$$[\zeta]_n = \sum_{1 \le j \le n} 1$$

$$[\zeta(s)]_n = \sum_{1 \le j \le n} j^{-s}$$

$$[\zeta_k(0)]_n = \sum_{1 \le j^{\frac{1}{k}} \le n} 1$$

$$[\zeta_{\frac{1}{k}}(s \cdot a)]_n = \sum_{1 \le j^k \le n} j^{-sa}$$

$$[\zeta_{\frac{1}{k}}(s)]_n = [\zeta(s \cdot k)]_{n^{\frac{1}{k}}}$$

$$[\zeta_{\frac{1}{k}}(s)]_n = [\zeta(k)]_{n^{\frac{1}{k}}}$$

$$[\zeta_k(s)]_n = [\zeta(k)]_{n^k} \qquad \sum_{1 \le j^2 \le n} j^{-2s} = [\zeta_{\frac{1}{2}}(2s)]_n = \sum_{1 \le j \le n^{\frac{1}{2}}} j^{-2s} = [\zeta(2s)]_{n^{\frac{1}{2}}}$$

FIX. LOOK INTO THIS AGAIN. NOT DONE.

$$[\zeta_k(s)] = [\zeta_k(s)]_e = [\zeta(k)]_{e^k}$$

$$[\zeta_{\log n}(s)] = [\zeta(k)]_{e^{\log n}} = [\zeta(k)]_n$$

$[\nabla \zeta(0)^2]_n = d(n)$
$[\nabla \zeta(0)^k]_n = d_k(n)$
$[\nabla \frac{1}{\zeta(0)}]_n = \mu(n)$
$[\nabla (\prod_{k=1} \zeta_{1/k}(0))]_n = a(n)$ (abelian group)
$[\nabla (\zeta(0) * \zeta(0*2)^{*-\frac{1}{2}} * \zeta(0*3)^{*-\frac{1}{3}} * \zeta(0*5)^{*-\frac{1}{5}} * \zeta(0*6)^{*\frac{1}{6}} * ...)]_n = ...$ (log of this is primeQ)
$[\nabla (\frac{\zeta_{1/2}(0)}{\zeta(0)})]_n = \lambda(n)$
$[\nabla (\frac{\zeta(0)}{\zeta_{\frac{1}{2}}(0)})]_n = |\mu(n)|$
$[\nabla (\zeta(-a) \cdot \zeta(0))]_n = \sigma_a(n)$
$[\nabla (\frac{\zeta(-1)}{\zeta(0)})]_n = \varphi(n)$
$[\nabla (\frac{\zeta(-a)}{\zeta(0)})]_n = J_a(n)$

$[\nabla \log \zeta(0)]_n = \frac{\Lambda(n)}{\log n}$
$[\nabla \lim_{s \to 0} \frac{\partial}{\partial s} \log \zeta(s)]_n = \Lambda(n)$
$[\nabla \log(\frac{\zeta_{\frac{1}{2}}(0)}{\zeta(0)})]_n = -[\nabla \log \zeta(0)]_n + [\nabla \log \zeta_{\frac{1}{2}}(0)]_n$
$[\nabla \log(\zeta(-a) \cdot \zeta(0))]_n = [\nabla \log \zeta(-a)]_n + [\nabla \log \zeta(0)]_n$

$$[\nabla \log(\frac{\zeta(-a)}{\zeta(0)})]_n = [\nabla \log \zeta(-a)]_n - [\nabla \log \zeta(0)]_n$$

$$[\nabla \log(\sum_{j=1}^{n} \chi_k(j))]_n = \chi_k(n) \cdot [\nabla \log \zeta(0)]_n$$

NOTE: $[\log\zeta_{1/2}(2s)]_n=[\log\zeta(2s)]_{n^{\frac{1}{2}}}$, and $[\log\zeta_{1/k}(s)]_n=[\log\zeta(k\cdot s)]_{n^{\frac{1}{k}}}$ … There is some question here. FIX. LOOK INTO THIS AGAIN. NOT DONE.

$$[\log(\frac{\zeta_{1/2}(0)}{\zeta(0)})]_n=[\log\zeta_{1/2}(0)]_n-[\log\zeta(0)]_n$$

$$[\log(\frac{\zeta(0)}{\zeta_{1/2}(0)})]_n=[\log\zeta(0)]_n-[\log(\zeta_{1/2}(0))]_n$$

$$[\log(\zeta(-a)\cdot\zeta(0))]_n=[\log\zeta(-a)]_n+[\log\zeta(0)]_n$$

$$[\log(\frac{\zeta(-a)}{\zeta(0)})]_n=[\log\zeta(-a)]_n-[\log\zeta(0)]_n$$

$$[\log(\zeta_{1/2}(0)\cdot\zeta(0))]_n=[\log\zeta(0)]_n+[\log(\zeta_{1/2}(0))]_n$$

$$[\log(\prod_{k=1}\zeta_{1/k}(0))]_n=\sum_{k=1}[\log\zeta_{1/k}(0)]_n$$

$$[\log(\prod_{k=1}\zeta_{1/k}(0)^{\frac{\mu(k)}{k}})]_n=\pi(n)$$

$$[\log(\sum_{j=1}^{n}\chi_k(j))]_n=\sum_{j=1}^{n}\chi_k(j)\cdot[\nabla\log\zeta(0)]_j$$

$$\log(\sum_{j=1}^{n}\chi_k(j))=\sum_{s=0}^{t}\chi_k(s)\cdot\sum_{j=0}^{\lfloor\frac{n}{t}\rfloor}\nabla*\log(j\cdot t+s)$$

$$[\log(\zeta_{1/2}(0)\cdot\zeta_{1/2}(0))]_n=2[\log(\zeta_{1/2}(0))]_n$$

$$[\log(\zeta(0)\cdot\zeta(0))]_n=[\log\zeta(0)]_n+[\log\zeta(0)]_n$$

$$[\log(\zeta(0)^z)]_n=z\cdot[\log\zeta(0)]_n$$

$$[\log(t\cdot\zeta(s))]_n=\log t+[\log\zeta(s)]_n$$

$$[\log((1-x^{1-s})\zeta(s))]_n=-\sum_{k=1}^{\lfloor\frac{\log n}{\log x}\rfloor}\frac{x^k\cdot x^{-sk}}{k}+[\log\zeta(s)]_n$$

$$[(((1-x^{1-s})\zeta(s))^z-1)^k]_n=\sum_{j=2}^{n}[\nabla((1-x^{1-s})\zeta(s))^z]_j\cdot[(((1-x^{1-s})\zeta(s))^z-1)^{k-1}]_{n\cdot j^{-1}}$$

$$[\log((1-x^{1-s})\zeta(s)^z)]_n=-z\cdot\sum_{k=1}^{\lfloor\frac{\log n}{\log x}\rfloor}\frac{x^{(1-s)k}}{k}+z\cdot[\log\zeta(s)]_n$$

```
bin[z_, k_] := Product[z - j, {j, 0, k - 1}]/k!
```

```
E2[n_, k_] :=  E2[n, k] = Sum[(-1)^(j + 1)  E2[Floor[n/j], k - 1], {j, 2, n}]
E2[n_, 0] := UnitStep[n - 1]
Etz[n_, z_] := Sum[ bin[ z, k] E2[n, k], {k, 0, Log[2, n]}]
etz[n_, z_] := Etz[n, z] - Etz[n - 1, z]
D1xD[n_, k_, z2_] := D1xD[n, k, z2] = Sum[etz[j, z2]  D1xD[n/j, k - 1, z2], {j, 2, n}]
D1xD[n_, 0, z2_] := UnitStep[n - 1]
E1[n_, z_] := Sum[ (-1)^(k + 1)/k D1xD[n, k, z], {k, 1, Log2@n}]
fo[n_] := -Sum[ 2^k/k, {k, 1, Log2@n}]
pr[n_] := Sum[ PrimePi[ n^(1/k)]/k, {k, 1, Log2@n}]
DiscretePlot[ E1[n, 2] - (2 pr[n] + 2 fo[n]), {n, 1, 100}]
```

$$\log((1-x^{1-s})\zeta(s))=\log(1-x^{1-s})+\log\zeta(s)$$

$$\log((1-x^{1-s})\zeta(s)^2)=\log(1-x^{1-s})+\log(\zeta(s)^2)$$

$$[(\frac{\zeta(0)}{\zeta_{\log_n m}(0)})]_n=\sum_{j\cdot k^{\frac{\log m}{\log n}}\le n}\mu(k)$$

$$[(\frac{\zeta(s)}{\zeta_{\log_n m}(s)})^z]_n=\sum_{j=1}^{n}\sum_{k=1}^{\lfloor\frac{m}{j^{\frac{\log m}{\log n}}}\rfloor}[\nabla\zeta(s)^z]_j\cdot[\nabla\zeta(s)^{-z}]_k$$

$$[(\frac{\zeta(s)}{\zeta_{\log_n m}(s)})^z]_n=\sum_{j\cdot k^{\frac{\log n}{\log m}}\le n}[\nabla\zeta(s)^z]_j\cdot[\nabla\zeta(s)^{-z}]_k$$

$$[\log(\frac{\zeta(s)}{\zeta_{\log_n m}(t)})]_n=[\log\zeta(s)]_n-[\log\zeta_{\log_n m}(t)]_n=[\log\zeta(s)]_n-[\log\zeta(t)]_m$$

$$[\log(\frac{\zeta(s)}{\zeta_{1/2}(2s)})]_n=[\log\zeta(s)]_n-[\log(\zeta_{1/2}(2s))]_n=[\log\zeta(s)]_n-[\log(\zeta(2s))]_{n^{\frac{1}{2}}}$$

$$[(\frac{\zeta_{\log n}(s)}{\zeta_{\log m}(s)})^z]=\sum_{\frac{\log j}{\log n}+\frac{\log k}{\log m}\le 1}[\nabla\zeta_{\log n}(s)^z]_j\cdot[\nabla\zeta_{\log m}(s)^{-z}]_k$$

```
bin[z_, k_] := Product[z - j, {j, 0, k - 1}]/k!
FI[n_] := FactorInteger[n]; FI[1] := {}
dz[n_, z_] := dz[n, z] = Product[(-1)^p[[2]] bin[-z, p[[2]]], {p, FI[n]}]
ddz1[n_, m_, s_, z_] := Sum[a^-s b^-s dz[a, z] dz[b, -z], {a, 1, n}, {b, 1, (m/a^(N@Log[m]/Log[n]))}]
ldz[n_, m_, s_] := D[ ddz1[n, m, s, z], z] /. z -> 0
```

$$\boxed{[\zeta(s)^z]_n = \sum_{j=0} \sum_{k=0} (-1)^j \binom{-z}{j} \binom{z}{k} x^{j(1-s)} \cdot [((1-x^{1-s})\zeta(s)-1)^k]_{n \cdot x^{-j}}}$$

$$[(\zeta_n(s)^x)^z]_n = \sum_{j=0} \sum_{k=0} (-1)^j \binom{-z}{j} \binom{z}{k} x^{j(1-s)} \cdot [((1-x^{1-s})\zeta(s)^x-1)^k]_{n \cdot x^{-j}}$$

$$\boxed{[\zeta(s)^z]_n = \sum_{j=0} (-1)^j \binom{-z}{j} \cdot x^{j(1-s)} \cdot [((1-x^{1-s})\zeta(s))^z]_{n \cdot x^{-j}}}$$

$$[(1+y^{s-1}\cdot\zeta(s,1+y))^{z}]_n=\sum_{k=0}^{\infty}\binom{z}{k}[(y^{s-1}\cdot\zeta(s,1+y))^{k}]_n$$

$$[\log(1+y^{s-1}\cdot\zeta(s,1+y))]_n=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}[(y^{s-1}\cdot\zeta(s,1+y))^{k}]_n$$

$$\boxed{[(y^{s-1}\cdot\zeta(s,1+y))^{k}]_n=y^{-1}\cdot\sum_{j=1}(1+j\,y^{-1})^{-s}[(y^{s-1}\cdot\zeta(s,1+y))^{k-1}]_{n(1+j\,y^{-1})^{-1}}}$$

$$\boxed{[(y^{s-1}\cdot\zeta(s,1+y))^{k}]_n=y^{s-1}\cdot\sum_{j=1}(j+y)^{-s}[(y^{s-1}\cdot\zeta(s,1+y))^{k-1}]_{n\cdot y(j+y)^{-1}}}$$

$$\boxed{[(\zeta(s)-1)^{k}]_n=\sum_{m=0}\frac{1}{m!}(\lim_{x\to 0}\frac{\partial^m}{\partial x^m}\frac{x}{\log(1+x)})\cdot[(\zeta(s)-1)^{k-1+m}\cdot\log\zeta(s)]_n}$$

$$\boxed{[y^{s-1}\cdot\zeta(s,1+y)]_n=\sum_{m=0}\frac{1}{m!}(\lim_{x\to 0}\frac{\partial^m}{\partial x^m}\frac{x}{\log(1+x)})\cdot[(y^{s-1}\cdot\zeta(s,1+y))^{m}\cdot\log(1+y^{s-1}\cdot\zeta(s,1+y))]_n}$$ YEP!!!

