```
Clear[dis, dis2]
dis[j_, x_] := dis[j, x] = dis[j - 1, x] + j^(-1 / 2) (-1)^(j) N@Sin[x Log[j]]
dis[0, x_] := 0
dis2[j_, x_] := dis2[j, x] = dis2[j - 1, x] + j^(-1 / 2) N@Sin[x Log[j]]
dis2[0, x_] := 0
```

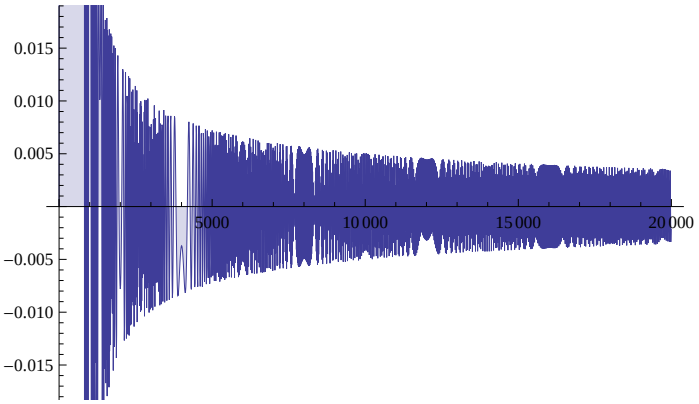
```
N@dis[20 000, N@Im@ZetaZero@1]
```

```
0.00347674
```

```
$RecursionLimit = 10 000 000
```

```
10 000 000
```

```
DiscretePlot[Re@dis[n, N@Im@ZetaZero@2000], {n, 1, 20 000, 10}]
```




```
FullSimplify[Sum[j^(-1 / 2) (-1)^(j) Sin[x Log[j]], {j, 1, n}]]
```

$$-i\,2^{-\frac{3}{2}-i x}\left(-\text{Zeta}\left[\frac{1}{2}+i x\right]+2^{2 i x}\left(\text{Zeta}\left[\frac{1}{2}-i x\right]-\text{Zeta}\left[\frac{1}{2}-i x,\frac{3}{2}\right]+(-1)^n\left(\text{Zeta}\left[\frac{1}{2}-i x,\frac{1+n}{2}\right]-\text{Zeta}\left[\frac{1}{2}-i x,\frac{2+n}{2}\right]\right)\right)+\text{Zeta}\left[\frac{1}{2}+i x,\frac{3}{2}\right]-(-1)^n\,\text{Zeta}\left[\frac{1}{2}+i x,\frac{1+n}{2}\right]+(-1)^n\,\text{Zeta}\left[\frac{1}{2}+i x,\frac{2+n}{2}\right]\right)$$

```
FullSimplify[Sum[ (-1)^(j) Sin[x Log[j]], {j, 1, n}]]
```

$$-i\,2^{-1-i x}\left(2^{i x}\left(-1+2^{1+i x}\right)\text{Zeta}[-i x]+\left(-2+2^{i x}\right)\text{Zeta}[i x]+(-1)^n\left(4^{i x}\left(\text{Zeta}\left[-i x,\frac{1+n}{2}\right]-\text{Zeta}\left[-i x,\frac{2+n}{2}\right]\right)-\text{Zeta}\left[i x,\frac{1+n}{2}\right]+\text{Zeta}\left[i x,\frac{2+n}{2}\right]\right)\right)$$

```
Limit[-i 2^(-3/2 - i x) (-Zeta[1/2 + i x] +
    2^(2 i x) (Zeta[1/2 - i x] - Zeta[1/2 - i x, 3/2] + (-1)^n (Zeta[1/2 - i x, (1 + n)/2] - Zeta[1/2 - i x, (2 + n)/2])) +
    Zeta[1/2 + i x, 3/2] - (-1)^n Zeta[1/2 + i x, (1 + n)/2] + (-1)^n Zeta[1/2 + i x, (2 + n)/2]), n -> Infinity]
```

```
Limit[-i 2^(-3/2 - i x) (-Zeta[1/2 + i x] +
    2^(2 i x) (Zeta[1/2 - i x] - Zeta[1/2 - i x, 3/2] + (-1)^n (Zeta[1/2 - i x, (1 + n)/2] - Zeta[1/2 - i x, (2 + n)/2])) +
    Zeta[1/2 + i x, 3/2] - (-1)^n Zeta[1/2 + i x, (1 + n)/2] + (-1)^n Zeta[1/2 + i x, (2 + n)/2]), n -> ∞]
```

```
FullSimplify[Sum[j^(-1/2) (-1)^(j) Sin[x Log[j]], {j, 1, Infinity}]]
```

$$\sum_{j=1}^{\infty} \frac{(-1)^j \text{Sin}[x \text{Log}[j]]}{\sqrt{j}}$$

```
Expand[FullSimplify[Sum[j^(-1/2) (-1)^(j) Sin[x Log[j]], {j, 1, n}] /. x -> Im@ZetaZero@2]]
```

$$i\ (-1)^n\ 2^{-2+\text{ZetaZero}[2]}\ \text{Zeta}\left[1-\text{ZetaZero}[2],\ 1+\frac{n}{2}\right] -$$
$$i\ (-1)^n\ 2^{-2+\text{ZetaZero}[2]}\ \text{Zeta}\left[1-\text{ZetaZero}[2],\ \frac{1+n}{2}\right] -$$
$$i\ (-1)^n\ 2^{-1-\text{ZetaZero}[2]}\ \text{Zeta}\left[\text{ZetaZero}[2],\ 1+\frac{n}{2}\right] + i\ (-1)^n\ 2^{-1-\text{ZetaZero}[2]}\ \text{Zeta}\left[\text{ZetaZero}[2],\ \frac{1+n}{2}\right]$$

```
N@Zeta[ZetaZero[1], 60 000]
```

```
17.3061 - 0.659029 i
```

```
FullSimplify[Sum[j^(-1/2) (-1)^(j) Cos[x Log[j]], {j, 1, n}]]
```

$$2^{-\frac{3}{2}-i x}\left(\left(-2^{\frac{1}{2}+i x}+2^{1+2 i x}\right)\text{Zeta}\left[\frac{1}{2}-i x\right]-\left(-2+2^{\frac{1}{2}+i x}\right)\text{Zeta}\left[\frac{1}{2}+i x\right]+\right.$$
$$(-1)^n\left(2^{2 i x}\left(\text{Zeta}\left[\frac{1}{2}-i x,\ \frac{1+n}{2}\right]-\text{Zeta}\left[\frac{1}{2}-i x,\ \frac{2+n}{2}\right]\right)+\right.$$
$$\left.\left.\left.\text{Zeta}\left[\frac{1}{2}+i x,\ \frac{1+n}{2}\right]-\text{Zeta}\left[\frac{1}{2}+i x,\ \frac{2+n}{2}\right]\right)\right)\right)$$

```
Expand@FullSimplify@Sum[j^(-1/2) (-1)^(j) Sin[x Log[j]], {j, 1, n}]
```

