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Solitary Waves in Saturated Films of Superfluid ^{4}He

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It is shown that there should exist in saturated films of superfluid ^{4}He at low temperature a nonlinear surface wave described by the Korteweg–de Vries equation. The effect of surface tension makes the solitary wave cold, in contrast to very thin films discussed in a preceding paper. An electrostatic method is proposed to generate cold solitary waves. The asymptotic behavior can be found theoretically by the inverse scattering method. Estimates of parameters imply the possibility of observing each solitary wave separately.

1. INTRODUCTION

In a preceding paper[1] (referred to as I hereafter), we applied two-fluid hydrodynamics with appropriate nonlinear terms to a thin film of superfluid ^{4}He adsorbed on a solid substrate and derived a Korteweg–de Vries (KdV) type of equation[2] to describe the motion of the film thickness. Thus there should exist in the superfluid film a solitary wave[3] very similar to the case of a shallow water canal. In practice, however, we could only generate and detect a train of solitary waves with an almost continuous velocity spectrum, since the usual generators and detectors of third sound[4] have much greater spatial and temporal scales than the single *soliton*, i.e., the individual solitary wave. To make the damping effect negligible, the film should be kept at a temperature below 1 K, where the evaporation rate at the film–vapor interface is exponentially small.

In the present paper, we extend the theory to a so-called saturated film of superfluid ^{4}He, whose thickness is of the order of 10^{-6} cm. The most important point is then that the surface tension, completely neglected in I, now plays a decisive role and makes the characteristic length of the soliton even longer than the film thickness. The velocity of propagation (third-sound velocity) is lower in our thicker film, so that the characteristic time of the soliton is also longer. Thus the soliton in the saturated film is more extended and therefore easier to generate and detect than in the very thin film we dealt with in I.

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In the present paper, we also correct a number of weak points in I. In particular, in discussing experimental conditions to observe solitary waves, we have overlooked the fact that the KdV equation describes the motion with reference to a system of coordinates moving uniformly with respect to the laboratory system.

In Section 2, we briefly repeat the derivation of the KdV equation including the surface tension, but neglecting the thermomechanical force. The resulting KdV equation gives a "cold" solitary wave, in contrast to the very thin film in I. In Section 3, we estimate the characteristic length and time of our solitary wave, taking a typical experimental arrangement for observing third sound in saturated films. In Section 4, we suggest an electrostatic method for generating cold solitary waves in such a way that the asymptotic behavior at a large distance from the generator may be found theoretically by the method of inverse scattering.[5] It does not seem to be difficult to observe a single soliton. In Section 5, we summarize our conclusions.

2. KDV EQUATION WITH SURFACE TENSION

Suppose we have a saturated film of superfluid ^{4}He with a uniform equilibrium thickness $d \sim 10^{-6}$ cm. As we have shown in I, there should exist in our superfluid film a solitary wave similar to the case of a shallow canal. The van der Waals attraction from the solid substrate plays the same role as gravity acting on the water in the canal, but it depends on the distance from the bottom, in contrast to the gravitational force. Furthermore, the superfluid is accelerated by the temperature gradient, but this only gives a small correction in our low-temperature film, as seen in I. In the present paper, we neglect it for simplicity. We should still assume, however, that thermal conduction is good enough to keep the temperature uniform in the direction (y axis) of the film thickness and poor enough to make the motion adiabatic in the direction (x axis) parallel to the plane surface of the substrate.

In our saturated film, the phonon–roton excitation gas is still immobilized by its viscosity and we are concerned only with the superflow. Since this is irrotational, we describe it in terms of the velocity potential ϕ. We neglect the thermomechanical effect, as mentioned above. We see from the formalism given in I that our film may then be regarded as being virtually at zero temperature and assumed to be incompressible. Thus the continuity equation in bulk gives

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{1}$$

The continuity equation at the film–vapor interface takes the form

$$\frac{\partial y_1}{\partial t}+\left(\frac{\partial \phi}{\partial x}\right)_1\frac{\partial y_1}{\partial x}-\left(\frac{\partial \phi}{\partial y}\right)_1=0 \tag{2}$$

Here $y_1 = d + \eta(x, t)$ defines the film–vapor interface and the index 1 refers to this surface. Finally, the equation of motion of superfluid is

$$\partial\phi/\partial t+\mu=0 \tag{3}$$

where μ is the chemical potential per unit mass.

