-\pi^{\frac{1}{2}-s}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}\cdot g(1-s)\right)^{-1}\zeta(s,n)-\left(g(s)-\pi^{\frac{1}{2}-s}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}\right)^{-1}\zeta(1-s,n)}$$

$$0=\left(1-\frac{b(s)}{a(s)}\right)^{-1}\zeta(s,n)-(a(s)-b(s))^{-1}\zeta(1-s,n)$$

$$\boxed{\lim_{n\to\infty}(1-s)n^{s-\frac{1}{2}}H_n^{(s)}-s\cdot n^{\frac{1}{2}-s}H_n^{(1-s)}}$$

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}((1-s)\cdot\sum_{j=1}^{n}(\frac{n}{j})^{s}-s\cdot\sum_{j=1}^{n}(\frac{n}{j})^{1-s})$$

$$\boxed{\lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{j=1}^{n}(1-s)\cdot(\frac{n}{j})^{s}-s\cdot(\frac{n}{j})^{1-s}}$$

$$\boxed{\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot((1-2s)\cosh((\frac{1}{2}-s)\cdot\log\frac{j}{n})+\sinh((\frac{1}{2}-s)\cdot\log\frac{j}{n}))}$$

This one is particularly interesting BECAUSE it converges to 0 at zeta 0s, and doesn't converge anywhere else, at all.

So go back and look at this:

$$\zeta(s)=\lim_{n\to\infty}\frac{s\cdot n^{1-s}H_n^{(1-s)}-(1-s)\cdot n^{s}\cdot H_n^{(s)}}{s\cdot n^{1-s}\cdot 2^{1-s}\cdot\pi^{-s}\cos(\frac{\pi s}{2})\cdot\Gamma(s)-(1-s)\cdot n^{s}}$$

$$\zeta(s)=\lim_{n\to\infty}\frac{(1-s)\cdot n^{s-1}\cdot H_n^{(s)}-s\cdot n^{-s}H_n^{(1-s)}}{(1-s)\cdot n^{s-1}-s\cdot n^{-s}\cdot 2^{1-s}\cdot\pi^{-s}\cos(\frac{\pi s}{2})\cdot\Gamma(s)}$$

$$\zeta(s)=\lim_{n\to\infty}\frac{(1-s)\cdot n^{s-\frac{1}{2}}\cdot H_n^{(s)}-s\cdot n^{\frac{1}{2}-s}H_n^{(1-s)}}{(1-s)\cdot n^{s-\frac{1}{2}}-s\cdot n^{\frac{1}{2}-s}\cdot(2^{1-s}\cdot\pi^{-s}\cos(\frac{\pi s}{2})\cdot\Gamma(s))}$$

$$\alpha(s)=(-\frac{1}{2}\cdot\pi^{-\frac{s}{2}}\cdot\Gamma(\frac{s}{2})\cdot s\cdot(1-s))^{-1}$$

$$\xi(s)=\lim_{n\to\infty}\frac{(1-s)n^{s-\frac{1}{2}}\cdot H_n^{(s)}-s\,n^{\frac{1}{2}-s}\cdot H_n^{(1-s)}}{(1-s)n^{s-\frac{1}{2}}\alpha(s)-s\,n^{\frac{1}{2}-s}\cdot\alpha(1-s)}$$

Actually, it looks like the above could be multiplied through by n^0 through < n^1/2 and would still retain the property of zeta zeros converging to 0 and everything else diverging. But the above is particularly neat because values on the line .5 + t i keep stay bounded. Everything else grows without limit. At n^1/2, zeta zeros converge to their imaginary portions, more or less. Go past n^1/2, and nothing converges.

BUT BUT BUT

What happens when we bring in

$$f(s)=\frac{n^{\frac{1}{2}}}{\frac{1}{1-s}\sum_{k=0}^{\infty}\binom{1-s}{k}\cdot B_k\cdot n^{1-s-k}}$$

$$\zeta(s)=\lim_{n\to\infty}\frac{f(s)\cdot H_n^{(s)}-f(1-s)H_n^{(1-s)}}{f(s)-f(1-s)\cdot(2^{1-s}\cdot\pi^{-s}\cos(\frac{\pi s}{2})\cdot\Gamma(s))}$$

In this case, what is noteworthy is that $f(s)\cdot H_n^{(s)}$ has an imaginary component of 0 at the zeta zeros. So it would appear that the n^(1/2) component could maybe get as big as desired.

$$f(s)=\frac{(1-s)^a}{s^b}\cdot(\sum_{k=0}^{d}\binom{1-s}{k}\cdot B_k\cdot n^{c-s-k})^{-1}$$

$$\zeta(s)=\lim_{n\to\infty}\frac{f(s)\cdot H_n^{(s)}-f(1-s)H_n^{(1-s)}}{f(s)-f(1-s)\cdot(2^{1-s}\cdot\pi^{-s}\cos(\frac{\pi s}{2})\cdot\Gamma(s))}$$

$$\alpha(s)=(-\frac{1}{2}\cdot\pi^{-\frac{s}{2}}\cdot\Gamma(\frac{s}{2})\cdot s\cdot(1-s))^{-1}$$

$$\zeta(s)=\lim_{n\to\infty}\frac{f(s)\cdot H_n^{(s)}-f(1-s)H_n^{(1-s)}}{f(s)-f(1-s)\frac{\alpha(1-s)}{\alpha(s)}}$$

$$\xi(s)=\lim_{n\to\infty}\frac{f(s)\cdot H_n^{(s)}-f(1-s)H_n^{(1-s)}}{f(s)\alpha(s)-f(1-s)\alpha(1-s)}$$

$$\zeta(s)=\lim_{n\to\infty}(1-\frac{\alpha(1-s)}{\alpha(s)}\cdot\frac{f(1-s)}{f(s)})^{-1}\cdot\sum_{j=1}^{n}\frac{1}{j^s}-(\frac{f(s)}{f(1-s)}-\frac{\alpha(1-s)}{\alpha(s)})^{-1}\cdot\sum_{j=1}^{n}\frac{1}{j^{1-s}}$$

…

$$\alpha(s)=2^{1-s}\cdot\pi^{-s}\cos(\frac{\pi s}{2})\cdot\Gamma(s)$$

$$f(s)=\frac{(1-s)^a}{s^b}\cdot\frac{\sum_{k=0}^{d}\binom{s}{k}\cdot B_k\cdot n^{c+s-k}}{\sum_{k=0}^{d}\binom{1-s}{k}\cdot B_k\cdot n^{c+1-s-k}}$$

