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Solitons in Superfluid ^{4}He Films

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The Korteweg–de Vries (KdV) equation is derived from Landau two-fluid hydrodynamics applied to the thickness oscillation of the superfluid ^{4}He film at low temperatures, where the main restoring force is van der Waals attraction from the substrate and the thermomechanical force due to phonons is a small correction. Since the usual third-sound generators and detectors are far wider than the individual solitons, the asymptotic solution of the KdV equation provided by the inverse scattering method is coarse-grained by regarding it as a continuous train of sharp pulses. The envelope so obtained still shows a singular front proportional to $(t-t_0)^{-1/2}$, where t_0 is the arrival time of the fastest soliton, and should therefore be observable with the appropriate experimental arrangement.

1. INTRODUCTION

In this paper, we concern ourselves with the surface oscillations of superfluid ^{4}He film adsorbed on a solid substate. We assume the film thickness to be of the order of 10^{-7} cm. The gas of elementary excitations is thus immobilized by its viscosity and the main restoring force of the superfluid oscillation is van der Waals attraction from the substrate. The thermomechanical force proportional to the temperature gradient should also be taken into account, but its effect depends on the condition of thermal conduction. Above 1 K and with a free film–vapor interface, evaporation and condensation are known to be important and, in the limit of small amplitude, we have isothermal third sound.[1,*]

In the present paper, we restrict ourselves to the low-temperature range, where the evaporation rate is exponentially low and the entropy of elementary excitations, phonons in this case, is also small. The effect of the thermomechanical force is a small correction and our situation becomes very similar to what we find in classical hydrodynamics for shallow water canals; the gravity force there corresponds to the van der Waals attraction in

*For the relation to fifth sound, see Williams *et al.*[2]

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our case. There must therefore exist in our superfluid film a nonlinear wave which is similar to the solitary wave of the shallow canal described by the Korteweg–de Vries (KdV) equation.[3]

The possibility of solitons in the superfluid film has been pointed out in fact by Huberman.[4] His argument starts from the Pitaevskii–Gross equation of the Bose condensate at zero temperature,[5] but the derivation of the KdV equation from there is not given in detail. As the solution of the KdV equation, the train of solitons given by the inverse scattering method[6] is assumed, but the question of how it should actually be observed is not clearly answered. This is an important point since the characteristic length and time of an individual soliton are the film thickness itself ($\sim 10^{-7}$ cm) and this length divided by the third-sound velocity ($\sim 10^{3}$ cm sec^{-1}), respectively. Usual third-sound detectors are too insensitive to see individual solitons.

In the present paper, we first derive the KdV equation of motion of the film thickness by including appropriate nonlinear terms in the usual two-fluid hydrodynamics.[7] Underlying physical assumptions are stated in Section 2 before we go into the mathematical details in Section 3.

In Section 4, we also make use of the asymptotic solution given by the inverse scattering method, but emphasize that the generator and detector of the pulse in the actual film are much wider in spatial extension and slower in response time than the characteristic length and time of an individual soliton. We thus assume a train of solitons whose velocity spectrum is almost continuous and take the time (or space) average over an interval containing many solitons. The resulting envelope, which we claim is observable under the appropriate experimental arrangement, still shows a singular front proportional to $(t-t_0)^{-1/2}$, where t_0 is the arrival time of the fastest soliton.

A discussion of further details is given in Section 5.

2. ASSUMPTIONS AND BASIC EQUATIONS

We take the x axis parallel to the (plane) surface of the substrate, and the y axis in the direction of the film thickness, and consider one-dimensional waves propagating along the x axis. Since the superflow is rotation-free, we describe it in terms of the velocity potential ϕ. We may write its equation of motion as

$$\partial\phi/\partial t+\mu=0 \tag{1}$$

where μ is the chemical potential per unit mass.

We assume that local thermal equilibrium is established within the characteristic length and time of our soliton. This would mean that the phonon mean free path is not too long. Remembering that the phonon gas is clamped at the substrate, we have[7]

$$\mu=\mu_0(T, P)+\alpha\left(\frac{1}{d^3}-\frac{1}{y^3}\right)+\frac{\rho_s}{2\rho}\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right] \tag{2}$$

The first term on the right is the equilibrium value as a function of local temperature T and pressure P, the second term is the van der Waals potential due to the substrate, and the third term is the kinetic energy of the superflow, where ρ and ρ_s are total and superfluid mass densities, respectively.