$$\boxed{[y^{s-1}\cdot\zeta(s,1+y)]_n=\sum_{k=1}\sum_{m=0}\frac{(-1)^{k+1}}{k}\frac{1}{m!}(\lim_{x\to 0}\frac{\partial^m}{\partial x^m}\frac{x}{\log(1+x)})\cdot[(y^{s-1}\cdot\zeta(s,1+y))^{m}\cdot((y^{s-1}\cdot\zeta(s,1+y))^{k})]_n}$$

$$\boxed{[y^{s-1}\cdot\zeta(s,1+y)]_n=\sum_{k=1}\sum_{m=0}\frac{(-1)^{k+1}}{k}\frac{1}{m!}(\lim_{x\to 0}\frac{\partial^m}{\partial x^m}\frac{x}{\log(1+x)})\cdot[((y^{s-1}\cdot\zeta(s,1+y))^{m+k})]_n}$$

$$[\zeta(s,y+1)]_n = [[\zeta(s)]_n - [\zeta(s)]_y]$$

$$[([\zeta(s)]_n - [\zeta(s)]_y)^k] = \sum_{j=1} (j+y)^{-s} \cdot [([\zeta(s)]_{n(j+y)^{-1}} - [\zeta(s)]_y)^{k-1}]$$

$$[(1+y^{s-1} \cdot \zeta(s,1+y))^z]_n = \sum_{k=0}^{\infty} \binom{z}{k} [(y^{s-1} \cdot \zeta(s,1+y))^k]_n$$

$$[\log(1+y^{s-1} \cdot \zeta(s,1+y))]_n = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} [(y^{s-1} \cdot \zeta(s,1+y))^k]_n$$

$$[(1+y^{s-1} \cdot (\zeta(s) - [\zeta(s)]_y))^z]_n = \sum_{k=0}^{\infty} \binom{z}{k} [(y^{s-1} \cdot (\zeta(s) - [\zeta(s)]_y))^k]_n$$

$$[(1+y^{s-1} \cdot (\zeta(s) - [\zeta(s)]_y))^z]_n = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{z}{k} y^{k(s-1)} \cdot (-1)^{k-j} \cdot \binom{k}{j} [\zeta(s)^j \cdot [\zeta(s)]_y^{k-j}]_n$$

$$[\log(1+y^{s-1} \cdot ([\zeta(s)] - [\zeta(s)]_y))]_n = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} [(y^{s-1} \cdot ([\zeta(s)] - [\zeta(s)]_y))^k]_n$$

$$[\log(y^{s-1} \cdot (\zeta(s) - [\zeta(s)]_y))]_n = \sum_{k=0}^{\infty} \sum_{j=1}^{k} \frac{(-1)^{k+1}}{k} y^{k(s-1)} \cdot (-1)^{k-j} \cdot \binom{k}{j} [[\zeta(s)^j]_{n \cdot y^j} \cdot [\zeta(s)^{k-j}]_y]_n$$

$$[\zeta(s,y+1)^k]_n = \sum_{j=0}^{k} (-1)^j \binom{k}{j} y^{-j \cdot s} [\zeta(s,y)^{k-j}]_{n(y+1)^{-j}}$$

$$f_k(n)=\int_1^n f_{k-1}\left(\frac{n}{x}\right)dx \text{ and } f_0(n)=1 \text{ : Here}$$

$$f_k(n)=\int_1^n f_{k-1}(n)\,dx \text{ and } f_0(n)=1 \text{ : Here } f_k(n)=(n-1)^k$$

$$f_k(n)=\int_1^n f_{k-1}(x)\,dx \text{ and } f_0(n)=1 \text{ : Here } f_k(n)=\frac{(n-1)^k}{k\,!}$$

$$\boxed{\lim_{x \to 1^+} \sum_{k=1}^{\lfloor \frac{\log n}{\log x} \rfloor} \frac{x^{k(1-s)}-1}{k} = li\left(n^{1-s}\right) - \log\log n^{1-s} - \gamma}$$

$$\boxed{\sum_{k=1}^{\infty} \frac{x^{k(1-s)}}{k} = -\log\left(1-x^{(1-s)}\right)}$$

AND...

$$\boxed{\lim_{x \to 1^+} \sum_{k=1}^{\lfloor \frac{\log n}{\log x} \rfloor} \frac{x^k-1}{k} = li(n) - \log\log n - \gamma}$$

$$\boxed{\sum_{k=1}^{\infty} \frac{x^k}{k} = -\log(1-x)}$$

$$\boxed{\lim_{x \to 1^+} \sum_{k=1}^{\lfloor \frac{\log n}{\log x} \rfloor} x^k \cdot \log x = n-1}$$

$$\boxed{\sum_{k=1}^{\infty} x^k \cdot \log x = -\frac{x \log x}{x-1}}$$

$$\boxed{\Pi(n)=li(n)-\log\log n-\gamma+\lim_{x\to 1+}[\log((1-x^{1-0})\zeta(0))]_n+H_{\lfloor\frac{\log n}{\log x}\rfloor}}$$

$$\boxed{\Pi(n)=li(n)-\log\log n-\gamma-\int_1^\infty \frac{\partial}{\partial y}[\log(1+y^{-1}\cdot\zeta(0,1+y))]_n\,dy}$$

$$\boxed{\lim_{x\to 1+}[\log((1-x)\zeta(0))]_n+H_{\lfloor\frac{\log n}{\log x}\rfloor}=-\int_1^\infty \frac{\partial}{\partial x}[\log(1+x^{-1}\cdot\zeta(0,1+x))]_n\,dx}$$

$$L_0(x)=1$$

$$L_1(x)=1-x$$

$$L_{z+1}(x)=\frac{(2z+1-x)L_z(x)-z\,L_{z-1}(x)}{z+1}$$

$$L_{-z}(x)=e^x\cdot L_{z-1}(-x)$$

$$L_0(\log x)=1$$

$$L_1(\log x)=1-\log x$$

$$L_{z+1}(\log x)=\frac{(2z+1-\log x)L_z(\log x)-z\,L_{z-1}(\log x)}{z+1}$$

$$L_{-z}(\log x)=x\cdot L_{z-1}(-\log x)$$

$$L_z(\log x)=x\cdot L_{-z-1}(-\log x)$$

$$x\cdot L_{-z-2}(-\log x)=\frac{(2z+1-\log x)(x\cdot L_{-z-1}(-\log x))-z\,x\cdot L_{-z}(-\log x)}{z+1}$$

$$L_{-z-2}(-\log x)=\frac{(2z+1-\log x)(L_{-z-1}(-\log x))-z\cdot L_{-z}(-\log x)}{(z+1)}$$

$$\boxed{\pi(n)=\sum_{k=1}^{\lfloor \log_2 n \rfloor} \frac{1}{k}\cdot\mu(k)[\log \zeta_{\frac{1}{k}}(0)]_n}$$

$$\boxed{[\log(\prod_{k=1} \zeta_{\frac{1}{k}}(0)^{\frac{1}{k}\cdot[\nabla\zeta(0)^{-1}]_k})]_n=\sum_{k=1} \frac{1}{k}\cdot[\nabla\zeta(0)^{-1}]_k\cdot[\log \zeta_{\frac{1}{k}}(0)]_n}$$

$$\boxed{[\log(\prod_{k=1} \zeta_{\frac{1}{k}}(s\cdot k)^{\frac{1}{k}\cdot[\nabla\zeta(0)^{-1}]_k})]_n=\sum_{k=1} \frac{1}{k}\cdot[\nabla\zeta(0)^{-1}]_k\cdot[\log \zeta_{\frac{1}{k}}(s\cdot k)]_n}$$

$$\boxed{[\log \zeta_{\frac{1}{k}}(s\cdot k)]_n=\sum_{k=1} \frac{1}{k}\cdot[\nabla\zeta(0)]_k\cdot[\log(\prod_{k=1} \zeta_{\frac{1}{k}}(s\cdot k)^{\frac{1}{k}\cdot[\nabla\zeta(0)^{-1}]_k})]_n}$$

$$\boxed{[f]_n=[\prod_{k=1} \zeta_{\frac{1}{k}}(0)^{\frac{1}{k}\cdot[\nabla\zeta(0)^{-1}]_k}]_n}$$

$$\boxed{[\zeta(0)]_n=[\prod_{k=1} f_{\frac{1}{k}}(0)^{\frac{1}{k}}]_n}$$

$$\sum_{a\cdot b^2\cdot c^3\cdot d^4\cdot\ldots\leq n} f_1(a)\cdot f_{\frac{1}{2}}(b)\cdot f_{\frac{1}{3}}(c)\cdot f_{\frac{1}{4}}(d)\cdot\ldots=n$$

$$[(x^{1-s}\zeta(s))^k]_n = x\sum_{j=1}(jx)^{-s}\cdot[(x^{1-s}\cdot\zeta(s))^{k-1}]_{n\cdot(j\cdot x)^{-1}}$$

$$[(1+x^{1-s}\zeta(s))^k]_n = [(1+x^{1-s}\cdot\zeta(s))^{k-1}]_n + x\sum_{j=1}(jx)^{-s}\cdot[(x^{1-s}\cdot\zeta(s))^{k-1}]_{n(jx)^{-1}}$$

$$[(x^{1-s}\cdot\zeta(s,a+1))^k]_n = x\sum_{j=1}(jx+a)^{-s}\cdot[(x^{1-s}\cdot\zeta(s,a+1))^{k-1}]_{n(jx+a)^{-1}}$$

$$[(1+x^{1-s}\cdot\zeta(s,a+1))^k]_n = [(1+x^{1-s}\cdot\zeta(s,a+1))^{k-1}]_n + x\sum_{j=1}(jx+a)^{-s}\cdot[(1+x^{1-s}\cdot\zeta(s,a+1))^{k-1}]_{n(jx+a)^{-1}}$$

...

$$[((1-x^{1-s})\zeta(s)-1)^k]_n = (\sum_{j=1} j^{-s}[((1-x^{1-s})\zeta(s)-1)^{k-1}]_{n\cdot j^{-1}} - x\cdot(jx)^{-s}[((1-x^{1-s})\zeta(s))^{k-1}]_{n(jx)^{-1}}) - [((1-x^{1-s})\zeta(s)-1)^{k-1}]_n$$

$$[((1-x^{1-s})\zeta(s))^k]_n = \sum_{j=1} j^{-s}\cdot[((1-x^{1-s})\zeta(s))^{k-1}]_{n\cdot j^{-1}} - x\cdot(jx)^{-s}[((1-x^{1-s})\zeta(s))^{k-1}]_{n\cdot(jx)^{-1}}$$

$$[(\zeta(0)\cdot\zeta_{\frac{1}{2}}(0)\cdot\zeta_{\frac{1}{3}}(0)\cdot\zeta_{\frac{1}{4}}(0)\cdot\zeta_{\frac{1}{5}}(0)\cdot\ldots)]_n=\sum_{j=1}^{n} a(j) \text{ (abelian group)}$$

$$lf(n)=\sum_{k=1} \Pi(n^{\frac{1}{k}})$$

$$lf(n)=\Pi(n)+\Pi(n^{\frac{1}{2}})+\Pi(n^{\frac{1}{3}})+\Pi(n^{\frac{1}{4}})+\ldots$$

$$lf(n)-lf(n^{\frac{1}{2}})=\Pi(n)+\Pi(n^{\frac{1}{3}})+\ldots$$

$$\sum_{k=1} \mu(k)\, lf(n^{\frac{1}{k}})=\Pi(n)$$

$$[f]_n=[\prod_{k=1} \zeta_{\frac{1}{k}}(0)]_n$$

$$[\zeta(0)]_n=[\prod_{k=1} f_{\frac{1}{k}}(0)^{\mu(k)}]_n=n$$

$$[\prod_{k=1} f_{\frac{1}{k}}(0)^{\mu(k)}]_n=n$$

$$[\nabla(\zeta(0)\cdot\zeta_{\frac{1}{2}}(0)\cdot\zeta_{\frac{1}{3}}(0)\cdot\zeta_{\frac{1}{4}}(0)\cdot\zeta_{\frac{1}{5}}(0)\cdot\ldots)]_n=a(n) \text{ (abelian group)}$$

$$\sum_{j=1}^{n}\sum_{k=1}^{(\frac{n}{j})^{\frac{1}{2}}}\sum_{l=1}^{(\frac{n}{jk^2})^{\frac{1}{3}}}\sum_{m=1}^{(\frac{n}{jk^2l^3})^{\frac{1}{4}}}\sum_{o=1}^{(\frac{n}{jk^2l^3m^4})^{\frac{1}{5}}}\ldots 1=\sum_{j=1}^{n} a(j)$$

$$(\sum_{j=1}^{n}\sum_{k=1}^{(\frac{n}{j})^{\frac{1}{2}}}\sum_{l=1}^{(\frac{n}{jk^2})^{\frac{1}{3}}}\sum_{m=1}^{(\frac{n}{jk^2l^3})^{\frac{1}{4}}}\sum_{o=1}^{(\frac{n}{jk^2l^3m^4})^{\frac{1}{5}}}\ldots 1)-(\sum_{j=1}^{n-1}\sum_{k=1}^{(\frac{n-1}{j})^{\frac{1}{2}}}\sum_{l=1}^{(\frac{n-1}{jk^2})^{\frac{1}{3}}}\sum_{m=1}^{(\frac{n-1}{jk^2l^3})^{\frac{1}{4}}}\sum_{o=1}^{(\frac{n-1}{jk^2l^3m^4})^{\frac{1}{5}}}\ldots 1)=a(n)$$

$$\boxed{[[f]_n \cdot [f]_m] = \sum_{j=1}^{n} \sum_{k=1}^{\lfloor \frac{m}{j^{\log_n m}} \rfloor} \nabla f(j) \cdot \nabla f(k)}$$

$$\frac{\zeta(s)^4}{\zeta(2s)} = \sum_{j=1} d(j)^2 \cdot j^{-s}$$

$$\frac{\zeta(s)^3}{\zeta(2s)} = \sum_{j=1} d(j^2) \cdot j^{-s}$$

$$\frac{\zeta(s)^2}{\zeta(2s)} = \sum_{j=1} 2^{\omega(j)} \cdot j^{-s} \text{ where } \omega(j) \text{ is the number of prime factors of j}$$

$$\frac{\zeta(s)}{\zeta(2s)} = \sum_{j=1} |\mu(j)| \cdot j^{-s}$$

$$\ldots$$

$$[f(s)]_n = [\frac{\zeta(s)}{\zeta_{\frac{1}{2}}(2s)}]_n$$

$$\sum_{a \cdot b^2 \le n} a^{-s} \cdot \mu(b) \cdot b^{-2s} = [f(s)]_n$$

$$[\zeta(s)]_n = [\prod_{k=0} f_{\frac{1}{2^k}}(2^k \cdot s)]_n$$

$$\sum_{a \cdot b^2 \cdot c^4 \cdot d^8 \cdot \ldots \le n} |\mu(a)| \cdot |\mu(b)| \cdot |\mu(c)| \cdot |\mu(d)| \cdot \ldots = n$$

$$\sum_{a \cdot b^2 \cdot c^4 \cdot d^8 \cdot \ldots \le n} |\mu(a)| \cdot a^{-s} \cdot |\mu(b)| \cdot b^{-2s} \cdot |\mu(c)| \cdot c^{-4s} \cdot |\mu(d)| \cdot d^{-8s} \cdot \ldots = [\zeta(s)]_n$$

```
a2[n_, s_] :=  a2[n, s] =   Sum[ j^-s k^(-2 s) MoebiusMu[k], {j, 1, n}, {k, 1, (n/j)^(1/2)}]
fa[n_, s_] := fa[n, s] = a2[n, s] - a2[n - 1, s]
ia2[n_, s_] := Sum[ fa[a, s] fa[b, 2 s] fa[c, 4 s] fa[d, 8 s] fa[e, 16 s] fa[f, 32 s], {a, 1, n}, {b, 1, (n/a)^(1/2)}, {c, 1, (n/(a b^2))^(1/4)}, {d, 1, (n/(a b^2 c^4))^(1/8)}, {e, 1, (n/(a b^2 c^4 d^8))^(1/16)}, {f, 1, (n/(a b^2 c^4 d^8 e^16))^(1/32)}]
```

$$\frac{\zeta(s)^3}{\zeta(2s)} = \sum_{j=1} d(j^2) \cdot j^{-s}$$

$$[f]_n = [\frac{\zeta(s)^3}{\zeta_{\frac{1}{2}}(2s)}]_n$$

$$[\zeta(s)]_n = [\prod_{k=0} f_{\frac{1}{2^k}}(2^k \cdot s)^{\frac{1}{3^k}}]_n$$

$$f_z(n) = \nabla \sum_{k=0} \binom{z}{k} \mathit{stuff} \ldots$$

$$\sum_{a \cdot b^2 \cdot c^4 \cdot d^8 \cdot \ldots \le n} f_{\frac{1}{3}}(a) \cdot f_{\frac{1}{9}}(b) \cdot f_{\frac{1}{27}}(c) \cdot f_{\frac{1}{81}}(d) \cdot \ldots = n$$

