9: Partial Sums of the Dirichlet $\eta(s)$ function and Exponential-Style Dirichlet Convolutions

9.1

9. The Difference between $li(n)$ and $\Pi(n)$: A Partial Sum Equivalence to the Dirichlet Eta Function

This section shows another way to express the exact difference between $\Pi(n)$ and the logarithmic integral $li(n)$, here mirroring certain relationships between the Riemann Zeta function $\zeta(s)$ and the Dirichlet eta function $\eta(s)$, but in a partial sum, Dirichlet convolution context.

$$[\eta(s)^k]_n = \sum_{j=1} (-1)^{j+1} j^{-s} [\eta(s)^{k-1}]_{n \cdot j^{-1}} \tag{8.13}$$

$$\begin{aligned}
[\eta(s)]_n &= \sum_{j=1}^{n} (-1)^{j+1} \cdot j^{-s} \\
[\eta(s)^2]_n &= \sum_{j=1}^{n} \sum_{k=2}^{\lfloor \frac{n}{j} \rfloor} (-1)^{j+k} \cdot (j \cdot k)^{-s} \\
[\eta(s)^3]_n &= \sum_{j=1}^{n} \sum_{k=2}^{\lfloor \frac{n}{j} \rfloor} \sum_{l=2}^{\lfloor \frac{n}{j \cdot k} \rfloor} (-1)^{j+k+l+1} \cdot (j \cdot k \cdot l)^{-s}
\end{aligned} \tag{8.13}$$

The limit of this as n approaches infinity, if $\Re(s) > 0$, is

$$\lim_{n \to \infty} [\eta(s)^k]_n = \eta(s)^k \tag{8.13}$$

```
Dm2D[n_,0]:=UnitStep[n-1]
Dm2D[n_,k_]:= Sum[ (-1)^(j+1)Dm2D[n/j,k-1],{j,1,n}]
Table[ Dm2D[ n, k ], { n, 1, 50 }, { k, 1, 7 } ] // TableForm
```

We will also use

$$[(\eta(s)-1)^k]_n = \sum_{j=1} (j+1)^{-s} (-1)^j [(\eta(s)-1)^{k-1}]_{n \cdot (j+1)^{-1}} \tag{8.14}$$

$$[\eta(s)-1]_n=\sum_{j=2}^{n}(-1)^{j+1}\cdot j^{-s}$$

$$[(\eta(s)-1)^2]_n=\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}(-1)^{j+k}\cdot(j\cdot k)^{-s}$$

$$[(\eta(s)-1)^3]_n=\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor\frac{n}{j\cdot k}\rfloor}(-1)^{j+k+l+1}\cdot(j\cdot k\cdot l)^{-s} \tag{8.14}$$

and so on.

```
Dm12D[n_,k_]:= Sum[ (-1)^(j+1)Dm12D[n/j,k-1],{j,2,n}];Dm12D[n_,0]:=UnitStep[n-1]
Table[ Dm12D[ n, k ], { n, 1, 50 }, { k, 1, 7 } ] // TableForm
```

The limit of this as n approaches infinity, if $\Re(s)>0$, is

$$\lim_{n\to\infty}[(\eta(s)-1)^k]_n=(\eta(s)-1)^k$$

9.2

$$\theta(n)=\frac{\Lambda(n)}{\log n}-\frac{n}{\log_2 n}\cdot(1+\lfloor\lfloor\log_2 n\rfloor-\log_2 n\rfloor)$$

$$[(\log\eta(s))^k]_n=\sum_{j=1} j^{-s}\cdot\theta(j)[(\log\eta(s))^{k-1}]_{n\cdot j^{-1}}$$

$$[\log\eta(s)]_n=\sum_{j=1}^{n}\theta(j)\cdot j^{-s}$$

$$[(\log\eta(s))^2]_n=\sum_{j=1}^{n}\sum_{k=1}^{\lfloor\frac{n}{j}\rfloor}\theta(j)\cdot\theta(k)\cdot(j\cdot k)^{-s}$$

$$[(\log\eta(s))^3]_n=\sum_{j=1}^{n}\sum_{k=1}^{\lfloor\frac{n}{j}\rfloor}\sum_{l=1}^{\lfloor\frac{n}{j\cdot k}\rfloor}\theta(j)\cdot\theta(k)\cdot\theta(l)\cdot(j\cdot k\cdot l)^{-s}$$

$$[\log\eta(s)]_n=\sum_{j=2}^{n}\frac{\Lambda(j)}{\log j}\cdot j^{-s}+\sum_{j=1}^{\lfloor\log_2 n\rfloor}\frac{2^{j(1-s)}}{j}$$