$$-i\ 2^{-\frac{3}{2}+i x}\ \text{Zeta}\left[\frac{1}{2}-i x\right] + i\ 2^{-\frac{3}{2}-i x}\ \text{Zeta}\left[\frac{1}{2}+i x\right] + i\ 2^{-\frac{3}{2}+i x}\ \text{Zeta}\left[\frac{1}{2}-i x,\ \frac{3}{2}\right] -$$
$$i\ (-1)^n\ 2^{-\frac{3}{2}+i x}\ \text{Zeta}\left[\frac{1}{2}-i x,\ \frac{1+n}{2}\right] + i\ (-1)^n\ 2^{-\frac{3}{2}+i x}\ \text{Zeta}\left[\frac{1}{2}-i x,\ \frac{2+n}{2}\right] -$$
$$i\ 2^{-\frac{3}{2}-i x}\ \text{Zeta}\left[\frac{1}{2}+i x,\ \frac{3}{2}\right] + i\ (-1)^n\ 2^{-\frac{3}{2}-i x}\ \text{Zeta}\left[\frac{1}{2}+i x,\ \frac{1+n}{2}\right] - i\ (-1)^n\ 2^{-\frac{3}{2}-i x}\ \text{Zeta}\left[\frac{1}{2}+i x,\ \frac{2+n}{2}\right]$$

```
fr[n_, x_] := {-i 2^(-3/2+i x) Zeta[1/2 - i x], +i 2^(-3/2-i x) Zeta[1/2 + i x],
  +i 2^(-3/2+i x) Zeta[1/2 - i x, 3/2], -i (-1)^n 2^(-3/2+i x) Zeta[1/2 - i x, (1+n)/2],
  +i (-1)^n 2^(-3/2+i x) Zeta[1/2 - i x, (2+n)/2], -i 2^(-3/2-i x) Zeta[1/2 + i x, 3/2],
  +i (-1)^n 2^(-3/2-i x) Zeta[1/2 + i x, (1+n)/2], -i (-1)^n 2^(-3/2-i x) Zeta[1/2 + i x, (2+n)/2]}
```

```
Chop@N@fr[100 000 000, .5]
```

```
{0.264488 + 0.268174 i, 0.264488 - 0.268174 i, -0.216139 - 0.461478 i, -2974.74 - 1910.75 i,
 2974.74 + 1910.75 i, -0.216139 + 0.461478 i, -2974.74 + 1910.75 i, 2974.74 - 1910.75 i}
```

```
N@Table[ sin[ Mod[N@Im@ZetaZero@100 Log[j], 2 Pi]], {j, 1, 200}] // TableForm
```

```
sin[0.]
sin[0.583285]
sin[2.23783]
sin[1.16657]
sin[3.67994]
sin[2.82111]
sin[1.58237]
sin[1.74985]
sin[4.47566]
sin[4.26323]
sin[1.67365]
sin[3.4044]
sin[3.48688]
sin[2.16566]
sin[5.91777]
sin[2.33314]
sin[4.10596]
sin[5.05894]
sin[5.28078]
sin[4.84651]
sin[3.8202]
sin[2.25694]
sin[0.204488]
sin[3.98768]
sin[1.0767]
sin[4.07017]
sin[0.430298]
sin[2.74894]
sin[4.7657]
sin[0.217871]
sin[1.69027]
sin[2.91642]
sin[3.91148]
sin[4.68925]
sin[5.26232]
sin[5.64223]
sin[5.83956]
sin[5.86406]
sin[5.72471]
sin[5.4298]
sin[4.98701]
```

```
sin[4.40348]
sin[3.68583]
sin[2.84022]
sin[1.87241]
sin[0.787773]
sin[5.87451]
sin[4.57097]
sin[3.16474]
sin[1.65999]
sin[0.0606035]
sin[4.65345]
sin[2.87563]
sin[1.01358]
sin[5.3536]
sin[3.33223]
sin[1.23542]
sin[5.34899]
sin[3.10905]
sin[0.801156]
sin[4.71074]
sin[2.27356]
sin[6.05803]
sin[3.49971]
sin[0.88364]
sin[4.49477]
sin[1.7684]
sin[5.27253]
sin[2.44232]
sin[5.8456]
sin[2.91743]
sin[6.22551]
sin[3.20478]
sin[0.139661]
sin[3.31453]
sin[0.164162]
sin[3.25603]
sin[0.0248087]
sin[3.0379]
sin[6.01308]
sin[2.66813]
sin[5.5703]
sin[2.15411]
sin[4.98677]
sin[1.50272]
sin[4.26912]
sin[0.720346]
sin[3.42351]
sin[6.09613]
sin[2.4557]
sin[5.06925]
sin[1.37106]
sin[3.9281]
sin[0.174611]
sin[2.67754]
sin[5.15425]
sin[1.32212]
```

```
sin[3.74803]
sin[6.14931]
sin[2.24327]
sin[4.59677]
sin[0.643888]
sin[2.95146]
sin[5.23674]
sin[1.21696]
sin[3.45891]
sin[5.67981]
sin[1.59687]
sin[3.77683]
sin[5.93688]
sin[1.7942]
sin[3.91551]
sin[6.01796]
sin[1.8187]
sin[3.88443]
sin[5.93227]
sin[1.67935]
sin[3.69234]
sin[5.68833]
sin[1.38444]
sin[3.34731]
sin[5.29402]
sin[0.941657]
sin[2.85684]
sin[4.75665]
sin[0.358127]
sin[2.22789]
sin[4.08299]
sin[5.92366]
sin[1.46693]
sin[3.27938]
sin[5.07805]
sin[0.579964]
sin[2.35169]
sin[4.11024]
sin[5.85582]
sin[1.30542]
sin[3.0256]
sin[4.73336]
sin[0.1457]
sin[1.82915]
sin[3.50071]
sin[5.16054]
sin[0.52561]
sin[2.16246]
sin[3.78806]
sin[5.40257]
sin[0.722946]
sin[2.31571]
sin[3.89782]
sin[5.46941]
sin[0.747447]
sin[2.29843]
```

```
sin[3.83931]
sin[5.37022]
sin[0.608094]
sin[2.11944]
sin[3.62118]
sin[5.11345]
sin[0.313183]
sin[1.78686]
sin[3.25141]
sin[4.70695]
sin[6.15358]
sin[1.30824]
sin[2.73739]
sin[4.15796]
sin[5.57005]
sin[0.690578]
sin[2.086]
sin[3.47325]
sin[4.8524]
sin[6.22356]
sin[1.30363]
sin[2.65907]
sin[4.00679]
sin[5.34688]
sin[0.396228]
sin[1.7213]
sin[3.03898]
sin[4.34937]
sin[5.65254]
sin[0.665379]
sin[1.95434]
sin[3.23632]
sin[4.51139]
sin[5.77962]
sin[0.757896]
sin[2.01267]
sin[3.26082]
sin[4.50242]
sin[5.73754]
sin[0.683052]
sin[1.9054]
sin[3.12147]
sin[4.33131]
sin[5.535]
sin[0.44941]
sin[1.64097]
sin[2.82656]
```

```
N@Mod[7, 2 Pi]
```

```
7.
```

```
Animate[DiscretePlot[ Mod[x Log[j], 2 Pi], {j, 1, 500}], {x, 100, 103}]
```

```
Animate[DiscretePlot[ Sin[Mod[x Log[j], 2 Pi]], {j, 1, 500}], {x, 100, 103}]
```

```
1 / I
```

$-i$

```
(-I x (E^(I (x Log[n] + c)) - E^(-I (x Log[n] + c))) +
   (1 / 2) (E^(I (x Log[n] + c)) + E^(-I (x Log[n] + c))))
 (1 / (j^(1 / 2))) (1 / (2 I)) (E^(I (x Log[n] + c)) - E^(-I (x Log[n] + c)))
```

$$-\frac{1}{2\sqrt{j}}$$

$$i\left(-e^{-i\,(c+x\,\mathrm{Log}[n])}+e^{i\,(c+x\,\mathrm{Log}[n])}\right)\left(\frac{1}{2}\left(e^{-i\,(c+x\,\mathrm{Log}[n])}+e^{i\,(c+x\,\mathrm{Log}[n])}\right)-i\left(-e^{-i\,(c+x\,\mathrm{Log}[n])}+e^{i\,(c+x\,\mathrm{Log}[n])}\right)x\right)$$

```
(x (E^(I (x Log[n] + c)) + E^(-I (x Log[n] + c))) +
   (1 / (2 I)) (E^(I (x Log[n] + c)) - E^(-I (x Log[n] + c))))
 (1 / (j^(1 / 2))) (1 / (2)) (E^(I (x Log[n] + c)) + E^(-I (x Log[n] + c)))
```

$$\frac{1}{2\sqrt{j}}\left(e^{-i\,(c+x\,\mathrm{Log}[n])}+e^{i\,(c+x\,\mathrm{Log}[n])}\right)$$

$$\left(-\frac{1}{2}i\left(-e^{-i\,(c+x\,\mathrm{Log}[n])}+e^{i\,(c+x\,\mathrm{Log}[n])}\right)+\left(e^{-i\,(c+x\,\mathrm{Log}[n])}+e^{i\,(c+x\,\mathrm{Log}[n])}\right)x\right)$$

```
FullSimplify[(-I x (E^(I (x Log[n] + c)) - E^(-I (x Log[n] + c))) +
      (1 / 2) (E^(I (x Log[n] + c)) + E^(-I (x Log[n] + c))))
   (1 / (j^(1 / 2))) (1 / (2 I)) (E^(I (x Log[n] + c)) - E^(-I (x Log[n] + c))) +
  (x (E^(I (x Log[n] + c)) + E^(-I (x Log[n] + c))) +
      (1 / (2 I)) (E^(I (x Log[n] + c)) - E^(-I (x Log[n] + c))))
   (1 / (j^(1 / 2))) (1 / (2)) (E^(I (x Log[n] + c)) + E^(-I (x Log[n] + c)))]
```

$$\frac{i\,e^{-2ic}\,n^{-2ix}-i\,e^{2ic}\,n^{2ix}+4x}{2\sqrt{j}}$$

```
N[(i e^(-2 i c) n^(-2 i x) - i e^(2 i c) n^(2 i x) + 4 x) / (2 Sqrt[j]) /. n -> 30 /. j -> 4 /. x -> 3 /. c -> 2]
```

