```
rr14a[n_, m_, d_] := Sum[j^-m Cosh[d Log[j]], {j, 1, n}] -
  Coth[ArcTanh[d/(-1 + m)] + d Log[n]] Sum[j^-m Sinh[d Log[j]], {j, 1, n}]
rr14[n_, s_, t_] := rr14a[n, (s + t) / 2, (s - t) / 2]
Expand[rr14[n, s, t]]
```

$$\sum_{j=1}^{n} j^{\frac{1}{2}(-s-t)} \operatorname{Cosh}\left[\frac{1}{2}(s-t)\operatorname{Log}[j]\right] -$$

$$\operatorname{Coth}\left[\operatorname{ArcTanh}\left[\frac{s-t}{2\left(-1+\frac{s+t}{2}\right)}\right]+\frac{1}{2}(s-t)\operatorname{Log}[n]\right] \sum_{j=1}^{n} j^{\frac{1}{2}(-s-t)} \operatorname{Sinh}\left[\frac{1}{2}(s-t)\operatorname{Log}[j]\right]$$

$$\sum_{j=1}^{n}\left(\frac{1}{2}\left(j^{-s}+j^{-t}\right)\right) - \operatorname{Coth}\left[\operatorname{ArcTanh}\left[\frac{s-t}{2\left(-1+\frac{s+t}{2}\right)}\right]+\frac{1}{2}(s-t)\operatorname{Log}[n]\right] \sum_{j=1}^{n}\left(\frac{1}{2}\left(-j^{-s}+j^{-t}\right)\right)$$

$$\text{FullSimplify}\left[\text{TrigToExp}\left[-\operatorname{Coth}\left[\operatorname{ArcTanh}\left[\frac{s-t}{2\left(-1+\frac{s+t}{2}\right)}\right]+\frac{1}{2}(s-t)\operatorname{Log}[n]\right]\right]\right]$$

$$1+\frac{2}{-1+\frac{n^{-s+t}(-1+t)}{-1+s}}$$

$$\sum_{j=1}^{n}\left(\frac{1}{2}\left(j^{-s}+j^{-t}\right)\right)+\left(1+\frac{2}{-1+\frac{n^{-s+t}(-1+t)}{-1+s}}\right)\sum_{j=1}^{n}\left(\frac{1}{2}\left(-j^{-s}+j^{-t}\right)\right)$$

$$\sum_{j=1}^{n}\left(\frac{1}{2}\left(j^{-s}+j^{-t}\right)\right)+\left(1+\frac{2}{-1+\frac{n^{-s+t}(-1+t)}{-1+s}}\right)\sum_{j=1}^{n}\left(\frac{1}{2}\left(-j^{-s}+j^{-t}\right)\right)$$

$$\frac{1}{2}\left(1+\frac{2}{-1+\frac{n^{-s+t}(-1+t)}{-1+s}}\right)(-\text{HarmonicNumber}[n, s]+\text{HarmonicNumber}[n, t])+$$
$$\frac{1}{2}(\text{HarmonicNumber}[n, s]+\text{HarmonicNumber}[n, t])$$

$$\text{az}[n\_, s\_, t\_] := \frac{1}{2}\left(1+\frac{2}{-1+\frac{n^{-s+t}(-1+t)}{-1+s}}\right)(-\text{HarmonicNumber}[n, s]+\text{HarmonicNumber}[n, t])+$$
$$\frac{1}{2}(\text{HarmonicNumber}[n, s]+\text{HarmonicNumber}[n, t])$$

$$\text{az2}[n\_, s\_, t\_] := -\frac{\text{HarmonicNumber}[n, s]}{-1+\frac{n^{-s+t}(-1+t)}{-1+s}}+\text{HarmonicNumber}[n, t]+\frac{\text{HarmonicNumber}[n, t]}{-1+\frac{n^{-s+t}(-1+t)}{-1+s}}$$

$$\text{az3}[n\_, s\_, t\_] := \frac{n^{s}(-1+s)\,\text{HarmonicNumber}[n, s]-n^{t}(-1+t)\,\text{HarmonicNumber}[n, t]}{n^{s}(-1+s)-n^{t}(-1+t)}$$

$$\text{ax}[n\_, s\_, t\_] := -\frac{\text{HarmonicNumber}[n, s]}{-1+\frac{n^{-s+t}(-1+t)}{-1+s}}$$

```
az3[1000000000, .3, .7 + I]
0.284392 - 0.841583 i
```

```
Zeta[.7 + I]
```