Say, for example, for a=1,b=1,c=1/2,d=0

$$f(s)=\frac{1-s}{s}\cdot n^{1-2s}$$

$$\zeta(s)=\lim_{n\to\infty}\frac{1}{1-\frac{\alpha(s)}{f(s)}}\cdot\sum_{j=1}^{n}\frac{1}{j^s}-\frac{1}{f(s)-\alpha(s)}\cdot\sum_{j=1}^{n}\frac{1}{j^{1-s}}$$

$$\alpha(s)=\left(-\frac{1}{2}\cdot\pi^{-\frac{s}{2}}\cdot\Gamma\left(\frac{s}{2}\right)\cdot s\cdot(1-s)\right)^{-1} \qquad f(s)=\frac{n^s}{s}$$

$$t(s)=\left(\alpha(s)-\alpha(1-s)\cdot\frac{f(1-s)}{f(s)}\right)^{-1}\cdot H_n^{(s)}$$

$$t(s)=\frac{f(s)H_n^{(s)}}{f(s)\alpha(s)-\alpha(1-s)\cdot f(1-s)}$$

$$\xi(s)=\lim_{n\to\infty}\frac{f(s)H_n^{(s)}-f(1-s)H_n^{(1-s)}}{f(s)\alpha(s)-\alpha(1-s)\cdot f(1-s)}$$

$$\xi(s)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{f(s)j^{-s}-f(1-s)j^{s-1}}{f(s)\alpha(s)-\alpha(1-s)\cdot f(1-s)}$$

$$\xi(s)=\lim_{n\to\infty} t(s)+t(1-s)$$

$$\zeta(s)=\lim_{n\to\infty}\alpha(s)\cdot(t(s)+t(1-s))$$

$$\ldots$$

$$\lim_{n\to\infty} t(s)+t(1-s)=0$$

$$\lim_{n\to\infty} t(s)=-t(1-s)$$

$$\lim_{n\to\infty}\left(\alpha(s)-\alpha(1-s)\cdot\frac{f(1-s)}{f(s)}\right)^{-1}\cdot H_n^{(s)}=-\left(\alpha(1-s)-\alpha(s)\cdot\frac{f(s)}{f(1-s)}\right)^{-1}\cdot H_n^{(1-s)}$$

$$\lim_{n\to\infty}\frac{1}{\alpha(s)-\alpha(1-s)\cdot\frac{f(1-s)}{f(s)}}\cdot H_n^{(s)}=-\frac{1}{\alpha(1-s)-\alpha(s)\cdot\frac{f(s)}{f(1-s)}}\cdot H_n^{(1-s)}$$

$$\lim_{n\to\infty}-\frac{\alpha(1-s)-\alpha(s)\cdot\frac{f(s)}{f(1-s)}}{\alpha(s)-\alpha(1-s)\cdot\frac{f(1-s)}{f(s)}}\cdot H_n^{(s)}=H_n^{(1-s)}$$

$$\lim_{n\to\infty}-\frac{\frac{\alpha(1-s)\cdot f(1-s)}{f(1-s)}-\frac{\alpha(s)\cdot f(s)}{f(1-s)}}{\frac{\alpha(s)\cdot f(s)}{f(s)}-\frac{\alpha(1-s)\cdot f(1-s)}{f(s)}}\cdot H_n^{(s)}=H_n^{(1-s)}$$

$$\lim_{n\to\infty}-\frac{f(s)}{f(1-s)}\cdot\frac{\alpha(1-s)\cdot f(1-s)-\alpha(s)f(s)}{\alpha(s)\cdot f(s)-\alpha(1-s)\cdot f(1-s)}\cdot H_n^{(s)}=H_n^{(1-s)}$$

$$\lim_{n\to\infty}\frac{f(s)}{f(1-s)}\cdot\frac{\alpha(s)\cdot f(s)-\alpha(1-s)f(1-s)}{\alpha(s)\cdot f(s)-\alpha(1-s)\cdot f(1-s)}\cdot H_n^{(s)}=H_n^{(1-s)}$$

$$\lim_{n\to\infty}\frac{f(s)}{f(1-s)}\cdot H_n^{(s)}=H_n^{(1-s)}$$

NEW! Improved! Exciting!

$$\zeta^*(s)=\lim_{n\to\infty}\frac{f(s)\cdot H_n^{(s)}-f(1-s)H_n^{(1-s)}}{f(s)-f(1-s)}$$ If re(s)> .5, otherwise = zeta(1-s)

and an inverse,

$$\zeta^{-*}(s)=\lim_{n\to\infty}\frac{f(s)\cdot H_n^{(s)}-f(1-s)H_n^{(1-s)}}{f(s)\frac{\alpha(s)}{\alpha(s-1)}-f(1-s)\frac{\alpha(1-s)}{\alpha(s)}}$$ If re(s)< .5, otherwise = zeta(1-s)

...

$$\zeta(s)=\lim_{n\to\infty}\frac{f(s)\cdot H_n^{(s)}-f(1-s)H_n^{(1-s)}}{f(s)-f(1-s)\frac{\alpha(1-s)}{\alpha(s)}}$$

$$\zeta(1-s)=\frac{\alpha(1-s)}{\alpha(s)}\cdot\zeta(s)$$

…

$$\zeta^*(s)=\lim_{n\to\infty}\frac{f(s)\cdot H_n^{(s)}-f(1-s)H_n^{(1-s)}}{f(s)-f(1-s)}$$

$$\zeta^*(s)=\lim_{n\to\infty}\frac{(1-s)n^s\cdot H_n^{(s)}-s\,n^{1-s}H_n^{(1-s)}}{(1-s)n^s-s\,n^{1-s}}$$

$$\zeta^*(s)=\lim_{n\to\infty}\frac{(1-s)n^s\cdot H_n^{(s)}-s\,n^{1-s}H_n^{(1-s)}}{(1-s)n^s-s\,n^{1-s}}$$

$$\zeta^*(\frac{1}{2}-s)=\lim_{n\to\infty}\frac{(s+\frac{1}{2})n^{\frac{1}{2}-s}\cdot H_n^{(\frac{1}{2}-s)}+(s-\frac{1}{2})n^{\frac{1}{2}+s}H_n^{(1-(\frac{1}{2}-s))}}{(s+\frac{1}{2})n^{\frac{1}{2}-s}+(s-\frac{1}{2})n^{\frac{1}{2}+s}}$$

