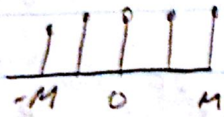
frequency response "windowing"

1. moving average: $y[n] = \frac{1}{2M+1} \sum_{m=-M}^{M} x[n-m]$

$h[n] = \frac{1}{2M+1} \sum_{m=-M}^{M} \delta[n-m]$  "window"

"window" at zero is low pass filter

$$H(e^{jw}) = \sum_{k=-\infty}^{\infty} h[k] e^{-jwk} = \frac{1}{2M+1} \sum_{k=-M}^{M} e^{-jwk}$$

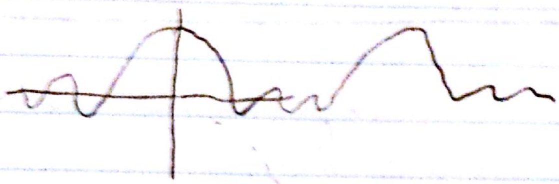
$$H(e^{jw}) = \left(\frac{e^{jwM}}{2M+1}\right)\left(\frac{e^{-jw(2M+1)} - 1}{e^{-jw} - 1}\right) = \left(\frac{e^{-jw(M+1)} - e^{jwM}}{e^{-jw} - 1}\right)\left(\frac{1}{2m+1}\right)$$

$$= \left(\frac{1}{2M+1}\right)\left(\frac{e^{-jw/2}\left(e^{-jw(M+1/2)} - e^{jw(M+1/2)}\right)/2}{e^{-jw/2}\left(e^{-jw/2} - e^{jw/2}\right)/2}\right)$$

$$= \frac{1}{2M+1} \frac{\sin[(M+1/2)w]}{\sin(w/2)} \quad \text{when } w \simeq 0$$

$\sin(w/2) \approx w/2$

$$\boxed{H(e^{jw}) \approx \frac{1}{2M+1} \frac{\sin((M+1/2)w)}{w/2}}$$

DIRICHLET KERNAL 

2π periodic

lecture 8/31/15

z- and laplace dont give physical intuition

fourier: if $x[n] \in \mathbb{C}$ is discrete then we formally define the fourier transform of x to be

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

problem: need conditions $X(e^{j\omega})$ exists

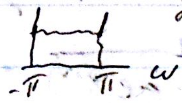
sufficient condition:

1. $\sum_{n=-\infty} |x[n]| < \infty$ → absolutely summable

2. finite energy $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \Big| \underbrace{\sum_{n=-N}^{N} x[n] e^{-j\omega n}}_{\text{finite sum}} - X(e^{j\omega}) \Big|^2 d\omega = 0$$

note: GIBBS



inverse fourier: given $X(e^{j\omega})$ we can reconstruct $x[n]$ using

$$X(e^{-j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

so,

$$e^{-j\omega n} = e^{-j(\omega+2\pi)n}$$
$$= e^{-j\omega n} \underbrace{e^{-j2\pi n}}_{1}$$

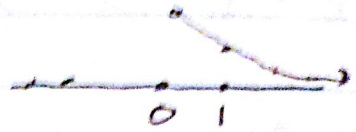
note frequency domain is continuous so,

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \quad \text{(synthesis)}$$

impulse response example:

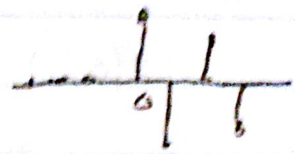
$h[n] = a^n u[n] \quad |a| < 1$

@ $a = \frac{1}{2}$



$Y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$

@ $a = -0.9$



@ $a = \frac{1}{2}$ $= \sum_{k=0}^{\infty} a^k x[n-k] = x[n] + \frac{1}{2} x[n-1] + \frac{1}{4} x[n-2]$

@ $a = -0.9$ $= x[n] - 0.9\, x[n-1]$ ⟶ high pass filter

averaging the past but exponentially forgetting the past

proof:

