CSCI 3022

intro to data science with probability & statistics

Lecture 6
February 2, 2018

0. HW1 due by 5PM today to Moodle.

1. Bayes' Rule
2. Random variables and probability distributions

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Bayes' Rule

- Recall the probability that two events intersect: $P(A \cap B) = P(A|B)P(B)$
- But we can write it the other way, too: $P(A \cap B) = P(B|A)P(A)$
- And then… $P(A|B)P(B) = P(B|A)P(A)$ $P(B) > 0$
- And this, is Bayes' Rule: $$\boxed{P(A|B) = \frac{P(B|A)P(A)}{P(B)}}$$
- [Bonus: suppose A and B are independent…]

$P(A|B) = P(A)$, $P(B|A) = P(B)$

$\cancel{P(A)} = \frac{\cancel{P(B)}\,\cancel{P(A)}}{\cancel{P(B)}}$

Bayes' Rule + Law of T. P.

- Use Law of Total Probability to rewrite the denominator:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

LTP

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

- Or, if B can be broken into K disjoint events:

$$P(A_1|B) = \frac{P(B|A_1)\,P(A_1)}{P(B)} \overset{\text{LTP}}{=} \frac{P(B|A_1)P(A_1)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \dots + P(B|A_K)\,P(A_K)}$$

$$= \frac{P(B|A_1)\,P(A_1)}{\sum_{k=1}^{K} P(B|A_k)\,P(A_k)}$$

sum notation

Bayes' Rule + Law of T. P.

- Use Law of Total Probability to rewrite the denominator:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \qquad P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^C)P(A^C)}$$

- Or, if B can be broken into K disjoint events:

$$P(A_1|B) = \frac{P(B|A_1)P(A_1)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \ldots}$$

$$P(A_1|B) = \frac{P(B|A_1)P(A_1)}{\sum_{k=1}^{K} P(B|A_k)P(A_k)}$$

Drugs, Prostates, TSA

- One of the most common applications of Bayes' rule is when we develop a test for something, but **the test is not always accurate**.

- Imagine that 1% of math professors are using a drug *Bayes Salts*. There is a test that detects Bayes Salts on the breath of professors 98% of the time, and only gives a false positive 1% of the time.

- Suppose we test Professor Charles Xavier… he's **positive** for Bayes Salts! What is the probability that Professor Xavier is **actually** on Bayes Salts?

Drugs, Prostates, TSA

2% is the False Negative Rate

1% is the False Positive Rate

- 1% of math professors *Bayes Salts* users.
- If on *Bayes Salts*, test says **+** 98% of the time
- If not on *Bayes Salts*, test says **+** only 1% of the time.
- What is probability that Prof. X is a user if he tests positive for *Bayes Salts*?

$$P(\text{user} \mid \text{test} \oplus) \overset{\text{Bayes}}{=} \frac{P(\text{test} \oplus \mid \text{user})\, P(\text{user})}{P(\text{test} \oplus)} \overset{\text{LTP}}{=} \frac{P(\text{test} \oplus \mid \text{user})\, P(\text{user})}{P(\text{test} \oplus \mid \text{user})\, P(\text{user}) + P(\text{test} \oplus \mid \text{not use})\, P(\text{not use})}$$

$$= \frac{0.98 \cdot \cancel{0.01}}{0.98 \cdot \cancel{0.01} + \cancel{0.01} \cdot 0.99} = \frac{0.98}{0.98 + 0.99} = \boxed{0.497}$$

Teach the controversy! 🫢

- Should we test men for prostate cancer?
- Bayes' Rule allows us to write down the probability that someone who tests positive for prostate cancer *actually has* prostate cancer.
- False positives may cause huge amounts of stress, heartache, and even unnecessary surgery!
- On the other hand, if you don't test for cancer, you may not discover it until it's too late.
- Things are slightly more complicated than this: age, PSA cutoffs, etc.