Already we can use this to see that

$$\boxed{\Pi(n)=\sum_{k=1}\frac{1}{k}(2^k[((1-2^{1-0})\zeta(0)-1)^0]_{n\cdot 2^{-k}}+(-1)^k[((1-2^{1-0})\zeta(0)-1)^k]_n)}$$

```
RiemannPrimeCount[n_]:=Sum[PrimePi[n^(1/j)]/j,{j,1,Log[2,n]}]
Dm12D[n_,k_]:= Sum[ (-1)^(j+1)Dm12D[n/j,k-1],{j,2,n}];Dm12D[n_,0]:=UnitStep[n-1]
CountAlt[n_]:=Sum[1/k( 2^k Dm12D[n/2^k,0]+(-1)^(k+1) Dm12D[n,k]),{k,1,Log[2,n]}]
Table[ RiemannPrimeCount[ n ]-CountAlt[n] , { n, 1,100 } ] // TableForm
```

Compare this to

$$\log\zeta(s)=\sum_{k=1}^{\infty}\frac{1}{k}(2^k((1-2^{1-s})\zeta(s)-1)^0+(-1)^{k+1}((1-2^{1-s})\zeta(s)-1)^k)$$

```
FullSimplify[{Log[Zeta[s]],Sum[(2^(1-s))^j((1-2^(1-s))Zeta[s]-1)^0/j,{j,1,Infinity}]+Sum[(-1)^(k-1)/k((1-2^(1-s))Zeta[s]-1)^k,{k,1,Infinity}]}/.s->0]
```

Expanded out, this identity can also be written as

$$\begin{aligned}\Pi(n)=&\sum_{j=1}^{\lfloor\log_2 n\rfloor}\frac{2^j}{j}\\ &+\sum_{j=2}^{n}(-1)^{j+1}\\ &-\frac{1}{2}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}(-1)^{j+1}\cdot(-1)^{k+1}\\ &+\frac{1}{3}\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor\frac{n}{j\cdot k}\rfloor}(-1)^{j+1}\cdot(-1)^{k+1}\cdot(-1)^{l+1}\\ &-\frac{1}{4}\ldots\end{aligned}$$

9.3

$$\boxed{\alpha_x(n)=1-x\cdot(\lfloor\frac{n}{x}\rfloor-\lfloor\frac{n-1}{x}\rfloor)}$$

$$\boxed{[((1-x^{1-s})\zeta(s))^k]_n=\sum_{j=1}\alpha_x(j)[((1-x^{1-s})\zeta(s))^{k-1}]_{n j^{-1}}}$$

$$[(1-x^{1-s})\zeta(s)]_n=\sum_{j=1}^{n}\alpha_x(j)\cdot j^{-s}$$

$$[((1-x^{1-s})\zeta(s))^2]_n=\sum_{j=1}^{n}\sum_{k=1}^{\lfloor\frac{n}{j}\rfloor}\alpha_x(j)\cdot\alpha_x(k)\cdot(j\cdot k)^{-s}$$

$$[((1-x^{1-s})\zeta(s))^3]_n=\sum_{j=1}^{n}\sum_{k=1}^{\lfloor\frac{n}{j}\rfloor}\sum_{l=1}^{\lfloor\frac{n}{j\cdot k}\rfloor}\alpha_x(j)\cdot\alpha_x(k)\cdot\alpha_x(l)\cdot(j\cdot k\cdot l)^{-s}$$

$$\boxed{[((1-x^{1-s})\zeta(s)-1)^k]_n=\sum_{j=2}\alpha_x(j)[((1-x^{1-s})\zeta(s)-1)^{k-1}]_{n j^{-1}}}$$

$$[(1-x^{1-s})\zeta(s)-1]_n=\sum_{j=2}^{n}\alpha_x(j)\cdot j^{-s}$$

$$[((1-x^{1-s})\zeta(s)-1)^2]_n=\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\alpha_x(j)\cdot\alpha_x(k)\cdot(j\cdot k)^{-s}$$

$$[((1-x^{1-s})\zeta(s)-1)^3]_n=\sum_{j=2}^{n}\sum_{k=2}^{\lfloor\frac{n}{j}\rfloor}\sum_{l=2}^{\lfloor\frac{n}{j\cdot k}\rfloor}\alpha_x(j)\cdot\alpha_x(k)\cdot\alpha_x(l)\cdot(j\cdot k\cdot l)^{-s}$$