$H(e^{jw}) = \sum_{n=0}^{\infty} h[n] e^{-jwn} = \sum_{n=0}^{\infty} (a e^{-jw})^n = \frac{1}{1-ae^{-jw}}$

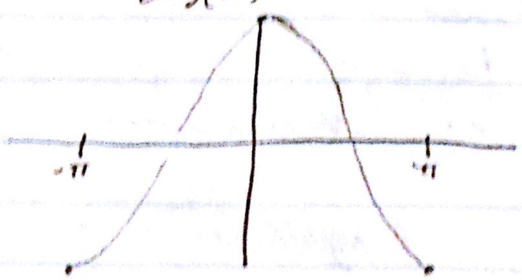
1. compute magnitude $H(e^{jw}) \cdot H(e^{jw})^*$

$= \frac{1}{(1-ae^{-jw})} \cdot \frac{1}{1-a^* e^{jw}}$

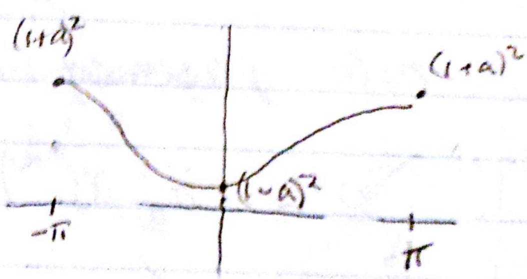
$= \frac{1}{1+|a|^2 - 2a\cos(w)}$ if $a > 0$

$= \frac{1}{1+a^2 - 2a\cos(w)}$ → mag is real so $H(e^{jw})$ is eve

$\cos(w)$



$1+a^2 - 2a\cos(w)$



$H(e^{jw})$ is low pass / high pass

example 2: $x[n] = 1$ for all n $\qquad \Sigma|x[n]| = \infty$

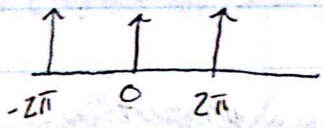
$\Sigma|x[n]|^2 = \infty$

claim: the fourier transform is a $\delta[n]$

proof 1:

$$X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

↑ needs to be periodic



$$= \frac{1}{2\pi} \cdot 2\pi \int_{-\pi}^{\pi} \delta(\omega) e^{j\omega} d\omega$$

$$= \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega} d\omega = \int_{-\infty}^{\infty} \delta(\omega)\, d\omega = 1$$

proof 2: (informal) $\quad x[n] = 1$ for all n

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} e^{-j\omega n} = 1 + \sum_{n=-\infty}^{-1} e^{-j\omega n} + \sum_{n=1}^{\infty} e^{-j\omega n}$$

$$= 1 + \underbrace{\sum_{n=1}^{\infty} e^{j\omega n} + \sum_{n=1}^{\infty} e^{-j\omega n}}_{\text{eulers}}$$

$$= 1 + 2\sum_{n=1}^{\infty} \cos(n\omega)$$

@ $\omega = 0$ $\quad X(e^{j\omega}) = \infty$

@ $\omega = 2\pi$ $\quad X(e^{j\omega}) = \infty$ $\qquad$ for $\omega = 2\pi k$, $X(e^{j\omega}) = \infty$

@ $\omega = \pi$ $\quad X(e^{j\omega}) = ?$ $\qquad \sum_{n=1}^{\infty} = -1 + 1 - 1 + 1$

$0 < r < 1 \qquad \sum_{r=1}^{\infty} (-r)^n = -\frac{1}{2}$ so,

$1 + 2(-\frac{1}{2}) = 0$

DSP 9/2/15

$\mathcal{F}\left(\sum_{n=-\infty}^{\infty} x_1[n]\right) = \sum \mathcal{F}(x)$ should know $\mathcal{F}(x'(t))$

$= \sum x[n] e^{j\omega} = X(e^{j0})$ $\qquad j\omega X(j\omega)$

but

$\mathcal{F}\left(\frac{x[n]}{y[n]}\right) \neq \frac{\mathcal{F}x[n]}{\mathcal{F}y[n]}$

properties of the fourier transform :

linearity $\quad y[n] = \alpha x_1[n] + \beta x_2[n] \quad \alpha, \beta \in \mathbb{C}$

