```
a1[n_, s_] := -8 s / (1 + 4 s^2) n + 2 Sum[(n / j) ^ (1 / 2) Sin[s Log[n / j]], {j, 1, n}] +
  I ( n^ (1 / 2 + s I) Zeta[1 / 2 + s I] - n^ (1 / 2 - s I) Zeta[1 / 2 - s I])
b1[n_, s_] := -4 / (1 + 4 s^2) n + 2 Sum[(n / j) ^ (1 / 2) Cos[s Log[n / j]], {j, 1, n}] -
  ( n^ (1 / 2 + s I) Zeta[1 / 2 + s I] + n^ (1 / 2 - s I) Zeta[1 / 2 - s I]) - 1

b1[10 000, -12 + - .3 I]

-8.33333 × 10^-6 - 1.66551 × 10^-11 i

a2[n_, s_] :=
 (2 s) ^ (-1 / 2) (-8 s / (1 + 4 s^2) n + 2 Sum[(n / j) ^ (1 / 2) Sin[s Log[n / j]], {j, 1, n}] +
    I ( n^ (1 / 2 + s I) Zeta[1 / 2 + s I] - n^ (1 / 2 - s I) Zeta[1 / 2 - s I]))
b2[n_, s_] := (2 s) ^ (1 / 2) (-4 / (1 + 4 s^2) n + 2 Sum[(n / j) ^ (1 / 2) Cos[s Log[n / j]], {j, 1, n}] -
    ( n^ (1 / 2 + s I) Zeta[1 / 2 + s I] + n^ (1 / 2 - s I) Zeta[1 / 2 - s I]) - 1)

b2[10 000, -120 + - .3 I]

-1.60464 × 10^-7 + 0.000129104 i

ab1[n_, s_] :=
 (2 s) ^ (-1 / 2) (-8 s / (1 + 4 s^2) n + 2 Sum[(n / j) ^ (1 / 2) Sin[s Log[n / j]], {j, 1, n}] +
     I ( n^ (1 / 2 + s I) Zeta[1 / 2 + s I] - n^ (1 / 2 - s I) Zeta[1 / 2 - s I])) - (
   (2 s) ^ (1 / 2) (-4 / (1 + 4 s^2) n + 2 Sum[(n / j) ^ (1 / 2) Cos[s Log[n / j]], {j, 1, n}] -
      ( n^ (1 / 2 + s I) Zeta[1 / 2 + s I] + n^ (1 / 2 - s I) Zeta[1 / 2 - s I]) - 1))

ab1[10 000, -120 + - .3 I]

-9.19412 × 10^-10 - 4.10627 × 10^-9 i

ab2[n_, s_] := (2 s) ^ (-1 / 2) (2 Sum[(n / j) ^ (1 / 2) Sin[s Log[n / j]], {j, 1, n}] +
     I ( n^ (1 / 2 + s I) Zeta[1 / 2 + s I] - n^ (1 / 2 - s I) Zeta[1 / 2 - s I])) - (
   (2 s) ^ (1 / 2) (2 Sum[(n / j) ^ (1 / 2) Cos[s Log[n / j]], {j, 1, n}] -
      ( n^ (1 / 2 + s I) Zeta[1 / 2 + s I] + n^ (1 / 2 - s I) Zeta[1 / 2 - s I]) - 1))

ab2[10 000, -120 + - .3 I]

-9.19681 × 10^-10 - 4.10666 × 10^-9 i

ab3[n_, s_] :=
 (2 Sum[(n / j) ^ (1 / 2) (2 s) ^ (-1 / 2) Sin[s Log[n / j]], {j, 1, n}] + I ( n^ (1 / 2 + s I)
        (2 s) ^ (-1 / 2) Zeta[1 / 2 + s I] - n^ (1 / 2 - s I) (2 s) ^ (-1 / 2) Zeta[1 / 2 - s I])) - (
   (2 Sum[(n / j) ^ (1 / 2) (2 s) ^ (1 / 2) Cos[s Log[n / j]], {j, 1, n}] -
     ( n^ (1 / 2 + s I) (2 s) ^ (1 / 2) Zeta[1 / 2 + s I] + n^ (1 / 2 - s I) (2 s) ^ (1 / 2) Zeta[1 / 2 - s I]) -
     (2 s) ^ (1 / 2)))

ab3[10 000, -120 + - .3 I]

-9.44276 × 10^-10 - 4.15094 × 10^-9 i

ab4[n_, s_] := 2 Sum[(n / j) ^ (1 / 2) (2 s) ^ (-1 / 2) Sin[s Log[n / j]], {j, 1, n}] +
  I n^ (1 / 2 + s I) (2 s) ^ (-1 / 2) Zeta[1 / 2 + s I] - I n^ (1 / 2 - s I) (2 s) ^ (-1 / 2) Zeta[1 / 2 - s I] +
  -2 Sum[(n / j) ^ (1 / 2) (2 s) ^ (1 / 2) Cos[s Log[n / j]], {j, 1, n}] +
  n^ (1 / 2 + s I) (2 s) ^ (1 / 2) Zeta[1 / 2 + s I] +
  n^ (1 / 2 - s I) (2 s) ^ (1 / 2) Zeta[1 / 2 - s I] + (2 s) ^ (1 / 2)

ab4[10 000, -120 + - .3 I]

-9.45874 × 10^-10 - 4.15093 × 10^-9 i
```

```
ab5[n_, s_] :=
 2 Sum[(n / j)^(1 / 2) ((2 s)^(-1 / 2) Sin[s Log[n / j]] - (2 s)^(1 / 2) Cos[s Log[n / j]]),
    {j, 1, n}] +
  I n^(1 / 2 + s I) (2 s)^(-1 / 2) Zeta[1 / 2 + s I] -
  I n^(1 / 2 - s I) (2 s)^(-1 / 2) Zeta[1 / 2 - s I] +
  + n^(1 / 2 + s I) (2 s)^(1 / 2) Zeta[1 / 2 + s I] +
  n^(1 / 2 - s I) (2 s)^(1 / 2) Zeta[1 / 2 - s I] + (2 s)^(1 / 2)
```

```
ab5[10 000, -120 + -.3 I]
```