9.4

Now we'll generalize this notion of alternating series to the rationals. Essentially, we replace the function $(-1)^{n+1}$ with the function

$$\alpha_{\frac{a}{b}}(n)=b\cdot(\lfloor\frac{n}{b}\rfloor-\lfloor\frac{n-1}{b}\rfloor)-a\cdot(\lfloor\frac{n}{a}\rfloor-\lfloor\frac{n-1}{a}\rfloor)$$

where *a* and *b* are the numerator and denominator of some rational number *x*. You can verify that $\alpha_{\frac{2}{1}}(n)=(-1)^{n+1}$.

```
al[n_,a_, b_]:=b (Floor[n/b]-Floor[(n-1)/b])-a (Floor[n/a]-Floor[(n-1)/a])
Table[{n,al[n,2,1],(-1)^(n+1)},{n,1,50}]//TableForm
```

(As an aside, $\alpha_{\frac{a}{b}}(n)$ can also be used to generalize the very well known $\sum_{n=1}^{\infty}\frac{(-1)^{j+1}}{n}=\log 2$ to

$$\sum_{n=1}^{\infty}\frac{\alpha_{\frac{a}{b}}(n)}{n}=\log\frac{a}{b}$$

```
al[n_,a_, b_]:=b (Floor[n/b]-Floor[(n-1)/b])-a (Floor[n/a]-Floor[(n-1)/a])
Grid[Table[{Sum[N[al[n,a,b]/n],{n,1,100000}],N[Log[a/b]]},{a,1,10},{b,1,6}]]
```

)

At any rate, continuing the pattern laid out previously with the zeta function, we can mirror the generalization of $\eta(s,\frac{a}{b})^{z}$ from (8.8) with the following function, where *c* is some rational constant fraction of the form $c=\frac{a}{b}, a>b$. Then

$$[((1-(\frac{a}{b})^{1-s})\zeta(s))^{k}]_{n}=\frac{1}{b}\sum_{j=1}\alpha_{\frac{a}{b}}(j)[((1-(\frac{a}{b})^{1-s})\zeta(s))^{k-1}]_{n\cdot b\cdot j^{-1}} \tag{8.15}$$

$$[(1-(\frac{a}{b})^{1-s})\zeta(s)]_{n}=\sum_{j=1}^{n}\alpha_{\frac{a}{b}}(j)$$

$$[((1-(\frac{a}{b})^{1-s})\zeta(s))^{2}]_{n}=\sum_{j=1}^{n}\sum_{k=1}^{\lfloor\frac{n}{j}\rfloor}\alpha_{\frac{a}{b}}(j)\cdot\alpha_{\frac{a}{b}}(k)$$

$$[((1-(\frac{a}{b})^{1-s})\zeta(s))^{3}]_{n}=\sum_{j=1}^{n}\sum_{k=1}^{\lfloor\frac{n}{j}\rfloor}\sum_{l=1}^{\lfloor\frac{n}{j\cdot k}\rfloor}\alpha_{\frac{a}{b}}(j)\cdot\alpha_{\frac{a}{b}}(k)\cdot\alpha_{\frac{a}{b}}(l)$$

```
num[c_] := Numerator[c]; den[c_] := Denominator[c]
alpha[n_,c_]:=den[c] (Floor[n/den[c]]-Floor[(n-1)/den[c]])-num[c] (Floor[n/num[c]]-Floor[(n-1)/num[c]])
E1[n_,k_,c_]:=(1/den[c])Sum[If[alpha[j,c] == 0,0,alpha[j,c]E1[(den[c] n)/j,k-1,c]],{j,1,den[c] n}];E1[n_,0,c_]:=UnitStep[n-1]
```