Flipping around a previous problem

- **Suppose I have two bags of marbles.** The first bag contains 6 white marbles and 4 black marbles. The second bag contains 3 white marbles and 7 black marbles. Now suppose I put the two bags in a box. If I close my eyes, grab a bag from the box, reach into the bag and pull out a white marble. What is the probability that I picked Bag 1?

$$P(\text{bag }1 \mid \text{white}) \overset{\text{Bayes}}{=} \frac{P(\text{white} \mid \text{bag }1)\,P(\text{bag }1)}{P(\text{white})}$$

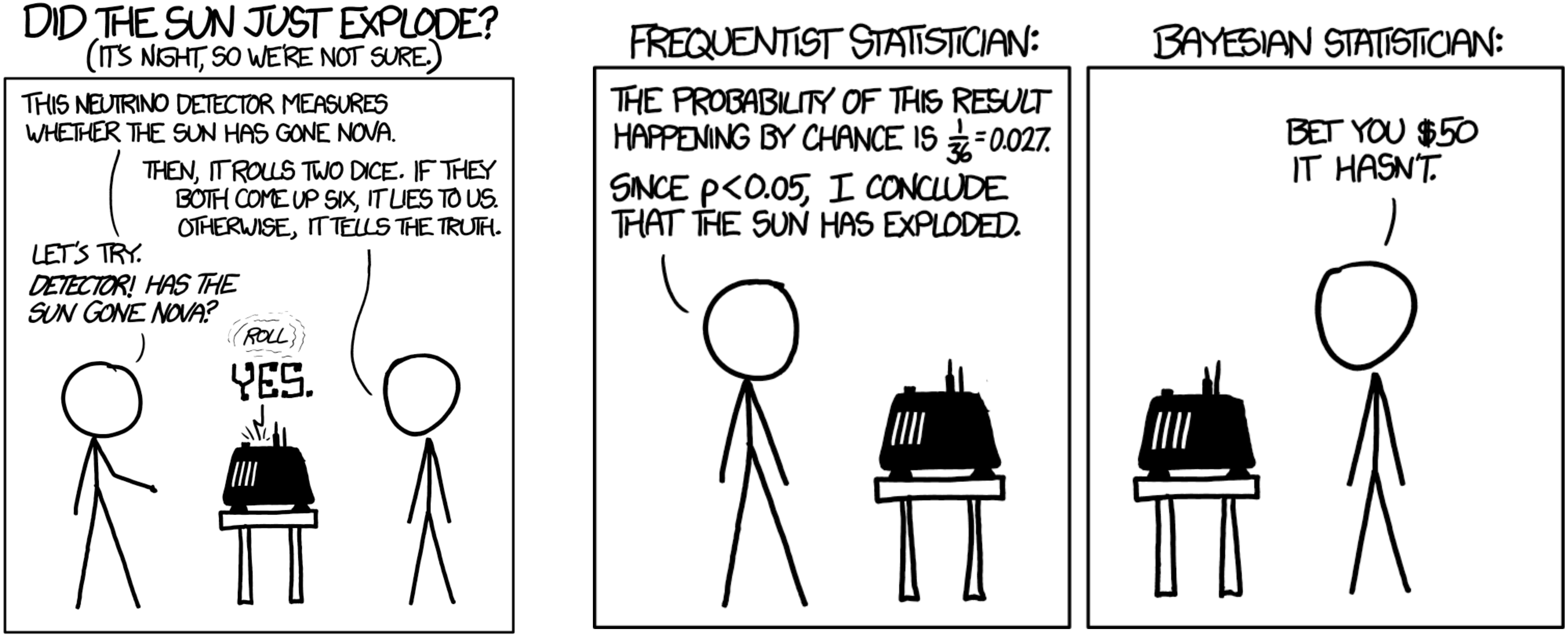
$$\overset{\text{LTP}}{=} \frac{P(\text{white} \mid \text{bag }1)\,P(\text{bag }1)}{P(\text{white} \mid \text{bag }1)\,P(\text{bag }1) + P(\text{white} \mid \text{bag }2)\,P(\text{bag }2)}$$

$$= \frac{6/10 \cdot 1/2}{6/10 \cdot 1/2 + 3/10 \cdot 1/2} = \frac{6}{9} = \boxed{\frac{2}{3}}$$

Bayes' Rule in the wild ☀️

xkcd
v funny

- Bayes' Rule is very helpful because it helps us incorporate our **prior knowledge** about probabilities into our conclusions.



Bayes' Rule in the wild

- Bayes' Rule is very helpful because it helps us incorporate our **prior knowledge** about probabilities into our conclusions.

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

prior knowledge.

- When we calculated the probability that Professor X was on Bayes Salts, which one of these terms was our prior knowledge of the background rate of Salts use?

1% $\quad P(\text{Bayes Salts})$

Bayes' Rule in machine learning

- Often, we have a **model with parameters** M and we have **data** D.

