STUDENT IDENTIFICATION

Matthew Zeller
Phone: 208-871-2474
zell1874@vandals.uidaho.edu
16991

MATH428
Exam 1

University of Idaho
College of Engineering
Engineering Outreach
875 Perimeter Dr MS 1014
Moscow, ID 83844-1014
Phone: 208-885-6373
Fax: 208-885-6165
outreach@uidaho.edu
http://eo.uidaho.edu

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EXAMINATION IDENTIFICATION:

Course Number: **Math 428/529/Engr 428 "Numerical Methods"**
Instructor: **Lyudmyla Barannyk**
Exam number: **1** (11 pages) After Session: 19

Proctor Certification (initial each below):

X I am not a friend, family member, work subordinate of the student, or UI Student
X Student ID was verified before test was released
X The exam was completed following all **exam instructions**:

- Closed book, 1 single sided page notes
- 80 minute time limit
- Calculator allowed. No other electronic equipment (e.g. cell phones, tablets, Google Glass, etc) allowed

General Proctoring Instructions:

- Student should contact you regarding setting up a testing time for each exam during the semester.
- Release exam to the student only for the amount of time allowed. Students should not receive a copy of the exam either before the examination period or after the exam is completed.
- Provide a location to administer timed exams **under your direct supervision and observation**
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I certify I am the approved proctor and have followed all above instructions:

Testing Start Time: 2:00 pm
Testing Completion Time: 3:20 pm
Date Exam Taken: 7-24-18
Proctor's Signature:
Printed Name: Ahchata Stanley

STUDENT CERTIFICATION:

Please read and sign the following statement before you begin: I fully understand that I am on my honor to do my own work on this examination. If I violate this confidence, I may receive a letter grade of "F" for this exam or for the course, or be expelled from the program. Further, I certify that I have neither given nor received any aid on this test.

Signed:

Print Name: MATT ZELLER Date: 07/24/18

INSTRUCTOR SIGNATURE: GRADE: 74/100

MATH/PHYS/ENGR 428 and MATH 529/PHYS 528:
Midterm Exam

SUMMER 2018

Prof. Lyudmyla Barannyk

Due by Friday, July 6, 2018

NAME: MATT ZELLER

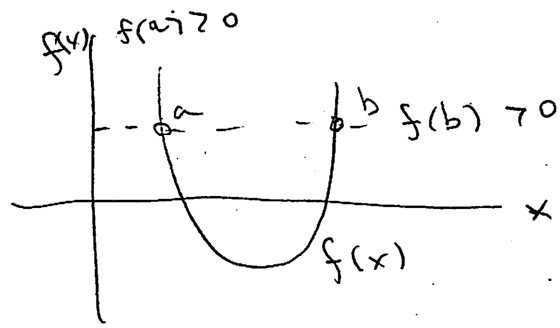
For each problem clearly $\boxed{\text{BOX}}$ your final answer. If you need additional space, continue your work on the back of the page or extra sheet at the end of the exam. There is a **crib sheet** attached at the end.

Problem	**Points Possible**	**Points Earned**
1	50	35
2	10	6
3	15	10
4	25	23
Total	100	74
Bonus/MATH 529/PHYS 528	5	—

1. **[50 pts]** True or False? (Justify your answer for full credit, if false - give a counter example)

   (a) **[5 pts]** $f(x)$ is a continuous function on $[a, b]$ satisfying both $f(a) > 0$ and $f(b) > 0$, then $f(x)$ does not have a root in $[a, b]$.

$\boxed{\text{FALSE,}}$



✓

5

   (b) **[10 pts]** Let $x = (x_1, x_2)^T$ be a vector, then $||x|| = |4x_1 + x_2|$ defines a vector norm.

$\boxed{\text{TRUE}}$

X

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$|4x_1 + x_2| \geq 0$ FOR $\forall$ $x \in \mathbb{R}$ ✓

$x_1, x_2 = 0 \rightarrow 4x + x_2 = 0$

$\not\Rightarrow$

$||\lambda x|| = |\lambda| \, ||x||$

$|\lambda(4x_1 + x_2)| = |\lambda||4x_1 + x_2|$

$||y|| = |y_1 + y_2|$, ✓

$||x + y|| = ||x|| + ||y||$

$|(4x_1 + x_2) + (y_1 + y_2)| \leq \ldots$

$\ldots |4x_1 + x_2| + |y_1 + y_2|$ ✓

What if

$(x_1, x_2) = (-1, 4) \neq \vec{0}$

but

$|4x_1 + x_2| = 0$

violates

$||\vec{x}|| = 0$ iff $\vec{x} = 0$

6

(c) [5 pts] When using Gaussian elimination to compute the solution of $A\mathbf{x} = \mathbf{b}$, partial pivoting is not needed if $\det(A) \neq 0$.

