

# Constructive Epistemic Semantics in Guarded String Models

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## Abstract

Constructive and computable multi-agent epistemic possible worlds models are defined, where possible worlds models are guarded string models in an epistemic extension of Kleene Algebra with Tests. The account is framed as a formal language Epik (Epistemic KAT) for defining such models. The language is interpreted by translation into the finite state calculus, and alternatively by modeling propositions as lazy lists. The syntax-semantics interface for a fragment of English is defined by a categorial grammar.

## 1 Introduction and Related Work

Linguistic semantics in the Montague tradition proceeds by assigning propositional *semantic values* to disambiguated sentences of a natural language. A proposition is a set or class of *possible worlds*. These worlds are often assumed to be things with the same nature and complexity as the world we occupy (Lewis 1986). But alternatively, one can work with small idealized models, in order to illustrate and test ideas. To build such models, spaces of worlds and individuals are stipulated as small finite sets, and semantic values of lexical items are constructed as functions or relations from these small sets. Such toy or idealized models are useful in research and in teaching, in that it is possible to represent propositions finitely and explicitly, and to calculate with them. The point of this paper is to scale up toy or idealized models to countable sets of worlds, and to constructive and computable modeling of epistemic alternatives for agents. We describe a certain systematic way of defining such models, and illustrate how to apply them in natural language semantics. The focus on epistemic semantics and clausal embedding. The fundamental move is to identify possible worlds with strings primitive events, so that propositions are sets of strings. An

advantage in this is that it allows for a mathematical description of an algebra of a propositions, coupled with a computational representation using either lazy lists of strings, or finite state machines that describe sets of strings.

The approach taken here synthesizes five antecedents in a certain way. John McCarthy's *Situation Calculus* is the source of the idea of constructing possible worlds as event sequences (McCarthy 1963, Reiter 2001). The algebraic theory of *Kleene Algebra with Tests* characterizes algebras with elements corresponding to propositions and event types in our application (Kozen 2003). The models we propose are an epistemic extension of guarded string models for KAT, where a unary operation interpreted as an existential epistemic modality is included for each agent. *Action models* in dynamic epistemic semantics introduced the technique of constructing epistemic models from primitive alternative relations on events, in order to capture the epistemic consequences of perceptual events (Baltag, Moss, and Solecki 1998). This is the basis for our construction of epistemic alternative relations. Literature on finite state methods in linguistic semantics has used event strings and sets of event strings to theorize about tense and aspect in natural language semantics (Fernando 2003, Carlson 2009) and to express propositions (Fernando 2017). Literature on finite state intensional semantics has shown how to do the semantics of intensional complementation including indirect questions in a setting where propositions are represented using finite state machines, and compositional semantics is expressed in a finite state calculus (Rooth 2017, Collard 2018). We adopt this in our syntax-semantics interface for English.

We begin with examples of event-sequence models. *The Elevator*. An elevator moves up and down in a four-story building, with floors numbered in the European fashion as 0,1,2,3. There are primi-

tive events  $u$  (the elevator going up one floor), and  $d$  (the elevator going down on floor). In worlds  $v_1$  and  $v_2$ , the events shown in (1) transpire. The truth values for English sentences shown in (2) are observed.

- (1)  $v_1$   $u$  it goes up from 0 to 1  
 $u$  it goes up from 1 to 2  
 $d$  it goes down from 2 to 1  
 $u$  it goes up from 1 to 2  
 $v_2$   $u$  it goes up from 0 to 1  
 $u$  it goes up from 1 to 2  
 $u$  it goes up from 2 to 3

- (2)  $v_1$   $v_2$  Sentence  
false true It's on floor 3.  
true true It has gone up.  
true false It has gone down.  
true false It could go up.

*The Concealed Coin.* Amy and Bob are seated at a table. There is a coin on the table under a cup, heads up (H). The coin could be H (heads) or T (tails), and neither agent knows which it is. This initial situation is possible world  $w_1$ . Two additional worlds  $w_2$  and  $w_3$  are defined by sequencing events after the initial state, with events interpreted as in (3). The truth values for English sentences shown in (5) are observed.

- (3)  $a_1$  Amy peeks at H, by tipping the cup.  
Bob sees she's peeking, but not what she sees.  
 $b_1$  Bob peeks at H.

- (4)  $w_1$   
 $w_2 = w_1 a_1$   
 $w_3 = w_1 a_1 b_1$

- (5)  $w_1$   $w_2$   $w_3$  Sentence  
false true true Amy knows it's H.  
false false true Bob knows it's H.  
false false true Bob knows Amy knows it's H.  
false true true Bob knows Amy knows whether it's H or T.