$$lf(n)=lp(n)+lp(n^{\frac{1}{2}})$$

$$lf(n)-lp(n^{\frac{1}{2}})+lp(n^{\frac{1}{4}})=lp(n)+lp(n^{\frac{1}{8}})$$

$$\sum_{k=0}(-1)^k\cdot lf(n^{\frac{1}{2^k}})=lp(n)$$

$$[f]_n=[\zeta(s)\cdot\zeta_{\frac{1}{2}}(2s)]_n$$

$$[\zeta(0)]_n=[\prod_{k=0} f_{\frac{1}{2^k}}(0)^{(-1)^k}]_n$$

$$\ldots$$

$$lf(n)=lp(n)+lp(n^{\frac{1}{3}})$$

$$lp(n)=\sum_{k=0}(-1)^k\cdot lf(n^{\frac{1}{3^k}})$$

$$[f]_n=[\zeta(0)\cdot\zeta_{\frac{1}{3}}(0)]_n$$

$$[\zeta(0)]_n=[\prod_{k=0} f_{\frac{1}{3^k}}(0)^{(-1)^k}]_n$$

$$\ldots$$

$$lf(n)=lp(n)-lp(n^{\frac{1}{3}})$$

$$lp(n)=\sum_{k=0} lf(n^{\frac{1}{3^k}})$$

$$[f]_n=[\frac{\zeta(0)}{\zeta_{\frac{1}{3}}(0)}]_n$$

$$[\zeta(0)]_n=[\prod_{k=0} f_{\frac{1}{3^k}}(0)]_n$$

$$lf(n)=lp(n^{\frac{1}{2}})-lp(n^{\frac{1}{3}})$$

$$\sum_{a=1}^{n}\sum_{b=1}^{\frac{n}{a}} gcd(a,b)=[\frac{\zeta(0)^2\cdot\zeta_{\frac{1}{2}}(-2)}{\zeta_{\frac{1}{2}}(0)}]_n$$

$$[\log\sum_{a=1}^{n}\sum_{b=1}^{\frac{n}{a}} gcd(a,b)]_n=2[\log\zeta(0)]_n+[\log\zeta(-1)]_{n^{\frac{1}{2}}}-[\log\zeta(0)]_{n^{\frac{1}{2}}}$$

$$\sum_{a=1}^{n}\sum_{b=1}^{\frac{n}{a}} gcd(a,b)=\sum_{j=1}^{n}\sum_{k=1}^{(\frac{n}{j})^{\frac{1}{2}}} d(j)\varphi(k)$$

$$\sum_{a=1}^{n}\sum_{b=1}^{\frac{n}{a}}\sum_{c=1}^{\frac{n}{ab}} gcd(a,b,c)=[\frac{\zeta(0)^3\cdot\zeta_{\frac{1}{3}}(-3)}{\zeta_{\frac{1}{3}}(0)}]_n$$

$$\sum_{a=1}^{n}\sum_{b=1}^{\frac{n}{a}}\sum_{c=1}^{\frac{n}{ab}}\sum_{d=1}^{\frac{n}{abc}} gcd(a,b,c,d)=[\frac{\zeta(0)^4\cdot\zeta_{\frac{1}{4}}(-4)}{\zeta_{\frac{1}{4}}(0)}]_n$$

And another one. Suppose we define g(n) as

$$\sum_{j=1}^{n} gcd(j,n)$$

Then

$$\sum_{k=1}^{n}\sum_{j=1}^{k} gcd(j,k)=[\frac{\zeta(-1)^2}{\zeta(0)}]_n$$

…

$$[\log\sum_{a=1}^{n}\sum_{b=1}^{\frac{n}{a}} lcm(a,b)]_n=2[\log\zeta(-1)]_n+[\log\zeta(-2)]_{n^{\frac{1}{2}}}-[\log\zeta(-1)]_{n^{\frac{1}{2}}}$$

$$\sum_{a=1}^{n}\sum_{b=1}^{\frac{n}{a}} lcm(a,b)=[\frac{\zeta(-1)^2\cdot\zeta_{\frac{1}{2}}(-2)}{\zeta_{\frac{1}{2}}(-4)}]_n$$

$$\frac{\zeta(s)^4}{\zeta(2s)}=\sum_{j=1} d(j)^2\cdot j^{-s}$$

$$\frac{\zeta(s)^3}{\zeta(2s)}=\sum_{j=1} d(j^2)\cdot j^{-s}$$

Suppose we have

$$D_k(n)=\sum_{j=1}^{n} d_z(j^a)^b\cdot D_{k-1}(\frac{n}{j})\ with\ D_0(n)=1$$
$$D_k'(n)=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}D_j(n)$$
$$logD(n)=\sum_{k=1}^{\log_2 n}\frac{(-1)^{k+1}}{k}D_k'(n)$$

So what is $D_1(n)$? I already have the two identities listed above. From testing, I know that changing z, a, and b all produce logs that only change on prime powers. So what is the general formula here?

Also, maybe fun to play with lcm as well.

$$\log\left(\left(1-x^{1-s}\right)\zeta(s)\right)=-\sum_{k=1}^{\infty}\frac{x^{k(1-s)}}{k}+\log\zeta_n(s)$$

$$[\log\left(\left(1-x^{1-s}\right)\zeta(s)\right)]_n=-\sum_{k=1}^{\lfloor\frac{\log n}{\log x}\rfloor}\frac{x^{k(1-s)}}{k}+[\log\zeta(s)]_n$$

$$\lim_{s\to 1}\left(1-x^{1-s}\right)\zeta(s)=\log x$$

$$\lim_{x\to 1}\lim_{s\to 1}[\left(1-x^{1-s}\right)\zeta(s)]_n=???$$

$$\lim_{x\to 1}\lim_{s\to 1}[\left(1-x^{1-s}\right)\zeta(s)-1]_n=???$$

$$[\log\left(\left(1-x^{1-s}\right)f(s)\right)]_n=???+[f(s)]_n$$

$$\sum_{a\cdot b\leq n} 1$$

$$\sum_{\frac{\log a}{\log n}+\frac{\log b}{\log n}\leq 1} 1$$

$$D_2(s)=\sum_{\log_n j\leq s} D_1(s-\log_n j)$$

$$[[\zeta(0)]_n\cdot[\zeta(0)]_m]=[\zeta_{\log n}(0)\cdot\zeta_{\log m}(0)]$$

$$[\zeta(0)^2]_n=[\zeta_{\log n}(0)^2]$$

$$[\zeta(0)]_n=[\zeta_{\log n}(0)]$$

$$\boxed{[\zeta(s)^k]_n=\sum_{j=1} j^{-s}\cdot[\zeta(s)^{k-1}]_{n j^{-1}}}$$

$$\boxed{[\zeta_{\log n}(s)^k]=\sum_{j=1} j^{-s}\cdot[\zeta_{\log n\cdot j^{-1}}(s)^{k-1}]}$$

$$\boxed{[\zeta_{\log n}(s)]=\sum_{\frac{\log j}{\log n}\leq 1} j^{-s}}$$

$$\boxed{[\zeta_{\log n}(s)^2]=\sum_{\frac{\log j}{\log n}+\frac{\log k}{\log n}\leq 1} (j\cdot k)^{-s}}$$

$$\boxed{\lim_{n\to\infty}[\zeta_{\log n}(s)]=\lim_{n\to\infty}\sum_{\frac{\log j}{\log n}\leq 1} j^{-s}=\zeta(s)}$$

$$[\zeta(0)]_n=[\zeta_{\log n}(0)]$$

$$D_k(l)=\sum_{0\leq\log_n j\leq l} D_{k-1}(l-\log_n j)$$
$$D_0(l)=UnitStep(l)$$

$$\log_n j\leq l$$
$$\frac{\log j}{\log n}\leq l$$
$$\log j\leq l\cdot\log n$$
$$e^{\log} j\leq e^{l\cdot\log n}$$
$$j\leq e^{l\cdot\log n}$$

$$D_k(l)=\sum_{j=0}^{\lfloor n^l\rfloor} D_{k-1}(l-\log_n j)$$
$$D_0(l)=UnitStep(l)$$

In this version, l is initialized to 1, and n is log n.

$$D_k(l)=\sum_{j=0}^{\lfloor e^{l\cdot n}\rfloor} D_{k-1}(l-\frac{\log j}{n})$$
$$D_0(l)=UnitStep(l)$$

$$[\log(\frac{\zeta_{\frac{1}{2}\log n}(0)}{\zeta_{\log n}(0)})]=[\log\zeta_{\frac{1}{2}\log n}(0)]-[\log\zeta_{\log n}(0)]$$

$$[\zeta(0)\cdot\zeta_{\frac{1}{2}}(0)]_n=\sum_{a\cdot b^2\leq n}1=\sum_{\frac{a\cdot b^2}{n}\leq 1}1=\sum_{\log a+2\log b-\log n\leq 0}1=\sum_{\frac{\log a+2\log b}{\log n}\leq 1}1$$

$$[\zeta(0)\cdot\zeta(0)]_n=\sum_{a\cdot b\leq n}1=\sum_{\frac{a\cdot b}{n}\leq 1}1=\sum_{\log a+\log b-\log n\leq 0}1=\sum_{\frac{\log a+\log b}{\log n}\leq 1}1$$

$$[\zeta(s)\cdot\zeta_{\frac{1}{2}}(2s)]_n=\sum_{a\cdot b^2\leq n}1\,a^{-s}\cdot b^{-2s}$$

…………..

$$[\zeta_{\log n}(0)\cdot\zeta_{\log m}(0)]=\sum_{\frac{\log a}{\log n}+\frac{\log b}{\log m}\leq 1}1$$

$$[\zeta_{\log n}(s)\cdot\zeta_{\log m}(s)]=\sum_{\frac{\log a}{\log n}+\frac{\log b}{\log m}\leq 1}a^{-s}\cdot b^{-s}$$

$$\boxed{[((1-x^{1-s})\zeta(s))^z]_n = \sum_{k=0} \binom{z}{k} [((1-x^{1-s})\zeta(s)-1)^k]_n}$$

$$[(1+y^{s-1}\cdot\zeta(s,1+y))^z]_n$$

$$[(1+y^{s-1}\cdot\zeta(s,1+y))^z]_n = [((1-(n+1)^{1-s})\zeta(s))^z]_n$$

???

$$\frac{1}{b}\sum_{j=b+1}^{b\cdot n}\alpha_{\frac{a}{b}}(j)-\frac{1}{2}\cdot\frac{1}{b^2}\cdot\sum_{j=b+1}^{b\cdot n}\sum_{k=b+1}^{\lfloor\frac{b^2\cdot n}{j}\rfloor}\alpha_{\frac{a}{b}}(j)\cdot\alpha_{\frac{a}{b}}(k)+\frac{1}{3}\cdot\frac{1}{b^3}\cdot\sum_{j=b+1}^{b\cdot n}\sum_{k=b+1}^{\lfloor\frac{b^2\cdot n}{j}\rfloor}\sum_{l=b+1}^{\lfloor\frac{b^3\cdot n}{j\cdot k}\rfloor}\alpha_{\frac{a}{b}}(j)\cdot\alpha_{\frac{a}{b}}(k)\cdot\alpha_{\frac{a}{b}}(l)-\frac{1}{4}\ldots$$

$$1+\binom{z}{1}\cdot\frac{1}{b}\sum_{j=b+1}^{b\cdot n}\alpha_{\frac{a}{b}}(j)+\binom{z}{2}\cdot\frac{1}{b^2}\cdot\sum_{j=b+1}^{b\cdot n}\sum_{k=b+1}^{\lfloor\frac{b^2\cdot n}{j}\rfloor}\alpha_{\frac{a}{b}}(j)\cdot\alpha_{\frac{a}{b}}(k)+\binom{z}{3}\cdot\frac{1}{b^3}\cdot\sum_{j=b+1}^{b\cdot n}\sum_{k=b+1}^{\lfloor\frac{b^2\cdot n}{j}\rfloor}\sum_{l=b+1}^{\lfloor\frac{b^3\cdot n}{j\cdot k}\rfloor}\alpha_{\frac{a}{b}}(j)\cdot\alpha_{\frac{a}{b}}(k)\cdot\alpha_{\frac{a}{b}}(l)+\binom{z}{4}\ldots$$

…

$$\alpha_{\frac{a}{b}}(n)=b\cdot(\lfloor\frac{n}{b}\rfloor-\lfloor\frac{n-1}{b}\rfloor)-a\cdot(\lfloor\frac{n}{a}\rfloor-\lfloor\frac{n-1}{a}\rfloor)$$

$$1+b^{-1}\cdot\binom{z}{1}\cdot\sum_{j=1+b^{-1}}^{n}\alpha_{\frac{a}{b}}(j)+b^{-2}\cdot\binom{z}{2}\cdot\sum_{j=b^{-1}+1}^{n}\sum_{k=b^{-1}+1}^{\lfloor\frac{n}{j}\rfloor}\alpha_{\frac{a}{b}}(j)\cdot\alpha_{\frac{a}{b}}(k)+b^{-3}\cdot\binom{z}{3}\cdot\sum_{j=b^{-1}+1}^{n}\sum_{k=b^{-1}+1}^{\lfloor\frac{n}{j}\rfloor}\sum_{l=b^{-1}+1}^{\lfloor\frac{n}{j\cdot k}\rfloor}\alpha_{\frac{a}{b}}(j\cdot b)\cdot\alpha_{\frac{a}{b}}(k\cdot b)\cdot\alpha_{\frac{a}{b}}(l\cdot b)+\binom{z}{4}\ldots$$

all sums incremented by b^-1

...

$$t(n)=d\cdot(\lfloor\frac{n\,d}{d}\rfloor-\lfloor\frac{n\,d-1}{d}\rfloor)-a\cdot(\lfloor\frac{n\,d}{a}\rfloor-\lfloor\frac{n\,d-1}{a}\rfloor)$$

$$1+d^{-1}\cdot\binom{z}{1}\cdot\sum_{j=1+d^{-1}}^{n}t(j)+d^{-2}\cdot\binom{z}{2}\cdot\sum_{j=d^{-1}+1}^{n}\sum_{k=d^{-1}+1}^{\lfloor\frac{n}{j}\rfloor}t(j)\cdot t(k)+d^{-3}\cdot\binom{z}{3}\cdot\sum_{j=d^{-1}+1}^{n}\sum_{k=d^{-1}+1}^{\lfloor\frac{n}{j}\rfloor}\sum_{l=d^{-1}+1}^{\lfloor\frac{n}{j\cdot k}\rfloor}t(j)\cdot t(k)\cdot t(l)+\binom{z}{4}\ldots$$

all sums incremented by d^-1

...