Since the third-sound velocity is one order of magnitude smaller than the first-sound velocity, we assume a constant value ρ_0 of the total density. As for ρ_s, on the other hand, we assume that T and $\sigma=\rho_s/\rho$ are independent of y, but do depend on x, t. Thus

$$\rho(x, y, t)=\rho_0\Theta(y_1(x, t)-y) \tag{3}$$

$$\rho_s(x, y, t)=\rho_0\sigma(x, t)\Theta(y_1(x, t)-y) \tag{4}$$

where $\Theta(x)$ is the Heaviside step function and $y_1(x, t)$ defines the film–vapor interface. Hereafter the subscript 1 will refer to values at this interface.

Substituting (3), (4) in the total mass conservation law,[7] in which we assume the mass current only through the superflow, we obtain

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial\phi}{\partial x}\frac{\partial}{\partial x}\log\sigma=0 \tag{5}$$

$$\sigma^{-1}\frac{\partial y_1}{\partial t}+\frac{\partial y_1}{\partial x}\left(\frac{\partial\phi}{\partial x}\right)_1-\left(\frac{\partial\phi}{\partial y}\right)_1=0 \tag{6}$$

Note that ϕ is *not* harmonic, because of the compressibility of ρ_s.

As we have mentioned, the elementary excitations in our low-temperature range are phonons, which are quanta of third sound in the limit of thin films. We denote the phonon thermal energy per unit mass by $E=\beta T^3$. The entropy per unit mass is thus given by $S=(3/2)\beta T^2$. There also exist bound pairs of quantized vortices, but their number is exponentially small unless T is close to the onset temperature of two-dimensional superfluidity.[8] We neglect their contribution. From Landau's formula,[9] then, we have $\sigma=1-(3E/2c_3^2)$, where c_3 is the third-sound velocity at $T=0$.

We have already neglected the evaporation and the drift of the phonon gas as a whole. As for the ordinary thermal conduction of the phonon gas, we assume that it is good enough to make T uniform along the y axis, but poor enough to make the motion in the x direction approximately adiabatic. From the entropy conservation law, then, Sy_1 should be independent of time. We will use this adiabatic condition in the form

$$T = T_0[1+(\eta/d)]^{-1/2} \tag{7}$$

where T_0 and d are the temperature and thickness of the film at rest, respectively, and η is the small fluctuation of the film thickness. Hence

$$\sigma = 1-\nu[1+(\eta/d)]^{-3/2} \tag{8}$$

where $\nu = (3\beta T_0^3/2c_3^2)$.

Finally, assuming that P_1 is equal to the negligibly small vapor pressure, we write the equation of motion (1) at the film–vapor interface as

$$\begin{aligned}\frac{\partial}{\partial x}\Bigg\{\left(\frac{\partial\phi}{\partial t}\right)_1 - a\left(1+\frac{\eta}{d}\right)^{-3} - 2b\left(1+\frac{\eta}{d}\right)^{-3/2} \\ +\tfrac{1}{2}\left[1-\nu\left(1+\frac{\eta}{d}\right)^{-3/2}\right]\left[\left(\frac{\partial\phi}{\partial x}\right)_1^2 + \left(\frac{\partial\phi}{\partial y}\right)_1^2\right]\Bigg\} = 0\end{aligned} \tag{9}$$

Here $a = \alpha d^{-3}$ is the van der Waals potential at the film surface and $b = (1/4)\beta T_0^3$ characterizes the thermomechanical force.