Compare this to $(n-x\,n)^{k}$

and corresponding to $(\eta(s,\frac{a}{b})-1)^{k}$, from (8.9), is

$$[((1-(\frac{a}{b})^{1-s})\zeta(s)-1)^{k}]_{n}=\frac{1}{b}\sum_{j=b+1}\alpha_{\frac{a}{b}}(j)[((1-(\frac{a}{b})^{1-s})\zeta(s)-1)^{k-1}]_{n\cdot b\cdot j^{-1}} \tag{8.16}$$

Examples of the function include

$$[((1-(\frac{a}{b})^{1-s})\zeta(s)-1)]_n=\frac{1}{b}\sum_{j=2}^{b\cdot n}\alpha_{\frac{a}{b}}(j)$$
$$[((1-(\frac{a}{b})^{1-s})\zeta(s)-1)^2]_n=\frac{1}{b^2}\sum_{j=2}^{b\cdot n}\sum_{k=2}^{\lfloor\frac{b^2\cdot n}{j}\rfloor}\alpha_{\frac{a}{b}}(j)\cdot\alpha_{\frac{a}{b}}(k)$$
$$[((1-(\frac{a}{b})^{1-s})\zeta(s)-1)^3]_n=\frac{1}{b^3}\sum_{j=2}^{b\cdot n}\sum_{k=2}^{\lfloor\frac{b^2\cdot n}{j}\rfloor}\sum_{l=2}^{\lfloor\frac{b^3\cdot n}{j\cdot k}\rfloor}\alpha_{\frac{a}{b}}(j)\cdot\alpha_{\frac{a}{b}}(k)\cdot\alpha_{\frac{a}{b}}(l)$$

(8.16)

and so on.

```
num[c_] := Numerator[c]; den[c_] := Denominator[c]
alpha[n_,c_]:=den[c] (Floor[n/den[c]]-Floor[(n-1)/den[c]])-num[c] (Floor[n/num[c]]-Floor[(n-1)/num[c]])
E2[n_,k_,c_]:=(1/den[c])Sum[If[alpha[j,c] == 0,0,alpha[j,c]E2[(den[c] n)/j,k-1,c]],{j,den[c]+1,den[c]
n}];E2[n_,0,c_]:=UnitStep[n-1]
```

$$\Pi(n)=\sum_{j=1}^{\lfloor\frac{\log n}{\log a-\log b}\rfloor}\frac{(\frac{a}{b})^j}{j}$$
$$+\frac{1}{b}\sum_{j=b+1}^{b\cdot n}\alpha_{\frac{a}{b}}(j)$$
$$-\frac{1}{2}\cdot\frac{1}{b^2}\cdot\sum_{j=b+1}^{b\cdot n}\sum_{k=b+1}^{\lfloor\frac{b^2\cdot n}{j}\rfloor}\alpha_{\frac{a}{b}}(j)\cdot\alpha_{\frac{a}{b}}(k)$$
$$+\frac{1}{3}\cdot\frac{1}{b^3}\cdot\sum_{j=b+1}^{b\cdot n}\sum_{k=b+1}^{\lfloor\frac{b^2\cdot n}{j}\rfloor}\sum_{l=b+1}^{\lfloor\frac{b^3\cdot n}{j\cdot k}\rfloor}\alpha_{\frac{a}{b}}(j)\cdot\alpha_{\frac{a}{b}}(k)\cdot\alpha_{\frac{a}{b}}(l)$$
$$-\frac{1}{4}\ldots$$

9.5

$$[((1-x^{1-s})\zeta(s))^k]_n=\sum_{j=1} j^{-s}\cdot[((1-x^{1-s})\zeta(s))^{k-1}]_{n\cdot j^{-1}}-x\cdot(j\cdot x)^{-s}\cdot[((1-x^{1-s})\zeta(s))^{k-1}]_{n\cdot(j\cdot x)^{-1}}$$

$$[(1-x^{1-s})\zeta_n(s)-1]^{*k}=\sum_{j=1}(j+1)^{-s}\cdot[((1-x^{1-s})\zeta(s)-1)^{k-1}]_{n\cdot(j+1)^{-1}}-x\cdot(j\cdot x)^{-s}\cdot[((1-x^{1-s})\zeta(s)-1)^{k-1}]_{n\cdot(j\cdot x)^{-1}}$$

$$[\log((1-x^{1-s})\zeta(s))]_n=-\sum_{k=1}^{\lfloor\frac{\log n}{\log x}\rfloor}\frac{x^{k(1-s)}}{k}+[\log\zeta(s)]_n$$

