

$$\begin{aligned}
 \frac{3}{9} \quad P_4(x) &= -\frac{1}{6}e^{x_0}(x^3 - 6x^2 + 11x - 6) + \frac{1}{2}e^{x_1}(x^3 - 5x^2 + 6x) + \dots \\
 &\quad \dots - \frac{1}{2}e^{x_2}(x^3 - 4x^2 + 7x) + \frac{1}{6}e^{x_3}(x^3 - 3x^2 + 3x) \\
 &= x^3 \left(-\frac{1}{6}e^{x_0} + \frac{1}{2}e^{x_1} - \frac{1}{2}e^{x_2} + \frac{1}{6}e^{x_3} \right) + \dots \\
 &\quad \dots + x^2 \left(e^{x_0} - \frac{5}{2}e^{x_1} + 2e^{x_2} - \frac{1}{2}e^{x_3} \right) + \dots \\
 &\quad \dots + x \left(-\frac{11}{6}e^{x_0} + 3e^{x_1} - \frac{7}{2}e^{x_2} + \frac{1}{2}e^{x_3} \right) + \dots \\
 &\quad \dots + e^{x_0}
 \end{aligned}$$

$$P_4(x) \approx -11.18x^3 - 1.060x^2 - 9.497x + 1$$

EITHER THIS $P_4(x)$

OR THE NEXT
IN 3b ARE

INCORRECT ... OR BOTH...
BUT THEY SHOULD BE EQUAL

3
a

$$f(x) = e^x$$

KNOWN AT

$$x_0 = 0$$

$$x_1 = 1$$

$$x_2 = 2$$

$$x_3 = 4$$

WRITE THE INTERPOLATING
POLY IN LAGRANGE
FORM

$$P_n(x) = \sum_{k=0}^n f(x_k) l_k(x) = \sum_{k=0}^3 e^{x_k} l_k(x)$$

$$= e^{x_0} l_0 + e^{x_1} l_1 + e^{x_2} l_2 + e^{x_3} l_3$$

$$l_k = \prod_{\substack{i=0 \\ i \neq k}}^n \frac{x - x_i}{x_k - x_i}$$

$$l_0 = \prod_{\substack{i=1 \\ i \neq 0}}^3 \frac{x - x_i}{x_0 - x_i} = \frac{x - x_1}{x_0 - x_1} \cdot \frac{x - x_2}{x_0 - x_2} \cdot \dots \cdot \frac{x - x_3}{x_0 - x_3}$$

$$= \frac{x-1}{0-1} \cdot \frac{x-2}{0-2} \cdot \frac{x-4}{0-4}$$

$$l_0 = -\frac{1}{8}(x-1)(x-2)(x-4)$$

$$l_1 = \prod_{\substack{i=0 \\ i \neq 1}}^3 \frac{x - x_i}{x_1 - x_i} = \frac{x - x_0}{x_1 - x_0} \cdot \frac{x - x_2}{x_1 - x_2} \cdot \frac{x - x_3}{x_1 - x_3} = \frac{x-0}{1-0} \cdot \frac{x-2}{1-2} \cdot \frac{x-4}{1-4}$$

$$l_1 = \frac{1}{3} x(x-2)(x-4)$$

$$\frac{3}{2} l_2 = \prod_{i=1}^3 \frac{x - x_i}{x_2 - x_i} = \frac{x - x_0}{x_2 - x_0} \frac{x - x_1}{x_2 - x_1} \frac{x - x_3}{x_2 - x_3}$$

$$= \frac{x - 0}{2 - 0} \frac{x - 1}{2 - 1} \frac{x - 4}{2 - 4}$$

$$l_2 = -\frac{1}{4} x (x-1)(x-4)$$

$$l_3 = \prod_{i=1}^3 \frac{x - x_i}{x_3 - x_i} = \frac{x - x_0}{x_3 - x_0} \frac{x - x_1}{x_3 - x_1} \frac{x - x_2}{x_3 - x_2}$$

