1

$$S_0(x) = \frac{a_0}{6h}(x_1 - x)^3 + \frac{a_1}{6h}(x - x_0)^3 + b_0(x_1 - x) + c_0(x - x_0)$$

$$= -4x^3 + b_0(1/2 - x) + c_0 x$$

$$b_0 = \frac{f_0}{h} - \frac{a_0 h}{6}, \quad c_0 = \frac{f_1}{h} - \frac{a_1 h}{6}$$

$$b_0 = 0, \quad c_0 = 2 + 1 = 3$$

$$\boxed{S_0(x) = -4x^3 + 3x}$$

$$S_1(x) = \frac{a_1}{6h}(x_2 - x)^3 + \frac{a_2}{6h}(x - x_1)^3 + b_1(x_2 - x) + c_1(x - x_1)$$

$$= -4(1 - x)^3 + b_1(1 - x) + c_1(x - 1/2)$$

$$\left.\begin{aligned} b_1 &= \frac{f_1}{h} - \frac{a_1 h}{6} \\ &= 2 + 1 = 3 \end{aligned}\right\} \boxed{S_1(x) = -4(1-x)^3 + 3(1-x)}$$

1 INTERPOLATION

$S_0(0) = 0$

$S_0(1/2) = 1$

$S_1(1/2) = 1$

$S_1(1) = 0$

$$S_0(0) = 0 + 0 = 0$$

$$S_0(1/2) = -4\left(\frac{1}{2}\right)^3 + 3 \cdot \frac{1}{2}$$

$$= -\frac{1}{2} + \frac{3}{2} = 1$$

$$S_1(1/2) = -4\left(1-\frac{1}{2}\right)^3 + 3\left(1-\frac{1}{2}\right)$$

$$= -\frac{1}{2} + \frac{3}{2} = 1$$

$$S_1(1) = -4(1-1)^3 + 3(1-1) = 0$$

1 CONTINUITY OF DERIVATIVES

$$S_0'(1/2) = S_1'(1/2)$$

$$S_0''(1/2) = S_1''(1/2)$$

$$S_0' = -12x^2 + 3$$

$$S_1' = 12(1-x)^2 - 3$$

$$-12\left(\tfrac{1}{2}\right)^2 + 3 = 12\left(1-\tfrac{1}{2}\right)^2 - 3$$

$$-\frac{12}{4} + 3 = \frac{12}{4} - 3$$

$$0 = 0$$

$$S_0''(x) = -24x$$

$$S_1''(x) = -24(1-x)$$

$$-24 \cdot \frac{1}{2} = -24 \cdot \frac{1}{2}$$

$$-12 = -12$$

3 a

$$\int_a^b f(x)\,dx = T(h) - \frac{(b-a)h^2}{12} f''(\xi)$$

$f(x) = 1$

$$\int_a^b 1\,dx = T(h) - \frac{(b-a)h^2}{12} f''(\xi)$$

$$b - a = T(h)$$

$$\frac{b-a}{n} = h$$

$$hn = h\left(\frac{1}{2} + 1 + \cdots + 1 + \frac{1}{2}\right)$$

$$hn = hn$$

3 a

$$\int_a^b f(x)\,dx = T(h) - \frac{(b-a)h^2}{12} f''(\xi)$$

$$\int_0^h f(x)\,dx \sim C_0 f(0) + C_1 f(h)$$

$f(x) = x$

$$\int_0^h x\,dx = \frac{1}{2}x^2\Big|_0^h = \frac{h^2}{2} = C_0 f(0) + C_1 f(h)$$