$2.66822 + 0.\,i$

```
FullSimplify[(2 x Sin[x Log[n] + c] + Cos[x Log[n] + c]) (1 / (j^(1 / 2))) Sin[x Log[j] + c] +
  (2 x Cos[x Log[n] + c] - Sin[x Log[n] + c]) (1 / (j^(1 / 2))) Cos[x Log[j] + c]]
```

$$\frac{2x\,\mathrm{Cos}[x\,(\mathrm{Log}[j]-\mathrm{Log}[n])]+\mathrm{Sin}[x\,(\mathrm{Log}[j]-\mathrm{Log}[n])]}{\sqrt{j}}$$

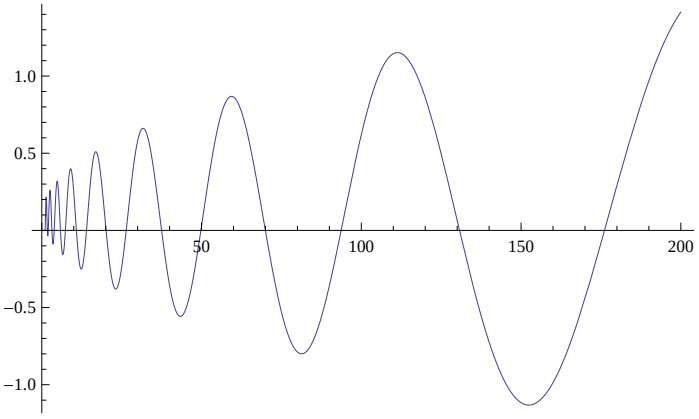
```
N[(2 x Cos[x (Log[j] - Log[n])] + Sin[x (Log[j] - Log[n])]) / Sqrt[j] /. n -> 30 /. j -> 4 /. x -> 3 /. c -> 2]
```

3.03321

```
Integrate[j^(-1 / 2) Sin[x Log[j]], {j, 1, n}]
```

$$\mathrm{ConditionalExpression}\left[\frac{4x+2\sqrt{n}\;(-2x\,\mathrm{Cos}[x\,\mathrm{Log}[n]]+\mathrm{Sin}[x\,\mathrm{Log}[n]])}{1+4x^2},\ \mathrm{Re}[n]\geq 0\ ||\ n\notin\mathrm{Reals}\right]$$

```
ag[n_, x_] := (4 x + 2 Sqrt[n] (-2 x Cos[x Log[n]] + Sin[x Log[n]])) / (1 + 4 x^2)

Plot[ag[n, 10], {n, 1, 200}]
```




```
Integrate[ j^(-1 / 2) (2 x Cos[x Log[j / n]] + Sin[x Log[j / n]]), {j, 1, n}]
```

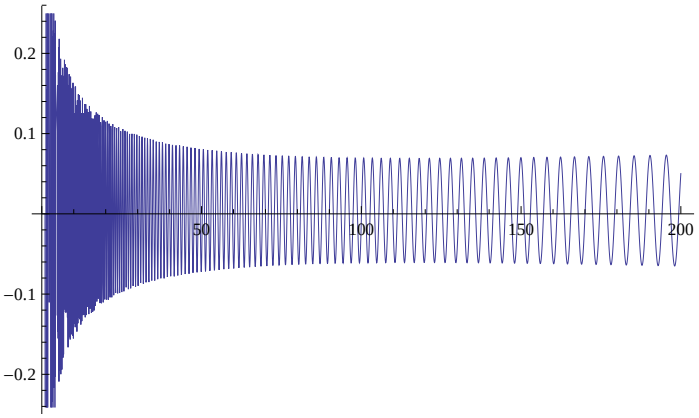
$$\text{ConditionalExpression}\left[-2 \sin\left[x \log\left[\frac{1}{n}\right]\right], \text{Re}[n] \geq 0 \,||\, n \notin \text{Reals}\right]$$

```
ag2[n_, x_] := -2 Sin[x Log[1 / n]]

Integrate[ j^(-1 / 2) Sin[x Log[j] + c], {j, 1, n}]
```

$$\text{ConditionalExpression}\left[\frac{1}{1+4x^2}\left(4x\cos[c] - 2\sin[c] + 2\sqrt{n}\,(-2x\cos[c + x\log[n]] + \sin[c + x\log[n]])\right), \text{Re}[n] \geq 0 \,||\, n \notin \text{Reals}\right]$$

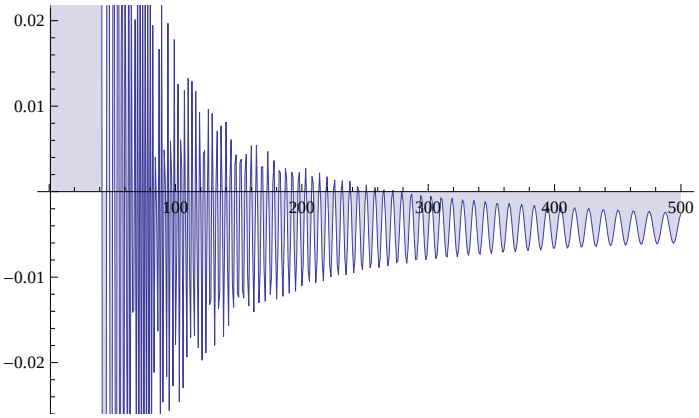
```
ag3[n_, x_, c_] := 1 / (1 + 4 x^2)
  (4 x Cos[c] - 2 Sin[c] + 2 Sqrt[n] (-2 x Cos[c + x Log[n]] + Sin[c + x Log[n]]))
ag3as[j_, x_, c_] := j^(-1 / 2) Sin[x Log[j] + c]
ag3a[n_, x_, c_] :=
  1 / (1 + 4 x^2) (4 x Cos[c] - 2 Sin[c] + 2 Sqrt[n] (-2 x Cos[c + x Log[n]] + Sin[c + x Log[n]])) -
   (Sin[c + x Log[n]] / Sqrt[n] + Sin[c]) / 2

ag3as[1, x, c]
Sin[c]
```

```
Plot[ag3a[n, N@Im@ZetaZero@100, 0], {n, 1, 200}]
```




```
ag4[n_, x_, c_] := Sum[ j^(-1 / 2) Sin[x Log[j] + c], {j, 2, n}]
```

```
DiscretePlot[
 {ag4[n - 1, N@Im@ZetaZero@100, 0] - ag3a[n, N@Im@ZetaZero@100, 0]}, {n, 1, 500}]
```




```
Integrate[ j^(-1 / 2) (2 x Cos[x Log[j / n]] + Sin[x Log[j / n]]), {j, 0, n}]
```

$$\text{ConditionalExpression}\left[0\,\text{Sin}[x\,(-\infty)],\ -\frac{1}{2} < \text{Im}[x] < \frac{1}{2}\right]$$

```
Integrate[ j^(-1 / 2) (2 x Cos[x Log[j / n]] + Sin[x Log[j / n]]), j]
```

$$2\sqrt{j}\ \text{Sin}\left[x\,\text{Log}\left[\frac{j}{n}\right]\right]$$

```
FullSimplify[D[j^(-1 / 2) Sin[x Log[j] + c], {j, 0}]]
```

