```
ClearAll["Global`*"]
```

```
E1x[n_, k_, b_] := Sum[ Binomial[k, j2] E2a[n, k - j2, b], {j2, 0, k}]
DDx[n_, k_, b_] := Sum[Binomial[k + j - 1, k - 1] b^j E1x[n / b^j, k, b], {j, 0, Log[b, n]}]
D2x[n_, k_, b_] := Sum[(-1)^j Binomial[k, j] DDx[n, k - j, b], {j, 0, k}]

DDy[n_, k2_, b_] := Sum[Binomial[k2 + j3 - 1, k2 - 1] b^j3
    (Sum[ Binomial[k2, j2] E2a[n / b^j3, k2 - j2, b], {j2, 0, k2}]), {j3, 0, Log[b, n]}]
D2y2[n_, k_, b_] := Sum[(-1)^j Binomial[k, j] DDy[n, k - j, b], {j, 0, k}]
D2y[n_, k_, b_] := Sum[(-1)^j Binomial[k, j]
    (Sum[Binomial[(k - j) + j3 - 1, (k - j) - 1] b^j3 (Sum[ Binomial[(k - j), j2]
          E2a[n / b^j3, (k - j) - j2, b], {j2, 0, (k - j)}]), {j3, 0, Log[b, n]}]), {j, 0, k}]
D2z[n_, k_, b_] := Sum[(-1)^j Binomial[k, j] Binomial[(k - j) + j3 - 1, (k - j) - 1]
    b^j3 Binomial[(k - j), j2] E2a[n / b^j3, (k - j) - j2, b],
  {j, 0, k}, {j3, 0, Log[b, n]}, {j2, 0, (k - j)}]
D2za[n_, k_, b_] := Sum[(-1)^j Binomial[k, j] Binomial[(k - j) + a - 1, (k - j) - 1]
    b^a Binomial[(k - j), m] Et2a[n / b^a, (k - j) - m, b],
  {a, 0, Log[b, n]}, {j, 0, k}, {m, 0, (k - j)}]
D2zaa[n_, k_, b_, a_] := Sum[(-1)^j Binomial[k, j] Binomial[(k - j) + a - 1, (k - j) - 1]
    b^a Binomial[(k - j), m] Et2a[n / b^a, (k - j) - m, b], {j, 0, k}, {m, 0, (k - j)}]
D2zaax[n_, k_, b_, a_] := Sum[(-1)^j Binomial[k, j] Binomial[(k - j) + a - 1, (k - j) - 1]
    b^a Binomial[(k - j), m] E2a[n / b^a, (k - j) - m, b], {j, 0, k}, {m, 0, (k - j)}]
D2za2[n_, k_, b_] := Sum[ D2zaax[n, k, b, a], {a, 0, Log[b, n]}]
```

```
E2a[n_, k_, a_] :=
 E2a[n, k, a] = Sum[ E2a[n / j, k - 1, a], {j, 2, n}] - a Sum[ E2a[n / (a j), k - 1, a], {j, 1, n / a}];
E2a[n_, 0, a_] := 1
E1a[n_, k_, a_] :=
 E1a[n, k, a] = Sum[ E1a[n / j, k - 1, a], {j, 1, n}] - a Sum[ E1a[n / (a j), k - 1, a], {j, 1, n / a}];
E1a[n_, 0, a_] := 1
DDa[n_, k_] := DDa[n, k] = Sum[DDa[Floor[n / j], k - 1], {j, 1, n}]; DDa[n_, 0] := 1
D2a[n_, k_] := D2a[n, k] = Sum[D2a[Floor[n / j], k - 1], {j, 2, n}]; D2a[n_, 0] := 1
D2b[n_, k_] := Sum[(-1)^j Binomial[k, j] DDa[n, k - j], {j, 0, k}]
DDb[n_, k_] := Sum[Binomial[k, j] D2a[n, k - j], {j, 0, k}]
E2b[n_, k_, b_] := Sum[(-1)^j Binomial[k, j] E1a[n, k - j, b], {j, 0, k}]
E1b[n_, k_, b_] := Sum[ Binomial[k, j] E2a[n, k - j, b], {j, 0, k}]
DDc[n_, k_, b_] := Sum[Binomial[k + j - 1, k - 1] b^j E1a[n / b^j, k, b], {j, 0, Log[b, n]}]
E1c[n_, k_, b_] := Sum[(-1)^j Binomial[k, j] b^j DDa[n / b^j, k], {j, 0, k}]
E2c[n_, k_, b_] :=
 Sum[(-1)^j b^j Binomial[k, j] Binomial[j, m] D2a[n / b^j, k - m], {j, 0, k}, {m, 0, j}]
D2E2[n_, k_, b_] := Sum[ (-1)^j b^j Binomial[k, j]
   Sum[ Binomial[j, m] If[n / b^j < 1, 0, D2a[n / b^j, k - m]], {m, 0, j}], {j, 0, k}]
E2D2[n_, k_, b_] := (-1)^k + Sum[b^a / ((k - 1)!) Binomial[k, j]
    Pochhammer[a - k + j + 1, k - 1] E2a[b^-a n, j, b], {a, 0, Log[b, n]}, {j, 0, k}]
```

```
{D2z[100, 4, 3], D2y[100, 4, 3], D2x[100, 4, 3],
 D2a[100, 4], D2za2[100, 4, 3], E2D2[100, 4, 3]}
```

```
{184, 184, 184, 184, 184, 184}
```

