Firm Debt-relief in Financial Downturn

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Abstract

I study the aggregate implications of firm-specific debt-relief in a dynamic, stochastic general equilibrium model. The model is calibrated to the skewed size distribution, the age-size distribution, and key moments observed in U.S. data. Firms face persistent idiosyncratic risk and financial frictions that give rise to an endogenous distribution over capital, debt, and productivity, while leading to capital misallocation and life-cycle effects. This creates a role for policy in a financial crisis: government can borrow on behalf of firms that are financially constrained in and provide debt-relief. I find policy that targets firms with the highest level of excess return to capital investment to be the best in improving aggregates in a crisis, cutting the drop in output by over 50%. However, this may not be a readily observable policy target. I further consider firm size and age as alternative targets. Contrary to motivation for policy directed at small firms, I find that policy that targets medium size firms outperforms these other targets in dampening a recession, though they all fall short of the excess return policy.

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I Introduction

Does firm-targeted debt-relief dampen financial downturn? Around the time of economic downturn, the idea of debt-relief, or "bailouts," becomes more common[1] in political and economic discourse. These policies are usually idiosyncratic, varying in size, scope, and may also allow for changes in market structures. In recent U.S. history, debt relief policies have been implemented during the Great Recession and the COVID-19 pandemic. Motivation for policy during the Great Recession was driven by concerns of larger firms, specifically the "Big 3[2]" in U.S. Auto, laying off workers and a negative wealth effect spreading throughout the economy. Furthermore, larger firms are responsible for a greater share of aggregate economic activity than smaller firms[3]. On the other hand, the Paycheck Protection Program (PPP) established during the pandemic was chiefly concerned with small business, with the main target being firms with fewer than 500 employees[4]. Smaller firms are, on average, more likely to face higher marginal returns to investment[5]. Besides identifying which end of the size distribution to focus resources on, it remains to be seen if policy that targets size leads to the best improvements in aggregates during financial downturn.

To answer the question, I build a dynamic stochastic general equilibrium model with persistent firm-level heterogeneity and financial frictions. The model is calibrated to match key moments in the United States economic data, including the heavily skewed unconditional size distribution of firms, as well as the age-size distribution. The heterogeneity in production creates differences in optimal investment decision rules, while financial frictions can constrain

[1]Blau et al. (2013) measure this with two approximations for political engagement, lobbying and connections maintained.

[2]General Motors, Ford Motor Company and Chrysler, at the time.

[3]Crouzet and Mehrotra (2020) conclude that cyclical behavior in large firms firms may have implications for aggregate fluctuations, while small firms' behavior may not. This, along with he positively skewed employment distribution in the United States shown in the Business Dynamics Statistics (BDS) data may suggest the opposite targeting is appropriate.

[4]The second of three "draws" targeted 300 employees. Other financial targets were also implemented such as limiting net worth and hedge funds.

[5]Banz (1981) find that firm size is negatively correlated with risk adjusted returns; much work has been done discussing this correlation and whether or not causality exists (Berk, 1997; Asness et al., 2018). Gertler and Gilchrist (1994) state "While size per se may not be a direct determinant, it is strongly correlated with the primitive factors that do matter," suggesting policy would be better off targeting smaller firms.

some of these decisions by limiting firms' access to debt markets. The result is a dynamic distribution of firms, which provides a breadth of firm-level variables to target for policy analysis. I then use this model as a quantitative laboratory for policy experiments to study which debt-relief policy targets lead to the greatest improvements in aggregate outcomes.

I model financial market frictions by assuming collateralized borrowing. Investment among firms with insufficient collateral is limited by their inability to borrow, leading to a misallocation of capital. Young, productive firms are not large enough to post sufficient collateral, nor do they have enough retained earnings to reach their efficient investment level, slowing their growth. The financial frictions in the model endogenously produce these life-cycle characteristics. A shock to firms' access to credit exacerbates the misallocation already present in the model as growing firms have a stronger reliance on external financing. This leads to a deterioration of the aggregate capital stock. The recovery path is a function of the distribution of firms, which is inherently a slow moving object, and if capital requires "*time-to-build*," the recovery is slowed further, as firms are not able to update their capital immediately. This shock generates a recession with features similar to that of the 2008 Financial Crisis: an initial rise in consumption, a steep fall in investment, and a slow initial fall in output.

Financial frictions interact with firm-level productivity, a stochastic process drawn from a Pareto distribution. While the slowed growth seen in young and small firms can also be reproduced in models using the more common log-normal distribution (Khan and Thomas, 2013) for idiosyncratic productivity, this model delivers something further: many large firms are inefficiently smaller than they should be. Though these firms are considered large by their employment share, a significant portion of them have still not reached their optimal investment level implied by the long right tail of the productivity process. Thus, the model is well suited to study size and age debt-relief targets.

I divide debt-relief policy targets into two general categories: unobservable targets and observable targets. Unobservable targets rely on an omniscient policymaker knowing the entire state space in order to be implemented. Moreover, they magnify the forces underlying effective debt-relief policy. Observable targets are readily measurable firm-level variables that can be used as criteria for policy eligibility, for example, firm size, as well as age.

Debt-relief that aims to decrease and equalize the highest levels of excess return to capital investment across firms yields the best results, cutting the drop in output by over 50%. Excess return to capital investment is defined as the expected lifetime marginal value of investing in capital today, minus its cost. Targeting this measure of inefficiency moves resources to far more productive locations, causing a boom in endogenous TFP. Understanding that this may be a difficult target to measure, I consider readily observable policy targets, firm size and age.

Many policies[6] in the United States and abroad focus on financial assistance to small and young firms. However, I find that debt-relief policy that targets medium size firms[7] provides the best results when comparing recession troughs across observable policy regimes. Most aggregate variables are 0.5% closer to steady state values under this policy, while investment is 2% closer, compared to a scenario without relief. Consequently, the aggregate capital stock reaches its no policy half-life[8] roughly 1 year sooner than the no policy regime. However, these benefits are not without costs. In the model, the government issues bonds to pay for debt-relief and, eventually, must raise taxes to pay off public debt, leading to a slower recovery.

Policy that targets small and large firms, along with age-based policy for young firms age 0-5[9] are also considered and compared to an untargeted debt-relief policy where all indebted firms are eligible. I find that only the medium size policy target outperforms the untargeted

[6] Horvath and Lang (2021), Jo and Senga (2019), Guner et al. (2008)

[7] Firm size bins are defined by data from Census Business Dynamics Statistics, the quantitative details of which are discussed in the Calibration IV section.

[8] The level of capital corresponding to the point where it is halfway back to its steady state value, during a recession without relief policy.

[9] I consider an age target of 5 years, formally borrowing this definition of a young firm from Goetz and Stinson's (2021) Census report. They define a young firm as "firms with positive employment for five years or less." Moreover, Babina et al. (2019) find similar employment dynamics between firms of age 5 and firms of ages 4 and 6.

policy in improving aggregates in the heart of a financial crisis. The effect of decreasing returns to scale production is strong. Keeping the marginal value of policy resources higher across more firms boosts the impact of untargeted policy. However, there is a unique benefit to targeting young firms, giving firms the resources to grow faster increases labor demand, decreasing the amount of payroll taxes needed to cover government debt.

Specificity of targets must also be considered. Concentrating public funds on important variables sounds appealing at fist, but in a decreasing returns to scale environment, keeping the marginal value of a "*bailout-unit*" high may require less specific targets. Furthermore, keeping resources concentrated on a small subset of firms may lead to misallocation in and of itself. If firms receive more resources than what would be required for optimal investment, the rest will may be stored as savings when it could have been used as investment by another firm. Moving to the other extreme and eliminating targets all together can lessen these issues, but at the cost of foregoing focusing on important variables. Thus an untargeted policy is needed for comparison.

Debt-relief allows firms that suffer the most to access funds that the private sector will not provide, mitigating the effects of a financial crisis, providing a potential role for policy. Debt-relief that relaxes a firm's budget constraint affords them more investment, which dampens the impact of credit shocks, reducing the severity and the duration of a recession. In the model, policy is conducted by a government, which issues bonds in order to fund debt-relief. Taxes are then raised in order to pay off its debt. Here, the government's role is that of one in Woodford (1990), to borrow on behalf of agents that are financially constrained. The benefit of extended borrowing capacity depends on to whom it is extended. Thus, it is important to consider these potential targets to discover if targeted policy, or policy in general, is worth the cost of higher public debt.

In the model environment, firms invest in their capital stock through cash-on-hand and issuing bonds. Aside from collateral constraints, they also face a lower bound on dividends. The dividends restriction gives the collateral constraint its 'bite.' If a firm cannot afford to finance investment and they cannot provide enough collateral for a loan, this firm may not make up the difference by issuing sufficiently negative dividends[10]. This modeling choice allows me to capture some of the effects described in the corporate finance literature on the preference of debt financing over equity financing[11] in a tractable manner, while still focusing a quantitatively robust analysis on debt channels.

The remainder of the paper is prepared as follows. Section II provides a review of the literature. Section III describes the model and provides analysis. Section IV begins the quantitative exercise with calibrations. Section V quantifies measures of efficiency in the model, while section VI discusses policy to remedy the effects of a financial crisis. Section VII concludes.

II Literature Review

There is a large literature on fiscal policy improving macroeconomic outcomes. Bianchi (2016) studies a model of systemic and idiosyncratic debt-relief, where systemic policy is a function of only the aggregate state, while idiosyncratic policy takes firm-level variables as arguments, as well. Debt-relief is intended to alleviate working capital constraints, a la Jermann and Quadrini (2012). Under an idiosyncratic policy regime, the potential for debt-relief enters into the firm's decision rules, creating a potential moral hazard problem. Firms are incentivized to over-borrow and may be unable to cover the cash flow mismatch between their outlays and revenue next period. In this setting, systemic policy outperforms idiosyncratic policy.

