2 formulas you need to know

1) $X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j\Omega - \frac{2\pi k}{T})$ $\mathcal{F}\{\}$ $x[n]\delta(t-nT)$

2) $X(e^{j\omega}) = X_s(\frac{j\omega}{T})$

$\omega = \Omega T$

$x_s(t) = \sum_n x[n]\delta(t-nT) \xrightarrow{CTFT} X_s(j\Omega) = \sum_n x[n]e^{-jnT\Omega} = X(e^{j\Omega T})$

$$\boxed{\begin{aligned} X(j\Omega) &= X(e^{j\Omega T}) \\ X_s(\tfrac{j\omega}{T}) &= X(e^{j\omega}) \end{aligned}}$$

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T sampling period $\Omega_s = \frac{2\pi}{T}$

4.1

$$x[n] \leftrightarrow X(e^{J\omega}) \qquad x[n] = x_c(nT)$$

$$x(t) \leftrightarrow X(J\Omega) \qquad X(e^{J\omega}) = X(e^{J\Omega T})$$

$$x_s(t) = x(nT) \leftrightarrow X(e^{J\Omega T}) \qquad \omega = \Omega T$$

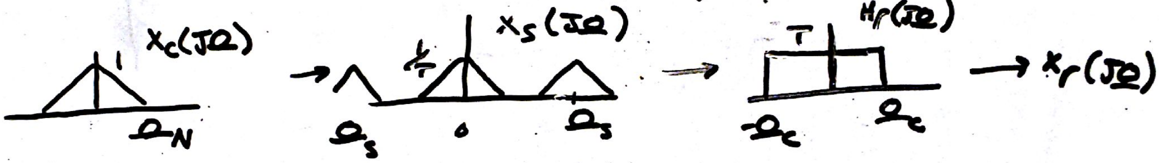
$$s(t) \leftrightarrow S(J\Omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)$$

$$x_s(t) \leftrightarrow X_s(J\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(J(\Omega - k\Omega_s))$$

$$X_c(J\Omega) = \begin{cases} 1 & \text{if } |\Omega| < \Omega_N \\ 0 & \text{else} \end{cases}$$

4.2

if $\Omega_s \geq 2\Omega_N$ then no aliasing



if $\Omega_N \leq \Omega_c \leq (\Omega_s - \Omega_N)$ then $X_r(J\Omega) = X_c(J\Omega)$

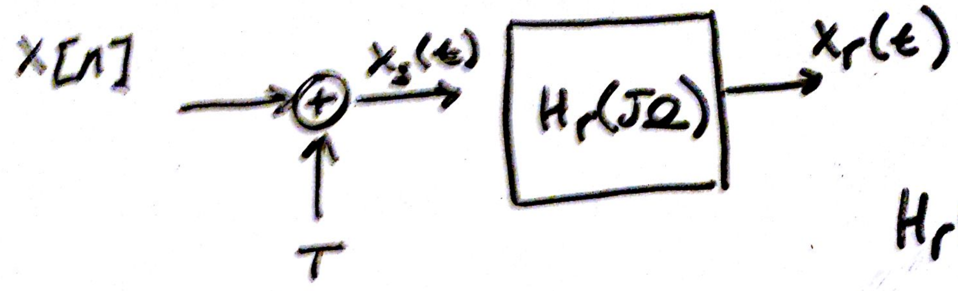
Nyquist-Shannon theorem:

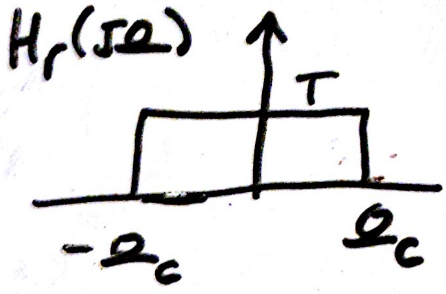
$$X_s(J\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT)\, e^{-J\Omega T}$$

$$X_s(J\Omega) = X(e^{J\omega})\Big|_{\omega = \Omega T} = X(e^{J\Omega T})$$

$$X(e^{J\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(J(\Omega - k\Omega_s))$$

4.3 reconstruction of bandlimited signal from its samples.





$$x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\,\delta(t-nT)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]\,h_r(t-nT)$$

$$X_r(j\Omega) = H_r(j\Omega)\sum_{n=-\infty}^{\infty} x[n]\,e^{-j\Omega nT}$$

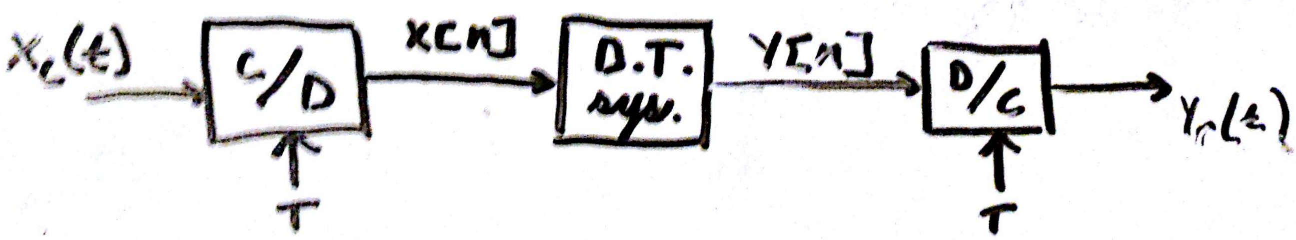
$$X_r(j\Omega) = H_r(j\Omega)\,X(e^{j\Omega T})$$

4.4 discrete time processing of cont. time signal

$$x[n] = x_c(nT)$$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{n=-\infty}^{\infty} x[n]\,e^{-j\omega n} = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(\Omega-\Omega_s k)$$

$$y_r(t) = \sum_{n=-\infty}^{\infty} y[n]\,\frac{\sin\left(\pi(t-nT)/T\right)}{\pi(t-nT)/T}$$



