```
Clear[zeta, zetb, zetc]
bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
zeta[n_, s_, z_, k_] :=
 zeta[n, s, z, k] = 1 + ((z + 1) / k - 1) Sum[j^-s zeta[Floor[n / j], s, z, k + 1], {j, 2, n}]
zetb[n_, s_, z_, k_] := zetb[n, s, z, k] =
  1 + ((z) / k) Sum[j^-s zetb[Floor[n / j], s, z, k + 1], {j, 2, n}]
zetc[n_, s_, z_, k_] := zetc[n, s, z, k] =
  1 + ((z - 1) / k + 1) Sum[j^-s zetc[Floor[n / j], s, z, k + 1], {j, 2, n}]
zetam1[n_, s_, 0] := UnitStep[n - 1]
zetam1[n_, s_, k_] := Sum[j^-s zetam1[n / j, s, k - 1], {j, 2, n}]
altlogzeta[n_, s_] := Sum[k^-1 zetam1[n, s, k], {k, 1, Log[2, n]}]
altd[n_,  s_, z_]  :=  Sum[ (-1)^(k) bin[z, k] zetam1[n, s, k], {k, 0, Log[2, n]}]
maind[n_,  s_, z_]  :=  Sum[ bin[z, k] zetam1[n, s, k], {k, 0, Log[2, n]}]
FI[n_] := FactorInteger[n]; FI[1] := {}
dz[n_, z_] := Product[(-1)^p[[2]] Binomial[-z, p[[2]]], {p, FI[n]}]
altdz[n_, z_] := Product[ Binomial[z, p[[2]]], {p, FI[n]}]


Expand[zeta[100, 0, z, 1]]

Expand[altd[100, 0, z]]
```

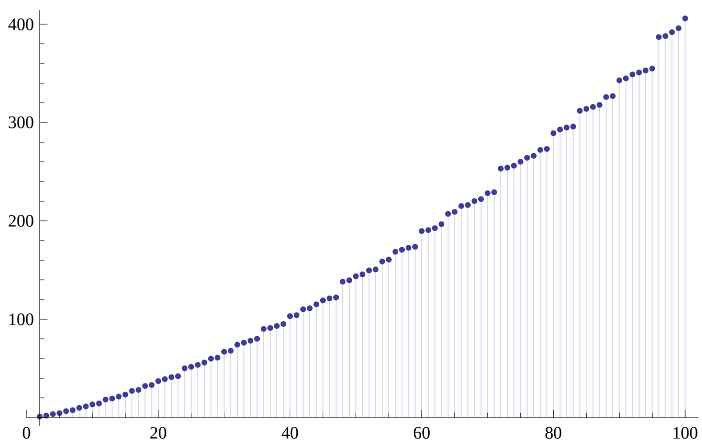
$$1+\frac{428\,z}{15}+\frac{16\,289\,z^2}{360}+\frac{331\,z^3}{16}+\frac{611\,z^4}{144}+\frac{67\,z^5}{240}+\frac{7\,z^6}{720}$$

$$1-\frac{6088\,z}{15}+\frac{148\,229\,z^2}{360}-\frac{1873\,z^3}{16}+\frac{1835\,z^4}{144}-\frac{137\,z^5}{240}+\frac{7\,z^6}{720}$$

```
Expand[zetc[100, 0, z, 1]]
```

$$1+\frac{6088\,z}{15}+\frac{148\,229\,z^2}{360}+\frac{1873\,z^3}{16}+\frac{1835\,z^4}{144}+\frac{137\,z^5}{240}+\frac{7\,z^6}{720}$$

```
DiscretePlot[ D[Expand[zetc[n, 0, z, 1]], z] /. z → 0, {n, 2, 100}]
```



```
Table[{n, D[altd[n, 0, z] - altd[n - 1, 0, z], z] /. z → 0,
   D[ maind[n, 0, z] - maind[n - 1, 0, z], z] /. z → 0}, {n, 2, 30}] // TableForm
```

