```
g[f_, k_] := Sum[Binomial[k, j] f[j], {j, 0, k}]
h[f_, k_] := Sum[(-1)^(k - j) Binomial[k, j] g[f, j], {j, 0, k}]
pl[n_] := 1 / (n + 1)

Table[g[pl, k], {k, 0, 20}]
Table[h[pl, k], {k, 0, 20}]
```

$$\left\{1, \frac{3}{2}, \frac{7}{3}, \frac{15}{4}, \frac{31}{5}, \frac{21}{2}, \frac{127}{7}, \frac{255}{8}, \frac{511}{9}, \frac{1023}{10}, \frac{2047}{11}, \frac{1365}{4}, \frac{8191}{13}, \frac{16383}{14}, \frac{32767}{15}, \frac{65535}{16}, \frac{131071}{17}, \frac{29127}{2}, \frac{524287}{19}, \frac{209715}{4}, \frac{299593}{3}\right\}$$

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \frac{1}{13}, \frac{1}{14}, \frac{1}{15}, \frac{1}{16}, \frac{1}{17}, \frac{1}{18}, \frac{1}{19}, \frac{1}{20}, \frac{1}{21}\right\}$$

```
bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
D2[n_, 0] := UnitStep[n - 1]
D2[n_, k_] := D2[n, k] = Sum[D2[Floor[n / j], k - 1], {j, 2, n}]
Dz[n_, z_] := Sum[(-1)^k bin[z, k] D2[n, k], {k, 0, Log2@n}]

Expand@Dz[100, z]
```

$$1 - \frac{6088\, z}{15} + \frac{148229\, z^2}{360} - \frac{1873\, z^3}{16} + \frac{1835\, z^4}{144} - \frac{137\, z^5}{240} + \frac{7\, z^6}{720}$$

```
Integrate[D[t^z, t] f[x], {t, 0, x}]
```

ConditionalExpression$[x^z f[x], \mathrm{Re}[z] > 0]$

```
D[t^z, t] /. z → -1
```

$$-\frac{1}{t^2}$$

Integrate$\left[t^{-1+z}\, z\, f[x], \{t, 0, x\}\right]$

ConditionalExpression$[x^z f[x], \mathrm{Re}[z] > 0]$

```
ff[n_] := 1
iff[n_] := n
g[f_, x_, k_] := Integrate[ t^(k - 1) / (k - 1)! f[x - t], {t, 0, x}]

g[iff, n, -1 / 2]
```

Integrate::idiv : Integral of $-\frac{n}{2\sqrt{\pi}\, t^{3/2}} + \frac{1}{2\sqrt{\pi}\,\sqrt{t}}$ does not converge on $\{0, n\}$. ≫

$$\int_0^n -\frac{n - t}{2\sqrt{\pi}\, t^{3/2}}\, dt$$

```
ml[n_] := 1
vs[n_] := 1; ws[n_] := (-1) ^ (n + 1)
vs2[n_] := -1; ws2[n_] := -1
vs3[n_] := (-1) ^ (n + 1) / n; ws3[n_] := 1 / n!

Clear[F, ff, g, G, h, F2]
F[f_, n_, 0] := UnitStep[n - 1]
F[f_, n_, k_] := F[f, n, k] = Sum[f[j] F[f, Floor[n / j], k - 1], {j, 2, n}]
ff[f_, n_, k_] := F[f, n, k] - F[f, n - 1, k]
g[f_, v_, n_] := g[f, v, n] = Sum[v[k] ff[f, n, k], {k, 1, Log2@n}]
G[f_, v_, n_, 0] := UnitStep[n - 1]
G[f_, v_, n_, k_] := G[f, v, n, k] = Sum[g[f, v, j] G[f, v, Floor[n / j], k - 1], {j, 2, n}]
gg[f_, v_, n_, k_] := G[f, v, n, k] - G[f, v, n - 1, k]
h[f_, v_, w_, n_] := h[f, v, w, n] = Sum[w[k] gg[f, v, n, k], {k, 1, Log2@n}]
F2[f_, v_, w_, n_, 0] := UnitStep[n - 1]
F2[f_, v_, w_, n_, k_] :=
 F2[f, v, w, n, k] = Sum[h[f, v, w, j] F2[f, v, w, Floor[n / j], k - 1], {j, 2, n}]
```

```
Table[g[ml, vs, n], {n, 1, 20}]
```

