```
bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
Clear[dd]
dd[n_, s_, y_, k_] := dd[n, s, y, k] = Sum[ j^-s dd[Floor[n / j], s, y, k - 1], {j, y, n}]
dd[n_, s_, y_, 0] := UnitStep[n - 1]
dz[n_, s_, y_, z_] := Sum[ bin[z, k] dd[n, s, y, k], {k, 0, Log[y, n]}]
dc[n_, y_, z_] := If[ n < y, 1, Sum[ bin[z, k] dc[n / y^k , y + 1, z - k], {k, 0, Log[y, n]}]]
ds[n_, s_, y_, z_] :=
 If[ n < y, 1, Sum[ bin[z, k] (y^(-s k)) ds[n / y^k , s, y + 1, z - k], {k, 0, Log[y, n]}]]
dt[n_, s_, y_, z_] := If[ n < y - 1, 1,
  Sum[ (-1)^k bin[z, k] ((y - 1)^(-s k)) dz[n / (y - 1)^k , s, y - 1, z - k], {k, 0, Log[y - 1, n]}]]

Expand@dc[100, 2, z]
```

$$1 + \frac{428\,z}{15} + \frac{16\,289\,z^2}{360} + \frac{331\,z^3}{16} + \frac{611\,z^4}{144} + \frac{67\,z^5}{240} + \frac{7\,z^6}{720}$$

```
binomial[z_, k_] := binomial[z, k] = Product[z - j, {j, 0, k - 1}] / k!
Ds[n_, 0, s_, a_] := UnitStep[n - 1]
Ds[n_, 1, s_, a_] := Ds[n, 1, s, a] = HarmonicNumber[Floor[n], s] - HarmonicNumber[a, s]
Ds[n_, 2, s_, a_] := Ds[n, 2, s, a] =
  Sum[(m^(-2 s)) + 2 (m^-s) (Ds[Floor[n / m], 1, s, m]), {m, a + 1, Floor[n^(1 / 2)]}]
Ds[n_, k_, s_, a_] := Ds[n, k, s, a] =
  Sum[(m^(-s k)) + k (m^(-s (k - 1))) Ds[Floor[n / (m^(k - 1))], 1, s, m] +
    Sum[binomial[k, j] (m^-s)^j Ds[Floor[n / (m^j)], k - j, s, m], {j, 1, k - 2}],
   {m, a + 1, Floor[n^(1 / k)]}]

Dnsz[n_, s_, 1] := Expand@Sum[binomial[1, k] Ds[n, k, s, 1], {k, 0, 1}]
Dnsz[n_, s_, z_] := Expand@Sum[binomial[z, k] Ds[n, k, s, 1], {k, 0, Log2@n}]

Dnsz[100, -1, z]
```

$$1 + \frac{69\,389\,z}{60} + \frac{170\,473\,z^2}{72} + \frac{14\,521\,z^3}{12} + \frac{21\,091\,z^4}{72} + \frac{314\,z^5}{15} + \frac{8\,z^6}{9}$$

```
Expand@ds[100, -2, 3, z]
```

$$1 + \frac{654\,409\,z}{12} + \frac{1\,716\,153\,z^2}{8} + \frac{828\,221\,z^3}{12} + \frac{2187\,z^4}{8}$$

```
Expand@dt[100, -2, 3, z]
```

$$1 + \frac{654\,409\,z}{12} + \frac{1\,716\,153\,z^2}{8} + \frac{828\,221\,z^3}{12} + \frac{2187\,z^4}{8}$$

```
Expand@dz[100, -2, 3, z]
```