$\boxed{\text{FALSE}}$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$ ✓

5

(d) [10 pts] Consider the fixed point iteration scheme $x_{n+1} = x_n + 2\cos(\pi x_n)$, which corresponds to the function $f(x) = -2\cos(\pi x)$. The iterations converge to the point $\alpha = \frac{1}{2}$ for all initial iterates $x_0 \in \mathbb{R}$.

$\boxed{\text{FALSE}}$

$$g(x) = x + 2\cos(\pi x)$$ ✓

$$g'(x) = -2\pi \sin(\pi x)$$ ✓

$$|g'(x)| = |-2\pi \sin(\pi x)|$$

$|-2\pi \sin(\pi x)|$ IS NOT $\leq k < 1$ FOR $\forall\, x \in \mathbb{R}$ ... ✓

$$x = 0 \longrightarrow |g(x)| = 2\pi$$ ✓

10

(e) **[10 pts]** The following data comes from evaluating a finite difference approximation to the derivative of a function $f(x)$ at the point $x = 0$.

TRUE

h	Df
0.40	1.026881
0.20	1.006680
0.10	1.001668
0.05	1.000417

$\frac{0.03}{0.4} = 0.075$ 0.01

$\frac{0.007}{0.2} = 0.0014$ not about the same

The exact value is $f'(0) = 1$. The order of accuracy of this approximation is $O(h)$. Hint: examine the error as h decreases.

ERROR $\sim C \cdot h^p$ $\Rightarrow \frac{\text{error}}{h^p} \sim \text{const}$

P SHOULD BE 1 IF THIS IS TRUE...

$C \cdot h^p \approx f'(x) - Df$

$p \log h \approx \log(f'(x) - D_f)$

$p \approx \frac{1}{h} \log(\varepsilon)$

6

(f) **[10 pts]** The value of $10^3 - ((4 \times 10^{-1} + 9 \times 10^{-3}) + 5 \times 10^{-4})$ computed using three-digit, decimal, rounding arithmetic is 1000.

FALSE, IF SMALL TERMS ARE OPERATED ON BEFORE LARGE TERMS, THE ANSWER MAY BE $\sim \pm 1$

3

need to work this out like in HW and round after each arithmetic operation

2. **[10 pts]** The following MATLAB code is intended to perform LU factorization and store the results in the original matrix A (assuming that no pivoting is needed). The code does not yield the correct LU decomposition. Fix it.

$n = 3$

```
1   for k=1:n-1
2       for i=k+1:n
3           xm=a(i,k)/a(k,k);
4           a(i,k)=xm;
5           for j=k+1:n
6               a(i,j)=a(i,j)+xm*a(k,k);
7           end
8       end
9   end
```

$$a = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

SET TO k ✗

j

SHOULD BE MINUS ✓

OR

$XM = [$

$k = 1:$

FOR $i = 2:3$

$i = 2:$

$$XM = \frac{a(2,1)}{a(1,1)}$$

$$a(2,1) = \frac{a(2,1)}{a(1,1)}$$

FOR $j = 2:3$

$j = 2:$

$$a(2,2) = a_{22} + \frac{a_{21}}{a_{11}} \cdot a_{11}$$

$\boxed{\text{REPLACE LINE 5 W/ FOR } j = k\text{:}n}$ +1 ✓

~~REMOVE LINE 4~~

$\boxed{\text{REPLACE LINE 6 W/ } a(i,j) = a(i,j) - xm*a(k,\cancel{k})}$ ✗ j

OR

$\boxed{\text{REPLACE LINE 3 W/ } xm = -a(i,k)/a(k,k)}$

6

3. **[15 pts]** Suppose that the bisection method is applied to find a root α of a function $f(x)$ and that after 4 iterations we have $|\alpha - x_4| < 10^{-2}$. How many additional iterations are needed to ensure that the error will be $< 10^{-4}$?