$$\zeta^*(\frac{1}{2}-s)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{(s+\frac{1}{2})n^{\frac{1}{2}-s}\cdot j^{s-\frac{1}{2}}+(s-\frac{1}{2})n^{\frac{1}{2}+s}j^{-s-\frac{1}{2}}}{(s+\frac{1}{2})n^{\frac{1}{2}-s}+(s-\frac{1}{2})n^{\frac{1}{2}+s}}$$

$$\zeta^*(\frac{1}{2}-s)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\frac{(s+\frac{1}{2})(\frac{n}{j})^{-s}+(s-\frac{1}{2})(\frac{n}{j})^{s}}{(s+\frac{1}{2})n^{-s}+(s-\frac{1}{2})n^{s}}$$

$$\boxed{\zeta(\frac{1}{2}+s)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\frac{2s\cosh(s\cdot\log\frac{n}{j})-\sinh(s\cdot\log\frac{n}{j})}{2s\cosh(s\cdot\log n)-\sinh(s\cdot\log n)}}$$ for re(s) > 0

$$\zeta(\frac{1}{2}+t\,i)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\frac{2\,t\cos(t\cdot\log\frac{n}{j})-\sin(t\cdot\log\frac{n}{j})}{2\,t\cos(t\cdot\log n)-\sin(t\cdot\log n)}\text{ for im(t) > 0}$$

$$\zeta(\frac{1}{2}+t\,i)=\lim_{n\to\infty}\frac{(2\,t\sin(t\log n)+\cos(t\log n))}{2\,t\cos(t\cdot\log n)-\sin(t\cdot\log n)}\cdot(\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\sin(t\log j))+ \frac{(2\,t\cos(t\log n)-\sin(t\log n))}{2\,t\cos(t\cdot\log n)-\sin(t\cdot\log n)}\cdot(\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\cos(t\log j))=0\quad\text{im(t) > 0}$$

$$\zeta(\frac{1}{2}+t\,i)=\lim_{n\to\infty}\frac{(2\,t\sin(t\log n)+\cos(t\log n))}{2\,t\cos(t\cdot\log n)-\sin(t\cdot\log n)}\cdot(\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\sin(t\log j))+\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\cos(t\log j)=0\quad\text{im(t) > 0}$$

$$\boxed{\zeta(s+\frac{1}{2})=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\frac{s\cosh(s\cdot\log\frac{n}{j})-\frac{1}{2}\sinh(s\cdot\log\frac{n}{j})}{s\cosh(s\cdot\log n+s\log\pi+\log\Gamma(\frac{1}{4}-\frac{s}{2})-\log\Gamma(\frac{1}{4}+\frac{s}{2}))-\frac{1}{2}\sinh(s\cdot\log n+s\log\pi+\log\Gamma(\frac{1}{4}-\frac{s}{2})-\log\Gamma(\frac{1}{4}+\frac{s}{2}))}}$$

for re(s) < 0

…

$$\boxed{\zeta(s+\frac{1}{2})=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\frac{s\cosh(s\cdot\log\frac{n}{j})-\frac{1}{2}\sinh(s\cdot\log\frac{n}{j})}{s\cosh(s\cdot\log n)-\frac{1}{2}\sinh(s\cdot\log n)}}$$

for re(s) > 0

…

$$\zeta^*(s)=\lim_{n\to\infty}\frac{(1-s)n^s\cdot H_n^{(s)}-s\,n^{1-s}H_n^{(1-s)}}{(1-s)n^s-s\,n^{1-s}}$$

$$\zeta^*(s)=\lim_{n\to\infty}\frac{1}{1-\frac{s}{1-s}n^{1-2s}}\cdot H_n^{(s)}-\frac{1}{\frac{1-s}{s}n^{2s-1}-1}H_n^{(1-s)}$$

$$\zeta^*(s)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{1-\frac{s}{1-s}n^{1-2s}}\cdot j^{-s}-\frac{1}{\frac{1-s}{s}n^{2s-1}-1}j^{s-1}$$

…

Incorrect negative sign in here somewhere :/

$$\zeta(\frac{1}{2}+t\,i)=\lim_{n\to\infty}\frac{(2\,t\sin(t\log n)+\cos(t\log n))}{2\,t\cos(t\cdot\log n)-\sin(t\cdot\log n)}\cdot(\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\sin(t\log j))+\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\cos(t\log j)=0\quad\text{im(t) > 0}$$

$$\lim_{n\to\infty}(2\,t\cos(t\cdot\log n)-\sin(t\cdot\log n))\,\zeta(\frac{1}{2}+t\,i)= (2\,t\sin(t\log n)+\cos(t\log n))\cdot(\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\sin(t\log j))+ (2\,t\cos(t\cdot\log n)-\sin(t\cdot\log n))\cdot\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\cos(t\log j)=0$$

for im(t) > 0

$$\zeta\left(\frac{1}{2}+s\right)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\cosh(s\log j)+\frac{\left(-s\sinh(s\log n)+\frac{1}{2}\cosh(s\log n)\right)}{s\cosh(s\cdot\log n)-\frac{1}{2}\sinh(s\cdot\log n)}\cdot\left(\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\sinh(s\log j)\right)\ \text{re(s)} > 0$$

$$\boxed{\zeta\left(\frac{1}{2}+t\cdot i\right)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\cos(t\log j)+\frac{t\sin(t\log n)+\frac{1}{2}\cos(t\log n)}{t\cos(t\cdot\log n)-\frac{1}{2}\sin(t\cdot\log n)}\cdot\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\sin(t\log j)}\ \text{re(t)} > 0$$

$$\zeta\left(\frac{1}{2}+t\cdot i\right)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\left(\cos(t\log j)+\frac{t\sin(t\log n)+\frac{1}{2}\cos(t\log n)}{t\cos(t\cdot\log n)-\frac{1}{2}\sin(t\cdot\log n)}\cdot\sin(t\log j)\right)\ \text{re(t)} > 0$$

$$\boxed{\zeta\left(\frac{1}{2}+t\cdot i\right)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\left(\cos(t\log j)+\tan\left(t\log n+\cot^{-1}(2x)\right)\cdot\sin(t\log j)\right)}\ \text{re(t)} > 0$$