(8.16)

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\log((1-x^{1-s})\zeta(s))=-\sum_{k=1}^{\infty}\frac{x^{k(1-s)}}{k}+\log\zeta_n(s)$$

(8.16)

and, because

$$-\sum_{k=1}^{\infty}\frac{x^{k(1-s)}}{k}=\log(1-x^{1-s})$$ (8.16)

thus

$$\log((1-x^{1-s})\zeta(s))=\log(1-x^{1-s})+\log\zeta(s)$$ (8.16)

exactly as expected.

9.6

With these definitions in place, a similar approach, the details of which we'll skip over here, gives us the following. As before, c is some rational constant fraction of the form $c=\frac{a}{b}, a>b$.

Filling the role, roughly, of (8.6), we can express (8.15) in terms of (8.16) as

$$\boxed{[((1-x^{1-s})\zeta(s))^z]_n=\sum_{k=0}\binom{z}{k}[((1-x^{1-s})\zeta(s)-1)^k]_n}$$ (8.18)

```
DxD[n_,k_,x_]:=Sum[DxD[n/j,k-1,x]-x DxD[n/(x j),k-1,x],{j,1,n}]; DxD[n_,0,x_]:=UnitStep[n-1]
D1xD[n_,k_,x_]:=D1xD[n,k,x]=Sum[D1xD[n/(j+1),k-1,x]-x D1xD[n/(x j),k-1,x],{j,1,n}]; D1xD[n_,0,x_]:=UnitStep[n-1]
DxDAlt[n_,z_,x_]:=Sum[Binomial[z,k] D1xD[n,k,x],{k,0,Log[x,n]}]
Grid[Table[Chop[DxD[n,3,(b+1)/b]-DxDAlt[n,3,(b+1)/b]],{n,10,80,10},{b,1,6}]]
```

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$((1-x^{1-s})\zeta(s))^z=\sum_{k=0}^{\infty}\binom{z}{k}((1-x^{1-s})\zeta(s)-1)^k$$ (8.18)

```
FullSimplify[(((1-x^(1-s))Zeta[s])^z)-(Sum[Binomial[z,k] ((1-x^(1-s))Zeta[s]-1)^k,{k,0,Infinity}])]
```

Our function from (8.15) can be expressed in terms of the generalized divisor summatory function as

$$\boxed{[((1-x^{1-s})\zeta(s))^z]_n=\sum_{j=0}(-1)^j\binom{z}{j}x^{j(1-s)}[\zeta(s)^z]_{n\cdot x^{-j}}}$$ (8.17)

```
Dz[n_,s_, z_,k_]:=1+((z+1)/k-1)Sum[j^-s Dz[n/j,s,z,k+1],{j,2,n}]
D1xD[n_,s_, k_,x_]:=Sum[(j+1)^-s D1xD[n/(j+1),s,k-1,x]-x (j x)^-s D1xD[n/(x j),s,k-1,x],{j,1,n}]
D1xD[n_,s_, 0,x_]:=UnitStep[n-1]
DxD[n_,s_, z_,x_]:=Sum[Binomial[z,k] D1xD[n,s,k,x],{k,0,If[x<2,Log[x,n],Log[2,n]]}]
DxDAlt[n_,s_, z_,x_]:=Sum[(-1)^j Binomial[z,j] x^(j(1-s)) Dz[n/x^j,s,z,1],{j,0,Log[x,n]}]
```