0.284305 − 0.841353 ⅈ

$$\text{Expand}\left[\frac{1}{2}\left(1+\frac{2}{-1+\frac{n^{-s+t}(-1+t)}{-1+s}}\right)(-\text{HarmonicNumber}[n,s]+\text{HarmonicNumber}[n,t])+\frac{1}{2}(\text{HarmonicNumber}[n,s]+\text{HarmonicNumber}[n,t])\right]$$

$$-\frac{\text{HarmonicNumber}[n,s]}{-1+\frac{n^{-s+t}(-1+t)}{-1+s}}+\text{HarmonicNumber}[n,t]+\frac{\text{HarmonicNumber}[n,t]}{-1+\frac{n^{-s+t}(-1+t)}{-1+s}}$$

$$\text{FullSimplify}\left[-\frac{\text{HarmonicNumber}[n,s]}{-1+\frac{n^{-s+t}(-1+t)}{-1+s}}+\text{HarmonicNumber}[n,t]+\frac{\text{HarmonicNumber}[n,t]}{-1+\frac{n^{-s+t}(-1+t)}{-1+s}}\right]$$

$$\frac{n^s(-1+s)\,\text{HarmonicNumber}[n,s]-n^t(-1+t)\,\text{HarmonicNumber}[n,t]}{n^s(-1+s)-n^t(-1+t)}$$

$$\text{rr14a}[n\_,m\_,d\_]:=\text{Sum}[j^{-m}\,\text{Cosh}[d\,\text{Log}[j]],\{j,1,n\}]-\text{Coth}\left[\text{ArcTanh}\left[\frac{d}{-1+m}\right]+d\,\text{Log}[n]\right]\text{Sum}[j^{-m}\,\text{Sinh}[d\,\text{Log}[j]],\{j,1,n\}]$$

$$\text{FullSimplify}\left[\text{Sum}\left[\frac{j^{-d-m}}{2}+\frac{j^{d-m}}{2},\{j,1,n\}\right]+\left(\frac{1+d-m-(-1+d+m)\,n^{2d}}{1+d-m+(-1+d+m)\,n^{2d}}\right)\text{Sum}\left[-\frac{1}{2}j^{-d-m}+\frac{j^{d-m}}{2},\{j,1,n\}\right]\right]$$

$$\left((1+d-m)\,\text{HarmonicNumber}[n,-d+m]+(-1+d+m)\,n^{2d}\,\text{HarmonicNumber}[n,d+m]\right)/\left(1+d-m+(-1+d+m)\,n^{2d}\right)$$

$$\text{FullSimplify}\left[\text{TrigToExp}\left[-\text{Coth}\left[\text{ArcTanh}\left[\frac{d}{-1+m}\right]+d\,\text{Log}[n]\right]\right]\right]$$

$$\frac{1+d-m-(-1+d+m)\,n^{2d}}{1+d-m+(-1+d+m)\,n^{2d}}$$

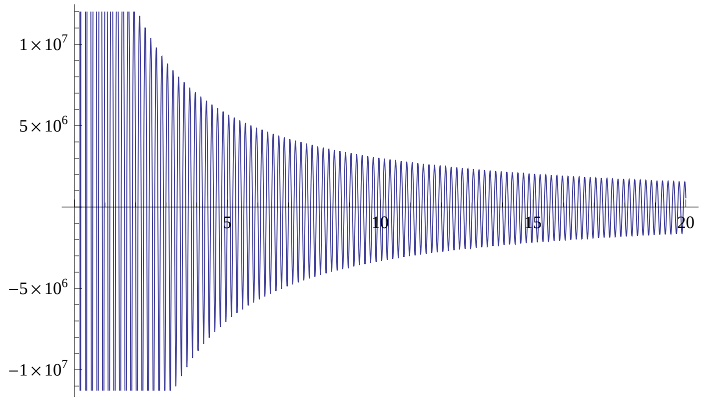
```
az2[1 000 000 000 000 000, N@ZetaZero@1 - .5, N@ZetaZero@1]
```