This paper goes beyond the identical firm framework, bringing more depth to idiosyncratic policy. Indeed, I find the opposite result; targeted policy outperforms untargeted policy. Given the non-degenerate distribution of firms and the specific targets tested, most eligible

[10] See Bianchi (2016) as an example of setting a lower bound on dividends as a way to balance tractability with quantitative robustness.

[11] Shleifer and Vishny's 1997 research survey and Brav et al.'s 2005 managerial survey show that some firms will choose external capital or even give up positive NPV investments in order to not decrease dividends.

firms are already at (or near) their borrowing limit anyway. Thus even after the policy takes effect, they are not responsible for more debt than before.

In this spirit, I turn to Buera et al. (2013). Agents are given the choice to be laborers or entrepreneurs and the government targets high productivity, low wealth agents with a free capital policy. While there are short run benefits of subsidizing productive entrepreneurs, in the long run that productivity will eventually decay. Long run costs accrue from policymakers being unable to adjust the original subsidy policies (for a myriad of potential reasons) and end up funding unproductive entrepreneurs. I depart from Buera et al. in a few ways. First, this work concerns itself with potential stimulus during recession. Policy is intended to counteract the effects of failing credit markets. Although many well-intended policies have inefficient inertia, I study those that exist only for a short period, as in the motivating examples. Moreover, these policies do not distribute free capital. If a firm receives debt-relief, the still must pay for investment. Related to this, the interest rate will endogenously change with the aggregate state, rather than being set by a large open economy as in the two aforementioned papers.

Jo and Senga (2019) study fiscal responses to credit market imperfections. As they point out, there are important general equilibrium effects to consider intrinsic in extended access to debt, and therefore capital. Increased factor prices make investment more difficult for untargeted firms. Moreover, these prices effect entry and continuation decisions that are made on extensive margins. The general equilibrium effect may dominate the benefits of policy depending on the idiosyncratic shock process assumed. While their paper focuses on long-run resource allocation through steady state comparisons, this paper will abstract away from endogenous entry and exit and focus on transitional dynamics.

As mentioned previously, in this environment, and as in Woodford (1990), the government's role is to extend its access to debt markets to firms that are financially constrained. Holmström and Tirole (1998) expand this idea in the presence of liquidity shocks. Angeletos et al. (2023) solve, non-linearly, for optimal dynamics of tax and public debt in the presence of collateral constraints. In order to focus more on the relevant asset distribution and gains from targeting, optimal debt levels will be left to future work. I do, however, corroborate their finding that tax smoothing, from slower debt repayment, provides smoother consumption over time.

Finally, this work leans on literature studying financial frictions, namely collateralized borrowing as in Khan and Thomas (2013). The collateralized borrowing allows me to replicate important firm life-cycle aspects that are relevant for policy analysis. Young firms grow as their ability to borrow rises[12]. Including persistent, yet uncertain, idiosyncratic productivity shocks means optimal (unconstrained) capital choices will vary across firms, and change dynamically, creating a richer distribution of firms over excess returns. This allows me to explore different dimensions of policy targeting.

III Model

III.a Firms

The model economy is populated with a unit mass of firms that use labor, n, and predetermined capital, k, to produce a homogeneous consumption good through a decreasing returns to scale, twice differentiable, production function, $y = z\varepsilon f(k, n)$, where z is the aggregate productivity level. Each firms' idiosyncratic productivity, ε, is stochastic. I assume ε is drawn from a bounded Pareto distribution with bounds ε_L and ε_H, shape η, and discrete points N_ε, such that $\varepsilon \in \mathbf{E} \equiv \{\varepsilon_1, \dots \varepsilon_{N_\varepsilon}\}$, where $\Pr(\varepsilon' = \varepsilon_j \mid \varepsilon = \varepsilon_i) = \pi^{\varepsilon}_{i,j}$ and $\sum_{j=1}^{N_\varepsilon} \boldsymbol{\pi}^{z}_{i,j} = 1$ holds. Each period, there is a ρ_ε probability that firms keep their current value of ε. With probability $(1 - \rho_\varepsilon)$, they draw a new value from the ergodic Pareto distribution.

In order to fund investments, i, for next period's capital stock, k', firms use their revenues

[12] See Albuquerque and Hopenhayn (2004) for results driven by limited enforceability, Clementi and Hopenhayn (2006) for private information, and Bernanke and Gertler (1989) for agency costs.

and issue a risk-free bond, b', and b represents their current financial assets[13]. They face collateralized borrowing constraints and may not borrow more than a given fraction, ζ, of their current capital stock[14] each period. Collateralized borrowing is a reduced form way to capture the health of financial markets. A higher ζ reflects lenders' positive beliefs in the market, thus they are willing to lend more to firms. On the other hand, a lower ζ reflects just the opposite.

As firms age, their age being denoted by a, their borrowing capacity increases with capital accumulation. In order to prevent the whole economy from outgrowing financial conditions, each period a fraction, $\pi_d(a)$, of firms are exogenously forced to exit the market permanently at the end of a period. To reflect the negative correlation between age and exit rates seen in the BDS, this exit shock is a function of firm age. Thus, a given firm, (k, b, ε, a), is defined by its current capital stock, $k \in \mathbf{K} \subset \mathbf{R}_+$, debt level, $b \in \mathbf{B} \subset \mathbf{R}$, idiosyncratic shock realization, and age, $a \in \mathbf{A} \subset \mathbf{R}_+$. An exiting firm is replaced by an entrant with a capital stock of k_0, a total debt value of b_0, and idiosyncratic productivity, ε_0 drawn from the ergodic distribution implied by π_{ij}, from an infinite pool of potential entrants to ensure the mass of firms will not change. The resulting entrant rate is π_e.

At the start of each period, firms are made aware of their idiosyncratic productivity and whether or not they will exit the market. There is no penalty for exiting the market; firms know if they will exit and, if so, simply optimize their labor choice given their remaining capital stock and productivity, produce output, and settle any outstanding financial obligations, which had an upper bound given from last period's financial state. They do not borrow or invest. This timing is crucial, as it insures all private debt is risk-free.

Along with government debt, θ, to be described in Section III.b below, I can now characterize the state of the economy. The distribution of firms over (k, b, ε, a) is described by the probability measure μ defined on the Borel algebra, $\mathcal{S}$, generated by the open subsets

[13] The convention used is $b > 0$ represents positive bonds issued (*in debt*) and $b < 0$ represents net savings.

[14] Many papers study future capital Kiyotaki and Moore (1997), or earnings-based Lian and Ma (2021) collateral. This will not meaningfully change the analysis, subject to a properly calibrated ζ.

of the product space, $\mathbf{S} = \mathbf{K} \times \mathbf{B} \times \mathbf{E} \times \mathbf{A}$. The distribution of firms evolves over time according to the mapping, Γ, from the current aggregate state, $\mu' = \Gamma(\theta, \mu)$. This process is known to all agents in the economy. Let S represent the aggregate state, $S = (\theta, \mu)$.

III.a.1 The Firms' Problem

With the aggregate state described, I now begin to characterize a given firm's problem. Their cash-on-hand is is their output, less taxed wage bill, plus undepreciated capital stock, minus debt obligations, with the addition of a payroll tax, $\tau(S)$ and debt-relief, $g(\Theta; S)$. Capital depreciates at rate δ and takes 1 period to build. Therefore, cash-on-hand is written as:

$$x(k, b, \varepsilon_i, a; S) = z\varepsilon_i F(k, n) - \boldsymbol{(1 + \tau(S))} w(S)n + (1 - \delta)k - \boldsymbol{(1 - g(\Theta; S))} b \qquad (1)$$

where $(1 - g(\Theta; S))$ is the fraction of debt the firm is still responsible for and Θ is a vector of firm-level variables that may be targeted for policy. This style of debt relief, where the government pays for a fraction of firm debt is akin to that of Bianchi (2016).

When entering a period, the firm does not yet know if it will be forced to exit or be allowed to continue. Before making any decisions, its value is then given by $v_0(k, b, \varepsilon_i, a; S)$, the probability-weighted sum of exiting with cash-on-hand, or continuing on to the next period and receiving continuation value, $v(k, b, \varepsilon_i, a; S)$:

$$v_0(k, b, \varepsilon_i, a; S_l) = \pi_d(a)x(k, b, \varepsilon_i; S_l) + (1 - \pi_d(a))v(k, b, \varepsilon_i; S_l) \qquad (2)$$

Once the period starts and firms are made aware of their continuation status, they choose their dividend payments, D, to borrow or save b', at the loan discount factor $q(S)$, and capital for next period, k', while not exceeding their collateral constraint of $b' \leq \zeta k$. The law of motion for capital is $k' = i + (1 - \delta)k$. Taking as given the transition conditions of the idiosyncratic states, the continuation value is given by:

$$v(k, b, \varepsilon_i, a; S) = \max_{k', b', D} \left[D + d(S) \sum_{j=1}^{N_\varepsilon} \pi^{\varepsilon}_{i,j} v_0(k', b', \varepsilon_j, a'; S') \right] \tag{3}$$

subject to:

$$\begin{aligned} &\underline{D} \leq D \leq x(k, b, \varepsilon_i, a; S) + q(S)b' - k' \\ &b' \leq \zeta k \\ &\mu' = \Gamma(S) \end{aligned} \tag{4}$$

where $d(S)$ is the stochastic discount factor on the next period's value of the firm, $\underline{D}$ is the lower bound on dividends paid out to the household, and $q(S)$ is the risk-free loan discount factor.

Notice there are no inter-temporal considerations for labor demand and the wage bill is paid out of production, meaning all firms with the same (k, ε) will have the same labor demand. Thus, the firm's labor decision rule is $g_N(k, \varepsilon; S)$. Following from this, a firm's output is $y(k, \varepsilon; S)$. In an environment such as the one laid out thus far, decision rules are age invariant. Once age-based policy is introduced, capital and debt decisions rules require arguments from the full individual state; they are $g_K(k, b, \varepsilon, a; S)$ and $g_B(k, b, \varepsilon, a; S)$, respectively.

