Epistemic Semantics in Guarded String Models

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Abstract

Constructive and computable multi-agent epistemic possible worlds models are defined, where possible worlds models are guarded string models in an epistemic extension of Kleene Algebra with Tests. The account is framed as a formal language Epik (Epistemic KAT) for defining such models. The language is interpreted by translation into the finite state calculus, and alternatively by modeling propositions as lazy lists in Haskell. The syntax-semantics interface for a fragment of English is defined by a categorial grammar.

1 Introduction and Related Work

Linguistic semantics in the Montague tradition proceeds by assigning propositional *semantic values* to disambiguated sentences of a natural language. A proposition is a set or class of *possible worlds*. These worlds are often assumed to be things with the same nature and complexity as the world we occupy (Lewis, 1986). But alternatively, one can work with small idealized models, in order to illustrate and test ideas. To build such models, spaces of worlds and individuals are stipulated as small finite sets, and semantic values of lexical items are constructed as functions or relations from these small sets. Such toy or idealized models are useful in research and in teaching, in that it is possible to represent propositions finitely and explicitly, and to calculate with them. The point of this paper is to scale up toy or idealized models to countable sets of worlds, and to constructive and computable modeling of epistemic alternatives for agents. We describe a certain systematic way of defining such models, and illustrate how to apply them in natural language semantics. The focus on epistemic semantics and clausal embedding. The fundamental move is to identify possible worlds with strings primitive events, so that propositions are sets of strings. An advantage in this is that it allows for a mathematical description of an algebra of a propositions, coupled with a computational representation using either lazy lists of strings, or finite state machines that describe sets of strings.

The approach taken here synthesizes five antecedents in a certain way. John McCarthy's *Situation Calculus* is the source of the idea of constructing possible worlds as event sequences (McCarthy, 1963; Reiter, 2001). The algebraic theory of *Kleene Algebra with Tests* characterizes algebras with elements corresponding to propositions and event types in our application (Kozen, 2001). The models we propose are an epistemic extension of guarded string models for KAT, where a unary operation interpreted as an existential epistemic modality is included for each agent. *Action models* in dynamic epistemic semantics introduced the technique of constructing epistemic models from primitive alternative relations on events, in order to capture the epistemic consequences of perceptual events (Baltag et al., 1999). This is the basis for our construction of epistemic alternative relations. Literature on finite state methods in linguistic semantics has used event strings and sets of event strings to theorize about tense and aspect in natural language semantics (Fernando, 2004, 2007; Carlson, 2009) and to express intensionality (Fernando, 2017). Literature on finite state intensiononal semantics has discussed how to do the semantics of intensional complementation including indirect questions in a setting where compositional semantics is expressed in a finite state calculus (Rooth, 2017; Collard, 2018). We adopt this in our syntax-semantics interface for English.

2001 or 2003?

We begin with examples of event-sequence models. *The Elevator*. An elevator moves up and down in a four-story building, with floors numbered in the European fashion as 0,1,2,3. There are primitive events u (the elevator going up one floor), and

d (the elevator going down on floor). In worlds v_1 and v_2, the events shown in (1) transpire. The truth values for English sentences shown in (2) are observed.

(1)

v_1	u	it goes up from 0 to 1
	u	it goes up from 1 to 2
	d	it goes down from 2 to 1
	u	it goes up from 1 to 2
v_2	u	it goes up from 0 to 1
	u	it goes up from 1 to 2
	u	it goes up from 2 to 3

(2)

v_1	v_2	Sentence
false	true	It's on floor 3.
true	true	It has gone up.
true	false	It has gone down.
true	false	It could go up.

The Concealed Coin. Amy and Bob are seated at a table. There is a coin on the table under a cup, heads up (H). The coin could be H (heads) or T (tails), and neither agent knows which it is. This initial situation is possible world w_1. Two additional worlds w_2 and w_2 are defined by sequencing events after the initial state, with events interpreted as in (3). The truth values for English sentences shown in (5) are observed.

(3)

a_1	Amy peeks at H, by tipping the cup. Bob sees she's peeking, but not what she sees.
b_1	Bob peeks at H.

(4)

w_1

$w_2 = w_1\, a_1$

$w_2 = w_1\, a_1\, b_1$

(5)

w_1	w_2	w_3	Sentence
false	true	true	Amy knows it's H.
false	false	true	Bob knows it's H.
false	false	true	Bob knows Amy knows it's H.
false	true	true	Bob knows Amy knows whether it's H or T.