The events in the examples come with preconditions. The elevator can not go up if it is already on floor 3, so  $u$  has the pre-condition of the elevator being of floor 0, 1, or 2. Similarly  $d$  has the precondition that the elevator is on floor 1, 2 or 3. Amy can peek at heads only if the coin is heads up, so  $a_1$  has the precondition of the coin being

heads up. Let  $h$  be the Boolean proposition that the coin is heads up. In the other example, let  $q$  be the proposition that the elevator is on a high floor (2 or 3), and  $p$  be the proposition that it is on an odd floor (1 or 3). Then preconditions can be described by Boolean formulas, with  $h$  being the precondition of  $a_1$ , and  $!(pq)$  being the precondition of  $u$ . Juxtaposition is used for Boolean conjunction, and the exclamation point for Boolean negation. Events come as well with a relation between prior and following state, for instance with  $u$  incrementing the floor. This is expressed using an operator “:” (read “and next”) that pairs Boolean formulas. The first line in (6) describes  $a_1$  (Amy looking at heads) as happening only in an  $h$  state, and as not changing the state. Symmetrically,  $a_0$  (Amy looking at tails) can happen only in a not- $h$  state, and does not change the state. The third line says that  $u$  increments the floor, and can happen only on floors 0, 1, and 2. The fourth line describes  $d$  in similar terms. Plus is disjunction.

- (6)  $a_1$   $h : h$   
 $a_0$   $(!h):(!h)$   
 $u$   $(!q!p):(!qp) + (!qp):(q!p) + (q!p):(qp)$   
 $d$   $(!qp):(!q!p) + (q!p):(!qp) + (qp):(q!p)$

## 2 Epistemic guarded string models

Figure 1 shows an Epik program that describes a possible worlds model for two agents with information about one coin, and events of the agents semi-privately looking at the coin. The line beginning with `state` lists the basic stative propositions. To illustrate syntax, a second proposition  $t$  (tails) is included. The line beginning with `constraint` defines compatibilities among the propositions: the coin is heads or tails and not both. The lines beginning with `event` declare events, their preconditions, and their effect on state, following the format in (6). Finally the lines beginning with `agent` define *event alternative* relations for agents. Each clause with an arrow has a single event symbol on the left, and a disjunction of alternative events on the right of the arrow. The interpretation of Amy's alternatives for  $b_1$  (Bill peeks at heads), is that when  $b_1$  happens, for Amy either  $b_1$  or  $b_0$  (Bill peeks at tails) could be happening.

Kleene Algebra with Tests is an algebraic theory that is defined by equations and inequalities, which has model classes including guarded string models, relational models, finite models, and matrix models. This paper focuses on defining a family of