$$= 0 + C_1 h$$

$$\longrightarrow C_1 = \frac{h}{2}$$

3 a

$$\int_a^b f(x)dx = T(h) - \frac{(b-a)h^2}{12} f''(\xi)$$

$$\int_0^h f(x) \sim C_0 f(0) + C_2 f(h)$$

$$f(x) = x^2$$

$$\int_0^h x^2 dx = \frac{1}{3}x^3 \Big|_0^h = \frac{1}{3}h^3$$

$$\frac{h}{3} = C_0 f(0) + C_2 f(h)$$

$$= 0 + x^2 C_2$$

SO THE TRAPEZOIDAL RULE
HAS DEGREE OF PRECISION 1

$$\int_a^b f(x)\,dx = S(x)$$

$$f(x) = 1$$

$$\int_0^{2h} dx = 2h = S(x)$$

$$= h\left(\frac{1}{3}f(x_0) + \frac{4}{3}f(x_1) + \frac{1}{3}f(x_2)\right)$$

$$= h\left(\frac{1}{3} + \frac{4}{3} + \frac{1}{3}\right)$$

$$= h\left(\frac{6}{3}\right)$$

$$2h = 2h$$

3a

$$\int_a^b f(x)\,dx = S(x)$$

$f(x) = x$

$$\int_0^{2h} x\,dx = S(x)$$

$$x^2 \Big|_0^{2h} = h\left(\frac{1}{3}x_0 + \frac{4}{3}f(x_1) + \frac{2}{3}f(x_2)\right)$$

$$2h^2 = h\left(\frac{4}{3}h + \frac{4}{3}h\right)$$

$$2h^2 = 2h^2$$

3a

$$\int_a^b f(x)\,dx = S(x)$$

$$f(x) = x^2$$

$$\int_0^{2h} x^2\,dx = h\left(\frac{1}{3}f(x_0) + \frac{4}{3}f(x_1) + \frac{1}{3}f(x_2)\right)$$

$$\frac{1}{3}x^3\Big|_0^{2h} = h\left(0 + \frac{4}{3}h^2 + \frac{4}{3}h^2\right)$$

$$\frac{1}{3}(2h)^3 = h\left(\frac{8}{3}h^2\right)$$

$$\frac{8}{3}h^3 = \frac{8}{3}h^3$$

3a

$$\int_a^b f(x)\,dx = S(x)$$

$$f(x) = x^3$$

$$\int_0^{2h} x^3\,dx = h\left(\frac{1}{3}f(x_0) + \frac{4}{3}(x_1) + \frac{1}{3}(x_2)\right)$$

$$\frac{1}{4}x^4\Big|_0^{2h} = h\left(0 + \frac{4}{3}h^3 + \frac{1}{3}(2h)^3\right)$$

$$\frac{1}{4}(2h)^4 = h\left(\frac{4}{3}h^3 + \frac{8}{3}h^3\right)$$

$$4h^4 = 4h^4$$

3a

$$\int_a^b f(x)\,dx = S(x)$$

$f(x) = x^4$

$$\int_0^{2h} x^4\,dx = h\left(\frac{1}{3}f(x_0) + \frac{4}{3}(x_1) + \frac{1}{3}(x_2)\right)$$

$$\frac{1}{5}x^5\Big|_0^{2h} = h\left(0 + \frac{4}{3}h^4 + \frac{1}{3}(2h)^4\right)$$

$$\frac{1}{5}(2h)^5 = h\left(\frac{4}{3}h^4 + \frac{16}{3}h^4\right)$$

$$\frac{32}{5}h^5 \neq \frac{20}{3}h^5$$

SO, SIMPSON'S RULE HAS DEGREE OF PRECISION 3

3b $T(h) - \int_0^{2\pi} e^{-x} \sin x \, dx = \frac{2\pi h^2}{12} f''(\xi) \leq 10^{-4}$

$$\frac{2\pi h^2}{12} f''(\xi) = 10^{-4}$$

$$\frac{2\pi \left(\frac{2\pi}{n}\right)^2}{12} f''(\xi) = 10^{-4}$$

$$\frac{8\pi^2}{12 n^2} f''(\xi) = 10^{-4}$$

$$n = \sqrt{\frac{8\pi^2}{12 \cdot 10^{-4}} f''(\xi)}$$

$$n = \sqrt{-\frac{8\pi^2}{12 \cdot 10^{-4}} 2e^{-\xi} \cos(\xi)}$$

$\xi = 0$

$n \approx 362.76$

$\boxed{\hat{n} = 363}$

$f(x_0) \rightarrow f(x_n)$

$\hat{n} + 1 = 364$ FUNCTION EVALUATIONS

3b $S(h) - \int_0^{2\pi} e^{-x} \sin x \, dx = \frac{2\pi \, h^4}{180} f^{(4)}(\xi) \leq 10^{-4}$

