# Homework 1

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September 19, 2017

## 1 Introduction

Mandelbrot set images are created by sampling a set of complex numbers and determining whether or not they meet a certain condition. By making a plot, differentiating between those who do and those who don't, one can get an astonishing, infinitely repeating, fractal pattern.

## 2 Theory

### 2.1 Mandelbrot Set

The Mandelbrot set is defined as a set of complex numbers c, for which the function:

$$z' = z^2 + c \tag{1}$$

does not diverge when iterated through from z=0. The condition set for the function not diverging, is that $|z| < 2$ for 100 iterations. The plot shown below is achieved by iterating through values of x and y from -2 to 2, in 1000 steps, such that $c = x + iy$, starting with z=0 and checking for every value if 1 holds or not. As an example, setting c=1, gives the sequence 0,1,2,5,26 ... which tends to infinity. On the other hand, c=-1 gives 0,-1,0,-1,0 ... and thus -1 is in the set. For the figure below, note that the axis are scaled. To retrieve the actual values, perform the operation $\times \frac{4}{1000} - 2$ on all ticks. The scaling is due to the fact that the data-points have to be stored in integer-values in a Numpy array.

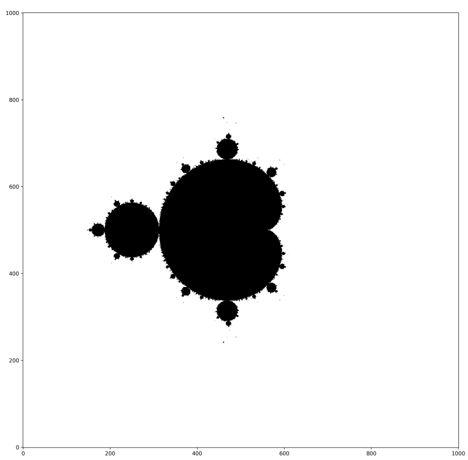


Figure 1: Mandelbrot Set

## 2.2 Least Squares Fit

When dealing with experimental data, it is often the case that we expect a linear relation between in the dataset. However, as with all experiments, there is always uncertainty and points will rarely (if ever) fall perfectly on a straight line. A way to create a good linear fit is through the method of least squares. To do this, we make a "guess" of the parameters m and c in a straight line of form y=mx+c. If we have N data points, we want the (squared) distance between this line and our points to be a minimum. To do this, we compute $\chi^2$, defined as follows:

$$\chi^2 = \sum_{i=1}^{N}(mx_i + c - y_i)^2 = \sum_{i=1}^{N}(m^2x_i^2 + 2cmx_i - 2mx_iy_i - 2cy_i + y_i^2) \tag{2}$$

To find the best fit in terms of the parameters m and c, we simply take the partial derivative with respect to m and c. Doing this and dividing by 2, gives:

$$m\sum_{i=1}^{N}x_i^2 + c\sum_{i=1}^{N}x_i - \sum_{i=1}^{N}x_iy_i = 0$$
$$m\sum_{i=1}^{N}x_i + cN - \sum_{i=1}^{N}y_i = 0$$

Defining the following quantities for simplicity:

$$E_x = \frac{1}{N}\sum_{i=1}^{N}x_i \qquad E_y = \frac{1}{N}\sum_{i=1}^{N}y_i \qquad E_{xx} = \frac{1}{N}\sum_{i=1}^{N}x_i^2 \qquad E_{xy} = \frac{1}{N}\sum_{i=1}^{N}x_iy_i$$

Leaves us with:

$$mE_{xx} + cE_x = E_{xy}$$
$$mE_x + c = E_y$$

which solved simultaneously for m and c give:

$$m = \frac{E_{xy} - E_xE_y}{E_{xx} - E_x^2} \qquad c = \frac{E_{xx}E_y - E_xE_{xy}}{E_{xx} - E_x^2}$$

The 'E' quantities are easily computed in Python and thus we simply retrieve the slope and intercept of our best fit. The data with the best fit is plotted here:

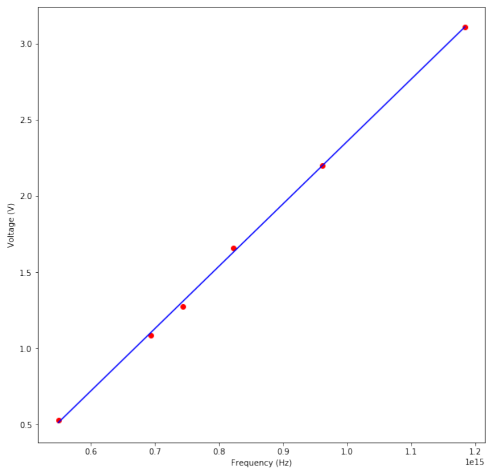


Figure 2: Least Squares Fit

The data stems from Millikan's experiment that measured the photoelectric effect. Photons are sent toward a slab of metal, where if the wavelength is appropriate, electrons can get ejected.

The energy of the electron is equal to that of the photon, minus some energy lost to releasing the electron, called the metal's work function, $\phi$. The energy of the electron can be measured by finding the voltage it takes to stop the electron. Thus we get the following, linear equation:

$$V = \frac{h}{e}\nu - \phi$$

The data in the plot above is of $V$ vs $\nu$, and thus, as Millikan did, we can get an experimental value of Planck's constant by multiplying the slope of our best fit by the electron charge. Our best-fit slope was $m = 4.08822735852e - 15$, which multiplied by $e = 1.602 \times 10^{-19}$ gives $h = 6.549e - 34$. The modern accepted value for Planck's constant is $6.626e - 34$ and thus our fractional error is 0.01156, or a little above 1%.

# 3 Discussion

## 3.1 Mandelbrot Set

The figure created clearly depicts a Mandelbrot Set. The main cardioid is clearly visible, consisting of all points of form $c = \frac{a}{2}(1 - \frac{a}{2})$ for all a within the open unit disk. Three perfect circles tangent to the main cardioid are visible, and there are in fact infinitely many circles tangent to the cardioid. One thing that is not visible in the plot is the famous fractal behaviour of the Mandelbrot set. For that, the resolution is too low. An interesting further effort would be to create a high-resolution animation, magnifying some part of the image to view some of the fractal behaviour.

## 3.2 Least Squares Fit

The method of least squares is one of the simplest and most intuitive ways of perform a fit to data. As can be seen from Millikan's data, the fit gives a decently accurate value for Planck's constant, considering the experiment was performed in 1914. What's even more impressing is that Millikan did this by hand! Of course, having more data available would also able us to get a better fit and an improved prediction.

One downside of the method of least squares is that it only takes into account error in the vertical direction from the line to the points. A way to improve this is by the method of Total Least Squares, taking into account error in both the x- and y-direction.