2	-1	1
3	-1	1
4	$-\frac{3}{2}$	$\frac{1}{2}$
5	-1	1
6	-2	0
7	-1	1
8	$-\frac{7}{3}$	$\frac{1}{3}$
9	$-\frac{3}{2}$	$\frac{1}{2}$
10	-2	0
11	-1	1
12	-4	0
13	-1	1
14	-2	0
15	-2	0
16	$-\frac{15}{4}$	$\frac{1}{4}$
17	-1	1
18	-4	0
19	-1	1
20	-4	0
21	-2	0
22	-2	0
23	-1	1
24	-8	0
25	$-\frac{3}{2}$	$\frac{1}{2}$
26	-2	0
27	$-\frac{7}{3}$	$\frac{1}{3}$
28	-4	0
29	-1	1
30	-6	0

```
(-1)^(1) bin[-z, 1] (-1)^(1) bin[-z, 1] (-1)^(1) bin[-z, 1]
```

z^3

```
FullSimplify[z + 3 (-1 + z) z + (-2 + z) (-1 + z) z]
```

z^3

```
Expand[-z + 3 (-1 + z) z - (-2 + z) (-1 + z) z]
```

$-6 z + 6 z^2 - z^3$

```
Expand[-z + (-1 + z) z]
```

$-2 z + z^2$

```
altd[4, 0, z] - altd[3, 0, z]
```

$$-z + \frac{1}{2} (-1 + z) z$$

```
altd[30, 0, z] - altd[29 - 1, 0, z]
```

$-2 z + 3 (-1 + z) z - (-2 + z) (-1 + z) z$

```
tt[k_] := Sum[ (-1)^(j) bin[k - 1, j - 1] bin[ z, j], {j, 1, k}]
```

```
Expand[tt[5] tt[2]]
```

$$\frac{93 z^2}{10} - \frac{559 z^3}{40} + \frac{113 z^4}{16} - \frac{73 z^5}{48} + \frac{11 z^6}{80} - \frac{z^7}{240}$$

```
Sum[ (-1)^(j) bin[3 - 1, j - 1] bin[ z, j], {j, 1, 3}]
```

$$-z + (-1 + z)\, z - \frac{1}{6} (-2 + z) (-1 + z)\, z$$

```
Expand[zeta[10, 0, z, 1]]
```

$$1 + \frac{16 z}{3} + \frac{7 z^2}{2} + \frac{z^3}{6}$$

```
Expand[zeta[10 000, 0, z, 1]]
```

$$1 + \frac{56\,175\,529 z}{45\,045} + \frac{5\,304\,616\,687 z^2}{1\,663\,200} + \frac{64\,238\,883\,431 z^3}{19\,958\,400} + \frac{3\,688\,608\,229 z^4}{2\,177\,280} + \frac{11\,603\,252\,491 z^5}{21\,772\,800} + \frac{4\,483\,862\,353 z^6}{43\,545\,600} + \frac{557\,009\,347 z^7}{43\,545\,600} + \frac{2\,872\,319 z^8}{2\,903\,040} + \frac{688\,397 z^9}{14\,515\,200} + \frac{58\,651 z^{10}}{43\,545\,600} + \frac{8339 z^{11}}{479\,001\,600} + \frac{17 z^{12}}{95\,800\,320} + \frac{z^{13}}{6\,227\,020\,800}$$

```
Expand[zeta[500, -1, z, 1]]
```

$$1 + \frac{1\,878\,019 z}{84} + \frac{120\,118\,007 z^2}{2520} + \frac{6\,961\,123 z^3}{180} + \frac{1\,657\,477 z^4}{120} + \frac{45\,367 z^5}{18} + \frac{3131 z^6}{15} + \frac{3571 z^7}{315} + \frac{26 z^8}{315}$$

```
Expand[zeta[50, 1, z, 1]]
```

$$1 + \frac{36\,227\,089\,580\,823\,978\,984\,163 z}{18\,594\,267\,025\,475\,980\,238\,400} + \frac{2\,722\,987\,611\,283 z^2}{2\,248\,776\,129\,600} + \frac{1\,770\,229 z^3}{5\,702\,400} + \frac{41 z^4}{1440} + \frac{13 z^5}{11\,520}$$

```
Expand[zeta[20, N[ZetaZero[1]], z, 1]]
```

$$1 - (2.06564 - 0.208919\, i)\, z + (1.0225 - 0.278071\, i)\, z^2 - (0.268006 - 0.181271\, i)\, z^3 + (0.000833781 - 0.0103832\, i)\, z^4$$

```
N[Expand[D[zeta[80, s, z, 1], s]] /. s → 0]
```

$$-79.4645 z - 120.818 z^2 - 62.7616 z^3 - 9.80744 z^4 - 0.815798 z^5 - 0.00577623 z^6$$

```
N[ZetaZero[1]]
```