$1.36952\times10^{-8}+3.59523\times10^{-9}\,ⅈ$

```
Zeta[.7]
```

$$\text{aza}[n\_,s\_,t\_]:=\frac{1}{2}\left(1+\frac{2}{-1+\frac{n^{-s+t}(-1+t)}{-1+s}}\right)(-\text{HarmonicNumber}[n,s])+\frac{1}{2}(\text{HarmonicNumber}[n,s])$$

```
Plot[Re@ aza[1 000 000 000 000 000, .5, .5 + s I ], {s, 0, 20}]
```



```
eh[n_, s_] := Sum[ j^(-1 / 2) Cos[s Log[n / j] + ArcCot[2 s]] / Cos[s Log[n]], {j, 1, n}]
eh2[n_, s_] := Sum[j^(-1 / 2)  Cos[s Log[n] + ArcCot[2 s]], {j, 1, n}]
```

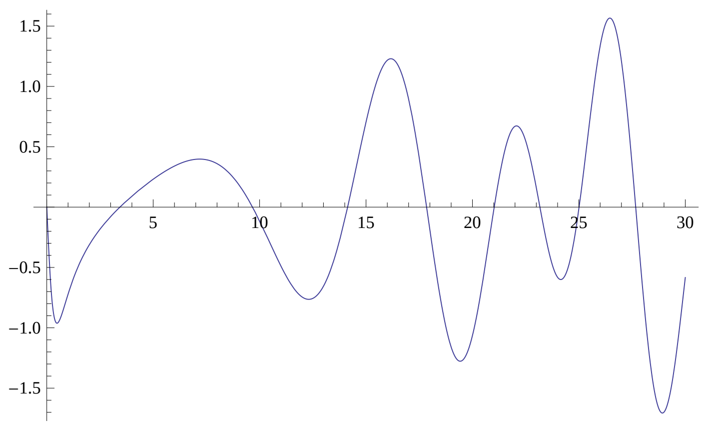
```
eh[100 000, 60.3518119691]
```

0.534093

```
Zeta[.5 + 60.3518119691 I]
```

$0.538586 + 1.97639 \times 10^{-11}\, i$

```
Plot[Im[Zeta[.5 + I s]], {s, 0, 30}]
```



$$\text{rrx}[n\_, a\_, b\_] := \left(1 - i\,\text{Cot}\left[\text{ArcTan}\left[\frac{b}{-1 + a}\right] + b\,\text{Log}[n]\right]\right) \text{HarmonicNumber}[n, a + i\, b]$$

$$\text{rr2}[n\_, a\_, b\_] := \frac{2}{1 - n^{-2\, i\, b\, \frac{((a-1) - i\, b)}{(a-1) + i\, b}}}\ \text{HarmonicNumber}[n, a + i\, b]$$

$$\text{rr2z}[n\_, a\_, b\_] := \frac{2\, n\text{^}(I\, b)\ ((a - 1) + i\, b)}{n\text{^}(I\, b)\ ((a - 1) + i\, b) - n^{-i\, b}\ ((a - 1) - i\, b)}\ \text{HarmonicNumber}[n, a + i\, b]$$

$$\text{rr3}[n\_, s\_] := 2\left(1 - n^{(\text{Conjugate}[s] - s)}\ \frac{\text{Conjugate}[s] - 1}{s - 1}\right)\text{^}{-1}\ \text{HarmonicNumber}[n, s]$$