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state h t
constraint h!t + t!h
event a1 h:h
event a0 t:t
event b1 h:h
event b0 t:t
agent aly
  a1 -> a1
  a0 -> a0
  b1 -> b1 + b0
  b0 -> b1 + b0
agent bob
  b1 -> b1
  b0 -> b0
  a1 -> a1 + a0
  a0 -> a1 + a0

```

Figure 1: Epik program describing a possible-worlds event sequence model for two agents with information about one coin, and events of the agents semi-privately looking at the coin.

concrete guarded string algebras, the elements of which are sets of guarded strings. Definitions and notation mostly follow Kozen (2003). Additional syntax and semantics is included to model multi-agent epistemic semantics. Guarded strings over a finite alphabet  $E$  are like ordinary strings, but with truth assignments to a set  $T$  of primitive propositions (primitive tests) alternating with the symbols from  $E$ . In the algebra described by Figure 1,  $E$  is the set of events  $\{a_1, a_0, b_1, b_0\}$ , and in the elevator example,  $\{u, d\}$ . In the elevator example,  $T$  is  $\{p, q\}$ , and in the coin example it is  $\{h, t\}$ . We write truth assignments to the primitive propositions that observe the constraint as Boolean vectors. In the coin example, we get the vectors 01, 10 and in the elevator example, the vectors 00, 01, 10, 11. Guarded strings are strings of events, alternating with such vectors, and starting and ending with vectors. (7) gives the encoding as guarded strings of the worlds in (1) and (3).

$$\begin{aligned}
 (7) \quad v_1 & 00u01u10d01u10 \\
 v_2 & 00u01u10u11 \\
 w_1 & 01 \\
 w_2 & 01a_001 \\
 w_3 & 01a_001b_001
 \end{aligned}$$

The discussion of (4) mentioned building worlds by incrementing worlds with events. This is accomplished in guarded string models with fusion prod-

uct, a partial operation that combines two guarded strings, subject to the condition that the truth assignment at the end of the the first argument is identical to the truth assignment at the start of the second one. (8) gives some examples.

$$\begin{aligned}
 (8) \quad & 00u01u10d01 \cdot 01u10 = 00u01u10d01u10 \\
 & 00u01u10d01 \cdot 10u11 = \text{undefined} \\
 & 01 \cdot 01a_001 = 01a_001 \\
 & 01 \cdot 10a_110 = \text{undefined}
 \end{aligned}$$

Rather than guarded strings, elements of a guarded string model for KAT are sets of guarded strings. In the application, these elements have the interpretation of propositions (sets of possible worlds) and/or event types. An event such as  $u$  in the guarded string model corresponds to the set of guarded strings where the bare event is flanked by compatible truth assignments, as defined in an Epik program by the event declaration and the constraint declaration. These have the status of event types, in that they can “happen” in different possible worlds. Here happening corresponds to incrementing the world with some compatible element of the event type. Where  $e$  is a bare event symbol,  $\hat{e}$  is the corresponding set of guarded strings, consisting of the bare event decorated with compatible truth assignments. See the examples in (9).

$$(9) \quad \begin{array}{c|c} e & \hat{e} \\ \hline u & \{00u01, 01u10, 10u11\} \\ d & \{01d00, 10d01, 11d10\} \\ a_1 & \{10a_110\} \\ a_0 & \{01a_001\} \\ b_1 & \{10b_110\} \\ b_0 & \{01b_001\} \end{array}$$

Kleene Algebra with Tests has the algebraic signature  $\langle K, +, \cdot, *, \bar{\cdot}, 0, 1 \rangle$ . To this we add an unary operation  $\diamond_a$  for each agent, and a unary complement operation  $\cdot^c$  on propositions. This results in the signature  $\langle K, +, \cdot, *, \bar{\cdot}, 0, 1, \cdot^c, \diamond_a, \diamond_b \rangle$  for epistemic KAT with two agents. In a guarded string model in our construction, the elements of  $K$  are sets of guarded strings. The operation  $+$  is set union. The constant 0 is the empty set. The constant 1 is the set of all truth assignments obeying the constraint on truth assignments, i.e.  $\{10, 10\}$  in the coin example. The operation  $\cdot$  is fusion product raised to sets:  $x \cdot y$  is the set of all defined fusion products of an element of  $x$  with an element of  $y$ . The operation  $*$  is Kleene star, with  $x^* = \bigcup_{i \geq 0} x^i$ , where  $x^i$  is the  $i$ -times product of  $x$  with itself and  $x^0 = 1$ .

Subsets of 1 are also elements of  $K$ , and these form the Boolean algebra of tests as a set Boolean algebra. The overbar operation is complement in this Boolean algebra. The complement operation  $^c$  is complement at the level of sets of guarded strings, with  $x^c = 0^c - x$ , where the operation on the right is set difference. The epistemic modality  $\Diamond_a$  is interpreted using Kripke semantics, as pre-image relative to a fixed relation  $R_a$  between guarded strings,  $\Diamond_a x = \{u \mid \exists v. v \in x \wedge u R_a v\}$ . Here  $u$  and  $v$  are guarded strings, while  $x$  is an element of  $K$ .

It remains to define the Kripke relation on guarded strings from an agent specification as in Figure 1. An agent specification pairs each bare event with a set of bare events, and so determines a relation between bare events, call it relation  $R_a$  for an agent  $e$ . This determines a relation  $\hat{R}_a$  between decorated events, see (10). It is generalized to a relation  $\hat{R}$  on arbitrary guarded strings by relational Kleene star, as defined in (11). The operation  $\cdot$  in the definition is KAT product, which enforces matching of tests. The number of terms  $n$  is construed as satisfying  $n \geq 1$ . *What is desired for  $n = 0$ ?*

$$(10) \hat{R} =_{\text{def.}} \{ \langle u, v \rangle \mid \exists c \exists d. c R d \wedge u \in c \wedge v \in d \}$$

$$(11) \hat{R} =_{\text{def.}} \{ \langle x, y \rangle \mid \exists u_1 \dots \exists u_n \exists v_1 \dots \exists v_n. \\ u_1 \hat{R} v_1 \wedge \dots \wedge u_n \hat{R} v_n \wedge \\ x = u_1 \cdot \dots \cdot u_n \wedge \\ y = v_1 \cdot \dots \cdot v_n \}$$

This defines an epistemic alternative to world  $x$  to be a world of the same length, where each component event in the alternative is an event-alternative to the event in corresponding position in the base world. Fusion product enforces preconditions and a correspondence between pre-states and post-states of events on both sides of the epistemic alternative relation. This provides for finitely specifiable construction of epistemic models that reflect intuitions about information exchange and epistemic consequences of perceptual events. See Section 6 for linguistic examples. Since an epistemic alternative has the same length as its base world, it follows from the construction that agents know how many events have transpired in their base worlds.

Universal box modalities are defined as duals,  $\Box_a x = (\Diamond_a x^c)^c$ . For instance, Aly is certain that the coin is heads if and only if she does not consider it possible that it is not heads.

Summing up, given an Epik program with  $n$

agents, we construct a concrete guarded string model  $\langle K, +, \cdot, *, \bar{\cdot}, 0, 1, ^c, \Diamond_1, \dots, \Diamond_n \rangle$ . The elements of  $K$  are interpreted as propositions.  $0^c$  is the set of worlds, and it may be countably infinite.  $\Diamond_i$  is an epistemic modality for the  $i$ th agent. Or referring to the Kripke relations  $\hat{R}_i$ , the construction defines a multi-agent Kripke frame  $\langle 0^c, \hat{R}_1, \dots, \hat{R}_n \rangle$  (usually a countable one) from an Epik specification. The frame consists of a set of worlds, and an epistemic-alternative relation for each agent. These models are used as a target for natural-language interpretation in Section 5 and Section 6, where we obtain semantic values such as  $\llbracket \text{Amy knows that it's heads and Bob knows that Amy knows whether it is heads or tails, and does not know that it's heads} \rrbracket$  as elements of  $K$ . Concretely the propositions are sets of guarded strings (usually countable ones), construed as sets of worlds as they figure in possible worlds semantics for natural language.

### 3 Translation into the finite state calculus

The finite state calculus is an algebra of regular sets of strings and regular relations between strings that was designed for use in computational phonology and morphographemics (cite Karttunen etc.) Current implementations allow for the definition of functions with the status of defined operators on regular sets and relations. Such definitions are used to define an embedding of epistemic KAT in a string algebra. The methodology follows Section 2 closely. Let  $\mathcal{K}$  be an epistemic algebra as described in Section 2. A given element of  $\mathcal{K}$  is represented in the the string algebra by the very same set of strings, i.e. by a set of strings that have the form of a sequence of bare event symbols, with interleaved Boolean vectors. Product in the KAT can not be modeled as concatenation in the string algebra, because this would not enforce identity of states, and would result in lengthening Boolean vectors at the concatenation point. Instead, KAT product and KAT Kleene star are defined operations in the string algebra, see Figure 2. The operations concatenate in the string algebra, delete strings with non-matching tests, and then delete the second of two tests create a well-formed guarded string.

A given bare event such as  $a_1$  (Aly looks at heads) is in the KAT algebra a set of bare events decorated with compatible tests on each side, semantically  $\{10a_110\}$  in this case. This is a unit set



St Tests such as 0 1 1 0. The length is the number of generators.

UnequalStPair Sequence of two unequal tests such as 0 1 1 0 0 1 1 1, differing in one or more positions.