$$t(n)=d\cdot(\lfloor\frac{n\,d}{d}\rfloor-\lfloor\frac{n\,d-1}{d}\rfloor)-(d+1)\cdot(\lfloor\frac{n\,d}{d+1}\rfloor-\lfloor\frac{n\,d-1}{d+1}\rfloor)$$

$$1+d^{-1}\cdot\binom{z}{1}\cdot\sum_{j=1+d^{-1}}^{n}t(j)+d^{-2}\cdot\binom{z}{2}\cdot\sum_{j=d^{-1}+1}^{n}\sum_{k=d^{-1}+1}^{\lfloor\frac{n}{j}\rfloor}t(j)\cdot t(k)+d^{-3}\cdot\binom{z}{3}\cdot\sum_{j=d^{-1}+1}^{n}\sum_{k=d^{-1}+1}^{\lfloor\frac{n}{j}\rfloor}\sum_{l=d^{-1}+1}^{\lfloor\frac{n}{j\cdot k}\rfloor}t(j)\cdot t(k)\cdot t(l)+\binom{z}{4}\ldots$$

all sums incremented by d^-1

$$\lim_{z\to 0}\frac{\partial}{\partial z}([\zeta(s)^z]_n\cdot[\zeta(s)^{-z}]_m)=[\log\zeta(s)]_n-[\log\zeta(s)]_m$$

$$\lim_{z\to 0}\frac{\partial}{\partial z}(D_z(n)\cdot D_{-z}(m))=\Pi(n)-\Pi(m)$$

$$\lim_{s\to 0}\frac{\partial}{\partial s}\lim_{z\to 0}\frac{\partial}{\partial z}([\zeta(s)^z]_n\cdot[\zeta(s)^{-z}]_m)=\psi(n)-\psi(m)$$

AND ALSO

$$\lim_{z\to 0}\frac{\partial}{\partial z}(\frac{[\zeta(s)^z]_n}{[\zeta(s)^z]_m})=[\log\zeta(s)]_n-[\log\zeta(s)]_m$$

$$\lim_{z\to 0}\frac{\partial}{\partial z}(\frac{D_z(n)}{D_z(m)})=\Pi(n)-\Pi(m)$$

$$\lim_{s\to 0}\frac{\partial}{\partial s}\lim_{z\to 0}\frac{\partial}{\partial z}(\frac{[\zeta(s)^z]_n}{[\zeta(s)^z]_m})=\psi(n)-\psi(m)$$

AND THUS

$$\lim_{z\to 0}\frac{\partial}{\partial z}(\frac{[\zeta(s)^z]_n}{[\zeta(s)^z]_{n-1}})=\kappa(n)\cdot n^{-s}$$

$$\lim_{z\to 0}\frac{\partial}{\partial z}(\frac{D_z(n)}{D_z(n-1)})=\kappa(n)$$

$$\lim_{s\to 0}\frac{\partial}{\partial s}\lim_{z\to 0}\frac{\partial}{\partial z}(\frac{[\zeta(s)^z]_n}{[\zeta(s)^z]_{n-1}})=\Lambda(n)$$

SO. Basic question. How much further does this hold?

$$\lim_{z\to 0}\frac{\partial}{\partial z}([\zeta(s)^{a\cdot z}]_n\cdot[\zeta(t)^{-z}]_m)=a[\log\zeta(s)]_n-[\log\zeta(t)]_m$$

(check on some point how this accords with the laguerreL stuff)

$$\lim_{z\to 0}\frac{\partial}{\partial z}(\frac{L_{-z}(\log n)}{L_{-z}(\log m)})=(li(n)-\log\log n-\gamma)-(li(m)-\log\log m-\gamma)$$

$$\lim_{z\to 0}\frac{\partial}{\partial z}(L_{-a\cdot z}(\log n)\cdot L_{-b\cdot z}(\log m))=a(li(n)-\log\log n-\gamma)+b(li(m)-\log\log m-\gamma)$$

$$\lim_{z\to 0}\frac{\partial}{\partial z}(\frac{[(1-2^{1-s})\zeta(s)^z]_n}{[(1-2^{1-s})\zeta(s)^z]_m})=[\log((1-2^{1-s}))\zeta(s)]_n-[\log((1-2^{1-s})\zeta(s))]_m$$

$$\lim_{z\to 0}\frac{\partial}{\partial z}\left(\frac{[\zeta(s)^z]_n}{[\zeta(s)^z]_{n-1}}\right)=\lim_{z\to 0}\frac{\partial}{\partial z}\left([\zeta(s)^z]_n-[\zeta(s)^z]_{n-1}\right)$$

$$\lim_{z\to 0}\frac{\partial}{\partial z}\left(\frac{D_z(n)}{D_z(n-1)}\right)=\lim_{z\to 0}\frac{\partial}{\partial z}\left(D_z(n)-D_z(n-1)\right)$$

$$\lim_{s\to 0}\frac{\partial}{\partial s}\lim_{z\to 0}\frac{\partial}{\partial z}\left(\frac{[\zeta(s)^z]_n}{[\zeta(s)^z]_{n-1}}\right)=\lim_{s\to 0}\frac{\partial}{\partial s}\lim_{z\to 0}\frac{\partial}{\partial z}\left([\zeta(s)^z]_n-[\zeta(s)^z]_{n-1}\right)$$

$$\boxed{[((1-x^{1-s})\zeta(s))^z]_n = \sum_{j=0} (-1)^j \binom{z}{j} x^{j(1-s)} [\zeta(s)^z]_{n\cdot x^{-j}}}$$

$$[\zeta(0,y+1)^0]=1_{[1,\infty)}(n)$$
$$[\zeta(0,y+1)]_n=\lfloor n\rfloor-y$$
$$[\zeta(0,y+1)^2]_n=y^2-\lfloor n^{\frac{1}{2}}\rfloor^2+2\sum_{b=y+1}^{\lfloor n^{\frac{1}{2}}\rfloor}\lfloor\frac{n}{b}\rfloor$$
$$[\zeta(0,y+1)^3]_n=-y^3+\lfloor n^{\frac{1}{3}}\rfloor^3+3\sum_{b=y+1}^{\lfloor n^{\frac{1}{3}}\rfloor}\lfloor\frac{n}{b^2}\rfloor-3\sum_{b=y+1}^{\lfloor n^{\frac{1}{3}}\rfloor}\lfloor(\frac{n}{b})^{\frac{1}{2}}\rfloor^2+6\sum_{b=y+1}^{\lfloor n^{\frac{1}{3}}\rfloor}\sum_{c=b+1}^{\lfloor(\frac{n}{b})^{\frac{1}{2}}\rfloor}\lfloor\frac{n}{b\,c}\rfloor$$

$$[\zeta(0,y+1)]_n-[\zeta(0,y)]_n=-1$$

$$[\zeta(0,y+1)^2]_n-[\zeta(0,y)^2]_n=2(y-\lfloor\frac{n}{y}\rfloor)-1$$

$$[\zeta(0,y+1)^3]_n-[\zeta(0,y)^3]_n=-3y^2+3y-1+3\lfloor\frac{n}{y^2}\rfloor-3\lfloor(\frac{n}{y})^{\frac{1}{2}}\rfloor^2+6\sum_{c=y+1}^{\lfloor(\frac{n}{y})^{\frac{1}{2}}\rfloor}\lfloor\frac{n}{y\,c}\rfloor$$

$$[(1+\zeta(0,y))^z]_n-[(1+\zeta(0,y+1))^z]_n=???$$

$$[\zeta(s,y+1)^k]_n=\sum_{j=0}^{k}(-1)^j\binom{k}{j}y^{-j\cdot s}[\zeta(s,y)^{k-j}]_{n(y+1)^{-j}}$$

$$[\zeta(0,b+1)^k]_n=\sum_{j=1}^{k}\sum_{y=1}^{b}(-1)^j\binom{k}{j}[\zeta(0,y)^{k-j}]_{n(y+1)^{-j}}$$

$$[\zeta(0,b+1)^k]_n=-k\cdot\sum_{y=1}^{b}[\zeta(0,y)^{k-1}]_{\frac{n}{y+1}}+\sum_{j=2}^{k}\sum_{y=1}^{b}(-1)^j\binom{k}{j}[\zeta(0,y)^{k-j}]_{n(y+1)^{-j}}$$

$$[(1+y^{s-1}\cdot\zeta(s,1+y))^z]_n=\sum_{k=0}^{\infty}\binom{z}{k}y^{k(s-1)}\cdot[(\zeta(s,1+y))^k]_{n\cdot y^k}$$

$$\Pi(n)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}1^{-k}\cdot[(\zeta(0,1+1))^k]_{n\cdot 1^k}$$

$$li(n)-\log\log n-\gamma=\lim_{y\to\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}y^{-k}\cdot[(\zeta(0,1+y))^k]_{n\cdot y^k}$$

$$M(x)=M(u)-\sum_{m\leq u}\mu(m)\cdot\sum_{\frac{u}{m}<n\leq\frac{x}{m}}M\left(\frac{x}{mn}\right)$$

$$[\zeta(0)^{-1}]_x=[\zeta(0)^{-1}]_u-\sum_{m\leq u}\nabla[\zeta(0)^{-1}]_m\cdot\sum_{\frac{u}{m}<n\leq\frac{x}{m}}[\zeta(0)^{-1}]_{\frac{x}{mn}}$$

$$[\zeta(0)^{-1}]_n=[\zeta(0)^{-1}]_t-\sum_{j\leq t}\nabla[\zeta(0)^{-1}]_j\cdot\sum_{\frac{t}{j}<k\leq\frac{n}{j}}[\zeta(0)^{-1}]_{\frac{n}{jk}}$$

$$[\zeta(0)^{-1}]_n=[\zeta(0)^{-1}]_t-\sum_{j=1}^{t}\sum_{k=\lfloor\frac{t}{j}\rfloor+1}^{\lfloor\frac{n}{j}\rfloor}\nabla[\zeta(0)^{-1}]_j\cdot[\zeta(0)^{-1}]_{n\cdot(jk)^{-1}}$$

$$[f(0)]_n = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} [f(s)]_\infty \cdot \frac{n^s}{s}\, ds$$

$$[y^{s-1}\cdot\zeta(s,1+y)]_n = y^{-1}\sum_{j=1}(1+\frac{j}{y})^{-s}$$

$$[(y^{s-1}\cdot\zeta(s,1+y))^2]_n = y^{-2}\sum_{j=1}\sum_{k=1}((1+\frac{j}{y})\cdot(1+\frac{k}{y}))^{-s}$$

$$[(y^{s-1}\cdot\zeta(s,1+y))^3]_n = y^{-3}\sum_{j=1}\sum_{k=1}\sum_{l=1}((1+\frac{j}{y})\cdot(1+\frac{k}{y})\cdot(1+\frac{l}{y}))^{-s}$$

$$\sum_{(1+\frac{j}{y})\cdot(1+\frac{k}{y})\le n}\frac{1}{y}\cdot\frac{1}{y}$$

$$\sum_{1+\frac{j}{y}+\frac{k}{y}+\frac{j\cdot k}{y^2}\le n}\frac{1}{y}\cdot\frac{1}{y}$$

$$\sum_{\log(1+\frac{j}{y})+\log(1+\frac{k}{y})\le\log n}\frac{1}{y}\cdot\frac{1}{y}$$

$$\sum_{\frac{\log(1+\frac{j}{y})}{\log n}+\frac{\log(1+\frac{k}{y})}{\log n}\le 1}\frac{1}{y}\cdot\frac{1}{y}$$

$$\sum_{\frac{\log(\frac{y+j}{y})}{\log n}+\frac{\log(\frac{y+k}{y})}{\log n}\le 1}\frac{1}{y}\cdot\frac{1}{y}$$

$$\sum_{\frac{\log(y+j)-\log y}{\log n}+\frac{\log(y+k)-\log y}{\log n}\le 1}\frac{1}{y}\cdot\frac{1}{y}$$

$$\sum_{\log_n(y+j)+\log_n(y+k)-2\log_n y\le 1}\frac{1}{y}\cdot\frac{1}{y}$$

$$f_y(n) = (\sum_{(1+\frac{j}{y})\le n}\frac{1}{y}) - \frac{1}{2}(\sum_{(1+\frac{j}{y})\cdot(1+\frac{k}{y})\le n}\frac{1}{y}\cdot\frac{1}{y}) + \frac{1}{3}(\sum_{(1+\frac{j}{y})\cdot(1+\frac{k}{y})\cdot(1+\frac{l}{y})\le n}\frac{1}{y}\cdot\frac{1}{y}\cdot\frac{1}{y}) - \frac{1}{4}\cdots$$

$$f_1(n) = \Pi(n)$$

$$\lim_{y\to\infty} f_y(n) = li(n) - \log\log n - \gamma$$

$$f_y(n) - f_{y-\epsilon}(n) =$$

$$(\sum_{(1+\frac{j}{y})\le n}\frac{1}{y} - \sum_{(1+\frac{j}{y-\epsilon})\le n}\frac{1}{y-\epsilon})$$

$$-\frac{1}{2}(\sum_{(1+\frac{j}{y})\cdot(1+\frac{k}{y})\le n}\frac{1}{y}\cdot\frac{1}{y} - \sum_{(1+\frac{j}{y-\epsilon})\cdot(1+\frac{k}{y-\epsilon})\le n}\frac{1}{y-\epsilon}\cdot\frac{1}{y-\epsilon})$$

$$+\frac{1}{3}(\sum_{(1+\frac{j}{y})\cdot(1+\frac{k}{y})\cdot(1+\frac{l}{y})\le n}\frac{1}{y}\cdot\frac{1}{y}\cdot\frac{1}{y} - \sum_{(1+\frac{j}{y-\epsilon})\cdot(1+\frac{k}{y-\epsilon})\cdot(1+\frac{l}{y-\epsilon})\le n}\frac{1}{y-\epsilon}\cdot\frac{1}{y-\epsilon}\cdot\frac{1}{y-\epsilon})$$

$$-\frac{1}{4}\cdots$$

$$\sum_{(1+\frac{j}{y})\le n}\frac{1}{y}-\sum_{(1+\frac{j}{y-\epsilon})\le n}\frac{1}{y-\epsilon}$$

$$\sum_{(1+\frac{j}{y})\cdot(1+\frac{k}{y})\le n}\frac{1}{y}\cdot\frac{1}{y}-\sum_{(1+\frac{j}{y-\epsilon})\cdot(1+\frac{k}{y-\epsilon})\le n}\frac{1}{y-\epsilon}\cdot\frac{1}{y-\epsilon}$$

$$\sum_{(1+\frac{j}{y})\cdot(1+\frac{k}{y})\cdot(1+\frac{l}{y})\le n}\frac{1}{y}\cdot\frac{1}{y}\cdot\frac{1}{y}-\sum_{(1+\frac{j}{y-\epsilon})\cdot(1+\frac{k}{y-\epsilon})\cdot(1+\frac{l}{y-\epsilon})\le n}\frac{1}{y-\epsilon}\cdot\frac{1}{y-\epsilon}\cdot\frac{1}{y-\epsilon}$$