$$|\alpha - x_n| \le \left(\frac{1}{2}\right)^n |b_0 - a_0|$$

$$\left(\frac{1}{2}\right)^4 |b_0 - a_0| \sim 10^{-2}$$

✓

LINEAR CONVERGENCE

$$\cancel{\frac{10^{-2}}{4} \sim \frac{10^{-4}}{n} \quad n \sim 4 \frac{10^{-4}}{10^{-2}} =}$$

$$\frac{-2}{4} \sim \frac{-4}{n} \qquad n \sim \frac{-16}{2} = 8$$

$\boxed{\text{4 ADDITIONAL ITERATIONS}}$ X

$$\left(\frac{1}{2}\right)^n |b-a| = \underbrace{\left(\frac{1}{2}\right)^4 |b-a|}_{\sim 10^{-2}} \cdot \left(\frac{1}{2}\right)^{n-4} < 10^{-4}$$

$$\Rightarrow \left(\frac{1}{2}\right)^{n-4} < 10^{-2} \Rightarrow 2^{n-4} > 10^2 = 100$$

$$2^7 = 128 > 100$$

$$\Rightarrow n-4 > 7 \Rightarrow \boxed{n > 11}$$

10

③ CHECKING $PA = LU$

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 0 & 3/2 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix} \checkmark$$

4. **[25 pts]** Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 0 \\ 2 & 1 & 2 \end{pmatrix}.$$

(a) **[13 pts]** Find the LU-decomposition of matrix A using partial pivoting technique, and state the permutation matrix (or permutation vector) P. Check your decomposition.

①

$$P_1 A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$$E_1 P_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 3/2 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$E_2 P_2 E_1 PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 0 & 3/2 & -1 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 3/2 & -1 \\ 0 & 3/2 & 0 \end{bmatrix}$$

13

$$E_3 P_3 \cdots A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 0 & 3/2 & -1 \\ 0 & 3/2 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 1 & 2 \\ 0 & 3/2 & -1 \\ 0 & 0 & 1 \end{bmatrix} = U} \checkmark$$

②

$$\boxed{L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 1 & 1 \end{bmatrix}} \checkmark \qquad \boxed{P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}} \checkmark$$

CHECKING ✓

$Ly = b$ $\quad P \mid Ax = b$ $\quad$ $PA = LU$

$Ux = y$ $\quad$ $LUx = b \} Ux = y$ $\quad$ $\Rightarrow \underbrace{PA}_{LU} x = Pb$ etc.

$Ly = b$

(b) [**12 pts**] Use L, U and P from part (a) to solve $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = [-1,\ 2,\ 1]^T$. Check your answer.

$LU = A$

$Ux = y$

$Ly = x$

$PA = LU$

① $Ly = Pb$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$y_1 = -1$

$\frac{1}{2}y_1 + y_2 = 2 \rightarrow y_2 = 2 - \frac{1}{2}y_1 = 2 + \frac{1}{2} = 5/2$

$\frac{1}{2}y_1 + y_2 + y_3 = 1 \rightarrow y_3 = 1 - y_2 - \frac{1}{2}y_1$

$= 1 - \frac{5}{2} + \frac{1}{2}$

$= \frac{2}{2} - \frac{4}{2} = -1$

② $y = \begin{bmatrix} -1 \\ 5/2 \\ -1 \end{bmatrix}$ X

~~$Ux = y$~~

~~$\begin{bmatrix} 2 & 1 & 2 \\ 0 & 3/2 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$~~

③ $Ux = y$

$$\begin{bmatrix} 2 & 1 & 2 \\ 0 & 3/2 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 5/2 \\ -1 \end{bmatrix}$$

10

$x_3 = -1$

$\frac{3}{2}x_2 - x_3 = 5/2 \rightarrow x_2 = \left(\frac{5}{2} + x_3\right)\frac{2}{3}$

$= \left(\frac{5}{2} - \frac{2}{2}\right)\frac{2}{3} = 1$

$2x_1 + x_2 + 2x_3 = -1$

$x_1 = \frac{1}{2}(-1 - x_2 - 2x_3)$

$= \frac{1}{2}\left(-\frac{2}{2} - \frac{2}{2} + 2\frac{2}{2}\right) = \frac{1}{2}\left(-\frac{4}{2} + \frac{4}{2}\right) = 0$

④ $x = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ X

⑤ $PAx = Pb \longrightarrow \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1-2 \\ 2 \\ 2-1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ ✓

Check. $Ax = b$ X

$PAx = b$ is a different system

Bonus problem/Additional problem for MATH 529/PHYS 528 students

[5 pts] If B is an approximation of the matrix A^{-1} and $AB = I + E$, show that the relative error in B is bounded by $||E||$.