$$1 + \frac{654\,409\,z}{12} + \frac{1\,716\,153\,z^2}{8} + \frac{828\,221\,z^3}{12} + \frac{2187\,z^4}{8}$$

```
binomial[z_, k_] := binomial[z, k] = Product[z - j, {j, 0, k - 1}] / k!
Ds[n_, 0, s_, a_] := UnitStep[n - 1]
Ds[n_, 1, s_, a_] := Ds[n, 1, s, a] = HarmonicNumber[Floor[n], s] - HarmonicNumber[a, s]
Ds[n_, 2, s_, a_] := Ds[n, 2, s, a] =
  Sum[(m^(-2 s)) + 2 (m^-s) (Ds[Floor[n / m], 1, s, m]), {m, a + 1, Floor[n^(1 / 2)]}]
Ds[n_, k_, s_, a_] := Ds[n, k, s, a] =
  Sum[(m^(-s k)) + k (m^(-s (k - 1))) Ds[Floor[n / (m^(k - 1))], 1, s, m] +
    Sum[binomial[k, j] (m^-s)^j Ds[Floor[n / (m^j)], k - j, s, m], {j, 1, k - 2}],
   {m, a + 1, Floor[n^(1 / k)]}]
Dnsz[n_, s_, z_] := Expand@Sum[binomial[z, k] Ds[n, k, s, 1], {k, 0, Log2@n}]

Dns112z[n_, s_, z_] :=
 Expand@Sum[(-1)^j binomial[z, j] 2^(j (1 - s)) Dnsz[n / (2^j), s, z], {j, 0, Log2@n}]

es[n_, s_, y_, z_] := If[ n < y, 1,
  Sum[ bin[z, k] (y^(-s k)) (-1)^((y + 1) k) es[n / y^k , s, y + 1, z - k], {k, 0, Log[y, n]}]]
Dns112z[100, -1, z]
```

$$1+\frac{10\,301\,z}{60}-\frac{235\,459\,z^2}{360}+\frac{2363\,z^3}{4}-\frac{12\,797\,z^4}{72}+\frac{286\,z^5}{15}-\frac{32\,z^6}{45}$$

```
Expand@es[100, -1, 2, z]
```

$$1+\frac{10\,301\,z}{60}-\frac{235\,459\,z^2}{360}+\frac{2363\,z^3}{4}-\frac{12\,797\,z^4}{72}+\frac{286\,z^5}{15}-\frac{32\,z^6}{45}$$

```
Clear[dz]
bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
FI[n_] := FactorInteger[n]; FI[1] := {}
dz[n_, z_] := dz[n, z] = Product[(-1)^p[[2]] bin[-z, p[[2]]], {p, FI[n]}]
dsz[n_, s_, z_] := dz[n, z] n^-s

lz[n_, s_, z_] := Sum[ dsz[j, 2 s, z] dsz[k, s, -z], {j, 1, n^(1 / 2)}, {k, 1, (n / (j^2))}]
ls[n_, s_, y_, z_] := If[ n < y, 1, Sum[
    bin[z, k] (y^(-s k)) LiouvilleLambda[y]^k ls[n / y^k , s, y + 1, z - k], {k, 0, Log[y, n]}]]
Expand[lz[100, -1, z]]
```

$$1-\frac{19\,993\,z}{20}+\frac{99\,217\,z^2}{72}-\frac{4915\,z^3}{12}+\frac{12\,643\,z^4}{72}-\frac{34\,z^5}{15}+\frac{8\,z^6}{9}$$

```
Expand@ls[100, -1, 2, z]
```

$$1-\frac{19\,993\,z}{20}+\frac{99\,217\,z^2}{72}-\frac{4915\,z^3}{12}+\frac{12\,643\,z^4}{72}-\frac{34\,z^5}{15}+\frac{8\,z^6}{9}$$

```
ds[n_, s_, y_, z_] :=
 If[ n < y, 1, Sum[ bin[z, k] (y^(-s k)) ds[n / y^k , s, y + 1, z - k], {k, 0, Log[y, n]}]]
dss[n_, s_, y_, z_, x_] :=
 If[n < y, 1, Sum[bin[z, k] (x y^-s)^k dss[n / y^k, s, y + x, z - k, x], {k, 0, Log[y, n]}]]
dsp[n_, s_, y_, z_, x_] := dss[n, s, y, z, x] - dss[n - x^Log[ 1 + x, n], s, y, z, x]


binomial[z_, k_] := binomial[z, k] = Product[z - j, {j, 0, k - 1}] / k!
Ds[n_, 0, s_, a_] := UnitStep[n - 1]
Ds[n_, 1, s_, a_] := Ds[n, 1, s, a] = HarmonicNumber[Floor[n], s] - HarmonicNumber[a, s]
Ds[n_, 2, s_, a_] := Ds[n, 2, s, a] =
  Sum[(m^(-2 s)) + 2 (m^-s) (Ds[Floor[n / m], 1, s, m]), {m, a + 1, Floor[n^(1 / 2)]}]
Ds[n_, k_, s_, a_] := Ds[n, k, s, a] =
  Sum[(m^(-s k)) + k (m^(-s (k - 1))) Ds[Floor[n / (m^(k - 1))], 1, s, m] +
    Sum[binomial[k, j] (m^-s)^j Ds[Floor[n / (m^j)], k - j, s, m], {j, 1, k - 2}],
   {m, a + 1, Floor[n^(1 / k)]}]

Ddy[n_, s_, y_, k_] := y^(k (s - 1)) Ds[n y^k, k, s, y]

Dnsyz[n_, s_, y_, z_] := Expand@Sum[binomial[z, k] Ddy[n, s, y, k], {k, 0, Log[(y + 1) / y, n]}]


Expand@dss[100, 0, 2, z, 1]
```

