Quantum Computing and Graph Theory
Task 2

2025–02–24

Introduction

Complete the task below and compile a corresponding note containing

1. A detailed description of your (current) understanding of the problem.
2. A complete set of calculations with each step explained in as much detail as possible.
3. A collection of results with discussion.
4. A collection of any questions regarding this task at the end of the document.

After you have compiled your note, add a short summary (less than one page) stating what you have done, what results you obtained and a short comment regarding your confidence in the results.

You should complete this task by hand and then typeset your work in LaTeX. You should find `Mathematica` a helpful tool for this task, in particular the `Graph` and `AdjacencyMatrix` functions are generally useful. However, you are free to use any software package that you already know to complete this task.

Sub Task A

Consider a graph G with n vertices, labelled $\{v_1, \ldots, v_n\}$, and k edges, labelled $\{e_1, \ldots, e_k\}$. Permuting the vertex labels on G produces a new graph H with the same structure as G. This is an example of a graph *homomorphism*. We say that G is homomorphic to H and write $G \sim H$. For each graph G, there exists mupltiple such graphs H that are homomorphic to G. Let $\mathbb{H}_G$ be the collection of all graphs that are homomorphic to G.

Answer the following questions.

1. Show that $G \sim G$; give as much detail as you can.
2. Suppose that $G \sim H$ then show that $H \sim G$; give as much detail as you can.
3. Suppose that $G \sim H$ and $H \sim K$, then prove that $G \sim K$; give as much detail as you can.

Sub Task B

Now define $\mathbb{H}(n)$ to be the collection of all graphs with n vertices that are not homomorphic. Then a graph $K \in \mathbb{H}_G$ is a representative of the collection of graphs with a structure identical to that of G and we can think of $\mathbb{H}(n)$ as the set of such representatives.

Answer the following questions.

1. Let $n = 1$ fix the number of vertices.

(a) Construct $\mathbb{H}(1)$.

(b) Determine the number of elements in $\mathbb{H}(1)$.

(c) For each $G \in \mathbb{H}(1)$, determine the number of elements in $\mathbb{H}_G$.

(d) Check that you have accounted for all graphs with $n = 1$; explain how you checked this.

(e) List all pairs $(k,l) \in \mathbb{Z}^2$ for which $R(k,l)$ is well defined.

(f) Construct all two colour Ramsey numbers $R(k,l)$.

2. Let $n = 2$ fix the number of vertices.

(a) Construct $\mathbb{H}(2)$.

(b) Determine the number of elements in $\mathbb{H}(2)$.

(c) For each $G \in \mathbb{H}(2)$, determine the number of elements in $\mathbb{H}_G$.

(d) Check that you have accounted for all graphs with $n = 2$; explain how you checked this.

(e) List all pairs $(k,l) \in \mathbb{Z}^2$ for which $R(k,l)$ is well defined.

(f) Construct all two colour Ramsey numbers $R(k,l)$.

3. Let $n = 3$ fix the number of vertices.

(a) Construct $\mathbb{H}(3)$.

(b) Determine the number of elements in $\mathbb{H}(3)$.

(c) For each $G \in \mathbb{H}(3)$, determine the number of elements in $\mathbb{H}_G$.

(d) Check that you have accounted for all graphs with $n = 3$; explain how you checked this.

(e) List all pairs $(k,l) \in \mathbb{Z}^2$ for which $R(k,l)$ is well defined.

(f) Construct all two colour Ramsey numbers $R(k,l)$.

4. Let $n = 4$ fix the number of vertices.

(a) Construct $\mathbb{H}(4)$.

(b) Determine the number of elements in $\mathbb{H}(4)$.

(c) For each $G \in \mathbb{H}(4)$, determine the number of elements in $\mathbb{H}_G$.

(d) Check that you have accounted for all graphs with $n = 4$; explain how you checked this.

(e) List all pairs $(k,l) \in \mathbb{Z}^2$ for which $R(k,l)$ is well defined.

(f) Construct all two colour Ramsey numbers $R(k,l)$.

5. Describe a general procedure that you might attempt to compute $R(k,l)$ for an arbitrary n and list the difficulties you might expect to encounter when implementing such a procedure.