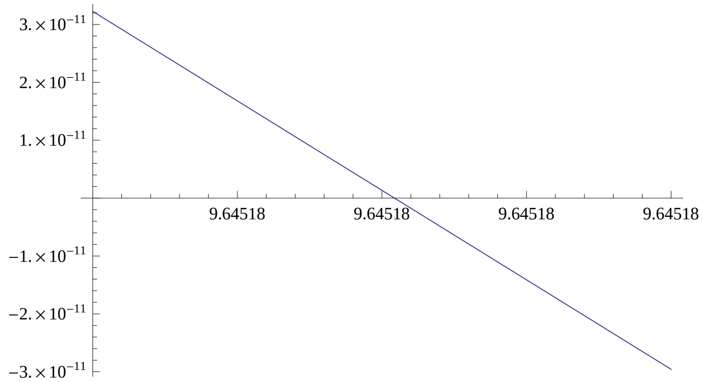
```
rr3[1 000 000 000, .5 + 60.3518119691 I]
```

$0.538603 + 556.17\, i$

```
rr2z[1 000 000 000, .5, 60.3518119691]
```

0.538603 + 556.17 i

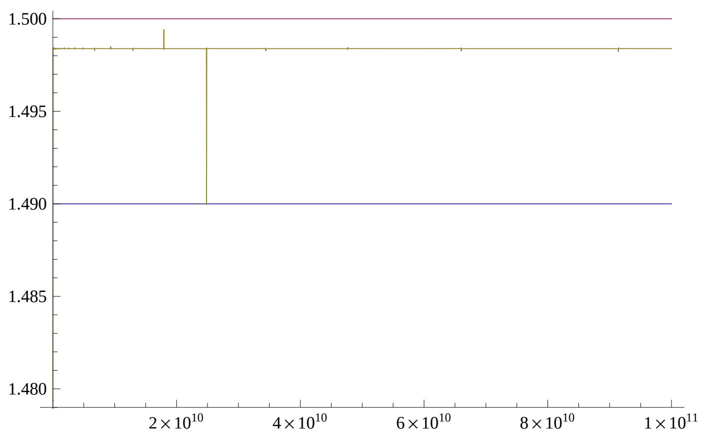
```
Plot[Im[Zeta[.6 + s I]], {s, 9.6451796488, 9.645179649}]
```




```
Zeta[.6 + 9.6451796488 I]
```

$1.49839 + 3.22501 \times 10^{-11}\, i$

```
Plot[ {1.49, 1.5, Re[ rr3[n, .6 + 9.6451796488 I]]}, {n, 1, 100 000 000 000}]
```



$$\text{FullSimplify}\left[\frac{2}{1 - n^{-2\, i\, b\, \frac{((a-1) - i\, b)}{(a-1) + i\, b}}} \text{ /. } a \to 1/4\right]$$

$$\frac{2}{1 + \frac{(-3\, i + 4\, b)\, n^{-2\, i\, b}}{3\, i + 4\, b}}$$

```
Expand[FullSimplify[Expand[(1 / 2 - I b) ^2]]]
```

$$\frac{1}{4} - i\, b - b^2$$

```
FullSimplify[Expand[(1 / 2 - I b) (1 / 2 + I b)]]
```

$$\frac{1}{4} + b^2$$

$$\frac{2\ ((a-1)+\mathbb{i}\,b)\ n^\wedge(I\,b)}{n^\wedge(I\,b)\ ((a-1)+\mathbb{i}\,b)-n^{-\mathbb{i}\,b}\ ((a-1)-\mathbb{i}\,b)}$$

$$\frac{2\ (-1+a+\mathbb{i}\,b)\ n^{\mathbb{i}\,b}}{-(-1+a-\mathbb{i}\,b)\ n^{-\mathbb{i}\,b}+(-1+a+\mathbb{i}\,b)\ n^{\mathbb{i}\,b}}$$