$\frac{2\pi h^4}{180} f^{(4)}(\xi) = 10^{-4}$

$-\frac{2\pi \left(\frac{2\pi}{n}\right)^4}{180} 4 e^{-\xi} \sin(\xi) = 10^{-4}$

$-\frac{128\pi^5}{180 n^4} e^{-\xi} \sin(\xi) = 10^{-4}$

$n = \sqrt[4]{-\frac{128\pi^5}{180 \cdot 10^{-4}} e^{-\xi} \sin(\xi)}$

$\approx \sqrt[4]{2.1761 \times 10^6 \cdot e^{-\xi} \sin(\xi)}$

$\xi = \frac{3\pi}{2}$

$n \cong \sqrt[4]{2.1761 \times 10^6 \cdot 1.1132 \times 10^2}$

$n \cong 1.24776 \times 10^2$

$\longrightarrow \hat{n} = 125$

4a

$$N_0(h) = M - K_1 h - K_2 h^2 - K_3 h^3 - \cdots$$

$$N_0(h/3) = M - K_1 \frac{h}{3} - K_2 \frac{h^2}{9} - K_3 \frac{h^3}{27} - \cdots$$

$$N_1(h/3) = \frac{3N_0(h/3) - N_0(h)}{2}$$

$$= \frac{1}{2}\left[2M + K_1 h(1-1) + K_2 h^2\left(\frac{1}{3} - 1\right) + K_3 h^3\left(\frac{1}{9} - 1\right)\right]$$

$$N_1(h/3) = M + \frac{1}{3}K_2 h^2 + \frac{4}{9}K_3 h^3 + \cdots$$

$$N_1(h/9) = \frac{3N_0(h/9) - N_0(h/3)}{2}$$

$$= \frac{1}{2}\left[2M + K_1 \frac{h}{3}(0) + K_2 h^2\left(\frac{1}{9} - \frac{3}{81}\right) + \cdots\right.$$

$$\left.\cdots + K_3 h^3\left(\frac{1}{27} - \frac{3}{729}\right) + \cdots\right]$$

4b

$$\frac{df(x_0)}{dx} = N_2(h/9) + \frac{1}{27}K_3h^3 + \cdots$$

$$= N_2(h/9) + C_3h^3 + C_4h^4 + C_5h^5 + \cdots$$

$$= N_2(h/9) + O(h^3)$$

SAY $\dfrac{f(x_0+h) - f(x_0)}{h} = N_0(h)$

THEN $N_2(h/9) = \dfrac{9N_1(h/3) - N_1(h/9)}{8}$

$$= \frac{1}{8}\left[9\left(\frac{3N_0(h) - N_0(h/3)}{2}\right) - \left(\frac{3N_0(h/3) - N_0(h/9)}{2}\right)\right]$$

$$= \frac{9}{16}\left[3\left(\frac{f(x_0+h) - f(x_0)}{h}\right) - \left(\frac{f(x_0+h/3) - f(x_0)}{h/3}\right) - \cdots\right.$$

$$\left.\cdots - 3\left(\frac{f(x_0+h/3) - f(x_0)}{h/3}\right) + \left(\frac{f(x_0+h/9) - f(x_0)}{h/9}\right)\right]$$