$$\frac{\text{Sin}[c + x\,\text{Log}[j]]}{\sqrt{j}}$$

```
FullSimplify[D[j^(-1 / 2) Sin[x Log[j] + c], {j, 1}]]
```

$$-\frac{-2x\,\text{Cos}[c + x\,\text{Log}[j]] + \text{Sin}[c + x\,\text{Log}[j]]}{2\,j^{3/2}}$$

```
FullSimplify[D[j^(-1 / 2) Sin[x Log[j] + c], {j, 2}]]
```

$$\frac{-8x\,\text{Cos}[c + x\,\text{Log}[j]] + \left(3 - 4x^2\right)\text{Sin}[c + x\,\text{Log}[j]]}{4\,j^{5/2}}$$

```
FullSimplify[D[j^(-1 / 2) Sin[x Log[j] + c], {j, 3}]]
```

$$\frac{\left(46\,x - 8\,x^3\right)\,\text{Cos}[c + x\,\text{Log}[j]] + 3\,\left(-5 + 12\,x^2\right)\,\text{Sin}[c + x\,\text{Log}[j]]}{8\,j^{7/2}}$$

```
FullSimplify[D[j^(-1 / 2) Sin[x Log[j] + c], {j, 4}]]
```

$$\frac{1}{16\,j^{9/2}}\left(32\,x\,\left(-11 + 4\,x^2\right)\,\text{Cos}[c + x\,\text{Log}[j]] + \left(105 - 344\,x^2 + 16\,x^4\right)\,\text{Sin}[c + x\,\text{Log}[j]]\right)$$

```
Integrate[ j^(-1 / 2) Sin[x Log[j] + c], j]
```

$$-\frac{2\,\sqrt{j}\;\left(2\,x\,\text{Cos}[c + x\,\text{Log}[j]] - \text{Sin}[c + x\,\text{Log}[j]]\right)}{1 + 4\,x^2}$$

```
Limit[-(2 Sqrt[j] (2 x Cos[c + x Log[j]] - Sin[c + x Log[j]]))/(1 + 4 x^2), j -> 0]
```

$$0$$

```
-(2 Sqrt[j] (2 x Cos[c + x Log[j]] - Sin[c + x Log[j]]))/(1 + 4 x^2) /. j -> n
```

$$-\frac{2\,\sqrt{n}\;\left(2\,x\,\text{Cos}[c + x\,\text{Log}[n]] - \text{Sin}[c + x\,\text{Log}[n]]\right)}{1 + 4\,x^2}$$

```
Integrate[ j^(-1 / 2) Sin[x Log[j] + c], {j, 1, n}]
```

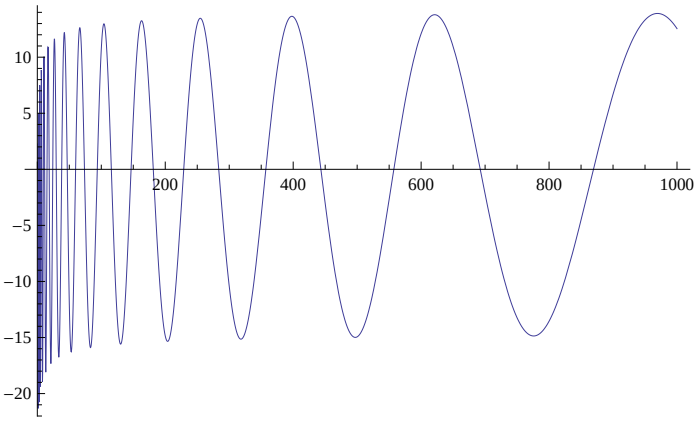
$$\text{ConditionalExpression}\left[\frac{1}{1 + 4\,x^2}\left(4\,x\,\text{Cos}[c] - 2\,\text{Sin}[c] + 2\,\sqrt{n}\;\left(-2\,x\,\text{Cos}[c + x\,\text{Log}[n]] + \text{Sin}[c + x\,\text{Log}[n]]\right)\right), \text{Re}[n] \geq 0 \;||\; n \notin \text{Reals}\right]$$

```
FullSimplify[
 Integrate[ j^(-1 / 2) (2 x Cos[x Log[j / (n + 1)]] + Sin[x Log[j / (n + 1)]]), {j, 1, n + 1}] -
  ( ((n + 1)^(-1 / 2) (2 x Cos[x Log[(n + 1) / (n + 1)]] + Sin[x Log[(n + 1) / (n + 1)]])) +
     (1^(-1 / 2) (2 x Cos[x Log[1 / (n + 1)]] + Sin[x Log[1 / (n + 1)]]))) / 2]
```

$$\text{ConditionalExpression}\left[-\frac{x}{\sqrt{1 + n}} - x\,\text{Cos}\left[x\,\text{Log}\left[\frac{1}{1 + n}\right]\right] - \frac{5}{2}\,\text{Sin}\left[x\,\text{Log}\left[\frac{1}{1 + n}\right]\right], \text{Re}[n] \geq -1 \;||\; n \notin \text{Reals}\right]$$

```
p[n_, x_] := -x/Sqrt[1 + n] - x Cos[x Log[1/(1 + n)]] - 5/2 Sin[x Log[1/(1 + n)]]
```

```
Plot[p[n, Im@ZetaZero@1], {n, 1, 1000}]
```



$$2\sqrt{j}\ \text{Sin}\left[x\ \text{Log}\left[\frac{j}{n}\right]\right]\ /.\ j \to 0$$

$$0\ \text{Sin}[x\ (-\infty)]$$

```
FullSimplify[Sum[ j^(-1 / 2) Sin[x j + c], {j, 1, n}]]
```

$$-\frac{1}{2}\,i\,e^{-i\,(c+x+n\,x)}\left(\text{LerchPhi}\left[e^{-i\,x},\ \frac{1}{2},\ 1+n\right]-e^{2\,i\,(c+x+n\,x)}\ \text{LerchPhi}\left[e^{i\,x},\ \frac{1}{2},\ 1+n\right]+e^{i\,(1+n)\,x}\left(-\text{PolyLog}\left[\frac{1}{2},\ e^{-i\,x}\right]+e^{2\,i\,c}\ \text{PolyLog}\left[\frac{1}{2},\ e^{i\,x}\right]\right)\right)$$

```
Sum[ j^(-1 / 2) Sin[x j + c], {j, 1, Infinity}]
```

$$-\frac{1}{2}\,i\,e^{-i\,c}\left(-\text{PolyLog}\left[\frac{1}{2},\ e^{-i\,x}\right]+e^{2\,i\,c}\ \text{PolyLog}\left[\frac{1}{2},\ e^{i\,x}\right]\right)$$

```
Sum[1 / j Sin[x j + c], {j, 1, n}]
```

$$\frac{1}{2}\,i\,e^{-i\,c-i\,x}\left(-\left(e^{-i\,x}\right)^{n}\ \text{LerchPhi}\left[e^{-i\,x},\ 1,\ 1+n\right]+e^{2\,i\,c+2\,i\,x}\left(e^{i\,x}\right)^{n}\ \text{LerchPhi}\left[e^{i\,x},\ 1,\ 1+n\right]+e^{2\,i\,c+i\,x}\ \text{Log}\left[1-e^{i\,x}\right]-e^{i\,x}\ \text{Log}\left[e^{-i\,x}\left(-1+e^{i\,x}\right)\right]\right)$$

```
FullSimplify@Integrate[j^(-1 / 2) Sin[x j + c], {j, 1, n}]
```

$$\text{ConditionalExpression}\left[\frac{1}{\sqrt{x}}\sqrt{2\,\pi}\left(\text{Cos}[c]\left(-\text{FresnelS}\left[\sqrt{\frac{2}{\pi}}\ \sqrt{x}\right]+\text{FresnelS}\left[\sqrt{n}\ \sqrt{\frac{2}{\pi}}\ \sqrt{x}\right]\right)+\left(-\text{FresnelC}\left[\sqrt{\frac{2}{\pi}}\ \sqrt{x}\right]+\text{FresnelC}\left[\sqrt{n}\ \sqrt{\frac{2}{\pi}}\ \sqrt{x}\right]\right)\text{Sin}[c]\right),\ \text{Re}[n]\geq 0\ ||\ n\notin\text{Reals}\right]$$

```
Integrate[ j^(-1 / 2) Sin[x j + c], {j, 1, Infinity}]
```

$$\frac{1}{x^2}\sqrt{\frac{\pi}{2}}\left(x\left(x^2\right)^{1/4}\text{Cos}[c]+\left(x^2\right)^{3/4}\text{Sin}[c]-2\,x^{3/2}\left(\text{Cos}[c]\ \text{FresnelS}\left[\sqrt{\frac{2}{\pi}}\ \sqrt{x}\right]+\text{FresnelC}\left[\sqrt{\frac{2}{\pi}}\ \sqrt{x}\right]\text{Sin}[c]\right)\right)$$

```
Integrate[ j^(-1) Sin[x j + c], {j, 1, Infinity}]
```