{1, 2, 3, 8, 5, 18, 7, 32, 18, 30, 11, 96, 13, 42, 45, 128, 17, 144, 19, 160}

```
Table[h[ml, vs, ws, n], {n, 1, 20}]
```

{-1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20}

```
G[ml, vs3, 100, 1]
```

$\frac{428}{15}$

```
F2[ml, vs3, ws3, 100, 1]
```

99

```
F[ml, 100, 1]
```

99

```
ml[n_] := n
vs[n_] := 1; ws[n_] := (-1)^(n+1)
vs2[n_] := -1; ws2[n_] := -1
vs3[n_] := (-1)^(n+1) / n; ws3[n_] := 1 / n!

Clear[aF, aff, ag, aG, ah, aF2]
aF[f_, n_, 0] := UnitStep[n]
aF[f_, n_, k_] := aF[f, n, k] = Sum[f[j] aF[f, n-j, k-1], {j, 1, n}]
aff[f_, n_, k_] := aF[f, n, k] - aF[f, n-1, k]
ag[f_, v_, n_] := ag[f, v, n] = Sum[v[k] aff[f, n, k], {k, 1, n}]
aG[f_, v_, n_, 0] := UnitStep[n]
aG[f_, v_, n_, k_] := aG[f, v, n, k] = Sum[ag[f, v, j] aG[f, v, n-j, k-1], {j, 1, n}]
agg[f_, v_, n_, k_] := aG[f, v, n, k] - aG[f, v, n-1, k]
ah[f_, v_, w_, n_] := ah[f, v, w, n] = Sum[w[k] agg[f, v, n, k], {k, 1, n}]
aF2[f_, v_, w_, n_, 0] := UnitStep[n]
aF2[f_, v_, w_, n_, k_] :=
 aF2[f, v, w, n, k] = Sum[ah[f, v, w, j] aF2[f, v, w, n-j, k-1], {j, 1, n}]
```

```
aF2[ml, vs, ws, 10, 1]
```

55

```
aF[ml, 10, 1]
```

55

```
aG[ml, vs2, 10, 1]
```

-10 945

```
bs[n_] := Binomial[2, n]
al[f_, 1, k_] := f[k]
al[f_, n_, k_] := Sum[al[f, 1, j] al[f, n-1, k-j+1], {j, 1, k}]
b[f_, 1] := 1 / al[f, 1, 1]
b[f_, k_] := -1 / al[f, k, 1] Sum[b[f, j] al[f, j, k-j+1], {j, 1, k-1}]
```

```
bo2[n_] := Binomial[2, n]
{Table[b[bo2, n], {n, 1, 10}], Table[D[(1+x)^(1/2), {x, k}] / k! /. x → 0, {k, 1, 10}]}
```

$$\left\{\left\{\frac{1}{2}, -\frac{1}{8}, \frac{1}{16}, -\frac{5}{128}, \frac{7}{256}, -\frac{21}{1024}, \frac{33}{2048}, -\frac{429}{32\,768}, \frac{715}{65\,536}, -\frac{2431}{262\,144}\right\}, \left\{\frac{1}{2}, -\frac{1}{8}, \frac{1}{16}, -\frac{5}{128}, \frac{7}{256}, -\frac{21}{1024}, \frac{33}{2048}, -\frac{429}{32\,768}, \frac{715}{65\,536}, -\frac{2431}{262\,144}\right\}\right\}$$

```
bo3[n_] := Binomial[3, n]
{Table[b[bo3, n], {n, 1, 10}], Table[D[(1+x)^(1/3), {x, k}] / k! /. x → 0, {k, 1, 10}]}
```

$$\left\{\left\{\frac{1}{3}, -\frac{1}{9}, \frac{5}{81}, -\frac{10}{243}, \frac{22}{729}, -\frac{154}{6561}, \frac{374}{19\,683}, -\frac{935}{59\,049}, \frac{21\,505}{1\,594\,323}, -\frac{55\,913}{4\,782\,969}\right\}, \left\{\frac{1}{3}, -\frac{1}{9}, \frac{5}{81}, -\frac{10}{243}, \frac{22}{729}, -\frac{154}{6561}, \frac{374}{19\,683}, -\frac{935}{59\,049}, \frac{21\,505}{1\,594\,323}, -\frac{55\,913}{4\,782\,969}\right\}\right\}$$

```
bo32[n_] := Binomial[3.2, n]
{Table[b[bo32, n], {n, 1, 10}], Table[D[(1 + x)^(1 / (3.2)), {x, k}] / k! /. x → 0, {k, 1, 10}]}
```