$-9.24047\times10^{-10} - 4.30737\times10^{-9}\,i$

```
ab6[n_, s_] :=
 2 Sum[(n / j)^(1 / 2) ((2 s)^(-1 / 2) Sin[s Log[n / j]] - (2 s)^(1 / 2) Cos[s Log[n / j]]),
    {j, 1, n}] +
  I n^(1 / 2 + s I) (2 s)^(-1 / 2) Zeta[1 / 2 + s I] -
  I n^(1 / 2 - s I) (2 s)^(-1 / 2) Zeta[1 / 2 - s I] +
  + n^(1 / 2 + s I) (2 s)^(1 / 2) Zeta[1 / 2 + s I] +
  n^(1 / 2 - s I) (2 s)^(1 / 2) Zeta[1 / 2 - s I] + (2 s)^(1 / 2)
```

```
ab6[10 000, -120 + -.3 I]
```

$-9.24047\times10^{-10} - 4.30737\times10^{-9}\,i$

```
((2 s)^(-1 / 2) Sin[s Log[n / j]] - (2 s)^(1 / 2) Cos[s Log[n / j]]) /. s → .3 /. j → 13 /. n → 27
```

-0.475242

```
Cos[s Log[n / j] + ArcCot[2 s]] /. s → .3 /. j → 13 /. n → 27
```

0.315661

```
TrigToExp[((2 s)^(-1 / 2) Sin[s Log[n / j]] - (2 s)^(1 / 2) Cos[s Log[n / j]])]
```

$$\frac{i\left(\frac{n}{j}\right)^{-i s}}{2\sqrt{2}\sqrt{s}} - \frac{i\left(\frac{n}{j}\right)^{i s}}{2\sqrt{2}\sqrt{s}} - \frac{\left(\frac{n}{j}\right)^{-i s}\sqrt{s}}{\sqrt{2}} - \frac{\left(\frac{n}{j}\right)^{i s}\sqrt{s}}{\sqrt{2}}$$

$$(1/2)\left(\frac{i\left(\frac{n}{j}\right)^{-i s}}{\sqrt{2 s}} - \frac{i\left(\frac{n}{j}\right)^{i s}}{\sqrt{2 s}} - \left(\frac{n}{j}\right)^{-i s}\sqrt{2 s} - \left(\frac{n}{j}\right)^{i s}\sqrt{2 s}\right) \text{ /. s → .3 /. j → 13 /. n → 27}$$