The events in the examples come with preconditions. The elevator can not go up if it is already on floor 3, so u has the pre-condition of the elevator being of floor 0, 1, or 2. Similarly d has the precondition that the elevator is on floor 1, 2 or 3. Amy can peek at heads only if the coin is heads up, so a_1 has the precondition of the coin being heads up. Let h be the Boolean proposition that the coin is heads up. In the other example, let q be the proposition that the elevator is on a high floor (2 or 3), and p be the proposition that it is on an odd floor (1 or 3). Then preconditions can be described by Boolean formulas, with h being the precondition of a_1, and $\overline{pq}$ being the precondition of u. Juxtaposition is used for Boolean conjunction, and the overbar for Boolean negation. Events come as well with a relation between prior and following state, for instance with u incrementing the floor. This is expressed using an operator ":" (read "and next") that pairs Boolean formulas. The first line in (6) describes a_1 (Amy looking at heads) as happening only in an h state, and as not changing the state. Symmetrically, a_0 (Amy looking at tails) can happen only in a not-h state, and does not change the state. The third line says that u increments the floor, and can happen only on floors 0, 1, and 2. The fourth line describes d in similar terms. Plus is disjunction.

(6)

a_1	$h : h$
a_0	$\bar{h}{:}\bar{h}$
u	$(\bar{q}\bar{p}){:}(\bar{q}p) + (\bar{q}p){:}(q\bar{p}) + (q\bar{p}){:}(qp)$
d	$(\bar{q}p){:}(\bar{q}\bar{p}) + (q\bar{p}){:}(\bar{q}p) + (qp){:}(q\bar{p})$

In this discussion, a sequence such as $\bar{q}p$ can be viewed as a valuation of the primitive propositions q and p, which is used to describe world state. The primitives are listed in fixed order, and left unmarked (indicating true) or marked with the overbar (indicating false). Suppose that in the coin example, we had an additional primitive stative proposition (or atomic test) t, interpreted as tails. Since a coin is heads or tails but not both, we want to allow the valuations $h\bar{t}$ and $\bar{h}t$, and disallow ht and $\bar{h}\bar{t}$. This is enforced by a *state formula*, which is a simply Boolean formula, in this case (7a). Where B is a set of atomic tests and ρ is a state constraint over B, $\mathcal{A}^{\rho}_{\mathsf{B}}$ is the set of valuations of B that make formula ρ true. Following (Kozen, 2001), we call the valuations *atoms*.

Formulas like the ones in (6) that describe pre- and post-conditions are *effect formulas*. They are interpreted as defining relations between valuations, as defined in Figure 1. The valuations they relate are constrained by the state formula as well. For the heads-tails example, let the state formula and the effect formula for a_1 (Aly peeking at heads) be as specified in (7). Then the set of valuations determined by φ and the relation on valuations for the event u are $\mathcal{A}^{\varphi}_{\mathsf{B}}$ and $[\![\zeta]\!]^{\varphi}$ as given at the bottom

state formulas $(a \in \mathsf{B})$

$\varphi, \psi \quad ::= \quad a \mid 0 \mid 1 \mid \varphi{+}\psi \mid \varphi\ \psi \mid \bar{\varphi}$

effect formulas

$\zeta, \eta \quad ::= \quad \varphi : \sigma \mid \zeta{+}\eta \mid \zeta\ \&\ \eta \mid \bar{\zeta}$

$$
\begin{array}{rcl}
[\![\psi_1 : \psi_2]\!]^{\varphi} & = & \mathcal{A}_{\mathsf{B}}^{(\varphi\ \psi_1)} \times \mathcal{A}_{\mathsf{B}}^{(\varphi\ \psi_2)} \\
[\![\zeta{+}\eta]\!]^{\varphi} & = & [\![\zeta]\!]^{\varphi} \cup [\![\eta]\!]^{\varphi} \\
[\![\zeta\&\eta]\!]^{\varphi} & = & [\![\zeta]\!]^{\varphi} \cap [\![\eta]\!]^{\varphi} \\
[\![\bar{\zeta}]\!]^{\varphi} & = & V \times V \setminus [\![\zeta]\!]^{\varphi}
\end{array}
$$

Figure 1: Syntax and semantics of state formulas and effect formulas. Effect formulas are evaluated relative to as set Boolean valuations V, and denote relations between Boolean valuations.

in (**??**). Given a state formula ρ, the the relation characterizing pre- and post-conditions of an event with effect formula ζ is defined to be $[\![\zeta]\!]^{[\![\rho]\!]}$.

(7) State formula: $\varphi = h\bar{t} + \bar{h}t$
Effect formula for a_1: $\zeta = h : h$
$\mathcal{A}_{\mathsf{B}}^{\varphi} = \{h\bar{t}, \bar{h}t\}$
$[\![\zeta]\!]^{\varphi} = \{\langle h\bar{t}, h\bar{t}\rangle\}$

2 Epistemic guarded string models

Figure 2 shows an Epik program that describes a possible worlds model for two agents with information about one coin, and events of the agents semi-privately looking at the coin. The line beginning with `state` lists the basic stative propositions. To illustrate syntax, a second proposition t (tails) is included. The line beginning with `constraint` defines compatibilities among the propositions: the coin is heads or tails and not both. The lines beginning with `event` declare events, their preconditions, and their effect on state, following the format in (6). Finally the lines beginning with `agent` define *event alternative* relations for agents. Each clause with an arrow has a single event symbol on the left, and a disjunction of alternative events on the right of the arrow. The interpretation of Amy's alternatives for b_1 (Bill peeks at heads), is that when b_1 happens, for Amy either b_1 or b_0 (Bill peeks at tails) could be happening.