$$= \frac{x - 0}{4 - 0} \frac{x - 1}{4 - 1} \frac{x - 2}{4 - 2}$$

$$= \frac{1}{14} x (x-1)(x-2)$$

$$\rightarrow P_n(x) = -\frac{1}{8} e^0 (x-1)(x-2)(x-4) + \dots$$

$$\dots + \frac{1}{3} e^1 x (x-2)(x-4) + \dots$$

$$\dots - \frac{1}{4} e^2 x (x-1)(x-4) + \dots$$

$$\dots - \frac{1}{14} e^4 x (x-1)(x-2)$$

3
a

$$P_n(x) = \sum_{k=0}^n f(x_k) l_k(x)$$

x_0	x_1	x_2	x_3
0	1	2	3

$$P_4(x) = f(x_0) l_0(x) + \dots + f(x_4) l_4(x)$$

$$= f(x_0) \cdot \frac{x-x_1}{x_0-x_1} \cdot \dots \cdot \frac{x-x_4}{x_0-x_4} + \dots + f(x_4) \cdot \frac{x-x_0}{x_4-x_0} \cdot \dots \cdot \frac{x-x_3}{x_4-x_3}$$

$$f(x) = e^{x^2}$$

$$P_4(x) = e^{x_0^2} \cdot \frac{x-1}{0-1} \cdot \frac{x-2}{0-2} \cdot \frac{x-3}{0-3} + e^{x_1^2} \cdot \frac{x-0}{1-0} \cdot \frac{x-2}{1-2} \cdot \frac{x-3}{1-3} + \dots$$

$$\dots + e^{x_2^2} \cdot \frac{x-0}{2-0} \cdot \frac{x-1}{2-1} \cdot \frac{x-3}{2-3} + e^{x_3^2} \cdot \frac{x-0}{3-0} \cdot \frac{x-1}{3-1} \cdot \frac{x-2}{3-2}$$

$$= -e^{x_0^2} \frac{1}{6} (x-1)(x-2)(x-3) + e^{x_1^2} \frac{1}{2} x(x-2)(x-3) + \dots$$

$$\dots - e^{x_2^2} \frac{1}{2} x(x-1)(x-3) + e^{x_3^2} \frac{1}{6} x(x-1)(x-2)$$

$$= -e^{x_0^2} \frac{1}{6} (x-1)(x^2-5x+6) + e^{x_1^2} \frac{1}{2} x(x^2-5x+6) + \dots$$

$$\dots - e^{x_2^2} \frac{1}{2} x(x^2-4x+7) + e^{x_3^2} \frac{1}{6} x(x^2-3x+3)$$

3
a

$$f(x) = e^x$$

KNOWN AT

$$x_0 = 0$$

$$x_1 = 1$$

$$x_2 = 2$$

$$x_3 = 4$$

WRITE THE INTERPOLATING
POLY IN LAGRANGE
FORM

$$P_n(x) = \sum_{k=0}^n f(x_k) l_k(x) = \sum_{k=0}^3 e^{x_k} l_k(x)$$

$$= e^{x_0} l_0 + e^{x_1} l_1 + e^{x_2} l_2 + e^{x_3} l_3$$

$$l_n = \prod_{\substack{i=0 \\ i \neq n}}^n \frac{x - x_i}{x_n - x_i}$$

$$l_0 = \prod_{\substack{i=0 \\ i \neq 0 \\ i \neq 1 \\ i \neq 2 \\ i \neq 3}}^3 \frac{x - x_i}{x_0 - x_i} = \frac{x - x_1}{x_0 - x_1} \cdot \frac{x - x_2}{x_0 - x_2} \cdot \dots$$