4.4.1 $\quad Y(e^{j\omega}) = H(e^{j\omega})\,X(e^{j\omega})$

$$Y_r(j\Omega) = H_r(j\Omega)\,H(e^{j\Omega T})\,X(e^{j\omega})$$

$$Y_r(j\Omega) = H_r(j\Omega)\,H(e^{j\Omega T})\,\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left[j\left(\Omega - \frac{2\pi k}{T}\right)\right]$$

$$Y_r(j\Omega) = \begin{cases} H(e^{j\Omega T})X_c(j\Omega) & \text{if } |\Omega| < \pi/T \\ 0 & \text{else} \end{cases}$$

if $X_c(j\Omega)$ is band limited and nyquest req:

$$\Omega_s \geq 2\Omega_N \iff \frac{2\pi}{T} \geq 2\Omega_N$$

$$Y_r(j\Omega) = H_{eff}(j\Omega)\,X_c(j\Omega) \text{ where } H_{eff}(j\Omega) = \begin{cases} H(e^{j\Omega T}) & \text{for } |\Omega| < \pi/T \\ 0 & \text{for } |\Omega| \geq \pi/T \end{cases}$$

4.6 changing the sampling rate using discrete time processing

$$x[n] = x_c(nT)$$

reconstruction formula: $x_r(t) = \sum_{n=-\infty}^{\infty} x[n]\,h_r(t-nT)$

$x_1[n] = x_c[nT_1] \rightarrow$ change sampling rate

4.6.1

downsampling: $x[n] \rightarrow \boxed{\downarrow M} \rightarrow$

$x_d[n] = x[nM]$ $\quad$ if $|\Omega| \geq \Omega_N$ then $x_d[n]$ if

$T_d = MT$ $\quad \frac{2\pi}{T_d} = \frac{2\pi}{MT} \geq 2\Omega_N$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left[j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right]$$ similarly $x_d[n] = x[nM] = x_c(n$

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1}\left\{\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left[j\left(\frac{\omega}{MT} - \frac{2\pi k}{T} - \frac{2\pi i}{MT}\right)\right]\right\}$$

* we know this

Eq(4.71) $$X(e^{j(\omega - 2\pi i)/M}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left[j\left(\frac{\omega - 2\pi i}{MT} - \frac{2\pi k}{T}\right)\right]$$

$$\boxed{X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\omega - 2\pi i/M)})}$$

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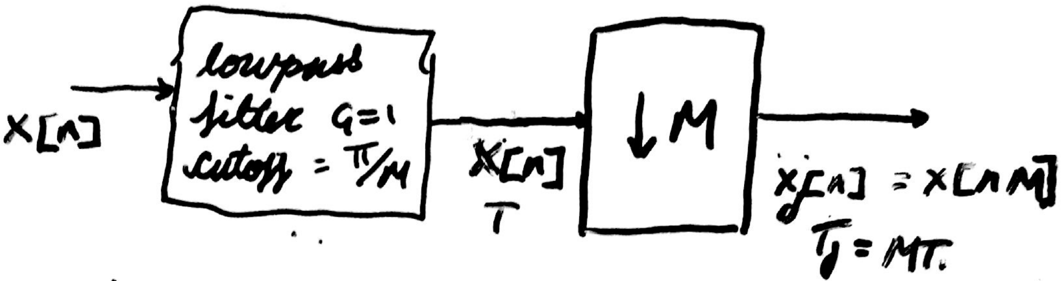
general system for sampling rate reduction



4.6.2 increasing sampling by integer

$x_i[n] = x_c(nT_i)$

upsample by factor L

$T_i = T/L$

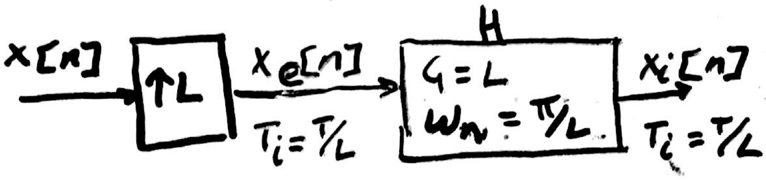
from sequence of samples

$x[n] = x_c(nT)$

$$x[n] = \begin{cases} x[n/L] & , n = 0, \pm L, \pm 2L \ldots \\ 0 & , \text{else} \end{cases}$$

* upsampling is same as interpolation

* works if $x[n]$ was obtained $x[n] = x_c(nT)$ without aliasing



$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-kL]$$

$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-kL]\right)e^{-j\omega n} = \sum_{k=-\infty}^{\infty} x[k]\,e^{-j\omega Lk} = X(e^{j\omega L})$$

- we have expanded the frequency scale
- $\omega = \omega L$ so ω is now normalized by $\omega = \Omega T_i$
- $\omega_n = \pi/L$
- $h_i[n] = \dfrac{\sin(\pi n/L)}{\pi n/L}$
- interpolation eq $x_i[n] = x[k]\dfrac{\sin[\pi(n-kL)/L]}{\pi(n-kL)/L}$

POST TEST 1

DSP lecture 9/23/15

sampling therom:

$x(t)$ = signal $\quad x_s(t)$ = sampled signal

$$x_s(t) = x(t) \sum_{n \in \mathbb{Z}} \delta(t - nt)$$

what happens to $x_s(t)$ in fourier?

can we recover $x(t)$ from $x_s(t)$

Q1: $X_s(j\Omega)$ = fourier transform $\quad X_s(t) = \int_{-\infty}^{\infty} x_s(t) e^{-j\Omega t} dt$

$$X_s(j\Omega) = \frac{1}{2\pi} X(j\Omega) * S(j\Omega) \qquad \text{S - for sampling}$$

where

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \qquad \text{T - sampling period}$$

find F.T.