The basic equations (1)–(3) have exactly the same form as those describing the surface wave of the water canal. The difference exists only in the chemical potential, in which the gravitational potential is replaced by the van der Waals potential $-\alpha y^{-3}$ due to the solid substrate:

$$\mu_1=\text{const}+\frac{1}{\rho}(P_1-P_0)-\alpha\left(\frac{1}{y_1^3}-\frac{1}{d^3}\right)+\frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)_1^2+\left(\frac{\partial\phi}{\partial y}\right)_1^2\right] \tag{4}$$

Here ρ is the density of superfluid and P_0 is the vapor pressure, which we assume to be related with the film pressure P_1 by Pascal's law. With use of the surface tension σ we have

$$P_1-P_0=-\sigma\,\partial^2\eta/\partial x^2 \tag{5}$$

We expand the van der Waals potential in (4) in powers of η and retain terms up to second order. The final form of the equation of motion at the surface is

$$\left(\frac{\partial\phi}{\partial t}\right)_1+\frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)_1^2+\left(\frac{\partial\phi}{\partial y}\right)_1^2\right]-\frac{\sigma}{\rho}\frac{\partial^2\eta}{\partial x^2}-g_1\eta-\frac{1}{2}g_2\frac{\eta^2}{d}=0 \tag{6}$$

where

$$g_1=3\alpha/d^4, \qquad g_2=12\alpha/d^4 \tag{7}$$

As we have shown in I, we can derive the KdV equation of η from (1), (2), (6) in almost the same way as in the classical approach.[2] For instance, (1) may be satisfied by

$$\frac{\partial\phi}{\partial x}=\left(1-\frac{1}{2!}y^2\frac{\partial^2}{\partial x^2}+\frac{1}{4!}y^4\frac{\partial^4}{\partial x^4}+\cdots\right)U(x,t) \tag{8}$$

where U is to be determined by (2), (6). It is convenient to suppose that $\partial/\partial x$, η, $\partial/\partial t$ are proportional to a small parameter ε, ε^2, and ε^3, respectively. For instance, we write

$$U=U_0+U_1(\eta/d)+U_2(\gamma/d)+\cdots \tag{9}$$

where U_0, U_1, U_2, ... are constant and $\gamma(x, t)$ is of the order of ε^4. From (2) and (6) to the order of ε^3 we obtain $U_1 = -U_0$ and

$$U_0 = \pm(g_1 d)^{1/2} \tag{10}$$

This is the *adiabatic* third-sound velocity.

From (2) and (6) to the order of ε^5 we obtain the KdV equation

$$\frac{\partial\eta}{\partial t} + \frac{1}{2U_0}\left[(g_2 - 3g_1)\eta\frac{\partial\eta}{\partial x} + \left(\frac{\sigma d}{\rho} - \frac{1}{3}g_1 d^3\right)\frac{\partial^3\eta}{\partial x^3}\right] = 0 \tag{11}$$

In the corresponding classical equation[2] g_1 is replaced by the gravitational acceleration and $g_2 = 0$, so that the coefficient of the nonlinear term has the opposite sign, since $g_2 = 4g_1$ in our case. The sign of our dispersive term, on the other hand, depends similarly on the size of the surface tension. The sign changes at the critical thickness

$$d_c = [\rho\alpha/\sigma]^{1/2} \tag{12}$$

which is of the order of 10^{-7} cm. Hence our theory given in I, where the surface tension is completely neglected, applies only to a very thin film consisting of a few atomic layers.

3. ESTIMATES OF PARAMETERS

In the present paper, we concern ourselves with a saturated film whose thickness d is larger than the critical value (12). For the sake of definiteness, let us take the negative sign in (10). This means that our KdV equation (11) describes the motion with reference to the x axis moving with the negative velocity U_0 with regard to the laboratory system.