$$1 + \frac{428\,z}{15} + \frac{16\,289\,z^2}{360} + \frac{331\,z^3}{16} + \frac{611\,z^4}{144} + \frac{67\,z^5}{240} + \frac{7\,z^6}{720}$$

```
Expand[dss[100, 0, 1 + 1 / 2, z, 1 / 2]]
```

$$1 + \frac{202\,986\,703\,z}{7\,096\,320} + \frac{68\,602\,319\,z^2}{1\,612\,800} + \frac{622\,902\,011\,z^3}{29\,030\,400} + \frac{2\,091\,660\,979\,z^4}{371\,589\,120} + \frac{52\,801\,531\,z^5}{74\,317\,824} + \frac{21\,461\,041\,z^6}{353\,894\,400} + \frac{5\,689\,681\,z^7}{2\,477\,260\,800} + \frac{16\,259\,z^8}{247\,726\,080} + \frac{739\,z^9}{743\,178\,240} + \frac{37\,z^{10}}{7\,431\,782\,400} + \frac{z^{11}}{81\,749\,606\,400}$$

```
Dnsyz[100, 0, 2, z]
```

$$1 + \frac{202\,986\,703\,z}{7\,096\,320} + \frac{68\,602\,319\,z^2}{1\,612\,800} + \frac{622\,902\,011\,z^3}{29\,030\,400} + \frac{2\,091\,660\,979\,z^4}{371\,589\,120} + \frac{52\,801\,531\,z^5}{74\,317\,824} + \frac{21\,461\,041\,z^6}{353\,894\,400} + \frac{5\,689\,681\,z^7}{2\,477\,260\,800} + \frac{16\,259\,z^8}{247\,726\,080} + \frac{739\,z^9}{743\,178\,240} + \frac{37\,z^{10}}{7\,431\,782\,400} + \frac{z^{11}}{81\,749\,606\,400}$$

```
N@Log[ 1 + 1 / 2, 30]
```

```
8.38838
```

```
Expand@dsp[27 × 16, 0, 1 + 1 / 2, z, 1 / 2]
```

$$\frac{3845\,z^2}{43\,008} + \frac{99\,413\,z^3}{516\,096} + \frac{57\,703\,z^4}{387\,072} + \frac{82\,669\,z^5}{1\,474\,560} + \frac{5623\,z^6}{491\,520} + \frac{193\,z^7}{147\,456} + \frac{85\,z^8}{1\,032\,192} + \frac{3\,z^9}{1\,146\,880} + \frac{z^{10}}{30\,965\,760}$$

```
Expand@(dsp[16, 0, 1 + 1 / 2, z, 1 / 2] dsp[27, 0, 1 + 1 / 2, z, 1 / 2])
```

$$\frac{55\,z^2}{2048} + \frac{5411\,z^3}{73\,728} + \frac{17\,789\,z^4}{221\,184} + \frac{14\,083\,z^5}{294\,912} + \frac{5075\,z^6}{294\,912} + \frac{563\,z^7}{147\,456} + \frac{25\,z^8}{49\,152} + \frac{11\,z^9}{294\,912} + \frac{z^{10}}{884\,736}$$