And at the imaginary parts of zeta zeros

$$\boxed{\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\left(\cos\left(t\log n+\cot^{-1}(2t)\right)\cos(t\log j)+\sin\left(t\log n+\cot^{-1}(2t)\right)\cdot\sin(t\log j)\right)=0}$$

$$\boxed{\lim_{n\to\infty}\cos\left(t\log n+\cot^{-1}(2t)\right)\cdot\sum_{j=1}^{n}\frac{\cos(t\log j)}{\sqrt{j}}+\sin\left(t\log n+\cot^{-1}(2t)\right)\cdot\sum_{j=1}^{n}\frac{\sin(t\log j)}{\sqrt{j}}=0}$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\left(\cos\left(t\log n+\cot^{-1}(2t)\right)\cos(t\log j)+\sin\left(t\log n+\cot^{-1}(2t)\right)\cdot\sin(t\log j)\right)=0$$

$$\boxed{\lim_{n\to\infty}\sum_{j=1}^{n}\frac{\cos\left(t\log\frac{n}{j}+\cot^{-1}(2t)\right)}{\sqrt{j}}=0}$$

$$\boxed{\zeta\left(\frac{1}{2}+t\cdot i\right)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\frac{\cos\left(t\log\frac{n}{j}+\cot^{-1}(2t)\right)}{\cos\left(t\log n+\cot^{-1}(2t)\right)}}\ \text{re(t)} > 0$$

$$\boxed{\zeta\left(\frac{1}{2}+t\cdot i\right)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\frac{\cos\left(t\log\frac{n}{j}+\cot^{-1}(2t)\right)}{\cos\left(t\log n+\cot^{-1}(2t)\right)+i\log\left(\pi^{is}\cdot\frac{\Gamma\left(\frac{1}{4}-\frac{is}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{is}{2}\right)}\right)}}\ \text{re(t)} < 0$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}\frac{\cos\left(t\log\frac{n}{j}+\cot^{-1}(2t)\right)}{\sqrt{j}}=0$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\cos\left(t\log\frac{n}{j}+\cot^{-1}(2t)\right)=0$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}\left(\frac{n}{j}\right)^{\frac{1}{2}}\cdot\cos\left(t\log\frac{n}{j}+\cot^{-1}(2t)\right)\approx\frac{1}{2}$$

$$\cdots$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}\left(\frac{n}{j}\right)^{\frac{1}{2}}\cdot\frac{1}{2}\left(e^{i\left(t\log\frac{n}{j}+\cot^{-1}(2t)\right)}+e^{-i\left(t\log\frac{n}{j}+\cot^{-1}(2t)\right)}\right)\approx\frac{1}{2}$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}\left(\frac{n}{j}\right)^{\frac{1}{2}}\cdot\frac{1}{2}\left(e^{i\left(t\log\frac{n}{j}\right)}e^{\left(i\cot^{-1}(2t)\right)}+e^{-i\left(t\log\frac{n}{j}\right)}e^{\left(-i\cot^{-1}(2t)\right)}\right)\approx\frac{1}{2}$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}\left(\frac{n}{j}\right)^{\frac{1}{2}}\cdot\frac{1}{2}\left(\left(\frac{n}{j}\right)^{it}e^{\left(i\cot^{-1}(2t)\right)}+\left(\frac{n}{j}\right)^{-it}e^{\left(-i\cot^{-1}(2t)\right)}\right)\approx\frac{1}{2}$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}\left(\frac{n}{j}\right)^{\frac{1}{2}}\cdot\frac{1}{2}\left(\left(\frac{n}{j}\right)^{it}e^{\left(i\cot^{-1}(2t)\right)}\right)+\sum_{j=1}^{n}\left(\frac{n}{j}\right)^{\frac{1}{2}}\cdot\frac{1}{2}\left(\left(\frac{n}{j}\right)^{-it}e^{\left(-i\cot^{-1}(2t)\right)}\right)\approx\frac{1}{2}$$

$$\lim_{n\to\infty}\frac{e^{i\cot^{-1}(2t)}}{2}n^{\frac{1}{2}+it}\sum_{j=1}^{n}j^{-\frac{1}{2}-it}+\frac{e^{\left(-i\cot^{-1}(2t)\right)}}{2}n^{\frac{1}{2}-it}\sum_{j=1}^{n}j^{-\frac{1}{2}+it}\approx\frac{1}{2}$$

$$\lim_{n\to\infty}\frac{e^{i\cot^{-1}(2t)}}{2}n^{\frac{1}{2}+it}H_n^{\left(\frac{1}{2}+it\right)}+\frac{e^{\left(-i\cot^{-1}(2t)\right)}}{2}n^{\frac{1}{2}-it}H_n^{\left(\frac{1}{2}-it\right)}\approx\frac{1}{2}$$

$$\zeta(s)=\lim_{n\to\infty}\frac{s\cdot n^{\frac{1}{2}-s}H_n^{\left(\frac{1}{2}-s\right)}-(1-s)\cdot n^{s-\frac{1}{2}}\cdot H_n^{\left(s-\frac{1}{2}\right)}}{s\cdot n^{\frac{1}{2}-s}\cdot 2^{1-s}\cdot\pi^{-s}\cos\left(\frac{\pi s}{2}\right)\cdot\Gamma(s)-(1-s)\cdot n^{s-\frac{1}{2}}}$$

How this actually works:

$$\lim_{n\to\infty}\frac{f(s)\cdot H_n^{(s)}-f(1-s)H_n^{(1-s)}}{a(s)f(s)-b(1-s)f(1-s)}=\begin{cases}\frac{\zeta(s)}{a(s)}\text{ if }(\Re(s)>\frac{1}{2})\\ \frac{\zeta(1-s)}{b(1-s)}\text{ if }(\Re(s)>\frac{1}{2})\\ \text{oscillating between these values if }\Re(s)=\frac{1}{2}\end{cases}$$

$$g(s)=\lim_{n\to\infty}\frac{f(s)\cdot H_n^{(s)}-f(1-s)H_n^{(1-s)}}{f(s)\cdot a(s)-f(1-s)\cdot b(s)}$$

$$g(s)=\lim_{n\to\infty}\frac{f(s)\cdot(\zeta(s)-\zeta(s,n+1))-f(1-s)(\zeta(1-s)-\zeta(1-s,n+1))}{f(s)\cdot a(s)-f(1-s)\cdot b(s)}$$

$$g(s)=\lim_{n\to\infty}\frac{f(s)\cdot\zeta(s)-f(1-s)\zeta(1-s)}{f(s)\cdot a(s)-f(1-s)\cdot b(s)}-\frac{f(s)\cdot\zeta(s,n+1)-f(1-s)\zeta(1-s,n+1)}{f(s)\cdot a(s)-f(1-s)\cdot b(s)}$$

$$g(s)=\lim_{n\to\infty}\frac{f(s)\cdot\zeta(s)-f(1-s)\zeta(1-s)}{f(s)\cdot a(s)-f(1-s)\cdot b(s)}$$

