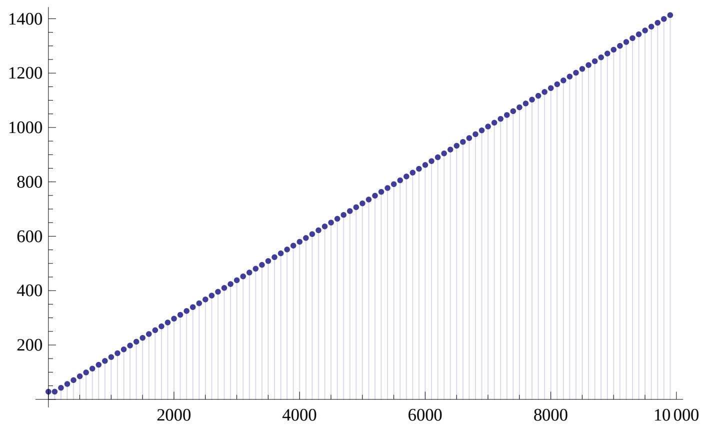
```
ff[n_, s_] := s n^s HarmonicNumber[n, s] - (1 - s) n^(1 - s) HarmonicNumber[n, 1 - s]
DiscretePlot[ Im@ff[n, N@ZetaZero@1], {n, 1, 10 000, 100}]
```




```
10 000^(.5)
```

100.

```
N@Im@ZetaZero@1 (1 000 000) / 100
```

141 347.

```
ff[10 000 000, N@ZetaZero@1] - N@Im@ZetaZero@1 (10 000 000) / 100 I
```

0. − 271.716 i

```
N[(Im@ZetaZero@1)^2]
```

199.79

```
Gamma[s / 2] /. s → 10
```

24

```
(s / 2 - 1) ! /. s → 10
```

24

```
FullSimplify[(1 - s) / 2 - 1]
```

$\frac{1}{2}(-1-s)$

```
4 ! × 5
```

120

```
FullSimplify[
 (1 / Zeta[s] (n^s / s HarmonicNumber[n, s] + n^(1 - s) / (1 - s) HarmonicNumber[n, 1 - s]) -
    n^s / s) / (-Pi^(1 / 2 - s) n^(1 - s) / (1 - s))]
```

$\frac{1}{n\,s\,\text{Zeta}[s]}\pi^{-\frac{1}{2}+s}\left(-n\,s\,\text{HarmonicNumber}[n,1-s]-n^{2s}(-1+s)\,\text{HurwitzZeta}[s,1+n]\right)$

$\frac{1}{s\,\text{Zeta}[s]}\pi^{-\frac{1}{2}+s}\left(-s\,\text{HarmonicNumber}[n,1-s]-n^{2s-1}(-1+s)\,\text{HurwitzZeta}[s,1+n]\right)$

$\frac{1}{s\,\text{Zeta}[s]}\pi^{-\frac{1}{2}+s}\left(-s\,\text{HarmonicNumber}[n,1-s]-n^{-1+2s}(-1+s)\,\text{HurwitzZeta}[s,1+n]\right)$

```
Limit[1/(s Zeta[s])
  π^(-1/2+s) (-s HarmonicNumber[n, 1 - s] - n^(-1+2 s) (-1 + s) HurwitzZeta[s, 1 + n]), n → Infinity]
```

$$\text{Limit}\left[\frac{1}{\text{s Zeta[s]}}\pi^{-\frac{1}{2}+s}\left(-s\,\text{HarmonicNumber}[n,1-s]-n^{-1+2s}(-1+s)\,\text{HurwitzZeta}[s,1+n]\right),\ n\to\infty\right]$$

```
1/(s n^-s Zeta[s]) π^(-1/2+s) (-s n^-s HarmonicNumber[n, 1 - s] - n^(-1+s) (-1 + s) HurwitzZeta[s, 1 + n]) /.
  s → .5 + 3 I /. n → 10 000 000 000
```

65 613.9 - 102 979. i

```
zetr[n_, s_] := (n^s s HarmonicNumber[n, s] + n^(1-s) (1 - s) HarmonicNumber[n, 1 - s]) /
  (n^s / s - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] n^(1 - s) / (1 - s))
```

```
Zeta[.51 + 5.5 I]
```

0.764717 + 0.289062 i

```
al4[s_] := -1 / ((1 / 2) Pi^(-s / 2) Gamma[s / 2])
ssosub3[n_, s_] := (s^-1 n^s)/(al4[s] s^-1 n^s - al4[1 - s] (1 - s)^-1 n^(1 - s)) HarmonicNumber[n, s]
sso10cc[n_, s_] := ssosub3[n, s] + ssosub3[n, 1 - s]
szet10cc[n_, s_] := al4[s] sso10cc[n, s]
ssosub3dd[n_, s_] := (s^-1 n^s)/(al4[s] s^-1 n^s - al4[1 - s] (1 - s)^-1 n^(1 - s)) HarmonicNumber[n, s]
szet10dd[n_, s_] :=
 al4[s] (((s^-1 n^s)/(al4[s] s^-1 n^s - al4[1 - s] (1 - s)^-1 n^(1 - s)) HarmonicNumber[n, s]) +
    (((1 - s)^-1 n^(1 - s) ) / (al4[(1 - s)] (1 - s)^-1 n^(1 - s) -
         al4[1 - (1 - s)] (1 - (1 - s))^-1 n^(1 - (1 - s))) HarmonicNumber[n, (1 - s)]))
szet10ee[n_, s_] := (((s^-1 n^s)/(s^-1 n^s - al4[1 - s] / al4[s] (1 - s)^-1 n^(1 - s)) HarmonicNumber[n, s]) -
   (((1 - s)^-1 n^(1 - s))/(s^-1 n^(s) - al4[(1 - s)] / al4[s] (1 - s)^-1 n^(1 - s)) HarmonicNumber[n, (1 - s)]))
szet10ff[n_, s_] := (((s^-1 n^s HarmonicNumber[n, s])/(s^-1 n^s - al4[1 - s] / al4[s] (1 - s)^-1 n^(1 - s))) -
   (((1 - s)^-1 n^(1 - s) HarmonicNumber[n, (1 - s)])/(s^-1 n^(s) - al4[(1 - s)] / al4[s] (1 - s)^-1 n^(1 - s))))
zet10gg[n_, s_] := (n^s HarmonicNumber[n, s] / s -
    n^(1 - s) HarmonicNumber[n, (1 - s)] / (1 - s)) /
  (n^s / s - al4[1 - s] / al4[s] n^(1 - s) / (1 - s))
zet10hh[n_, s_] := (n^s HarmonicNumber[n, s] / s -
    n^(1 - s) HarmonicNumber[n, (1 - s)] / (1 - s)) /
  (n^s / s - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] n^(1 - s) / (1 - s))
```

```
zet10hh[100 000, .51 + 5.5 I]
```

```
0.767294 + 0.2896 i
```

```
g1a[n_, s_] := (n^s HarmonicNumber[n, s] / s - n^(1 - s) HarmonicNumber[n, (1 - s)] / (1 - s)) /
   (n^s / s - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] n^(1 - s) / (1 - s))
g1b[n_, s_] := Zeta[s]
g2a[n_, s_] :=
  (n^s HarmonicNumber[n, s] / s - n^(1 - s) HarmonicNumber[n, (1 - s)] / (1 - s)) / Zeta[s]
g2b[n_, s_] := (n^s / s - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] n^(1 - s) / (1 - s))
g3a[n_, s_] :=
  ((n^s HarmonicNumber[n, s] / s - n^(1 - s) HarmonicNumber[n, (1 - s)] / (1 - s)) / Zeta[s] -
     n^s / s) / (-Pi^(1 / 2 - s) n^(1 - s) / (1 - s))
g3b[n_, s_] := Gamma[s / 2] / Gamma[(1 - s) / 2]
```

$$\texttt{g4a[n\_, s\_] :=} \frac{1}{\texttt{Zeta[s]}} \pi^{-\frac{1}{2}+s} \left(\texttt{HarmonicNumber[n, 1 - s]} - n^{2s-1}\,(-1+s)\,/\,s\ \texttt{HurwitzZeta[s, 1 + n]}\right)$$

```
g4b[n_, s_] := Gamma[s / 2] / Gamma[(1 - s) / 2]
```