FullSimplify[Expand[D2za[n, 3, b]]]

$$\sum_{a=0}^{\frac{\text{Log}[n]}{\text{Log}[b]}} \sum_{j=0}^{3} \sum_{m=0}^{3-j} (-1)^j b^a \text{Binomial}[3, j] \text{Binomial}[3-j, m] \text{Binomial}[2+a-j, 2-j] \text{Et2a}[b^{-a} n, 3-j-m, b]$$

FullSimplify[D2zaa[n, 1, b, a]]

$$b^a (\text{Et2a}[b^{-a} n, 0, b] + \text{Et2a}[b^{-a} n, 1, b])$$

FullSimplify[D2zaa[n, 2, b, a]]

$$b^a ((-1+a) \text{Et2a}[b^{-a} n, 0, b] + 2a \text{Et2a}[b^{-a} n, 1, b] + (1+a) \text{Et2a}[b^{-a} n, 2, b])$$

FullSimplify[D2zaa[n, 3, b, a]]

$$\frac{1}{2} b^a ((-2+a)(-1+a) \text{Et2a}[b^{-a} n, 0, b] + 3(-1+a) a \text{Et2a}[b^{-a} n, 1, b] + (1+a)(3a \text{Et2a}[b^{-a} n, 2, b] + (2+a) \text{Et2a}[b^{-a} n, 3, b]))$$

FullSimplify[D2zaa[n, 4, b, a]]

$$\frac{1}{6} b^a ((-3+a)(-2+a)(-1+a) \text{Et2a}[b^{-a} n, 0, b] + 4(-2+a)(-1+a) a \text{Et2a}[b^{-a} n, 1, b] + (1+a)(6(-1+a) a \text{Et2a}[b^{-a} n, 2, b] + (2+a)(4a \text{Et2a}[b^{-a} n, 3, b] + (3+a) \text{Et2a}[b^{-a} n, 4, b])))$$

FullSimplify[Expand[D2zaa[n, 5, b, a]]]

$$\frac{1}{24} b^a ((-4+a)(-3+a)(-2+a)(-1+a) \text{Et2a}[b^{-a} n, 0, b] + 5(-3+a)(-2+a)(-1+a) a \text{Et2a}[b^{-a} n, 1, b] + (1+a)(5a(2(-1+a)((-2+a) \text{Et2a}[b^{-a} n, 2, b] + (2+a) \text{Et2a}[b^{-a} n, 3, b]) + (2+a)(3+a) \text{Et2a}[b^{-a} n, 4, b]) + (2+a)(3+a)(4+a) \text{Et2a}[b^{-a} n, 5, b]))$$

$$(-4+a)(-3+a)(-2+a)(-1+a) \text{Et2a}[b^{-a} n, 0, b]$$

$5(-3+a)(-2+a)(-1+a) a \text{Et2a}[b^{-a} n, 1, b]$

$$5(-3+a)(-2+a)(-1+a) a \text{Et2a}[b^{-a} n, 1, b]$$

$(1+a)(5a(2(-1+a)((-2+a) \text{Et2a}[b^{-a} n, 2, b] + (2+a) 0 + (2+a)(3+a) 0) + (2+a)(3+a)(4+a) 0))$

$$10(-2+a)(-1+a) a (1+a) \text{Et2a}[b^{-a} n, 2, b]$$

$(1+a)(5a(2(-1+a)((-2+a) 0 + (2+a) \text{Et2a}[b^{-a} n, 3, b]) + (2+a)(3+a) 0) + (2+a)(3+a)(4+a) 0)$

$$10(-1+a) a (1+a)(2+a) \text{Et2a}[b^{-a} n, 3, b]$$

$(1+a)(5a(2(-1+a)((-2+a) 0 + (2+a) 0) + (2+a)(3+a) \text{Et2a}[b^{-a} n, 4, b]) + (2+a)(3+a)(4+a) 0)$

$$5a(1+a)(2+a)(3+a) \text{Et2a}[b^{-a} n, 4, b]$$

```
(1 + a)
 (5 a (2 (-1 + a) ((-2 + a) 0 + (2 + a) 0) + (2 + a) (3 + a) 0) + (2 + a) (3 + a) (4 + a) Et2a[b^-a n, 5, b])
(1 + a) (2 + a) (3 + a) (4 + a) Et2a[b^-a n, 5, b]
```

```
(-4 + a) (-3 + a) (-2 + a) (-1 + a) Et2a[b^-a n, 0, b]
5 (-3 + a) (-2 + a) (-1 + a) a Et2a[b^-a n, 1, b]
10 (-2 + a) (-1 + a) a (1 + a) Et2a[b^-a n, 2, b]
10 (-1 + a) a (1 + a) (2 + a) Et2a[b^-a n, 3, b]
5 a (1 + a) (2 + a) (3 + a) Et2a[b^-a n, 4, b]
(1 + a) (2 + a) (3 + a) (4 + a) Et2a[b^-a n, 5, b]
```

```
b^a / (4!) Binomial[5, 0] Pochhammer[a - 4, 4]
```