$$s(j\Omega) = \sum_{n \in \mathbb{Z}} e^{-jnT\Omega}$$

remember

$$\sum_{n=-\infty}^{\infty} e^{-jnw} = 2\pi \sum_{k=-\infty}^{\infty} \delta(w - 2\pi k)$$

if $w = \Omega T$

$$\delta(\Omega T - 2\pi k) = \delta\left(T\left(\Omega - \frac{2\pi k}{T}\right)\right)$$

$$= \frac{1}{T} \delta\left(\Omega - \frac{2\pi k}{T}\right)$$

so

$$s(j\Omega) = \frac{2\pi}{T} \sum \delta\left(\Omega - \frac{k 2\pi}{T}\right)$$

go back to $x(j\Omega)$

$\boxed{\text{* when sampling, the F.T. must be periodic}}$

$$X(j\Omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} x(j\Omega) * \delta\left(\Omega - \frac{2\pi k}{T}\right)$$

$$x(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} x\left(j\Omega - \frac{2\pi k}{T}\right) \leftarrow \frac{2\pi}{T} \text{ periodic}$$

What is the connection between

$$X_s(j\Omega) = \frac{1}{T}\sum_k X(j(\Omega - \tfrac{2\pi k}{T})) \rightarrow P = \frac{2\pi}{T}$$

and

$$X(e^{j\omega}) = \sum x[n]\, e^{-j\omega n} \rightarrow P = 2\pi$$

but

$$x_s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nt) = \sum_{n=-\infty}^{\infty} x(nt)\,\delta(t-nt) \rightarrow x(t) @ nT$$

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x(nT)\, e^{-jnT\Omega} \rightarrow x[n] = x(nT)$$

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jn\Omega T} \rightarrow \omega = \Omega T$$

ω ↑ C.T. Ω ↑ D.T. T ↖ period

Ex1: $x(t) = \cos(4000\pi t)$ sample $T = \frac{1}{6000}$ s

What is $X_s(j\Omega)$?

$$\cos(4000\pi t) = \frac{e^{4000\pi t}}{2} + \frac{e^{4000\pi t}}{2}$$

$\omega = 4000\pi$

$\frac{2\pi}{T} = 12000\pi$

F.T

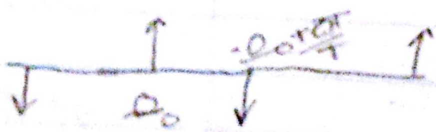
$$X(j\Omega) = \pi(\delta(\Omega - 4000\pi) + \delta(\Omega + 4000\pi))$$

$$X_s(j\Omega) = 6000\pi \sum_k \delta(\Omega - 4000\pi - k\pi 12000) + \delta(\Omega + 4000\pi - k\pi 12000)$$

$x(t) = \sin(\Omega_0 t)$ $X(j\Omega) = \frac{\pi}{j}(\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0))$

can we sample at exactly $\boxed{T = \frac{\pi}{\Omega_0}}$

$$-\Omega_0 + \frac{2\pi}{T} \leftarrow -\Omega_0 + \frac{2\Omega_0\pi}{\pi} = \Omega_0$$



$X_s(j\Omega) = 0$ in time,

so we shift it

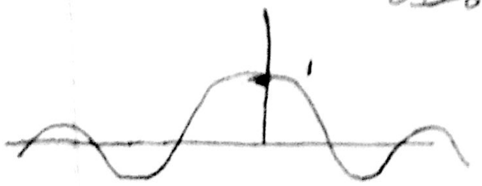
DSP LECTURE 9/25/15

EX 3: $x(t) = \frac{\sin(\Omega_0 t)}{\Omega_0 t}$ $X(j\Omega)$

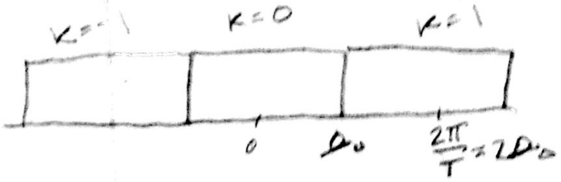
copy every $\frac{2\pi}{T}$ so we want $\frac{\pi}{T} = \Omega_0$

$\sin\left(\frac{n\pi}{\Omega_0} \times \Omega_0\right) = \sin(n\pi) = 0$

@ $n=0$ $\frac{\sin\left(\frac{0\pi}{\Omega_0}\Omega_0\right)}{0\Omega_0} = 1$ the sampled function is $\delta[n]$

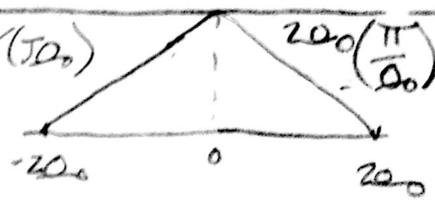


paradox: if we know $x[n] = \delta[n]$ is a sampled at some period we do not expect that the original signal is a sinc



$\frac{2\pi}{T} = 2\Omega_0 \rightarrow \sum \delta[n] e^{-j\omega n} = 1$ for all ω

Ex 4: $x(t) = \left[\frac{\sin(\Omega_0 t)}{\Omega_0 t}\right]^2$ $X(j\Omega_0)$



if we sample every $\frac{2\pi}{T}$,

$T = \frac{\pi}{2\Omega_0}$ and $\frac{2\pi}{T} = 4\Omega_0$

What if we sample w/ larger period $\left(\frac{\pi}{\Omega_0}\right)$ → smaller frequency

→ aliasing.

$x[n] = x\left(\frac{n\pi}{\Omega_0}\right) = \delta[n] = \delta[n]$

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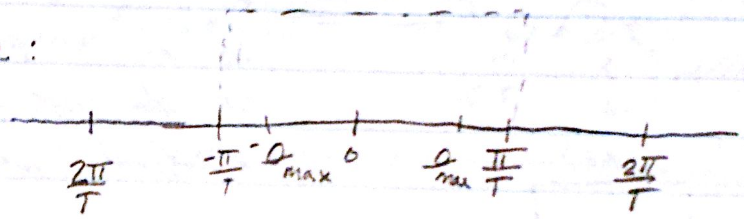
reconstruction theorm (nyquist):
"whitaker 1915"

• signal must be bandlimited

$X(t)$ limited on $[-\Omega_{max}, \Omega_{max}] \rightarrow X(j\Omega) = 0$ if $|\Omega| > \Omega_{max}$