{{0.3125, -0.107422, 0.0604248, -0.0405979, 0.029941, -0.0233914, 0.0190055, -0.0158874, 0.0135705, -0.0117894}, {0.3125, -0.107422, 0.0604248, -0.0405979, 0.029941, -0.0233914, 0.0190055, -0.0158874, 0.0135705, -0.0117894}}

```
bn2[n_] := Pochhammer[2, n] / n!
{Table[b[bn2, n], {n, 1, 10}], Table[D[-1 / (1 + x)^(1 / 2), {x, k}] / k! /. x → 0, {k, 1, 10}]}
```

$$\left\{\left\{\frac{1}{2}, -\frac{3}{8}, \frac{5}{16}, -\frac{35}{128}, \frac{63}{256}, -\frac{231}{1024}, \frac{429}{2048}, -\frac{6435}{32\,768}, \frac{12\,155}{65\,536}, -\frac{46\,189}{262\,144}\right\}, \left\{\frac{1}{2}, -\frac{3}{8}, \frac{5}{16}, -\frac{35}{128}, \frac{63}{256}, -\frac{231}{1024}, \frac{429}{2048}, -\frac{6435}{32\,768}, \frac{12\,155}{65\,536}, -\frac{46\,189}{262\,144}\right\}\right\}$$

```
bn3[n_] := Pochhammer[3, n] / n!
{Table[b[bn3, n], {n, 1, 10}], Table[D[-1 / (1 + x)^(1 / 3), {x, k}] / k! /. x → 0, {k, 1, 10}]}
```

$$\left\{\left\{\frac{1}{3}, -\frac{2}{9}, \frac{14}{81}, -\frac{35}{243}, \frac{91}{729}, -\frac{728}{6561}, \frac{1976}{19\,683}, -\frac{5434}{59\,049}, \frac{135\,850}{1\,594\,323}, -\frac{380\,380}{4\,782\,969}\right\}, \left\{\frac{1}{3}, -\frac{2}{9}, \frac{14}{81}, -\frac{35}{243}, \frac{91}{729}, -\frac{728}{6561}, \frac{1976}{19\,683}, -\frac{5434}{59\,049}, \frac{135\,850}{1\,594\,323}, -\frac{380\,380}{4\,782\,969}\right\}\right\}$$

```
Sum[x^k, {k, 0, Infinity}]
```

$$\frac{1}{1-x}$$

```
Table[(-1)^k Binomial[-1, k], {k, 0, 10}]
```

{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

```
Sum[ Pochhammer[z, k] / k! x^k, {k, 0, Infinity}]
```

$(1-x)^{-z}$

```
Sum[ Pochhammer[z, k] / k! Binomial[x, k], {k, 0, Infinity}]
```

Hypergeometric2F1[-x, z, 1, -1]

```
Sum[ Pochhammer[z, k] / k! x^k / k!, {k, 0, Infinity}]
```

Hypergeometric1F1[z, 1, x]

```
bin[z_, k_] := Product[z - j, {j, 0, k - 1}] / k!
D2[n_, 0] := UnitStep[n - 1]
D2[n_, k_] := D2[n, k] = Sum[D2[Floor[n / j], k - 1], {j, 2, n}]
Dz[n_, z_] := Sum[ bin[z, k] D2[n, k], {k, 0, Log2@n}]
Cz[n_, z_] := Sum[Pochhammer[z, k] / k! D2[n, k], {k, 0, Log2@n}]
cz[n_, z_] := cz[n, z] = Cz[n, z] - Cz[n - 1, z]
C2z[n_, z_] := Sum[Pochhammer[z, k] / k! D2[n, 2 k], {k, 0, Log2@n}]
c2z[n_, z_] := c2z[n, z] = C2z[n, z] - C2z[n - 1, z]
C3z[n_, z_] := Sum[Pochhammer[z, k] / k! D2[n, 3 k], {k, 0, Log2@n}]
c3z[n_, z_] := c3z[n, z] = C3z[n, z] - C3z[n - 1, z]
```

```
Expand@Dz[100, z]
```