Think more about that function connected to $\pi(n)$

$$\nabla[(\prod_{k=1}\zeta_{1/k}(0)^{\mu(k)\cdot k^{-1}})^{z}]_n=\sum_{p^a|n}\frac{z^a}{a!}$$

$$[(\prod_{k=1}\zeta_{1/k}(0)^{\mu(k)\cdot k^{-1}})^{z}]_n=\sum_{j=1}^{n}\sum_{p^a|j}\frac{z^a}{a!}$$

$$\boxed{[(\prod_{k=1}\zeta_{1/k}(s)^{\mu(k)\cdot k^{-1}})^{z}]_n=\sum_{k=0}^{\infty}\binom{z}{k}[(\prod_{k=1}\zeta_{1/k}(0)^{\mu(k)\cdot k^{-1}}-1)^{k}]_n}$$

$$\boxed{[(\prod_{k=1}\zeta_{1/k}(s)^{\mu(k)\cdot k^{-1}})^{z}]_n=\sum_{k=0}^{\infty}\frac{z^k}{k!}[(\log\prod_{k=1}\zeta_{1/k}(s)^{\mu(k)\cdot k^{-1}})^{k}]_n}$$

$$\boxed{[(\prod_{k=1}\zeta_{1/k}(s)^{\mu(k)\cdot k^{-1}})^{z}]_n=\sum_{a=0}^{\frac{\log n}{\log 2}}\frac{z^a}{a!}\cdot 2^{-as}\cdot\sum_{b=0}^{\frac{\log n-a\log 2}{\log 3}}\frac{z^b}{b!}\cdot 3^{-bs}\cdot\sum_{c=0}^{\frac{\log n-a\log 2-b\log 3}{\log 5}}\frac{z^c}{c!}\cdot 5^{-cs}\cdot\sum_{d=0}^{\frac{\log n-a\log 2-b\log 3-c\log 5}{\log 7}}\frac{z^d}{d!}\cdot 7^{-ds}\cdot\ldots}$$

$$\boxed{[\prod_{k=1}\zeta_{1/k}(0)^{\mu(k)\cdot k^{-1}}]_n=\sum_{a=0}^{\frac{\log n}{\log 2}}\frac{1}{a!}\cdot\sum_{b=0}^{\frac{\log n-a\log 2}{\log 3}}\frac{1}{b!}\cdot\sum_{c=0}^{\frac{\log n-a\log 2-b\log 3}{\log 5}}\frac{1}{c!}\cdot\sum_{d=0}^{\frac{\log n-a\log 2-b\log 3-c\log 5}{\log 7}}\frac{1}{d!}\cdot\ldots}$$

$$[(\prod_{k=1}\zeta_{1/k}(s)^{\mu(k)\cdot k^{-1}})^{z}]_n=f_1(n,1)\quad\text{where}\quad f_k(n,j)=\begin{cases}p_j^{-s}(\frac{z}{k}\cdot f_{k+1}(\frac{n}{p_j},j)+f_1(n,j+1)) & \text{if } n\geq p_j\\ 1 & \text{otherwise}\end{cases}$$

$$[(\prod_{k=1}\zeta_{1/k}(0)^{\mu(k)\cdot k^{-1}})^{z}]_n=f_1(n,1)\quad\text{where}\quad f_k(n,j)=\begin{cases}\frac{z}{k}\cdot f_{k+1}(\frac{n}{p_j},j)+f_1(n,j+1) & \text{if } n\geq p_j\\ 1 & \text{otherwise}\end{cases}$$

What is the relationship between $[(1+\zeta(s,y))^z]_n$ and $[(1+\zeta(s,y+1))^z]_n$?

$$[(1+\zeta(s,y))^z]_n=\sum_{k=0}\binom{z}{k}[\zeta(s,y)^k]_n$$

$$[(1+\zeta(s,y+1))^z]_n=\sum_{k=0}\binom{z}{k}[\zeta(s,y+1)^k]_n$$

$$\dots$$

Because

$$[\zeta(s,y)^k]_n=\sum_{j=0}^{k}\binom{k}{j}\cdot[\zeta(s,y+1)^j]_{n\cdot y^{j-k}}$$

and

$$[\zeta(s,y+1)^k]_n=\sum_{j=0}^{k}(-1)^{k-j}\cdot\binom{k}{j}\cdot[\zeta(s,y)^j]_{n\,y^{j-k}}$$

it must be that

$$[(1+\zeta(s,y+1))^z]_n=\sum_{k=0}\binom{z}{k}\sum_{j=0}^{k}(-1)^{k-j}\cdot\binom{k}{j}\cdot[\zeta(s,y)^j]_{n\,y^{j-k}}$$

and

$$[(1+\zeta(s,y))^z]_n=\sum_{k=0}\binom{z}{k}\sum_{j=0}^{k}\binom{k}{j}\cdot[\zeta(s,y+1)^j]_{n\cdot y^{j-k}}$$

Move on from there to

$$[(1+\zeta(s,y+1))^z]_n=\sum_{k=0}\binom{z}{k}\sum_{j=0}^{k}(-1)^{k-j}\cdot\binom{k}{j}\cdot\sum_{m=0}^{j}(-1)^{(j-m)}\binom{j}{m}[(1+\zeta(s,y))^m]_{n\,y^{j-k}}$$

and

$$[(1+\zeta(s,y))^z]_n=\sum_{k=0}\binom{z}{k}\sum_{j=0}^{k}\binom{k}{j}\cdot\sum_{m=0}^{j}(-1)^{(j-m)}\binom{j}{m}[(1+\zeta(s,y+1))^m]_{n\cdot y^{j-k}}$$

Shift to

$$[(1+\zeta(s,y+1))^z]_n=\sum_{k=0}\sum_{j=0}^{k}\sum_{m=0}^{j}(-1)^{k-m}\cdot\binom{z}{k}\binom{k}{j}\cdot\binom{j}{m}[(1+\zeta(s,y))^m]_{n\,y^{j-k}}$$

and

$$[(1+\zeta(s,y))^z]_n=\sum_{k=0}\sum_{j=0}^{k}\sum_{m=0}^{j}(-1)^{(j-m)}\binom{z}{k}\binom{k}{j}\cdot\binom{j}{m}[(1+\zeta(s,y+1))^m]_{n\cdot y^{j-k}}$$

$$[(1+\zeta(s,y))^z]_n=\sum_{k=0}(\genfrac{}{}{0pt}{}{z}{k})\sum_{j=0}^{k}(\genfrac{}{}{0pt}{}{k}{j})\cdot[\zeta(s,y+1)^j]_{n\cdot y^{j-k}}$$

$$[(1+\zeta(s,y))^z]_n=\sum_{k=0}(\genfrac{}{}{0pt}{}{z}{k})([\zeta(s,y+1)^k]_n+\sum_{j=0}^{k-1}(\genfrac{}{}{0pt}{}{k}{j})\cdot[\zeta(s,y+1)^j]_{n\cdot y^{j-k}})$$

$$[(1+\zeta(s,y))^z]_n=[(1+\zeta(s,y+1))^z]_n+\sum_{k=0}(\genfrac{}{}{0pt}{}{z}{k})\sum_{j=0}^{k-1}(\genfrac{}{}{0pt}{}{k}{j})\cdot[\zeta(s,y+1)^j]_{n\cdot y^{j-k}}$$

$$[(1+\zeta(s,y))^z]_n=[(1+\zeta(s,y+1))^z]_n+\sum_{k=0}(\genfrac{}{}{0pt}{}{z}{k})\sum_{j=0}^{k-1}(\genfrac{}{}{0pt}{}{k}{j})\cdot[\zeta(s,y+1)^j]_{n\cdot y^{j-k}}$$

$$[(1+\zeta(s,y))^z]_n=[(1+\zeta(s,y+1))^z]_n+(\genfrac{}{}{0pt}{}{z}{1})[(1+\zeta(s,y+1))^{z-1}]_{n/y}+\sum_{j=0}^{k-2}(\genfrac{}{}{0pt}{}{k}{j})\cdot[\zeta(s,y+1)^j]_{n\cdot y^{j-k}}$$

...

$$\boxed{[(1+\zeta(0,y))^z]_n=\sum_{k=0}(\genfrac{}{}{0pt}{}{z}{k})\cdot[1+\zeta(0,y+1)^{z-k}]_{n/y^k}}$$

$$[\zeta(0)^z]=[(1+\zeta(0,2))^z]_n$$

(up to $k\le\log_y n$)

…

And also,

$$\boxed{[(1+\zeta(0,y))^z]_n=\sum_{k=0}(-1)^k(\genfrac{}{}{0pt}{}{z}{k})\cdot[1+\zeta(0,y-1)^{z-k}]_{n/(y-1)^k}}$$

(up to $k\le\log_{y-1} n$)

HOORAY. HOORAY. HOORAY.

Now, account for the s.

And make sure this identity works for all the variations.

Generally:

$$[(1+f(s,y))^z]_n=\sum_{k=0}\binom{z}{k}\cdot(\nabla[(1+f(s,y))]_y)^k\cdot[(1+f(s,y+1))^{z-k}]_{n/y^k}$$

...

$$[(1+\zeta(s,y))^z]_n=\sum_{k=0}\binom{z}{k}\cdot y^{-sk}\cdot[(1+\zeta(s,y+1))^{z-k}]_{n/y^k}$$

$$[(1+\zeta(s,y))^z]_n=\sum_{k=0}(-1)^k\cdot(y-1)^{-sk}\cdot\binom{z}{k}\cdot[1+\zeta(s,y-1)^{z-k}]_{n/(y-1)^k}$$

...

$$[((1-2^{1-s})(1+\zeta(s,y)))^z]_n=\sum_{k=0}\binom{z}{k}\cdot(y^{-s}\cdot(-1)^{y+1})^k\cdot[((1-2^{1-s})(1+\zeta(s,y+1)))^{z-k}]_{n/y^k}$$

...

$$[(1+\frac{\zeta_{1/2}(2s,y)}{\zeta(s,y)})^z]_n=\sum_{k=0}\binom{z}{k}\cdot(y^{-s}\cdot\lambda(y))^k\cdot[(1+\frac{\zeta_{1/2}(2s,y+1)}{\zeta(s,y+1)})^{z-k}]_{n\cdot y^{-k}}$$

...

$$[(1+\frac{\zeta(s,y)}{\zeta_{1/2}(2s,y)})^z]_n=\sum_{k=0}\binom{z}{k}\cdot(y^{-s}\cdot|\mu(y)|)^k\cdot[(1+\frac{\zeta(s,y+1)}{\zeta_{1/2}(2s,y+1)})^{z-k}]_{n\cdot y^{-k}}$$

...

$$[(1+\zeta(s-a,y)\cdot\zeta(s,y))^z]_n=\sum_{k=0}\binom{z}{k}\cdot(y^{-s}\cdot\sigma_a(y))^k\cdot[(1+\zeta(s-a,y+1)\cdot\zeta(s,y+1))^{z-k}]_{n\cdot y^{-k}}$$

...

$$[(1+\frac{\zeta(s-a,y)}{\zeta(s,y)})^z]_n=\sum_{k=0}\binom{z}{k}\cdot(y^{-s}\cdot J_a(y))^k\cdot[(1+\frac{\zeta(s-a,y+1)}{\zeta(s,y+1)})^{z-k}]_{n\cdot y^{-k}}$$

...

$$[(1+\prod_{k=1}\zeta_{1/k}(ks,y))^z]_n=\sum_{k=0}\binom{z}{k}\cdot(y^{-s}\cdot a(y))^k\cdot[(1+\prod_{k=1}\zeta_{1/k}(ks,y+1))^{z-k}]_{n\cdot y^{-k}}$$

...

$$[(1+\prod_{k=1}\zeta_{1/k}(ks)^{\frac{\mu(k)}{k}},y)^z]_n=\sum_{k=0}\binom{z}{k}\cdot(y^{-s}\cdot(\sum_{p^a|y}\frac{z^a}{a!}))^k\cdot[(1+\prod_{k=1}\zeta_{1/k}(ks)^{\frac{\mu(k)}{k}},y+1)^{z-k}]_{n\cdot y^{-k}}$$

...

Yep $[(\zeta(s)^y)^z]_n$ and $[(t\cdot\zeta(s))^z]_n$

FOR:

$$[(1+y^{s-1}\cdot\zeta(s,1+y))^z]_n$$

$$\boxed{[(1+x\cdot\zeta(s,y))^z]_n=\sum_{k=0}\binom{z}{k}\cdot(x\,y^{-s})^k\cdot[(1+x\cdot\zeta(s,y+x))^{z-k}]_{n/y^k}}$$

with initial value

$$[(1+x\cdot\zeta(s,1+x))^z]_n$$

NEWER NOTATION:

$$\boxed{[(1+x^{1-s}\cdot\zeta(s,y))^z]_n=\sum_{k=0}\binom{z}{k}\cdot(x^{1-s}\cdot y^{-s})^k\cdot[(1+x^{1-s}\cdot\zeta(s,1+y))^{z-k}]_{n/(x\cdot y)^k}}$$

with initial value of

$$\boxed{[(1+y^{1-s}\cdot\zeta(s,1+y^{-1}))^z]_n}$$

OLD:

$$\boxed{f(n,z,y)=\sum_{k=0}^{\log_y n}\binom{z}{k}x^k\cdot f(n\cdot y^{-k},z-k,y+x)}$$

with initial value of

$$\boxed{f(n,z,1+x)}$$

Math checks out. Verify the notation, however. Identities?

```
dss[n_, s_, y_, z_, x_] :=
If[ n < y, 1, Sum[ bin[z, k] (x y^-s)^k dss[n/y^k , s, y + x, z - k, x], {k, 0, Log[y, n]}]]
dss[100,0,1+1/3,z,1/3]
```

$$\nabla[(1+x\cdot\zeta(s,1+x))^z]_n=[(1+x\cdot\zeta(s,1+x))^z]_n-[(1+x\cdot\zeta(s,1+x))^z]_{n-x^{\frac{\log n}{\log 1+x}}}$$

$$[(y^{1-s}\cdot\zeta(s,1+y^{-1}))^k]_n=y^{k(1-s)}[\zeta(s,1+y^{-1})^k]_{n\cdot y^k}$$

$$[(1+y^{1-s}\cdot\zeta(s,1+y^{-1}))^z]_n=\sum_{k=0}^{\infty}\binom{z}{k}[(y^{1-s}\cdot\zeta(s,1+y^{-1}))^k]_n$$

$$[(1+y^{1-s}\cdot\zeta(s,1+y^{-1}))^z]_n=\sum_{k=0}^{\infty}\binom{z}{k}y^{k(1-s)}[\zeta(s,1+y^{-1})^k]_{n\cdot y^k}$$

$$[(1+y^{1-s}\cdot\zeta(s,1+y^{-1}))^k]_n=[(1+y^{1-s}\cdot\zeta(s,1+y^{-1}))^{k-1}]_n+y\sum_{j=1}(1+j\cdot y)^{-s}[(1+y^{1-s}\cdot\zeta(s,1+y^{-1}))^{k-1}]_{n(1+j\cdot y)^{-1}}$$

$$[(1+x^{1-s}\cdot\zeta(s,y))^k]_n=[(1+x^{1-s}\cdot\zeta(s,y))^{k-1}]_n+x\sum_{j=0}((y+j)\cdot x)^{-s}[(1+x^{1-s}\cdot\zeta(s,y))^{k-1}]_{n((y+j)\cdot x)^{-1}}$$

$$\boxed{[(1+\zeta(0,y))^z]_n = \sum_{k=0} (-1)^k \binom{z}{k} \cdot [1+\zeta(0,y-1)^{z-k}]_{n/(y-1)^k}}$$

$$\boxed{\nabla[(1+\zeta(0,3))^z]_n = \sum_{k=0} (-1)^k \binom{z}{k} \cdot ([1+\zeta(0,2)^{z-k}]_{n/2^k} - [1+\zeta(0,2)^{z-k}]_{(n-1)/2^k})}$$

$$\boxed{\nabla[(1+\zeta(0,3))^z]_n = \nabla[1+\zeta(0,2)^z]_n - \sum_{k=1} (-1)^k \binom{z}{k} \cdot ([1+\zeta(0,2)^{z-k}]_{n/2^k} - [1+\zeta(0,2)^{z-k}]_{(n-1)/2^k})}$$

$$\boxed{\nabla[(1+\zeta(0,3))^z]_n = \sum_{k=0}^{2^k|n} (-1)^k \binom{z}{k} \cdot (\nabla[1+\zeta(0,2)^{z-k}]_{n/2^k})}$$

$$\boxed{\nabla[(1+\zeta(0,y+1))^z]_n = \sum_{k=0}^{y^k|n} (-1)^k \binom{z}{k} \cdot (\nabla[1+\zeta(0,y)^{z-k}]_{n/y^k})}$$