$$\dots \cdot \frac{x - x_3}{x_0 - x_3}$$

$$= \frac{x-1}{0-1} \cdot \frac{x-2}{0-2} \cdot \frac{x-4}{0-4}$$

$$l_0 = -\frac{1}{8}(x-1)(x-2)(x-4)$$

$$l_1 = \prod_{\substack{i=0 \\ i \neq 1 \\ i \neq 2 \\ i \neq 3}}^3 \frac{x - x_i}{x_1 - x_i} = \frac{x - x_0}{x_1 - x_0} \cdot \frac{x - x_2}{x_1 - x_2} \cdot \frac{x - x_3}{x_1 - x_3} = \frac{x-0}{1-0} \cdot \frac{x-2}{1-2} \cdot \frac{x-4}{1-4}$$

$$l_1 = \frac{1}{3} x(x-2)(x-4)$$

2

$$f(x+s) \approx f(x) + J_f(x) s$$

$$f(x) = \begin{bmatrix} x^2 + y^2 \\ x^2 - y^2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$J_f(x) = \begin{bmatrix} 2x & 2y \\ 2x & -2y \end{bmatrix}$$

$$X_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$f(X_0) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$J_f(X_0) = \begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix}$$

$$J_f(X_0) \cdot s_0 = -f(X_0)$$

$$s_0 = J_f(X_0)^{-1} \cdot -f(X_0)$$

$$= \frac{1}{8} \begin{bmatrix} -2 & -2 \\ -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/4 \end{bmatrix}$$

$$x_1 = X_0 - s_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 3/4 \end{bmatrix}$$

$$f(X_1) = \begin{bmatrix} 1/4 + 9/16 \\ 1/4 - 9/16 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -5/16 \end{bmatrix}$$

$$\frac{3}{a} \quad l_2 = \prod_{i=1}^3 \frac{x-x_i}{x_2-x_i} = \frac{x-x_0}{x_2-x_0} \frac{x-x_1}{x_2-x_1} \frac{x-x_3}{x_2-x_3}$$

$$= \frac{x-0}{2-0} \frac{x-1}{2-1} \frac{x-4}{2-4}$$

$$l_2 = -\frac{1}{4} x (x-1)(x-4)$$

$$l_3 = \prod_{i=1}^3 \frac{x-x_i}{x_3-x_i} = \frac{x-x_0}{x_3-x_0} \frac{x-x_1}{x_3-x_1} \frac{x-x_2}{x_3-x_2}$$

$$= \frac{x-0}{4-0} \frac{x-1}{4-1} \frac{x-2}{4-2}$$

$$= \frac{1}{14} x (x-1)(x-2)$$

$$\rightarrow P_n(x) = -\frac{1}{2} e^0 (x-1)(x-2)(x-4) + \dots$$

$$\dots + \frac{1}{3} e^1 x (x-2)(x-4) + \dots$$

$$\dots - \frac{1}{4} e^2 x (x-1)(x-4) + \dots$$

$$\dots - \frac{1}{14} e^4 x (x-1)(x-2)$$

3
a

$$P_4(x) = -\frac{1}{6}e^{x_0}(x^3 - 6x^2 + 11x - 6) + \dots$$

$$\dots + \frac{1}{2}e^{x_1}(x^3 - 5x^2 + 6x) + \dots$$

$$\dots - \frac{1}{2}e^{x_2}(x^3 - 4x^2 + 7x) + \dots$$

$$\dots + \frac{1}{6}e^{x_3}(x^3 - 3x^2 + 3x)$$

$$= -\frac{1}{6}e^{x_0}x^3 + e^{x_0}x^2 - \frac{11}{6}e^{x_0}x + e^{x_0} + \dots$$