$$\texttt{g5a[n\_, s\_] :=} \left(\pi^{-\frac{1}{2}+s}\ \texttt{HarmonicNumber[n, 1 - s]}\right) \Big/ \texttt{Zeta[s]} - \pi^{-\frac{1}{2}+s}\, n^{2s-1}\,(-1+s)\,/\,s\ \texttt{HurwitzZeta[s, 1 + n] / Zeta[s]}$$

```
g5b[n_, s_] := Gamma[s / 2] / Gamma[(1 - s) / 2]
```

$$\texttt{g6a[n\_, s\_] :=} \left(\pi^{-\frac{1}{2}+s}\ \texttt{Zeta[1 - s]}\right) \Big/ \texttt{Zeta[s]} - \left(\pi^{-\frac{1}{2}+s}\ \texttt{Zeta[1 - s, n + 1]}\right) \Big/ \texttt{Zeta[s]} - \pi^{-\frac{1}{2}+s}\, n^{2s-1}\,(-1+s)\,/\,s\ \texttt{Zeta[s, 1 + n] / Zeta[s]}$$

```
g6b[n_, s_] := Gamma[s / 2] / Gamma[(1 - s) / 2]
```

$$\texttt{g7a[n\_, s\_] :=} \pi^{-\frac{1}{2}+s}\ \texttt{Zeta[1 - s] / Zeta[s]}$$

```
g7b[n_, s_] := Gamma[s / 2] / Gamma[(1 - s) / 2]
{g7a[n, s], g7b[n, s]} /. n → 1 000 000 /. s → .3 + 4 I
```

```
{-0.368269 - 0.789246 i, -0.368269 - 0.789246 i}
```

```
FullSimplify[
  ((n^s HarmonicNumber[n, s] / s - n^(1 - s) HarmonicNumber[n, (1 - s)] / (1 - s)) / Zeta[s] -
     n^s / s) / (-Pi^(1 / 2 - s) n^(1 - s) / (1 - s))]
```

$$\frac{1}{n\, s\ \texttt{Zeta[s]}} \pi^{-\frac{1}{2}+s} \left(n\, s\ \texttt{HarmonicNumber[n, 1 - s]} - n^{2s}\,(-1+s)\ \texttt{HurwitzZeta[s, 1 + n]}\right)$$

```
Zeta[.3] - Zeta[.3, 1 + 10]
```

```
6.5046
```

```
HarmonicNumber[10, .3]
```

```
6.5046
```

$$\texttt{g6ar[n\_, s\_] :=} \left\{\left(\pi^{-\frac{1}{2}+s}\ \texttt{Zeta[1 - s]}\right) \Big/ \texttt{Zeta[s]},\ -\left(\pi^{-\frac{1}{2}+s}\ \texttt{Zeta[1 - s, n + 1]}\right) \Big/ \texttt{Zeta[s]} - \pi^{-\frac{1}{2}+s}\, n^{2s-1}\,(-1+s)\,/\,s\ \texttt{Zeta[s, 1 + n] / Zeta[s]}\right\}$$

```
Chop@g6ar[n, s] /. n → 100 000 000 000 000 /. s → .3 + 4 I
```

$$\{-0.368269 - 0.789246\, i,\ 3.00133 \times 10^{-11} + 2.7012 \times 10^{-10}\, i\}$$

```
h1a[n_, s_] := (n^s HarmonicNumber[n, s] / s - n^(1 - s) HarmonicNumber[n, (1 - s)] / (1 - s)) /
  (n^s / s - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] n^(1 - s) / (1 - s))
h1b[n_, s_] := n^s HarmonicNumber[n, s] / s /
    (n^s / s - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] n^(1 - s) / (1 - s)) -
  n^(1 - s) HarmonicNumber[n, (1 - s)] / (1 - s) /
    (n^s / s - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] n^(1 - s) / (1 - s))
h1c[n_, s_] := n^s (Zeta[s] - Zeta[s, n + 1]) / s /
    (n^s / s - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] n^(1 - s) / (1 - s)) -
  n^(1 - s) (Zeta[1 - s] - Zeta[1 - s, n + 1]) / (1 - s) /
    (n^s / s - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] n^(1 - s) / (1 - s))
h1d[n_, s_] := n^s (Zeta[s]) / s /
    (n^s / s - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] n^(1 - s) / (1 - s)) -
  n^s (Zeta[s, n + 1]) / s / (n^s / s - Pi^(1 / 2 - s)
       Gamma[s / 2] / Gamma[(1 - s) / 2] n^(1 - s) / (1 - s)) -
  n^(1 - s) (Zeta[1 - s]) / (1 - s) / (n^s / s - Pi^(1 / 2 - s)
       Gamma[s / 2] / Gamma[(1 - s) / 2] n^(1 - s) / (1 - s)) +
  n^(1 - s) (Zeta[1 - s, n + 1]) / (1 - s) / (n^s / s - Pi^(1 / 2 - s)
       Gamma[s / 2] / Gamma[(1 - s) / 2] n^(1 - s) / (1 - s))
h1dx[n_, s_] := {n^s (Zeta[s]) / s / (n^s / s - Pi^(1 / 2 - s)
       Gamma[s / 2] / Gamma[(1 - s) / 2] n^(1 - s) / (1 - s)), -
    n^s (Zeta[s, n + 1]) / s / (n^s / s - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2]
       n^(1 - s) / (1 - s)), -
    n^(1 - s) (Zeta[1 - s]) / (1 - s) / (n^s / s - Pi^(1 / 2 - s)
       Gamma[s / 2] / Gamma[(1 - s) / 2] n^(1 - s) / (1 - s)),
  n^(1 - s) (Zeta[1 - s, n + 1]) / (1 - s) / (n^s / s - Pi^(1 / 2 - s)
       Gamma[s / 2] / Gamma[(1 - s) / 2] n^(1 - s) / (1 - s))}
```

```
h1dx[1 000 000 000 000 000 000, .6 + 30 I]
```

{0.0224625 − 0.566477 i, −324 573. + 416 745. i,
 −0.000163445 − 0.0000318504 i, 324 573. − 416 745. i}

```
Zeta[.6 + 30 I]
```

0.0222991 − 0.566509 i

```
n^s Zeta[s] / s / (n^s / s - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] n^(1 - s) / (1 - s))
```

$$\frac{n^s\,\mathrm{Zeta}[s]}{s\left(\frac{n^s}{s}-\frac{n^{1-s}\,\pi^{\frac{1}{2}-s}\,\mathrm{Gamma}\left[\frac{s}{2}\right]}{(1-s)\,\mathrm{Gamma}\left[\frac{1-s}{2}\right]}\right)}$$

```
N[n^s / s / (n^s / s - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] n^(1 - s) / (1 - s)) /.
   s → 1 / 2 + I /. n → 1 000 000 000 000 000]
```

0.5 − 0.669235 i

```
Zeta[s] / (1 - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] n^(1 - 2 s) s / (1 - s))
```

$$\frac{\mathrm{Zeta}[s]}{1-\frac{n^{1-2s}\,\pi^{\frac{1}{2}-s}\,s\,\mathrm{Gamma}\left[\frac{s}{2}\right]}{(1-s)\,\mathrm{Gamma}\left[\frac{1-s}{2}\right]}}$$

$$\text{Limit}\left[\frac{n^{1-2s}\,\pi^{\frac{1}{2}-s}\,s\,\text{Gamma}\left[\frac{s}{2}\right]}{(1-s)\,\text{Gamma}\left[\frac{1-s}{2}\right]}\ /.\ s\to 2/3,\ n\to \text{Infinity}\right]$$

0

$$\text{N}\left[\frac{\pi^{\frac{1}{2}-s}\,s\,\text{Gamma}\left[\frac{s}{2}\right]}{(1-s)\,\text{Gamma}\left[\frac{1-s}{2}\right]}\ /.\ s\to \text{N@ZetaZero@1}\right]$$