NOT GENERALLY MULTIPLICATIVE, UNFORTUNATELY

$$\boxed{\nabla[(1+\zeta(0,y))^z]_n = ???}$$

$$\boxed{\nabla[(1+\zeta(0,2))^z]_n = \prod_{p^a\|n} \frac{z^{(a)}}{a!}}$$

```
bin[z_, k_] := Product[z - j, {j, 0, k - 1}]/k!
dd[n_, s_, y_, k_] := dd[n, s, y, k] = Sum[ j^-s dd[Floor[n/j], s, y, k - 1], {j, y, n}]
dd[n_, s_, y_, 0] := UnitStep[n - 1]
dz[n_, s_, y_, z_] := Sum[ bin[z, k] dd[n, s, y, k], {k, 0, Log[y, n]}]
ddz[n_, s_, y_, z_] := dz[n, s, y, z] - dz[n - 1, s, y, z]
de[n_, k_, y_, z_] := bin[z, k] ddz[n, 0, y - 1, z - k] - If[ Mod[n, y - 1] == 0, de[n/(y - 1), k + 1, y, z], 0]
```

$$\boxed{f(n,y,z)=\sum_{k=0}^{\log_y n}\frac{z^k}{k!}\cdot\left(y^{-s}\cdot\kappa(y)\right)^k\cdot f\left(n\cdot y^{-k},y+1,z\right)}$$

$$\boxed{\left[\left(1+\zeta(s,2)\right)^z\right]_n=f(n,2,z)}$$

$$\boxed{[(1+x\cdot\zeta(s,y))^z]_n=\sum_{k=0}\binom{z}{k}\cdot(x\,y^{-s})^k\cdot[(1+x\cdot\zeta(s,y+x))^{z-k}]_{n/y^k}}$$

$$\boxed{[(1+x^{-s}\cdot\zeta(s,y\cdot x^{-1}))^z]_n=\sum_{k=0}\binom{z}{k}\cdot(x\,y^{-s})^k\cdot[(1+x^{-s}\cdot\zeta(s,y\cdot x^{-1}+1))^{z-k}]_{n/y^k}}$$

$$\boxed{[(1+x^{1-s}\cdot\zeta(s,y))^z]_n=\sum_{k=0}\binom{z}{k}\cdot(x^{1-s}\cdot y^{-s})^k\cdot[(1+x^{1-s}\cdot\zeta(s,y+1))^{z-k}]_{n/(x\cdot y)^k}}$$

with initial value of

$$\boxed{[(1+x^{1-s}\cdot\zeta(s,1+x^{-1}))^z]_n}$$

$$\boxed{[(1+x\cdot\zeta(s,y))^2]_n=1+2\cdot\frac{1}{x}\cdot\sum_{(y+\frac{a_1}{x})\le n}\left(y+\frac{a_1}{x}\right)^{-s}+\frac{1}{x^2}\cdot\sum_{(y+\frac{a_1}{x})\cdot(y+\frac{a_2}{x})\le n}\left(y+\frac{a_1}{x}\right)^{-s}\cdot\left(y+\frac{a_2}{x}\right)^{-s}}$$

$$\boxed{[1+2x\cdot\zeta(s,y)+x^2\cdot\zeta(s,y)^2]_n=1+2\cdot\frac{1}{x}\cdot\sum_{(y+\frac{a_1}{x})\le n}\left(y+\frac{a_1}{x}\right)^{-s}+\frac{1}{x^2}\cdot\sum_{(y+\frac{a_1}{x})\cdot(y+\frac{a_2}{x})\le n}\left(y+\frac{a_1}{x}\right)^{-s}\cdot\left(y+\frac{a_2}{x}\right)^{-s}}$$

$$1+\zeta(s,5)=1+\sum_{j=0}(5+j)^{-s}$$

$$(1+\zeta(s,5))^2=1+2\sum_{j=0}(5+j)^{-s}+\sum_{j=0}\sum_{k=0}(5+j)^{-s}\cdot(5+k)^{-s}$$

$$1+\zeta(s,5)=1+\sum_{j=0}(5+j)^{-s}$$

$$2^s\cdot\zeta(s,10)=\sum_{j=0}\left(5+\frac{j}{2}\right)^{-s}$$

$$(2^s\cdot\zeta(s,10))^2=\sum_{j=0}\sum_{k=0}\left(5+\frac{j}{2}\right)^{-s}\cdot\left(5+\frac{k}{2}\right)^{-s}$$

$$(1+2^s\cdot\zeta(s,10))^2=1+2\cdot\sum_{j=0}\left(5+\frac{j}{2}\right)^{-s}+\sum_{j=0}\sum_{k=0}\left(5+\frac{j}{2}\right)^{-s}\cdot\left(5+\frac{k}{2}\right)^{-s}$$

$$\boxed{f(n,y,z)=\sum_{k=0}\binom{z}{k}\cdot(x(x\cdot y)^{-s})^k\cdot f\left(\frac{n}{(x\cdot y)^k},y+1,z-k\right)}$$

with initial value of

$$\boxed{f(n,1+x^{-1},z)}$$

```
dsr[n_, s_, y_, z_, x_] := If[n < x y, 1, Sum[binomial[z, k] (x (x y)^-s)^k dsr[n/(x y)^k, s, y + 1, z - k, x], {k, 0, Log[(x y), n]}]]
```

$$\boxed{[(1+x^{1-s}\cdot\zeta(s,y))^z]_n=\sum_{k=0}\binom{z}{k}\cdot(x^{1-s}\cdot y^{-s})^k\cdot[(1+x^{1-s}\cdot\zeta(s,1+y))^{z-k}]_{n/(x\cdot y)^k}}$$

with initial value of

$$\boxed{[(1+y^{1-s}\cdot\zeta(s,1+y^{-1}))^z]_n}$$

...

$$\boxed{[((1-2^{1-s})(1+\zeta(s,y)))^z]_n=\sum_{k=0}\binom{z}{k}\cdot(y^{-s}\cdot(-1)^{y+1})^k\cdot[((1-2^{1-s})(1+\zeta(s,y+1)))^{z-k}]_{n/y^k}}$$

with initial value of

$$\boxed{[(1-2^{1-s})(1+\zeta(s,2))^z]_n}$$

$$\boxed{[((1-x^{1-s})(1+\zeta(s,y)))^z]_n=\sum_{k=0}\binom{z}{k}\cdot(x^{1-s}\cdot y^{-s}\cdot t(y))^k\cdot[((1-x^{1-s})(1+\zeta(s,y+1)))^{z-k}]_{n/(x\cdot y)^k}}$$

with initial value of

$$\boxed{[(1-x^{1-s})(1+\zeta(s,1+y^{-1}))^z]_n}$$

WHAT IS THE RIGHT NOTATION FOR THIS?

Identities for multiplicative functions ^z

$$\boxed{\nabla[\zeta(s)^z]_n=\prod_{p^k|n}\frac{z^{(k)}}{k!}\cdot p^{-sk}}$$

$$\boxed{\nabla[(\prod_{k=1}\zeta_{1/k}(ks)^{\frac{\mu(k)}{k}})^z]_n=\prod_{p^k|n}\frac{z^k}{k!}\cdot p^{-sk}}$$

$$\boxed{\nabla[(\frac{\zeta_{1/2}(2s)}{\zeta(s)})^z]_n=\prod_{p^k|n}\frac{(-1)^k\cdot(-z)^{(k)}}{k!}\cdot p^{-sk}}$$

$$\boxed{\nabla[(\zeta(s-a)\cdot\zeta(s))^z]_n=\prod_{p^k|n}\frac{z^{(k)}}{k!}\cdot p^{-sk}\cdot{}_2F_1(-k;z;1-k-z;p^a)}$$

$$\boxed{\nabla[(\frac{\zeta(s-a)}{\zeta(s)})^z]_n=\prod_{p^k|n}\frac{(-z)^{(k)}}{k!}\cdot p^{-sk}\cdot{}_2F_1(-k;z;1-k+z;p^a)}$$

$$\boxed{\nabla[((1-2^{1-s})\zeta(s))^z]_n=\prod_{p^k|n}\begin{cases}p^{-sk}\cdot -z\cdot{}_2F_1(1-k;1-z;2;-1) & \text{if } p=2\\ p^{-sk}\cdot\frac{z^{(k)}}{k!} & \text{if } p\neq 2\end{cases}}$$

$$\boxed{\nabla[(\prod_{k=1}\zeta_{1/k}(0))^z]_n=\prod_{p^k|n}\sum_{j=1}^{k}\binom{z}{j}\cdot p_j(k)\ \text{where}\ p_k(n)=\begin{cases}\sum_{j=1}^{n-1}P(j)\cdot p_{k-1}(n-j) & \text{if } k>1\\ P(n) & \text{if } k=1\end{cases}}$$

OR

$$\boxed{\nabla[(\prod_{k=1}\zeta_{1/k}(0))^z]_n=\prod_{p^k|n}\sum_{j=1}^{k}\frac{z^j}{j!}\cdot p_j(k)\ \text{where}\ p_k(n)=\begin{cases}\sum_{j=1}^{n-1}\frac{\sigma_1(j)}{j}\cdot p_{k-1}(n-j) & \text{if } k>1\\ \frac{\sigma_1(n)}{n} & \text{if } k=1\end{cases}}$$

$$\boxed{[(\frac{\zeta(s-a)}{\zeta(s)})^{z}]_{n}=\sum_{j\cdot k\leq n}\nabla[\zeta(s-a)^{z}]_{j}\cdot\nabla[\zeta(s)^{-z}]_{k}}$$

$$\boxed{\nabla[(\frac{\zeta(s-a)}{\zeta(s)})^{z}]_{n}=\sum_{j\cdot k=n}\nabla[\zeta(s-a)^{z}]_{j}\cdot\nabla[\zeta(s)^{-z}]_{k}}$$

$$\boxed{\nabla[(\frac{\zeta(s-a)}{\zeta(s)})^{z}]_{n}=\sum_{j|n}n\nabla[\zeta(s-a)^{z}]_{j}\cdot\nabla[\zeta(s)^{-z}]_{\frac{n}{j}}}$$

$$\boxed{[(\zeta(s-a)\cdot\zeta(s))^{z}]_{n}=\sum_{j\cdot k\leq n}\nabla[\zeta(s-a)^{z}]_{j}\cdot\nabla[\zeta(s)^{z}]_{k}}$$

$$\boxed{\nabla[(\zeta(s-a)\cdot\zeta(s))^{z}]_{n}=\sum_{j\cdot k=n}\nabla[\zeta(s-a)^{z}]_{j}\cdot\nabla[\zeta(s)^{z}]_{k}}$$

$$\boxed{\nabla[(\zeta(s-a)\cdot\zeta(s))^{z}]_{n}=\sum_{j|n}\nabla[\zeta(s-a)^{z}]_{j}\cdot\nabla[\zeta(s)^{z}]_{\frac{n}{j}}}$$

$$\boxed{\nabla[(\zeta(s)\cdot\zeta(s))^{z}]_{n}=\sum_{j|n}\nabla[\zeta(s)^{z}]_{j}\cdot\nabla[\zeta(s)^{z}]_{\frac{n}{j}}=\nabla[\zeta(s)^{2z}]_{n}}$$

$$\boxed{\nabla[(\zeta(s-a)\cdot\zeta(s))^{z}]_{n}=\sum_{j|n}\nabla[\zeta(s-a)^{z}]_{j}\cdot\nabla[\zeta(s)^{z}]_{\frac{n}{j}}=\sum_{j|n}(\prod_{p^{k}|j}\frac{z^{(k)}}{k!}\cdot p^{(a-s)k})(\prod_{p^{k}|\frac{n}{j}}\frac{z^{(k)}}{k!}\cdot p^{-sk})}$$

$$\boxed{\nabla[(\zeta(s-a)\cdot\zeta(s))^{z}]_{p^{k}}=\sum_{i=0}^{k}\nabla[\zeta(s-a)^{z}]_{p^{i}}\cdot\nabla[\zeta(s)^{z}]_{p^{k-i}}=\sum_{i=0}^{k}(\frac{z^{(i)}}{i!}\cdot p^{(a-s)i})(\frac{z^{(k-i)}}{(k-i)!}\cdot p^{-s(k-i)})}$$

$$\boxed{\nabla[(\zeta(s-a)\cdot\zeta(s))^{z}]_{p^{k}}=\sum_{i=0}^{k}\nabla[\zeta(s-a)^{z}]_{p^{i}}\cdot\nabla[\zeta(s)^{z}]_{p^{k-i}}=\sum_{i=0}^{k}(\frac{z^{(i)}}{i!}\cdot p^{(a-s)i})(\frac{z^{(k-i)}}{(k-i)!}\cdot p^{-s(k-i)})}$$

$$\boxed{\nabla[(\zeta(s-a)\cdot\zeta(s))^{z}]_{n}=\prod_{p^{k}|n}\frac{z^{(k)}}{k!}\cdot p^{-sk}\cdot{}_{2}F_{1}(-k,z,1-k-z,p^{a})}$$

$$\boxed{\nabla[(\frac{\zeta(s-a)}{\zeta(s)})^{z}]_{p^{k}}=\sum_{i=0}^{k}\nabla[\zeta(s-a)^{z}]_{p^{i}}\cdot\nabla[\zeta(s)^{-z}]_{p^{k-i}}=\sum_{i=0}^{k}(\frac{z^{(i)}}{i!}\cdot p^{(a-s)i})(\frac{(-z)^{(k-i)}}{(k-i)!}\cdot p^{-s(k-i)})}$$

$$\boxed{\nabla[(\frac{\zeta(s-a)}{\zeta(s)})^{z}]_{n}=\prod_{p^{k}|n}\frac{(-z)^{(k)}}{k!}\cdot p^{-sk}\cdot{}_{2}F_{1}(-k,z,1-k+z,p^{a})}$$

$$\boxed{\nabla[(\zeta(s-a)\cdot\zeta(s))^{z}]_{p^{k}}=\sum_{j=0}^{k/2}\nabla[\zeta(2s)^{z}]_{p^{j}}\cdot\nabla[\zeta(s)^{z}]_{p^{k-2j}}=\sum_{i=0}^{k/2}(\frac{z^{(i)}}{i!}\cdot p^{(-2s)i})(\frac{z^{(k-2i)}}{k!}\cdot p^{-s(k-i)})}$$

$$[\nabla(\prod_{k=1}\zeta_{1/k}(s))]_{n}=a(n)\cdot n^{-s}$$

$$\nabla[\log(\prod_{k=1}\zeta_{1/k}(0))]_{n}=\sum_{k=1}\nabla[\log\zeta_{1/k}(0)]_{n}$$

The Laguerre polynomials $L_z(n)$ are orthonormal, with $\int_{x=0}^{\infty} e^{-x} L_a(x) \cdot L_b(x)\, dx = 0$ unless a = b.