$-0.475242 + 0.\,i$

```
Limit[1 / n Sum[(j / n)^(-1 / 2) f[s Log[j / n]], {j, 1, n}], n → Infinity]
```

$$\text{Limit}\left[\frac{\sum_{j=1}^{n}\frac{f\left[s\,\text{Log}\left[\frac{j}{n}\right]\right]}{\sqrt{\frac{j}{n}}}}{n},\ n\to\infty\right]$$

```
Integrate[ Cos[s Log[x] - ArcCot[2 s]] / x^(1 / 2), {x, 0, 1}]
```

$$\text{ConditionalExpression}\left[0,\ -\frac{1}{2} < \text{Im}[s] < \frac{1}{2}\right]$$

TrigToExp[Cos[x]]

$$\frac{e^{-i x}}{2}+\frac{e^{i x}}{2}$$

FullSimplify[TrigToExp[E^(- I ArcCot[2 t])]]

$$\frac{\sqrt{4+\frac{1}{t^2}}\ t}{i+2 t}$$

FullSimplify[TrigToExp[E^(I ArcCot[2 t])]]

$$\frac{\sqrt{4+\frac{1}{t^2}}\ t}{-i+2 t}$$

FullSimplify $\left[\sqrt{4+\frac{1}{t^2}}\right]$

$$\sqrt{4+\frac{1}{t^2}}$$

Expand[(2 - 1 / t I) (2 + 1 / t I)]

$$4+\frac{1}{t^2}$$

Integrate[Sin[s Log[x] + ArcTan[2 s]] / x^(1 / 2), {x, 0, 1}]

ConditionalExpression $\left[0, -\frac{1}{2} < \text{Im}[s] < \frac{1}{2}\right]$

2 s Sin[s Log[x]] + Cos[s Log[x]] /. s → 1 /. x → 22

Cos[Log[22]] + 2 Sin[Log[22]]

(2 s) ^ (1 / 2) Sin[s Log[x] + ArcTan[(2 s)]] /. s → 1 /. x → 22

$\sqrt{2}$ Sin[ArcTan[2] + Log[22]]

Integrate[Sin[s Log[x] + c] / x^(1 / 2), {x, 0, 1}]

$$\frac{2\ (-2\ s\ \text{Cos}[c]+\text{Sin}[c])}{1+4\ s^2}$$

$\frac{2\ (-2\ s\ \text{Cos}[c]+\text{Sin}[c])}{1+4\ s^2}$ **/. c → ArcTan[2 s]**

0

Cos[ArcTan[2 s]]

$$\frac{1}{\sqrt{1+4\ s^2}}$$

Sin[ArcTan[2 s]]