III.b Government

The government is endowed with two sets of resources: revenue from a payroll tax, and funds borrowed, θ', from the household at the risk-free rate[15]. It uses these to pay for government expenditures, $\overline{G}$, public debt, θ, and private debt-relief, $T(\Theta; S)$, which is the

[15] I assume the sovereign cannot default.

sum of all relief provided:

$$T(\Theta;S) = \int_{\mathbf{S}} g(\Theta;S) b\mu(d[k \times b \times \varepsilon \times a]) \tag{5}$$

Bonds issued by the government are either repaid through tax revenues, or rolled over to the next period at the risk-free rate. Public debt is a type of consol debt, where the speed at which it is paid is governed by the parameter ϕ, the fraction of debt paid. Conditional on making a payment, $(1-\phi)$ is the fraction of debt rolled over to next period. Finally, expenditures, $\overline{G}$, are *not* valued by households, *nor* are they used in production. The government's budget constraint is:

$$\tau(S)w(S) \int_{\mathbf{S}} [g_N(k,\varepsilon;S)]\mu(d[k \times b \times \varepsilon]) + q(S)\theta' \geq \overline{G} + \theta + T(\Theta;S) \tag{6}$$

where θ' is determined by:

$$\theta' = \begin{cases} (1-\phi)\theta & \textit{make payment} \\ \frac{1}{q(S)}\big(G(S) + \theta + T(\Theta;S) - \tau(\phi,S)w(S)N(S)\big) & \textit{debt rollover} \end{cases} \tag{7}$$

The functionality of this set up is that it pins down a singular path of taxes over time for a given ϕ, as the lowest tax rate that satisfies the budget condition. Varying ϕ allows for the analysis of different repayment horizons.

III.c Household

There is a representative household endowed with ownership of the firms and a unit of time that may be divided between labor and leisure. The household supplies labor to the firms in exchange for a wage,w, transfers resources intertemporally by purchasing bonds, κ', and shares, λ', and chooses consumption, c, to maximize its lifetime value, $W(\lambda,\kappa;S)$. Note in this set up, government and corporate bonds are perfect substitutes for the household.

The household discounts the future at rate $\beta \in (0,1)$ per period and maximizes its periodic utility over consumption and leisure, $U(c, 1-n)$, subject to their budget constraint. The full household problem[16] is therefore:

$$W(\lambda, \kappa; S) = \max_{c, n^h, \lambda', \kappa'} \left[U(c, n^h) + \beta W(\lambda', \kappa'; S')\right] \tag{8}$$

subject to:

$$\begin{aligned} c + q(S)\kappa' + \int \rho_1(k', b', \varepsilon', a'; S)\lambda'(d[k' \times b' \times \varepsilon' \times a') \leq \\ w(S)n^h + \kappa + \int \rho_0(k, b, \varepsilon, a; S)\lambda(d[k \times b \times \varepsilon \times a]) \\ \mu' = \Gamma(S) \end{aligned} \tag{9}$$

where $\rho_1(k', b', \varepsilon', a'; S_l)$ is ex-dividend price of a share and $\rho_0(k, b, \varepsilon, a; S_l)$ is dividend-inclusive value of a share. Let $h^c(\lambda, \kappa; S)$ be the household's decision rule for consumption, $h^n(\lambda, \kappa; S)$ be the decision rule for labor hours, and $h^\kappa(\lambda, \kappa; S)$ be the decision rule for bonds. Finally, let $h^\lambda(k', b', \varepsilon', a', \lambda, \kappa; S)$ be the decision rule for shares in firms with next period values of k' capital, b' debt, and ε' idiosyncratic productivity, of age a[17].

III.d Recursive Equilibrium

A *recursive competitive equilibrium* is a set of functions:

$$\left(w, q, \{d_m\}_{m=1}^{N_s}, \rho_0, \rho_1, v_0, g_N, g_K, g_B, g_D, W, h^c, h^N, h^\lambda, h^\kappa\right)$$

that solve the firms' and household's problems and clear the asset, labor and output markets, such that:

[16] The household has access to a complete set of state-contingent claims. These assets are in zero net supply in equilibrium; for simplicity I do not model them here.

[17] The household can choose shares of these firm types in equilibrium, since it knows the transition probabilities of the aggregate and idiosyncratic states and the law of large numbers applies.

(i) v_0 solves (2-3) subject to (4), g_N is the decision rule for exiting firms, and (g_N, g_K, g_B, g_D) are the decision rules for continuing firms

(ii) The government's budget constraint (6), subject to (7) is satisfied

(iii) W solves (8) subject to (9) and $(h^c, h^N, h^\lambda, h^\kappa)$ are the household's decision rules

(iv) The share market clears: $h^\lambda(k', b', \varepsilon_j, a', \lambda, \kappa; S) = \mu'(k', b', \varepsilon_j, a'; S)$ for $(k', b', \varepsilon_j, a') \in \mathbf{S}$

(v) The labor market clears: $h^N(\lambda, \kappa; S) = \int_{\mathbf{S}} [g_N(k, \varepsilon; S)] \mu(d[k \times b \times \varepsilon \times a])$

(vi) The goods market clears: $C(\lambda, \kappa; S) = \int_{\mathbf{S}} \Big[z\varepsilon F(k, n(k, \varepsilon, a; S)) - (1 - \pi_d(a)) \big(g_K(k, b, \varepsilon; S) - (1 - \delta)k \big) + \pi_e \big((1 - \delta)k - k_0 \big) \Big] \mu(d[k \times b \times \varepsilon]) - \overline{G}$

(vii) $\forall (A, \varepsilon_j) \in \mathcal{S}$ defines Γ, where $\chi(k_0) = \{1 \text{ if } (k_0, 0) \in A; 0 \text{ otherwise}\}$

$\mu'(A, \varepsilon_j) = (1 - \pi_d(a)) \int_{\{(k,b,\varepsilon_i,a) | (g^K(k,b,\varepsilon_i,a;s,\mu), g^B(k,b,\varepsilon_i,a;s,\mu)) \in A\}} \pi_{ij} \mu \left(d \left[k \times b \times \varepsilon_i \times a \right] \right) + \pi_e \chi(k_0) H(\varepsilon_j)$

(viii) The bond market clears through Walras's Law: $h^\kappa(\lambda, \kappa; S) = \int_{\mathbf{S}} [g_B(k, b, \varepsilon, a; S)] \mu(d[k \times b \times \varepsilon \times a]) + \theta'$

III.e Prices

I begin my analysis by deriving a set of optimality conditions from the household's problem that may be subsumed into the firms' problem. Let C and N represent the optimal household choices of consumption and labor hours, respectively. Furthermore, let C' and N' be the optimal choices of consumption and labor next period when the aggregate state is S'. Then the real wage (10) is the marginal rate of substitution between leisure and consumption. Since the household owns the firms, the firms' stochastic discount factor (11) of future value is the household's marginal rate of substitution between consumption across states. The loan discount factor (12), is the conditional expectation of the marginal rate of substitution between consumption across states[18][19].

$$w(S) = \frac{D_2U(C, 1-N)}{D_1U(C, 1-N)} \tag{10}$$

$$d(S) = \beta \frac{D_1U\left(C', 1-N'\right)}{D_1U(C, 1-N)} \tag{11}$$

$$q(S) = \beta \frac{D_1U(C', 1-N')}{D_1U(C, 1-N)} \tag{12}$$

Finally, let $p(S)$ be the household's marginal valuation of output. Again, since the household owns the firms, I can multiply the firms' value (3) by $p(S)$ to price all (current and future) output in units of marginal utility without altering decision rules. Letting $V_0 = p(S)v_0$ and $V = p(S)v$, this changes equations (2-3, 10-12) such that:

$$V(k, b, \varepsilon_i; S_l, a) = \max_{k', b', D} \left[p(S)D + \beta \sum_{j=1}^{N_\varepsilon} \pi^{\varepsilon}_{i,j} V_0(k', b', \varepsilon_j, a'; S') \right] \tag{13}$$

$$w(S) = \frac{D_2U(C, 1-N)}{p(S)} \tag{14}$$

$$d(S) = \beta \frac{D_1U\left(C', 1-N'\right)}{p(S)} \tag{15}$$

$$q(S) = \beta \frac{D_1U\left(C', 1-N'\right)}{p(S)} \tag{16}$$

where:

$$p(S) = D_1U(C, 1-N) \tag{17}$$

III.f Allocations

A firm's labor choice is static. It chooses $n^*(k, \varepsilon) = z\varepsilon D_2F(k, n) = (1+\tau)w$. Output is then $y(k, \varepsilon) = z\varepsilon F(k, z\varepsilon D_2F(k, n))$, which can be used to define cash-on-hand in (1).

[18] As there is no aggregate uncertainty, the next state is known, therefore the probability weight of it occurring is 1.

[19] The risk-free interest rate on bonds is: $r = \frac{1}{q(S)} - 1$.

Turning from choices involving only static variables to those that involve dynamic ones, I divide the analysis into two groups of firms to illustrate the potential for policy to improve capital allocations. The first group can reach their optimal capital level unimpeded by the collateral constraint. Let $k'^{*}(\varepsilon) = \text{argmax}_{k'} V(k, b, \varepsilon, a; S)$ for this group, which I will also use as a reference point for the next group. The second group cannot post sufficient collateral to afford optimal investment.