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$((1-x^{1-s})\zeta(s))^z=\sum_{j=0}(-1)^j\binom{z}{j}x^{j(1-s)}\zeta(s)^z$$ (D13)

```
{((1-x^(1-s))Zeta[s])^z,FullSimplify[Sum[ (-1)^j Binomial[ z,j ] x^(j(1-s)) Zeta[s]^z, {j,0,Infinity}]]}
```

and its inverse, a relationship similar to that of (8.5), expresses $[\zeta(s)^z]_n$ as

$$[\zeta(s)^z]_n = \sum_{j=0} (-1)^j \binom{-z}{j} x^{j(1-s)} [((1-x^{1-s})\zeta(s))^z]_{n\cdot x^{-j}} \tag{D13}$$

```
Dz[n_,z_,k_]:=1+((z+1)/k-1)Sum[Dz[n/j,z,k+1],{j,2,n}]
D1xD[n_,k_,x_]:=D1xD[n,k,x]=Sum[D1xD[n/(j+1),k-1,x]-x D1xD[n/(x j),k-1,x],{j,1,n}]
D1xD[n_,0,x_]:=UnitStep[n-1]
DxD[n_,z_,x_]:=Sum[Binomial[z,k] D1xD[n,k,x],{k,0,Log[x,n]}]
DzAlt[n_,z_,x_]:=Sum[(-1)^j Binomial[-z,j] x^j DxD[n/x^j,z,x],{j,0,Log[x,n]}]
Grid[Table[Chop[Dz[a=111,s+t I,1]-DzAlt[a,s+t I,5/4]],{s,-1.3,4,.7},{t,-1.3,4,.7}]]
```

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\zeta(s)^z = \sum_{j=0} (-1)^j \binom{-z}{j} x^{j(1-s)} ((1-x^{1-s})\zeta(s))^z \tag{8.18}$$

```
Table[ n^z-Sum[(-1)^j Binomial[ -z, j] x^j(n - x n)^z ,{j,0,Infinity}],{z,-3,6}]
```

And finally, corresponding to (8.9) is this second identity for the generalized divisor summatory function $[\zeta(s)^z]_n$ in terms of $[((1-x^{1-s})\zeta(s)-1)^k]_n$ from (8.16)

$$[\zeta(s)^z]_n = \sum_{j=0} \sum_{k=0} (-1)^j \binom{-z}{j} \binom{z}{k} x^{j(1-s)} [((1-x^{1-s})\zeta(s)-1)^k]_{n\cdot x^{-j}} \tag{D14}$$

```
Dz[n_,z_,k_]:=1+((z+1)/k-1)Sum[Dz[n/j,z,k+1],{j,2,n}]
D1xD[n_,k_,x_]:=D1xD[n,k,x]=Sum[D1xD[n/(j+1),k-1,x]-x D1xD[n/(x j),k-1,x],{j,1,n}]; D1xD[n_,0,x_]:=UnitStep[n-1]
DzAlt[n_,z_,x_]:=Sum[(-1)^j Binomial[-z,j] Binomial[z,k] x^j D1xD[n /x^j,k,x],{j,0,Log[x,n]},{k,0,Log[x,n/x^j]}]
Grid[Table[Dz[123,j+1/3,1]-DzAlt[123,j+1/3,(b+1)/b],{j,1,5},{b,1,5}]]
```

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\zeta(s)^z = \sum_{j=0} \sum_{k=0} (-1)^j \binom{-z}{j} \binom{z}{k} x^{j(1-s)} ((1-x^{1-s})\zeta(s)-1)^k \tag{8.18}$$

```
FullSimplify[Table[Zeta[s]^z-Sum[(-1)^j Binomial[-z,j] Binomial[z,k] x^(j(1-s)) ((1-x^(1-s))Zeta[s]-1)^k,{j,0,Infinity},
{k,0,Infinity}],{z,-3,3}]]
```

9.6

$$[((1-x^{1-s})\zeta(s)-1)^k]_n = \sum_{j=0}^{k} \sum_{m=0}^{j} (-1)^j \binom{k}{j} \binom{j}{m} x^{j(1-s)} [(\zeta(s)-1)^{k-m}]_{n\cdot x^{-j}}$$

$$[(\zeta(s)-1)^k]_n = (-1)^k + \sum_{j=0} \sum_{m=0}^{k} \binom{k}{m} \binom{m+j-1}{k-1} x^{j(1-s)} [((1-x^{1-s})\zeta(s)-1)^m]_{nx^{-j}}$$

9.6

The pair of binomials in (D14) simplify when z is an integer. In particular, if s is 0 and z is 2, we have the Dirichlet Divisor function $D(n)$, and if z is -1, we have the Mertens function $M(n)$.