$$\frac{2\ s}{\sqrt{1+4\ s^2}}$$

```
vb1[n_, s_] := -4 / (1 + 4 s^2) n + 2 Sum[(n / j) ^ (1 / 2) Cos[s Log[n / j]], {j, 1, n}] -
  (n^(1 / 2 + s I) Zeta[1 / 2 + s I] + n^(1 / 2 - s I) Zeta[1 / 2 - s I])
vb1a[n_, s_] := {-4 / (1 + 4 s^2) n, 2 Sum[(n / j) ^ (1 / 2) Cos[s Log[n / j]], {j, 1, n}],
  - n^(1 / 2 + s I) Zeta[1 / 2 + s I], -n^(1 / 2 - s I) Zeta[1 / 2 - s I]}
vb3[n_, s_] := 2 Sum[(n / j) ^ (1 / 2) Sin[s Log[n / j] - ArcTan[2 s]], {j, 1, n}] +
  I (E^(-I ArcTan[2 s]) n^(1 / 2 + s I) Zeta[1 / 2 + s I] -
     E^(I ArcTan[2 s]) n^(1 / 2 - s I) Zeta[1 / 2 - s I])
(* starts here *)
vb4[n_, s_] := 2 Sum[(n / j) ^ (1 / 2) Sin[s Log[n / j] - ArcTan[2 s]], {j, 1, n}] +
  I (E^(-I ArcTan[2 s]) n^(1 / 2 + s I) Zeta[1 / 2 + s I] -
     E^(I ArcTan[2 s]) n^(1 / 2 - s I) Zeta[1 / 2 - s I]) + Sin[ArcTan[2 s]]
vb5[n_, s_] := 2 Sum[(n / j) ^ (1 / 2) Sin[s Log[n / j] - ArcTan[2 s]], {j, 1, n}] +
```

$$\frac{i\sqrt{1-2 i s}}{\sqrt{1+2 i s}}\ \texttt{n^(1 / 2 + s I) Zeta[1 / 2 + s I]} -$$

$$\frac{i\sqrt{1+2 i s}}{\sqrt{1-2 i s}}\ \texttt{n^(1 / 2 - s I) Zeta[1 / 2 - s I]} + \frac{2 s}{\sqrt{1+4 s^2}}$$

```
vb6[n_, s_] := 2 Sum[(n / j) ^ (1 / 2) Sin[s Log[n / j] - ArcTan[2 s]], {j, 1, n}] +
```

$$\texttt{I}\left(\frac{\sqrt{1/2 - i s}}{\sqrt{1/2 + i s}}\ \texttt{n^(1 / 2 + s I) Zeta[1 / 2 + s I]} - \frac{\sqrt{1/2 + i s}}{\sqrt{1/2 - i s}}\ \texttt{n^(1 / 2 - s I) Zeta[1 / 2 - s I]}\right) +$$

$$\frac{2 s}{\sqrt{1+4 s^2}}$$

$$\texttt{vb7[n_, s_] := 2 Sum}\Big[\texttt{(n / j) ^ (1 / 2) Sin}\Big[\texttt{s Log[n / j]} - \frac{1}{2}\, i\, \texttt{Log}\Big[\frac{\frac{1}{2} - i s}{\frac{1}{2} + i s}\Big]\Big], \texttt{\{j, 1, n\}}\Big] +$$

$$\texttt{I}\left(\frac{\sqrt{1/2 - i s}}{\sqrt{1/2 + i s}}\ \texttt{n^(1 / 2 + s I) Zeta[1 / 2 + s I]} - \frac{\sqrt{1/2 + i s}}{\sqrt{1/2 - i s}}\ \texttt{n^(1 / 2 - s I) Zeta[1 / 2 - s I]}\right) +$$

$$\frac{s}{\sqrt{(1/2 - s\,I)\ (1/2 + s\,I)}}$$

$$\texttt{vb7[n_, s_] := 2 Sum}\Big[\texttt{(n / j) ^ (1 / 2) Sin}\Big[\texttt{s Log[n / j]} - \frac{1}{2}\, i\, \texttt{Log}\Big[\frac{\frac{1}{2} - i s}{\frac{1}{2} + i s}\Big]\Big], \texttt{\{j, 1, n\}}\Big] +$$

$$\texttt{I}\left(\frac{\sqrt{1/2 - i s}}{\sqrt{1/2 + i s}}\ \texttt{n^(1 / 2 + s I) Zeta[1 / 2 + s I]} - \frac{\sqrt{1/2 + i s}}{\sqrt{1/2 - i s}}\ \texttt{n^(1 / 2 - s I) Zeta[1 / 2 - s I]}\right) +$$