Firms unimpeded by their collateral invest so $\frac{\beta}{p(S)}\left(\mathbb{E}_{\pi^s}\,\mathbb{E}_{\pi^\varepsilon}\, D_1 V_0(k', b', \varepsilon_j, a'; S'_m)\right) = 1$. The expected marginal discounted (recall $p(S'_m)$ is embedded in V_0) value of an extra unit of capital equals its price, 1 output unit. Applying the Benveniste and Scheinkman condition, this is:

$$\begin{aligned} \frac{\beta}{p(S)}\Bigg[p(S')\sum_{j=1}^{N_\varepsilon}\pi^{\varepsilon}_{i,j}\Big(z\varepsilon_j D_1 F\big(k'^{*}(\varepsilon), n^*(k'^{*}(\varepsilon), \varepsilon_j)\big) & \qquad (18) \\ + \big[1 - \big(1 + \tau(S')\big)w(S')\big] D_1 n^*(k'^{*}(\varepsilon), \varepsilon_j) + (1-\delta)\Big)\Bigg] &= 1 \end{aligned}$$

These firms have no excess returns to capital investment. Any debt-relief policy will produce no extra output. Another way to see this is that neither the collateral constraint, nor debt enter into this optimality condition. Any relief will change neither labor, nor capital, and output will remain the same. Their choice of debt is then $b' \in \left[\frac{\underline{D} + k'^{*}(\varepsilon) - x(k,b,\varepsilon,a;S)}{q(S)}, \zeta k\right]$ from the conditions in (4), to be discussed further below.

Moving to firms bound by their collateral, I take advantage of the monotonically increasing value function in k to simplify the problem. Firms will borrow until $b' = \zeta k$ and invest such that:

$$k'(k, b, \varepsilon, a) = x(k, b, \varepsilon, a) + q(S)\zeta k - \underline{D} < k'^{*}(\varepsilon) \qquad (19)$$

The strict inequality is the center of the misallocation problem that policy would seek to correct. These firms cannot reach their desired capital level and are unable to hire as much labor or produce as much output next period, compared to an environment without these financial frictions. Moreover, given decreasing returns to scale production, these firms have a higher shadow price on their collateral and higher returns on potential investment than firms without binding constraints. Again using the properties of the value function and going back to (18), substituting in $k'(k,b,\varepsilon)$ for $k'^{*}(\varepsilon)$ returns:

$$\frac{\beta}{p(S)}\Bigg[p(S')\sum_{j=1}^{N_\varepsilon}\pi_{i,j}^{\varepsilon}\Big(z\varepsilon_j D_1 F\big(k'(k,b,\varepsilon),n^*(k'(k,b,\varepsilon),\varepsilon_j)\big) \\ +\big[1-\big(1+\tau(S')\big)w(S')\big]D_1 n^*(k'(k,b,\varepsilon),\varepsilon_j)+(1-\delta)\Big)\Bigg]>1 \tag{20}$$

Through (19), it is easy to see how a credit shock exacerbates these conditions, as the more $k'^{*}(\varepsilon)-k'(k,b,\varepsilon,a)$ increases, the more the inefficiencies grow. However, this is where the role for policy exists. Debt-relief policy that increases a firm's cash-on-hand can shrink, or even eliminate, the magnitude of the inequality in (19), as well as the difference between (20) and (18). Thus policy may increase output, investment, and labor hours compared to the same scenario without policy.

There is still the matter of issuing dividends and the unconstrained firms' debt. From here out, I assume a *zero-dividends* policy, meaning $\underline{D}=D=0$. Firms use as much of their retained earnings as needed to invest. Those that need to borrow for optimal investment, but are unconstrained in doing so, will borrow to make up the difference. Funds leftover after investment will be used as savings. Constrained firms take this $\underline{D}$ as given in (19). Given market clearing, the household is indifferent to the firm paying dividends, or retaining any resources leftover after investment as savings; this policy assigns the decision rule to the latter. Therefore, this is an optimal dividend policy for all firms[20].

Finally, I set all steady state quantities in the government's budget to 0. Since this paper is concerned with policy responses to shocks, the level of these steady state variables will have little to no impact on the insights derived from the results and are just a calibration

[20]The (weak) optimality of the zero-dividend policy should not be confused with multiplicity of equilibria, which does not exist in this model. This is a result of the firm's value function being linear in D and b.

exercise.

IV Calibration

The model has an annual frequency. The common DRS production function is Cobb-Douglas: $z\varepsilon F(k,n) = z\varepsilon k^{\alpha} n^{v}$. Entrant firms' capital supply is a fixed at χ percent of the steady state aggregate stock: $k_0 = \chi \int k\tilde{\mu}(d[k \times b \times \varepsilon \times a])$. Household utility, $U(c, 1-n^h) = \ln(c) + \psi(1-n^h)$, is the result of Hansen (1985) - Rogerson (1988) indivisible labor.

The household discount factor, β is set to 0.96, to imply a 4% risk-free real interest rate, as found in Gomme et al. (2011). The parameter governing labor's share of output, ν is set to 0.60 to match Cooley et al. (1995). I set the depreciation rate, δ, to imply an average investment-to-capital rate of 0.069, which matches the the BEA Fixed Asset Tables data. The parameter governing the marginal utility of leisure, ψ, is set to 2.14, so that in equilibrium, 1/3 of the household's time endowment is spend on market labor. χ is set to 0.208, so that the relative employment size of the average entrant is 0.260 of the average firm, matching that in the BDS from 1990-2006. The entrant rate, π_e is set to 0.106, in order to keep the mass of firms constant at 1, given the vector of exit rates by age, $\pi_d(a)$, taken from the BDS (1990-2006), in Table 1 below.

Table 1: Exit Hazard by Age

Age:	1	2	3	4	5	6-10	11+
Hazard:	0.2478	0.1640	0.1356	0.1174	0.1062	0.0840	0.0655

Note: Exit rates taken from BDS (1990-2006).

The next parameters are jointly calibrated to match the following moments. The parameter governing capital's share of output, α is set to 0.28 to target a capital output ratio of 2.25, from the BEA Fixed Asset Tables 1954 - 2006. The probability of a firm maintaining its ε draw is 0.99, to target a standard deviation of investment rates across firms of 0.337, from Cooper and Haltiwanger (2006). During normal economic times, the collateral constraint will be $\zeta_o = 0.981$, to target a debt-to-asset ratio of 0.372, as found from 1954-2006 nonfarm, nonfinancial businesses in the Flow of Funds. Finally, during a credit crunch, this value changes to $\zeta_l = 0.647$ in order to facilitate a peak-to-trough decrease in debt of about 26%. Khan and Thomas (2013) report this GDP deflated real lending decrease in Commercial and Industrial loans in their data from 2008Q4 - 2011Q4. Calibrated parameters are reported on Table 2, along with their closest related data target.

Table 2: Parameters

	Parameter		Target		Model
β	discount factor	$= 0.96$	real interest rate	$= 4.0\%$	4.16%
ν	labor share	$= 0.600$	labor share	$= 0.600$	0.600
χ	fraction of entrant K	$= 0.208$	entrant average N share	$= 0.260$	0.260
δ	depreciation	$= 0.069$	investment/capital	$= 0.069$	0.069
ψ	leisure preference	$= 2.150$	labor hours	$= 0.333$	0.332
$\frac{b_0}{k_0}$	entrant leverage	$= 0.400$	entrant leverage	$= 0.400$	0.400
π_e	entrant rate	$= 0.106$	firm mass	$= 1.000$	1.000
α	capital share	$= 0.280$	capital/output	$= 2.250$	2.305
ρ_ε	maintain ε	$= 0.990$	stand dev i/k	$= 0.337$	0.358
ζ_o	collateral fraction	$= 0.981$	debt/assets	$= 0.372$	0.372
ζ_l	credit crunch	$= 0.647$	decrease in debt	$= 26.0\%$	26.0%

IV.a Size Distributions

The model captures the size and age-size distributions in the U.S. through the use of a bounded Pareto distribution for firms' idiosyncratic shock process, discretized using $N_\varepsilon = 7$ values. The process of defining firm sizes bins in the model comes from Jo and Senga (2019). First, I define the threshold for small, medium, and large firms to be 1-19, 20-499, and 500+ employees respectively. To map this to the model, I use each bin's corresponding employment share, calculated from the BDS (1990-2006), to identify thresholds, $\underline{n}$ and $\overline{n}$ in the model such that firms that account for an employment share $\leq \underline{n}$ are small, firms that account for an employment share $\in [\underline{n}, \overline{n}]$ are medium, and firms that account for an employment share $\geq \overline{n}$ are large. These thresholds are defined such that the employment share bins they create exactly match the BDS data[21]. Then I check the resulting population share in each

[21]Since a firm's (k, ε) state determines its labor demand, it is possible that the employment share given by a (k, ε) makes a given bin too large compared to the BDS data. In that case, I allocate enough mass to

bin and compare that to the data. Using the population share of each firm size as calibration targets, I set the last remaining parameters to $\varepsilon_L = 0.497, \varepsilon_H = 0.937,$ and $\eta = 5.5$ to target the unconditional size distribution. These size bins will be used as eligibility thresholds for size-based policy in Section VI.b.1. The results are reported on Table 3.

Table 3: Size Distribution

Employment Bins	Emp. Share	Data Pop. Share	Model Pop. Share
Small (1-19)	0.201	0.885	0.880
Med. (20-499)	0.319	0.112	0.101
Large (500+)	0.480	0.003	0.019

Given χ is chosen to match the average entrant's relative share of employment, the model generates a age-size distribution comparable to what is observed in the U.S. for firms age 0-5, as well as the population weighted mean of each age group. They are reported below on Table 4.