$x(t)$ is uniquely characterized by the samples @ nT $n \in \mathbb{Z}$ where $T \leq \frac{\pi}{\Omega_{max}}$ ← sample fast enough

reconstruction:



we cookie cut the $k=0$ term of the infinite sum.

filter: $H_r(j\Omega) = \begin{cases} T & \text{if } |\Omega| \leq \frac{\pi}{T} \\ 0 & \text{else} \end{cases}$

$$X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right)$$

$X_s(j\Omega) H_r(j\Omega) = X(j\Omega) \rightarrow x_r(t) = h_r * x_s(t)$

$h_r(t) = \dfrac{\sin\left(\frac{\pi t}{T}\right)}{\pi t/T}$ $\quad x_s(t) = \sum x(nT)\,\delta(t - nT)$

$$h_r * x_s = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin(\pi(t - nT)/T)}{\pi(t - nT)/T}$$

interpolation:
* when you plug in a multiple of the period you get the sample

$$= \sum x[n] \frac{\sin(\pi(t/T - n))}{\pi(t/T - n)}$$

$$= \sum_{n=-\infty}^{\infty} x[n] \frac{\sin(\pi(t/T - n))}{\pi(t/T - n)}$$ if $t = mT$ we get a $\delta[m-n]$

observation $\quad X(j\Omega)\, H_r(j\Omega) = X(j\Omega)$

$$\downarrow$$

DTFT $\rightarrow \quad X(e^{j\Omega T})\, H_r(j\Omega) = X(j\Omega)$

$$\uparrow$$
D2A

Discrete time implamentation of analog filter

$$x(t) \longrightarrow \underbrace{\boxed{A/D}}_{\text{sampling}} \longrightarrow \boxed{H(z)} \xrightarrow{y[n]} \boxed{D/A} \longrightarrow y(t)$$

Question: what is the overall filter that maps $x(t)$ to $y(t)$

$$y(t) = h_{eff} * x$$

$$Y(e^{j\omega}) = H(e^{j\omega})\, X(e^{j\omega}) \qquad\qquad X(e^{j\omega T}) = X_s(j\Omega)$$

$$X(e^{j\omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\Omega - \frac{2\pi k}{T}\right)$$

$$H_{eff}(\Omega) = \begin{cases} H(e^{j\Omega T}) & \text{if } |\Omega| \le \frac{\pi}{T} \\ 0 & \text{else} \end{cases}$$

$$\begin{aligned} Y_r(j\Omega) &= H_r(j\Omega)\, Y(e^{j\omega T}) \\ &= H_r(\Omega)\, H(e^{j\omega T})\, X(e^{j\omega T}) \\ &= H_r(\Omega)\, H(e^{j\omega T}) \left(\frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\Omega - \frac{2\pi k}{T}\right)\right) \end{aligned}$$

$$Y_r(\Omega) = \begin{cases} H(e^{j\omega T})\, X(\Omega) & \text{if } |\Omega| \le \frac{\pi}{T} \\ 0 & \text{else} \end{cases}$$

js

DSP lecture 9/28/15

HW PB 1 $x(t)$ bandlimited

sampling rate for recovery is $T = \frac{1}{2}$

what if we sample 2 and $2+a$ but $T=1$
$[0, 2\pi]$ and $[-2\pi, 0]$ because you are breaking symmetry

X_s is $X_s(j\Omega) = \sum x(j(\Omega - 2\pi k))$
$X_a(j\Omega) =$

sample with a shifted train of impulses

$s_a(t) = \sum_{n=-\infty}^{\infty} \delta(t-n-a)$ $\qquad FT\{x(t)s_a(t)\} = \frac{1}{2\pi} \; x(j\Omega) * S_a(j$

$S_a(j\Omega) = e^{-j\Omega a} S(j\Omega)$ $\qquad S(j\Omega) = 2\pi \sum_k \delta(\Omega - 2\pi k)$

$F.T(x(t)s_a(t)) - x(j\Omega) * e^{-j\Omega a} \sum_k \delta(\Omega - 2\pi k)$

property:

$* \sum_k e^{-j\Omega a} \delta(\Omega - 2\pi k)$

delta forces the function to be zero everywhere except at delta

$\sum e^{-ja2\pi k} \delta(\Omega - 2\pi k)$

$F.T\{x(t)\,s_a(t)\} = \sum_{k=-\infty}^{\infty} e^{-jka2\pi} \; x(\Omega - 2\pi k)$

$[0, 2\pi]$ $\quad X_s(\Omega) = x(j\Omega) + x(j(\Omega - 2\pi))$
$\quad X_a(\Omega) = x(j\Omega) + e^{-j2\pi a} x(j(\Omega - 2\pi))$

goal: solve for $x(j\Omega)$

summary $\quad x(j\Omega) = A(\Omega) X_s(j\Omega) + A(\Omega) X_a(j\Omega)$

$A(j\Omega) = \frac{1}{\sin(\pi a)} \begin{cases} e^{j\pi(a - \frac{1}{2})} & \Omega < 0 \\ e^{-j\pi(a-\frac{1}{2})} & \Omega > 0 \end{cases}$

$H(\Omega) = 1$ if $|\Omega| < 2\pi$ claim for every $\Omega \in [-\infty, \infty]$

$X(j\Omega) = H(\Omega)\{A(\Omega)\, X_s(j\Omega) + A^*\, X_a(j\Omega)$

DSP lecture 9/30/15

HW PB 2

$\delta[n]$

$$\frac{\sin \pi n}{\pi n}$$

prove $\sum_{k=-\infty}^{\infty} F(\Omega - 2\pi k) = 1$ $\longrightarrow$ same as $F(e^{j\omega}) = 1$ then $f[n] = \begin{cases} 1 & \text{if } n \\ 0 & \text{else} \end{cases}$

F.T. of a sampled function

$X_s(j\Omega) = \frac{1}{T} \sum X(\Omega - \frac{2\pi k}{T})$

$T = 1$

say f: is the interpolation function

$f[n] = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{else} \end{cases}$ $\qquad F(e^{j\omega}) = 1 \qquad F(e^{j\Omega}) = f(j\Omega) = 1$