…

$$g(n,s,t)=\lim_{n\to\infty}\frac{f(s)\cdot H_n^{(s)}-f(t)H_n^{(t)}}{f(s)-f(t)}$$

If re(s) > re(t)

or even, if re(s) > re(t),

$$g(n,s,t)=\lim_{n\to\infty}\frac{f(s)\cdot H_n^{(s)}-f(t)H_n^{(t)}}{f(s)}$$

$$g(n,s,t)=\lim_{n\to\infty}H_n^{(s)}-\frac{f(t)}{f(s)}H_n^{(t)}$$

$$g(n,s,t)=\lim_{n\to\infty}\frac{(1-s)n^s\cdot H_n^{(s)}-(1-t)n^t H_n^{(t)}}{(1-s)n^s-(1-t)n^t}$$

$$g(n,s,t)=\lim_{n\to\infty}\frac{(1-s)n^s\cdot H_n^{(s)}}{(1-s)n^s-(1-t)n^t}-\frac{(1-t)n^t H_n^{(t)}}{(1-s)n^s-(1-t)n^t}$$

$$g(n,s,t)=\lim_{n\to\infty}\frac{1}{1-\frac{1-t}{1-s}n^{t-s}}H_n^{(s)}-\frac{1}{\frac{1-s}{1-t}n^{s-t}-1}H_n^{(t)}$$

$$g(n,s,t)=\begin{cases}\frac{\zeta(s)}{a(s)}\text{ if }\Re(s)>\Re(t)\\ \frac{\zeta(t)}{b(t)}\text{ if }\Re(t)>\Re(s)\\ \text{oscillating between these values if }\Re(s)=\Re(t)\end{cases}$$

But either re(s) or re(t) needs to be > 0.

…

$$\zeta(s)=\lim_{n\to\infty}\frac{f(s)\cdot H_n^{(s)}-f(1-s)H_n^{(1-s)}}{(1-s)n^{\frac{1}{2}+s}}$$

$$\zeta(\frac{1}{2}+s)=\lim_{n\to\infty}\frac{n^{-s}}{s-\frac{1}{2}}\cdot\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot(2s\cosh(s\cdot\log\frac{n}{j})-\sinh(s\cdot\log\frac{n}{j}))\text{ for re(s) > 0}$$

….

$$\lim_{n\to\infty}\sum_{j=1}^{n}(\frac{1}{2}i+x)\cdot(\frac{n}{j})^{\frac{1}{2}+xi}-(\frac{1}{2}i-x)\cdot(\frac{n}{j})^{\frac{1}{2}-xi}$$

$$\lim_{n\to\infty}(\frac{1}{2}i+x)\cdot n^{\frac{1}{2}+ix}\cdot H_n^{(\frac{1}{2}+ix)}-(\frac{1}{2}i-x)\cdot n^{\frac{1}{2}-ix}\cdot H_n^{(\frac{1}{2}-ix)}$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}\sqrt{\frac{n}{j}}\cdot(2x\cos(x\cdot\log\frac{n}{j})-\sin(x\cdot\log\frac{n}{j}))=x$$

$$\lim_{n\to\infty}\sqrt{n}(2x\sin(x\log n)+\cos(x\log n))\cdot(\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\sin(x\log j))+$$
$$\sqrt{n}(2x\cos(x\log n)-\sin(x\log n))\cdot(\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\cos(x\log j))$$

…

$$\zeta(s)=\lim_{n\to\infty}\frac{(1-s)\cdot n^{s}\cdot H_n^{(s)}-s\cdot n^{1-s}H_n^{(1-s)}}{(1-s)\cdot n^{s}-s\cdot n^{1-s}\cdot 2^{1-s}\cdot\pi^{-s}\cos(\frac{\pi s}{2})\cdot\Gamma(s)}$$

$$\zeta(\frac{1}{2}+xi)=\lim_{n\to\infty}\frac{(1-(\frac{1}{2}+xi))\cdot n^{(\frac{1}{2}+xi)}\cdot H_n^{((\frac{1}{2}+xi))}-(\frac{1}{2}+xi)\cdot n^{1-(\frac{1}{2}+xi)}H_n^{(1-(\frac{1}{2}+xi))}}{(1-(\frac{1}{2}+xi))\cdot n^{(\frac{1}{2}+xi)}-(\frac{1}{2}+xi)\cdot n^{1-(\frac{1}{2}+xi)}\cdot 2^{1-(\frac{1}{2}+xi)}\cdot\pi^{-(\frac{1}{2}+xi)}\cos(\frac{\pi(\frac{1}{2}+xi)}{2})\cdot\Gamma((\frac{1}{2}+xi))}$$

$$\zeta(\frac{1}{2}+xi)=\lim_{n\to\infty}\frac{(\frac{1}{2}-xi)\cdot n^{(\frac{1}{2}+xi)}\cdot H_n^{(\frac{1}{2}+xi)}-(\frac{1}{2}+xi)\cdot n^{\frac{1}{2}-xi}H_n^{(\frac{1}{2}-xi)}}{(\frac{1}{2}-xi)\cdot n^{(\frac{1}{2}+xi)}-(\frac{1}{2}+xi)\cdot n^{\frac{1}{2}-xi}\cdot 2^{\frac{1}{2}-xi}\cdot\pi^{-\frac{1}{2}-xi}\cos(\frac{\pi}{4}+\frac{\pi xi}{2})\cdot\Gamma(\frac{1}{2}+xi)}$$

sb

```
$$f(n,x)=\sum_{j = 1}^n ({1 \over 2} i + x) \cdot ({n \over j})^{{1 \over 2} + x i }-({1 \over 2} i - x)\cdot ({n \over j})^{{1 \over 2} - x i}
$$
```

asb

$$f(n,x) = \sqrt n(2 x \sin(x \log n)+\cos(x \log n))\sum_{j=1}^n {{1 \over \sqrt j}\cdot \sin x \log j)} \\ +\sqrt n(2 x \cos x \log n)-\sin x \log n))\sum_{j=1}^n {{1 \over \sqrt j}\cdot \cos(x \log j)}$$

asb

$$\lim_{n \rightarrow \infty} f(n,s + t i)$$

...