$$0.5 + 14.1347\, i$$

```
Clear[K, zeta, zetb]
K[n_] := K[n] = FullSimplify[ MangoldtLambda[n] / Log[n]]
zeta[n_, s_, z_, k_] := zeta[n, s, z, k] =
  Expand[1 + ((z + 1) / k - 1) Sum[j^-s zeta[Floor[n / j], s, z, k + 1], {j, 2, n}]]
zetb[n_, s_, z_, k_] := zetb[n, s, z, k] =
  Expand[1 + z / k Sum[If[ K[j] == 0, 0, K[j] j^-s zetb[Floor[n / j], s, z, k + 1]], {j, 2, n}]]
```

```
Timing[zeta[100 000, 0, z, 1]]
```

$$\left\{7.051, 1+\frac{991892879\, z}{102960}+\frac{16611877533197\, z^2}{605404800}+\frac{27613425421567\, z^3}{864864000}+\frac{8883298064606291\, z^4}{435891456000}+\frac{82938597121\, z^5}{10264320}+\frac{12123475378339\, z^6}{5748019200}+\frac{987114594581\, z^7}{2612736000}+\frac{6832898553167\, z^8}{146313216000}+\frac{53237749\, z^9}{13063680}+\frac{1772592397\, z^{10}}{7315660800}+\frac{20466961\, z^{11}}{2052864000}+\frac{30323737\, z^{12}}{114960384000}+\frac{841\, z^{13}}{186810624}+\frac{9773\, z^{14}}{209227898880}+\frac{71\, z^{15}}{373621248000}+\frac{17\, z^{16}}{20922789888000}\right\}$$

```
Timing[zetb[100 000, 0, z, 1]]
```

$$\left\{7.473, 1+\frac{991892879\, z}{102960}+\frac{16611877533197\, z^2}{605404800}+\frac{27613425421567\, z^3}{864864000}+\frac{8883298064606291\, z^4}{435891456000}+\frac{82938597121\, z^5}{10264320}+\frac{12123475378339\, z^6}{5748019200}+\frac{987114594581\, z^7}{2612736000}+\frac{6832898553167\, z^8}{146313216000}+\frac{53237749\, z^9}{13063680}+\frac{1772592397\, z^{10}}{7315660800}+\frac{20466961\, z^{11}}{2052864000}+\frac{30323737\, z^{12}}{114960384000}+\frac{841\, z^{13}}{186810624}+\frac{9773\, z^{14}}{209227898880}+\frac{71\, z^{15}}{373621248000}+\frac{17\, z^{16}}{20922789888000}\right\}$$

```
D[z^3 / 6, z]
```

$$\frac{z^2}{2}$$

```
D[z, z]
```

1

```
D[x^z, z]
```

$x^z \operatorname{Log}[x]$

```
D[x^z, {z, 2}]
```

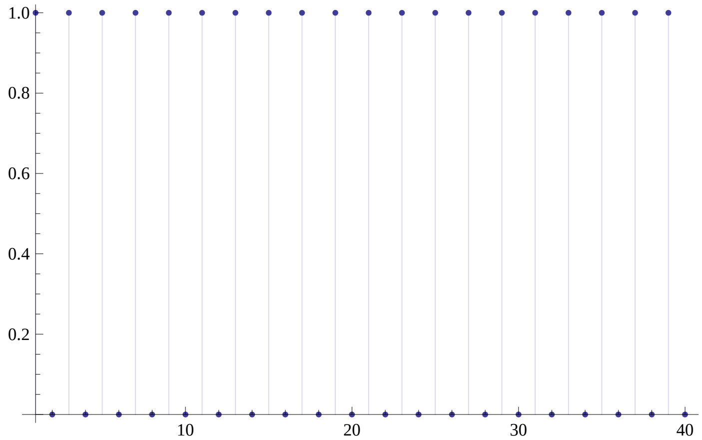
$x^z \operatorname{Log}[x]^2$

```
D[x^z, {z, 3}]
```