- Our goal is to learn the parameters M from the data. Yet we also have some beliefs about the parameters, and no particular beliefs about the data.

Likelihood

Prior beliefs about parameters.

$$P(M|D) = \frac{P(D|M)\,P(M)}{P(D)}$$

posterior distribution

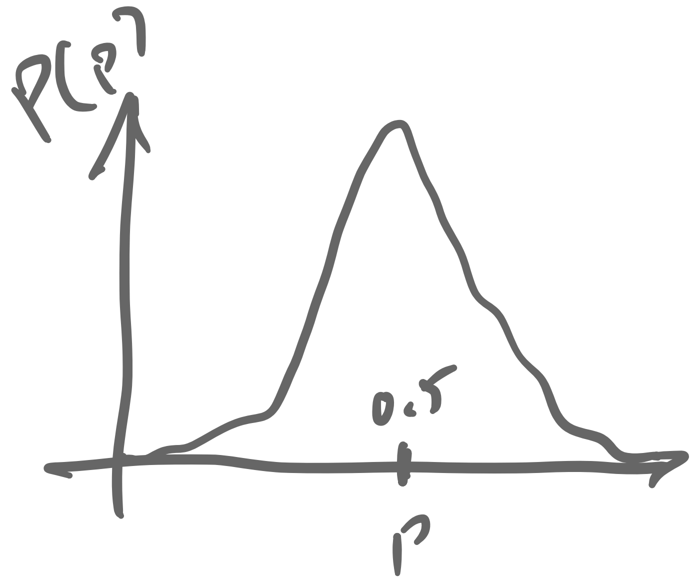
D data.

Flip coin 9H, 1T

Compute Likelihood data | p

but, I tell you, "I'm pretty sure it's not biased."

Prior suggest p=0.5



Random variables

- Say I roll two dice.
- What's the most likely outcome? all same
- What's the most likely sum?

  7

Random variables

$\Omega = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$

- Say I roll two dice. What's the sample space? What are the tables of sums, differences, and maxima?

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

sums

$P(\text{sum}=7) = \frac{6}{36} = \frac{1}{6}$

$P(\text{sum}=2) = \frac{1}{36}$

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	2	3	4	5	6
3	3	3	3	4	5	6
4	4	4	4	4	5	6
5	5	5	5	5	5	6
6	6	6	6	6	6	6

Random variables

- **The key**: the dice are random, so the sum is random!
- Let's *sidestep* the sample space entirely and just go straight to the thing we care about: the sum.
- We call the sum of the dice a *random variable*.

Discrete random variables

- **Definition**: a discrete random variable is a function that maps the elements of a sample space Ω to a finite number of values $a_1, a_2, \ldots, a_N$ or an infinite number of values $a_1, a_2, \ldots$

- Note: even if there are an infinite number, the values must be discrete.

- **Examples** of discrete random variables:
  - Sum of dice; difference of dice; maximum of dice.
  - Number of flips until we get a heads.

Probability mass functions

- **Definition**: a *probability mass function* is the map between the random variable's values and the probabilities of those values.

$$f(a) = P(X = a)$$

PMF Random variable outcome

- Called a "probability mass" function (PMF) because each of the random variable's values has some probability mass (or weight) attached to it.

- Since the PMF is a probability function, what is the sum of all the masses?

$P(\Omega) = \underline{1}$ (def'n of prob. function)

$a \in \Omega$

a is in the set omega

\in

$$\sum_{a \in \Omega} f(a) = 1$$

Probability mass functions

- **Question**: what is the probability mass function for the number of coin flips until a biased coin comes up heads?

Want: function f(n) that tells me the prob that it takes n flips to get a H.

$f(1) = p$

$f(2) = (1-p)p$

$f(3) = (1-p)(1-p)p = (1-p)^2 p$

$f(n) = (1-p)^{n-1} p$

Cumulative distribution functions

- **Definition**: a *cumulative distribution function* (CDF) is a function whose value at point *a* is the cumulative sum of probability masses up until *a*.

$$\underbrace{F}_{\text{CDF}}(a) = P(X \leq a)$$

- **Question**: what's the relationship between the PMF and the CDF?

$$\underbrace{F(a)}_{\text{CDF}} = \underbrace{\sum_{x=-\infty}^{a}}_{\text{everything up to } a} \underbrace{f(x)}_{\text{PMF}}$$

Cumulative distribution functions

- **Example**: What is the probability that I roll two dice and they add up to at most 9?