$$\zeta((x-{1 \over 2})i) = \lim_{n \rightarrow \infty} {f(n,x)\over{}}$$

…

$$\lim_{n\to\infty}(2x\sin(x\log n+c)+\cos(x\log n+c))\cdot(\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\sin(x\log j+c))+$$
$$(2x\cos(x\log n+c)-\sin(x\log n+c))\cdot(\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\cos(x\log j+c))=0$$

$$\lim_{n\to\infty}\frac{(2x\sin(x\log n+c)+\cos(x\log n+c))}{(2x\cos(x\log n+c)-\sin(x\log n+c))}\cdot(\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\sin(x\log j+c))+\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\cos(x\log j+c)$$

…

$$\lim_{n\to\infty}\sum_{j=1}^{n}(1-s)n^{s}\cdot j^{-s}-s\,n^{1-s}j^{s-1}$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}(1-s)(\frac{n}{j})^{s}-s(\frac{n}{j})^{1-s}$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}(1-s)n^{s}\cdot(\frac{n}{j})^{s}-s(\frac{n}{h})^{1-s}$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}(\frac{1}{2}-s)n^{(s-\frac{1}{2})}\cdot(\frac{n}{j})^{s-\frac{1}{2}}-(s-\frac{1}{2})(\frac{n}{j})^{\frac{1}{2}-s}$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}(\frac{\frac{1}{2}-s}{\frac{1}{2}+s})^{\frac{1}{2}}(\frac{n}{j})^{\frac{1}{2}+s}-(\frac{\frac{1}{2}-s}{\frac{1}{2}+s})^{-\frac{1}{2}}(\frac{n}{j})^{\frac{1}{2}-s}$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}(\frac{n}{j})^{\frac{1}{2}}((\frac{\frac{1}{2}-s}{\frac{1}{2}+s})^{\frac{1}{2}}(\frac{n}{j})^{s}-(\frac{\frac{1}{2}-s}{\frac{1}{2}+s})^{-\frac{1}{2}}(\frac{n}{j})^{-s})$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}(\frac{n}{j})^{\frac{1}{2}}((\frac{\frac{1}{2}-s}{\frac{1}{2}+s})^{\frac{1}{2}}(\frac{n}{j})^{s}-((\frac{\frac{1}{2}-s}{\frac{1}{2}+s})^{\frac{1}{2}}(\frac{n}{j})^{s})^{-1})$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}\left(\frac{n}{j}\right)^{\frac{1}{2}}\left(\left(\left(\frac{\frac{1}{2}-s-ti}{\frac{1}{2}+s+ti}\right)^{\frac{1}{2}}\left(\frac{n}{j}\right)^{s+ti}-\left(\left(\frac{\frac{1}{2}-s-ti}{\frac{1}{2}+s+ti}\right)^{\frac{1}{2}}\left(\frac{n}{j}\right)^{s+ti}\right)^{-1}\right)\right)$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}\left(\frac{n}{j}\right)^{\frac{1}{2}}\left(\left(\frac{\frac{1}{2}-s-ti}{\frac{1}{2}+s+ti}\right)^{\frac{1}{2}}\left(\frac{n}{j}\right)^{s+ti}-\left(\frac{\frac{1}{2}-s-ti}{\frac{1}{2}+s+ti}\right)^{-\frac{1}{2}}\left(\frac{n}{j}\right)^{-s-ti}\right)$$

$$\left(\frac{\frac{1}{2}-s-ti}{\frac{1}{2}+s+ti}\right)^{\frac{1}{2}}\left(\frac{n}{j}\right)^{s+ti}=\left(\frac{\frac{1}{2}-s-ti}{\frac{1}{2}+s+ti}\right)^{\frac{1}{2}}\left(\frac{n}{j}\right)^{s}\cdot\left(\frac{n}{j}\right)^{ti}=\left(\frac{\frac{1}{2}-s-ti}{\frac{1}{2}+s+ti}\right)^{\frac{1}{2}}\left(\frac{n}{j}\right)^{s}\cdot\left(\cos\left(t\log\frac{n}{j}\right)+i\sin\left(t\log\frac{n}{j}\right)\right)$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}\left(\frac{n}{j}\right)^{\frac{1}{2}}\left(\left(\frac{\frac{1}{2}-s-ti}{\frac{1}{2}+s+ti}\right)^{\frac{1}{2}}\left(\frac{n}{j}\right)^{s}\left(\cos\left(t\log\frac{n}{j}\right)+i\sin\left(t\log\frac{n}{j}\right)\right)-\left(\frac{\frac{1}{2}-s-ti}{\frac{1}{2}+s+ti}\right)^{-\frac{1}{2}}\left(\frac{n}{j}\right)^{-s}\left(\cos\left(t\log\frac{n}{j}\right)-i\sin\left(t\log\frac{n}{j}\right)\right)\right)$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}\left(\frac{n}{j}\right)^{\frac{1}{2}}\left(\left(\frac{\frac{1}{2}-s-ti}{\frac{1}{2}+s+ti}\right)^{\frac{1}{2}}\left(\frac{n}{j}\right)^{s}\left(\cos\left(t\log\frac{n}{j}\right)\right)-\left(\frac{\frac{1}{2}-s-ti}{\frac{1}{2}+s+ti}\right)^{-\frac{1}{2}}\left(\frac{n}{j}\right)^{-s}\left(\cos\left(t\log\frac{n}{j}\right)\right)\right)$$
$$+\sum_{j=1}^{n}\left(\frac{n}{j}\right)^{\frac{1}{2}}\left(\left(\frac{\frac{1}{2}-s-ti}{\frac{1}{2}+s+ti}\right)^{\frac{1}{2}}\left(\frac{n}{j}\right)^{s}\left(i\sin\left(t\log\frac{n}{j}\right)\right)-\left(\frac{\frac{1}{2}-s-ti}{\frac{1}{2}+s+ti}\right)^{-\frac{1}{2}}\left(\frac{n}{j}\right)^{-s}\left(-i\sin\left(t\log\frac{n}{j}\right)\right)\right)$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}\left(\frac{n}{j}\right)^{\frac{1}{2}}\left(\left(\left(\frac{\frac{1}{2}-s-ti}{\frac{1}{2}+s+ti}\right)^{\frac{1}{2}}\left(\frac{n}{j}\right)^{s}-\left(\frac{\frac{1}{2}-s-ti}{\frac{1}{2}+s+ti}\right)^{-\frac{1}{2}}\left(\frac{n}{j}\right)^{-s}\right)\cos\left(t\log\frac{n}{j}\right)\right)$$
$$+i\sum_{j=1}^{n}\left(\frac{n}{j}\right)^{\frac{1}{2}}\left(\left(\left(\frac{\frac{1}{2}-s-ti}{\frac{1}{2}+s+ti}\right)^{\frac{1}{2}}\left(\frac{n}{j}\right)^{s}+\left(\frac{\frac{1}{2}-s-ti}{\frac{1}{2}+s+ti}\right)^{-\frac{1}{2}}\left(\frac{n}{j}\right)^{-s}\right)\left(\sin\left(t\log\frac{n}{j}\right)\right)\right)$$