```

define Wf0 ~[$ UnequalStPair];
String that doesn't contain a non-matching test pair.

define Squash St -> 0 || St _;
Rewrite relation deleting the second of two tests.

define Cn(X,Y)
[[[X Y] & Wf0] .o. Squash].l;
KAT product in Fst, where & is intersection, .o. is relation composition, and .l is relation image.

define Kpl(X)
[[[X+] & Wf0] .o. Squash].l;

define Kst(X) St | Kpl(X);
KAT Kleene plus and Kleene star in Fst. The Fst operation | is union.

```

Figure 2: Translation into Fst of KAT product and KAT Kleene star.

rather than a guarded string, because elements of the KAT algebra are sets. Worlds in the KAT algebra are defined by sequencing events using *Kst*. The operation enforces compatibility of states, so that  $(a_1 + a_0)(b_1 + b_0)$  contains two worlds rather than four. The program in Figure 1 as interpreted in FST defines a countably infinite set of possible worlds by KAT Kleene closure as  $Kst(a_1 + a_0 + b_1 + b_0)$ , and an algebra of propositions as regular sets of strings drawn from this space of worlds.

It remains to define an epistemic alternative relation on worlds for each agent. The relevant information in Figure 1 is a relation between bare events for each agent. This determines a relation in the guarded string algebra a relation between bare events decorated with compatible tests. For agent Aly, this is the relation described in (12) as a set of ordered pairs.

$$(12) \left\{ \begin{array}{l} \langle 10a_110, 10a_110 \rangle, \\ \langle 01a_001, 01a_001 \rangle, \\ \langle 10b_110, 10b_110 \rangle, \\ \langle 10b_110, 01b_001 \rangle, \\ \langle 01b_001, 10b_110 \rangle, \\ \langle 01b_001, 01b_001 \rangle \end{array} \right\}$$

The relation on decorated events needs to be generalized to a relation of worlds. The principle for this is that an epistemic alternative to a world of the form *we* is a world of the form *vd*, where *v* is a

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define RelKpl(R) Squash.i.o.
Wf0.o.[R+].o.Wf0.o.Squash
a Relational Kleene plus in the string algebra
b Constrain domain and co-domain to contain no unmatched tests.
c Reduce doubled tests to a single test in the domain and co-domain.
Squash.i is the inverse of Squash.

define Kst(R) [St.x.St] | Kpl(X);
The Fst operation .x. is Cartesian product.