Table 4: Model Generated Age-Size Distribution

Age:	0	1	2	3	4	5	Pop. Weighted Average
Data:	0.260	0.338	0.378	0.417	0.451	0.477	0.368
Model:	0.260	0.302	0.362	0.427	0.506	0.595	0.377

V Quantifying Inefficiency

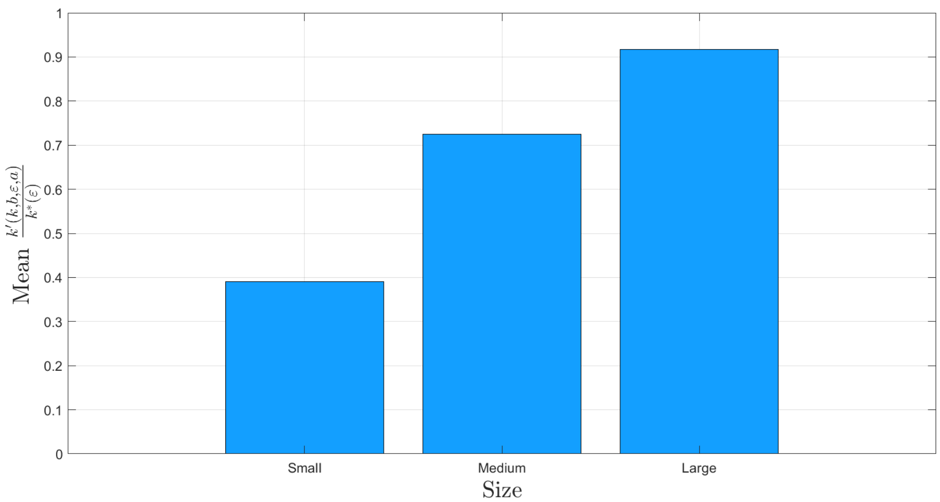
V.a Steady State Inefficiencies

As described in Section III.f, there are inefficiencies present in the model before any shocks. Financial frictions limit investment and capital accumulation. Small firms do not have a large enough capital stock to post collateral for a loan, nor do they have enough retained earnings to afford their optimal investment level outright. This creates a wedge between their choice of $k'(k, b, \varepsilon, a)$ and the optimal k'^{*}. Figure 1 shows the mean ratio of a

the smaller bin to match the data and place the rest in the next bin.

firm's capital choice to their optimal capital level, implied by their productivity, for different size bins. In a perfectly optimal environment, all firms would be able to reach their target capital, which would be shown as all three bars at $y = 1$ since $k'(k, b, \varepsilon, a) = k'^{*}(\varepsilon)$. However, given the financial frictions present, there is an inverse relationship between size and distance to optimal capital.

Figure 1: Average $\frac{k'(k,b,\varepsilon,a)}{k'^{*}(\varepsilon)}$ Ratio by Firm Size

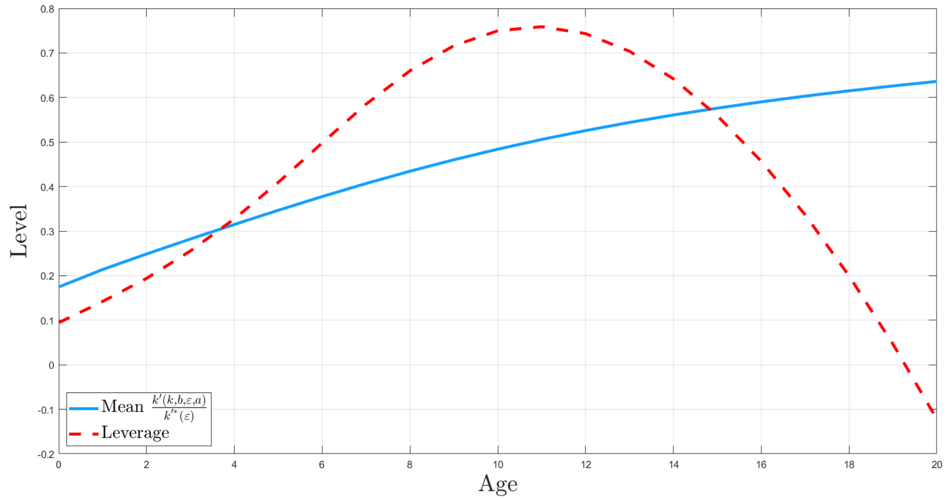


Note: Small firms do not have sufficient collateral or retained earnings to afford optimal investment, increasing the distance between their choice of capital and the optimal choice implied by their productivity level.

As evidenced by Table 4, firms grow as they age. They build up a stock of retained earnings to afford more capital investment. At first, this happens through external financing and taking on more debt. Building up their capital stock also allows them to post more collateral, giving them access to more external financing, further increasing their investment potential. As firms grow deeper into their life-cycle, they begin to deleverage as their retained earnings become larger, covering a greater share of investment. Thus, the older firms get, the more they are able to close the gap between their choice of capital, $k'(k, b, \varepsilon, a)$, and their efficient level, $k'^{*}(\varepsilon)$. This is a slow process; on average, firms to not reach half of their optimal capital stock until age 11 and do not become net savers until age 19. Figure 2 displays this trend.

As firms move away from their optimal capital choice, their excess return to capital

Figure 2: Average $\frac{k'(k,b,\varepsilon)}{k^*(\varepsilon)}$ and Leverage across Ages



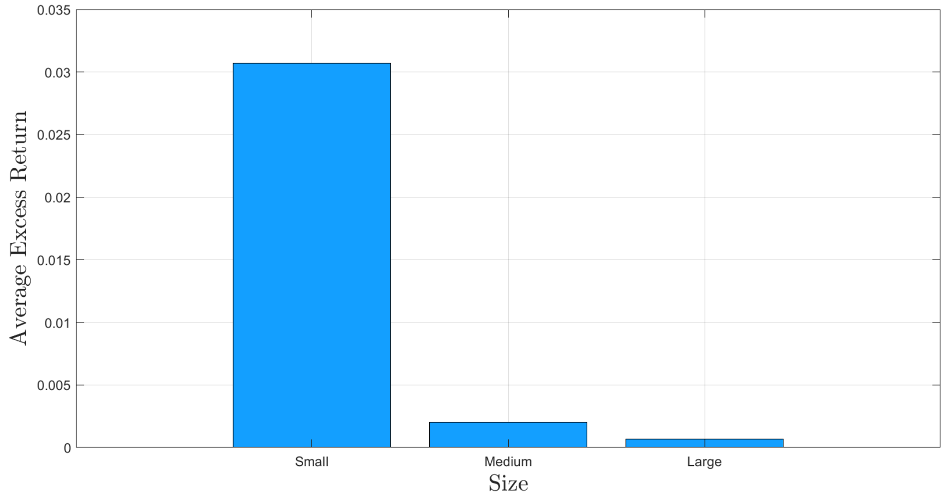
Note: As firms move through their life-cycle, they build up a stock of retained earnings to finance capital investment. They rely more on external financing earlier in their life-cycle, and deleverage as they age.

investment grows. From the parameterization of the model, (18) becomes:

$$\frac{\alpha}{1-\nu}\frac{\beta}{p(S)}\left(p(S')\sum_{j=1}^{N_\varepsilon}\pi^{\varepsilon}_{i,j}k'^{\frac{\alpha+\nu-1}{1-\nu}}\left[z_m\varepsilon_j\left(\frac{\nu z_m\varepsilon_j}{\big(1+\tau(S)\big)w(S)}\right)^{\frac{\nu}{1-\nu}}\right.\right.$$
$$\left.\left. -\big(1+\tau(S)\big)w(S)\left(\frac{\nu z_m\varepsilon_j}{\big(1+\tau(S)\big)w(S)}\right)^{\frac{1}{1-\nu}}\right]+(1-\delta)\right)=1 \tag{21}$$

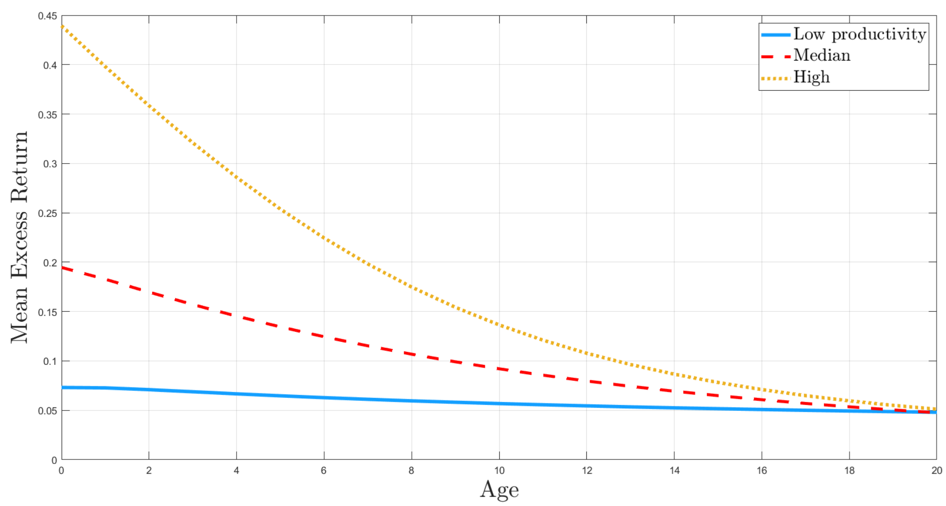
The further their capital choice is from the optimal level, the more the marginal lifetime benefit of capital investment increases and the Left Hand Side of the equation grows bigger than 1, the marginal cost of a unit of capital. On average, smaller and younger firms are farther from their optimal capital level than other firms, increasing their excess return to investment. Moreover, productive firms have a higher marginal product of capital, increasing their $k'^*(\varepsilon)$, and thus their excess return on investment as well. These results highlight the concentration of inefficiency found in small firms, as well as young and productive firms . Figures 3 and 4 illustrate these results.

Figure 3: Average Excess Return to Investment by Size



Note: Larger firms are able to offer more collateral, or use higher retained earnings to finance investment. They are closer to their optimal investment, thus on average face lower excess returns.

Figure 4: Average Excess Return to Investment by Productivity across Ages



Note: More productive firms have a higher optimal level of capital and thus a higher level of optimal investment. This increases average excess return to investment. Moreover, as firms age, the accumulate more capital increasing their access to external financing, as well at their retained earnings.