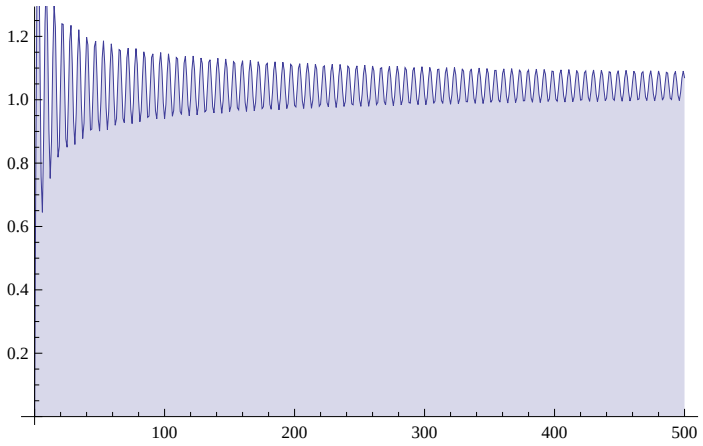
$$\frac{\pi \sqrt{x^2}\ \text{Cos}[c]}{2\,x} - \text{CosIntegral}[x]\ \text{Sin}[c] + \text{Log}[x]\ \text{Sin}[c] - \frac{1}{2}\ \text{Log}\left[x^2\right]\ \text{Sin}[c] - \text{Cos}[c]\ \text{SinIntegral}[x]$$

```
Integrate[ j^(-1) Sin[x j + c], {j, 1, Infinity}]
```

```
cp[n_, x_, c_] := -1/2 i e^(-i (c+x+n x)) (LerchPhi[e^(-i x), 1/2, 1 + n] -
    e^(2 i (c+x+n x)) LerchPhi[e^(i x), 1/2, 1 + n] + e^(i (1+n) x) (-PolyLog[1/2, e^(-i x)] + e^(2 i c) PolyLog[1/2, e^(i x)]))
```

```
cp2[n_, x_, c_] := 1/Sqrt[x] Sqrt[2 π] (Cos[c] (-FresnelS[Sqrt[2/π] Sqrt[x]] + FresnelS[Sqrt[n] Sqrt[2/π] Sqrt[x]]) +
    (-FresnelC[Sqrt[2/π] Sqrt[x]] + FresnelC[Sqrt[n] Sqrt[2/π] Sqrt[x]]) Sin[c])
```

```
DiscretePlot[ Re@cp[n, 1, 0], {n, 0, 500}]
```




```
FullSimplify[Sum[ j^(-1 / 2) Sinh[Log[x] j + Log[c]], {j, 1, Infinity}]] /. c → 1
```

$$-\frac{1}{2}\ \text{PolyLog}\left[\frac{1}{2}, \frac{1}{x}\right] + \frac{1}{2}\ \text{PolyLog}\left[\frac{1}{2}, x\right]$$

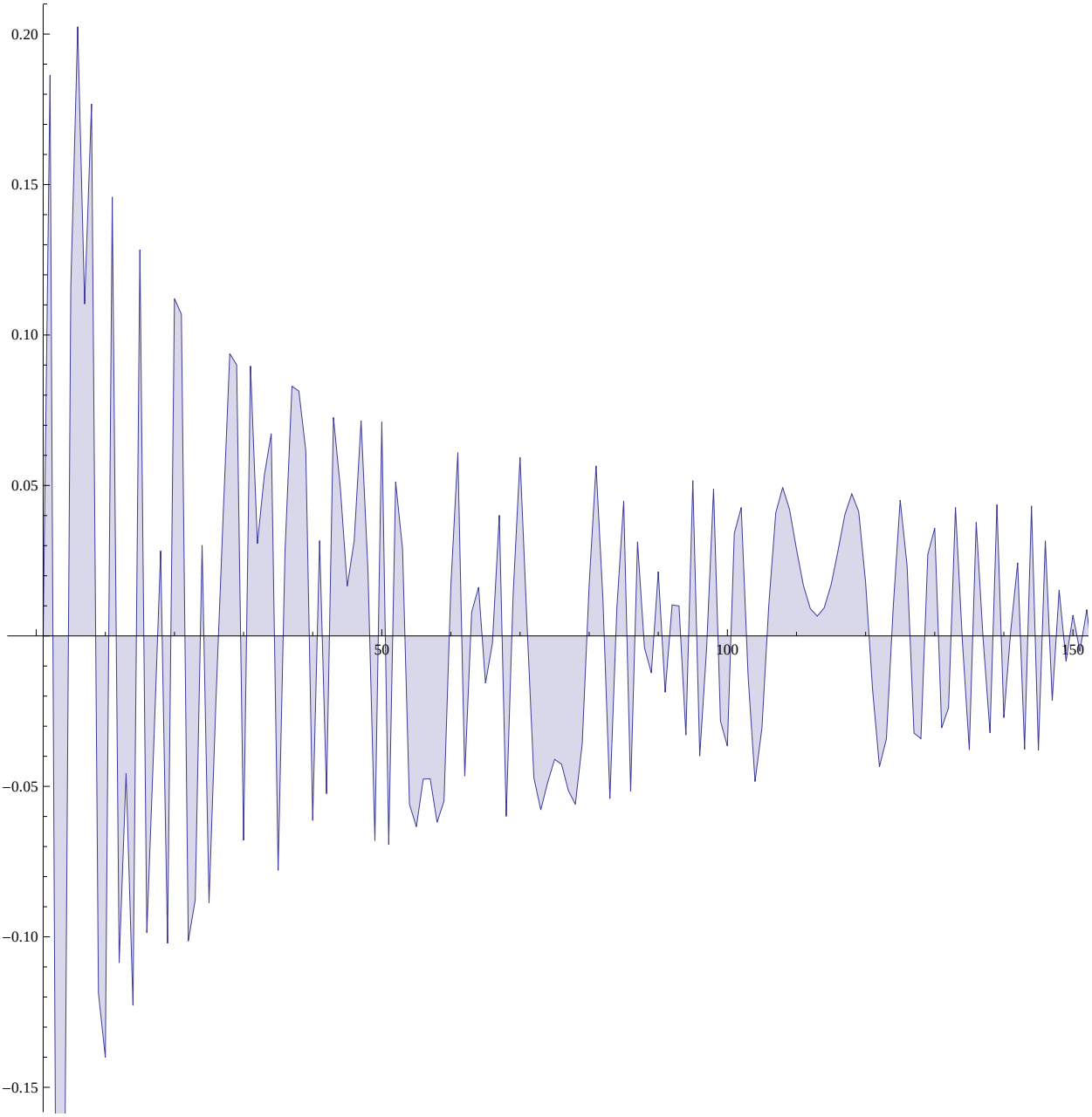
```
Integrate[ j^(-1 / 2) Sin[x Log[j] + c], {j, 1, n}]
```

$$\text{ConditionalExpression}\left[\frac{1}{1 + 4\,x^2}\left(4\,x\,\text{Cos}[c] - 2\,\text{Sin}[c] + 2\sqrt{n}\ \left(-2\,x\,\text{Cos}[c + x\,\text{Log}[n]] + \text{Sin}[c + x\,\text{Log}[n]]\right)\right), \text{Re}[n] \geq 0 \,||\, n \notin \text{Reals}\right]$$

```
FullSimplify[D[1/(1 + 4 x^2) (4 x Cos[c] - 2 Sin[c] + 2 Sqrt[n] (-2 x Cos[c + x Log[n]] + Sin[c + x Log[n]])) -
    (Sin[c + x Log[n]]/Sqrt[n] + Sin[c]) / 2, n] /. c → 0]
```