3. DERIVATION OF THE KDV EQUATION

We will derive the KdV equation from (5), (6), and (9) in a manner similar to the classical derivation.[3] We first expand ϕ in even powers of y and substitute in (5) to determine expansion coefficients in terms of the first term ϕ_0 or the velocity field $U = \partial\phi_0/\partial x$. Thus

$$\phi = \phi_0 + \frac{1}{2!}\phi_2 y^2 + \frac{1}{4!}\phi_4 y^4 + \cdots \tag{10}$$

$$\phi_{2n+2}(x, t) = -\frac{\partial^2\phi_{2n}}{\partial x^2} - \frac{\partial\phi_{2n}}{\partial x}\frac{\partial}{\partial x}\log\sigma \tag{11}$$

In the usual theory of third sound, we are concerned with wavelengths much longer than the film thickness, so that $\partial\phi/\partial x \approx U$ and $\partial\phi/\partial y \approx -d\,\partial U/\partial x$. On the other hand, U and η are regarded as infinitesimals of the same order. From (6) and (9), we then obtain the usual wave equation of third sound with the *adiabatic* sound velocity $[3(1-\nu)(a+b)]^{1/2}$.

In order to derive the KdV equation, we introduce replacements

$$\partial/\partial x \to \varepsilon\,\partial/\partial x, \qquad \partial/\partial t \to \varepsilon^3\,\partial/\partial t, \qquad \eta \to \varepsilon^2\eta \tag{12}$$

We regard ε as an infinitesimal, but it disappears in the final KdV equation, which is known to be invariant under the transformation (12) for *arbitrary* ε.

We also assume the form

$$U = -U_0 + \varepsilon^2 U_1 d^{-1}\eta + \varepsilon^4 d^{-1}\gamma + \cdots \tag{13}$$

Coefficients U_0, U_1 will be determined below and $\gamma(x, t)$ will not enter in the KdV equation of η, though we need γ to write down (6) and (9) separately up to the order of ε^5.

The lowest order terms in (6) are proportional to $\varepsilon^3\, \partial\eta/\partial x$. Demanding that their sum vanish, we obtain

$$U_1 = \left(1 + \frac{3\nu}{2(1-\nu)}\right) U_0 \tag{14}$$

Similarly from (9), we obtain

$$(1-\nu)U_1 U_0 = g_1 d + \tfrac{3}{4}\nu U_0^2 \tag{15}$$

where

$$g_1 = 3d^{-1}(a+b) \tag{16}$$

corresponds to the gravitational acceleration in the case of the shallow water canal. From (14) and (15)

$$U_0^2 = (1 - \tfrac{1}{4}\nu)^{-1} g_1 d \tag{17}$$

Note that $[(1-\nu)g_1 d]^{1/2}$ is the adiabatic third-sound velocity.

Thus (6) and (9) begin with terms of the order of ε^5. To this lowest order, they take the following form:

$$\begin{aligned} &\frac{\partial\eta}{\partial t} - (1-\nu)\frac{\partial\gamma}{\partial x} + C_1\eta\frac{\partial\eta}{\partial x} + C_2\frac{\partial^3\eta}{\partial x^3} = 0 \\ &\frac{\partial\eta}{\partial t} + (1-\nu)\frac{\partial\gamma}{\partial x} + C_3\eta\frac{\partial\eta}{\partial x} + C_4\frac{\partial^3\eta}{\partial x^3} = 0 \end{aligned} \tag{18}$$

We will not write down explicit expressions for C_n, since they are rather complicated in comparison with the classical case of the shallow canal. This is because the van der Waals force depends on the height and our superfluid is not incompressible.

Eliminating γ from (18), we obtain the KdV equation

$$\frac{\partial\eta}{\partial t} - 6t_m^{-1}\eta\frac{\partial\eta}{\partial x} + t_m^{-1}l_m^3\frac{\partial^3\eta}{\partial x^3} = 0 \tag{19}$$

Here

$$\begin{aligned} t_m &= 12(U_0^2/gd)(d/U_0) \\ (l_m/d)^3 &= 2(1-\nu)^{-1}(1+11\nu/4)(U_0^2/gd) \end{aligned} \tag{20a}$$

with

$$g = g_2 - 3g_1(1 - \tfrac{9}{8}\nu + \tfrac{1}{8}\nu^2)(1 - \tfrac{10}{8}\nu + \tfrac{1}{4}\nu^2)^{-1}$$
$$g_2 d = 12a + \tfrac{15}{2}b \tag{20b}$$

In the low-temperature range, where $b \ll a$ and $\nu \ll 1$, we have $g > 0$. For definiteness, let us take the positive square root of (17), so that $U_0 > 0$ and $t_m > 0$. Though expressions are complicated, l_m is of the same order of magnitude as the film thickness d and t_m is of the order of l_m divided by the third-sound velocity.