```

Figure 3: Definition in Fst of the Kleene concatenation closure of a relation between guarded strings.

world-alternative to *w*, *d* is an event-alternative to *e*, and *vd* is defined (i.e. the world alternative *v* satisfies the pre-conditions of the event alternative *d*). This principle is found in earlier literature (Moore 198x, Baltag, Moss and Solecki 20xx). In the construction in Fst, the definition of world alternatives takes a simple form. Where  $R_a$  is the relation on decorated events for agent *a*, the the corresponding relation on worlds in is the Kleene closure of  $R_a$ . Where *R* and *S* are relations, the concatenation product of *R* and *S* is the set of pairs of the form  $\langle x_1x_2, y_1y_2 \rangle$ , where  $\langle x_1, y_1 \rangle$  is in relation *R*, and  $\langle x_2, y_2 \rangle$  is in relation *S*. The Kleene closure of relation *R* is  $\cup_{n \geq 0} R^n$ , where  $R^n$  is the *n*-times concatenation product of *R* with itself. This is an operation in the finite state calculus. Figure ?? defines the corresponding operation in the guarded string algebra. The epistemic alternative relation on worlds for an agent is then defined as the concatenation closure of the event alternative relation for the agent.

Other operations in the guarded string algebra as defined in FST are simpler. Union is union in the string algebra. The complement of a proposition is complement relative to the set of worlds, as defined by set difference in the string algebra.

## 4 Interpretation using lazy lists of strings

## 5 Syntax-semantics interface

An architecture of interpretation by translation is employed, where English sentences are mapped to terms in a logical language via an interpreted grammar, and these terms are intreted in a guarded string model as propositions. For the latter, there

are options of translation into the finite state calculus, and representation in Haskell via lazy lists of guarded strings.

The grammar is a semantically interpreted multi-modal categorial grammar, consisting of a lexicon of words, their categorial types, and interpretations in a logical lambda language. (13) lists and exemplifies phenomena that are covered.

(13) *Phenomena*

Basic statives	It's heads. It's tails.
That-complement	Amy knows that it's heads.
Wh-complement	Amy knows whether its heads.
Negation	Bob doesn't know that it isn't heads.
Tensed and base verbal forms	Bob knows that it's heads. Bob doesn't know that it's heads.
Sentence conjunction	It's heads and Bob doesn't know that it's heads
Predicate conjunction	Bob knows that Amy knows whether it's heads and doesn't know that Amy knows that it's heads.

As illustrated towards the end, there is recursion through conjunction and verbal complementation, so that the language is infinite, and includes talk of beliefs about beliefs, or in general, talk of arbitrarily iterated belief.

(14) gives illustrative lexical entries. The grammar and semantics are in certain way optimized for the simple fragment of English concerned with clausal complementation. The agent names *Amy* and *Bob* contribute the epistemic alternative relations for those agents, rather than individuals. This is possible because the agents are never arguments of extensional predicates, so what matters about the agents is their epistemic alternative relations. The root verb *know* contributes existential modal force. The complementizers *that* and *whether* are the heads of their dominating clauses, and assemble an alternative relation, modal force, and proposition contributed by the complement, in part by introducing the dual of the existential

modal force to obtain universal modal force. These moves are bizarre, but are offered here as a way of constructing a compact interpreted grammar. They can easily be reformulated with more complicated constructions that are better motivated in more comprehensive interpreted grammar of English.

(14) *Lexicon (partial)*

Amy	$e$	$\hat{R}_a$
Bob	$e$	$\hat{R}_b$
heads	$d \backslash_D t$	$W \cdot h$
it's	$t / (d \backslash_D t)$	$\lambda p.p$

## 6 Examples and discussion

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