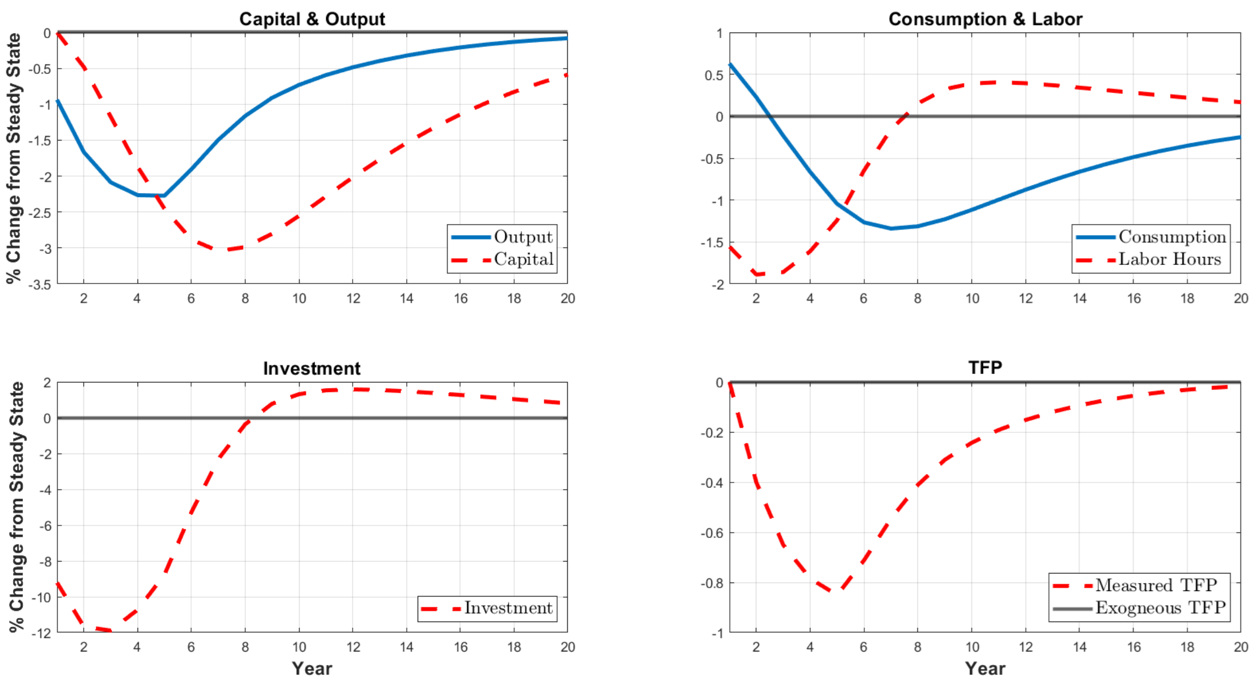
V.b Financial Crisis

I now being the dynamic analysis of this study with a discussion on financial shocks without fiscal policy. The shock in the model is a credit crunch, an economy-wide contraction of credit[22], calibrated to match the 26% fall in debt discussed in the Calibration, Section

[22] Ivashina and Scharfstein (2010) find a large decrease in lending at the start of the Great Recession.

IV. The shock in unexpected decrease in ζ_o to ζ_l and persist for 4 periods, then recovers at a rate of 31.25% per year, agreeing with Khan and Thomas (2013)'s calibration to the data presented in Reinhart and Rogoff (2009). Exogenous TFP does not change in this exercise.

Figure 5: Financial Crisis without Debt-relief



Note: Model response to an unexpected credit shock of ζ_o falling to ζ_l. Shock persists to date 4 and recovers at rate $\rho_\zeta = 31.25\%$ per year.

The response to the credit crunch shares defining characteristics with the Great Recession. There is an initial increase in consumption, as once the shock hits the household foresees lower returns to saving. Moreover, the fall in output is persistent as it takes time to rebuild the capital stock after the sharp downturn in investment. The persistent decline in endogenous TFP results from changes in the distribution of firms. Financial requirements have forced the majority of firms to become smaller, increasing misallocation of resources in the economy. Figure 5 displays these dynamics.

VI Debt-relief Policy

The purpose of debt-relief in the model is to relax the budget constraint of firms to enable them to afford more investment, offsetting the impact of a credit crunch. The response to the credit conditions laid out is idiosyncratic, implying the impact of different policies will be as well. Given a fixed supply of policy funds, policymakers will be concerned with the best return on the resources spent and will focus resources accordingly.

I divide debt-relief policy targets into two general categories: unobservable targets and observable targets. Unobservable targets require an omniscient policymaker to implement and serve as exposition for the underlying economic forces at hand. Observable policy targets are firm-level variables that are readily measurable, size and age. Moreover, firm size and age track well with investment return and growth, both in the micro-data and in the model and serve as policy recommendations that can approximate more efficient, but unobservable targets.

Common across all policy experiments will be the total amount of debt-relief paid out, $T(\Theta, S)$, to ensure that each policy has the same amount of resources being expended, as well as the government's debt repayment start date and repayment fraction, ϕ, to ensure no exogenous variations in the tax rates. Repayment begins in period 7, the half-life point of the output recover. This avoids raising taxes in the heart of a recession. All policies will be in response to the credit shock described above in Section V.b and will occur on impact date. The total resource cost will be 4% of steady state output and tarting at date $t = 7$, public debt will be paid down 5% per period, with the exception of one "free" policy shown to isolate the effect of taxation on the recovery path. This policy is not literally free; resources are not being created from nowhere, rather, the government rolls over its debt each period by borrowing more to pay for its current debt. These parameters are reported on Table 5.

VI.a Unobservable Targets

The first policy experiment considered directly targets firm level excess return to capital investment. This policy identifies firms with the highest level of excess returns and issues debt relief to them until they match the firms with the second highest level of excess returns. The policy algorithm proceeds in that manner until funds are exhausted[23].

This policy design has another added benefit; it naturally establishes a threshold for *eligible debt*. Some firms have debt that would be inefficient to be relieved. Consider a firm that has some positive level of debt, but not enough for the collateral constraint to be binding. This firm can reach its optimal investment level, so any debt-relief resources would just be held as savings by the firm (or passed on as dividends to shareholders in other specifications). Relief leads to no extra output, but still must be paid for by future tax increases. Thus optimal policy would set a threshold of eligible debt such that any relief payments being made will be used for investment by the firm. This then avoids concerns of funds not reaching their intended destination, a concern that has been raised for the Paycheck Protection Program (Li (2021), Autor et al. (2022)). This eligible debt threshold is:

$$\max\Big\{\min\big[\frac{k'^{*}(\varepsilon) - x(k, b, \varepsilon, a)}{q(S)}, b\big], 0\Big\} \tag{22}$$

This equation makes sure that the firm in question has debt to be relieved in the first place. Beyond that, the relief policy will only pay out to the firm until their cash-on-hand becomes sufficient to pay for their optimal investment level. At that point the relief turns off and the firm will not again become eligible, even if there is remaining $b > 0$. By following the excess returns policy program, this threshold is naturally enforced.

The process of solving for these transition paths involves two simultaneous guesses on the time vectors of consumption and taxation. The appendix provides a general overview of

[23]Computationally, one could implement this policy as described in the text. Another way to do so would be to realize there will be a level of excess returns that no indebted firm will exceed. A guess of this level implies a total resource cost for the government and this guess may be iterated on until convergence to the exogenously chosen resource cost.

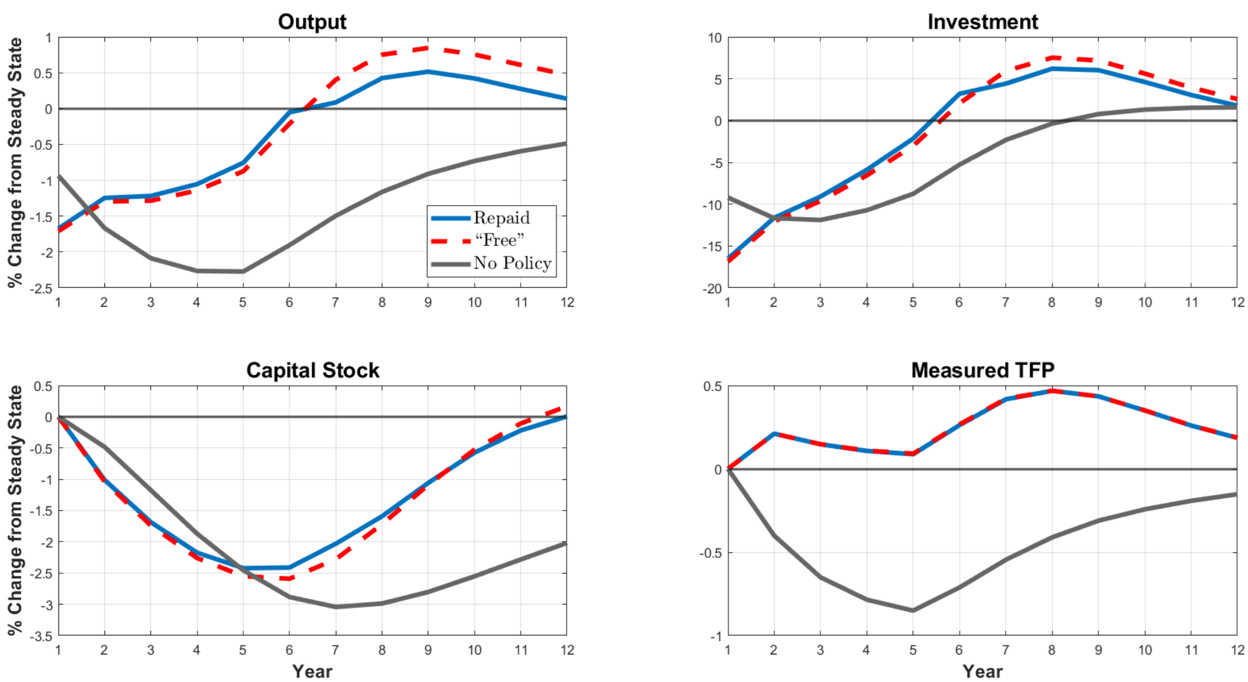
Table 5: Excess Returns Policy Parameters

Parameter		Value
Policy Cost	$\frac{T(\Theta;S)}{\overline{Y}}$	4%
Credit Shock Dates	t	1-4
Shock Recovery Rate	ρ_ζ	0.313
Relief Date	t	1
Repayment (θ) Start	t	7
Repayment (θ) Fraction	ϕ	0.05

Note: These parameters will be common across all policy experiments, save for one "free" policy to illustrate the effect of taxation on recovery.

this method, with details on updating the taxation vector.