We may write (11) in the standard form

$$\frac{\partial\eta}{\partial t} - 6\eta\frac{\partial\eta}{\partial x} - \frac{\partial^3\eta}{\partial x^3} = 0 \tag{13}$$

where units of length and time are now taken, respectively, as

$$l_s = d[2(d/d_c)^2 - 2]^{1/3}, \qquad t_s = 12d(d^3/3\alpha)^{1/2} \tag{14}$$

We thus see that $l_s \propto d^{5/3}$ when $d \gg d_c$, whereas $l_s \approx d$ in the case of very thin films discussed in I. As is well known, the KdV equation (13) has the single-soliton solution of the form

$$\eta = 2\varepsilon^2 \operatorname{sech}^2 \varepsilon(x + 4\varepsilon^2 t) \tag{15}$$

where the positive constant ε may be identified with the expansion parameter mentioned in Section 2. If the temperature is finite (though low) and thermal conduction is good enough in the direction of the film thickness, positive η means an increase of the superfluid density over the equilibrium value and therefore a decrease of temperature ("cold" soliton). We had a "hot" soliton in I and the difference comes from the change of sign of the dispersive term in (11). The soliton (15) moves to the left with the velocity $4\varepsilon^2$, but we should remember that (15) is the soliton as seen by an observer moving with velocity $-U_0 = (3\alpha/d^3)^{1/2}$, or

$$u = 12(d/l_s)/(t_s/t_s) \tag{16}$$

in units of (14), relative to the laboratory system. In the laboratory system, the soliton is moving to the right with the velocity $u - 4\varepsilon^2$.

To estimate parameters, we take from the review article of Atkins and Rudnick[4] a typical experimental arrangement to observe third sound in saturated films. The film is formed, for example, on the horizontal surface of a stainless steel mirror partially immersed in bulk superfluid ^{4}He. Supposing that the gravitational force is balanced by the van der Waals force from the substrate, we estimate the film thickness by

$$d = (\alpha/g)^{1/3} H^{-1/3} \tag{17}$$

where H is the height of the film above the bulk superfluid level and g is the gravitational acceleration. For definiteness, we will assume the value $(\alpha/g)^{1/3} = 3 \times 10^{-6}\ \mathrm{cm}^{4/3}$ suggested by Atkins and Rudnick.

From (14), then, we obtain $l_s \approx 3.5 \times 10^{-5}$ cm and $t_s \approx 6.6 \times 10^{-7}$ sec for $H = 1$ cm. The characteristic velocity is therefore $l_s/t_s \approx 5.3 \times 10$ cm sec^{-1}, and the dimensionless velocity (16) is $u \approx 1.0$. The time for a signal to travel 1 cm, or 2.9×10^4 in units of l_s, with this velocity is 1.9×10^{-2} sec. Comparing these values with $l_s \approx d \approx 10^{-7}$ cm, $t_s \approx 10^{-10}$ sec for very thin films as found in I, we see that saturated films are much more favorable for observing solitary waves.

4. GENERATION OF SOLITARY WAVES

According to the method of inverse scattering,[5] the KdV equation (13) has an asymptotic solution

$$\eta(x, t) = \sum 2E_n \operatorname{sech}^2 [E_n^{1/2}(x + 4E_n t + \theta_n)] \tag{18}$$

Here $-E_n$ is the "energy level" of the nth bound state of the one-dimensional Schrödinger equation whose potential is the initial value $\eta(x, 0)$:

$$[-d^2/dx^2 + E_n + \eta(x, 0)]\psi_n(x) = 0 \tag{19}$$

The phase θ_n in (18) is determined by the normalized eigenfunction ψ_n. The sum is to be taken over all the bound states of (19).

A simple choice is the potential well defined by $\eta(x, 0)=\eta_0>0$ for $|x|\leq D/2$ and $\eta(x, 0)=0$ otherwise. As is well known, (19) then has only one bound state if $\eta_0<(\pi/D)^2$. For $D=10^{-2}$ cm, or 2.9×10^2 in units of l_s, $\eta_0<1.2\times 10^{-4}$. For $\eta_0^{1/2}D\gg 1$, on the other hand,

$$E_n\approx\eta_0-(\pi n/D)^2, \qquad n=0, 1, 2, \ldots \tag{20}$$

and the number of bound states is approximately given by $(\eta_0^{1/2}D/\pi)$.