$$\frac{1}{24} (-4+a)(-3+a)(-2+a)(-1+a)\, b^a$$

```
b^a / (4!) Binomial[5, 2] Pochhammer[a - 2, 4]
```

$$\frac{5}{12} (-2+a)(-1+a)\, a\, (1+a)\, b^a$$

```
b^a / (k!) Binomial[k + 1, 5] Pochhammer[a - k + j, k] Et2a[b^-a n, j, b] /. {k → 4, j → 5}
```

$$\frac{1}{24} (1+a)(2+a)(3+a)(4+a)\, b^a\, \text{Et2a}[b^{-a} n, 5, b]$$

```
ff[k_] :=
 Sum[b^a / (k!) Binomial[k + 1, j] Pochhammer[a - k + j, k] Et2a[b^-a n, j, b], {j, 0, k + 1}]
ff[4]
```

$$\frac{1}{24} (-4+a)(-3+a)(-2+a)(-1+a)\, b^a\, \text{Et2a}[b^{-a} n, 0, b] +$$
$$\frac{5}{24} (-3+a)(-2+a)(-1+a)\, a\, b^a\, \text{Et2a}[b^{-a} n, 1, b] +$$
$$\frac{5}{12} (-2+a)(-1+a)\, a\, (1+a)\, b^a\, \text{Et2a}[b^{-a} n, 2, b] + \frac{5}{12} (-1+a)\, a\, (1+a)(2+a)\, b^a\, \text{Et2a}[b^{-a} n, 3, b] +$$
$$\frac{5}{24} a\, (1+a)(2+a)(3+a)\, b^a\, \text{Et2a}[b^{-a} n, 4, b] + \frac{1}{24} (1+a)(2+a)(3+a)(4+a)\, b^a\, \text{Et2a}[b^{-a} n, 5, b]$$

```
ff2[k_] := Sum[b^a / (k!) Binomial[k + 1, j] Pochhammer[a - k + j, k] Et2a[b^-a n, j, b],
   {a, 0, Log[b, n]}, {j, 0, k + 1}]
ff2[4]
```

```
$Aborted
```

```
ff3[n_, k_, b_] :=
 (-1)^k + Sum[b^a / ((k - 1)!) Binomial[k, j] Pochhammer[a - k + j + 1, k - 1] E2a[b^-a n, j, b],
   {a, 0, Log[b, n]}, {j, 0, k}]
```

```
D2za2[nn = 300, cc = 3, dr = 1.1] - ff3[nn, cc, dr]
```

$$-1.74623 \times 10^{-10}$$

```
Expand[FullSimplify[D2zaa[n, 3, b, a]]]
```

$$b^a\, Et2a[b^{-a} n, 0, b] - \frac{3}{2} a\, b^a\, Et2a[b^{-a} n, 0, b] + \frac{1}{2} a^2\, b^a\, Et2a[b^{-a} n, 0, b] - \frac{3}{2} a\, b^a\, Et2a[b^{-a} n, 1, b] + \frac{3}{2} a^2\, b^a\, Et2a[b^{-a} n, 1, b] + \frac{3}{2} a\, b^a\, Et2a[b^{-a} n, 2, b] + \frac{3}{2} a^2\, b^a\, Et2a[b^{-a} n, 2, b] + b^a\, Et2a[b^{-a} n, 3, b] + \frac{3}{2} a\, b^a\, Et2a[b^{-a} n, 3, b] + \frac{1}{2} a^2\, b^a\, Et2a[b^{-a} n, 3, b]$$

```
ffa[k_] :=
 Sum[b^a / ((k - 1)!) Binomial[k, j] Pochhammer[a - k + j + 1, k - 1] Et2a[b^-a n, j, b], {j, 0, k}]

Expand[FullSimplify[D2zaa[n, 2, b, a]]] - Expand[ffa[2]]
```

$$0$$

```
Expand[Binomial[k, j] Pochhammer[a - k + j + 1, k - 1]]
```

$$\text{Binomial}[k, j]\ \text{Pochhammer}[1 + a + j - k, -1 + k]$$

```
rr[k_, j_] := Pochhammer[1 + a + j - k, -1 + k]

rr[4, 2]
```

$$(-1 + a)\, a\, (1 + a)$$

```
rt[k_, j_] := Binomial[1 + a + j - k, -1 + k] (-1 + k)!

rt[4, 2]
```

$$(-3 + a)\, (-2 + a)\, (-1 + a)$$