$Y(e^{j\omega}) = \alpha X_1(e^{j\omega}) + \beta X_2(e^{j\omega})$

parity $\quad x[n] \in \mathbb{R} \quad X(e^{j\omega}) \in \mathbb{C}$: twice as much data

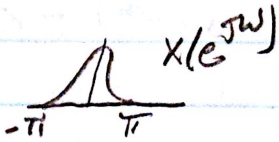
$\mathrm{Re}(X(e^{j\omega})) \rightarrow$ even $\qquad \mathrm{Im}(X(e^{j\omega})) \rightarrow$ odd functio

$|X(e^{j\omega})| \rightarrow$ even

$\angle X(e^{j\omega}) \rightarrow$ odd $\qquad X(e^{j\omega}) = |X(e^{j\omega})| e^{j\angle X(e^{j\omega})}$

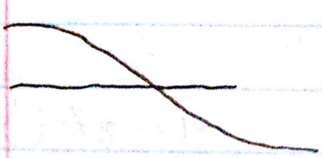
time shifting $\quad y[n] \triangleq x[n-n_0] \qquad Y(e^{j\omega}) = e^{-j\omega n_0} X(e$

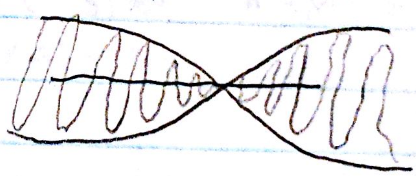
modulation $\quad y[n] = \triangleq x[n] e^{j\omega_0 n} \qquad \omega_0$ is close t



as $\omega \rightarrow \pi$,
make envelope

Amplitud modula





$Y(e^{j\omega}) = X(e^{j(\omega-\omega_0)})$

differentiation in time : ?

Z transform

① consider $u[n]$ and Dirac impulses to justify the existance of $\mathcal{F}\{u[n]\}$

② is there a better transform?

↳ can fix $u[n]$ and compute a fourier transfo

↳ introduce convergence factor

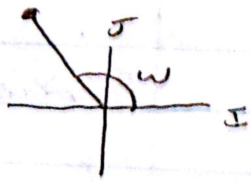
define: $\rho^n u[n] \quad |\rho| < 1$

$$= \sum_{n=-\infty}^{\infty} \rho^n u[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} u[n] \Big[\underbrace{\tfrac{1}{\rho} e^{j\omega}}_{z} \Big]^{-n} = V[z]$$

$V[z]$ depends on $\angle z = \omega$

$|z| = \frac{1}{\rho}$



HOPE: z transform inherits all properties of F.T.

- need to go back to unit circle
- if $z = e^{j\omega}$ we dont need ρ this is a F.T.

trial

$$\Sigma(x[n]) = \frac{\sum_{n=-\infty}^{\infty} n\, x[n]}{\sum_{n=-\infty}^{\infty} x[n]} \qquad \Sigma(e^{jw}) = \sum_{n=-\infty}^{\infty} m[n]\, e^{-jwn}$$

is it just

$$\Sigma(x/e^{jw}) = \sum_{n=-\infty}^{\infty} X$$

DSP lecture 9/4/15

z transform:

R.O.C : when the $\sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$ converges then, z is within the region of convergence of the z transform of x

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \qquad z \in R.O.C.(X)$$

note: if unit circle $\in$ ROC, we can recover the fourier transform of x by evaluating $X(z)$ on the unit circle

$$z = e^{jw} \qquad w \in [-\pi, \pi]$$

$$X(z)\big|_{z=e^{jw}} = X(e^{jw})$$

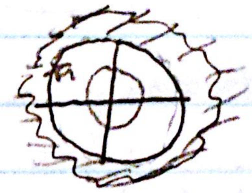
example: $x[n] = a^n u[n] \quad a \in \mathbb{C}$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