0.926247 - 0.376918 i

```
N[n^s Zeta[s, n + 1] / s /. n → 10 000 000]
```

$$\frac{1.\times 10^{7s}\,\text{Zeta}\left[s,\ 1.\times 10^{7}\right]}{s}$$

```
Limit[
 n^(1 - s) / (1 - s) / (n^s / s - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] n^(1 - s) / (1 - s)) /.
  s → 1 / 3, n → Infinity]
```

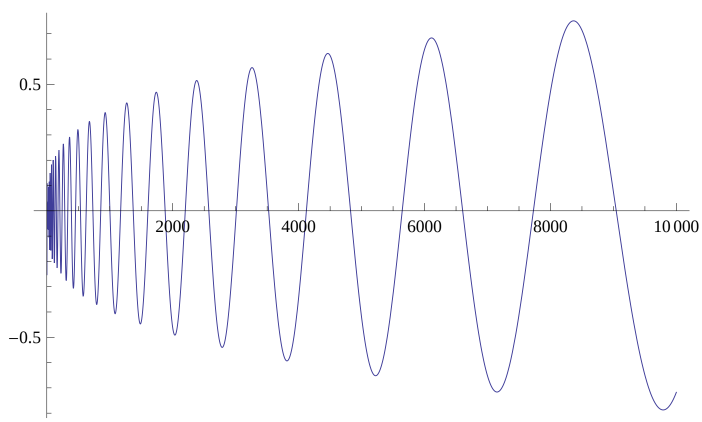
$$-\frac{3\,\text{Gamma}\left[\frac{4}{3}\right]}{\pi^{1/6}\,\text{Gamma}\left[\frac{1}{6}\right]}$$

```
pl[n_, s_] := HurwitzZeta[s, n + 1] / (n^(1 - s) / (s - 1))
plx[n_, s_] := HurwitzZeta[s, n + 1] - (n^(1 - s) / (s - 1))
pla[n_, s_] := HurwitzZeta[s, n + 1]
plb[n_, s_] := (n^(1 - s) / (s - 1))
```

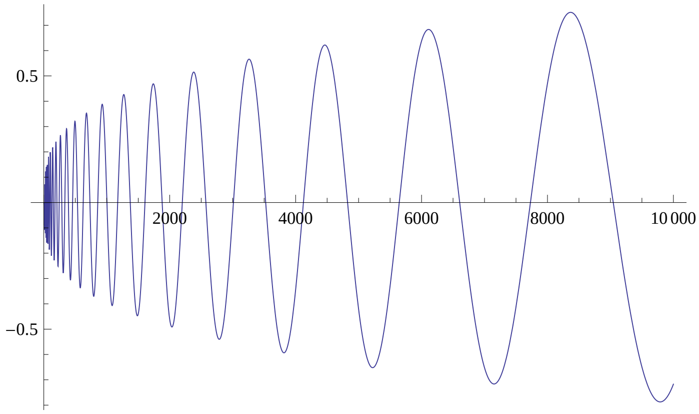
```
plx[1 000 000 000, 1.1 + 3 I]
```

$-4.96476\times 10^{-11} - 3.86956\times 10^{-11}$ i

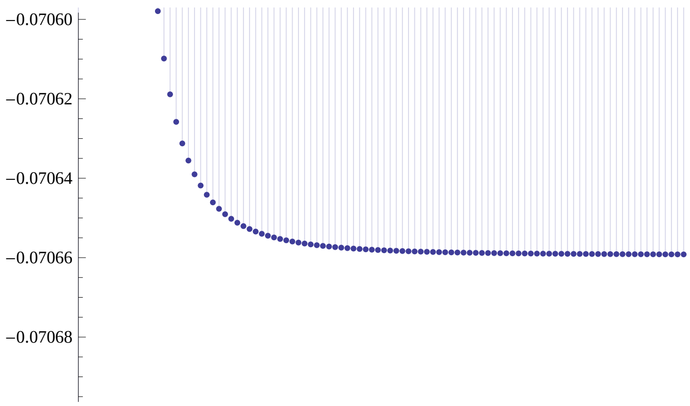
```
Plot[Re@pla[n, .7 + 20 I], {n, 1, 10 000}]
```



```
Plot[Re@plb[n, .7 + 20 I], {n, 1, 10 000}]
```




```
pb[n_, s_] := n^s HarmonicNumber[n, s] / s - n^(1 - s) HarmonicNumber[n, 1 - s] / (1 - s)
DiscretePlot[Im@pb[n, N@ZetaZero@1], {n, 1, 100}]
```




```
pb[1000, N@ZetaZero@1]
```

0. − 0.0706593 i

```
N[n^s / s - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] n^(1 - s) / (1 - s) /. s → 1 / 3]
```

$3.\ n^{1/3} - 3.77185\ n^{2/3}$

```
g1xa[n_, s_] := n^s HarmonicNumber[n, s] / s - n^(1 - s) HarmonicNumber[n, (1 - s)] / (1 - s)
g1xb[n_, s_] :=
  (n^s / s - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] n^(1 - s) / (1 - s)) Zeta[s]
```

```
{g1xa[n, s], g1xb[n, s]} /. n → 1 000 000 000 /. s → N@ZetaZero@10
```

$\{0. - 0.020089\ i,\ -2.54438\times 10^{-11} - 6.94188\times 10^{-12}\ i\}$

```
1. / (ZetaZero@1)
```

0.00249949 − 0.0706593 i

```
g1xa[1 000 000 000, N@ZetaZero@1]
```

0. − 0.0706591 i

```
g1xc[n_, s_] := s n^(1 - s) HarmonicNumber[n, (1 - s)] - (1 - s) n^s HarmonicNumber[n, s]
g1xc2[n_, s_] :=
 s n^(1 - s) (Zeta[1 - s] - Zeta[1 - s, n + 1]) - (1 - s) n^s ( Zeta[s] - Zeta[s, n + 1])
g1xc3[n_, s_] := s n^(1 - s) Zeta[1 - s] - s n^(1 - s) Zeta[1 - s, n + 1] -
  (1 - s) n^s Zeta[s] + (1 - s) n^s Zeta[s, n + 1]
g1xc3a[n_, s_] := {s n^(1 - s) Zeta[1 - s], -s n^(1 - s) Zeta[1 - s, n + 1],
  -(1 - s) n^s Zeta[s], (1 - s) n^s Zeta[s, n + 1]}
g1xc3a2[n_, s_] := {s n^(1 - s) HarmonicNumber[n, (1 - s)], -(1 - s) n^s HarmonicNumber[n, s]}
g1xc3c[n_, s_] := {n^(1 - s) Zeta[1 - s] / (1 - s),
  -n^(1 - s) Zeta[1 - s, n + 1] / (1 - s), -n^s Zeta[s] / s, n^s Zeta[s, n + 1] / s}
g1xc3c2[n_, s_] := {n^(1 - s) HarmonicNumber[n, (1 - s)] / (1 - s),
  -n^s HarmonicNumber[n, s] / s}
g1xd[n_, s_] := Sum[s (n / j)^(1 - s) - (1 - s) (n / j)^s, {j, 1, n}]
```

```
N@g1xd[50, ZetaZero@1 + 1 I]
```

$9.59233\times10^{-14} - 101.975\,i$

```
(N@ZetaZero@1 + 2 I) / 2
```

$0.25 + 8.06736\,i$

```
N@ZetaZero[1]
```

$0.5 + 14.1347\,i$

```
FullSimplify[((b[s] h[s] - b[1 - s] h[1 - s]) (b[1 - s] h[1 - s] - b[s] h[s]))^(1 / 2)]
```

$\sqrt{-(b[1-s]\,h[1-s]-b[s]\,h[s])^2}$

```
FullSimplify[Expand[(b[s] - a[1 - s] / a[s] b[1 - s]) (b[1 - s] - a[s] / a[1 - s] b[s])]]
```