FullSimplify[((n^(I b) ((a - 1) + ⅈ b) - n^{-ⅈ b} ((a - 1) - ⅈ b)) / (2 ((a - 1) + ⅈ b) n^(I b)))]

$$\frac{1}{2}-\frac{(-1+a-\mathbb{i}\,b)\ n^{-2\,\mathbb{i}\,b}}{2\ (-1+a+\mathbb{i}\,b)}$$

$$1\Big/\left(\frac{1}{2}-\frac{(-1+a-\mathbb{i}\,b)\ n^{-2\,\mathbb{i}\,b}}{2\ (-1+a+\mathbb{i}\,b)}\right)$$

$$\frac{1}{\frac{1}{2}-\frac{(-1+a-\mathbb{i}\,b)\ n^{-2\,\mathbb{i}\,b}}{2\ (-1+a+\mathbb{i}\,b)}}$$

$$\frac{1}{\frac{1}{2}-\frac{\left(-\frac{1}{2}-\mathbb{i}\,b\right)\ n^{-2\,\mathbb{i}\,b}}{2\ \left(-\frac{1}{2}+\mathbb{i}\,b\right)}}$$

$$\text{N}\left[\frac{(-1+a-\mathbb{i}\,b)\ n^{-2\,\mathbb{i}\,b}}{2\ (-1+a+\mathbb{i}\,b)}\ \text{//. } a\to 1/2\ \text{/. } b\to \text{N@Im@ZetaZero@1}\right]$$

$$(-0.49875+0.0353297\,\mathbb{i})\ n^{0.-28.2695\,\mathbb{i}}$$

$$\text{ExpToTrig}\left[1\Big/\left(1/2-\frac{(-1+a-\mathbb{i}\,b)\ n^{-2\,\mathbb{i}\,b}}{2\ (-1+a+\mathbb{i}\,b)}\right)\right]$$

$$\text{N}\left[\frac{1}{\frac{1}{2}-\frac{(-1+a-\mathbb{i}\,b)\ n^{-2\,\mathbb{i}\,b}}{2\ (-1+a+\mathbb{i}\,b)}}\ \text{/. } a\to .5\ \text{/. } b\to 10\ \text{/. } n\to 40\right]$$

$$1.-0.951135\,\mathbb{i}$$

$$\left(1-\mathbb{i}\,\text{Cot}\left[\text{ArcTan}\left[\frac{b}{-1+a}\right]+b\,\text{Log}[n]\right]\right)\ \text{/. } a\to .5\ \text{/. } b\to 10\ \text{/. } n\to 111\,114\,000\,000$$

$$1.-0.0809976\,\mathbb{i}$$

```
rrx[n_, a_, b_] := (1 - ⅈ Cot[ArcTan[b/(-1 + a)] + b Log[n]]) HarmonicNumber[n, a + ⅈ b]

rrx2[n_, a_, b_] := (1 - ⅈ Cot[ArcTan[b/(-1 + a)] + b Log[n]]) Sum[1 / j^(a + I b), {j, 1, n}]

rrx3[n_, a_, b_] := (1 - ⅈ Cot[ArcTan[b/(-1 + a)] + b Log[n]]) Sum[j^-a E^(-I b Log[j]), {j, 1, n}]

rrx4[n_, a_, b_] :=
  (1 - ⅈ Cot[ArcTan[b/(-1 + a)] + b Log[n]]) Sum[j^-a (Cos[b Log[j]] - I Sin[b Log[j]]), {j, 1, n}]

rrx5[n_, a_, b_] := (1 - ⅈ Cot[ArcTan[b/(-1 + a)] + b Log[n]])
  (Sum[j^-a Cos[b Log[j]], {j, 1, n}] - I Sum[j^-a Sin[b Log[j]], {j, 1, n}])

rrx6[n_, a_, b_] := Sum[j^-a Cos[b Log[j]], {j, 1, n}] -
  ⅈ Cot[ArcTan[b/(-1 + a)] + b Log[n]] Sum[j^-a Cos[b Log[j]], {j, 1, n}] -
  (Cot[ArcTan[b/(-1 + a)] + b Log[n]]) Sum[j^-a Sin[b Log[j]], {j, 1, n}] -
  I Sum[j^-a Sin[b Log[j]], {j, 1, n}]

rrx7[n_, a_, b_] :=
 Sum[j^-a (Cos[b Log[j]] - Cot[ArcTan[b/(-1 + a)] + b Log[n]] Sin[b Log[j]]), {j, 1, n}] -
  I Sum[j^-a (Sin[b Log[j]] + Cot[ArcTan[b/(-1 + a)] + b Log[n]] Cos[b Log[j]]), {j, 1, n}]

rrx7a[n_, a_, b_] := Sum[j^-a (Cos[b Log[j]] - Cot[ArcTan[b/(-1 + a)] + b Log[n]] Sin[b Log[j]]),
  {j, 1, n}]

rrx7b[n_, a_, b_] := Sum[
  j^-a (Cos[b Log[j]] + Tan[ArcTan[(1 - a) / b] + b Log[n]] Sin[b Log[j]]), {j, 1, n}]
```

```
rrx7b[10000, .5, N@Im@ZetaZero@1 + .3]
```