Kleene Algebra with Tests is an algebraic theory that is defined by equations and inequalities, which a has model classes including guarded string models, relational models, finite models, and matrix models. This paper focuses on defining a family of concrete guarded string algebras, the elements of which are sets of guarded strings. Definitions and notation mostly follow (Kozen, 2001). Additional syntax and semantics is included to model multi-agent epistemic semantics. Guarded strings over a finite alphabet P are like ordinary strings, but with truth assignments to a set T of primitive propositions (primitive tests) alternating with the symbols from E. In the algebra described by Figure 2, E is the set of events $\{a_1, a_0, b_1, b_0\}$, and in the elevator example, $\{u, d\}$. In the elevator example, T is $\{p, q\}$, and in the coin example it is $\{h, t\}$. We write atoms as the boolean formulae that uniquely specify the truth assignment. In the elevator example, we have atoms $\bar{q}\bar{p}$, $\bar{q}p$, $q\bar{p}$, qp, for notational simplicity we will write these as $\hat{0}$, $\hat{1}$, $\hat{2}$, $\hat{3}$ respectivelyto indicate the floor each atom represents. In the coin example, we get $h\bar{t}$ and $\bar{h}t$ (for which we use the shorthand H and T), as well as the bogus atoms $\bar{h}\bar{t}$ and ht! To preclude bogosity, the `constraint` declaration in Figure 2 describes a state formula φ which denotes the set of *feasible atoms* $\mathcal{A}_{\mathsf{B}}^{\varphi}$.

Guarded strings are strings of events, alternating with such atoms, and starting and ending with atoms. The length of a guarded string g, written $|g|$ is the number of events in g, ignoring the atoms. (8) gives the encoding as guarded strings of the worlds in (1) and (3).

(8)

	Guarded String	*Length*
v_1	$\hat{0}\ u\ \hat{1}\ u\ \hat{2}\ d\ \hat{1}\ u\ \hat{2}$	4
v_2	$\hat{0}\ u\ \hat{1}\ u\ \hat{2}\ u\ \hat{3}$	3
w_1	H	0
w_2	$H\ a_0\ H$	1
w_3	$T\ a_0\ T\ b_0\ T$	3

The discussion of (4) mentioned building worlds by incrementing worlds with events. This is accomplished in guarded string models with fusion product, a partial operation that combines two guarded strings, subject to the condition that the truth assignment at the end of the the first argument is identical to the truth assigment at the start of the second one. (9) gives some examples.

(9) $\hat{0}\ u\ \hat{1}\ u\ \hat{2}\ d\ \hat{1} \diamond \hat{1}\ u\ \hat{2} = \hat{0}\ u\ \hat{1}\ u\ \hat{2}\ d\ \hat{1}\ u\ \hat{2}$
$\hat{0}\ u\ \hat{1}\ u\ \hat{2}\ d\ \hat{1} \diamond \hat{2}\ u\ \hat{3} =$ *undefined*
$H \diamond H\ a_0\ H = H\ a_0\ H$
$T \diamond T\ a_1\ T =$ *undefined*

Rather than individual guarded strings, elements of a guarded string model for KAT are sets of guarded strings. In the application, these elements

```
state h t
constraint h!t + t!h
event a1 h:h
event a0 t:t
event b1 h:h
event b0 t:t
agent aly
  a1 -> a1
  a0 -> a0
  b1 -> b1 + b0
  b0 -> b1 + b0
agent bob
  b1 -> b1
  b0 -> b0
  a1 -> a1 + a0
  a0 -> a1 + a0
```

Figure 2: Epik program describing a possible-worlds event sequence model for two agents with information about one coin, and events of the agents semi-privately looking at the coin.

have the interpretation of propositions (sets of possible worlds) and/or event types. An event such as u in the guarded string model corresponds to the set of guarded strings where the bare event is flanked by compatible atoms. In standard KATs, every possible atom is allowed to surround every event, which leads to nonsensical strings like $\hat{1}\ u\ \hat{0}$ in the denotation of u, indicating that the elevator travelled up from floor 1 to floor 0 (a physical impossibility).

Fortunately, Epik provides a syntax to specify the input-output behavior via the `event` declarations. From these we construct a function $\mathcal{E}$ from agents to relations on atoms that specify the effects each action can have. Notably for every declaration `event` $e\ \zeta$, we have $\mathcal{E}^{\varphi}(e) = [\![\zeta]\!]^{\varphi}$. We can lift $\mathcal{E}^{\varphi}$ to $\hat{\mathcal{E}}$ which decorates the events with their permitted atoms, that is $\hat{\mathcal{E}}^{\varphi}(e) = \{\alpha\ e\ \beta \mid \alpha, \beta \in \mathcal{E}^{\varphi}(e)\}$. See the examples in (10).