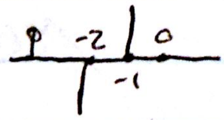
$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n \qquad \text{B.C} \quad a^n z^{-n} = \left(\frac{a}{z}\right)^n$$

$$X(z) = \frac{1}{1-\frac{a}{z}} \quad \text{iff } \left|\frac{a}{z}\right| < 1 \quad \text{or} \quad |a| < |z|$$

$$\boxed{X(z) = \frac{1}{1-\frac{a}{z}} = \frac{1}{1-az^{-1}}}$$



example: $x[n] = -a^n u[-n-1]$



$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{-1} a^n z^{-n}$$

get rid of neg. indicies $(-n \to m)$

$$= -\sum_{m=1}^{\infty} a^{-m} z^{n}$$

$$= \sum_{m=1}^{\infty} \left(\frac{z}{a}\right)^m = -\frac{z}{a}\left(\frac{1-0}{1-\frac{z}{a}}\right) \quad \text{true iff } \left|\frac{z}{a}\right| < 1$$

note:

$$\sum_{n=n_0}^{n_1} r^n = r^{n_0}\left(\frac{r^{(n_1-n_0+1)}-1}{r-1}\right) = \frac{r^{n_1-1}-r^{n_0}}{r-1}$$

$$X(z) = \frac{z}{a} \frac{1}{\frac{z}{a}-1} = \frac{(z/a)}{z/a - 1}$$

$$\boxed{X(z) = \frac{1}{1-\frac{a}{z}} = \frac{1}{1-az^{-1}}}$$

same z transform

R.O.C is inside $\left|\frac{z}{a}\right| < 1$

Common z transforms + R.O.C

$x[n]$		$X(z)$	ROC
$\delta[n]$	$\Leftrightarrow$	1	$\mathbb{C}$
$u[n]$	$\Leftrightarrow$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$ (causal)
$a^n u[n]$	$\Leftrightarrow$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\Leftrightarrow$	$\frac{1}{1-az^{-1}}$	$\lvert a\rvert > \lvert z\rvert$ (anti causal)

properties of z transform and R.O.C

1. the ROC of any seq is always a ring $R_1 < |z| < R_2$

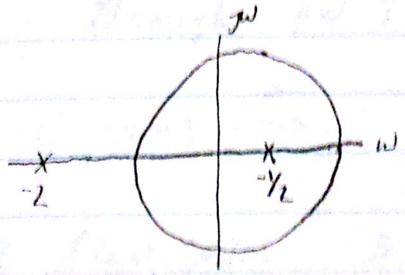
2. if $x[n]$ has a F.T. ROC contains unit circle

3. if $x[n]$ is finite, the ROC is all of $\mathbb{C}$
   ↳ You may need to remove 0

4. if there exists N_0 S.T. $x[n] = 0$ for all $n \le N_0$
   then the ROC has the form $|z| > R_1$

5. if there exists N_1 S.T. for all $n \ge N_1$ $x[n] = 0$
   then the ROC has the form $|z| < R_2$

Example :



pole-zero plot $X(z)$

What is $x[n]$? → there are 3 ROC's
$|z| > 2$
$\frac{1}{2} < |z| < 2$
$|z| < \frac{1}{2}$

$$X(z) = \frac{\quad}{(1-\frac{1}{2}z^{-1})(1+2z^{-1})}$$

DSP 9/11/15

pb 3.30 c

$x[n] = \cos(\frac{1}{2}\pi n) \quad n \in \mathbb{Z} \qquad H(z) = \frac{1 - z^{-1}}{1 - \frac{1}{4}z^{-2}}$

$y[n] = A \cos(\pi/2\, n + \phi)$ => always

if $h[n]$ is real, $A = |H(e^{j\pi/2})|$
and $e^{j\pi/2} = j$ $\quad \phi = \operatorname{atan}(H(j))$
$e^{-j\pi/2} = -j$

$\frac{|1+j|}{|1+\frac{1}{4}j^2|} = \frac{\sqrt{2}}{5/4} = \frac{4\sqrt{2}}{5}$ LTI: freq in = freq out

