

IN SUMMARY,

$$P_0 = 1$$

$$P_1 = x + \alpha_{10} P_0 = x - \frac{\langle x, P_0 \rangle}{\|P_0\|^2} P_0 = x$$

$$P_2 = x^2 + \alpha_{21} P_1 + \alpha_{20} P_0 = x^2 - \frac{\langle x^2, P_1 \rangle}{\|P_1\|^2} P_1 - \frac{\langle x^2, P_0 \rangle}{\|P_0\|^2} P_0 = x^2 - \frac{1}{3}$$

$$P_3 = x^3 + \alpha_{32} P_2 + \alpha_{31} P_1 + \alpha_{30} P_0 = x^3 - \frac{3}{5} x$$

$$P_4 = x^4 + \alpha_{43} P_3 + \alpha_{42} P_2 + \alpha_{41} P_1 + \alpha_{40} P_0$$

$$= x^4 - \frac{\langle x^4, P_3 \rangle}{\|P_3\|^2} P_3 - \frac{\langle x^4, P_2 \rangle}{\|P_2\|^2} P_2 - \frac{\langle x^4, P_1 \rangle}{\|P_1\|^2} P_1 - \frac{\langle x^4, P_0 \rangle}{\|P_0\|^2} P_0$$

$$\frac{\langle x^4, P_3 \rangle}{\|P_3\|^2} P_3 \xrightarrow{\text{orthogonal}} \langle x^4, P_3 \rangle = \int_{-1}^1 x^4 (x^3 - \frac{3}{5}x) dx = \int_{-1}^1 x^7 - \frac{3}{5}x^5 dx = 0$$

$$\rightarrow P_4 = x^4 - \frac{\langle x^4, P_2 \rangle}{\|P_2\|^2} P_2 - \frac{\langle x^4, P_0 \rangle}{\|P_0\|^2} P_0$$

$$\rightarrow \langle x^4, P_2 \rangle = \int_{-1}^1 x^4 (x^2 - \frac{1}{3}) dx = 2 \int_0^1 x^6 - \frac{1}{3}x^4 dx$$

$$\rightarrow \|P_2\|^2 = \int_{-1}^1 (x^2 - \frac{1}{3})^2 dx$$

$$= 2 \left[\frac{1}{7} - \frac{1}{12} \right] = \frac{1}{6} = \frac{1}{42}$$

$$\begin{aligned} &= 2 \int_0^1 x^4 - \frac{2}{3}x^2 + \frac{1}{9} \\ &= 2 \left[\frac{1}{5} - \frac{2}{6} + \frac{1}{9} \right] = -0.38 \end{aligned}$$