(10)

e	$\hat{\mathcal{E}}^{\varphi}(e)$
u	$\{\hat{0}\ u\ \hat{1}, \hat{1}\ u\ \hat{2}, \hat{2}\ u\ \hat{3}\}$
d	$\{\hat{1}\ d\ \hat{0}, \hat{2}\ d\ \hat{1}, \hat{3}\ d\ \hat{2}\}$
a_1	$\{H\ a_1\ H\}$
a_0	$\{T\ a_0\ T\}$
b_1	$\{H\ b_1\ H\}$
b_0	$\{T\ b_0\ T\}$

It remains to define the Kripke relation on guarded strings from an agent specification as in Figure 2. In Epik, an agent specification pairs each bare event with a set of (+-separated) bare events, and so determines a relation between bare events, call it relation R_a for an agent a. We can lift this to a relation $\hat{R}_a$ between guarded strings, as defined in (11). The operation $(;)$ in the definition is KAT product, which enforces matching of tests.

$$\begin{aligned}(11)\ \hat{R} \triangleq \{\langle x, y\rangle \mid\ & \exists u_1 ... \exists u_n \exists v_1 ... \exists v_n. \\ & u_1\ R\ v_1 \wedge ... \wedge u_n\ R\ v_n \wedge \\ & x \in [\![u_1; ...; u_n]\!]^{\varphi,\mathcal{E}} \wedge \\ & y \in [\![v_1; ...; v_n]\!]^{\varphi,\mathcal{E}}\ \}\end{aligned}$$

This defines an epistemic alternative to world x to be a world of the same length, where each component event in the alternative is an event-alternative to the event in corresponding position in the base world. Fusion product enforces preconditions and a correspondence between pre-states and post-states of events on both sides of the epistemic alternative relation. This provides for finitely specifiable construction of epistemic models that reflect intuitions about information exchange and epistemic consequences of perceptual events. See Section 6 for linguistic examples. Since an epistemic alternative has the same lenght as its base world, it follows from the construction that agents know how many events have transpired in their base worlds.

3 A Language for Epistemic Sets of Guarded Strings

To reason about epistemic alternatives, guarded strings are used to represent different possible worlds, so agents' knowledge/beliefs are represented by sets of guarded strings. The standard algebra for manipulating guarded strings is Kleene Algebra with Tests (KAT), which has the algebraic signature $\langle K, +, ;, *, \bar{}, 0, 1\rangle$ (Kozen, 2001). However, the KATs don't permit native reasoning about modal operators, so Epik's signature needs to subsume the KAT operators.

Figure 3 depicts the syntax of Epik's formal term language. As in KAT, 0 is the null element, 1 is the unit element, $p+q$ indicates disjuction, $p; q$ (or simply pq) indicates sequence, the boolean operators for tests φ are the standard ones (as in Figure **??**), and p^* indicates iteration of p. We add some additional operations to reason about epistemic alternatives: the term $\Diamond_a\ p$ represents epistemic possibility for agent a in worlds p, and complement $\neg p$ represents the worlds not described by p. We can

terms $\quad e \in \mathsf{P}$

$p, q ::= e \mid 0 \mid 1 \mid \phi \mid p + p \mid p; p \mid p^* \mid \neg p \mid \Diamond_a\, p$

Derived Operators

$\Box_a p \triangleq \neg\Diamond_a \neg p \quad p \& q \triangleq \neg(\neg p + \neg q) \quad \_ \triangleq \sum \mathsf{P}$

Figure 3: The language of Epik terms and key derived operators

encode $\Box_a\, p$, i.e. epistemic certainty for agent a in worlds p, by $\Box_a\, p = (\Diamond_a\, p^c)^c$.

We interpret each term in as a set of guarded strings as indicated in Figure 4. The function $[\![p]\!]^{\varphi,\mathcal{E}}$ interprets a term p as a set of guarded strings consistent with state constraints φ and effect relations $\mathcal{E}^\varphi$. A bare event e is interpreted as the set $\hat{\mathcal{E}}^\varphi(e)$. The constant 0 is interpreted as the empty set. The constant 1 is interpreted as the set $\mathcal{A}^\varphi_{\mathsf{B}}$, i.e. $\{h\bar{t}, \bar{h}t\}$ in the coin example. The operation $p + q$ is interpreted as the union of the interpretations of p and q. The operation $p; q$ interpreted as the fusion product of the interpretations of p and q raised to sets: $X \diamond Y = \{x \diamond y \mid x \in X, y \in Y, x \diamond y \textit{ defined}\}$. The operation $p*$ is Kleene star, which is interpreted to be least upper bound of the iterated fusion product of the denotation of p with itself. Subsets of 1 indicated by propositions ψ are also elements of K, and these form the Boolean algebra of tests, denoting all atoms admitted by the constraint φ and the boolean test ψ.