Taking l_m and t_m as units of length and time, respectively, we can write (19) in the standard form of the KdV equation:

$$\frac{\partial \eta}{\partial t} - 6\eta \frac{\partial \eta}{\partial x} + \frac{\partial^3 \eta}{\partial x^3} = 0 \tag{21}$$

As is well known, this has the soliton solution moving in the positive x direction:

$$\eta = -\tfrac{1}{2}v \,\mathrm{sech}^2\, \tfrac{1}{2}v^{1/2}(x - vt) \tag{22}$$

From (7), negative η means a hot soliton. From (13), our choice of $U_0 > 0$ means uniform flow in the negative x direction. We may choose the negative square root of (17) as well; we then have uniform flow in the positive x direction and hot solitons in the negative x direction.

4. AVERAGED TRAIN OF SOLITONS

As pointed out in Section 1, the usual third-sound generators and detectors have a very much greater width and are slower in response time on the scales given by (20). The actual heat pulse we can produce in the superfluid film must, at best, be a train of many solitons and we can only detect its time or space average. In this section, we present a general solution of such averaged soliton trains.

According to the inverse scattering method,[6] the asymptotic solution of (21) as $x \to \infty$, $t \to \infty$ is given by

$$\eta(x, t) = -\sum_{n=1}^{N} 2E_n \,\mathrm{sech}^2\, E_n^{1/2}(x - 4E_n t) \tag{23}$$

Here $-E_1, -E_2, \ldots, -E_N$ are eigenvalues of bound states given by the Schrödinger equation

$$[-d^2/dx^2 + E_n + \eta(x, 0)]\psi_n(x) = 0 \tag{24}$$

In view of the slow response of a typical detector, it is more realistic to have a coarse-grained formula for η. In the following, we derive such a formula, and show that it satisfies two basic requirements imposed by the KdV equation itself: the "charge" conservation law $(\partial/\partial t)\int dx\,\eta(x,t)=0$, and the scale invariance mentioned in the previous section.

In the coarse-grained picture, the nth contribution in (23)

$$\eta_n(x,t) = -2E_n \operatorname{sech}^2 E_n^{1/2}(x-4E_n t) \tag{25}$$

may be regarded as a delta function with respect to x or t:

$$\eta_n(x,t) = -(4E_n)^{1/2}\,\delta(x-4E_n t) \tag{26}$$

or

$$\eta_n(x,t) = -(4E_n)^{-1/2}\,\delta(t-x/4E_n) \tag{27}$$

where the coefficients have been obtained from the following integrals:

$$\int_{-\infty}^{\infty} dx\, 2E_n \operatorname{sech}^2 E_n^{1/2}(x-4E_n t) = (4E_n)^{1/2} \tag{28a}$$

$$\int_{-\infty}^{\infty} dt\, 2E_n \operatorname{sech}^2 E_n^{1/2}(x-4E_n t) = (4E_n)^{-1/2} \tag{28b}$$

The two expressions (26) and (27) lead, in fact, to the same equation

$$\eta_n(x,t) = -(1/2t)E_n^{1/2}\,\delta(E_n - x/4t) \tag{29}$$

due to the identity $\delta(ax)=|a|^{-1}\delta(x)$.