$$1 + \frac{428\,z}{15} + \frac{16\,289\,z^2}{360} + \frac{331\,z^3}{16} + \frac{611\,z^4}{144} + \frac{67\,z^5}{240} + \frac{7\,z^6}{720}$$

```
Expand@Cz[100, z]
```

$$1+\frac{6088\,z}{15}+\frac{148\,229\,z^2}{360}+\frac{1873\,z^3}{16}+\frac{1835\,z^4}{144}+\frac{137\,z^5}{240}+\frac{7\,z^6}{720}$$

```
Expand[C2z[100, z]]
```

$$1+\frac{1132\,z}{3}+\frac{191\,z^2}{2}+\frac{7\,z^3}{6}$$

```
Expand@C2z[100, z]
```

$$1+\frac{1132\,z}{3}+\frac{191\,z^2}{2}+\frac{7\,z^3}{6}$$

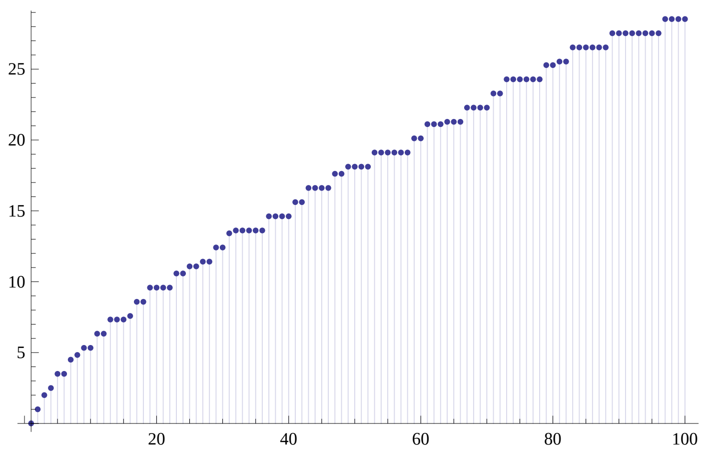
```
Table[FullSimplify@cz[n, z], {n, 1, 10}] // TableForm
Expand@Sum[cz[j, -z] c2z[k, z], {j, 1, 100}, {k, 1, 100 / j}]
```

$$1-\frac{428\,z}{15}+\frac{16\,289\,z^2}{360}-\frac{331\,z^3}{16}+\frac{611\,z^4}{144}-\frac{67\,z^5}{240}+\frac{7\,z^6}{720}$$

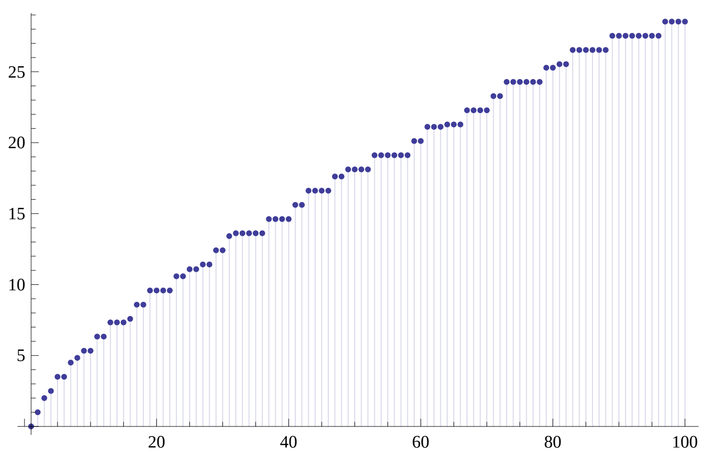
```
D[C2z[100, -z] + Cz[100, z], z] /. z → 0
```