$f(t) \xrightarrow{\text{HARD}} f(t)\, s(t) = f_s(t) \rightarrow F_s(j\Omega) = \frac{1}{T} \sum$

$\hookrightarrow \sum f[n]\delta(t - nT) \xrightarrow{\text{analog}} \sum f[n]\, e^{-j\Omega T n}$

$\xrightarrow{\text{D.T.}}$ easy $\rightarrow f[n] \rightarrow F(e^{j\omega}) = \sum f[n] e^{-j\omega n}$ $\quad \omega = \Omega T$

will be on final $\rightarrow$ lti cannot generate new frequencies

LTI $\cos(\Omega_0 t) \rightarrow h(t) \in \mathbb{R} \rightarrow |H(j\Omega)| \cos(\Omega_0 t + \measuredangle H(j\Omega))$

$|H(j\Omega_0)| \rightarrow$ system response at Ω_0

$\cos(\omega_0 n + \phi) \xrightarrow{H} |H(e^{j\omega_0})| \cos(\omega_0 n + \measuredangle H(e^{j\omega_0}) + \phi)$

MULTIRATE SIGNAL PROCESSING

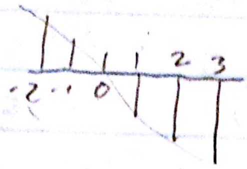
$x(t) \rightarrow \boxed{A/D} \rightarrow x[n]$

Here $x[n] \rightarrow$ up/down sample
change the sampling rate
$\rightarrow$ understand in fourier in z-domain

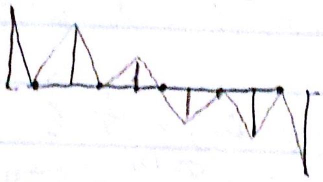
Downsampling / decimation $\quad x{\downarrow}M\,[n] = x[nM] \quad M$ integer ≥ 1
↳ linear but not time invariant

upsampling $\quad x{\uparrow}L\,[n] = \begin{cases} x[n/L] & \text{if } \frac{n}{L} \in \mathbb{Z} \\ 0 & \text{else} \end{cases}$

ex:



upsample by 2



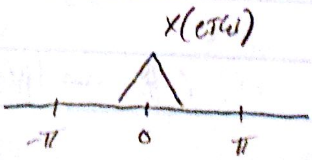
$\rightarrow$ F.T. high frequencies

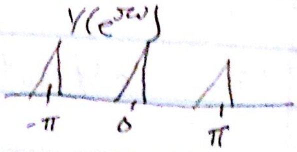
$Y[n] = \begin{cases} x[n/L] & \frac{n}{L} \in \mathbb{Z} \\ 0 & \text{else} \end{cases}$

$$Y(z) = \sum_{m=-\infty}^{\infty} x[m]\, z^{-mL}$$

$$= \sum_{n=-\infty}^{\infty} x[n] (z^L)^{-n} = X(z^L)$$

let $L = 2 \quad z = e^{j\omega}$
$Y(e^{j\omega}) = X(e^{j2\omega})$





↳ never upsample w/o low pass
↳ interpolate <moving avg>

z transform of downsampled signal

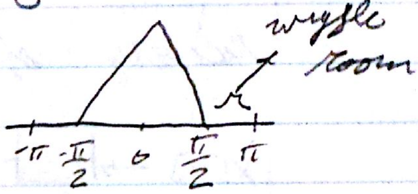
$M=2 \qquad Y(z) = \sum_{n=-\infty}^{\infty} y[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[2n] z^{-n}$

$Y(z^2) = \sum_{n=-\infty}^{\infty} x[2n] z^{-2n} \longrightarrow X(z) = \sum x[2n] z^{-2n} + \sum x[2n+1] z$

want to cancel odd

$Y(z^2) = \frac{1}{2}(X(z) + X(-z)) \rightarrow X(z) + X(-z) = 2 \sum_{n=-\infty}^{\infty} x[2n] z^{-2n}$

$Y(z) = \frac{1}{2}\left[X(z^{1/2}) + X(-z^{1/2})\right] \longrightarrow$ for $z = e^{j\omega}$

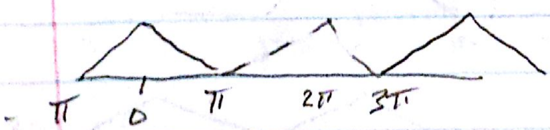


$\underbrace{Y(e^{j\omega})}_{2\pi \text{ periodic}} = \frac{1}{2}\left[X(e^{j\omega/2}) + X(e^{j(\omega - 2\pi)/2})\right]$

↑ 4π periodic

$y(t) = \sum_n x(t - nT) \rightarrow$ periodic on T

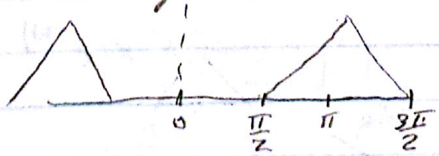
$\longrightarrow$ delay and copy



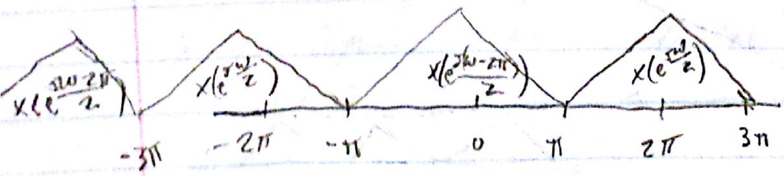
→ if there was no wiggle room we would get aliasing

→ the key is bandwidth → if $BW < \pi$ you can downsample w $M=2$

Ex:



symmetric



Note $\left(e^{\frac{j2\pi k}{M}}\right)^M = 1$

$M=3; \qquad Y(z) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(e^{\frac{-j2\pi k}{M}} z^{\frac{1}{M}}\right)$

$Y(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(e^{j\left(\frac{\omega - 2\pi k}{M}\right)}\right)$