$$F(9) = P(X \leq 9) = \sum_{y=-\infty}^{9} f(y) = \sum_{y=2}^{9} f(y)$$

$$= f(2) + f(3) + f(4) + \ldots + f(9)$$

$= \ldots$ do this

$$= \frac{5}{6}$$

Cumulative distribution functions

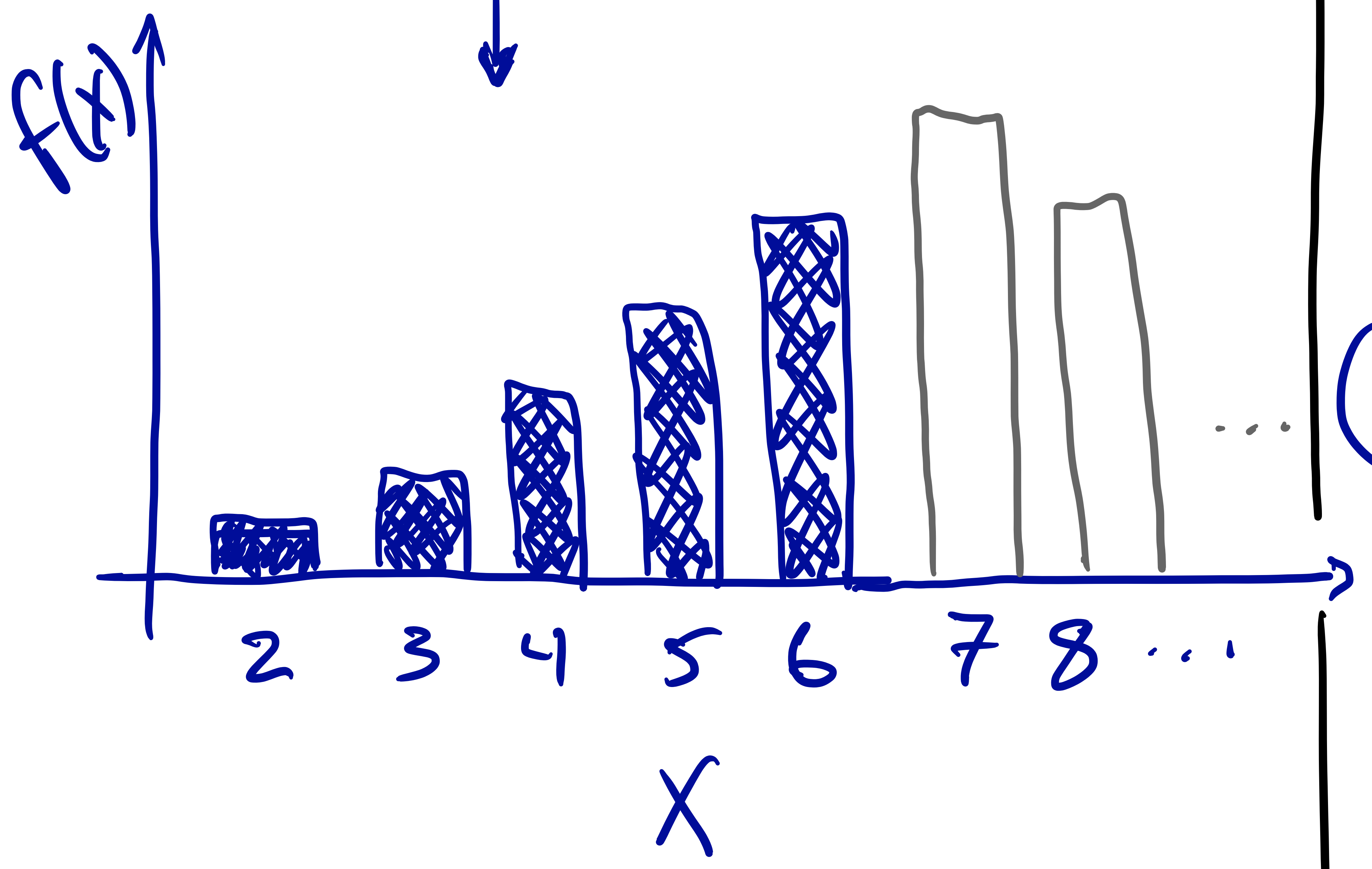
- **Example**: What is the probability that I roll two dice and they add up to between 3 and 6, inclusive?

Let X = the sum of two dice.

