Workshop #: IVPs with PDEs using Finite Differences
The Wave Equation

Reading: Numerical Recipes, Ch. 20.0 - 20.1

Consider a wire of length L with mass per unit length ρ that is is held under tension T. Small amplitude displacements of the string $u(x,t)$ are governed by the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}; \qquad c = \sqrt{T/\rho}; \qquad x \in \{0, L\}. \tag{1}$$

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1. **Dimensionless Variables –** Before solving the equation numerically, we should cast it in a convenient dimensionless form. This has the benefit of i) removing redundant parameters from the simulation (reducing the number of parameters), and ii) scaling numbers to $\sim \mathcal{O}(1)$, which can improve accuracy and numerical stability. By moving to dimensionless quantities

$$\tilde{x} = x/x_0 \qquad \tilde{t} = t/t_0 \qquad \tilde{u} = u/u_0 \tag{2}$$

show that for appropriate choices of x_0, t_0 and u_0, the wave equation can be expressed in a form containing no adjustable parameters.

2. **Boundary Conditions:** Code to get you started is provided in `waveEquation.cpp`, and `methods.cpp`. It uses the (unstable) FTCS method, and imposes Dirichlet boundary conditions $u(0) = u(L) = 0$. Modify the code to additionally implement:

i) Neumann boundary conditions[1]:	$u'(0) = u'(L) = 0$
ii) Mixed boundary conditions:	$u(0) = u'(L) = 0$
iii) Periodic boundary conditions:	$u(0) = u(L + \Delta x)$

3. **Travelling Waves**: To verify your boundary conditions, suppose the wire is "plucked" at the centre, such that initial state is a stationary disturbance of the form:

$$u(x,0) = \exp\left[-\left(\frac{x - 1/2}{0.05}\right)^2\right]; \qquad \partial_t u(x,0) = 0; \tag{3}$$

Create a space-time image showing the dynamics for the 4 boundary condition cases for $t \in [0, 2]$. Briefly explain the dynamics seen in each case. Do the same for the case where this initial disturbance is travelling to the right.

4. **Improving the Algorithm:** Writing $u(x_i, t_j) = u_i^j$, show that applying 2nd order finite differences to both the time and space derivatives yields the 2nd order leapfrog scheme:

$$u_i^1 = \Delta t g_i + (1 - \beta^2) u_i^1 + \tfrac{1}{2}\beta^2 (f_{i+1} + f_{i-1}) \tag{4}$$
$$u_i^{j+1} = -u_i^{j-1} + 2(1 - \beta^2) u_i^j + \beta^2 (u_{i+1}^j + u_{i-1}^j); \quad j > 0 \tag{5}$$

where $\beta = c\Delta t/\Delta x$, $g_i = \partial_t u_i^0$ and $f_i = u_i^0$. Von Neumann stability analysis shows this scheme is stable for $\beta < 1$. Implement Eqs. (4) and (5), for Dirichlet boundary conditions, and compare the difference between the initial condition and the final state, $|u(x,0) - u(x,2)|$ for the right-travelling pulse, for a few values of Δt and Δx. What happens for the special case $\beta = 1$?

[1] Use "ghost points" to ensure your Neumann conditions are accurate to the same order as the spatial derivatives.

Extra Problems (optional)

5. ***Boundary Conditions, again:** Implement the additional boundary conditions (as you did in problem 2) for the leapfrog scheme considered in problem 4.

6. ***Standing Waves:** Write down (or look up) the allowed standing wave solutions for either Dirichlet or Neumann boundary conditions, and the corresponding allowed values of the wavenumber k and their frequency ω. Simulate a few cases and verify that your program agrees with the expected analytical result.

7. ***Calculate the Energy:** Using the trapezoidal rule, add a function to your program calculate the energy (per unit mass)
$$E = K(t) + V(t) = \frac{1}{2L}\int_0^L dx \, \left(u_t^2 + c^2 u_x^2\right). \tag{6}$$
Verify E is conserved in the dynamics. Compare the error for different time stepping schemes and discretizations.

8. ***Waves on a Catenery** When a cable of uniform density ρ and length L is suspended under gravity, it forms a catenary shape with $y(x) = D\cosh(x/D)$, and $T(x) = T_0\cosh(x/D)$, where D is the minimum height of the cable above the ground and $x \in \{-L/2, L/2\}$. The disturbances $u(x)$ on top of the equilibrium background profile can be shown to obey the modified wave equation
$$\rho\frac{\partial^2 u}{\partial t^2} = T\frac{\partial^2 u}{\partial x^2} + \frac{\partial T}{\partial x}\frac{\partial u}{\partial x}. \tag{7}$$
Extend your code to solve Eq. (7) and explore what happens for the gaussian disturbance on a catenery.