Now,

$$\lim_{y \to \infty} [(1+y^{1-s} \cdot \zeta(0,1+y^{-1}))^{z}]_n = L_{-z}(\log n)$$

and so

$$\lim_{y \to \infty} [(1+y^{1-s} \cdot \zeta(0,1+y^{-1}))^{-z}]_{e^x} = L_z(x)$$

which means

$$\int_{x=0}^{\infty} e^{-x} (\lim_{y \to 0} [(1+y^{1-s} \cdot \zeta(0,1+y^{-1}))^{-a}]_{e^x} \cdot [(1+y^{1-s} \cdot \zeta(0,1+y^{-1}))^{-b}]_{e^x})\, dx = 0$$

Hmm.

$$\int_{x=0}^{\infty} e^{-x} (\lim_{y \to 0} [(1+y^{1-s} \cdot \zeta(0,1+y^{-1}))^{-a}]_{e^x} \cdot [(1+y^{1-s} \cdot \zeta(0,1+y^{-1}))^{-b}]_{e^x})\, dx = 0$$

$$\sum_{j+k=n} \frac{\sigma_1(j)}{j} \cdot \frac{\sigma_1(k)}{k}$$

$$\sum_{j+k=n} \sum_{a|j} \frac{a}{j} \cdot \sum_{b|k} \frac{b}{j}$$

$$\sum_{j<n} \sum_{a|j} \frac{a}{j} \cdot \sum_{b|n-j} \frac{b}{n-j}$$

....

$$\kappa_\varphi(p^k) = \kappa(p^k) \cdot p^k - \kappa(p^k)$$

$$\kappa_\varphi(p^k) = \frac{1}{k} \cdot p^k - \frac{1}{k}$$

$$\kappa_\varphi(p^k) = \frac{p^k - 1}{k}$$

$$\boxed{\frac{(-z)^{(k)}}{k!} \cdot p^{-sk} \cdot {}_2F_1(-k; z; 1-k+z; p)}$$

....

$$P(n)=\frac{1}{n}\sum_{k=0}^{n-1}\sigma_1(n-k)P(k)$$

$$P(n)=\frac{\sigma_1(n)}{n}+\frac{1}{2}\sum_{j+k=n}\frac{\sigma_1(j)}{j}\cdot\frac{\sigma_1(k)}{k}+\frac{1}{6}\sum_{j+k+l=n}\frac{\sigma_1(j)}{j}\cdot\frac{\sigma_1(k)}{k}\cdot\frac{\sigma_1(l)}{l}+\ldots$$

$$P(n)=\sum_{t=0}^{n-1}\frac{\sigma_1(n-t)}{n}\cdot(\frac{\sigma_1(t)}{t}+\frac{1}{2}\sum_{j+k=t}\frac{\sigma_1(j)}{j}\cdot\frac{\sigma_1(k)}{k}+\frac{1}{6}\sum_{j+k+l=t}\frac{\sigma_1(j)}{j}\cdot\frac{\sigma_1(k)}{k}\cdot\frac{\sigma_1(l)}{l}+\ldots)$$

$$P(n)=\sum_{t=0}^{n-1}(\frac{\sigma_1(n-t)}{n}\cdot\frac{\sigma_1(t)}{t}+\frac{1}{2}\sum_{j+k=t}\frac{\sigma_1(n-t)}{n}\cdot\frac{\sigma_1(j)}{j}\cdot\frac{\sigma_1(k)}{k}+\frac{1}{6}\sum_{j+k+l=t}\frac{\sigma_1(n-t)}{n}\cdot\frac{\sigma_1(j)}{j}\cdot\frac{\sigma_1(k)}{k}\cdot\frac{\sigma_1(l)}{l}+\ldots)$$

???

….

$$\frac{\sigma_1(n)}{n}=P(n)+\frac{B_2}{2!}\sum_{j+k=n}P(j)\cdot\frac{\sigma_1(k)}{k}+\frac{1}{6}\sum_{j+k+l=n}P(j)\cdot\frac{\sigma_1(k)}{k}\cdot\frac{\sigma_1(l)}{l}+\ldots$$

$$D_z(n,s,y,\frac{a}{b})=$$
$$1+\binom{z}{1}y^{(s-1)}\sum_{j=y+1}^{\lfloor ny\rfloor} t(j)\,j^{-s}+\binom{z}{2}y^{2(s-1)}\sum_{j=y+1}^{\lfloor \frac{ny^2}{y+1}\rfloor}\sum_{k=y+1}^{\lfloor \frac{ny^2}{j}\rfloor} t(j)t(k)(jk)^{-s}+\binom{z}{3}y^{3(s-1)}\sum_{j=y+1}^{\lfloor \frac{ny^3}{(y+1)^2}\rfloor}\sum_{k=y+1}^{\lfloor \frac{ny^3}{j(y+1)}\rfloor}\sum_{l=y+1}^{\lfloor \frac{ny^3}{jk}\rfloor} t(j)t(k)t(l)(jkl)^{-s}+\ldots$$

$$t_x(m)=(x_d\cdot(\lfloor\frac{m}{x_d}\rfloor-\lfloor\frac{m-1}{x_d}\rfloor)-x_n\cdot(\lfloor\frac{m}{x_n}\rfloor-\lfloor\frac{m-1}{x_n}\rfloor))$$

$$t_{a/b}(n)=(b\cdot(\lfloor\frac{n}{b}\rfloor-\lfloor\frac{n-1}{b}\rfloor)-a\cdot(\lfloor\frac{n}{a}\rfloor-\lfloor\frac{n-1}{a}\rfloor))$$

$$[((1-(x\cdot y)^{1-s})\zeta(s))^z]_n=$$
$$1+\binom{z}{1}y\sum_{j=1+y}^{n} t(j\cdot y)\,j^{-s}+\binom{z}{2}y^2\sum_{j=1+y}^{n}\sum_{k=1+y}^{\lfloor\frac{n}{j}\rfloor} t(j\cdot y)t(k\cdot y)(jk)^{-s}+\binom{z}{3}y^3\sum_{j=1+y}^{n}\sum_{k=1+y}^{\lfloor\frac{n}{j}\rfloor}\sum_{l=1+y}^{\lfloor\frac{n}{jk}\rfloor} t(j\cdot y)t(k\cdot y)t(l\cdot y)(jkl)^{-s}+\ldots$$

$$t_{3/2}(n)=(2\cdot(\lfloor\frac{n}{2}\rfloor-\lfloor\frac{n-1}{2}\rfloor)-3\cdot(\lfloor\frac{n}{3}\rfloor-\lfloor\frac{n-1}{3}\rfloor))$$

$$[((1-(3\cdot 2^{-1})^{1-s})\zeta(s))^z]_n=$$
$$1+\binom{z}{1}2^{-1}\sum_{j=1+2^{-1}}^{n} t(j\cdot 2)\,j^{-s}+\binom{z}{2}2^{-2}\sum_{j=1+2^{-1}}^{n}\sum_{k=1+2^{-1}}^{\lfloor\frac{n}{j}\rfloor} t(j\cdot 2)t(k\cdot 2)(jk)^{-s}+\binom{z}{3}2^{-3}\sum_{j=1+2^{-1}}^{n}\sum_{k=1+2^{-1}}^{\lfloor\frac{n}{j}\rfloor}\sum_{l=1+2^{-1}}^{\lfloor\frac{n}{jk}\rfloor} t(j\cdot 2)t(k\cdot 2)t(l\cdot 2)(jkl)^{-s}+\ldots$$

$$f_k(n)=y\cdot\sum_{j=1+y}^{n} t(j\cdot y^{-1})(\frac{1}{k}-f_{k+1}(\frac{n}{j}))$$

$$f(n,z,y)=\sum_{k=0}^{\log_y n}\binom{z}{k}x^k\cdot t(y\cdot x^{-1})^k\cdot f(n\cdot y^{-k},z-k,y+x)$$

with initial value of

$$f(n,z,1+x)$$

$$[((1-(a\cdot x)^{1-s})\cdot\zeta(s,y))^z]_n=\sum_{k=0}\binom{z}{k}\cdot(x^{1-s}\cdot y^{-s}\cdot t_{a\cdot x}(y))^k\cdot[((1-(a\cdot x)^{1-s})\cdot\zeta(s,1+y))^{z-k}]_{n/(x\cdot y)^k}$$

with initial value of

$$[((1-(a\cdot x)^{1-s})\cdot\zeta(s,1+x^{-1}))^z]_n$$

$$[((1-(a\cdot x)^{1-s})\cdot\zeta(0,y))^z]_n=\sum_{k=0}\binom{z}{k}\cdot(x\cdot t_{a\cdot x}(y))^k\cdot[((1-(a\cdot x)^{1-0})\cdot\zeta(0,1+y))^{z-k}]_{n/(x\cdot y)^k}$$

Notation is still giving me fits :/

$$\nabla[\zeta(0)^z]_n=\nabla[(1+\zeta(0,2))^z]_n \text{ where } \nabla[(1+\zeta(0,y))^z]_n=\begin{cases}\sum_{k=0}^{y^k|n}\binom{z}{k}\cdot\nabla[(1+\zeta(0,y+1))^{z-k}]_{n/y^k} & \text{if } n\geq y\\ 1 & \text{if } n=1\\ 0 & \text{if } 1<n<y\end{cases}$$

$$\nabla[\zeta(0)^z]_n=\nabla[(1+\zeta(0,2))^z]_n \text{ where } \nabla[(1+\zeta(0,y))^z]_n=\begin{cases}\sum_{m|n\,;m\geq y}\sum_{k=1}^{m^k|n}\binom{z}{k}\cdot\nabla[(1+\zeta(0,m+1))^{z-k}]_{n/m^k} & \text{if } n\neq 1\\ 1 & \text{if } n=1\end{cases}$$

$$\kappa(n)=\sum_{y|n\,,\,y>1}\sum_{k=1}^{y^k|n}\frac{(-1)^{k+1}}{k}\cdot\nabla[(1+\zeta(0,y))^{-k}]_{\frac{n}{y^k}}$$

$$[\zeta(0)^z]_n=[(1+\zeta(0,2))^z]_n \text{ where } [(1+\zeta(0,y))^z]_n=1+\sum_{a=y}^{n}\sum_{k=1}^{\lfloor\frac{\log n}{\log a}\rfloor}\binom{z}{k}\cdot[(1+\zeta(0,a+1))^{z-k}]_{n/a^k}$$

$$\Pi(n)=\sum_{a=2}^{n}\sum_{k=1}^{\lfloor\frac{\log n}{\log a}\rfloor}\frac{(-1)^{k+1}}{k}\cdot[(1+\zeta(0,a+1))^{-k}]_{n/a^k}$$

$$[\zeta(0)^z]_n=1+\sum_{a=2}^{n}\sum_{j=1}^{\lfloor\frac{\log n}{\log a}\rfloor}\binom{z}{j}(1+\sum_{b=a+1}^{\lfloor\frac{n}{a^j}\rfloor}\sum_{k=1}^{\lfloor\frac{\log(n/a^j)}{\log b}\rfloor}\binom{z-j}{k}\cdot(1+\sum_{c=b+1}^{\lfloor\frac{n}{a^j b^k}\rfloor}\sum_{l=1}^{\lfloor\frac{\log(n/(a^j b^k))}{\log c}\rfloor}\binom{z-j-k}{l}(1+\sum_{d=c+1}^{\lfloor\frac{n}{a^j b^k c^l}\rfloor}\sum_{m=1}^{\lfloor\frac{\log(n/(a^j b^k c^l))}{\log d}\rfloor}\binom{z-j-k-l}{m}(1+...))))$$

$$[\zeta(0)^z]_n=\sum_{a=0}^{\lfloor\frac{\log n}{\log 2}\rfloor}\frac{z^{(a)}}{a!}\cdot\sum_{b=0}^{\lfloor\frac{\log n-a\log 2}{\log 3}\rfloor}\frac{z^{(b)}}{b!}\cdot\sum_{c=0}^{\lfloor\frac{\log n-a\log 2-b\log 3}{\log 5}\rfloor}\frac{z^{(c)}}{c!}\cdot\sum_{d=0}^{\lfloor\frac{\log n-a\log 2-b\log 3-c\log 5}{\log 7}\rfloor}\frac{z^{(d)}}{d!}\cdot...$$

$$\lim_{n\to\infty}[\log((1-x^{1-s})\zeta(s))]_n, \Re(s)>0 \quad = \quad \log((1-x^{1-s})\zeta(s))$$

$$\lim_{n\to\infty}[\log((1-x^{1-s})\zeta(s))]_n, \Re(s)>1 \quad = \quad \log\zeta(s)-\sum_{k=1}^{\infty}\frac{x^{k(1-s)}}{k}$$

$$\lim_{n\to\infty}[\log((1-x^{1-s})\zeta(s))]_n, \Re(s)>1 \quad = \quad \log\zeta(s)-\log(1-x^{1-s}) \text{ , for x>1}$$

……..

$$\boxed{[\zeta(s)^z]_n=\sum_{k=0}\frac{z^{(k)}}{k!}\cdot x^{k(1-s)}[((1-x^{1-s})\zeta(s))^z]_{n\cdot x^{-k}}}$$

$$\boxed{[((1-x^{1-s})\zeta(s))^z]_n=\sum_{k=0}\frac{(-z)^{(k)}}{k!}\cdot x^{k(1-s)}[\zeta(s)^z]_{n\cdot x^{-k}}}$$

……..

$$\boxed{[\zeta(s)^z]_n=[((1-x^{1-s})\zeta(s))^z]_n+\sum_{k=1}\frac{z^{(k)}}{k!}\cdot x^{k(1-s)}[((1-x^{1-s})\zeta(s))^z]_{n\cdot x^{-k}}}$$

$$\boxed{[((1-x^{1-s})\zeta(s))^z]_n=[\zeta(s)^z]_n+\sum_{k=1}\frac{(-z)^{(k)}}{k!}\cdot x^{k(1-s)}[\zeta(s)^z]_{n\cdot x^{-k}}}$$