-0.409413

```
rrx[n, 1 - a, -b]
```

(1 - ⅈ Cot[ArcTan[b/a] - b Log[n]]) HarmonicNumber[n, 1 - a - ⅈ b]

```
Zeta[.6 + 9.6451796488 I]
```

$1.49839 + 3.22501 \times 10^{-11}\, i$

```
rrx7a[100000, .6, 9.6451796488]
```

1.49722


```
Zeta[.6 + 9.6451796488 I]
```

$1.49839 + 3.22501 \times 10^{-11}\, i$

$$\mathrm{ArcTan}\left[\frac{b}{-1+a}\right]\ /.\ a\to 1/2$$

$$-\mathrm{ArcTan}[2\,b]$$

```
FullSimplify[j^-a (Cos[b Log[j]] - Tan[ArcTan[(a - 1) / b] - b Log[n]] Sin[b Log[j]])]
```

$$j^{-a}\left(\mathrm{Cos}[b\,\mathrm{Log}[j]]+\mathrm{Sin}[b\,\mathrm{Log}[j]]\,\mathrm{Tan}\left[\mathrm{ArcTan}\left[\frac{1-a}{b}\right]+b\,\mathrm{Log}[n]\right]\right)$$

```
Tan[ArcTan[(1 - a) / b]]
```

$$\frac{1-a}{b}$$

rrx7c[n_, a_, b_] :=

$$\mathrm{Sum}\left[j\verb|^|-a\left(\left(\frac{j^{-i b}}{2}+\frac{j^{i b}}{2}\right)+\left(\frac{i\left(-1+a-i b+(-1+a+i b)\,n^{2 i b}\right)}{-1+a-i b-(-1+a+i b)\,n^{2 i b}}\right)\left(\frac{1}{2}\,i\,j^{-i b}-\frac{1}{2}\,i\,j^{i b}\right)\right),\ \{j,1,n\}\right]$$

$$\mathrm{rrx7d}[n\_,a\_,b\_]:=\mathrm{Sum}\left[\frac{j^{-a-i b}\left((1-a+i b)\,j^{2 i b}+(-1+a+i b)\,n^{2 i b}\right)}{1-a+i b+(-1+a+i b)\,n^{2 i b}},\ \{j,1,n\}\right]$$

rrx7e[n_, a_, b_] :=

$$\left((1-a+i b)\,\mathrm{HarmonicNumber}[n,a-i b]+(-1+a+i b)\,n^{2 i b}\,\mathrm{HarmonicNumber}[n,a+i b]\right)\Big/\left(1-a+i b+(-1+a+i b)\,n^{2 i b}\right)$$

```
rrx7e[1000000000, .6, 9.6451796488]
```

$$1.49839-5.66214\times 10^{-14}\,i$$

```
Zeta[.6 + 9.6451796488 I]
```

$$1.49839+3.22501\times 10^{-11}\,i$$