$-\frac{(a[1-s]\,b[1-s]-a[s]\,b[s])^2}{a[1-s]\,a[s]}$

```
ap[s_] := 2 Pi^(s / 2) / Gamma[s / 2]
bt[n_, s_] := n^s / s
zt[n_, s_] := (bt[n, s] HarmonicNumber[n, s] - bt[n, 1 - s] HarmonicNumber[n, 1 - s]) /
  (bt[n, s] - ap[1 - s] / ap[s] bt[n, 1 - s])
zm[s_] := (Zeta[s] Zeta[1 - s])^(1 / 2)
```

```
zt[100 000, .5 + I]
```

$0.143215 - 0.718479\,i$

```
Zeta[.5 + I]
```

$0.143936 - 0.7221\,i$

```
zt[n, s]
```

$$\frac{-\frac{n^{1-s}\,\text{HarmonicNumber}[n,1-s]}{1-s}+\frac{n^{s}\,\text{HarmonicNumber}[n,s]}{s}}{\frac{n^{s}}{s}-\frac{n^{1-s}\,\pi^{\frac{1-s}{2}-\frac{s}{2}}\,\text{Gamma}\left[\frac{s}{2}\right]}{(1-s)\,\text{Gamma}\left[\frac{1-s}{2}\right]}}$$

**FullSimplify@Expand[zt[n, s] zt[n, 1 - s]]**

$$\left(\pi^{\frac{1}{2}+s}\,\text{Gamma}\left[\frac{1}{2}-\frac{s}{2}\right]\,\text{Gamma}\left[\frac{s}{2}\right]\,\left(n\,s\,\text{HarmonicNumber}[n,1-s]+n^{2\,s}\,(-1+s)\,\text{HarmonicNumber}[n,s]\right)^{2}\right)\Big/\left(-2\,n^{2\,s}\,\pi^{s}\,\text{Gamma}\left[\frac{3}{2}-\frac{s}{2}\right]+n\,\sqrt{\pi}\,s\,\text{Gamma}\left[\frac{s}{2}\right]\right)^{2}$$

$$\left(\left(\pi^{\frac{1}{2}+s}\,\text{Gamma}\left[\frac{1}{2}-\frac{s}{2}\right]\,\text{Gamma}\left[\frac{s}{2}\right]\right)\text{^}(1/2)\,\left(n\,s\,\text{HarmonicNumber}[n,1-s]+n^{2\,s}\,(-1+s)\,\text{HarmonicNumber}[n,s]\right)\right)\Big/\left(-2\,n^{2\,s}\,\pi^{s}\,\text{Gamma}\left[\frac{3}{2}-\frac{s}{2}\right]+n\,\sqrt{\pi}\,s\,\text{Gamma}\left[\frac{s}{2}\right]\right)$$

$$\left(\sqrt{\pi^{\frac{1}{2}+s}\,\text{Gamma}\left[\frac{1}{2}-\frac{s}{2}\right]\,\text{Gamma}\left[\frac{s}{2}\right]}\,\left(n\,s\,\text{HarmonicNumber}[n,1-s]+n^{2\,s}\,(-1+s)\,\text{HarmonicNumber}[n,s]\right)\right)\Big/\left(-2\,n^{2\,s}\,\pi^{s}\,\text{Gamma}\left[\frac{3}{2}-\frac{s}{2}\right]+n\,\sqrt{\pi}\,s\,\text{Gamma}\left[\frac{s}{2}\right]\right)$$

$$\text{zsq}[\text{n\_}, \text{s\_}] := \left(\left(\pi^{\frac{1}{2}+s}\, \text{Gamma}\left[\frac{1}{2}-\frac{s}{2}\right] \text{Gamma}\left[\frac{s}{2}\right]\right) \wedge (1/2)\right.$$
$$\left.\left(n\, s\, \text{HarmonicNumber}[n, 1-s] + n^{2\,s}\, (-1+s)\, \text{HarmonicNumber}[n, s]\right)\right) \Big/$$
$$\left(-2\, n^{2\,s}\, \pi^{s}\, \text{Gamma}\left[\frac{3}{2}-\frac{s}{2}\right] + n\sqrt{\pi}\, s\, \text{Gamma}\left[\frac{s}{2}\right]\right)$$

$$\text{zsq2}[\text{n\_}, \text{s\_}] := \left(\left(\pi^{\frac{1}{2}+s}\right) \wedge (1/2) \left(\text{Gamma}\left[\frac{1}{2}-\frac{s}{2}\right] \text{Gamma}\left[\frac{s}{2}\right]\right) \wedge (1/2)\right.$$
$$\left.\left(s\, n \wedge -s\, \text{HarmonicNumber}[n, 1-s] + n^{s-1}\, (-1+s)\, \text{HarmonicNumber}[n, s]\right)\right) \Big/$$
$$\left(-2\, n^{s-1}\, \text{Pi}^{s}\, \text{Gamma}\left[\frac{3}{2}-\frac{s}{2}\right] + \text{Pi} \wedge (1/2)\, s\, n \wedge -s\, \text{Gamma}\left[\frac{s}{2}\right]\right)$$

$$\text{zsq3}[\text{n\_}, \text{s\_}] := \left(\left(-\pi^{\frac{1}{4}+s/2}\right) \left(\text{Gamma}\left[\frac{1}{2}-\frac{s}{2}\right] \text{Gamma}\left[\frac{s}{2}\right]\right) \wedge (1/2)\right.$$
$$\left.\left(s\, n \wedge -s\, \text{HarmonicNumber}[n, 1-s] + n^{s-1}\, (-1+s)\, \text{HarmonicNumber}[n, s]\right)\right) \Big/$$
$$\left(-2\, n^{s-1}\, \text{Pi}^{s}\, \text{Gamma}\left[1+\frac{1-s}{2}\right] + \text{Pi} \wedge (1/2)\, s\, n \wedge -s\, \text{Gamma}\left[\frac{s}{2}\right]\right)$$

$$\text{zsq4}[\text{n\_}, \text{s\_}] := \left(\left(\text{Gamma}\left[\frac{1}{2}-\frac{s}{2}\right] \text{Gamma}\left[\frac{s}{2}\right]\right) \wedge (1/2)\right.$$
$$\left.\left(n \wedge (-s)\, \text{HarmonicNumber}[n, 1-s] / (1-s) - n^{s-1}\, \text{HarmonicNumber}[n, s] / s\right)\right) \Big/$$
$$\left(2/(1-s)\, n^{s-1}\, \pi^{-\frac{1}{4}+\frac{s}{2}}\, \text{Gamma}\left[1+\frac{1-s}{2}\right] \Big/ s - \pi^{\frac{1}{4}-\frac{s}{2}}\, n \wedge -s\, \text{Gamma}\left[\frac{s}{2}\right] \Big/ (1-s)\right)$$

$$\text{zsq5}[\text{n\_}, \text{s\_}] := n \wedge (1-s)\, \text{HarmonicNumber}[n, 1-s] / (1-s) - n^{s}\, \text{HarmonicNumber}[n, s] / s \Big/$$
$$\left(n^{s} / s\, \pi^{-\frac{1}{4}+s/2} \left(\text{Gamma}\left[\frac{1-s}{2}\right] \Big/ \text{Gamma}\left[\frac{s}{2}\right]\right) \wedge (1/2) -\right.$$
$$\left.\pi^{\frac{1}{4}-\frac{s}{2}}\, n \wedge (1-s) / (1-s) \left(\text{Gamma}\left[\frac{s}{2}\right] \Big/ \text{Gamma}\left[\frac{1}{2}-\frac{s}{2}\right]\right) \wedge (1/2)\right)$$

$$\text{zsq6}[\text{n\_}, \text{s\_}] := n \wedge (1-s)\, \text{HarmonicNumber}[n, 1-s] / (1-s) - n^{s}\, \text{HarmonicNumber}[n, s] / s \Big/$$
$$\left(n^{s} / s\, \pi^{-\frac{1}{4}+s/2} \left(\text{Gamma}\left[\frac{1-s}{2}\right] \Big/ \text{Gamma}\left[\frac{s}{2}\right]\right) \wedge (1/2) -\right.$$
$$\left.\pi^{\frac{1}{4}-\frac{s}{2}}\, n \wedge (1-s) / (1-s) \left(\text{Gamma}\left[\frac{s}{2}\right] \Big/ \text{Gamma}\left[\frac{1-s}{2}\right]\right) \wedge (1/2)\right)$$

```
Chop@zsq6[10 000 000, N@ZetaZero@1 + .1]
```