Thus, if t is large enough so that the solitons may be regarded as essentially nonoverlapping, the asymptotic solution (23) can be rewritten as follows:

$$\eta(x,t) = -\sum_n \frac{1}{2t} E_n^{1/2}\,\delta\left(E_n - \frac{x}{4t}\right) = -\frac{1}{2t}\int dE\, N(E)\sqrt{E}\,\delta\left(E-\frac{x}{4t}\right) \tag{30}$$

where $N(E)$ is the density of bound states in the Schrödinger equation (24). We therefore obtain the coarse-grained asymptotic solution:

$$\eta(x,t) = -(1/2t)(x/4t)^{1/2}N(x/4t) \tag{31}$$

As mentioned above, this formula satisfies two important requirements. First, since the KdV equation $\partial_t\eta+\partial_x(-3\eta^2+\partial_x^2\eta)=0$ has the form of a conservation law, the "charge" $Q=\int dx\,\eta(x,t)$ must be a constant of motion. This is in fact the case with our formula (31), since

$$Q = -\int_{-\infty}^{\infty} dx \frac{1}{2t}\left(\frac{x}{4t}\right)^{1/2} N\left(\frac{x}{4t}\right) = -2\int_{-\infty}^{\infty} dE\,\sqrt{E}\,N(E) \tag{32}$$

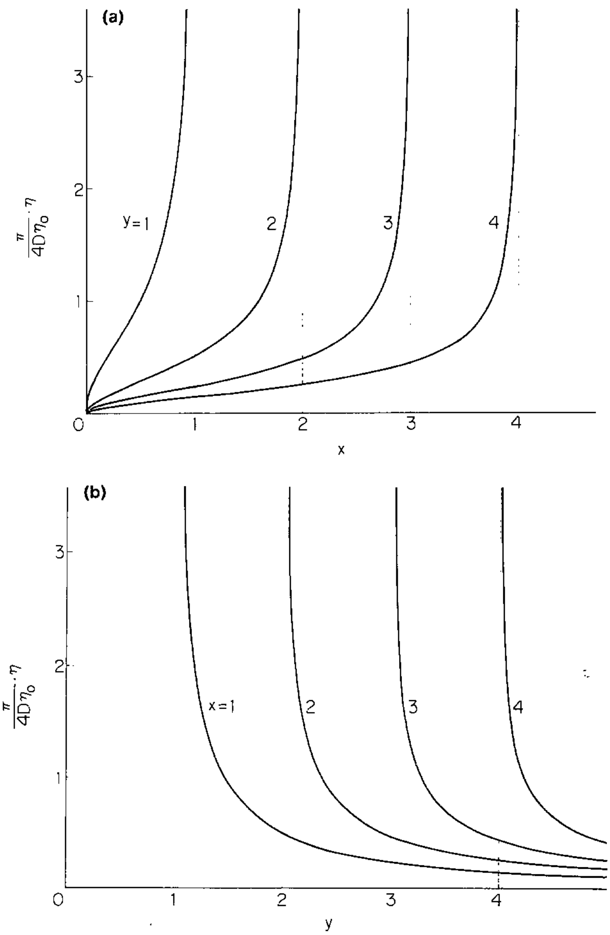

Fig. 1. Coarse-grained asymptotic solution η for an initial pulse of the square-well form. The time variable is written as $y \equiv 4\eta_0 t$. Space–time scales may be varied according to the scale transformation discussed in the text. (a) Spatial form of η. (b) The time dependence of η at fixed values of x.

is a constant determined solely by the initial value $\eta(x, 0)$. Second, the formula (31) is invariant under scale transformation. To see this we define, for any physical quantity A, a scale dimension $[A] = \alpha$ if A transforms like x^α under the scale transformation $x \to \varepsilon^{-1}x$. Then we obtain

$$[x] = 1, \qquad [t] = 3, \qquad [\eta] = -2, \qquad [E] = -2 \tag{33a}$$

and

$$[N(E)] = 2 \tag{33b}$$

As a simple model of the initial pulse, let us take the square well $\eta(x,0) = -\eta_0\Theta(D - |x|)$ with width $2D$ and depth η_0 (see Fig. 1). Let us assume $D \approx 10^{-3}$ cm, or 10^4 in our units, and $\eta_0 \approx 10^{-8}$ cm, or 10^{-1} in our units. The reason for this choice of D, which is somewhat smaller than the usual width of the third-sound generator, is to obtain the following picture of the soliton train.