IF s = 0,

$$\lim_{n\to\infty}\sum_{j=1}^{n}\left(\frac{n}{j}\right)^{\frac{1}{2}}\left(\left(\left(\frac{\frac{1}{2}-ti}{\frac{1}{2}+ti}\right)^{\frac{1}{2}}-\left(\frac{\frac{1}{2}-ti}{\frac{1}{2}+ti}\right)^{-\frac{1}{2}}\right)\cos\left(t\log\frac{n}{j}\right)\right)$$
$$+i\sum_{j=1}^{n}\left(\frac{n}{j}\right)^{\frac{1}{2}}\left(\left(\left(\frac{\frac{1}{2}-ti}{\frac{1}{2}+ti}\right)^{\frac{1}{2}}+\left(\frac{\frac{1}{2}-ti}{\frac{1}{2}+ti}\right)^{-\frac{1}{2}}\right)\left(\sin\left(t\log\frac{n}{j}\right)\right)\right)$$

$$a(j)=\left(\frac{\frac{1}{2}-s-ti}{\frac{1}{2}+s+ti}\right)^{\frac{1}{2}}\left(\frac{n}{j}\right)^{s}$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}((a(j)-\frac{1}{a(j)})\cos(t\log\frac{n}{j})+i(a(j)+\frac{1}{a(j)})\sin(t\log\frac{n}{j}))$$

IF s = 0,

$$\lim_{n\to\infty}(a(j)-\frac{1}{a(j)})\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cos(t\log\frac{n}{j})+i(a(j)+\frac{1}{a(j)})\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\sin(t\log\frac{n}{j})$$

…

$$\lim_{n\to\infty}\frac{e^{i\cot^{-1}(2t)}}{2}n^{\frac{1}{2}+it}H_n^{(\frac{1}{2}+it)}+\frac{e^{(-i\cot^{-1}(2t))}}{2}n^{\frac{1}{2}-it}H_n^{(\frac{1}{2}-it)}\approx\frac{1}{2}$$

…

$$\lim_{n\to\infty}\sum_{j=1}^{n}\frac{\cos(t\log\frac{n}{j}+\cot^{-1}(2t))}{\sqrt{j}}=0$$

$$\boxed{\lim_{n\to\infty}\sum_{j=1}^{n}\frac{\sin(t\log\frac{n}{j}+\tan^{-1}(2t))}{\sqrt{j}}=0}$$

$$\zeta(\frac{1}{2}+t\cdot i)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot(\cos(t\log j)+\tan(t\log n+\cot^{-1}(2t))\cdot\sin(t\log j))\ \text{re(t)}>0$$

…

$$\zeta(\frac{1}{2}+t)=\lim_{n\to\infty}\frac{1}{2}H_n^{(\frac{1}{2}+s)}\cdot(1-\tanh(\mathrm{Arctanh}(\frac{1}{2s})-s\log n))+\frac{1}{2}H_n^{(\frac{1}{2}-s)}\cdot(1+\tanh(\mathrm{Arctanh}(\frac{1}{2s})-s\log n))\text{ for re(t)>0}$$

$$\boxed{\Re(\lim_{n\to\infty}H_n^{(\frac{1}{2}+s)}\cdot(1-\tanh(\mathrm{Arctanh}(\frac{1}{2s})-s\log n)))=0}\text{ at zeta zeros-1/2}$$

which is

$$\Re(\lim_{n\to\infty}\frac{1}{1-\frac{\frac{1}{2}+s}{\frac{1}{2}-s}n^{-2s}}\cdot H_n^{(\frac{1}{2}+s)})=0\text{ at zeta zeros-1/2}$$

$$\Im(\lim_{n\to\infty}n^s(\frac{\frac{1}{2}-s}{\frac{1}{2}+s})^{\frac{1}{2}}H_n^{(\frac{1}{2}+s)})=0\text{ at zeta zeros – ½}$$

…

$$\zeta(s)=\lim_{n\to\infty}(\frac{1}{2}-\frac{1}{2}\tanh(\mathrm{Arccoth}(2s-1)+(\frac{1}{2}-s)\log n))H_n^{(s)}+(\frac{1}{2}+\frac{1}{2}\tanh(\mathrm{Arccoth}(2s-1)+(\frac{1}{2}-s)\log n))H_n^{(1-s)}\text{ for re(s)>1/2}$$

…

$$\Im(\lim_{n\to\infty}n^s(\frac{\frac{1}{2}-s}{\frac{1}{2}+s})^{\frac{1}{2}}H_n^{(\frac{1}{2}+s)})=0\text{ at zeta zeros – ½}$$

$$\Re(\lim_{n\to\infty}\frac{n^s(\frac{\frac{1}{2}-s}{\frac{1}{2}+s})^{\frac{1}{2}}}{n^s(\frac{\frac{1}{2}-s}{\frac{1}{2}+s})^{\frac{1}{2}}-n^{-s}(\frac{\frac{1}{2}-s}{\frac{1}{2}+s})^{-\frac{1}{2}}}H_n^{(\frac{1}{2}+s)})=0\text{ at zeta zeros – ½}$$