$k = 1$

$i = 2$

$$x_m = \frac{a_{21}}{a_{11}}$$

$$a_{21} = \frac{a_{21}}{a_{11}}$$

$j = 1$

$$a(2,1) = a(2,1) + \frac{a(2,1)}{a(1,1)} a(1,1)$$

$j = 2$

$$a(2,2) = a(2;2) +$$

$|g'(\alpha)| < 1 \rightarrow$ CONV.

$g'(\alpha) = 0, \quad g''(\alpha) = 0 \rightarrow Q$

$\qquad\qquad\quad g'(\alpha) \neq 0 \rightarrow L$

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \cdots + \frac{f^{(n)}(x)}{n!}h^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}h^{n+1}$$

$$= P_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}h^{n+1} = P_n(x) + R_n(x)$$

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} = D_+ f(x)$$

$$D_- f(x) = \frac{f(x) - f(x-h)}{h}$$

$$D_0 f(x) = \frac{f(x+h) - f(x-h)}{2h}$$

$$f'(x) = D_+ f(x) + O(h) \approx C \cdot h$$

$$= D_- f(x) + O(h)$$

$$= D_0 f(x) + O(h^2)$$

ERROR $\sim C \cdot h^p$

Number Systems

Base β

$$x = \pm(.a_1 a_2 a_3 \ldots a_i \ldots)_\beta \beta^e, \quad 1 \le a_i < \beta$$

Chopping

$$\tilde{x} = \pm(.a_1 a_2 a_3 \ldots a_i)_\beta \beta^e$$

Rounding

$$\tilde{x} = \begin{cases} +(.a_1 \ldots a_i)_\beta \beta^e, & a_{i+1} < \frac{\beta}{2} \\ \pm[(.a_1 \ldots a_i)_\beta \beta^e + (.0 \ldots 1)_\beta \beta^e], & a_{i+1} \ge \frac{\beta}{2} \end{cases}$$

Error

Error: $e(\tilde{x}) = |x - \tilde{x}|$

Relative Error: $re(\tilde{x}) = \left|\frac{x - \tilde{x}}{x}\right|$

Linear Systems, $Ax = b$

THM: Given a matrix A, the following are equivalent

1. The Equation $Ax = b$ has a unique solution
2. A is invertible.
3. $\det(A) \neq 0$
4. $Ax = 0$ has a unique solution, $x = 0$
5. The columns of A are linearly independent
6. The eignevalues, λ, of A are non-zero.

Gaussian Elimination: $A = LU$
Gaussian Elimination with pivoting: $PA = LU$

Norms

Properties of Vector Norms:

$\|x\| \ge 0,\ \|x\| = 0 \Rightarrow x = 0$

$\|\lambda x\| = |\lambda|\ \|x\|, \quad \lambda$ scalar

$\|x + y\| \le \|x\| + \|y\|$

Vector Norms:

$l_\infty: \quad \|x\|_\infty = \max_{1\le i\le n} |x_i|$

$l_1: \quad \|x\|_1 = \sum_{i=1}^n |x_i|$

$l_2: \quad \|x\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$

Matrix Norm:

$$\|A\| = \max_{\|u\|\neq 0} \{\|Au\|/\|u\| : u \in \mathbf{R}^n\}$$

Properties of Matrix Norms:

$\|A\| \ge 0,\ \|A\| = 0 \Leftrightarrow A = 0$

$\|\lambda A\| = |\lambda|\|A\|$

$\|A + B\| \le \|A\| + \|B\|$

$\|Ax\| \le \|A\|\ \|x\|$

$\|AB\| \le \|A\|\ \|B\|$

Examples of Matrix Norms:

l_∞ Matrix Norm: $\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^n |a_{i,j}|$

l_1 Matrix Norm: $\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^n |a_{i,j}|$

l_2 Matrix Norm: $\|A\|_2 = \sqrt{\rho(A^* A)}$

Stability

Condition Number: $\kappa(A) = \|A^{-1}\|\ \|A\|$
Residual: $r = b - A\tilde{x}$
THM:

$$\frac{1}{\kappa(A)}\frac{\|r\|}{\|b\|} \le \frac{\|e\|}{\|x\|} \le \kappa(A)\frac{\|r\|}{\|b\|}$$

Root Finding Methods

Newton's Methods: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Secant Methods: $x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$

Error Bound for Bisection Method:

$$|\alpha - x_n| \le \left(\frac{1}{2}\right)^n |b_0 - a_0|$$