The unary epistemic alternative operation $\Diamond_a$ for each agent a, is interpreted using using Kripke semantics, as pre-image relative to a fixed relation $\hat{R}_a$ between guarded strings, $\Diamond_a\, p = \{u \mid \exists v.v \in [\![p]\!]^{\phi,\mathcal{E}} \wedge u\ \hat{R}_a\ v\}$. Here u and v are guarded strings, while x is an element of K.

The complement operation p^c is complement at the level of sets of guarded strings, defined to every set the universe $\_^*$ except those denoted by p. Note that $[\![0^c]\!]^{\varphi,\mathcal{E}} = [\![\_^*]\!]^{\varphi,\mathcal{E}}$.

Summing up, given an Epik program with n agents, we construct state constraints φ, effect relations $\mathcal{E}$, and a concrete guarded string model for each term using $[\![-]\!]^{\varphi,\mathcal{E}}$. 0^c is the set of worlds, and it may be countably infinite. $\Diamond_{a_i}$ is an epistemic modality for the ith agent. Or referring to the Kripke relations $\hat{R}_i$, the construction defines a multi-agent Kripke frame $\langle 0^c, \hat{R}_1, ..., \hat{R}_n\rangle$ (usually a countable one) from an Epik specification. The frame consists of a set of worlds, and an epistemic-alternative relation for each agent. These models

$$
\begin{array}{lcl}
[\![0]\!]^{\varphi,\mathcal{E}} & \triangleq & \{\} \\
[\![1]\!]^{\varphi,\mathcal{E}} & \triangleq & \mathcal{A}^\varphi_{\mathsf{B}} \\
[\![e]\!]^{\varphi,\mathcal{E}} & \triangleq & \hat{\mathcal{E}}^\varphi(e) \\
[\![\psi]\!]^{\varphi,\mathcal{E}} & \triangleq & \mathcal{A}^{\varphi\ \psi} \\
[\![p + q]\!]^{\varphi,\mathcal{E}} & \triangleq & [\![p]\!]^{\varphi,\mathcal{E}} \cup [\![q]\!]^{\varphi,\mathcal{E}} \\
[\![p; q]\!]^{\varphi,\mathcal{E}} & \triangleq & [\![p]\!]^{\varphi,\mathcal{E}} \diamond [\![q]\!]^{\varphi,\mathcal{E}} \\
[\![p^*]\!]^{\varphi,\mathcal{E}} & \triangleq & \bigcup_n \underbrace{[\![p]\!]^{\varphi,\mathcal{E}} \diamond \ldots \diamond [\![p]\!]^{\varphi,\mathcal{E}}}_{n} \\
[\![\neg p]\!]^{\varphi,\mathcal{E}} & \triangleq & [\![\_^*]\!]^{\varphi,\mathcal{E}} \setminus [\![p]\!]^{\varphi,\mathcal{E}} \\
[\![\Diamond_a\, p]\!] & \triangleq & \{x \mid x\ \hat{R}_a\ y, y \in [\![p]\!]^{\varphi,\mathcal{E}}\}
\end{array}
$$

Figure 4: Interpretation of Epik terms as sets of guarded strings

are used as a target for natural-language interpretation in Section 5 and Section 6, where we obtain semantic values such as ⟦Amy knows that it's heads and Bob knows that Amy knows whether it is heads or tails, and does not know that it's heads⟧ as elements of K. Concretely the propositions are sets of guarded strings (usually countable ones), construed as sets of worlds as they figure in possible worlds semantics for natural language.

4 Translation into the finite state calculus

The finite state calculus is an algebra of regular sets of strings and regular relations between strings that was designed for use in computationonal phonology and morphographemics (Kaplan and Kay, 1994; Beesley and Karttunen, 2003). Current implementations allow for the definition of functions with the status of defined operators on regular sets and relations (Hulden, 2009; Karttunen, 2010). Such definitions are used to define an embedding of epistemic KAT in a string algebra. The methodology follows Section 2 closely. Let $\mathcal{K}$ be an epistemic algebra as described in Section 2. A given element of $\mathcal{K}$ is represented in the the string algebra by the very same set of strings, i.e. by a set of strings that have the form of a sequence of bare event symbols, with interleaved Boolean vectors. Product in the KAT can not be modeled as concatenation in the string algebra, because this would not enforce identity of states, and would result in lengthening Boolean vectors at the concatenation point. Instead, KAT product and KAT Kleene star are defined operations in the string algebra, see Figure 5. The operations concatenate in the string algebra, delete strings with non-matching

`St` Tests such as 0 1 1 0. The length is the number of generators.

`UnequalStPair` Sequence of two unequal tests such as 0 1 1 0 0 1 1 1, differing in one or more positions.

```
define Wf0 ~[$ UnequalStPair];
```
String that doesn't contain a non-matching test pair.

```
define Squash St -> 0 || St _;
```
Rewrite relation deleting the second of two tests.

```
define Cn(X,Y)
   [[[X Y] & Wf0] .o. Squash].l;
```
KAT product in Fst, where `&` is intersection, `.o.` is relation composition, and `.l` is relation image.

```
define Kpl(X)
   [[[X+] & Wf0] .o. Squash].l;
define Kst(X) St |  Kpl(X);
```
KAT Kleene plus and Kleene star in Fst. The Fst operation `|` is union.