Figure 6: Excess Return to Investment Policy Target



Note: Significant improvements with the excess return targeted policy. Policies relieves debt until the firms at the top of the excess returns distributions have this value equalized and lowered.

Figure 6 shows the impulse responses of aggregate variables to a credit shock with excess returns-targeted policy. There are two main components to the policy, debt-relief and repayment, implied by the second fall in aggregates. The debt-relief leads to major improvements in aggregates during a credit crisis. The output trough improves by over 50%. There is a massive initial improvement in endogenous TFP, which does not fall to steady state level even through the heart of the shock. Resources are being moved to their most productive location. In fact, the policy may work too well. The household expects massive improvements in wealth such that they reduce their labor hours supplied at first, borrowing against their future expected income. Thus, there is a dip in initial output from the policy.

Public debt repayment deserves specific consideration. Repayment can begin early in the recession, at which point taxes must be raised during economic downturn, or after most of the recovery has been completed. However, by that point, public debt keeps accruing interest, increasing the amount of taxes needed to pay it. For illustrative purposes, I chose a repayment start date when the output decay had already passed its half-life. The rise in taxes to pay for this policy induces a slight downturn and slower recovery, which is more evident in the coming policy experiments. The larger the share of public debt repaid each period, the higher taxes must be raised to pay off that amount. The knowledge of future tax increases does not appear to have much of an effect at the trough of a recession. However, the no-repayment path does begin to deviate from the repayment paths more as they approach the date of the tax increases.

VI.b Observable Targets

Identifying excess returns across the state space of an economy may not be feasible for policymakers, even if obtaining such information isn't cost prohibitive. I consider two readily observable policy targets: firm size and age. Results are compared to an untargeted policy at the end of the section.

VI.b.1 Size-based Targets

Since we are concerned with capital allocative efficiency, a good proxy for excess returns may be firm size. It was previously discussed how firm size may be correlated with excess returns as small firms may be further away from their optimal capital level, and thus face higher marginal returns.

I consider three sized-based policies targeted to small, medium, and large firms. These definitions can be found on on Table 3. As the total size of each policy will be held constant, and each targeted group is systematically different, the relief per firm must differ across polices. Naturally, the level of debt carried by firms increases with size. Small firms will have 23.7% of their debt relieved, medium firms will have 14.5%, and large firms will have 9.6%. These values are reported on Table 6, where value common across policies are listed in the bottom panel.

Table 6: Size-based Policy Parameters

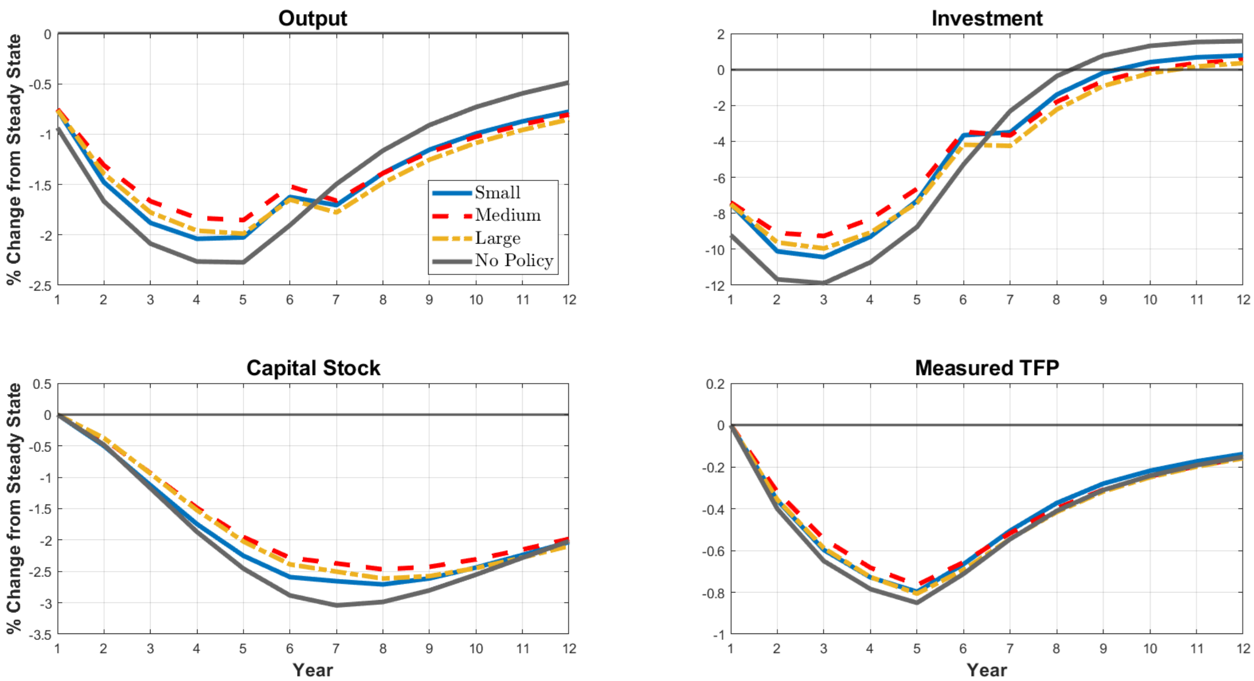
Parameter		Value
Small Target	μ_S	0.880
Medium Target	μ_M	0.101
Large Target	μ_L	0.019
Fraction of b paid (small)	$g(k, b, \varepsilon; S)$	0.237
Fraction of b paid (med)	$g(k, b, \varepsilon; S)$	0.145
Fraction of b paid (large)	$g(k, b, \varepsilon; S)$	0.096

Note: Different policies will relieve different fractions of firm debt to keep aggregate resource cost constant across experiments. $\mu_{S,M,L}$ represent the population share of firms implied by their employment share reported in Table 3.

Table 7 reports the response to a credit shock with a size-based policy. Given the results generated in Table 3, it is surprising that policy targeting medium size firms yields the best results rather than policy targeting small firms. Under the medium size policy, the trough of output remains a half percent closer to steady state and reaches its half life a year earlier when compared to the no policy scenario. The slower recovery due to the increase in taxes to pay for the policy is more noticeable here than in the excess return-based policy.

The benefit to targeting medium size firms is multi-factorial. To understand this outcome, it is helpful to discuss small and large firms first. The benefit of targeting small firms is that resources can be allocated to firms facing higher excess returns. However, these firms simply do not make up a large share of aggregate variables. The inverse is true for large firms; they can achieve a more efficient level of investment without policy, however, they make up the largest share of aggregate variables relative to their population share. Medium size firms act as a convex combination of these features, providing the best of both worlds. They make up a larger share of aggregates than smaller firms *and* fave higher excess returns than larger firms.

Figure 7: Response to Size-targeted Policy



Note: Aggregates have the highest improvement under medium size policy targets. Medium size firms face higher excess returns than large firms and make up a relatively larger share of aggregates than smaller firms.

VI.b.2 Age-based Targets

Beyond employment targeting, another proxy for excess returns may be firm age. In an environment absent financial frictions, entering firms would be able to immediately reach their optimal capital stock and excess returns across the economy would be 0. Given financial frictions, there is a slow growth process to $k'^{*}(\varepsilon)$ as firms age. This process of aging generates firm life-cycle results in the model, as young firms carry inefficiently low levels of capital. However, this also means that there is a relationship between age and excess returns, and thus potential gains from age-targeted policy.

Age-based policy will target young firms age 0-5 years old, a definition taken from Goetz and Stinson's (2021) Census report. They define a young firm as "firms with positive employment for five years or less." These firms will have 35.1% of their debt paid for. Policy parameters are listed on Table 7, along with the medium size firm target for reference.

Table 8 displays the response to a credit shock under a young firm targeted policy. Once again, the medium size firm target out performs other policy targets during the heart of the recession. However, there are additional benefits to targeting young firms. Speeding up the life-cycle process of firms when they are young helps them grow faster, increasing labor demand. The increased labor hours mean payroll taxes do not need to rise as much to cover the cost of public debt, reducing the aggregate impact of the tax burden. This phenomenon can be observed in the small firm policy results, since there is a lot of overlap between small and young firms. However, the effect is stronger for young firms because many small firms are purposefully small. They understand that their productivity is persistent and the probability of a significant increase in productivity is low. On the other hand, age is orthogonal to productivity; there are more firms in this category that would like to grow but are prevented from doing so by market conditions.