0.000713535 + 0.0793222 i

```
N@zm[ZetaZero@1 - .1]
```

0.000651963 - 0.0793368 i

```
Abs[Zeta[.4 + I]]
```

0.663341

```
Expand[(π^(1/2+s))^(1/2)]
```

$$\sqrt{\pi^{\frac{1}{2}+s}}$$

```
(3^(1/2+s))^(1/2) /. s → 3.2
```

7.63263

```
(3^(1/4+s/2)) /. s → 3.2
```

7.63263

```
FullSimplify[Pi^s / (-π^(1/4+s/2))]
```

$$-\pi^{-\frac{1}{4}+\frac{s}{2}}$$

```
FullSimplify[Pi^(1/2) / (-π^(1/4+s/2))]
```

$$-\pi^{\frac{1}{4}-\frac{s}{2}}$$

```
s (1 - s) /. s → N@ZetaZero@1
```

200.04 + 0. i

```
N@ZetaZero@1 / (1 / N@ZetaZero@1)
```

-199.54 + 14.1347 i

```
ffo[n_, s_] := (1 - s) n^s HarmonicNumber[n, s] - s n^(1 - s) HarmonicNumber[n, 1 - s]
ffp[n_, s_] := n^s / s HarmonicNumber[n, s] - n^(1 - s) / (1 - s) HarmonicNumber[n, 1 - s]
ffq[n_, s_] :=
  ((1 - s) n^s HarmonicNumber[n, s] - s n^(1 - s) HarmonicNumber[n, 1 - s]) / (s (1 - s))^(1 / 2)
ffr[n_, s_] := (s / (1 - s))^(1 / 2) n^(1 - s) HarmonicNumber[n, 1 - s] -
   ((1 - s) / s)^(1 / 2) n^s HarmonicNumber[n, s]
```

```
ffo[100 000, N@ZetaZero@2] / ffp[100 000, N@ZetaZero@2]
```

442.176 + 0. i

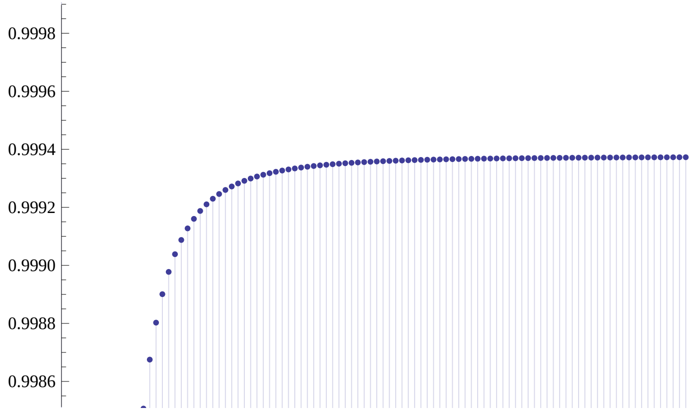
```
N@ffr[10 000 000, ZetaZero@1]
```

$1.16415 \times 10^{-10} + 0.999375\, i$

```
(s - s^2)^(1 / 2)
```

$$\sqrt{s - s^2}$$

```
DiscretePlot[Im[ffr[n, ZetaZero@1]], {n, 1, 100}]
```




```
i1a[n_, s_] := (n^s HarmonicNumber[n, s] / s - n^(1 - s) HarmonicNumber[n, (1 - s)] / (1 - s)) /
   (n^s / s - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] n^(1 - s) / (1 - s))
i1b[n_, s_] := ((1 - s) n^s HarmonicNumber[n, s] - s n^(1 - s) HarmonicNumber[n, (1 - s)]) /
   (n^s (1 - s) - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] n^(1 - s) s)
i1b2[n_, s_] := Zeta[s]
i1c[n_, s_] := (1 - s) n^s HarmonicNumber[n, s] - s n^(1 - s) HarmonicNumber[n, (1 - s)]
i1c2[n_, s_] := Zeta[s] (n^s (1 - s) - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] n^(1 - s) s)
i1c3o[n_, s_] :=
  (1 - s) n^s (Zeta[s] - Zeta[s, n + 1]) - s n^(1 - s) (Zeta[1 - s] - Zeta[1 - s, n + 1])
i1c3[n_, s_] := (1 - s) n^s (-Zeta[s, n + 1]) - s n^(1 - s) (-Zeta[1 - s, n + 1])
```

```
    {i1c[10 000 000 000, N@ZetaZero@1 + .00000001 I],
 i1c2[10 000 000 000, N@ZetaZero@1 + .00000001 I],
 i1c3[10 000 000 000, N@ZetaZero@1 + .00000001 I]}
```

$\{0. - 14.1542\, i,\ -7.1839 \times 10^{-9} - 0.019663\, i,\ 0. - 14.1346\, i\}$

```
{i1b[10 000 000 000, N@ZetaZero@1 + .00000001 I],
 i1b2[10 000 000 000, N@ZetaZero@1 + .00000001 I]}
```

$\{-8.97638 \times 10^{-7} + 5.63847 \times 10^{-6}\, i,\ -1.247 \times 10^{-9} + 7.83297 \times 10^{-9}\, i\}$

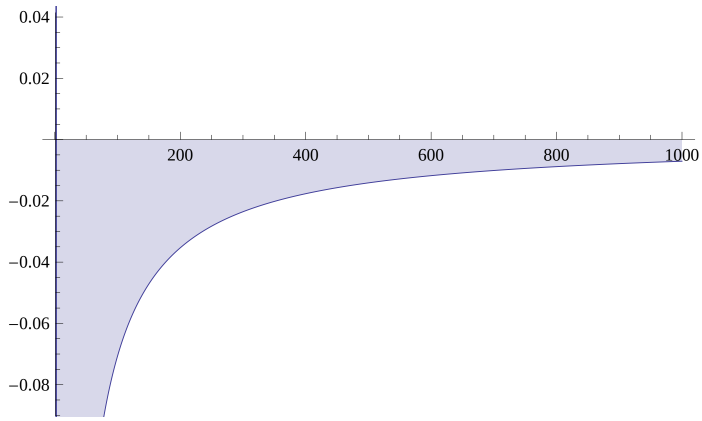
```
Expand[(((1 - s)^(1 / 2) / s^(1 / 2))) / (1 - s)]
```

$$\frac{1}{\sqrt{1-s}\ \sqrt{s}}$$

$$\text{N}\left[\frac{1}{\sqrt{1-s}\ \sqrt{s}}\ /.\ s \to \text{N@ZetaZero@1}\right]$$

$0.0707035 + 0.\, i$

```
pa[n_, s_] := (1 - s) n^s HarmonicNumber[n, s]
pax[n_, s_] := (1 - s) n^(s - 1) HarmonicNumber[n, s]
pa2[n_, s_] := n^s / s HarmonicNumber[n, s]
pa3[n_, s_] := ((1 - s) / s)^(1 / 2) n^s HarmonicNumber[n, s]
pa4[n_, s_] := (s / (1 - s))^(1 / 2) n^s HarmonicNumber[n, s]
pa5[n_, a_, s_] := ((1 - s)^a s^-(1 - a)) n^s HarmonicNumber[n, s]
DiscretePlot[Im[pax[n, N@ZetaZero@1]], {n, 1, 1000}]
```




```
pa[10 000 000, N@ZetaZero@1] + N@ZetaZero@1 / 2 - 1 / 2 - 10 000 000
```

$-1.67079\times10^{-6} - 3.39502\times10^{-7}\,i$

```
(N@ZetaZero@1)
```

0.5 + 14.1347 i

```
pa[100, N@ZetaZero@1]
```

100.083 - 7.06735 i

```
pa5[1000, .5, N@ZetaZero@4]
```

32.8691 - 0.499932 i

```
(1000 (1 / ((1 - N@ZetaZero@4) N@ZetaZero@4))^(1 / 2))
```