…

$$\Re(\lim_{n\to\infty}\frac{1}{1-\frac{\frac{1}{2}+s}{\frac{1}{2}-s}n^{-2s}}\cdot H_n^{(\frac{1}{2}+s)})=0\text{ at zeta zeros-1/2}$$

$$\Im\left(\lim_{n\to\infty} n^{s}\left(\frac{\frac{1}{2}-s}{\frac{1}{2}+s}\right)^{\frac{1}{2}} H_n^{\left(\frac{1}{2}+s\right)}\right)=0 \text{ at zeta zeros} - ½$$

$$-\sum_{\rho}\frac{x^{\rho}}{\rho}=-\sum_{\rho}\frac{x^{\frac{1}{2}+ti}}{\frac{1}{2}+ti}+\frac{x^{\frac{1}{2}-ti}}{\frac{1}{2}-ti}=-\sum_{\rho}x^{\frac{1}{2}}\left(\frac{x^{ti}}{\frac{1}{2}+ti}+\frac{x^{-ti}}{\frac{1}{2}-ti}\right)=-x^{\frac{1}{2}}\sum_{\rho}\left(\frac{x^{ti}}{\frac{1}{2}+ti}+\frac{x^{-ti}}{\frac{1}{2}-ti}\right)$$

$$=-\sum_{t}\left(t^2+\frac{1}{4}\right)^{-1}\cdot\sqrt{x}\cdot(2t\sin(t\log x)+\cos(t\log x))$$

which is quite similar to

$$\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\left(2x\cos\left(x\cdot\log\frac{j}{n}\right)+\sin\left(x\cdot\log\frac{j}{n}\right)\right)=0$$

which then became

$$\lim_{n\to\infty}(2x\sin(x\log n)+\cos(x\log n))\cdot\left(\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\sin(x\log j)\right)+$$
$$(2x\cos(x\log n)-\sin(x\log n))\cdot\left(\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\cos(x\log j)\right)=0$$

and

$$\zeta\left(\frac{1}{2}-t\cdot i\right)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\cdot\left(\cos(t\log j)+\tan\left(t\log n+\arctan\left(\frac{1}{2x}\right)\right)\cdot\sin(t\log j)\right)$$

…

also is

$$=-\sum_{t}\sqrt{x}\cdot(\cos(t\log x)+\cos(t\log x-2\arctan(2x)))$$

$$g(s)=\lim_{n\to\infty}\frac{f(s)\cdot H_n^{(s)}-f(1-s)H_n^{(1-s)}}{f(s)\cdot a(s)-f(1-s)\cdot b(s)}$$

$$g(s)=\lim_{n\to\infty}\frac{n^s(1-s)\cdot H_n^{(s)}-n^{(1-s)}s H_n^{(1-s)}}{n^s(1-s)\cdot a(s)-n^{(1-s)}s\cdot b(s)}$$

$$g(s)=\lim_{n\to\infty}\frac{n^{\frac{1}{2}+s}(\frac{1}{2}-s)\cdot H_n^{(\frac{1}{2}+s)}-n^{(\frac{1}{2}-s)}(\frac{1}{2}+s)H_n^{(\frac{1}{2}-s)}}{n^{\frac{1}{2}+s}(\frac{1}{2}-s)-n^{(\frac{1}{2}-s)}(\frac{1}{2}+s)}$$

$$g(s)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{n^{\frac{1}{2}+s}(\frac{1}{2}-s)\cdot j^{-(\frac{1}{2}+s)}-n^{(\frac{1}{2}-s)}(\frac{1}{2}+s)\,j^{-(\frac{1}{2}-s)}}{n^{\frac{1}{2}+s}(\frac{1}{2}-s)-n^{(\frac{1}{2}-s)}(\frac{1}{2}+s)}$$

$$g(s)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{(\frac{n}{j})^{\frac{1}{2}+s}(\frac{1}{2}-s)-(\frac{n}{j})^{(\frac{1}{2}-s)}(\frac{1}{2}+s)}{n^{\frac{1}{2}+s}(\frac{1}{2}-s)-n^{(\frac{1}{2}-s)}(\frac{1}{2}+s)}$$

$$g(s)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\frac{(\frac{n}{j})^{s}(\frac{1}{2}-s)-(\frac{n}{j})^{-s}(\frac{1}{2}+s)}{n^{s}(\frac{1}{2}-s)-n^{-s}(\frac{1}{2}+s)}$$

$$a(s)=\left(\frac{\frac{1}{2}-s}{\frac{1}{2}+s}\right)^{1/2}$$

$$g(s)=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\frac{(\frac{n}{j})^{s}a(s)-(\frac{n}{j})^{-s}a(s)^{-1}}{n^{s}a(s)-n^{-s}a(s)^{-1}}$$

$$\log a(s)=-\operatorname{arctanh}(s)$$

$$\zeta(s-\frac{1}{2})=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\frac{\sinh(s\log\frac{n}{j}-\operatorname{arctanh}(2s))}{\sinh(s\log n-\operatorname{arctanh}(2s))}$$

$$\zeta(s-\frac{1}{2})=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\frac{\sinh(s\log n-\operatorname{arctanh}(2s))\cosh(s\log j)-\cosh(s\log n-\operatorname{arctanh}(2s))\sinh(s\log j)}{\sinh(s\log n-\operatorname{arctanh}(2s))}$$

$$\zeta(s-\frac{1}{2})=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\sqrt{j}}\left(\cosh(s\log j)-\frac{\cosh(s\log n-\operatorname{arctanh}(2s))}{\sinh(s\log n-\operatorname{arctanh}(2s))}\sinh(s\log j)\right)$$

$$\boxed{\zeta(s-\frac{1}{2})=\lim_{n\to\infty}\sum_{j=1}^{n}\frac{\cosh(s\log j)}{\sqrt{j}}-\tanh(\log n-\operatorname{arctanh}(\frac{1}{2s}))\cdot\sum_{j=1}^{n}\frac{\sinh(s\log j)}{\sqrt{j}}}$$

$$g(n,s,t)=\lim_{n\to\infty}\frac{f(s)\cdot H_n^{(s)}-f(t)\,H_n^{(t)}}{f(s)}$$

$$g(n,s,t)=\lim_{n\to\infty}\frac{\dfrac{n^s}{s}\cdot H_n^{(s)}-\dfrac{n^t}{t}\,H_n^{(t)}}{\dfrac{n^s}{s}-\dfrac{n^t}{t}}$$

$$\Re\left(\lim_{n\to\infty}\left(1-\tanh\left(\operatorname{Arctanh}\left(\frac{1}{2s-1}\right)-\left(s-\frac{1}{2}\right)\log n\right)\right)\right)\cdot H_n^{(s)}=0$$