Computer Problem Set 2.1

The Black-Scholes hedging strategy

The present problem set is attached to Chapter 6 of the lectures notes. Let $T > 0$ *be a fixed maturity. For a positive integer* n*, we denote* $\Delta T := \frac{T}{n}$, $t_i^n := i\,\Delta T$, $i = 0, \ldots, n$. *We consider a Brownian motion* W*, and we introduce the process*

$$S_t \quad := \quad S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}, \quad t \geq 0,$$

where $\mu \in \mathbb{R}$ *and* $\sigma > 0$ *denote the drift and the volatility parameters.*

We recall that, in the context of the Black-Scholes model, the no-arbitrage price of a European call option on an underlying asset with price process $\{S_t, t \geq 0\}$ *is given by*

$$BS(S_0, K, T) := S_0\,\mathbf{N}\left(\mathbf{d}_+(S_0, Ke^{-rT}, \sigma^2 T)\right) - Ke^{-rT}\mathbf{N}\left(\mathbf{d}_-(S_0, Ke^{-rT}, \sigma^2 T)\right),$$

where r *is the instantaneous interest rate,* K, T *denote the strike and the maturity of the option, respectively, and*

$$\mathbf{d}_\pm(s, k, v) := \frac{\ln(s/k)}{\sqrt{v}} \pm \frac{1}{2}\sqrt{v}.$$

The corresponding optimal hedging strategy consists in holding Δ_t *shares of the underlying asset at each time* t*, with*

$$\Delta_t(K) = \mathit{Delta}(S_t, K, T-t) \quad := \quad \mathbf{N}\left(\mathbf{d}_+(S_t, Ke^{-r(T-t)}, \sigma^2(T-t))\right).$$

1. Build a program which produces a sample of $N = 1000$ copies of the discrete path $\{S_{t_i}, i = 0, \ldots, n\}$. Take $T = 1.5$, $S_0 = 100$, $\sigma = 0.3$, $r = 0.05$, use three values of μ: 0.05, 0.02 and 0.45. Compute the corresponding sample mean and variance. Comment the results.

2. Denote

$$e^{-rT} X_T^n(K) := \mathrm{BS}(S_0, K, T) + \sum_{i=1}^{n} \Delta_{t_{i-1}^n}(K)(e^{-rt_i^n} S_{t_i^n} - e^{-rt_{i-1}^n} S_{t_{i-1}^n})$$

   (a) Simulate a sample of $N = 1000$ copies of X_T^n for each value of μ. Use the values of $K \in \{100 \pm i, i = 0, \ldots, 20\}$.

   (b) Compute the corresponding *Profit and Loss*

$$\mathrm{PL}_T^n(K) := X_T^n(K) - (S_T - K)^+.$$

   (c) For each value of μ and K, compute the sample mean and variance of $\mathrm{PL}_T^n(K)$, and provide the corresponding plots in terms of the number of steps n and the strike K. Comment the results for different sample sizes.