$\measuredangle \frac{1+j}{1+\frac{1}{4}j} = \measuredangle(1+j) - \measuredangle\, 5/4$

$\phi = \pi/4 - 0$

today: invert z-transform
next week: sampling theorm

midterm 1: through HW 3 everything before sampling

know: $a^n u[n] \overset{z}{\Longleftrightarrow}$

inverse z transform 3 ways:

1. general formula is similar to $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega} d\omega$

2. identify taylor series expansion $z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots = \log(1+z)$

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☆ 3. $H(z) = \frac{P(z)}{Q(z)}$ } polynomials

P.F. decomp $\quad x(z) = \frac{\sum_0^M b_k z^*}{\sum_0^N a_k z^{-k}} = \frac{z^{-M} \sum_0^M b_k z^{M-k}}{z^{-N} \sum_0^N a_k z^{N-k}}$

write as product (factorization of a polynomial $\delta^\circ M$)

$$x(z) = \frac{z^{-M} b_0 \prod_{k=1}^{M} (z - \overset{\downarrow \text{zeros}}{z_k})}{a_0 z^{-N} \prod_{k=1}^{N} (z - \underset{\searrow \text{poles}}{p_k})} = \frac{b_0 \prod_{k=1}^{M} (1 - z_k z^{-1})}{a_0 \prod_{k=1}^{N} (1 - p_k z^{-1})}$$

1. check if $H(z)$ is proper / rational
   if $M \geq N$ (not proper) $\rightarrow$ do long division and collect poly in z

$$\sum_{r=0}^{M-N} C_r z^{-r} \rightarrow \sum_{r=0}^{M-N} C_r \delta[n-r]$$

example: $\quad x(z) = \frac{1 + 3z^{-1} + \frac{11}{6} z^{-2} + \frac{1}{3} z^{-3}}{1 + \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}} \qquad M = 3, \; N = 2$

want largest powers to match

$$1 + 3z^{-1} + \frac{11}{6} z^{-2} + \frac{1}{3} z^{-3} = 2z^{-2}\left(1 + \frac{5}{6} z^{-1} + \frac{1}{6} + z^{-2}\right) + \underbrace{1 + z^{-1} + \frac{1}{6} z^{-2}}_{\text{remainder}}$$

$$1 + z^{-1} + \frac{1}{6} z^{-2} = 1 + \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2} + \frac{1}{6} z^{-1}$$

$$x(z) = \underbrace{2z^{-1} + 1}_{\checkmark} + \frac{1/6\, z^{-1}}{1 + 5/6\, z^{-1} + 1/6\, z^{-2}}$$

if $N > M$ (proper) and if all poles are simple then

$$x(z) = \sum_{k=1}^{N \leftarrow \# \text{ of poles}} \frac{A_k}{1 - P_k z^{-1}} \qquad A_k = x(z)(1 - P_k z^{-1})\Big|_{z = P_k}$$

---

if $x(z)$ has a pole of order > 1

P_i is an order of "D"

$$x(z) = \sum_{\substack{k \neq i \\ k=1}}^{N} \frac{A_k}{1 - P_k z^{-1}} + \sum_{q=1}^{D} \frac{B_q}{(1 - P_i z^{-1})^q}$$ ← different

D = order q = index $w = z^{-1}$

$$B_q = \frac{1}{(D-q)!\,(-P_k)^{(D-q)}} \frac{d^{D-q}}{dw^{D-q}} \left[(1 - P_i w)^D x(w^{-1}) \right]$$

z transform: swiss army knife for fourier

$$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2} = \frac{A}{(1+z^{-1})} + \frac{B_1}{(1-z^{-1})} + \frac{B_2}{(1-z^{-1})}$$