Figure 5: Translation into Fst of KAT product and KAT Kleene star.

tests, and then delete the second of two tests create a well-formed guarded string.

A given bare event such as a_1 (Aly looks at heads) is in the KAT algebra a set of bare events decorated with compatible tests on each side, semantically $\{10a_110\}$ in this case. This is a unit set rather than a guarded string, because elements of the KAT algebra are sets. Worlds in the KAT algebra are defined by sequencing events using *Kst*. The operation enforces compatibility of states, so that $(a_1 + a_0)(b_1 + b_0)$ contains two worlds rather than four. The program in Figure 2 as interpreted in FST defines a countably infinite set of possible worlds by KAT Kleene closure as *Kst*$(a_1 + a_0 + b_1 + b_0)$, and an algebra of propositions as regular sets of strings drawn from this space of worlds.

It remains to define an epistemic alternative relation on worlds for each agent. The relevant information in Figure 2 is a relation between bare events for each agent. This determines a relation in the guarded string algebra a relation between bare events decorated with compatible tests. For agent Aly, this is the relation described in (12) as a set of ordered pairs.

```
define RelKpl(R) Squash.i.o.
                 (c)
  Wf0.o. [R+] .o. Wf0 .o. Squash
  (b)    (a)    (b)     (c)
```
- a Relational Kleene plus in the string algebra
- b Constrain domain and co-domain to contain no unmatched tests.
- c Reduce doubled tests to a single test in the domain and co-domain. `Squash.i` is the inverse of `Squash`.

```
define Kst(R) [St.x.St]|Kpl(X);
```
The Fst operation `.x.` is Cartesian product.

Figure 6: Definition in Fst of the Kleene concatenation closure of a relation between guarded strings.

$$(12)\quad \left\{ \begin{array}{l} \langle 10a_110, 10a_110\rangle, \\ \langle 01a_001, 01a_001\rangle, \\ \langle 10b_110, 10b_110\rangle, \\ \langle 10b_110, 01b_001\rangle, \\ \langle 01b_001, 10b_110\rangle, \\ \langle 01b_001, 01b_001\rangle \end{array} \right\}$$

The relation on decorated events needs to be generalized to a relation of worlds. The principle for this is that an epistemic alternative to a world of the form we is a world of the form vd, where v is a world-alternative to w, d is an event-alternative to e, and vd is defined (i.e. the world alternative v satisfies the pre-conditions of the event alternative d). This principle is found in earlier literature (Moore 198x, Baltag, Moss and Solecki 20xx). In the construction in Fst, the definition of world alternatives takes a simple form. Where R_a is the relation on decorated events for agent a, the the corresponding relation on worlds in is the Kleene closure of R_a. Where R and S are relations, the concatenation product of R and S is the set of pairs of the form $\langle x_1x_2, y_1y2\rangle$, where $\langle x_1, y_1\rangle$ is in relation R, and $\langle x_2, y_2\rangle$ is in relation S. The Kleene closure of relation R is $\cup_{n\geq 0}R^n$, where R^n is the n-times concatenation product of R with itself (the 0-times concatenation product is $[\![1]\!]^{\varphi,\mathcal{E}}$). This is an operation in the finite state calculus. Figure **??** defines the corresponding operation in the guarded string algebra. The epistemic alternative relation on worlds for an agent is then defined as the concatenation closure of the event alternative relation for the agent.

Other operations in the guarded string algebra as defined in FST are simpler. Union is union in the string algebra. The complement of a proposition is

complement relative to the set of worlds, as defined by set difference in the string algebra.

This scheme provides for an interpretation of the language of Epik terms that is defined in Figure 3. An Epik specifications such as Figure 1 is translated to a straight-line program of the finite state calculus that defines constants and functions, including the ones from Figures 2 and 3. In Xfst or Foma, which are interpreters for the finite state calculus, the program is read, and then propositional terms can be mapped to finite state machines that represent sets of guarded strings, interpreted as propositions.

5 Direct Bounded Interpretation

We also implement the semantics of Epik terms using lazy lists (**?**) in Haskell, rather than the direct interpretation as sets. Unfortunately, regular expressions, and hence also Epik, can denote infinite sets of strings, for example the term $\_^*$. Normally regular languages are represented using a finite coalgebra (). However the non-distributivity of $\diamond$ accross ; complicates this construction[1]. To sidestep this, we parameterize the interpretation function on a positive integer n and only produce guarded strings of length n or less.