Ex: $M = 3$ $\quad Y(e^{j\omega}) = \frac{1}{3} \sum_{k=0}^{3-1} X(e^{j\frac{(\omega - 2\pi k)}{3}})$

no aliasing if $BW < \frac{\pi}{3}$

$$Y(e^{j\omega}) = \frac{1}{3}\left[X(e^{j\omega/3}) + X(e^{j\frac{\omega - 2\pi}{3}}) + X(e^{j\frac{(\omega - 4\pi)}{3}})\right]$$

$T = 6\pi$
$BW = 2\pi$

→ you need 2 delays to make $Y(e^{j\omega})$ periodic on 2π

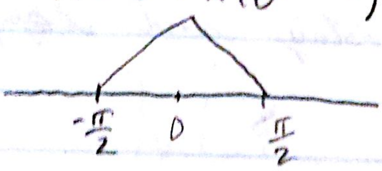
→ shift on factors of 2π to stay periodic

Recovery of a signal after upsampling

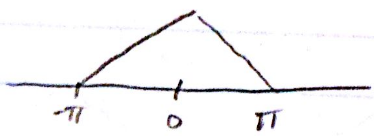
$x[n] \rightarrow \boxed{\uparrow 2} - \boxed{\downarrow 2} \rightarrow x[n]$

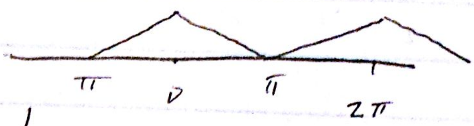
add 0's $\quad$ get rid of 0's

$Y(e^{j\omega}) = X(e^{j\omega 2})$



$X(e^{j\omega})$





$X(e^{j\omega/2})$

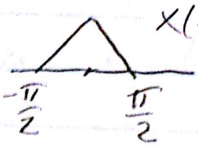
→ shift and add to recover original amplitude

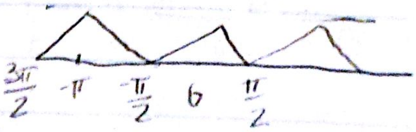
Recover after downsample

$x[n] \rightarrow \boxed{\downarrow 2} \rightarrow \boxed{\uparrow 2} -$

$Y(e^{j\omega}) = X(e^{j\omega/2})$

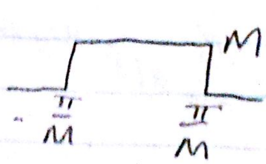
$X(e^{j\omega})$





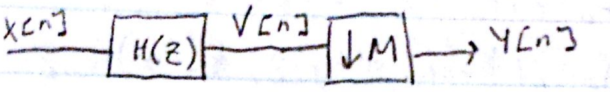
need Low Pass

H_r

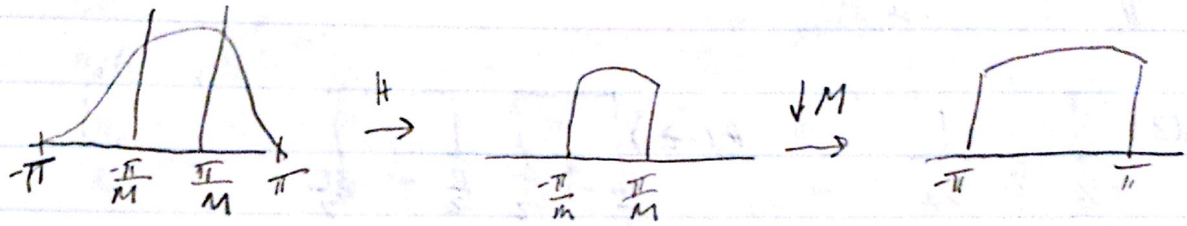


DSP lecture 10/5/15

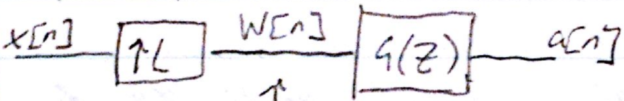
downsample and upsampling filters



no aliasing

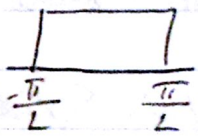


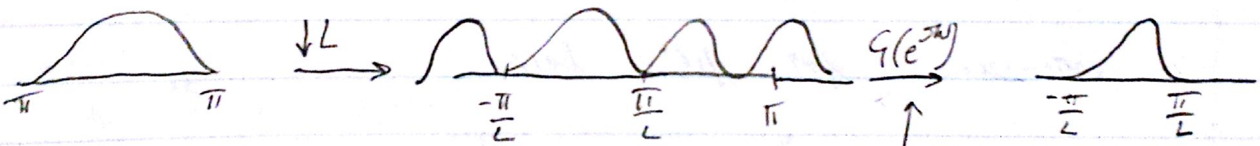
upsample



original + extra zeros

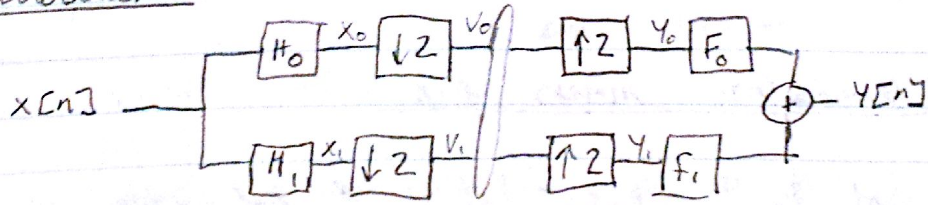
G(z)





interpolation filter

filterbanks: max decimated



processing

H_0 - ideal lowpass
H_1 - high pass

finite impulse response filter → work in z domain

$i \in 0,1$

$$X_i(z) = H_i(z)X(z)$$
$$V_i(z) = \tfrac{1}{2}\{X_i(z^{1/2}) + X_i(-z^{1/2})\}$$
$$Y_i(z) = V_i(z^2) = \tfrac{1}{2}\{X_i(z) + X_i(-z)\} = \tfrac{1}{2}\{H_i(z)X(z) + H_i(z)X(-z)\}$$