$$D(n)=[\zeta(0)^2]_n=\sum_{j=0}(j+1)x^j([((1-x^{1-0})\zeta(0)-1)^0]_{n\cdot x^{-j}}+2[(1-x^{1-0})\zeta(0)-1]_{n\cdot x^{-j}}+[((1-x^{1-0})\zeta(0)-1)^2]_{n\cdot x^{-j}})$$

$$M(n)=[\zeta(0)^{-1}]_n=\sum_{k=0}(-1)^k([((1-x^{1-0})\zeta(0)-1)^k]_n-x[((1-x^{1-0})\zeta(0)-1)^k]_{n\cdot x^{-1}})$$

```
D1xD[n_,k_,x_]:=D1xD[n,k,x]=-x D1xD[n/x,k-1,x]+Sum[D1xD[n/j,k-1,x]-x D1xD[n/(x j),k-1,x],{j,2,n}];
D1xD[n_,0,x_]:=UnitStep[n-1]
DAlt[n_,x_]:=Sum[(j+1)x^j (D1xD[n /x^j,0,x]+2 D1xD[n /x^j,1,x]+D1xD[n /x^j,2,x]),{j,0,Log[x,n]}]
MertensAlt[n_,x_]:=Sum[(-1)^k (D1xD[n,k,x]-x D1xD[n /x,k,x]),{k,0,Log[x,n]}]
Grid[Table[Sum[1,{j,1,n},{k,1,n/j}]-DAlt[n,(b+1)/b],{n,10,100,10},{b,1,7}]]
Grid[Table[Sum[MoebiusMu[j],{j,1,n}]-MertensAlt[n,(b+1)/b],{n,10,100,10},{b,1,5}]]
```

The limit of this as n approaches infinity is

$$\zeta(0)^2=\sum_{j=0}(j+1)x^{j(1-0)}\cdot(((1-x^{1-0})\zeta(0)-1)^0+2((1-x^{1-0})\zeta(0)-1)^1+((1-x^{1-0})\zeta(0)-1)^2)$$

```
{Zeta[0]^2,Sum[ (j+1)x^(j(1-s))(((1-x^(1-s))Zeta[s]-1)^0 + 2((1-x^(1-s))Zeta[s]-1)^1+((1-x^(1-s))Zeta[s]-1)^2),
{j,0,Infinity}]/.s->0}
```

The limit of this as n approaches infinity is

$$\frac{1}{\zeta(0)}=\sum_{k=0}(-1)^k(((1-x^{1-0})\zeta(0)-1)^k-x^{(1-0)}((1-x^{1-0})\zeta(0)-1)^k)$$

```
{1/Zeta[0],FullSimplify[Sum[(-1)^k(((1-x^(1-s))Zeta[s]-1)^k-x^(1-s) ((1-x^(1-s))Zeta[s]-1)^k),{k,0,Infinity}]]/.s->0}
```

The general hope for these last 3 equations would be to take the limit as $c\to 1^+$ and then transform the remaining sums in such a way that something interesting can be said about them. Although no such approach is illustrated here for $D(n)$ or $M(n)$, we can take exactly that approach for $\Pi(n)$, which follows below.

9.6

Continuing with these parallel identities from the beginning of this section, mapping to (8.11), if we start with (D14) and then use the identity $\Pi(n)=\lim_{z\to 0}\frac{[\zeta(0)^z]_n-1}{z}$, then the Riemann Prime counting function $\Pi(n)$ can be expressed in terms of $[((1-x^{1-0})\zeta(0)-1)^k]_n$, from (8.16), as

$$\boxed{[\log\zeta(s)]_n=\sum_{k=1}k^{-1}(x^{k(1-s)}[((1-x^{1-s})\zeta(s)-1)^0]_{n\cdot x^{-k}}+(-1)^{k+1}[((1-x^{1-s})\zeta(s)-1)^k]_n)} \tag{P13}$$