$$\frac{s}{\sqrt{(1/2 - s\,I)\ (1/2 + s\,I)}}$$

$$\texttt{vb7r[n_, s_] := 2 Sum}\Big[\texttt{(n / j) ^ (1 / 2) Sin}\Big[\texttt{s Log[n / j]} - \frac{1}{2}\, i\, \texttt{Log}\Big[\frac{\frac{1}{2} - i s}{\frac{1}{2} + i s}\Big]\Big], \texttt{\{j, 1, n\}}\Big] +$$

```
  Sin[ArcTan[2 s]]
```

```
Chop@vb7r[1000, 31.7179799547]
```

```
-40.9515
```

$$\text{FullSimplify}\left[\frac{s}{\sqrt{-\frac{1}{2}+s\ \sqrt{(1/2+s)}\ 2\verb|^|(1/2)}}\right]$$

$$\frac{s}{\sqrt{-\frac{1}{2}+s}\ \sqrt{1+2\,s}}$$

$$\frac{s}{\sqrt{(1/2-s\,I)\ (1/2+s\,I)}}\ \text{/. } s \to 2.\,I$$

$1.0328 + 0.\, i$

$$\frac{1}{2}\, i\, \text{Log}\left[\frac{\frac{1}{2}-i\,s}{\frac{1}{2}+i\,s}\right]\ \text{/. } s \to 2.\,I$$

$-1.5708 + 0.255413\, i$

$$\frac{1}{2}\, i\, \text{Log}\left[\frac{\frac{1}{2}-i\,s}{\frac{1}{2}+i\,s}\right]\ \text{/. } s \to 2.\,I$$

$-1.5708 + 0.255413\, i$

$$\frac{2\,s}{\sqrt{1+4\,s^2}}\ \text{/. } s \to 2.\,I$$

$1.0328 + 0.\, i$

```
N@2^(1/2)/2
```

0.707107

```
N@ -Sin[ArcTan[2 s]] /. s → (-1)^.75 + 1
```

$-1.12282 - 0.232262\, i$

```
N[(2 s)^(1/2) /. s → (-1)^.25]
```

$1.30656 + 0.541196\, i$

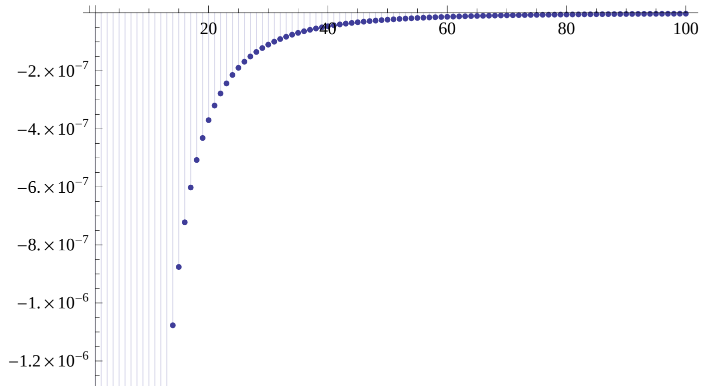
```
FullSimplify[TrigToExp[I E^(-I ArcTan[2 s])]]
```

$$\frac{i\,\sqrt{1-2\,i\,s}}{\sqrt{1+2\,i\,s}}$$

```
FullSimplify[TrigToExp[I E^(I ArcTan[2 s])]]
```

$$\frac{i\,\sqrt{1+2\,i\,s}}{\sqrt{1-2\,i\,s}}$$

```
DiscretePlot[Re@vb6[n, .3 + .7 I], {n, 1, 100}]
```



$$\frac{i\sqrt{1/2 - i\,s}}{\sqrt{1/2 + i\,s}} \text{ /. } s \to .3$$

0.514496 + 0.857493 i

```
TrigToExp[ArcTan[2 s]]
```

$$\frac{1}{2}\, i \,\text{Log}[1 - 2\, i\, s] - \frac{1}{2}\, i\, \text{Log}[1 + 2\, i\, s]$$

$$\frac{1}{2}\, i \,\text{Log}[1 - 2\, i\, s] - \frac{1}{2}\, i\, \text{Log}[1 + 2\, i\, s] \text{ /. } s \to .3 + .4\, I$$