We translate the Epik terms into a Haskell algebraic datatype that represents terms with the same signature as described in Section 2. We then parameterize the interpretation function, $(\!|p|\!)_n^{\varphi,\mathcal{E}}$ on an integer n that describes the maximum length of a string we will produce. The bounded interpretation into lists of strings is very similar to the unbounded interpretation into sets of strings, except for the bounds checking. The full details are shown in Figure **??**: we use list-comprehension notation, which is analogous to set-builder notation, except that it produces a list (potentially with duplicated elements) instead of a set, and so is written with square brackets []. We use $+$ to indicate list concatenation, and $\sum l$ to indicate the concatenation of a list of lists l. We overload the set membership symbol $\in$ to also indicate membership in a list. We also lift the fusion operator $\diamond$ to lists. For a list of guarded strings l, we write $l|_n$ as shorthand for the set $[g \mid g \in l, |g| \leq n]$. Finally, we write $\lfloor X \rfloor$ to convert a set X into a list containing same elements with an arbitrary order.

First of all, whenever $n = 0$, the set of guarded strings is empty. TODO ADD MORE TEXT.

$$
\begin{array}{rcl}
(\!|p|\!)_0^{\varphi,\mathcal{E}} & \triangleq & [] \\
(\!|0|\!)_n^{\varphi,\mathcal{E}} & \triangleq & [] \\
(\!|1|\!)_n^{\varphi,\mathcal{E}} & \triangleq & \lfloor \mathcal{A}_{\mathsf{B}}^{\varphi} \rfloor \\
(\!|e|\!)_n^{\varphi,\mathcal{E}} & \triangleq & \lfloor \hat{\mathcal{E}}^{\varphi}(e) \rfloor \\
(\!|\psi|\!)_n^{\varphi,\mathcal{E}} & \triangleq & \lfloor \mathcal{A}_{\mathsf{B}}^{\varphi\ \psi} \rfloor \\
(\!|p+q|\!)_n^{\varphi,\mathcal{E}} & \triangleq & (\!|p|\!)_n^{\varphi,\mathcal{E}} + (\!|q|\!)_n^{\varphi,\mathcal{E}} \\
(\!|p;q|\!)_n^{\varphi,\mathcal{E}} & \triangleq & \left((\!|p|\!)_n^{\varphi,\mathcal{E}} \cdot (\!|q|\!)_n^{\varphi,\mathcal{E}} \right) |_n \\
(\!|p^*|\!)_n^{\varphi,\mathcal{E}} & \triangleq & [] + \left((\!|p|\!)_n^{\varphi,\mathcal{E}} \diamond (\!|p^*|\!)_{n-i}^{\varphi,\mathcal{E}} \right) |_n \\
 & & \text{where } i = \max\{1, \min\{|g| \mid g \in [\![p]\!]^{\varphi,\mathcal{E}}\}\} \\
(\!|\neg p|\!)_n^{\varphi,\mathcal{E}} & \triangleq & [g \mid g \in [\![\_^*]\!]_n^{\varphi,\mathcal{E}}, g \notin [\![p]\!]_n^{\varphi,\mathcal{E}}] \\
(\!|\Diamond_a\ p|\!)_n^{\varphi,\mathcal{E}} & \triangleq & [g' \mid g \in g'\ \hat{R}_a\ g, g \in (\!|p|\!)^{\varphi,\mathcal{E}}]
\end{array}
$$

Figure 7: Bounded interpretation using lazy lists

To convert from a list of guarded strings l to a set of guarded strings, we simply write $\lceil l \rceil$. Note that for any p, φ, $\mathcal{E}$, and n, $\left\lceil (\!|p|\!)_n^{\varphi,\mathcal{E}} \right\rceil = \{g \mid g \in [\![p]\!]^{\varphi,\mathcal{E}}, |g| \leq n\}$.

TODO Describe advantages of Lazy Lists

6 Syntax-semantics interface

An architecture of interpretation by transtlation is employed, where English sentences are mapped to terms in the logical language (–) via an interpreted grammar, and these terms are in turn interpreted as propositions (sets of possible worlds). For the latter, there are options of translation into the finite state calculus in order to represent propositions as finite state machines (Section 3), and representation in Haskell via lazy lists of guarded strings (Section 4). The grammar is a semantically interpreted multi-modal categorial grammar, consisting of a lexicon of words, their categorial types, and interpretations in a logical lambda language. Figure 8 lists phenomena that are covered.

As illustrated towards the end, there is recursion through conjuction and verbal complementation, so that the language is infinite, and includes talk of beliefs about beliefs, or in general, talk of arbitrarily iterated belief.

Figure 9 gives illustrative lexical entries. The grammar and semantics are in certain way optimized for a simple fragment of English concerned with clausal complementation. The agent names *Amy* and *Bob* contribute the epistemic alternative relations for those agents, rather than individuals.

[1] Axiomatic and Coalgebraic models for Epik are open questions

Basic statives	It's heads. It's tails.
That-complement	Amy knows that it's heads.
Wh-complement	Amy knows whether its heads.
Negation	Bob doesn't know that it isn't heads.
Tensed and base verbal forms	Bob knows that it's heads. Bob doesn't know that it's heads.
Sentence conjunction	It's heads and Bob doesn't know that it's heads
Predicate conjunction	Bob knows that Amy knows whether it's heads and doesn't know that Amy knows that it's heads.