32.8634 + 0. i

```
pa3[100 000, N@ZetaZero@1]
```

7070.37 - 0.499687 i

```
pa5[100 000, .5, N@ZetaZero@1]
```

7070.37 - 0.499687 i

```
((ZetaZero@1) (1 - ZetaZero@1))^(-1 / 2) (100 000 + 1 / 2 - N@ZetaZero@1 / 2)
```

7070.37 - 0.499687 i

```
Chop[pa5[100 000, .5, N@ZetaZero@1] / pa5[100 000, 1, N@ZetaZero@1]]
```

0.0707035

```
N[ZetaZero@1^(1 / 2) (1 - ZetaZero@1)^(-1 / 2)]
```

0.0353518 + 0.999375 i

```
0.07070352773181221` pa5[100 000, 1, N@ZetaZero@1]
```

7070.37 − 0.499687 ⅈ

```
pa5[100 000, .5, N@ZetaZero@1]
```

7070.37 − 0.499687 ⅈ

```
1 / (N@ZetaZero@1 - 1 / 2)
```

0. − 0.0707477 ⅈ

```
N[(s - s^2) ^ (-1 / 2) (s / 2) /. s → .5 + 14.14 I]
```

0.0176693 + 0.499688 ⅈ

```
(s - s^2) ^ (-1 / 2) (s / 2)
```

$$\frac{s}{2\sqrt{s-s^2}}$$

```
N[(4 s - 4 s^2) ^ (-1 / 2) s /. s → .5 + 14.14 I]
```

0.0176693 + 0.499688 ⅈ

```
N[((1 - s) / s) ^ (-1 / 2) / 2 /. s → .5 + 14.14 I]
```

0.0176693 + 0.499688 ⅈ

```
(* so this thing has an abs of exactly 1/2 when re(s) is 1/2 *)
```

$$\frac{1}{2\sqrt{-1+\frac{1}{s}}}$$

```
FullSimplify[(s (1 - s)) ^ (-1 / 2) (1 / 2 - s / 2)]
```

$$\frac{\sqrt{1-s}}{2\sqrt{s}}$$

```
1 / 2 (s (1 - s)) ^ (-1 / 2)
```

$$\frac{1}{2\sqrt{(1-s)\, s}}$$

```
FullSimplify[1 / 2 (s (1 - s)) ^ (-1 / 2) - 1 / 2 ((1 - s) / s) ^ (-1 / 2)]
```

$$\frac{1}{2}\left(-\frac{1}{\sqrt{-1+\frac{1}{s}}}+\frac{1}{\sqrt{-(-1+s)\, s}}\right)$$

$$\frac{\sqrt{1-s}}{2\sqrt{s}} \text{ /. } s \to .5 + 2\, I$$

0.121268 − 0.485071 ⅈ

```
(s (1 - s)) ^ (-1 / 2) (1 / 2 - s / 2) /. s → .5 + 2 I
```

0.121268 − 0.485071 ⅈ

```
1 / 2 ((1 - s) / s) ^ (-1 / 2) /. s → .5 + 2 I
```

0.121268 + 0.485071 i

```
n (s (1 - s)) ^ (-1 / 2) + 1 / 2 (s / (1 - s)) ^ (1 / 2) /. s → N@ZetaZero@1 /. n → 100 000
```

7070.37 + 0.499687 i

```
pa3[100 000, N@ZetaZero@1]
```

7070.37 - 0.499687 i

```
Chop[pa5[100 000, 0, N@ZetaZero@1] / pa5[100 000, 1, N@ZetaZero@1]]
```

0.00499899

```
1 / (s (1 - s)) /. s → N@ZetaZero@1
```

0.00499899 + 0. i

```
pa5[100 000, 0, N@ZetaZero@1]
```

499.9 - 0.0353297 i

```
(n + (1 - s) / 2) / s / (1 - s) /. s → N@ZetaZero@1 /. n → 100 000
```

499.9 - 0.0353297 i

```
FullSimplify[((1 - s) / 2) / s / (1 - s)]
```

$$\frac{1}{2\,s}$$

```
(n) / s / (1 - s)
```

$$\frac{n}{(1-s)\,s}$$

```
n (s (1 - s)) ^ (-1 / 2) /. s → .5 + 5 I /. n → 100 000
```

19 900.7 + 0. i

```
(s / (1 - s)) ^ (1 / 2) / 2 /. s → .5 + 14.14 I
```

0.0176693 + 0.499688 i

```
n (s (1 - s)) ^ (-1 / 2) + 1 / 2 (s / (1 - s)) ^ (1 / 2) /. s → N@ZetaZero@1 /. n → 100
```

7.08803 + 0.499687 i

```
pa3[100, N@ZetaZero@1]
```

7.07624 - 0.499686 i

```
pa6[n_, s_] := (1 - s) n^s HarmonicNumber[n, s]
```

```
pa6[10 000 000, N@ZetaZero@1] - pa6[10 000 000, 1 - N@ZetaZero@1]
```

0. - 14.1347 i

```
n^(1 - s) / (1 - s) + n^-s / 2 - n^(-s - 1) s / 12 + n^(-s - 3) s (s + 1) (s + 2) / 720
```

$$\frac{n^{-s}}{2}+\frac{n^{1-s}}{1-s}-\frac{1}{12}\,n^{-1-s}\,s+\frac{1}{720}\,n^{-3-s}\,s\,(1+s)\,(2+s)$$

```
bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
bern[k_] := If[k == 1, 1 / 2, BernoulliB[k]]
bs[n_, s_, t_] := Sum[FullSimplify[ bin[1 - s, k] / (1 - s) bern[k] n^(1 - s - k)], {k, 0, t}]
```

```
bs[n, s, 10]
```

$$\frac{n^{-s}}{2}+\frac{n^{1-s}}{1-s}-\frac{1}{12}n^{-1-s}s+\frac{1}{720}n^{-3-s}s(1+s)(2+s)-\frac{n^{-5-s}s(1+s)(2+s)(3+s)(4+s)}{30\,240}+\frac{n^{-7-s}s(1+s)(2+s)(3+s)(4+s)(5+s)(6+s)}{1\,209\,600}-\frac{n^{-9-s}s(1+s)(2+s)(3+s)(4+s)(5+s)(6+s)(7+s)(8+s)}{47\,900\,160}$$

```
bs[n, s, 20]
```

$$\frac{n^{-s}}{2}+\frac{n^{1-s}}{1-s}-\frac{1}{12}n^{-1-s}s+\frac{1}{720}n^{-3-s}s(1+s)(2+s)-\frac{n^{-5-s}s(1+s)(2+s)(3+s)(4+s)}{30\,240}+\frac{n^{-7-s}s(1+s)(2+s)(3+s)(4+s)(5+s)(6+s)}{1\,209\,600}-\frac{n^{-9-s}s(1+s)(2+s)(3+s)(4+s)(5+s)(6+s)(7+s)(8+s)}{47\,900\,160}+\frac{1}{1\,307\,674\,368\,000}691\,n^{-11-s}s(1+s)(2+s)(3+s)(4+s)(5+s)(6+s)(7+s)(8+s)(9+s)(10+s)-\frac{1}{74\,724\,249\,600}n^{-13-s}s(1+s)(2+s)(3+s)(4+s)(5+s)(6+s)(7+s)(8+s)(9+s)(10+s)(11+s)(12+s)+\left(3617\,n^{-15-s}s(1+s)(2+s)(3+s)(4+s)(5+s)(6+s)(7+s)(8+s)(9+s)(10+s)(11+s)(12+s)(13+s)(14+s)\right)/10\,670\,622\,842\,880\,000-\left(43\,867\,n^{-17-s}s(1+s)(2+s)(3+s)(4+s)(5+s)(6+s)(7+s)(8+s)(9+s)(10+s)(11+s)(12+s)(13+s)(14+s)(15+s)(16+s)\right)/5\,109\,094\,217\,170\,944\,000+\left(174\,611\,n^{-19-s}s(1+s)(2+s)(3+s)(4+s)(5+s)(6+s)(7+s)(8+s)(9+s)(10+s)(11+s)(12+s)(13+s)(14+s)(15+s)(16+s)(17+s)(18+s)\right)/802\,857\,662\,698\,291\,200\,000$$

```
FullSimplify@Expand[bs[n, s, 10] / bs[n, 1 - s, 10]]
```

