Math/Phys/Engr 428, Math 529/Phys 528
Numerical Methods - Summer 2018

Homework 4

Due: **Friday, July 13, 2018**

1. **(Iterative Methods: Practice).** This problem extends #8 and #11 on page 235 of Bradie. Consider the linear system

$$\begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 10 \end{pmatrix}.$$

The optimal solution is $x = [1\ 2\ 3]^T$.

   (a) Solve the system numerically using Jacobi's Method, Gauss–Seidel, and SOR with (optimal) parameter $\omega = 2/(1+\sqrt{7/8})$. In each case, let $x^{(0)} = 0$. Terminate iteration after step k if $||x^{(k)} - x^{(k-1)}||_\infty < 10^{-7}$. Provide your code.

   (b) Make a `semilogy` plot that compares the evolution of the error $||x - x^{(k)}||_\infty$ for all three methods.

   (c) Use the formula

$$C \approx \frac{||x^{(k)} - x^{(k-1)}||_\infty}{||x^{(k-1)} - x^{(k-2)}||_\infty}$$

   to estimate the asymptotic error constant for each method. Tabulate these estimates along with the number of iterations required for each method to converge.

Discuss your results.

2. **(Nonlinear Systems)**

Write down Newton's method to solve the system $x^2 + y^2 = 4$, $x^2 - y^2 = 1$. Perform one step of Newton's method with initial guess $x_0 = 1$, $y_0 = 1$.

3. **(Interpolation)**

The function $f(x) = \mathrm{e}^x$ is given at the 4 points: $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$.

   (a) Write the interpolating polynomial in Lagrange form.

   (b) Write the interpolating polynomial in Newton form. (Recall that the interpolating polynomial is unique, so interpolating polynomial in Lagrange or Newton form should be the same after simplification.)

   (c) Evaluate $\mathrm{e}^{1.5}$ and e^4 using the interpolating polynomial. Note that the first value is interpolated while the second is extrapolated. Which approximate value is more accurate?

(d) Use the error formula to find an upper bound for the maximum error

$$||f - p_3||_\infty = \max_{0\le x\le 3} |f(x) - p_3(x)| .$$

4. The following data are taken from a polynomial $p(x)$ of degree ≤ 5. What is the degree of $p(x)$? Explain.

x	-2	-1	0	1	2	3
$p(x)$	-5	1	1	1	7	25

5. Show that $\sum_{k=0}^{n} \ell_k(x) = 1$ (Hint: consider the function $f(x) = 1$, a polynomial of degree 0! Don't forget to discuss the error term.)

6. We want to study the effect of different choices of interpolation points $\{x_0, x_1, \ldots, x_n\}$ on the function $w_n(x) = (x - x_0)(x - x_1)\ldots(x - x_n)$ in the formula for the error in interpolation polynomials. In particular, we want to study evenly spaced points and Chebyshev points in the interval $[-1, 1]$. Consider the following choices:

   (a) $x_i = -1 + \dfrac{2i}{n} \qquad i = 0, \ldots, n$

   (b) $x_i = -\cos\left[\dfrac{\pi}{n+1}\left(\dfrac{1}{2} + i\right)\right] \qquad i = 0, \ldots, n.$

   In each case, plot $w_n(x)$ in the interval [-1, 1] for $n = 10$ using fine enough resolution (with the number of points more than n). Also plot $w_n(x)$ with $n = 20$ and 30. Discuss the results.

7. Write a computer program to perform polynomial interpolation using equally spaced points and the Chebyshev points on the interval [-1, 1] for the function $f(x)$. Investigate the convergence of p_n to f by running the program for $n = 8,\ 16,\ 32$ in the following cases

$$f_1(x) = |x| \quad , \text{ and } \quad f_2(x) = \begin{cases} -1 & \text{if } \ x < 0, \\ 0 & \text{if } \ x = 0, \\ 1 & \text{if } \ x > 0. \end{cases}$$

   Use a fine mesh to plot your interpolating polynomials. Discuss the results. As n gets larger, is there pointwise convergence? Is convergence uniform in x?

   (in MATLAB $|x| = \text{abs}(x)$ and the second function is $\text{sign}(x)$, you can use the library function "polyfit" in MATLAB. Use "help polyfit" to find how to use it).

(**Hermite Interpolation**)

8. The theorem describing the error in using Hermite interpolation is as follows.

   Theorem: If $f \in \mathscr{C}^{2n+2}[a,b]$, then

   $$f(x) = H(x) + \frac{(x-x_0)^2 \dots (x-x_n)^2}{(2n+2)!} f^{(2n+2)}(\xi)$$

   for some ξ with $a < \xi < b$.

   Now, consider $f(x) = x \ln x$, $n = 1$, $x_0 = 1$, and $x_1 = 3$ (from problem # 3 on page 414).

   (a) Use linear interpolation and Hermite interpolation to approximate the value of $f(1.5)$. Which estimate is more accurate?

   (b) Verify that the error bound for Hermite interpolation holds for the Hermite polynomial found in (a).

9. Find a polynomial of least degree satisfying:

   $$p(1) = -1, \quad p'(1) = 2, \quad p''(1) = 0, \quad p(2) = 1, \quad p'(2) = -2$$

   Note: Extend the idea from Hermite interpolation to the case when more than two conditions are specified at the same point (see Lecture # 26).

Suggested / Additional problems for Math 529 / Phys 528 students:

10. Consider Hermite interpolation for $n = 1$, $x_0 = 1$, and $x_1 = 3$. Compute (by hand) $\tilde{h}_1(x)$ using the Lagrange polynomials and using the Newton form (from the divided difference table) and then plot $\tilde{h}_1(x)$.

11. **Iterative Methods**

    (a) Use both the Jacobi and the Gauss-Seidel method to solve the linear system of equations

    $$\begin{aligned} 4x_1 + x_2 + x_3 + x_4 &= -5 \\ x_1 + 8x_2 + 2x_3 + 3x_4 &= 23 \\ x_1 + 2x_2 - 5x_3 &= 9 \\ -x_1 + 2x_3 + 4x_4 &= 4 \end{aligned}$$

    Take $\mathbf{x}^{(0)} = \mathbf{0}$, and terminate iteration when $||\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}||_\infty$ falls below 5×10^{-6}. Record the number of iterations required to achieve convergence.

    (b) For the system in (a), generate a plot of the number of iterations required by the SOR method to achieve convergence as a function of the relaxation parameter w. Take $\mathbf{x}^{(0)} = \mathbf{0}$, and terminate iteration when $||\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}||_\infty$ falls below 5×10^{-6}. Over roughly what range of w values does the SOR method outperforms the Gauss-Seidel? the Jacobi method?

12. **Chebyshev polynomials**

The Chebyshev polynomials are defined for $x \in [-1, 1]$ by $\mathrm{T}_n(x) = \cos(n\theta)$, $x = \cos\theta$.

(a) Derive the 3-term recurrence relation,

$$\mathrm{T}_{n+1}(x) = 2x\mathrm{T}_n(x) - \mathrm{T}_{n-1}(x) \ .$$

(b) Given $\mathrm{T}_0(x) = 1$ and $\mathrm{T}_1(x) = x$, use the recurrence relation to find $\mathrm{T}_2(x)$ and $\mathrm{T}_3(x)$.

(c) What are the roots of $T_3(x)$?