$$\dots + \frac{1}{2}e^{x_1}x^3 - \frac{5}{2}e^{x_1}x^2 + 3e^{x_1}x + \dots$$

$$\dots - \frac{1}{2}e^{x_2}x^3 + 2e^{x_2}x^2 - 7e^{x_2}x + \dots$$

$$\dots + \frac{1}{6}e^{x_3}x^3 - \frac{1}{2}e^{x_3}x^2 + \frac{1}{2}e^{x_3}x$$

3
5

x_i

0

1

2

4

$f[.]$

e^0

e^1
 e^2
 e^3
 e^4

$f[.,.]$

$$\frac{e^1 - e^0}{1 - 0}$$

$$\frac{e^2 - e^1}{2 - 1}$$

$$\frac{e^4 - e^2}{4 - 2}$$

$$\frac{e^4 - e^2}{4 - 2}$$

1.71828

4.67077

23.6045

$f[.,.,.]$

$$\frac{4.67077 - 1.71828}{2 - 0}$$

$$\frac{23.6045 - 4.67077}{4 - 1}$$

1.47625

6.31124

$f[.,.,.,.]$

$$\frac{6.31124 - 1.47625}{4 - 0}$$

4.43499

3
b

$$\begin{array}{lcl}
 x_0 = 0 & e^{x_0} & \\
 x_1 = 1 & e^{x_1} & \\
 x_2 = 2 & e^{x_2} & \\
 x_3 = 3 & e^{x_3} &
 \end{array}
 \begin{array}{l}
 > e^{x_1} - e^{x_0} > \frac{1}{2}(e^{x_2} - 2e^{x_1} + e^{x_0}) > \frac{1}{6}(e^{x_3} - 3e^{x_2} + 3e^{x_1} - e^{x_0}) \\
 > e^{x_2} - e^{x_1} > \frac{1}{2}(e^{x_3} - 2e^{x_2} + e^{x_1}) \\
 > e^{x_3} - e^{x_2}
 \end{array}$$

$$\begin{aligned}
 p_n(x) &= e^{x_0} + (e^{x_1} - e^{x_0})(x - x_0) + \frac{1}{2}(e^{x_2} - 2e^{x_1} + e^{x_0})(x - x_0)(x - x_1) + \dots \\
 &\quad (e^{x_3} - 3e^{x_2} + 3e^{x_1} - e^{x_0})(x - x_0)(x - x_1)(x - x_2) \\
 &\approx 1 + 1.718x + 1.477(x^2 - 1) + \dots \\
 &\quad \dots + 5.073(x^4 - 3x^3 + 3x)
 \end{aligned}$$

$$\begin{aligned}
 p_n(x) &= 5.073x^4 - 15.22x^3 - 13.74x^2 + 16.94x + \dots \\
 &\quad \dots - 0.4770
 \end{aligned}$$

$\frac{3}{b}$

$$P_4(x) \approx 1 + 1.718x + 1.477(x^2 - x) + \dots$$
$$\dots + 5.073(x^3 - 3x^2 + 2x)$$

$$P_4(x) \approx 5.073x^3 - 13.74x^2 + 10.39x + 1$$

3/5

$$\rightarrow P_3(x) = f[x_0] + f[x_0, x_1](x - x_0) + \dots$$

$$\dots + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \dots$$

$$\dots + f[x_0, \dots, x_3](x - x_0)(x - x_1)(x - x_2)$$

$$= 1 + 1.71828(x - 1) + 1.47625(x - 1)(x - e) + \dots$$

$$\dots + 4.83499(x - 1)(x - e)(x - e^2)$$

$$P_3(x) = a_0 + (x - x_0) \left(a_1 + (x - x_1) \left(a_2 + a_3(x - x_2) \right) \right)$$

$$= 1 + (x - 1) \left(1.71828 + (x - e) \left(1.47625 + 4.83499(x - e^2) \right) \right)$$