$$-\left(n^{1-2s}s\left(239\,500\,800\,n^{10}-119\,750\,400\,n^{9}(-1+s)+19\,958\,400\,n^{8}(-1+s)s-332\,640\,n^{6}(-1+s)s(1+s)(2+s)+7920\,n^{4}(-1+s)s(1+s)(2+s)(3+s)(4+s)-198\,n^{2}(-1+s)s(1+s)(2+s)(3+s)(4+s)(5+s)(6+s)+5(-1+s)s(1+s)(2+s)(3+s)(4+s)(5+s)(6+s)(7+s)(8+s)\right)\right)/\left((-1+s)\left(239\,500\,800\,n^{10}+119\,750\,400\,n^{9}s+19\,958\,400\,n^{8}(-1+s)s-332\,640\,n^{6}(-3+s)(-2+s)(-1+s)s+7920\,n^{4}(-5+s)(-4+s)(-3+s)(-2+s)(-1+s)s-198\,n^{2}(-7+s)(-6+s)(-5+s)(-4+s)(-3+s)(-2+s)(-1+s)s+5(-9+s)(-8+s)(-7+s)(-6+s)(-5+s)(-4+s)(-3+s)(-2+s)(-1+s)s\right)\right)$$

```
HarmonicNumber[n, s] / bs[n, s, 10] - HarmonicNumber[n, 1 - s] / bs[n, 1 - s, 10] /. s → .95 + 3 I /.
 n → 1 000 000 000 000 000
```

$0.0692592+0.327965\,i$

```
Expand[Sum[j^2, {j, 1, n}]]
```

$$\frac{n}{6}+\frac{n^2}{2}+\frac{n^3}{3}$$

```
bs[100, N@ZetaZero@2, 10]
```

0.208036 − 0.429927 i

$$\text{fno}[n\_, s\_] := 1\Big/\left(\frac{n^{-s}}{2}+\frac{n^{1-s}}{1-s}-\frac{1}{12}n^{-1-s}s+\frac{1}{720}n^{-3-s}s(1+s)(2+s)\right)$$

$$\text{fno2}[n\_, s\_] := 1\Big/\left(\frac{n^{-s}}{2}+\frac{n^{1-s}}{1-s}-\frac{1}{12}n^{-1-s}s+\frac{1}{720}n^{-3-s}s(1+s)(2+s)-\frac{n^{-5-s}s(1+s)(2+s)(3+s)(4+s)}{30240}+\frac{n^{-7-s}s(1+s)(2+s)(3+s)(4+s)(5+s)(6+s)}{1209600}-\frac{n^{-9-s}s(1+s)(2+s)(3+s)(4+s)(5+s)(6+s)(7+s)(8+s)}{47900160}\right)$$

$$\text{fn}[n\_, s\_] := 1\Big/\left(\frac{n^{-s}}{2}+\frac{n^{1-s}}{1-s}-\frac{1}{12}n^{-1-s}s+\frac{1}{720}n^{-3-s}s(1+s)(2+s)-\frac{n^{-5-s}s(1+s)(2+s)(3+s)(4+s)}{30240}+\frac{n^{-7-s}s(1+s)(2+s)(3+s)(4+s)(5+s)(6+s)}{1209600}-\frac{n^{-9-s}s(1+s)(2+s)(3+s)(4+s)(5+s)(6+s)(7+s)(8+s)}{47900160}+\frac{1}{1307674368000}691\,n^{-11-s}s(1+s)(2+s)(3+s)(4+s)(5+s)(6+s)(7+s)(8+s)(9+s)(10+s)-\frac{1}{74724249600}n^{-13-s}s(1+s)(2+s)(3+s)(4+s)(5+s)(6+s)(7+s)(8+s)(9+s)(10+s)(11+s)(12+s)+\left(3617\,n^{-15-s}s(1+s)(2+s)(3+s)(4+s)(5+s)(6+s)(7+s)(8+s)(9+s)(10+s)(11+s)(12+s)(13+s)(14+s)\right)\Big/10670622842880000-\left(43867\,n^{-17-s}s(1+s)(2+s)(3+s)(4+s)(5+s)(6+s)(7+s)(8+s)(9+s)(10+s)(11+s)(12+s)(13+s)(14+s)(15+s)(16+s)\right)\Big/5109094217170944000+\left(174611\,n^{-19-s}s(1+s)(2+s)(3+s)(4+s)(5+s)(6+s)(7+s)(8+s)(9+s)(10+s)(11+s)(12+s)(13+s)(14+s)(15+s)(16+s)(17+s)(18+s)\right)\Big/802857662698291200000\right)$$

```
pm[n_, s_] := (fn[n, s] HarmonicNumber[n, s] - fn[n, 1 - s] HarmonicNumber[n, 1 - s]) /
  (fn[n, s] - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] fn[n, 1 - s])
pmx[n_, s_] := (1 - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] fn[n, 1 - s] / fn[n, s]) ^-1
   HarmonicNumber[n, s] -
  (fn[n, s] / fn[n, 1 - s] - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] ) ^-1
   HarmonicNumber[n, 1 - s]
pmo[n_, s_] := (fn[n, s] HarmonicNumber[n, s] - fn[n, 1 - s] HarmonicNumber[n, 1 - s])
fn2[n_, s_] := (1 - s) n^s
pm2[n_, s_] := (fn2[n, s] HarmonicNumber[n, s] - fn2[n, 1 - s] HarmonicNumber[n, 1 - s]) /
  (fn2[n, s] - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] fn2[n, 1 - s])
pmo2[n_, s_] := (fn2[n, s] HarmonicNumber[n, s] - fn2[n, 1 - s] HarmonicNumber[n, 1 - s])
```

```
pmx[10, .7 + 3 I]
```

0.571252 − 0.0923229 i

```
Zeta[.7 + 3 I]

0.571252 - 0.0923229 i

pm[10 000, s] - Zeta[s] /. s → N@ZetaZero@10 000 + .5 I

-3.76419 × 10^-11 + 3.47209 × 10^-11 i

pm[4, s] /. s → .7

-2.77839

Zeta[.7]

-2.77839

HarmonicNumber[4, s] /. s → .5

2.78446


pmo[1000, N@ZetaZero@100 + .1 I]

0. - 0.666732 i

bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
bern[k_] := If[k == 1, 1 / 2, BernoulliB[k]]
bs[n_, s_, t_] := 1 / (1 - s) Sum[FullSimplify[ bin[1 - s, k] bern[k] n^(1 - s - k)], {k, 0, t}]


fna[n_, s_] := (1 - s) n^s
pma[n_, s_] := (fna[n, s] HarmonicNumber[n, s] - fna[n, 1 - s] HarmonicNumber[n, 1 - s]) /
  (fna[n, s] - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] fna[n, 1 - s])
pma2[n_, s_] := {fna[n, s] HarmonicNumber[n, s], -fna[n, 1 - s] HarmonicNumber[n, 1 - s],
  fna[n, s], -Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] fna[n, 1 - s]}
pmx[n_, s_] := (1 - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] fna[n, 1 - s] / fn[n, s])^-1
   HarmonicNumber[n, s] -
  (fna[n, s] / fn[n, 1 - s] - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2])^-1
   HarmonicNumber[n, 1 - s]
pmx2[n_, s_] := {(1 - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] fna[n, 1 - s] / fn[n, s])^-1,
  HarmonicNumber[n, s],
  -(fna[n, s] / fn[n, 1 - s] - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2])^-1,
  HarmonicNumber[n, 1 - s]}


Chop@pma2[70 000 000 000, N@ZetaZero@1]

{7. × 10^10 - 7.06501 i, -7. × 10^10 - 7.06501 i,
 3.40939 × 10^6 - 1.54236 × 10^6 i, 3.71979 × 10^6 + 407 400. i}

pma[700, N@ZetaZero@1]

0.0451776 - 0.283781 i

Zeta[.7]

-2.77839
```