……..

$$\boxed{[((1-x^{1-s})\zeta(s))^z]_n=[((1-x^{1-s})(1+\zeta(s,2)))^z]_n=[(1-x^{1-s}+\zeta(s,2)-x^{1-s}\zeta(s,1+x^{-1}))^z]_n}$$

……..

$$[(1+x^{1-s}\cdot\zeta(s,1+x^{-1}))^z]_n$$

Alright. Suppose we start with

$$[(1-x^{1-s})^z]_n=\sum_{k=0}^{\lfloor\frac{\log n}{\log x}\rfloor} x^{k(1-s)}\cdot\frac{(-z)^{(k)}}{k!}$$

$$[\log(1-x^{1-s})]_n=\sum_{k=1}^{\lfloor\frac{\log n}{\log x}\rfloor}-\frac{x^{k(1-s)}}{k} \qquad [\log(1-x^{1-s})]_n=\sum_{x^k\le n}-\frac{x^{k(1-s)}}{k}$$

$$[(\log(1-x^{1-s}))^2]_n=\sum_{j=1}^{\lfloor\frac{\log n}{\log x}\rfloor}\sum_{k=1}^{\lfloor\frac{\log(n/x^j)}{\log x}\rfloor}(-\frac{x^{j(1-s)}}{j})\cdot(-\frac{x^{k(1-s)}}{k}) \qquad [(\log(1-x^{1-s}))^2]_n=\sum_{x^{j+k}\le n}(-\frac{x^{j(1-s)}}{j})\cdot(-\frac{x^{k(1-s)}}{k})$$

$$[(\log(1-x^{1-s}))^k]_n=\sum_{j=1}^{\frac{\log n}{\log x}}(-\frac{x^{j(1-s)}}{j})\cdot[(\log(1-x^{1-s}))^{k-1}]_{n/x^j}$$

$$[(1-x^{1-s})^z]_n=\sum_{k=0}^{\infty}\frac{z^k}{k!}[(\log(1-x^{1-s}))^k]_n$$

$$[(\log(1-x^{1-s}))^k]_n=\lim_{z\to 0}\frac{\partial}{\partial z}[(1-x^{1-s})^z]_n$$

...

$$[-\log(1-x^{1-s})]_n=\sum_{k=1}^{\lfloor\frac{\log n}{\log x}\rfloor}\frac{x^{k(1-s)}}{k}$$

$$[(1-x^{1-s})^{-z}]_n=\sum_{k=0}^{\infty}\frac{z^k}{k!}[(-\log(1-x^{1-s}))^k]_n$$

...

$$\log(1-x^{1-s})=\sum_{k=1}^{\infty}-\frac{x^{k(1-s)}}{k}$$

$$(\log(1-x^{1-s}))^2=\sum(-\frac{x^{j(1-s)}}{j})\cdot(-\frac{x^{k(1-s)}}{k})$$

$$(\log(1-x^{1-s}))^k=\sum(-\frac{x^{j(1-s)}}{j})\cdot(\log(1-x^{1-s}))^{k-1}$$

$$(\log(1-x^{1-s}))^k=\lim_{z\to 0}\frac{\partial}{\partial z}(1-x^{1-s})^z$$

$$(1-x^{1-s})^z=\sum_{k=0}^{\infty}\frac{z^k}{k!}(\log(1-x^{1-s}))^k$$

$$(1-x^{1-s})^z=\sum_{k=0}x^{k(1-s)}\cdot\frac{(-z)^{(k)}}{k!}$$

...

$$\lim_{n\to\infty}[(1-x^{1-s})^z]_n=(1-x^{1-s})^z \; if\,(s>0 \text{ or } z\in\mathbb{N}) \text{ and } x \; is\; a\; real>1$$

…

$$[(1-x^{1-s})^z]_n=\sum_{k=0}^{\lfloor\frac{\log n}{\log x}\rfloor} x^{k(1-s)}\cdot\frac{(-z)^{(k)}}{k!}$$

This will have $\log_x n$ roots for z.

For any x and n,

$$\lim_{s\to 1}[(1-x^{1-s})^z]_n=\prod_{k=1}^{\frac{\log n}{\log x}}(1-\frac{z}{k})$$

and

$$\lim_{s\to 1}[\log(1-x^{1-s})]_n=\sum_{k=1}^{\frac{\log n}{\log x}}-\frac{z}{k}=-H_{\frac{\log n}{\log x}}$$

Therefore

$$\lim_{s\to 1}(1-x^{1-s})^z=\prod_{k=1}(1-\frac{z}{k})=0 \text{ if } \Re(z)>0$$

$$[(1+x^{1-s})^z]_n=\sum_{k=0}^{\lfloor\frac{\log n}{\log x}\rfloor}\binom{z}{k}\cdot x^{k(1-s)}$$

$$[\log(1+x^{1-s})]_n=\sum_{k=1}^{\lfloor\frac{\log n}{\log x}\rfloor}(-1)^{k+1}\cdot\frac{x^{k(1-s)}}{k}$$

$$[(\log(1+x^{1-s}))^k]_n=\sum_{j=1}^{\frac{\log n}{\log x}}((-1)^{j+1}\frac{x^{j(1-s)}}{j})\cdot[(\log(1+x^{1-s}))^{k-1}]_{n/x^j}$$

$$[(1+x^{1-s})^z]_n=\sum_{k=0}^{\infty}\frac{z^k}{k!}[(\log(1+x^{1-s}))^k]_n$$

$$[(\log(1+x^{1-s}))^k]_n=\lim_{z\to 0}\frac{\partial}{\partial z}[(1+x^{1-s})^z]_n$$

$$\cdots$$

$$\log(1+x^{1-s})=\sum_{k=1}^{\infty}(-1)^{k+1}\cdot\frac{x^{k(1-s)}}{k}$$

$$(\log(1+x^{1-s}))^k=\sum((-1)^{j+1}\cdot\frac{x^{j(1-s)}}{j})\cdot(\log(1+x^{1-s}))^{k-1}$$

$$(\log(1+x^{1-s}))^k=\lim_{z\to 0}\frac{\partial}{\partial z}(1+x^{1-s})^z$$

$$(1+x^{1-s})^z=\sum_{k=0}^{\infty}\frac{z^k}{k!}(\log(1+x^{1-s}))^k$$

$$(1+x^{1-s})^z=\sum_{k=0}\binom{z}{k}x^{k(1-s)}$$

$$\cdots$$

$$\lim_{n\to\infty}[(1+x^{1-s})^z]_n=(1+x^{1-s})^z\ if\ ???$$

…

$$[(1+x^{1-s})^z]_n = \sum_{k=0}^{\lfloor \frac{\log n}{\log x} \rfloor} \binom{z}{k} \cdot x^{k(1-s)}$$

This will have $\log_x n$ roots for z.

…? And then what? So what?

…..

$$[\log 2]_n = \sum_{k=1}^{\lfloor \frac{\log n}{\log x} \rfloor} \frac{(-1)^{k+1}}{k}$$

$$[2^z]_n = \sum_{k=1}^{\lfloor \frac{\log n}{\log x} \rfloor} \binom{z}{k}$$

...

$$[\log 0]_n = \sum_{k=1}^{\lfloor \frac{\log n}{\log x} \rfloor} -\frac{1}{k} \text{ (harmonic number, obviously)}$$

$$[(\log 0)^k]_n = \sum_{k=1}^{\lfloor \frac{\log n}{\log x} \rfloor} -\frac{1}{k} \cdot [(\log 0)^{k-1}]_{n/x^j}$$

$$[0^z]_n = \sum_{k=0} \frac{z^k}{k!} \cdot [(\log 0)^k]_n$$

$$[0^z]_n = \sum_{k=0}^{\lfloor \frac{\log n}{\log x} \rfloor} \frac{(-z)^{(k)}}{k!}$$

$$[(\log 0)^k]_n = \lim_{z \to 0} \frac{\partial^k}{\partial z^k} [0^z]_n$$

$[0^z]_n$ will have $\log_x n$ roots, which will just be the positive integers

$$[0^z]_n = \prod_{k=1}^{\lfloor \frac{\log n}{\log x} \rfloor} \left(1 - \frac{z}{k}\right)$$

$$[\log 0]_n = \sum_{k=1}^{\lfloor \frac{\log n}{\log x} \rfloor} -\frac{1}{k} = -H_{\lfloor \frac{\log n}{\log x} \rfloor}$$

$$[\log 2]_n = \sum_{k=1}^{\lfloor \frac{\log n}{\log x} \rfloor} \frac{(-1)^{k+1}}{k}$$

$$[2^z]_n = \sum_{k=1}^{\lfloor \frac{\log n}{\log x} \rfloor} \binom{z}{k} = \sum_{k=0} \frac{z^k}{k!} \cdot [(\log 2)^k]_n$$

$$\ldots$$

$$\sum_{k=0} \frac{z^{(k)}}{k!} = 0 \; if \; \Re(z) < 0, \infty \; otherwise$$

$$\ldots$$

$$(-1)^k \frac{(-z)^{(k)}}{k!} = \binom{z}{k} = z \cdot \frac{(-1)^{k+1}}{k} + \frac{z^2}{2!} \cdot \sum_{a+b=k} \left(\frac{(-1)^{a+1}}{a}\right) \cdot \left(\frac{(-1)^{b+1}}{b}\right) + \frac{z^3}{3!} \cdot \sum_{a+b+c=k} \left(\frac{(-1)^{a+1}}{a}\right) \cdot \left(\frac{(-1)^{b+1}}{b}\right) \cdot \left(\frac{(-1)^{c+1}}{c}\right) + \frac{z^4}{4!} \cdot \ldots$$

$$\frac{z^{(k)}}{k!} = z \cdot \frac{1}{k} + \frac{z^2}{2!} \cdot \sum_{a+b=k} \frac{1}{a} \cdot \frac{1}{b} + \frac{z^3}{3!} \cdot \sum_{a+b+c=k} \frac{1}{a} \cdot \frac{1}{b} \cdot \frac{1}{c} + \frac{z^4}{4!} \cdot \ldots$$

$$\ldots$$

$$f(z,x,y) = z \cdot \frac{(-1)^{kx+y}}{k} + \frac{z^2}{2!} \cdot \sum_{a+b=k} \left(\frac{(-1)^{ax+y}}{a}\right) \cdot \left(\frac{(-1)^{bx+y}}{b}\right) + \frac{z^3}{3!} \cdot \sum_{a+b+c=k} \left(\frac{(-1)^{ax+y}}{a}\right) \cdot \left(\frac{(-1)^{bx+y}}{b}\right) \cdot \left(\frac{(-1)^{cx+y}}{c}\right) + \frac{z^4}{4!} \cdot \ldots$$

$$f(z,0,0) = \frac{z^{(k)}}{k!} \qquad f(z,0,1) = (-1)^k \binom{z}{k}$$

$$f(z,1,0) = (-1)^k \frac{z^{(k)}}{k!} \qquad f(z,1,1) = \binom{z}{k}$$

$$f(z,i,0) = (-1)^{ki} \frac{z^{(k)}}{k!} \qquad f(z,i,1) = (-1)^{k(1+i)} \binom{z}{k}$$

$$f(z,-i,0) = (-1)^{-ki} \frac{z^{(k)}}{k!} \qquad f(z,-i,1) = (-1)^{-k(1+i)} \binom{z}{k}$$

$$f(z,0,1/2) = (-1)^{k/2} \cdot (k!)^{-1} \cdot \prod_{k=0} (z - k \cdot (-1)^{1/2}) \qquad f(z,0,1/2i) = (-1)^{i/2} \cdot (k!)^{-1} \cdot \prod_{k=0} ((-1)^{i/2} z + k)$$

$$\boxed{\sum_{k=0} (-1)^{k(a+bi)} \cdot \frac{z^{(k)}}{k!} = (1 - (-1)^{a+bi})^{-z}} \qquad \boxed{\sum_{k=0} (-1)^{k(a+bi)} \cdot \binom{z}{k} = (1 + (-1)^{a+bi})^{z}}$$

$$\frac{z^{(k)}}{k!}=\frac{(z+k-1)!}{k!\cdot(z-1)!}$$

$$\binom{z}{k}=\frac{z!}{k!\cdot(z-k)!}$$

$$\binom{z}{k}=\frac{(-1)^k\cdot(-z)^{(k)}}{k!} \qquad \text{and} \qquad \frac{z^{(k)}}{k!}=(-1)^k\cdot\binom{-z}{k}$$

$$\frac{z^{(k)}}{k!}-\binom{z}{k}=\frac{z^{(k)}}{k!}-\frac{(-1)^k\cdot(-z)^{(k)}}{k!}$$

$$\frac{z^{(k)}}{k!}-\binom{z}{k}=(-1)^k\cdot\binom{-z}{k}-\binom{z}{k}$$

$$\sum_{k=0}^{n}\frac{z^{(k)}}{k!}=\frac{(z+1)^{(n)}}{n!}$$

$$1$$

$$\frac{1}{a}\cdot\sum_{j=0}^{a\,n} 1$$

$$\frac{1}{a^2}\cdot\sum_{j=0}^{a\,n}\sum_{k=0}^{a\,n-j} 1$$

$$\boxed{[((1-x^{1-(1)})\zeta(1))^z]_n = \sum_{j=0} \frac{(-z)^{(j)}}{j!}[\zeta(1)^z]_{n\cdot x^{-j}}}$$

$$[\log\zeta(1)]_n = [\log((1-x^{1-(1)})\zeta(1))]_n + H_{\lfloor\frac{\log n}{\log x}\rfloor}$$

$$\ldots$$

$$H_n - 1 = \log n + \int_0^1 \frac{\partial}{\partial y}[\zeta(1, 1+y^{-1})]_n \, dy$$

$$[\zeta(1, 1+y^{-1})]_n$$

$$[\zeta(s, 1+y^{-1})]_n = \sum_{j=1} \frac{1}{(y+j)}$$

$$\sum_{x=1+y;\, x+=y}^{n} \frac{y}{x}$$

$$[\zeta(1, 1+y^{-1})]_n = \sum_{x=1+y^{-1}}^{\frac{n}{y}} \frac{1}{x}$$

$$\lim_{s\to 1}[(y^{1-s}\cdot\zeta(s,1+y^{-1}))^k]_n=[\zeta(1,1+y^{-1})^k]_{n\cdot y^{-k}}$$

$$\lim_{y\to 1}[\zeta(1,1+y^{-1})^k]_{n\cdot y^{-k}}=[\zeta(1,2)^k]_n$$

$$\lim_{y\to 1/2}=[\zeta(1,3)^k]_{n\cdot 2^k}$$

$$\lim_{y\to 1/3}=[\zeta(1,4)^k]_{n\cdot 3^k}$$

$$[\zeta(1,3)^k]_{n\cdot 2^k}-[\zeta(1,2)^k]_n$$

$$[\zeta(1,4)^k]_{n\cdot 3^k}-[\zeta(1,3)^k]_{n\cdot 2^k}$$