$$\frac{-2\,x\,\text{Cos}[x\,\text{Log}[n]] + (1 + 4\,n)\,\text{Sin}[x\,\text{Log}[n]]}{4\,n^{3/2}}$$

```
DiscretePlot[1/(1 + 4 x^2) (4 x Cos[c] - 2 Sin[c] + 2 Sqrt[n] (-2 x Cos[c + x Log[n]] + Sin[c + x Log[n]])) -
    (Sin[c + x Log[n]]/Sqrt[n] + Sin[c]) / 2 /. c → 0 /. x -> 1419.4224809459956, {n, 1, 250}]
```



```
1 / j^x
```

$$j^{-x}$$

```
FullSimplify@Sum[ 2 (-1)^(j + 1) j^(-1 / 2) Sinh[x Log[j]], {j, 1, n}]
```

$$2^{-\frac{1}{2}-x}\left(\text{Zeta}\left[\frac{1}{2}+x\right]+4^x\left(-\text{Zeta}\left[\frac{1}{2}-x\right]+\text{Zeta}\left[\frac{1}{2}-x,\frac{3}{2}\right]+(-1)^n\left(-\text{Zeta}\left[\frac{1}{2}-x,\frac{1+n}{2}\right]+\text{Zeta}\left[\frac{1}{2}-x,\frac{2+n}{2}\right]\right)\right)-\text{Zeta}\left[\frac{1}{2}+x,\frac{3}{2}\right]+(-1)^n\,\text{Zeta}\left[\frac{1}{2}+x,\frac{1+n}{2}\right]-(-1)^n\,\text{Zeta}\left[\frac{1}{2}+x,\frac{2+n}{2}\right]\right)$$

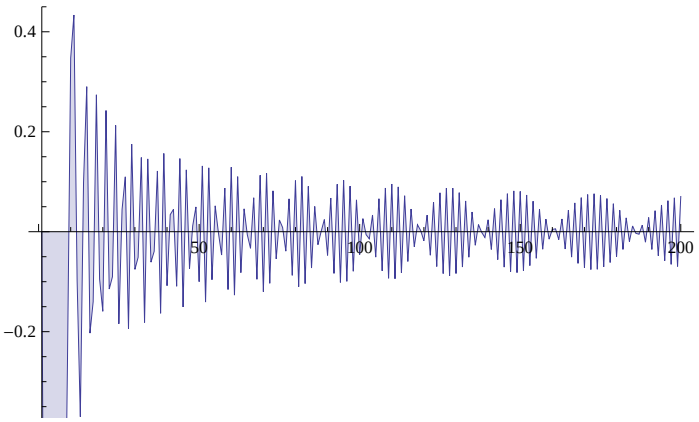
```
2^(-1/2-x) (Zeta[1/2 + x] +
    4^x (-Zeta[1/2 - x] + Zeta[1/2 - x, 3/2] + (-1)^n (-Zeta[1/2 - x, (1 + n)/2] + Zeta[1/2 - x, (2 + n)/2])) -
    Zeta[1/2 + x, 3/2] + (-1)^n Zeta[1/2 + x, (1 + n)/2] - (-1)^n Zeta[1/2 + x, (2 + n)/2]) /.
  x → 14.134725141734695` I /. n → 10 000 000 000.0
```

$$2.17031\times10^{-11}+9.52564\times10^{-6}\,i$$

```
N@ZetaZero@1
```

$$0.5+14.1347\,i$$

```
so[n_, x_] := Sum[ 2 (-1)^(j + 1) j^(-1 / 2) Sin[x Log[j]], {j, 1, n}]
DiscretePlot[so[n, Im@ZetaZero@2], {n, 1, 200}]
```




```
{((1 - 2^(1 / 2 - x))^-1), ((1 - 2^(1 / 2 + x))^-1)}
```

$$\left\{\frac{1}{1-2^{\frac{1}{2}-x}},\frac{1}{1-2^{\frac{1}{2}+x}}\right\}$$

```
FullSimplify[(((1 - 2^(1/2 - x))^-1) / (2^(-1/2)))]
```

$$\sqrt{2} + \frac{2}{-\sqrt{2} + 2^x}$$

```
j^(-1/2)
 (((1 - 2^(1/2 - x))^-1) ((-1)^(j + 1) / j^x) - ((1 - 2^(1/2 + x))^-1) ((-1)^(j + 1) / j^-x))
```

$$\frac{\frac{(-1)^{1+j}\, j^{-x}}{1-2^{\frac{1}{2}-x}} - \frac{(-1)^{1+j}\, j^{x}}{1-2^{\frac{1}{2}+x}}}{\sqrt{j}}$$

```
FullSimplify[1/(1 + 4 x^2) (4 x Cos[c] - 2 Sin[c] + 2 Sqrt[n] (-2 x Cos[c + x Log[n]] + Sin[c + x Log[n]]))]
```

$$\frac{1}{1 + 4\, x^2} \left(4\, x \cos[c] - 2 \sin[c] + 2 \sqrt{n}\, (-2\, x \cos[c + x \log[n]] + \sin[c + x \log[n]])\right)$$

```
Integrate[ j^(-1/2) Sin[x Log[j]], j]
```

$$\frac{2 \sqrt{j}\, (-2\, x \cos[x \log[j]] + \sin[x \log[j]])}{1 + 4\, x^2}$$

```
(Integrate[ j^(-1/2) Sin[x Log[j] + c], j] /. j → n) -
 (Integrate[ j^(-1/2) Sin[x Log[j] + c], j] /. j → 1)
```

$$\frac{2\, (2\, x \cos[c] - \sin[c])}{1 + 4\, x^2} - \frac{2 \sqrt{n}\, (2\, x \cos[c + x \log[n]] - \sin[c + x \log[n]])}{1 + 4\, x^2}$$

```
Integrate[ j^(-1/2) Sin[x Log[j] + C], {j, 1, n + 1}]
```

$$\text{ConditionalExpression}\left[\frac{1}{1 + 4\, x^2} \left(4\, x \cos[C] - 2 \sin[C] + 2 \sqrt{1 + n}\, (-2\, x \cos[C + x \log[1 + n]] + \sin[C + x \log[1 + n]])\right), \text{Re}[n] \geq -1 \,||\, n \notin \text{Reals}\right]$$

```
Integrate[ j^(-1/2) Sin[x Log[j]], j] /. j → n
```

$$\frac{2 \sqrt{n}\, (-2\, x \cos[x \log[n]] + \sin[x \log[n]])}{1 + 4\, x^2}$$

```
Integrate[ j^(-1/2) Cos[x Log[j]], {j, 1, n}]
```

$$\text{ConditionalExpression}\left[\frac{-2 + 2 \sqrt{n}\, (\cos[x \log[n]] + 2\, x \sin[x \log[n]])}{1 + 4\, x^2}, \text{Re}[n] \geq 0 \,||\, n \notin \text{Reals}\right]$$

```
ff[j_, x_] := j^(-1/2) Sin[x Log[j]]

Sum[ ff[j, x], {j, 1, b}]
```

$$\sum_{j=1}^{b} \frac{\sin[x \log[j]]}{\sqrt{j}}$$

```
Integrate[ ff[j, x], {j, a, b}] + (ff[b, x] + ff[a, x]) / 2
```

$$\text{ConditionalExpression}\left[\frac{1}{2}\left(\frac{\text{Sin}[x\,\text{Log}[a]]}{\sqrt{a}}+\frac{\text{Sin}[x\,\text{Log}[b]]}{\sqrt{b}}\right)+\frac{1}{1+4x^2}\left(4\sqrt{a}\,x\,\text{Cos}[x\,\text{Log}[a]]-2\sqrt{a}\,\text{Sin}[x\,\text{Log}[a]]+2\sqrt{b}\,(-2x\,\text{Cos}[x\,\text{Log}[b]]+\text{Sin}[x\,\text{Log}[b]])\right),\ ((\text{Im}[a]\geq\text{Im}[b]\ \&\&\ \text{Im}[b]\,\text{Re}[a]\leq\text{Im}[a]\,\text{Re}[b])\ ||\ (\text{Im}[b]\,\text{Re}[a]\geq\text{Im}[a]\,\text{Re}[b]\ \&\&\ \text{Im}[a]\leq\text{Im}[b]))\ \&\&\ \left(\left(\text{Re}\left[\frac{a}{-a+b}\right]\geq 0\ \&\&\ a^2\neq a\,b\right)\ ||\ \frac{a}{a-b}\notin\text{Reals}\ ||\ \text{Re}\left[\frac{a}{a-b}\right]\geq 1\right)\right]$$