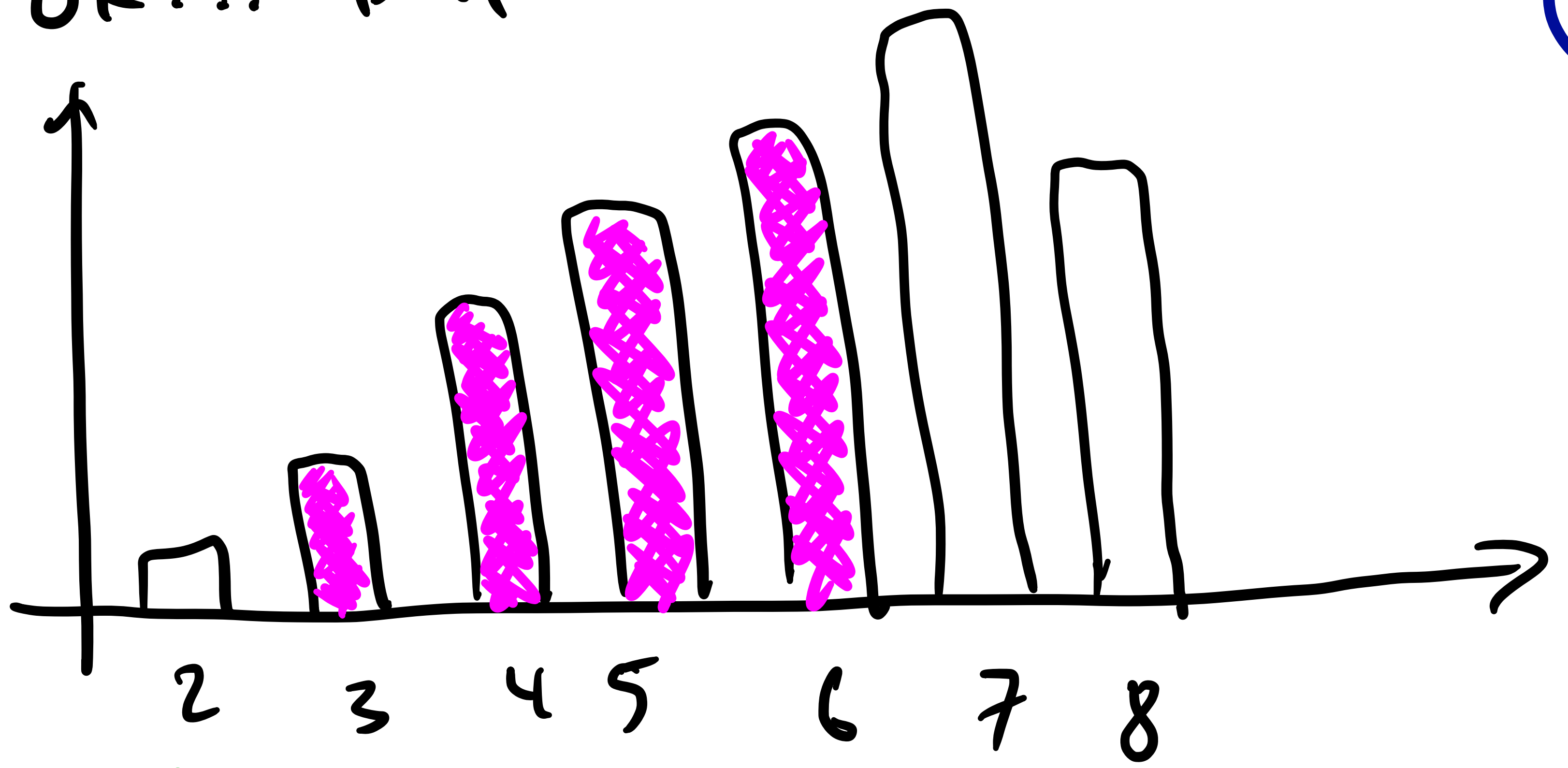
We want $P(X \le 6 \text{ and } X \ge 3)$

1 CDF gives us this:

$P(X \le 6) = F(6)$



ok... but we want this: 2



[More on next slide!]

3 so... we want $F(6)$, but minus this guy.

This guy is $F(2)$, however! so...