$$\frac{428}{15}$$

```
DiscretePlot[D[Sum[cz[j, z] c2z[k, -z], {j, 1, n}, {k, 1, n / j}], z] /. z → 0, {n, 1, 100}]
```




```
DiscretePlot[D[Cz[n, z] - C2z[n, z], z] /. z → 0, {n, 1, 100}]
```



```
Daz[n_, z_] := Sum[bin[z, k] bin[n, k], {k, 0, n}]
Caz[n_, z_] := Sum[Pochhammer[z, k] / k ! bin[n, k], {k, 0, n}]
caz[n_, z_] := caz[n, z] = Caz[n, z] - Caz[n - 1, z]
Ca2z[n_, z_] := Sum[Pochhammer[z, k] / k ! bin[n, 2 k], {k, 0, n}]
ca2z[n_, z_] := ca2z[n, z] = Ca2z[n, z] - C2z[n - 1, z]
```

```
Table[D[Expand@Caz[n, z], z] /. z → 0, {n, 1, 10}]
```

$$\left\{1, \frac{5}{2}, \frac{29}{6}, \frac{103}{12}, \frac{887}{60}, \frac{1517}{60}, \frac{18239}{420}, \frac{63253}{840}, \frac{332839}{2520}, \frac{118127}{504}\right\}$$

```
FullSimplify[Sum[(2^k - 1) / k, {k, 1, n}]]
```

$-i\pi - \text{HarmonicNumber}[n] - 2^{1+n}\,\text{LerchPhi}[2, 1, 1+n]$

```
ap[n_] := Sum[HarmonicNumber[Floor[n / 2^k]], {k, 0, 10}]
```

```
Table[FullSimplify[-i π - HarmonicNumber[n] - 2^(1+n) LerchPhi[2, 1, 1 + n]], {n, 1, 10}]
```

$$\left\{1, \frac{5}{2}, \frac{29}{6}, \frac{103}{12}, \frac{887}{60}, \frac{1517}{60}, \frac{18239}{420}, \frac{63253}{840}, \frac{332839}{2520}, \frac{118127}{504}\right\}$$

```
Table[ap[n], {n, 1, 10}]
```

$$\left\{1, \frac{5}{2}, \frac{17}{6}, \frac{55}{12}, \frac{287}{60}, \frac{317}{60}, \frac{2279}{420}, \frac{6133}{840}, \frac{18679}{2520}, \frac{3887}{504}\right\}$$

```
Table[D[Expand@ca2z[n, z], z] /. z → 0, {n, 1, 10}]
```

$$\left\{0, 1, 3, \frac{13}{2}, \frac{23}{2}, \frac{131}{6}, \frac{227}{6}, \frac{835}{12}, \frac{497}{4}, \frac{4509}{20}\right\}$$

```
D[Cz[100, z], z] /. z → 0
```

$$\frac{6088}{15}$$

```
Product[(1 + x^(2^k)), {k, 0, Infinity}]
```

$$\frac{1}{1-x}$$

```
FullSimplify@Sum[Binomial[z, k] Binomial[x, k], {k, 0, Infinity}]
```

$$\frac{\text{Gamma}[1+x+z]}{\text{Gamma}[1+x]\,\text{Gamma}[1+z]}$$

```
FullSimplify[Sum[Pochhammer[z, k] / k ! Pochhammer[x, k] / k !, {k, 0, Infinity}]]
```

$$\frac{\text{Gamma}[1-x-z]}{\text{Gamma}[1-x]\,\text{Gamma}[1-z]}$$

```
Sum[Pochhammer[z, k] / k ! Pochhammer[x, k] / k !, {k, 0, Infinity}] /. x → 3.1 /. z → 7.2
```

0.000230246

```
(-3.1 - 7.2) ! / (-3.1) ! / (-7.2) !
```

0.000230246

```
FullSimplify[Sum[Pochhammer[z, k] / k ! Binomial[x, k], {k, 0, Infinity}]]
```

Hypergeometric2F1[-x, z, 1, -1]

```
FullSimplify[Hypergeometric2F1[-x, z, 1, -1] - (Hypergeometric2F1[-x, z, 1, -1] /. x → x - 1)]
```

```
-Hypergeometric2F1[1 - x, z, 1, -1] + Hypergeometric2F1[-x, z, 1, -1]
```

```
FullSimplify[Gamma[1 + x + z]/(Gamma[1 + x] Gamma[1 + z]) - (Gamma[1 + x + z]/(Gamma[1 + x] Gamma[1 + z]) /. x → x - 1)]
```