3
5

x_i
0
1
2
4

$f[.]$
 e^0
 e^1
 e^2
 e^4

$f[.,.]$
 $e^1 - e^0$
 $e^2 - e^1$
 $e^4 - e^2$
 $4 - 2$

$f[.,.,.]$
 $4.67077 - 1.71828$
 $23.6045 - 4.67077$
 $4 - 1$

$f[.,.,.,.]$
 $6.31124 - 1.47625$
 $4 - 1$

1.71828
4.67077
23.6045

1.47625
6.31124

4.83499

$\frac{3}{c}$

$$P_{3L}(x) = 0.8455 x^3 - 1.060 x^2 + 1.933 x + 1$$

$$e^{1.5} \approx 0.8455 (1.5)^3 - 1.060 (1.5)^2 + 1.933 (1.5) + 1$$

$$\approx 4.368$$

$$e^4 \approx 0.8455 (4)^3 - \dots$$

$$\approx 45.88$$

$$P_{3N}(x) = 5.073 x^3 - 13.74 x^2 + 10.39 x - 1$$

$$e^{1.5} \approx 5.073 (1.5)^3 - 13.74 (1.5)^2 + 10.39 (1.5) - 1$$

$$\approx 2.791$$

I found $\frac{13.74}{1.060} \approx 2 \frac{10.39}{1.933} = 2 \frac{0.32}{0.36}$

$$\approx 2 \frac{5.073}{0.8455} = 2 \frac{0.12}{0.12}$$

WHAT IF I...

✓ $P_{3N}(x) \xrightarrow{\text{NOW,}} P'_{3N} = 5.073 x^3 - 13.74 x^2 + 10.39 x + 1$

$$e^{1.5} \approx 2.791 \xrightarrow{\text{LSHET}}$$

$$5.073 (1.5)^3 - 6.87 (1.5)^2 + 10.39 (1.5) + 1$$

$$17.12 - 15.46 + 15.59 + 1$$

4

$$l_h(x) = \prod_{\substack{i=0 \\ i \neq h}}^n \frac{x - x_i}{x_h - x_i}$$

x	-2	-1	0	1	2	3
$p(x)$	-5	1	1	1	7	25

BY THE UNIQUENESS THEOREM FROM LECTURE NOTES 21 ($\dots p(x) = l(x) \dots$),

$$l_h(x) = \prod_{\substack{i=0 \\ i \neq h}}^n \frac{x - x_i}{x_h - x_i} = p(x) \quad \text{FOR SOME } n \leq 5,$$

$$x = \{x_0 = -2, x_1 = -1, x_2 = 0, x_3 = 1, x_4 = 2, x_5 = 3\}$$

$$p(x) = \{-5, 1, 1, 1, 7, 25\}$$

OTHERWISE $p(x)$ DOES NOT HAVE DEGREE n

$$\longrightarrow n = 0$$

$$l_h(x) = \emptyset \neq p(x) \longrightarrow 1 \leq n \leq 5$$

$$n = 1$$

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x + 1}{-2 + 1} = -x - 1$$

3
b

$$\begin{aligned} \rightarrow P_3(x) &= f[x_0] + f[x_0, x_1](x - x_0) + \dots \\ &\quad \dots + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \dots \\ &\quad \dots + f[x_0, \dots, x_3](x - x_0)(x - x_1)(x - x_2) \end{aligned}$$

$$\begin{aligned} &= 1 + 1.71828(x-1) + 1.47625(x-1)(x-e) + \dots \\ &\quad \dots + 4.83499(x-1)(x-e)(x-e^2) \end{aligned}$$

$$\begin{aligned} P_3(x) &= a_0 + (x-x_0) \left(a_1 + (x-x_1) \left(a_2 + a_3(x-x_2) \right) \right) \\ &= \boxed{1 + (x-1) \left(1.71828 + (x-e) \left(1.47625 + 4.83499(x-e^2) \right) \right)} \end{aligned}$$