and, in particular, if s=0,

$$\boxed{\Pi(n)=\sum_{k=1}\frac{1}{k}(x^k[((1-x^{1-0})\zeta(0)-1)^0]_{n\cdot x^{-k}}+(-1)^{k+1}[((1-x^{1-0})\zeta(0)-1)^k]_n)} \tag{P13}$$

```
RiemanPrimeCount[n_]:=Sum[PrimePi[n^(1/k)]/k,{k,1,Log[2,n]}]
D1xD[n_,k_,x_]:=D1xD[n,k,x]=Sum[D1xD[n/(j+1),k-1,x]-x D1xD[n/(x j),k-1,x],{j,1,n}]; D1xD[n_,0,x_]:=UnitStep[n-1]
logD[n_,x_]:=Sum[x^j/j,{j,1,Log[x,n]}]+Sum[(-1)^(k+1)/k D1xD[n,k,x],{k,1,Log[If[x<2,x,2],n]}]
Table[{n,RiemanPrimeCount[n],logD[n,5/2],logD[n,3/2],logD[n,4/3]},{n,1,100}]//TableForm
```

The limit of this as n approaches infinity, if $\Re(s)>1$, is

$$\log\zeta(s)=\sum_{k=1}^{\infty}k^{-1}(x^k((1-x^{1-s})\zeta(s)-1)^0+(-1)^{k+1}((1-x^{1-s})\zeta(s)-1)^k)$$

(P13)

```
FullSimplify[{Log[Zeta[s]],Sum[(x^(1-s))^j((1-x^(1-s))Zeta[s]-1)^0/j,{j,1,Infinity}]+Sum[(-1)^(k-1)/k((1-x^(1-s))Zeta[s]-
1)^k,{k,1,Infinity}]}]
```

Now, it can be shown that

$$\boxed{\begin{aligned}&\lim_{x\to1^+}\sum_{k=1}\frac{1}{k}(x^{k(1-s)}[((1-x^{1-s})\zeta(s)-1)^0]_{n\cdot x^{-k}}-1)=\\&\lim_{x\to1^+}\sum_{k=1}^{\lfloor\frac{\log n}{\log x}\rfloor}\frac{x^{k(1-s)}-1}{k}=\\&li(n^{1-s})-\log\log n^{1-s}-\gamma\end{aligned}}$$

(P13)

$$\boxed{\begin{aligned}&\lim_{x\to1^+}\sum_{k=1}\frac{1}{k}(x^k[((1-x^{1-0})\zeta(0)-1)^0]_{n\cdot x^{-k}}-1)=\\&\lim_{x\to1^+}\sum_{k=1}^{\lfloor\frac{\log n}{\log x}\rfloor}\frac{x^k-1}{k}=\\&li(n)-\log\log n-\gamma\end{aligned}}$$

(8.19)

$$\lim_{x\to1^+}\sum_{k=1}\frac{1}{k}(x^k[((1-x^{1-(1)})\zeta(0)-1)^0]_{n\cdot x^{-k}}-1)=0$$

$$[\log\zeta(1)]_n=[\log((1-x^{1-(1)})\zeta(1))]_n+H_{\lfloor\frac{\log n}{\log x}\rfloor}$$

$$[\log\zeta(1)]_n=\lim_{x\to1+}[\log((1-x^{1-(1)})\zeta(1))]_n+H_{\lfloor\frac{\log n}{\log x}\rfloor}$$

```
Table[{n,Sum[N[(c^j-1)/j],{j,1,Floor[Log[n]/Log[c]]}]/.c->1.00001,N[LogIntegral[n]-Log[Log[n]]-EulerGamma]},
{n,5,100,5}]//TableForm
```

(which has a nice visual resemblance to the very well known $\lim_{k\to0}\frac{x^k-1}{k}=\log x$), meaning that if we take the limit in (P13) as c approaches 1 from above, similar to what we did in (8.12), then we finally have an equation expressing the relationship between $\Pi(n)$ and the logarithmic integral $li(n)$. With $c=\frac{b+1}{b}$, we have

$$\boxed{\Pi(n)=li(n)-\log\log n-\gamma+\lim_{x\to1+}\sum_{k=1}^{\lfloor\frac{\log n}{\log x}\rfloor}\frac{(-1)^{k-1}}{k}[((1-x^{1-0})\zeta(0)-1)^k]_n+\frac{1}{k}}$$

(P13)

$$\boxed{\Pi(n)=li(n)-\log\log n-\gamma+\lim_{x\to1+}[\log((1-x^{1-0})\zeta(0))]_n+H_{\lfloor\frac{\log n}{\log x}\rfloor}}$$

(P14)