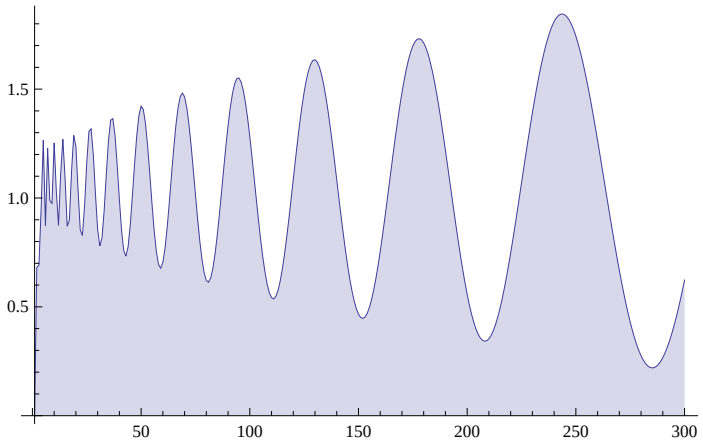
```
fr[x_, a_, b_] := 1/2 (Sin[x Log[a]]/Sqrt[a] + Sin[x Log[b]]/Sqrt[b]) + 1/(1 + 4 x^2)
  (4 Sqrt[a] x Cos[x Log[a]] - 2 Sqrt[a] Sin[x Log[a]] + 2 Sqrt[b] (-2 x Cos[x Log[b]] + Sin[x Log[b]]))
frs[x_, a_, b_] := Sum[ ff[j, x], {j, a, b}]
fr[x, 1, n]
```

$$\frac{\text{Sin}[x\,\text{Log}[n]]}{2\sqrt{n}}+\frac{4x+2\sqrt{n}\,(-2x\,\text{Cos}[x\,\text{Log}[n]]+\text{Sin}[x\,\text{Log}[n]])}{1+4x^2}$$

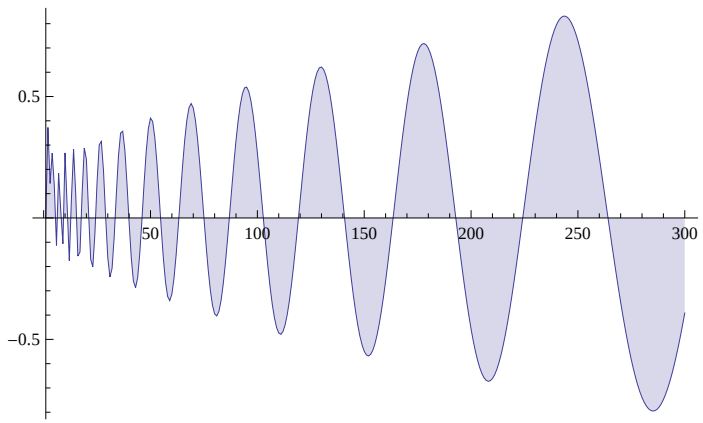
```
N@fr[20, 1, 20]
```

```
0.241714
```

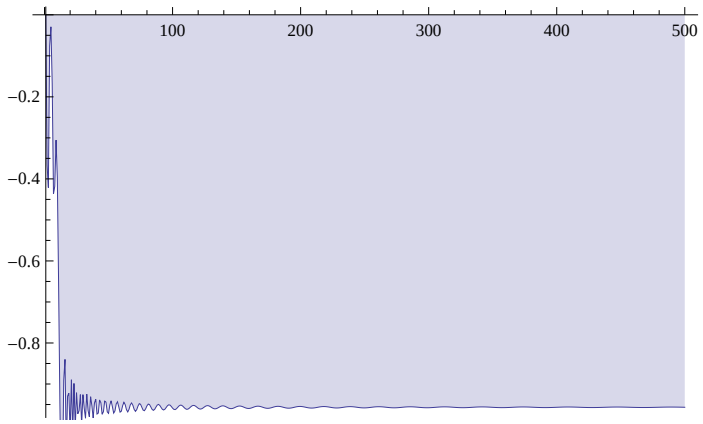
```
DiscretePlot[frs[20, 1, n], {n, 1, 300}]
```




```
DiscretePlot[fr[20, 1, n], {n, 1, 300}]
```



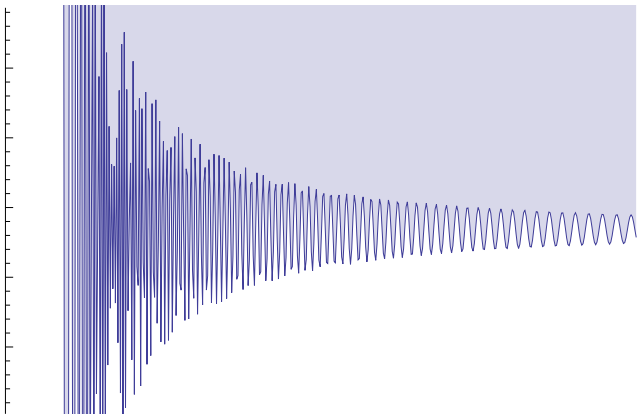
```
DiscretePlot[frs[70, 1, n] - fr[70, 1, n], {n, 1, 500}]
```




```
ff2[j_, x_, c_] := j^(-1 / 2) Sin[x Log[j] + c]
fr2[x_, a_, b_, c_] :=
 1/2 (Sin[c + x Log[a]]/Sqrt[a] + Sin[c + x Log[b]]/Sqrt[b]) + 1/(1 + 4 x^2) (4 Sqrt[a] x Cos[c + x Log[a]] -
    2 Sqrt[a] Sin[c + x Log[a]] + 2 Sqrt[b] (-2 x Cos[c + x Log[b]] + Sin[c + x Log[b]]))
frs2[x_, a_, b_, c_] := Sum[ ff2[j, x, c], {j, a, b}]
Integrate[ ff2[j, x, c], {j, a, b}] + (ff2[b, x, c] + ff2[a, x, c]) / 2
```

```
ConditionalExpression[
 1/2 (Sin[c + x Log[a]]/Sqrt[a] + Sin[c + x Log[b]]/Sqrt[b]) + 1/(1 + 4 x^2) (4 Sqrt[a] x Cos[c + x Log[a]] -
    2 Sqrt[a] Sin[c + x Log[a]] + 2 Sqrt[b] (-2 x Cos[c + x Log[b]] + Sin[c + x Log[b]])),
 ((Im[a] ≥ Im[b] && Im[b] Re[a] ≤ Im[a] Re[b]) || (Im[b] Re[a] ≥ Im[a] Re[b] && Im[a] ≤ Im[b])) &&
  ((Re[a/(-a + b)] ≥ 0 && a^2 ≠ a b) || a/(a - b) ∉ Reals || Re[a/(a - b)] ≥ 1)]
```

```
DiscretePlot[
 frs2[N@Im@ZetaZero@120, 1, n, 2] - fr2[N@Im@ZetaZero@120, 1, n, 2], {n, 1, 500}]
```



```
(D[ff2[b, x, c], {b, 2}] - D[ff2[a, x, c], {a, 2}]) / 12
```

$$\frac{1}{12}\left(\frac{x\cos[c+x\log[a]]}{a^{5/2}}-\frac{x\cos[c+x\log[b]]}{b^{5/2}}-\frac{3\sin[c+x\log[a]]}{4\,a^{5/2}}-\frac{-\frac{x\cos[c+x\log[a]]}{a^2}-\frac{x^2\sin[c+x\log[a]]}{a^2}}{\sqrt{a}}+\frac{3\sin[c+x\log[b]]}{4\,b^{5/2}}+\frac{-\frac{x\cos[c+x\log[b]]}{b^2}-\frac{x^2\sin[c+x\log[b]]}{b^2}}{\sqrt{b}}\right)$$

```
N[ZetaZero[1000]]
```