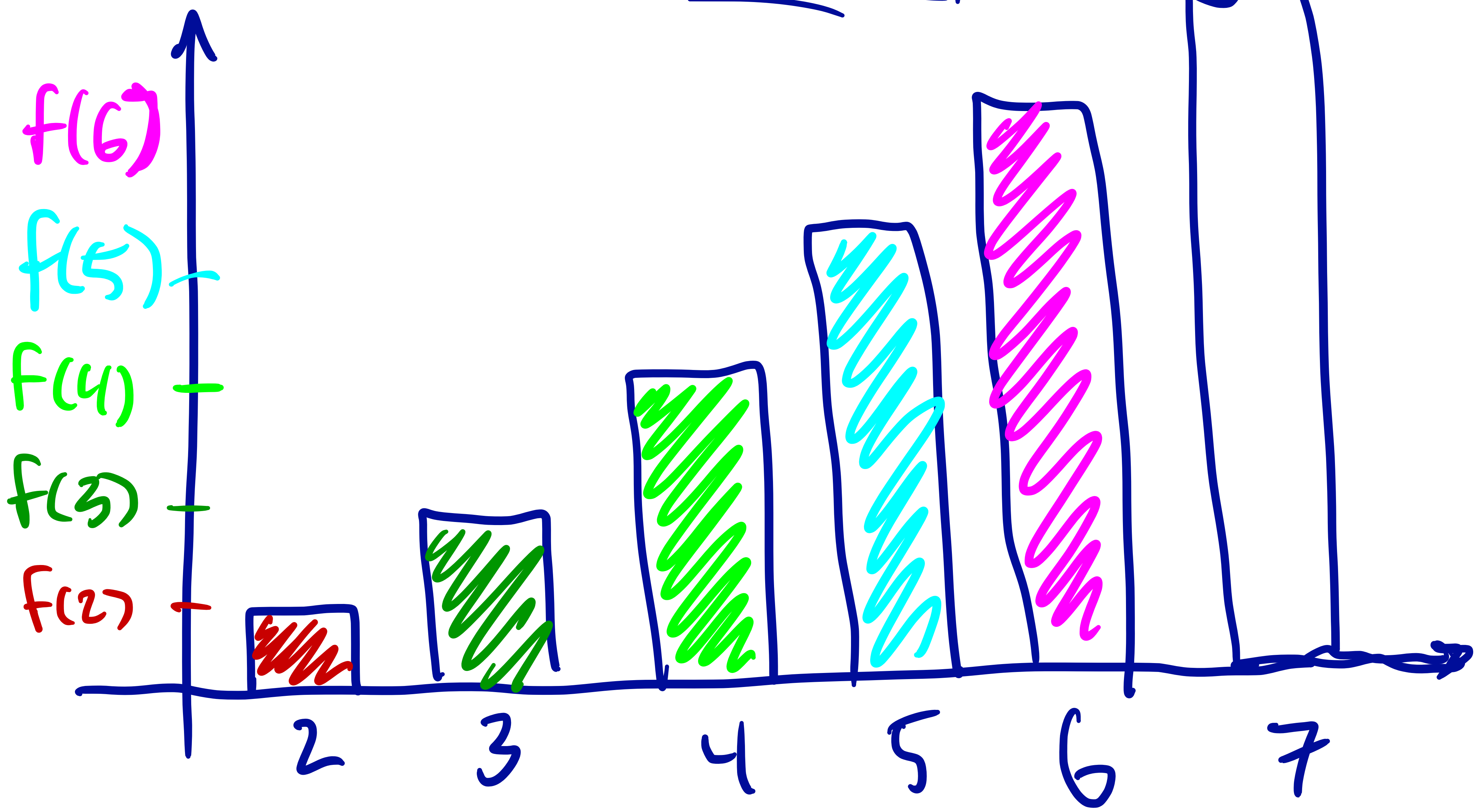
$F(6) - F(2)$ gives us what we want.

4 $P(X \le 6 \text{ and } X \ge 3) = F(6) - F(2)$

Cumulative distribution functions

- **Example**: What is the probability that I roll two dice and they add up to between 3 and 6, inclusive?

What are we actually doing?



$$F(6) = \sum_{k=2}^{6} f(k) = f(2) + f(3) + f(4) + f(5) + f(6)$$

$$F(2) = \sum_{k=2}^{2} f(k) = f(2)$$

So $F(6) - F(2)$ means that $f(2)$s cancel!

leaving $\underbrace{f(3) + f(4) + f(5) + f(6)}_{\text{this is what we wanted.}}$ ✓