$x^z \operatorname{Log}[x]^3$

```
bin[z_, k_] := bin[z, k] = Product[z - j, {j, 0, k - 1}] / k !
E2a[n_, k_, a_, s_] := E2a[n, k, a, s] = Sum[ j^(-s) E2a[n / j, k - 1, a, s], {j, 2, n}] -
   a Sum[ (j a)^(-s) E2a[n / (a j), k - 1, a, s], {j, 1, n / a}];
E2a[n_, 0, a_, s_] := UnitStep[n - 1]
E1b[n_, z_, b_, s_] := Sum[ bin[z, k] E2a[n, k, b, s], {k, 0, If[ b < 2, Log[b, n], Log[2, n]]}]
```

```
DiscretePlot[ E1b[n, 1, 2, 0], {n, 1, 40}]
```



```
Limit[(1 - 5^(1 - s)) Zeta[s], s → 2]
```

$$\frac{2\,\pi^2}{15}$$

```
N[2 Pi^2 / 15]
```

1.31595

```
N[Sum[ (Mod[n, 5] - Mod[n - 1, 5]) / n^2, {n, 1, 40}]]
```

1.31476

```
Expand[E1b[100, z, 5, 0]]
```

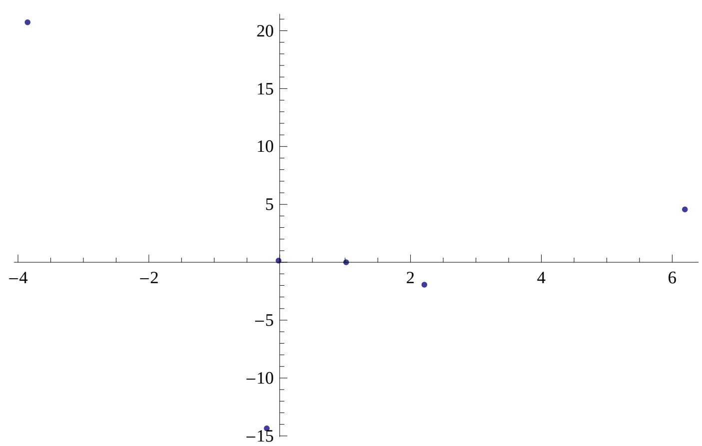
$$1+\frac{331\,z}{30}-\frac{7711\,z^2}{360}+\frac{403\,z^3}{48}+\frac{131\,z^4}{144}+\frac{17\,z^5}{240}+\frac{7\,z^6}{720}$$

```
zeros[n_, s_] := List @@ NRoots[Expand[E1b[n, z, 10, s]] == 0, z][[All, 2]]
```

```
zeros[105, N[ZetaZero[1]]]
```

{-18.223 - 4.89962 i, -0.447427 + 0.51221 i, 0.259219 + 21.386 i,
 0.828298 - 0.480269 i, 1.11297 + 3.10139 i, 1.14551 + 0.121388 i}

```
ListPlot[Table[{Re[n], Im[n]}, {n, zeros[104, N[ZetaZero[1]]]}]]
```



```
t[n_, x_, y_] := y (Floor[n / y] - Floor[(n - 1) / y]) - x (Floor[n / x] - Floor[(n - 1) / x])
Dd[n_, s_, z_, y_, x_, k_] := Expand[
  1 + y^(s - 1) ((z + 1) / k - 1) Sum[t[n, x, y] j^-s Dd[n / j, s, z, y, x, k + 1], {j, y + 1, n y}]]
```

```
Dd[100, 0, 1, 2, 1, 1]
```

100

```
t[7, 1, 2]
```

-1

```
Zeta[6]
```

$\frac{\pi^6}{945}$

```
n^p + Sum[ BernoulliB[k] p ! n^(p - k + 1) / (k ! (p - k + 1) !), {k, 0, p}] /. p → 4
```

$-\frac{n}{30}+\frac{n^3}{3}+\frac{n^4}{2}+\frac{n^5}{5}$

```
Expand[Sum[ j^(3 / 2), {j, 1, n}]]
```

HarmonicNumber$\left[n, -\frac{3}{2}\right]$

```
Expand[zeta[12, 0, z, 1]]
```

$1+\frac{19\,z}{3}+4\,z^2+\frac{2\,z^3}{3}$

```
zerosx[n_, s_] := List @@ NRoots[Expand[zeta[n, s, z, 1]] == 0, z][[All, 2]]
```

```
N[zerosx[100, 0]]
```

{-0.933809, -0.0372047, -11.1997 - 12.3982 ⅈ,
 -11.1997 + 12.3982 ⅈ, -2.67195 - 1.86184 ⅈ, -2.67195 + 1.86184 ⅈ}

```
N[zerosx[10 000, 0]]
```