Suppose that the soliton train is to be observed at a large distance $x \approx 10^7$ and around the time $t_0 = x/4\eta_0 \approx 3 \times 10^7$ in which the fastest soliton travels the distance x. The arrival time of the second soliton is $t_0 + \Delta t$, where $\Delta t \approx t_0\eta_0^{-1}(\pi/D)^2 \approx 30$ is somewhat longer than the duration of the single soliton $4\eta_0^{-3/2} \approx 10$. So, we would be able to see an individual soliton if the time resolution of the detector were high enough.

In the case of our square-well potential, we have $N(E) = (2D/\pi)[\eta_0 - E]^{1/2}$, provided that D is large enough to make the spectrum quasicontinuous. Thus

$$\eta(x,t) = -(D/\pi t)[4\eta_0(t/x) - 1]^{-1/2} \tag{34}$$

We should be able, under the appropriate experimental arrangement, to observe the heat pulse with the singular front proportional to $(t - t_0)^{-1/2}$, where $t_0 = x/4\eta_0$, and with a long-time tail proportional to $t^{-3/2}$.

5. DISCUSSION

In the previous sections we have shown in detail that nonlinear third sound obeys the KdV equation, and have given an asymptotic solution which has direct relevance to experiment. In particular we have predicted, for the square-well form of the initial deformation, a square-root singularity $(t - t_0)^{-1/2}$ and a long-time tail $t^{-3/2}$ in the time-dependent response. It should be noted that such a behavior is rather universal, although it might seem to be a consequence of the particular initial condition. The universality comes from the fact that the width $2D$ of the pulse generator is 10^5 times larger than the deformation η_0! Thus the initial deformation of the surface has little, if any, curvature, so that our initial condition should be applicable to real experimental situations irrespective of details. It should be remarked here again that the general formula (26) for a train of solitons satisfies the scale invariance required by the KdV equation (21).

Up to now, we have neglected dissipative processes such as evaporation and condensation at the liquid–vapor interface, motion of the normal component of the fluid, and so on. This is a good approximation at low temperatures, where these effects are strongly depressed. To be more realistic, however, let us consider what will happen in our system if dissipation is allowed. Since we are interested in the long-time behavior, it is natural to assume that the dissipation is characterized by a single relaxation time τ, the largest one corresponding to the slowest relaxation mechanism. Then we may generalize Eq. (21) to include a dissipative term as follows:

$$\left(\frac{\partial}{\partial t}+\frac{1}{\tau}\right)\eta-6\eta\frac{\partial\eta}{\partial x}+\frac{\partial^3\eta}{\partial x^3}=0 \tag{35}$$

Without solving the equation, we can see some general features of the dissipative system.

Let us introduce a new function $\zeta(x, t)$ defined by

$$\eta(x, t)=e^{-t/\tau}\zeta(x, t) \tag{36}$$

Then Eq. (35) is rewritten as

$$\frac{\partial\zeta}{\partial t}-6\,e^{-t/\tau}\zeta\frac{\partial\zeta}{\partial x}+\frac{\partial^3\zeta}{\partial x^3}=0 \tag{37}$$

In this form it is obvious that:

(i) For $t\ll\tau$, Eq. (37) is essentially identical with the usual KdV equation.

(ii) For $t\gg\tau$, it reduces to a *linear equation* $(\partial/\partial t+\partial^3/\partial x^3)\eta=0$ describing the usual third sound with damping: $\eta=\eta_0\,e^{-t/\tau}\cos k(x-k^2t)$.

Of course the "charge" $Q(t)=\int dx\,\eta(x, t)$ is no longer a constant of motion, but shows an exponential decay $Q(t)=Q(0)\,e^{-t/\tau}$.

Thus, in order to observe experimentally the nonlinear third sound which we have described, the relaxation time must be longer than the arrival time t_0 defined in Section 4. Although the arrival time t_0 cannot be made too short, to assure the validity of the asymptotic solution given in Eq. (23), the relaxation time τ can be made sufficiently long by lowering the temperature (for example, the evaporation–condensation mechanism dies away exponentially below 1 K).

We therefore believe that the characteristic behavior of the nonlinear third sound discussed in this paper should be observable. An experimental check of our theory is desirable, since if our theory is valid, it will provide a new tool for studies of both nonlinear waves and superfluid films.

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