0.785398 + 0.549306 i

$$\frac{1}{2}\, i \,\text{Log}[1/2 - i\, s] - \frac{1}{2}\, i\, \text{Log}[1/2 + i\, s] \text{ /. } s \to .3 + .4\, I$$

0.785398 + 0.549306 i

$$\frac{1}{2}\, i \,\text{Log}[(1/2 - i\, s) / (1/2 + i\, s)]$$

$$\frac{1}{2}\, i \,\text{Log}\left[\frac{\frac{1}{2} - i\, s}{\frac{1}{2} + i\, s}\right]$$

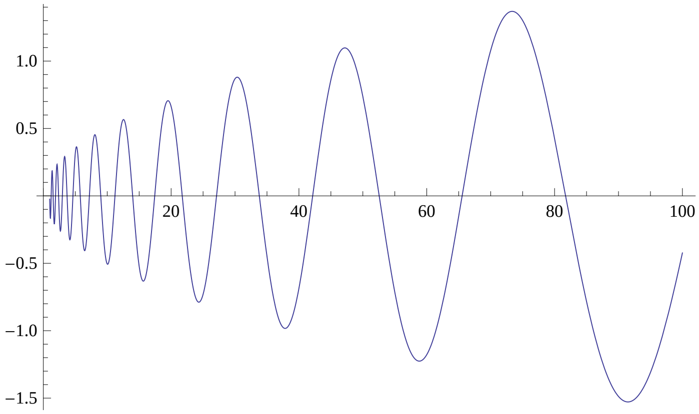
```
Expand[(1 - 2 s I) (1 + 2 s I)]
```

$1 + 4\, s^2$

$$\text{vb7x}[n\_, s\_] := I \left(\frac{\sqrt{1/2 - i\, s}}{\sqrt{1/2 + i\, s}}\; n\text{^}(1/2 + s\, I)\; \text{Zeta}[1/2 + s\, I] - \frac{\sqrt{1/2 + i\, s}}{\sqrt{1/2 - i\, s}}\; n\text{^}(1/2 - s\, I)\; \text{Zeta}[1/2 - s\, I] \right)$$

```
vb7x2[n_, s_] := I (E^(-I ArcTan[2 s]) n^(1/2 + s I) Zeta[1/2 + s I] -
    E^(I ArcTan[2 s]) n^(1/2 - s I) Zeta[1/2 - s I])
vb7x3[s_] := I (E^(-I ArcTan[2 s]) Zeta[1/2 + s I] - E^(I ArcTan[2 s]) Zeta[1/2 - s I])
```