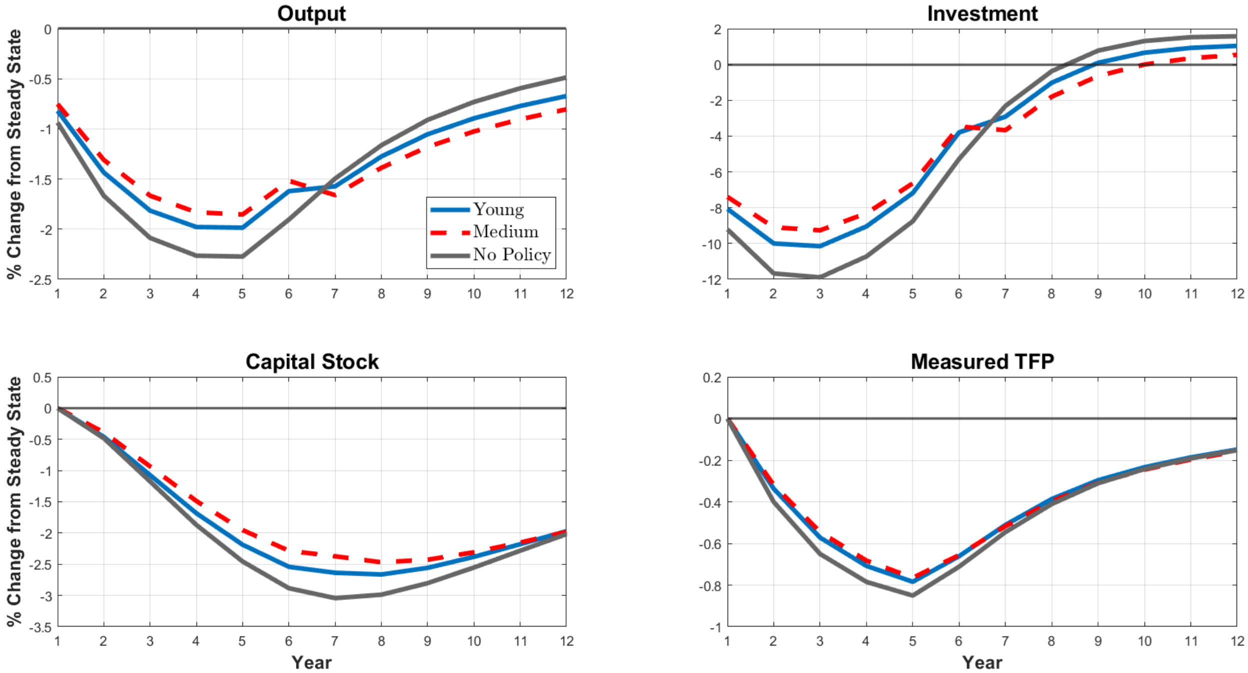
Weighing down the benefit to policy targeting young firms is their high exit rate, reported in Table 1. Firms that are receiving policy resources are systematically more likely to exit the economy than those not eligible for the policy. Ideally, policy would lower exit rates and improve results in that regard. However, these exit probabilities are taken from the BDS over 1990-2006. In a recession they will be higher than the values used here. It remains to be seen which effect would be stronger, and if one is stronger, will it be enough to significantly change the outcomes presented. I abstract from that discussion and leave it to future work.

Table 7: Age-based Policy Parameters

Parameter		Value
Age Target	a	[0-5]
Medium Target	μ_M	0.101
Fraction of b paid (young)	$g(k, b, \varepsilon, a; S)$	0.351
Fraction of b paid (medium)	$g(k, b, \varepsilon; S)$	0.145

Note: Age target is firms younger than 5 years old following Census Report from Goetz and Stinson (2021).

Figure 8: Response to Age-targeted Policy



Note: Medium size targets lead to better aggregate results than age targets as well. Age targeting does lead to a lower tax burden in the recovery.

VI.b.3 Untargeted Policy

I consider one final policy structure, a policy without targets. All indebted firms are eligible to receive policy funds and will have 4.6% of their debt paid. Table 8 displays the parameters and Figure 9 displays the transition paths for this policy case.

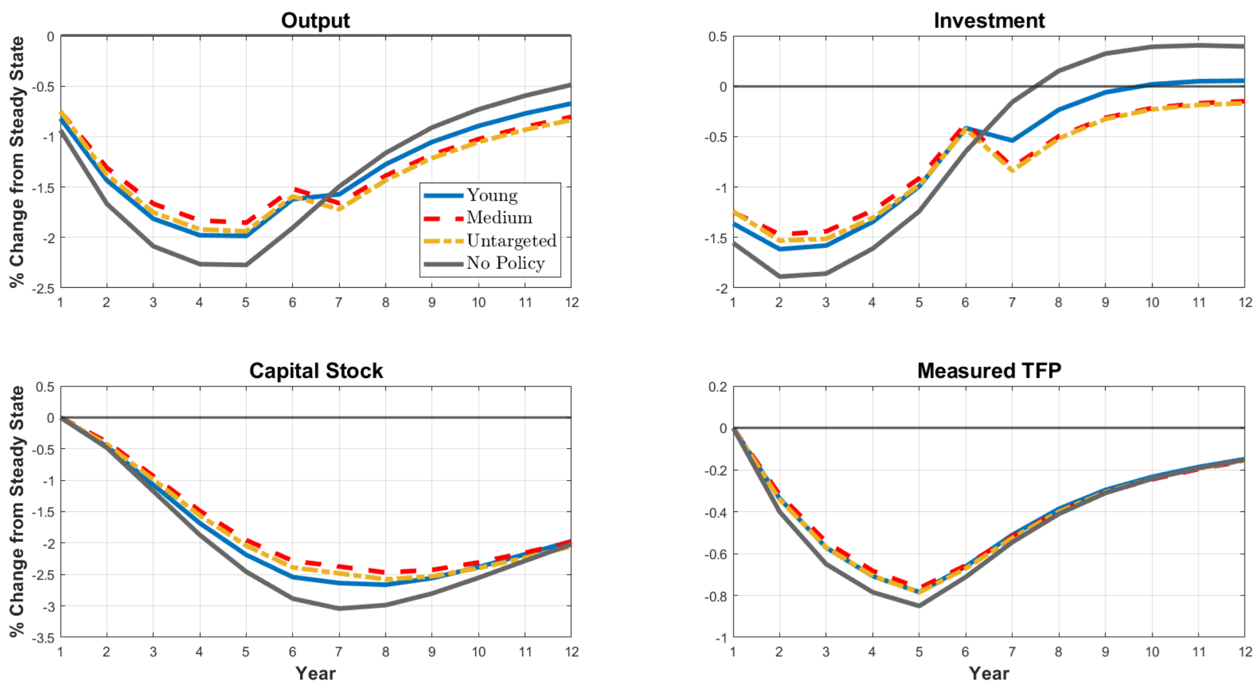
Untargeted policy performs well compared to policies conditioned on other observable targets. Only the medium size policy leads to better outcomes in aggregate variables. The benefit to untargeted policy is that it takes advantage of decreasing returns to scale production technology. Concentrating resources on a subsection of the distribution does allow policymakers to focus on key variables, but decreases the marginal return to a "bailout unit" quickly. On the other hand, spreading the relief out to more firms keeps this marginal value higher over a larger populaiton.

Table 8: Age-based Policy Parameters

Parameter		Value
Age Target	a	[0-5]
Medium Target	μ_M	0.101
Fraction of b paid (young)	$g(k, b, \varepsilon, a; S)$	0.351
Fraction of b paid (med)	$g(k, b, \varepsilon; S)$	0.145
Fraction of b paid (no target)	$g(b; S)$	0.046

Note: All indebted firms are eligible to relieve relief under untargeted policy.

Figure 9: Response to Age-targeted Policy



Note: Medium size targets lead to better aggregate results than no policy targets. However, having no policy targets for debt-relief outperforms all other observable targets.

VII Concluding Remarks

I develop a dynamic stochastic general equilibrium model where persistently heterogeneous firms face financial frictions and calibrate it to U.S. data. I employ a bounded Pareto distribution to allow the model to capture the heavily skewed size distribution and age size distribution seen in the United States. These frictions limit firm's access to debt, and thus investment, providing a role for debt-relief policy in a financial downturn. I use this model as a quantitative laboratory to study the aggregate effects of firm-targeted debt relief.

Policy that can decrease and equalize the highest levels of excess return to capital investment, the expected lifetime marginal value of capital investment minus its cost, yields the best results of those studied here. It allocates resources to more productive points in the distribution, dramatically cutting the fall in output by half at its no policy trough and drastically increasing endogenous TFP.

Understanding that this may not be a readily measurable policy target, I consider policies based on firm size and age, as these variables track well with excess returns. Policy that targets medium size firms performs the best out of these policies in terms of improving aggregates in the heart of a credit crunch. Output remains a half percent closer to its steady state value at the trough and reaches its half-life a year sooner. I also find that untargeted policy outperforms policy targeting small, large, and young firms. This is due to widening the eligibility pool, keeping the marginal return of a debt-relief unit higher across a larger population.

The government takes on a role similar to Woodford (1990); it borrows on behalf of firms that cannot. However, when it comes time to repay public debt, the increased taxes lead to a slower recovery. Policy that targeted young firms has a unique advantage here. Speeding up their growth over their life-cycle increases aggregate labor demand, decreasing the amount of payroll taxes necessary to pay for public debt.

Alternative observables that correlate with excess returns are a potential area for new work. Different tax structure can be studied to prevent the second dip in aggregates seen in the impulse responses. Endogenous entry and exit decision to allow for debt-relief of entrants can be studied as a comparison to the young firms target. I leave these ideas to future research.

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A Appendix

A.a Determining Transition Paths

1. Guess a vector of $\{\hat{\tau}\}_0^{T+1}$ and $\{\hat{C}\}_0^{T+1}$ (from this point forward, all variables are time-vectors)
   - τ' is needed for k' decision
   - $\hat{C}$ implies a $w(S)$, $q(S)$, and $p(S)$
2. Back-solve decision rules from date T
3. Forward-solve the distribution (and find aggregates) for each t
4. Back out $\tilde{C}$ implied by aggregate resource constraint
5. Back out $\tilde{\tau}$ implied by following:

   Define: $Balance \equiv \hat{\tau}(S)w(S)N(S) + q(S)\theta' - \overline{G} - \theta - T(\Theta, S)$

   Then, $\hat{\tau}(S)w(S)N(S) = Balance + q(S)\theta' - \overline{G} - \theta - T(\Theta, S)$

   Define: $\Delta\hat{\tau}(S)$ as change in $\hat{\tau}(S)$ such that:

   $(\hat{\tau}(S) + \Delta\hat{\tau}(S))w(S)N(S) = \overline{G} + \theta + T(\Theta, S) - q(S)\theta'$

   – This is the increase in $\hat{\tau}(S)$ needed to set $Balance = 0$

   Then: $\Delta\hat{\tau}(S) = \left(\left(\overline{G} + \theta + T(\Theta, S) - q(S)\theta'\right)\left(\frac{1}{w(S)N(S)}\right)\right) - \hat{\tau}(S)$

   So, $\tilde{\tau} = (\hat{\tau}(S) + \Delta\hat{\tau}(S))$
6. Check guess, set $\{\tilde{\tau}\}_0^{T+1} = \{\hat{\tau}\}_0^{T+1}$ and $\{\tilde{C}\}_0^{T+1} = \{\hat{C}\}_0^{T+1}$, and repeat.