$$\text{fn}[n\_,\, s\_] := 1\Big/\Bigg(\frac{n^{-s}}{2}+\frac{n^{1-s}}{1-s}-\frac{1}{12}n^{-1-s}s+\frac{1}{720}n^{-3-s}s(1+s)(2+s)-\frac{n^{-5-s}s(1+s)(2+s)(3+s)(4+s)}{30\,240}+\frac{n^{-7-s}s(1+s)(2+s)(3+s)(4+s)(5+s)(6+s)}{1\,209\,600}-\frac{n^{-9-s}s(1+s)(2+s)(3+s)(4+s)(5+s)(6+s)(7+s)(8+s)}{47\,900\,160}+\frac{1}{1\,307\,674\,368\,000}691\,n^{-11-s}s(1+s)(2+s)(3+s)(4+s)(5+s)(6+s)(7+s)(8+s)(9+s)(10+s)-\frac{1}{74\,724\,249\,600}n^{-13-s}s(1+s)(2+s)(3+s)(4+s)(5+s)(6+s)(7+s)(8+s)(9+s)(10+s)(11+s)(12+s)+\Big(3617\,n^{-15-s}s(1+s)(2+s)(3+s)(4+s)(5+s)(6+s)(7+s)(8+s)(9+s)(10+s)(11+s)(12+s)(13+s)(14+s)\Big)\Big/10\,670\,622\,842\,880\,000-\Big(43\,867\,n^{-17-s}s(1+s)(2+s)(3+s)(4+s)(5+s)(6+s)(7+s)(8+s)(9+s)(10+s)(11+s)(12+s)(13+s)(14+s)(15+s)(16+s)\Big)\Big/5\,109\,094\,217\,170\,944\,000+\Big(174\,611\,n^{-19-s}s(1+s)(2+s)(3+s)(4+s)(5+s)(6+s)(7+s)(8+s)(9+s)(10+s)(11+s)(12+s)(13+s)(14+s)(15+s)(16+s)(17+s)(18+s)\Big)\Big/802\,857\,662\,698\,291\,200\,000\Bigg)$$

```
pmxo[n_, s_] := (fn[n, s] HarmonicNumber[n, s] - fn[n, 1 - s] HarmonicNumber[n, 1 - s]) /
  (fn[n, s] - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] fn[n, 1 - s])
pmxa[n_, s_] := (1 - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] fn[n, 1 - s] / fn[n, s]) ^-1
   HarmonicNumber[n, s] -
  (fn[n, s] / fn[n, 1 - s] - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] ) ^-1
   HarmonicNumber[n, 1 - s]
pmxa2[n_, s_] := (1 - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] fn[n, 1 - s] / fn[n, s]) ^-1
   (Zeta[s] - Zeta[s, n + 1]) -
  (fn[n, s] / fn[n, 1 - s] - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] ) ^-1
   (Zeta[1 - s] - Zeta[1 - s, n + 1])
pmxa3[n_, s_] := {(1 - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] fn[n, 1 - s] / fn[n, s]) ^-1
   (Zeta[s]), -
    (1 - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] fn[n, 1 - s] / fn[n, s]) ^-1
   (Zeta[s, n + 1]), -
    (fn[n, s] / fn[n, 1 - s] - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] ) ^-1 (Zeta[1 - s]), +
    (fn[n, s] / fn[n, 1 - s] - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] ) ^-1
   (Zeta[1 - s, n + 1])}
pmxa4[n_, s_] := (1 - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] fn[n, 1 - s] / fn[n, s]) ^-1
   (Zeta[s]) -
  (fn[n, s] / fn[n, 1 - s] - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] ) ^-1 (Zeta[1 - s])
pmxa4a[n_, s_] := {(1 - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] fn[n, 1 - s] / fn[n, s]) ^-1,
  (Zeta[s]), -(fn[n, s] / fn[n, 1 - s] - Pi^(1 / 2 - s) Gamma[s / 2] / Gamma[(1 - s) / 2] ) ^-1,
  (Zeta[1 - s])}
```

```
pmxa3[5, .7]
```

```
{-4.51131, 9.02863, 1.73292, -9.02863}
```

```
Zeta[.9]
```

```
-9.43011
```

```
pmxa4[I, .9]
```

$-9.43011 + 5.86198 \times 10^{-14}\, i$

```
pmxa4a[I, .9]
```

{0.129863 - 0.892596 i, -9.43011, 13.6069 + 13.9581 i, -0.603038}

```
FullSimplify[((a[s] - b[s])^-1) / ((1 - b[s] / a[s])^-1 - 1)]
```

$\frac{1}{b[s]}$

```
pmxr[n_, s_] := (1 - Pi^(1/2 - s) Gamma[s/2] / Gamma[(1 - s) / 2] fn[n, 1 - s] / fn[n, s])^-1
    (Zeta[s] - Zeta[s, n + 1]) -
   (fn[n, s] / fn[n, 1 - s] - Pi^(1/2 - s) Gamma[s/2] / Gamma[(1 - s) / 2])^-1
    (Zeta[1 - s] - Zeta[1 - s, n + 1])
pmxa4[n_, s_] := (1 - Pi^(1/2 - s) Gamma[s/2] / Gamma[(1 - s) / 2] fn[n, 1 - s] / fn[n, s])^-1
    (Zeta[s]) -
   (fn[n, s] / fn[n, 1 - s] - Pi^(1/2 - s) Gamma[s/2] / Gamma[(1 - s) / 2])^-1 (Zeta[1 - s])
pmxa4s[n_, s_] := (1 - Pi^(1/2 - s) Gamma[s/2] / Gamma[(1 - s) / 2])^-1 (Zeta[s]) -
   (1 - Pi^(1/2 - s) Gamma[s/2] / Gamma[(1 - s) / 2])^-1 (Zeta[1 - s])
pmxa5[n_, s_] := -(1 - Pi^(1/2 - s) Gamma[s/2] / Gamma[(1 - s) / 2] fn[n, 1 - s] / fn[n, s])^-1
    (Zeta[s, n + 1]) +
   (fn[n, s] / fn[n, 1 - s] - Pi^(1/2 - s) Gamma[s/2] / Gamma[(1 - s) / 2])^-1 (Zeta[1 - s, n + 1])
pmxa5a[n_, s_] := {-(1 - Pi^(1/2 - s) Gamma[s/2] / Gamma[(1 - s) / 2] fn[n, 1 - s] / fn[n, s])^-1
    (Zeta[s, n + 1]),
   (fn[n, s] / fn[n, 1 - s] - Pi^(1/2 - s) Gamma[s/2] / Gamma[(1 - s) / 2])^-1 (Zeta[1 - s, n + 1])}
dis[n_, s_] := HarmonicNumber[n, s] - 1 / fn[n, s]
```

```
pmxa5a[60, -1.0]
```

{-0.000837346, 0.000837346}

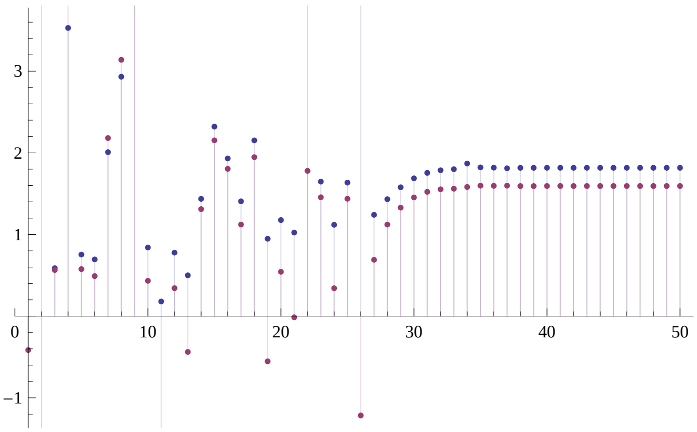
```
Zeta[.3 + 12 I]
```

1.04409 - 0.857521 i

```
pmxr[10, .3 + 12 I]
```

1.04409 - 0.857521 i

```
DiscretePlot[{Re@pmxr[n, .6 + 190 I], Im@pmxr[n, .6 + 190 I]}, {n, 1, 50}]
```






```
DiscretePlot[{Re@pmxr[n, .6 + 190 I], Im@pmxr[n, .6 + 190 I]}, {n, 1, 50}]
```