$$0.5+1419.42\,i$$

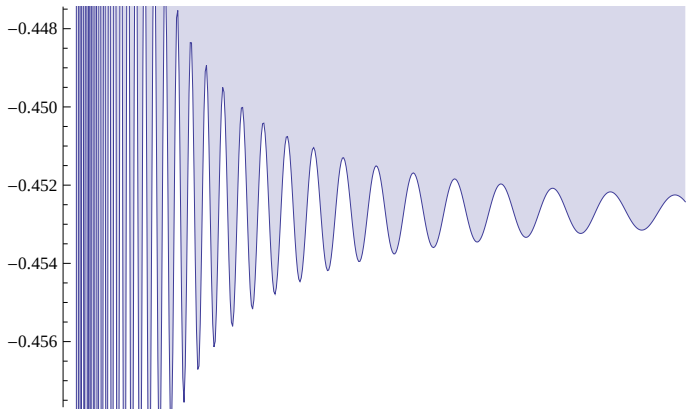
```
FullSimplify@Integrate[j^(-1/2) Sin[x Log[a j]], {j, 1, n}]
```

$$\text{ConditionalExpression}\left[\frac{1}{1+4x^2}\left(4x\cos[x\log[a]]-2\sin[x\log[a]]+2\sqrt{n}\,(-2x\cos[x\log[a\,n]]+\sin[x\log[a\,n]])\right),\ \text{Re}[n]\geq 0\ ||\ n\notin\text{Reals}\right]$$

```
ff4[j_, x_, c_] := j^(-1/2) Sin[x Log[c j]]
fr4[x_, a_, b_, c_] := 1/2 (Sin[x Log[a c]]/Sqrt[a] + Sin[x Log[b c]]/Sqrt[b]) + 1/(1 + 4 x^2)
  (4 Sqrt[a] x Cos[x Log[a c]] - 2 Sqrt[a] Sin[x Log[a c]] + 2 Sqrt[b] (-2 x Cos[x Log[b c]] + Sin[x Log[b c]]))
frs4[x_, a_, b_, c_] := Sum[ff4[j, x, c], {j, a, b}]
Integrate[ff4[j, x, c], {j, a, b}] + (ff4[b, x, c] + ff4[a, x, c]) / 2
```

$$\text{ConditionalExpression}\left[\frac{1}{2}\left(\frac{\sin[x\log[a\,c]]}{\sqrt{a}}+\frac{\sin[x\log[b\,c]]}{\sqrt{b}}\right)+\frac{1}{1+4x^2}\left(4\sqrt{a}\,x\cos[x\log[a\,c]]-2\sqrt{a}\,\sin[x\log[a\,c]]+2\sqrt{b}\,(-2x\cos[x\log[b\,c]]+\sin[x\log[b\,c]])\right),\ ((\text{Im}[a]\geq\text{Im}[b]\ \&\&\ \text{Im}[b]\,\text{Re}[a]\leq\text{Im}[a]\,\text{Re}[b])\ ||\ (\text{Im}[b]\,\text{Re}[a]\geq\text{Im}[a]\,\text{Re}[b]\ \&\&\ \text{Im}[a]\leq\text{Im}[b]))\ \&\&\ \left(\left(\text{Re}\left[\frac{a}{-a+b}\right]\geq 0\ \&\&\ a^2\neq a\,b\right)\ ||\ \frac{a}{a-b}\notin\text{Reals}\ ||\ \text{Re}\left[\frac{a}{a-b}\right]\geq 1\right)\right]$$

```
DiscretePlot[
 frs4[N@Im@ZetaZero@12, 1, n, .1] - fr4[N@Im@ZetaZero@12, 1, n, .1], {n, 1, 500}]
```

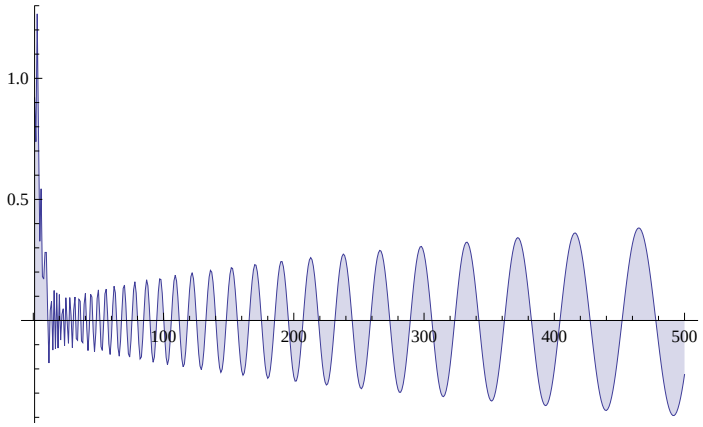


```
FullSimplify[fr4[x, 1, n, c]] /. c → a
```

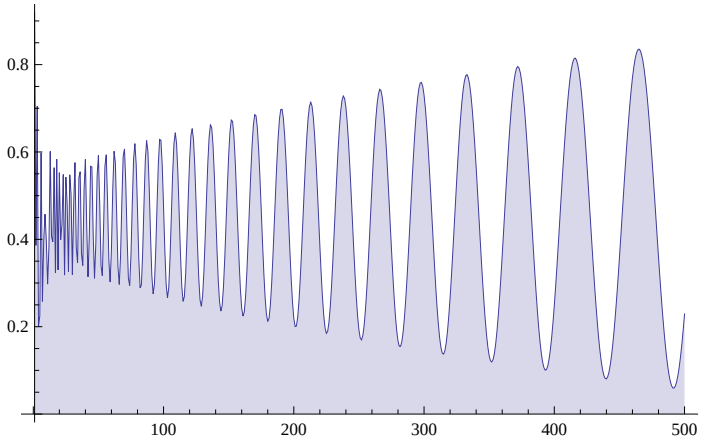
$$\frac{1}{2}\left(\text{Sin}[x\,\text{Log}[a]]+\frac{\text{Sin}[x\,\text{Log}[a\,n]]}{\sqrt{n}}\right)+\frac{1}{1+4\,x^2}$$
$$\left(4\,x\,\text{Cos}[x\,\text{Log}[a]]-2\,\text{Sin}[x\,\text{Log}[a]]+2\sqrt{n}\ (-2\,x\,\text{Cos}[x\,\text{Log}[a\,n]]+\text{Sin}[x\,\text{Log}[a\,n]])\right)$$

$$\frac{1}{2}\left(\text{Sin}[c]+\frac{\text{Sin}[c+x\,\text{Log}[a]]}{\sqrt{a}}\right)+\frac{1}{1+4\,x^2}$$
$$\left(4\sqrt{a}\ x\,\text{Cos}[c+x\,\text{Log}[a]]+2\,(-2\,x\,\text{Cos}[c]+\text{Sin}[c])-2\sqrt{a}\ \text{Sin}[c+x\,\text{Log}[a]]\right)$$

```
DiscretePlot[frs4[N@Im@ZetaZero@12, 1, n, .1], {n, 1, 500}]
```




```
DiscretePlot[fr4[N@Im@ZetaZero@12, 1, n, .1], {n, 1, 500}]
```




```
FullSimplify[1/2 (Sin[x Log[a]] + Sin[x Log[a n]]/Sqrt[n]) + 1/(1 + 4 x^2)
 (4 x Cos[x Log[a]] - 2 Sin[x Log[a]] + 2 Sqrt[n] (-2 x Cos[x Log[a n]] + Sin[x Log[a n]]))]
```

$$\frac{1}{2}\left(\text{Sin}[x\,\text{Log}[a]]+\frac{\text{Sin}[x\,\text{Log}[a\,n]]}{\sqrt{n}}\right)+\frac{1}{1+4\,x^2}$$
$$\left(4\,x\,\text{Cos}[x\,\text{Log}[a]]-2\,\text{Sin}[x\,\text{Log}[a]]+2\sqrt{n}\ (-2\,x\,\text{Cos}[x\,\text{Log}[a\,n]]+\text{Sin}[x\,\text{Log}[a\,n]])\right)$$