This note should be pretty self-explanatory.

I was looking into the general topic of, I guess, dirichlet convolution sums convolved to arbitrary complex powers, and the resulting polynomials (and the roots of those polynomials) – and then how those roots change if transformations are applied to the sums. Probably easier just to read what follows.

In retrospect I think the first section is kind of obvious, once you noticed that $G_z(n)=F_{z\cdot t}(n)$.

Anyway, for what follows, with my more recent notation, I would write

$[(F-1)^k]_n$ *rather than* $F_{k,2}(n)$

and

$[F^z]_n$ *rather than* $F_z(n)$

and

$\nabla[F^z]_n$ *rather than* $f_z(n)$

More notes on Roots of Generalized Divisor-Style Functions

Suppose we have some function $f(n)$. Then if we define this function,

$$F_{k,2}(n)=\sum_{j=2}^{\lfloor n\rfloor} f(j)F_{k-1,2}(\frac{n}{j}) \quad \text{with} \quad F_{0,2}(n)=1$$

we have the following generalized divisor-style sum.

$$F_z(n)=\sum_{k=0}^{\lfloor \log_2 n\rfloor} \binom{z}{k} F_{k,2}(n)$$

We can, in turn, express this sum in terms of its derivatives

$$F_z(n)=\sum_{k=0}^{\lfloor \log_2 n\rfloor} \frac{z^k}{k!}\frac{\partial^k}{\partial z^k}F_0(n)$$

which readily makes clear that it has $\log_2 n$ roots for z, which we will denote ρ_k. And so we can express our function as the product of its roots as

$$F_z(n)=\prod_{k=1}^{\lfloor \log_2 n\rfloor}(1-\frac{z}{\rho_k})$$

If $f(n)$ is a multiplicative function, then

$$-\sum_{k=1}^{\lfloor \log_2 n \rfloor} \frac{1}{\rho_k} = \lim_{z\to 0} \frac{F_z(n)-1}{z} = \frac{\partial}{\partial z} F_z(n) \, at \, z=0 = \sum_{k=1}^{\lfloor \log_2 n \rfloor} \frac{(-1)^{k-1}}{k} F_{k,2}(n) = \sum_{i=2}^{\lfloor n \rfloor} h(j)$$

for some function $h(n)$ determined by $f(n)$ that only takes non-zero values at prime powers.

If we define

$$f_z(n) = F_z(n) - F_z(n-1)$$

and we use that to define a related sum

$$G_{k,2}(n) = \sum_{j=2}^{\lfloor n \rfloor} f_t(j) G_{k-1,2}\left(\frac{n}{j}\right) \text{ with } G_{0,2}(n)=1 \text{ and } G_z(n) = \sum_{k=0}^{\lfloor \log_2 n \rfloor} \binom{z}{k} G_{k,2}(n)$$

where $t = r \cdot e^{i\theta}$ for some radius r and some angle θ, then we'll find that each root of $G_z(n)$, if we multiply it by $r \cdot e^{i\theta}$,will be a root of $F_z(n)$.

Another way of putting this is that if

$$F_z(n) = \prod_{k=1}^{\lfloor \log_2 n \rfloor} \left(1 - \frac{z}{\rho_k}\right)$$

then

$$G_z(n) = \prod_{k=1}^{\lfloor \log_2 n \rfloor} \left(1 - \frac{r \cdot e^{i\theta} z}{\rho_k}\right)$$

So that's kind of interesting.

Further Questions

Suppose we generalize our function as

$$F_{k,2}(n,s) = \sum_{j=2}^{\lfloor n \rfloor} j^{-s} f(j) F_{k-1,2}\left(\frac{n}{j}, s\right) \text{ with } F_{0,2}(n,s)=1$$

$$F_z(n,s) = \sum_{k=0}^{\lfloor \log_2 n \rfloor} \binom{z}{k} F_{k,2}(n,s)$$

I really want to know what happens to the zeros as s varies. It certainly does SOMETHING to them, possibly mostly continuous – visually, it seems as though s < 0 winds the zeros towards the negative real axis, and s > 0 winds them towards the positive real axis. In fact, although cursory experiments suggest that the real part of all the zeros is negative when s is 0 (as above), its easy to find examples of zeroes with positive real parts as s grows larger and larger.

Another interesting variant to explore would be

$$F_{k,2}(n,m)=\sum_{j=2,\, j\, p\le m\nmid j}^{\lfloor n\rfloor} f(j)\,F_{k-1,2}\left(\frac{n}{j},m\right) \quad \text{with} \quad F_{0,2}(n,m)=1$$

$$F_z(n,m)=\sum_{k=0}^{\lfloor \log_2 n\rfloor} \binom{z}{k} F_{k,2}(n,m)$$

where the sum in $F_{k,2}(n,m)$ runs over numbers less than n that don't have certain primes as factors (say, primes <= m). It could be interesting to see what impact that has on the zeros.