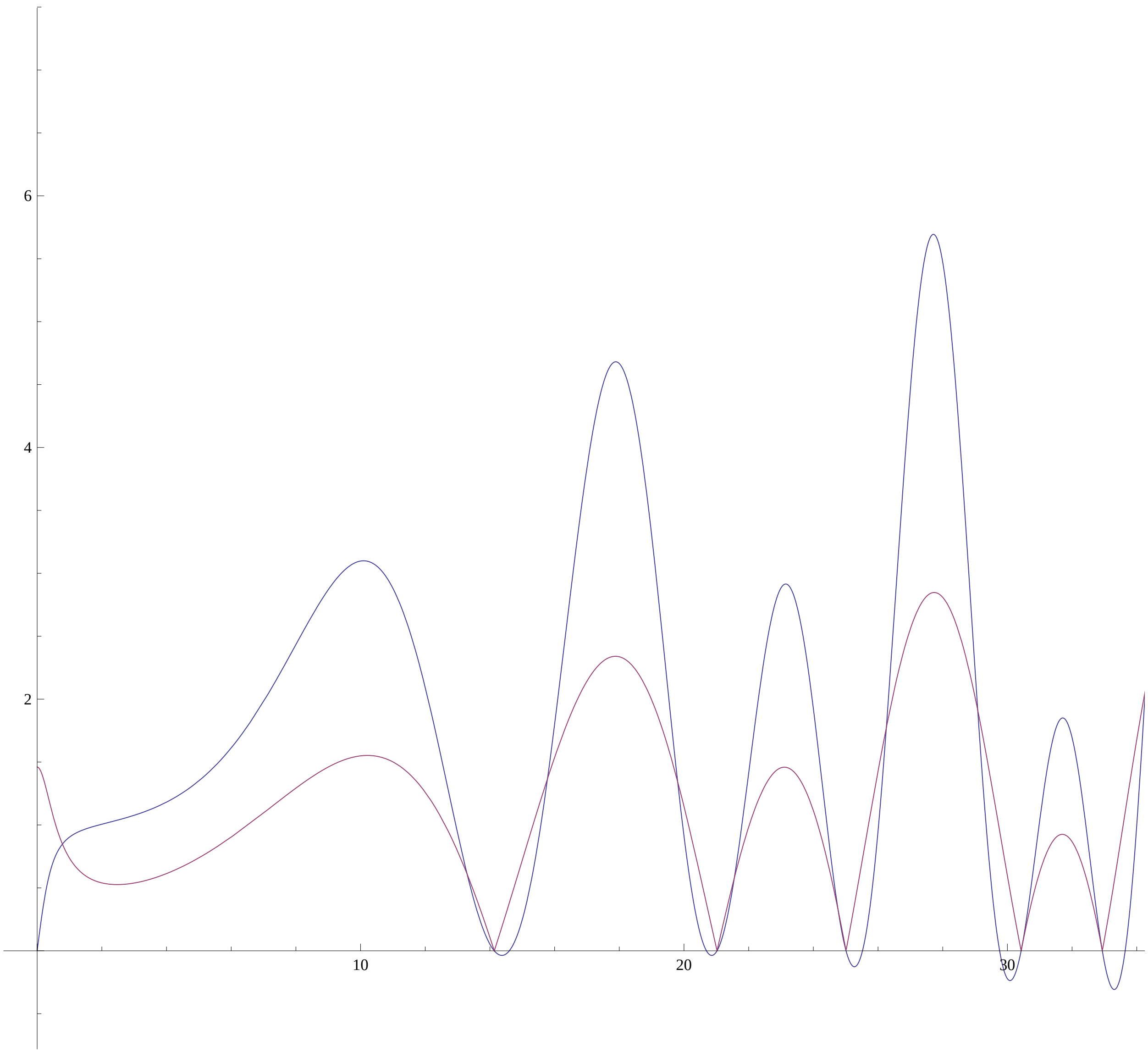
```
Plot[Re@vb7x2[n, N@Im@ZetaZero@1 + .1], {n, 1, 100}]
```




```
vb7x3[N@Im@ZetaZero@1 + .1 I]
```

−0.00610065 − 0.0305753 ⅈ

```
Plot[{Re@vb7x3[n], Abs@Zeta[.5 + n I]}, {n, 0, 50}]
```



$$\text{FullSimplify}\left[\frac{2\,(-2\,s\,\text{Cos}[c]+\text{Sin}[c])}{1+4\,s^2}\right]$$

$$\frac{2\,(-2\,s\,\text{Cos}[c]+\text{Sin}[c])}{1+4\,s^2}$$

$$-\frac{4\,s\,\text{Cos}[c]}{1+4\,s^2}+\frac{2\,\text{Sin}[c]}{1+4\,s^2}\ \text{/. } c\to 12.3\ \text{/. } s\to 7.2$$

-0.135874

$$-\frac{4\,s\,\text{Cos}[c]}{1+4\,s^2}+\frac{\text{Sin}[c]}{(1/2-s\,I)\,(1/2+s\,I)}\ \text{/. } c\to 12.3\ \text{/. } s\to 7.2$$

$-0.138401 + 0.\,i$

$$\frac{2\,(-2\,s\,\mathrm{Cos}[c]+\mathrm{Sin}[c])}{1+4\,s^2}\ /.\ c\to \mathrm{Pi}/2$$

$$\frac{2}{1+4\,s^2}$$