$$A_1 = \frac{1}{(1-z^{-1})^2}\Big|_{z=-1} = \frac{1}{4}$$

$d = 2$ (order of poles)
$q = 1$ (order of power
$P_2 = 1$

$$B_1 = \left[\frac{1}{(2-1)!(-1)}\right](1-w)^2\left(\frac{1}{(1+w)(1-w)^2}\right]\Big|_{@\, w = P_2^{-1} = 1}$$

$$B_1 = -\left(\frac{d}{dw}\frac{1}{(1+w)}\right)\Big|_{w=1} = \frac{1}{(1+w)^2}\Big|_{w=1} = \frac{1}{4}$$

$$B_2 = \frac{1}{1+w}\Big|_{w=1} = \frac{1}{2}$$

sampling theorm:

history:
whitaker 1915
kotel nikov 1933
shannon 1958
nyquist 1928
} regular sampling

NEW: irregular sampling
random sampling

$x(t)$ $t \in \mathbb{R}$
$x(t_n)$ $n \in \mathbb{Z}$

Question: under what condition on "x" can we replace $x(t)$, $t \in \mathbb{R}$ with a sequence $x(t_n)$, $n \in \mathbb{Z}$

$\mathbb{R}$ continuous ∞
$\mathbb{Z}$ discrete ∞

constraint: x cannot oscilate too much

condition: $\lim_{n\to\infty} = \infty$ $\lim_{n\to-\infty} = -\infty$

Intro: mathmatical representation of sampling

$$\frac{1}{\varepsilon}\int_{t_0-\varepsilon/2}^{t_0+\varepsilon/2} x(s)\,ds$$ → basically compute average value around t_0

$s = t_0 - v$

old dummy ↑ (s) new dummy ↑ (v)

$$\frac{1}{\varepsilon}\int_{-\varepsilon/2}^{\varepsilon/2} x(t_0 - v)\,dv = x * h_\varepsilon \qquad h_\varepsilon = \begin{matrix} 1/\varepsilon & \text{if } |v| \le \frac{\varepsilon}{2} \\ 0 & \text{else} \end{matrix}$$

moving average (non causal) $\qquad \lim_{\varepsilon \to 0} h_\varepsilon = \delta[t_0]$

the sampled function should be represented by

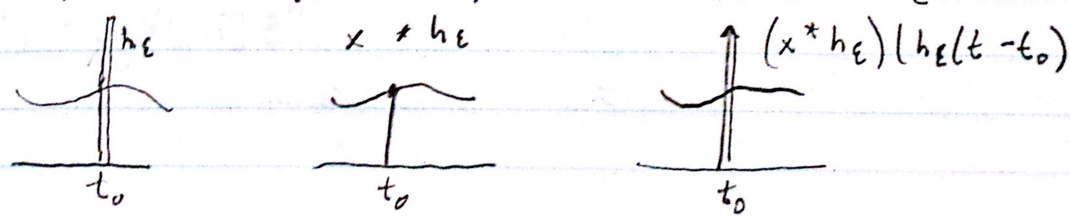
$(x * h_\varepsilon)(h_\varepsilon(t - t_0))$

moving average ↑ (x * h_ε) suspicious ↑ (h_ε(t - t_0))

- Why do we represent the sample signal by h_ε?
  - → we want the average value of the sampled signal be $x(t_0)$
  - → $x(t) \xrightarrow{\text{samp}} x * h_\varepsilon(t_0) \longrightarrow x_s(t) = x * h_\varepsilon(t_0)\, h_\varepsilon(t - t_0)$

$$\int x_s(t) = x * h_\varepsilon(t_0) \checkmark$$

1. find moving average @ t_0 → $x * h_\varepsilon = \#$



2 moving avg: $\int_{\varepsilon/2}^{\varepsilon/2} x(t_0 - v)\, h_\varepsilon(v)\, dv$

mathmatical representation of sampling

periodic impulse train: $s(t) = \sum_{n=-\infty}^{\infty} \delta[t - nT]$

sampling signal: $x_s(t) = x(t) \cdot s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$

$$= \sum_{n=-\infty}^{\infty} x(t)\,\delta(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t - nT)$$

sampled value @ time $t = nT$