$$\frac{\text{Gamma}[x+z]}{\text{Gamma}[1+x]\ \text{Gamma}[z]}$$

```
FI[n_] := FactorInteger[n]; FI[1] := {}
dz[n_, z_] := Product[(-1)^p[[2]] Binomial[-z, p[[2]]], {p, FI[n]}]
ap1[z_, x_] := Gamma[x + z]/(Gamma[1 + x] Gamma[z])
dza[n_, z_] := Product[ap1[z, p[[2]]], {p, FI[n]}]
ap2[z_, x_] := -Hypergeometric2F1[1 - x, z, 1, -1] + Hypergeometric2F1[-x, z, 1, -1]
cza[n_, z_] := Product[ap2[z, p[[2]]], {p, FI[n]}]
Clear[D2]
D2[n_, 0] := UnitStep[n - 1]
D2[n_, k_] := D2[n, k] = Sum[D2[Floor[n / j], k - 1], {j, 2, n}]
Cz[n_, z_] := Sum[Pochhammer[z, k] / k ! D2[n, k], {k, 0, Log2@n}]
cz[n_, z_] := cz[n, z] = Cz[n, z] - Cz[n - 1, z]
Clear[Dza, dza]
Dza[n_, z_, a_] := Dza[n, z, a] = Sum[bin[z, k] D2[n, 2^a k], {k, 0, Log2@n}]
dza[n_, z_, a_] := dza[n, z, a] = Dza[n, z, a] - Dza[n - 1, z, a]
```

```
rr[n_, z_] := Sum[dza[a, z, 0] dza[b, z, 1] dza[c, z, 2] dza[d, z, 3] dza[e, z, 4] dza[f, z, 5],
  {a, 1, n}, {b, 1, n / a}, {c, 1, n / a / b}, {d, 1, n / a / b / c},
  {e, 1, n / a / b / c / d}, {f, 1, n / a / b / c / d / e}]
Expand@rr[10, z]
```

$$1+\frac{40\,z}{3}+\frac{9\,z^{2}}{2}+\frac{z^{3}}{6}$$

```
Expand@Cz[10, z]
```

$$1+\frac{40\,z}{3}+\frac{9\,z^{2}}{2}+\frac{z^{3}}{6}$$

```
Table[Expand@Dza[4100, z, k], {k, 0, 4}] // TableForm
```

$$1+\frac{16\,042\,379\,z}{27\,720}+\frac{144\,719\,059\,z^{2}}{103\,950}+\frac{474\,203\,123\,z^{3}}{362\,880}+\frac{962\,678\,897\,z^{4}}{1\,555\,200}+\frac{35\,600\,083\,z^{5}}{207\,360}+\frac{1\,203\,158\,941\,z^{6}}{43\,545\,600}+\frac{1\,357\,597\,z^{7}}{483\,840}+\frac{2\,392\,231\,z^{8}}{14\,515\,200}+\frac{1607\,z^{9}}{290\,304}$$

$$1-\frac{464\,483\,z}{60}+\frac{1\,820\,813\,z^{2}}{90}+\frac{640\,999\,z^{3}}{48}+\frac{101\,555\,z^{4}}{144}+\frac{299\,z^{5}}{80}+\frac{z^{6}}{720}$$

$$1+\frac{729\,965\,z}{6}+8912\,z^{2}+\frac{z^{3}}{6}$$

$$1+17\,825\,z$$

$$1$$

```
D[Cz[100, z] - C2z[100, z], z] /. z → 0
```

$$\frac{428}{15}$$

```
D[Sum[Dza[100, z, k], {k, 0, 5}], z] /. z → 0
```

$$\frac{6088}{15}$$

```
D[Cz[100, z], z] /. z → 0
```

$$\frac{6088}{15}$$

```
FullSimplify[Pochhammer[z + 1, x] / x !]
```

$$\frac{\text{Pochhammer}[1+z, x]}{x!}$$

```
(z + x) ! / z ! / x ! = Multiset[z + 1, x] = Multiset[x + 1, z]

mset[z_, x_] := Pochhammer[z, x] / x !
mset2[z_, x_] := (z + x) ! / z ! / x !

mset[13 + 1, 5]
```

8568

```
mset[5 + 1, 13]
```

8568

```
mset2[5, 13]
```

8568

```
Table[Cz[100, z], {z, 0, 12}]
```

{1, 949, 3619, 9263, 19 591, 36 857, 63 952, 104 504, 162 985, 244 825, 356 533, 505 825, 701 759}