{-1005.17, -25.9197 - 61.2147 ⅈ, -25.9197 + 61.2147 ⅈ, -12.5619, -9.95084 - 13.237 ⅈ,
 -9.95084 + 13.237 ⅈ, -4.34989 - 4.84639 ⅈ, -4.34989 + 4.84639 ⅈ, -2.23696 - 1.84432 ⅈ,
 -2.23696 + 1.84432 ⅈ, -1.17804 - 0.181571 ⅈ, -1.17804 + 0.181571 ⅈ, -0.000803511}

```
Product[1 + 1 / j, {j, zerosx[12, 0]}]
```

-2.

```
19. / 3
```

6.33333

```
FI[n_] := FactorInteger[n]; FI[1] := {}
fz2[n_, z_] := Product[If[p[[1]] == 2, -z Hypergeometric2F1[1 - p[[2]], 1 - z, 2, -1],
   (-1)^p[[2]] Binomial[-z, p[[2]]]]], {p, FI[n]}]
fz3[n_, z_] := Product[If[p[[1]] == 2, Sum[ (-1)^(k) Binomial[p[[2]] - 1, k - 1]
     Binomial[z, k], {k, 1, p[[2]]}], (-1)^p[[2]] Binomial[-z, p[[2]]]]], {p, FI[n]}]
ff[n_, k_] := Sum[ (-1)^(j + 1) ff[Floor[n / j], k - 1], {j, 2, n}]
ff[n_, 0] := UnitStep[n - 1]
fz[n_, z_] := Sum[ bin[ z, k] ff[n, k], {k, 0, Log[2, n]}]
```

```
Table[ {n, FullSimplify[fz2[n, z] - (fz[n, z] - fz[n - 1, z])]}, {n, 2, 20}] // TableForm
```

2	0
3	0
4	0
5	0
6	0
7	0
8	0
9	0
10	0
11	0
12	0
13	0
14	0
15	0
16	0
17	0
18	0
19	0
20	0

```
Sum[ (-1)^(k) Binomial[p - 1, k - 1] Binomial[z, k], {k, 1, p}]
```

```
-z Hypergeometric2F1[1 - p, 1 - z, 2, -1]
```

```
Expand[(-1)^(k) Binomial[p - 1, k - 1] Binomial[z, k]]
```

$(-1)^{k}$ Binomial[-1 + p, -1 + k] Binomial[z, k]

```
Expand[Sum[ (-1)^k bin[p - 1, k - 1] bin[z, k], {k, 1, p}] /. p → 4]
```

$$-\frac{15 z}{4}+\frac{83 z^{2}}{24}-\frac{3 z^{3}}{4}+\frac{z^{4}}{24}$$

```
Expand[fz[16, z] - fz[15, z]]
```

$$-\frac{15 z}{4}+\frac{83 z^{2}}{24}-\frac{3 z^{3}}{4}+\frac{z^{4}}{24}$$

```
Expand[-z Hypergeometric2F1[1 - p, 1 - z, 2, -1] /. p → 0]
```

$$1-2^{z}$$

```
fz2a[n_, z_] := Product[If[p[[1]] == 2, -z Hypergeometric2F1[1 - p[[2]], 1 - z, 2, -1],
   (-1)^p[[2]] Binomial[-z, p[[2]]]], {p, FI[n]}]
fz3a[n_, z_] := Product[If[p[[1]] == 2, Sum[ (-1)^k Binomial[p[[2]] - 1, k - 1] Binomial[z, k],
    {k, 1, p[[2]]}], Binomial[z + p[[2]] - 1, p[[2]]]], {p, FI[n]}]
fz3x[n_, z_] := Product[Sum[ If[ p[[1]] == 2, (-1)^k, 1] Binomial[p[[2]] - 1, k - 1]
    Binomial[z, k], {k, 1, p[[2]]}], {p, FI[n]}]
fz3y[n_, z_] := Product[Sum[ (-1)^(k (Floor[p[[1]] / 2] - Floor[(p[[1]] - 1) / 2]))
    Binomial[p[[2]] - 1, k - 1] Binomial[z, k], {k, 1, p[[2]]}], {p, FI[n]}]
fz4a[p_, z_] := Table[ (-1)^k Binomial[p - 1, k - 1] Binomial[z, k], {k, 1, p}]
fz4b[p_, z_] := Binomial[z + p - 1, p]
Table[D[fz3y[n, z], z] /. z → 0, {n, 1, 20}] // TableForm
```