$$Y(z) = F_0(z)Y_0(z) + F_1(z)Y_1(z)$$

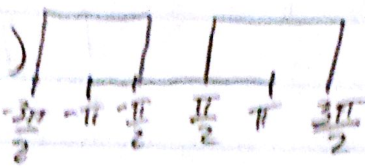
Scanned by CamScanner

$Y(z) = \frac{1}{2}X(z)\{F_0(z)H_0(z) + F_1(z)H_1(z)\} + \frac{1}{2}X(-z)\{F_0(z)H_0(-z) + F_1(z)H_1(-z)\}$

if $X(z) = Y(z)$ we need $\begin{cases} F_0(z)H_0(-z) + F_1(z)H_1(-z) = 0 \quad \forall z \\ (F_0(z)H_0(z) + F_1(z)H_1(z)) = 2 \end{cases}$

cancel the aliased term

1. $H_0 = F_0$, $H_1 = F_1$

$H_0(z)$  $H_0(-z)$ 

DSP lecture 10/7/15

HW. code can be shared. report needs to be unique

2 extensions for filter bank

What is extension? signal is defined from $[0, n]$ extend from 0 to $-n$ as a mirror semetric

conjugate Quadrature mirror filter

→ design of H_0 $H_1(z) = H_0(-z)$ ← not helpful

$H_1(z) = -z^{-2KN} H_0(-z^{-1})$

1. $H_1(z) = -z^{-N} H_0(-z^{-1})$ N: odd N: is the size of the filter
2. in addition $H_0(z)H_0(z^{-1}) + H_0(-z)H_0(-z^{-1}) = 2$
3. aliasing cancelation $F_0(z) = H_1(-z)$ $F_1(z) = -H_0(-z)$

If H_0, H_1, F_0 and F_1 are constructed using 1, 2, 3
then $Y(z) = z^{-N}X(z)$ $Y[n] = X[n-N]$
↑
output of filter

interpretation of ② :

$H_0(z)\,H_0(z^{-1}) + H_0(-z)\,H_0(-z^{-1}) = 2$

$H_0(e^{j\omega})\,H_0(e^{-j\omega}) + H(e^{j(\omega-\pi)})\,H_0(e^{-j(\omega-\pi)}) = 2$ note $H_0\, H_1\, f_0\, f_1$ real

$H_0(e^{-j\omega}) = \sum_{n=-\infty}^{\infty} h_0[n]\, e^{j\omega n} = [H_0(e^{j\omega})]^*$

so

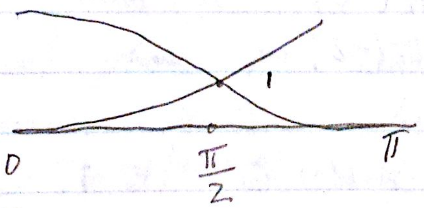
$H_0(e^{j\omega})\,H_0^*(e^{j\omega}) + H(e^{j(\omega-\pi)})\,H_0^*(e^{-j(\omega-\pi)}) = 2$

$|H_0(e^{j\omega})|^2 + |H_0(e^{j(\omega-\pi)})|^2 = 2$ $[0,\pi]$ $[-\frac{\pi}{2},\frac{\pi}{2}]$

$|H_0(e^{j(\frac{\pi}{2}-\lambda)})|^2 + |H_0(e^{j(\frac{\pi}{2}-\lambda)})|^2 = 2$ $\omega = \frac{\pi}{2} + \lambda$

$|H_0(e^{j\omega})|$ is even because $h_0[n]$ is real

at $\omega = \frac{\pi}{2}$ the $2|H_0(e^{j\pi/2})| = 2$



$|H_0(e^{j\omega})|^2 = \cos^2(\omega/2)$

$\cos^2(\omega - \pi/2) = \sin^2(\omega/2)$

solution

$H_0(e^{j\omega}) = \sqrt{2}\cos(\omega t)$

Euler $e^{-j\omega/2} + e^{j\omega/2} \to$ in time

time domain, ① ② ③ yield:

$h_1[n] = (-1)^n h_0(2k-1-n)$ $H_1(z) = -z^{-2k+1} H_0(-z^{-1})$

↑ time reversal $= -z^{-2k+1} \sum h_0[n](-1)^n z$

$F_0[n] = h_0[2k-1-n]$ $= \sum_n \underbrace{h_0[n](-1)^{(n-1)}}_{h_1[2k-1-n]} z^{-(2k-1-n)}$

$F_1[n] = h_1[2k-1-n]$

→ problem: construction prevents the filter to have linear phase.

→ they cannot be simetric in time

→ delays based on frequency

→ solution: use spline filter

filter banks using spline filter: H_0 and F_0 are needed

→ generalize the previous construction to allow linear phase

→ more degrees of freedom: carry 2 filters H_0, F_0

analysis ↑ ↖ reconstru[ction]

replace eq ① ② ③ by new equations

$$H_0(z)H_0(z^{-1}) + H_0(-z)H_0(-z^{-1}) = 2 \Rightarrow H_0(z)F_0(z) + H_0(-z)F_0(-z) = 2$$

$$H_1(z) = H_0(-z) \Rightarrow H_1(z) = z^{N} F_0(-z) \quad N: \text{odd}$$

$$F_1(z) = -H_0(-z) \Rightarrow F_1(z) = z^{-N} H_0(-z)$$

if we can design H_0 and F_0 s.t. ①, ②, ③ hold then output of filter bank is a pure delay $Y(z) = z^{-N} X(z)$ ← pure delay

plug and look for powers of

linear phase for all symmetric signals

Filter is LTI

DSP lecture 10/12/15

[...]sform analysis of Digital filters

idea: decipher properties of a filter

frequency response:

$x \to \boxed{H} \to y$ z domain $Y(z) = H(z)X(z)$

on unit circle $Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$

magnitude $|H(e^{j\omega})|$, $\angle H(e^{j\omega})$

effect of phase on $Y[n]$?