4

 $f[\dots]$ $f[\dots]$ $f[\dots]$ $f[\dots]$ (x_n) (-2) (-1) (0) (1) (2) (3)

$$\frac{1 + 5}{-1 + 2} = 6$$

$$\frac{1 - 1}{0 + 1} = \emptyset$$

$$\frac{1 - 1}{1 - 0} = \emptyset$$

$$\frac{7 - 1}{2 - 1} = 6$$

$$\frac{25 - 7}{3 - 2} = 18$$

$$\frac{\emptyset - 6}{0 + 2} = -\frac{6}{2} = -3$$

$$\emptyset = \emptyset$$

$$\frac{6 - 0}{2 - 0} = 3$$

$$\frac{18 - 6}{3 - 1} = \frac{12}{2} = 6$$

$$\frac{\emptyset + 3}{1 + 2} = 1$$

$$\frac{3 - \emptyset}{2 + 1} = 1$$

$$\frac{6 - 3}{3 - \emptyset} = 1$$

From the table,

$$\Rightarrow n = 3 \quad (a_4 = \emptyset, a_5 = \emptyset)$$

$$\text{For } p_n(x) = a_0 + a_1(x-x_0) + \dots$$

$$\dots > f[x_0, \dots, x_5] = \emptyset$$

 $(a_5 = \emptyset)$

$$4 \quad l_2(x) = \frac{1}{2}(x^2 + 3x + 2)$$

$$l_2(x_0) = \frac{1}{2}((-2)^2 + 3(-2) + 2) = \frac{1}{2}(4 - 6 + 2) = 0 \neq p(x_0)$$

$$\rightarrow 3 \leq n \leq 5$$

$n=3$

$$l_0(x) = \frac{x-x_1}{x_0-x_1} \cdot \frac{x-x_2}{x_0-x_2} \cdot \frac{x-x_3}{x_0-x_3} = \frac{x+1}{-2+1} \cdot \frac{x-0}{-2-0} \cdot \frac{x-1}{-2-1} = -\frac{1}{6}x(x+1)(x-1)$$

$$= -\frac{1}{6}(x^3 + x)$$

$$l_0(x_0) = -\frac{1}{6}((-2)^3 - 2) = -1 \neq p(x_0)$$

$$l_1(x) = \frac{x-x_0}{x_1-x_0} \cdot \frac{x-x_2}{x_1-x_2} \cdot \frac{x-x_3}{x_1-x_3} = \frac{x+2}{-1+2} \cdot \frac{x-0}{-1-0} \cdot \frac{x-1}{-1-1} = \frac{1}{2}x(x+2)(x-1)$$

$$= \frac{1}{2}(x^3 + x^2 - 2x)$$

$$l_1(x_0) = \frac{1}{2}((-2)^3 + (-2)^2 - 2(-2)) = 0 \neq p(x_0)$$

5

$$\sum_{h=0}^n l_h(x) = 1$$

$$p_n(x) = \sum_{h=0}^n f(x_h) l_h(x)$$

$$f(x) = p_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)(x-x_1)\dots(x-x_n)$$

$$\text{If } f(x) = 1 \rightarrow f \equiv 1, \quad f^{(n+1)}(\xi) = 0$$

$$1 = \sum_{h=0}^n 1 \cdot l_h(x) + 0$$

$$\sum_{h=0}^n l_h(x) = 0 \quad \therefore$$

2

$$f(x+s) \approx f(x) + J_f(x) s$$

$$f(x) = \begin{bmatrix} x^2 + y^2 \\ x^2 - y^2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$J_f(x) = \begin{bmatrix} 2x & 2y \\ 2x & -2y \end{bmatrix}$$

$$X_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$f(X_0) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$J_f(X_0) = \begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix}$$

$$J_f(X_0) \cdot s_0 = -f(X_0)$$

$$s_0 = J_f(X_0)^{-1} \cdot -f(X_0)$$

$$= \frac{1}{8} \begin{bmatrix} -2 & -2 \\ -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/4 \end{bmatrix}$$

$$x_1 = X_0 - s_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 3/4 \end{bmatrix}$$

$$f(X_1) = \begin{bmatrix} 1/4 + 9/16 \\ 1/4 - 9/16 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -5/8 \end{bmatrix}$$