Figure 8: Phenomena covered in the English grammar fragment.

This is possible because the agents are never arguments of extensional predicates, so what matters about the agents is their epistemic alternative relations. The root verb *know* contributes existential modal force. The complementizers *that* and *whether* are the heads of their dominating clauses, and assemble an alternative relation, modal force, and proposition contributed by the complement. These complementizers introduce the dual via two negations, in order to arrive universal modal force. These moves are offered here as a way of constructing a compact interpreted grammar. They can easily be reformulated in a more comprehensive interpreted grammar of English.

Multimodal categories such as $\backslash_D$ and $\backslash_M$ are used to control the derivation. For instance the category of *heads* $d\backslash_D t$. The dummy expletive subject *it* has category d, but the phrase *it heads* of category t can not be formed, because $\backslash_D$ is not syntactically active as a function. Instead *it is heads* can be formed with a predicator *is* of category $(d\backslash t)/(d\backslash_D t)$. (This uses Lambek/Bar-Hillel notation for slashes, so that $(d\backslash t)/(d\backslash_D t)$ combines with $d\backslash_D t$ on the right to give a value that combines with d on the left to give t.) Similarly *knows* has a category with the top-level slash $/_M$, and combines to form

Amy	e	R_a
Bob	e	R_b
it	d	d
heads	$d\backslash_D t$	$\lambda x.0^c{\cdot}h$
tails	$d\backslash_D t$	$\lambda x.0^c{\cdot}!h$
is	$(d\backslash t)/(d\backslash_D t)$	$\lambda P.\lambda x.Px$
knows	$(e\backslash t)/_M t$	$\lambda p.\lambda R.\Diamond Rp$
that	$(((e\backslash t)/_M t)\backslash(e\backslash t))/t$	$\lambda p.\lambda m.\lambda R.{\sim}(m({\sim}p)R)$
whether	$(((e\backslash t)/_M t)\backslash(e\backslash t))/t$	$\lambda p.\lambda m.\lambda R.{\sim}(m({\sim}p)R)$ $+{\sim}(mpR)$

Figure 9: Partial categorial grammar lexicon. The first column has a word form. the second column a categorial type, and third column a semantic translation in a logical language that extends the Epik term language with lambda.

a sentence as an argument of *that* or *whether*, which has a category that looks for the category of *know* on the left, after combining with a complement sentence on the right.

The semantic translations in the third column of Figure 9 use the Epik term language, incremented with lambda. The body of $\lambda x.0^c{\cdot}h$, which is the semantic lexical entry for *heads*, is a term denoting the set of all worlds where the coin is heads, represented as the set of all guarded strings that end with a Boolean valuation where the primitive proposition h (it's heads) is true. There is λx at the front because of a correspondence the grammar formalism uses a correspondence between syntactic and semantic types. However, it does not bind anything, because sentences such as *it isn't heads* have an expletive subject. The body $\Diamond Rp$ of $\lambda p.\lambda R.\Diamond Rp$, which is the semantic lexical entry of *knows*, is an Epik term denoting the pre-image of the set of worlds p according to the relation $\hat{\hat{R}}$ between guarded strings that is determined by the event-level relation R. This is not the right semantics for *Amy knows that it's heads*, because it is an existential modality $\Diamond_R p$, rather than an universal modality $\Box_R p$. This is corrected by the complementizer *that* or *whether*, which introduces the dual.

Sentences are parsed with a chart parser for categorial grammar. The semantics for complex phrases are obtained by syntactic application of semantic translations, accompanied by beta reduction. Semantic terms in the parsing formalism are expressions of untyped lambda calculus. The gram-

mar is set up so that lambda is eliminated by beta reduction in the semantic term corresponding to a sentence. In consequence, the semantic term translating a sentence is a term of the Epik language (–). Such a term designates a set of possible words (guarded strings) in the possible worlds model deterimened by an Epik specification such as the one in Figure 1. (13a) is an English sentence with predicate conjunction and three levels of clausal embedding. Using the grammar and parser, the sentence is mapped to the Epik term (**??**.) Using the result from Section 3, this term can be mapped in an implementation of the finite state calculus to a finite state machine that represents a countably infinite set of possible worlds, represented as guarded strings. Using the result from Section 4, it can be mapped to an infinite lazy list of guarded strings, representing the same set of possible worlds. Either of these is a concrete computational representation of the propositional semantic value ⟦Amy knows that Bob knows that Amy knows whether it is heads and knows that Bob does not know that Amy knows that it is tails⟧^{o}, in the familiar sense of Montague semantics for natural language.

(13) a. Amy knows that Bob knows that Amy knows whether it is heads and knows that Bob does not know that Amy knows that it is tails.

b.

7 Examples and discussion

Page breakdown

1,2	3.75	
3	1.25	FST translation
4	1.25	Haskell Epik
5	0.75	English CG fragment
6	1.0	Examples and discussion

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