0
-1
1
$-\frac{3}{2}$
1
0
1
$-\frac{7}{3}$
$\frac{1}{2}$
0
1
0
1
0
0
$-\frac{15}{4}$
1
0
1
0

```
fz4a[3, 4]
```

{-4, 12, -4}

```
fz4b[3, 4]
```

20

```
Sum[ Binomial[p - 1, k - 1] Binomial[z, k], {k, 1, p}] /. p → 3
```

$$\frac{\text{Gamma}[3+z]}{6\,\text{Gamma}[z]}$$

```
Sum[(-1)^k  Binomial[p - 1, k - 1] Binomial[z, k], {k, 1, p}] /. p → 3
```

$$-\frac{1}{6} z \left(14 - 9 z + z^2\right)$$

```
f[a_] := Floor[a / 2] - Floor[(a - 1) / 2]
```

```
Table[f[n], {n, 1, 20}]
```

{0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1}

```
bin[z_, k_] := bin[z, k] = Product[z - j, {j, 0, k - 1}] / k !
E2a[n_, k_, a_, s_] := E2a[n, k, a, s] = Sum[ j ^ (-s) E2a[n / j, k - 1, a, s], {j, 2, n}] -
   a Sum[ (j a) ^ (-s) E2a[n / (a j), k - 1, a, s], {j, 1, n / a}];
E2a[n_, 0, a_, s_] := UnitStep[n - 1]
E1b[n_, z_, b_, s_] := Sum[ bin[z, k] E2a[n, k, b, s], {k, 0, If[ b < 2, Log[b, n], Log[2, n]]}]
fzr[n_, s_, z_] := n^-s Product[If[p[[1]] == 2,
    -z Hypergeometric2F1[1 - p[[2]], 1 - z, 2, -1], (-1) ^p[[2]] bin[-z, p[[2]]]]], {p, FI[n]}]
Expand[Sum[ fzr[n, -1, z], {n, 1, 100}]]
```

$$1+\frac{10301\,z}{60}-\frac{235459\,z^2}{360}+\frac{2363\,z^3}{4}-\frac{12797\,z^4}{72}+\frac{286\,z^5}{15}-\frac{32\,z^6}{45}$$

```
Expand[E1b[100, z, 2, -1]]
```

$$1+\frac{10301\,z}{60}-\frac{235459\,z^2}{360}+\frac{2363\,z^3}{4}-\frac{12797\,z^4}{72}+\frac{286\,z^5}{15}-\frac{32\,z^6}{45}$$

```
Timing[Expand[Sum[ fzr[n, N[ZetaZero[1]], z], {n, 1, 10000000}]]]
```

$$\{2373.77,\ (1.+0.\,i)-(45.4558+48.4997\,i)\,z+(140.588+164.395\,i)\,z^2-(191.246+238.877\,i)\,z^3+(152.097+199.937\,i)\,z^4-(80.3319+108.801\,i)\,z^5+(30.3631+41.1341\,i)\,z^6-(8.62493+11.2783\,i)\,z^7+(1.90012+2.31031\,i)\,z^8-(0.329937+0.359479\,i)\,z^9+(0.0452778+0.0424413\,i)\,z^{10}-(0.00487763+0.00370671\,i)\,z^{11}+(0.00040802+0.000228966\,i)\,z^{12}-\left(0.0000261843+9.41988\times10^{-6}\,i\right)z^{13}+\left(1.26949\times10^{-6}+2.18011\times10^{-7}\,i\right)z^{14}-\left(4.554\times10^{-8}-3.03041\times10^{-12}\,i\right)z^{15}+\left(1.18086\times10^{-9}-1.83197\times10^{-10}\,i\right)z^{16}-\left(2.16758\times10^{-11}-6.08217\times10^{-12}\,i\right)z^{17}+\left(2.75011\times10^{-13}-9.65399\times10^{-14}\,i\right)z^{18}-\left(2.34225\times10^{-15}-8.61328\times10^{-16}\,i\right)z^{19}+\left(1.25981\times10^{-17}-4.23332\times10^{-18}\,i\right)z^{20}-\left(3.5056\times10^{-20}-1.60456\times10^{-20}\,i\right)z^{21}+\left(5.09704\times10^{-23}-3.36529\times10^{-23}\,i\right)z^{22}-\left(8.77986\times10^{-27}+1.0064\times10^{-26}\,i\right)z^{23}\}$$