IN SUMMARY,

$P_0 = 1$

$P_1 = x + \alpha_{10} P_0 = x - \frac{\langle x, P_0 \rangle}{\|P_0\|^2} P_0 = x$

$P_2 = x^2 + \alpha_{21} P_1 + \alpha_{20} P_0 = x^2 - \frac{\langle x^2, P_1 \rangle}{\|P_1\|^2} P_1 - \frac{\langle x^2, P_0 \rangle}{\|P_0\|^2} P_0 = x^2 - \frac{1}{3}$

NOW, THE REMAINING P_n ARE EASY...

$P_3 = x^3 + \alpha_{32} P_2 + \alpha_{31} P_1 + \alpha_{30} P_0$

$= x^3 - \frac{\langle x^3, P_2 \rangle}{\|P_2\|^2} P_2 - \frac{\langle x^3, P_1 \rangle}{\|P_1\|^2} P_1 - \frac{\langle x^3, P_0 \rangle}{\|P_0\|^2} P_0$

$\langle x^3, P_2 \rangle = \int_{-1}^{1} x^3 \left(x^2 - \frac{1}{3}\right) = \int_{-1}^{1} x^5 - \frac{1}{3} x^3 = \not{0}\ (\text{ODD})$

$\langle x^3, P_0 \rangle = \int_{-1}^{1} x^3 \cdot 1\, dx = \not{0}\ (\text{ODD})$

$= x^3 - \frac{\langle x^3, P_1 \rangle}{\|P_1\|^2} P_1$

$\langle x^3, P_1 \rangle = \int_{-1}^{1} x^3 \cdot x\, dx = \int_{-1}^{1} x^4 dx = \frac{1}{5} x^5 \Big|_{-1}^{1}$

$\|P_1\|^2 = \ldots ? = \langle P_1, P_1 \rangle = \int_{-1}^{1} x^2 dx = \frac{1}{3} + 1 = \frac{2}{5}$

$= \frac{2}{3} =$

$= x^3 - \frac{5}{2} \frac{3}{5} x$

$\boxed{P_3 = x^3 - \frac{3}{5} x}$