→ consider "narrow" band signal $x[n] = A\cos(\omega_0 n + \phi)$

$Y[n] = A|H(e^{j\omega})|\cos(\omega_0 n + \phi + \angle H(e^{j\omega}))$

→ if the phase is a linear function of the frequency, then there is τ s.t. $\angle H(e^{j\omega}) = -\tau\omega + \varphi_0$

→ $Y[n] = |H(e^{j\omega})| A\cos(\omega_0 n + \phi - \tau\omega_0 + \varphi_0)$
$= |H(e^{j\omega})| A\cos(\omega_0(n - \tau) + \phi + \varphi_0)$

↑ time shift/delay

if $\varphi_0 = 0$
not a constraint

linear phase: time shift/delay that does not depend on ω_0
: super important in audio processing

Ex: single zero/single pole

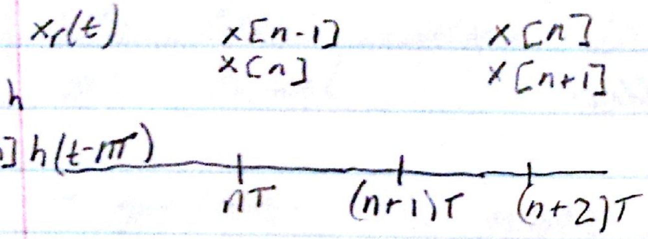
$H(z) = 1 - \alpha_0 z^{-1} = \frac{z}{z}(1 - \alpha_0 z^{-1}) = \frac{1}{z}(z - \alpha_0) \to$ zero @ α_0

$\alpha_0 = \rho e^{j\theta}$ $H(e^{j\omega}) = 1 - (\rho e^{j\theta} e^{-j\omega}) = 1 - \rho e^{-j(\omega-\theta)}$

$|H(e^{j\omega})|^2 = 1 + \rho^2 - 2\rho\left(\frac{e^{j(\omega-\theta)} + e^{-j(\omega-\theta)}}{2}\right) = 1 + \rho^2 - 2\rho\cos(\omega - \theta)$

H.P. $\theta = 0$ $\rho = 1$ | LP $\theta = \pi$ $\rho = 1$

DSP lecture 10/15/15 lecture before test 2



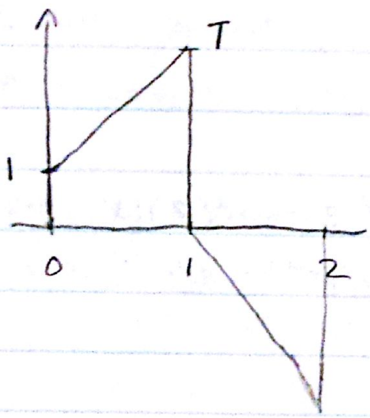
on $(nT, (n+1)T)$: $x[n] + \frac{1}{T}(x[n](t-nT$

$= x[n]\{1 + \frac{1}{T}(t - nT)\}$

$= x[n]\, h[t - nT]$

on $((n+1)T, (n+2)T)$:

the part of $x_r(t)$ that contains $x[n] - x[n]\frac{1}{T}(t-(n+1)T)$



Q: where did the minus sign come from

$$x_r(t) = x[n] + \frac{x[n] - x[n]}{T}(t - (n+1)T)$$

* you want to estimate slope so you borrow the previous term

$$H(j\Omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\Omega t}\, dt \quad = \int_0^1 \left(1 + \frac{t}{T}\right) e^{-j\Omega t} + \int_1^2$$

→ integration by parts

$$\int_0^1 t e^{-j\Omega t}\, dt = \frac{t e^{-j\Omega t}}{-j\Omega} + \frac{1}{j\Omega}\int_0^1 e^{-j\Omega t}$$

$$H(j\Omega) = T(1 + j\Omega T)\left[\frac{\sin(\Omega T/2)}{\Omega T/2}\right]^2$$

* not the sinc because filter is not perfect

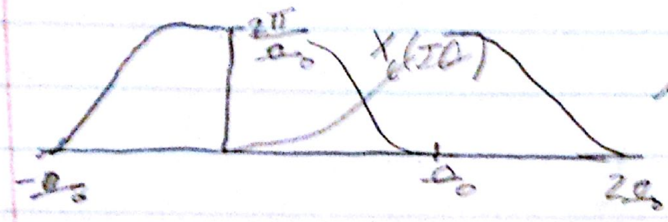
compute angle

$\angle H(j\Omega) = \angle(1 + j\Omega T) = \arctan(\Omega T)$

look up zero order hold

problem 4

$X_c(t) = ?$ $\quad X(j\Omega) = \frac{\pi}{\Omega_0}\left(1+\cos\left(\frac{\pi}{\Omega_0}\Omega\right)\right)$ if $|\Omega| \le \Omega_0$



sample @ $T = \frac{2\pi}{\Omega_0}$

$\frac{2\pi}{T} = \Omega_0$

* the F.T. = 1 time domain is a delta

$x_c(t) = \delta(t - nT)$ → what is the constant?

$1+\cos(2\theta)$ → what is $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$

$= 2\cos^2(\theta) - 1$

$\frac{1+\cos(2\theta)}{2} = \cos^2\theta$ so $1+\cos(2\theta) \to 2\cos^2\theta$

$X(j\Omega) = \frac{\pi}{\Omega_0} 2\cos^2\left(\frac{\pi\Omega}{2\Omega_0}\right)$

$X_c(j(\Omega-\Omega_0)) = \frac{\pi}{\Omega_0}\left(2\cos^2\left(\frac{\pi(\Omega-\Omega_0)}{2\Omega_0}\right)\right) = \cos^2\left(\frac{\pi}{2\Omega_0} - \frac{\pi}{2}\right) = \sin^2($

problem 5

$|X(j\Omega)| = 0$ if $|\Omega| \ge \Omega_m$

$y(t) \le x(t) \underbrace{\sum_{n=1}^{\infty} \lambda^n \sin(n\Omega_0 t)}_{a(t)}$

→ take x and move away from zero

→ if Ω_0 is large there is no cross talk between copies

$Y(j\Omega) = \frac{1}{2\pi} X * A(j\Omega)$

$A(j\Omega) = \sum \lambda^n j\pi(\delta(\Omega - n\Omega_0) - \delta(\Omega + n\Omega_0))$ → A.M.