The question is now how to set up such an initial distortion of the film surface experimentally. A possible method would be to make use of a capacitor formed by the substrate itself and a thin metal strip stretched above the film surface. The film surface within the capacitor will be lifted up by applying an electric field E, which is related to the increase η_0 of the film height by

$$E=\left\{\frac{8\pi\varepsilon}{\varepsilon-1}\frac{\rho\alpha}{d^3}\left[1-\left(1+\frac{\eta_0}{d}\right)^{-3}\right]\right\}^{1/2} \tag{21}$$

where $\varepsilon=1.057$ is the dielectric constant of liquid ^{4}He. For $\eta_0\ll d$, we have $E\approx 1.3\times 10^5(H\eta_0/d)^{1/2}$ V cm^{-1}, where H is measured in cm.

In this electrostatic method, our parameter D is the width of the capacitor. To set up the initial deformation $\eta(x, 0)$ we need to switch off the electric field quickly enough. The switching time should be shorter than, say, $0.1D$ divided by the third-sound velocity, i.e., 10^{-4} sec for $D=10^{-2}$ cm, $H=1$ cm.

Hereafter we assume these values of D and H, and take (14) as units of length and time unless explicitly stated otherwise. We also take $\eta_0=5\times 10^{-2}$ $(\eta_0/d\approx 0.6)$. Since $(\pi/D)^2\approx 1.2\times 10^{-4}$, (18) will give a train of about 20 solitary waves at a large distance from our generator. Take $\Delta x=1$ cm $=$ 2.9×10^4 as this distance.

The time for the *slowest* soliton to travel Δx is $t_0\approx\Delta x/(u-4\eta_0)=$ 3.6×10^4. Similarly, the travel time of the next slowest soliton is

$$t_1\approx t_0\left[1-\frac{1}{u-4\eta_0}\left(\frac{2\pi}{D}\right)^2\right]$$

so that $\Delta t_0=t_0-t_1\approx 21$ (14 μsec). This is longer than the duration time of the slowest soliton, $\delta t_0\approx[\eta_0^{1/2}(u-4\eta_0)]^{-1}\approx 5.6$. Hence the use of the asymptotic solution (18) is justified and the notion of soliton is meaningful. Note that phases θ_n in (18) are negligible, since they must be of the order of unity.

As for the question of how to detect solitons, it might be possible to use a usual metal strip at the superconducting transition point as the resistance thermometer,[4] provided that it is fast and sensitive enough. As pointed out in I, the situation becomes much easier if we can make the width D of the generator one order of magnitude smaller, so that (18) gives only two or three solitons. The time interval t_0 is of the order of 1 msec and we would be able to detect each soliton separately by using a detector of width 10^{-2} cm.

5. CONCLUSION

In this paper, we have derived the Korteweg–de Vries equation for the nonlinear motion of a saturated film of superfluid ^{4}He. In contrast to the case of very thin films discussed in our preceding paper,[1] we cannot neglect the surface tension, which makes the solitary wave "cold" and also more extended in space and time. It even seems possible to observe a single soliton with appropriate experimental arrangements.

We have suggested the use of a capacitor to generate cold solitary waves. If the width of the capacitor can be made as narrow as 10^{-3} cm, we can perhaps generate a pair of solitary waves whose separation is of the order of 1 msec. The film thickness is assumed to be 3×10^{-6} cm or so and the initial strain given by the capacitor to be 100%.

For thinner films, the number of solitary waves increases and we eventually reach the situation where we have a train of solitary waves with an almost continuous velocity spectrum. If the resolution of the detector is poor, what we observe will be not the precise form of the asymptotic solution (18), but its time or space average, as pointed out in the preceding paper. Let τ be a time interval such that $1 \gg \tau/t \gg u(\pi/D)^2$. Taking the average of (18) over this interval, we have

$$\frac{1}{\tau}\int_t^{t+\tau} \eta(x_L, t)\, dt = \frac{1}{4}\left(\frac{ut-x_L}{t^3}\right)^{1/2} N\left(\frac{ut-x_L}{4t}\right) \tag{22}$$

where $x_L = x + ut$ defines the laboratory system of coordinates and $N(E) = (D/\pi)(\eta_0 - E)^{-1/2}$ is the density of bound states of (19). Thus (22) is proportional to $[(t-t_f)/(t_0-t)]^{1/2}$, where $t_f = \Delta x/u$ is the arrival time of the fastest soliton. In other words, even in this limit of almost continuous solitary waves, we would have a singular backward accumulation of distortion. We reached a similar conclusion in I, but there the singularity appeared at the front.

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