4 a

$$N_1(h/9) = M + \frac{1}{27}K_2h^2 + \frac{4}{243}K_3h^3 + \cdots$$

$$N_2(h/9) = \frac{9N_1(h/9) - N_1(h/3)}{8}$$

$$= \frac{1}{8}\left[8M + K_2h^2\left(\frac{1}{3} - \frac{1}{3}\right) + K_3h^3\left(\frac{9\cdot 4}{243} - \frac{4}{9}\right) + \cdots\right]$$

$$\boxed{N_2(h/9) = M - \frac{1}{27}K_3h^3 - \cdots}$$

4b

$$N_2(h/9) = \frac{9}{16}\left[\frac{3f(x_0+h) - 3f(x)}{h} - \frac{3f(x_0+h/3) - 3f(x_0)}{h} - \cdots\right.$$

$$\cdots \left. - \frac{9f(x_0+h/3) - 9f(x_0)}{h} + \frac{9f(x_0+h/9) - 9f(x_0)}{h}\right]$$

$$= \frac{27}{16h}\left[f(x_0+h) - f(x) - f(x_0+h/3) + f(x_0) - \cdots\right.$$

$$\cdots \left. - 3f(x_0+h/3) + 3f(x_0) + 3f(x_0+h/9) - 3f(x_0)\right]$$

$$= \frac{27}{16h}\left[f(x_0+h) - 4f(x_0+h/3) + 3f(x_0+h/9)\right]$$

$$\frac{df(x_0)}{dx} = \boxed{\frac{27}{16h}\left(f(x_0+h) - 4f(x_0+h/3) + 3f(x_0+h/9)\right) + O(h^3)}$$

[a,b] = [1,0]

n	T	eh	e2h/eh
1	0.48986796623791	0.35985835918259	
2	0.71994619685185	0.12978012856865	0.36064225064397
4	0.80348570533171	0.04624062008879	0.35629969394216
8	0.83334214565712	0.01638417976338	0.35432439556228
16	0.84393436795018	0.00579195747032	0.35350915053191
32	0.84768044676220	0.00204587865830	0.35322750016539
64	0.84900377084043	0.00072255458007	0.35317567693285
128	0.84947111004251	0.00025521537799	0.35321259463560

[a,b] = [π/4, 9π/4]

n	T	eh	e2h/eh
1	0.00000000000000	1.27323954473516	
2	1.13843429252224	0.13480525221292	0.10587579750436
4	-1.26124388523432	0.01199565950084	0.08898510483771
8	-1.27232970400346	0.00090984073170	0.07584749564093
16	-1.27317790925396	0.00006163548120	0.06774315443652
32	-1.27323559135119	0.00000395338397	0.06414136619375
64	-1.27323929588531	0.00000024884985	0.06294603608860
128	-1.27323952915360	0.00000001558156	0.06261430351871

[a,b] = [π, 2π]

n	T	eh	e2h/eh
1	-1.51409455798888	0.34646544201112	
2	-1.77624020205602	0.08431979794398	0.24337145273286
4	-1.83963706667647	0.02092293332353	0.24813784939844
8	-1.85533630955249	0.00522369044751	0.24966338929329
16	-1.85925141520278	0.00130858479722	0.25050963688781
32	-1.86022957934930	0.00033042065070	0.25250228445412
64	-1.86047408202777	0.00008591797223	0.26002603665323
128	-1.86053520529857	0.00002479470143	0.28858573807671

All T values computed in MATLAB

6a

$$P_4 = x^4 + \alpha_{43}P_3 + \alpha_{42}P_2 + \alpha_{41}P_1 + \alpha_{40}P_0$$

$$= x^4 - \frac{\langle x^4, P_3\rangle}{\|P_3\|^2}P_3 - \frac{\langle x^4, P_2\rangle}{\|P_2\|^2}P_2 - \frac{\langle x^4, P_1\rangle}{\|P_1\|^2}P_1 - \frac{\langle x^4, P_0\rangle}{\|P_0\|^2}P_0$$

0

$$\langle x^4, P_0\rangle = \int_{-1}^{1} x^4 dx = \frac{1}{5}x^5\Big|_{-1}^{1} = \frac{2}{5}$$

$$\alpha_{40} = -\frac{1}{5}$$

$$\langle x^4, P_2\rangle = \int_{-1}^{1} x^6 dx - \frac{1}{3}\int_{-1}^{1} x^4 dx = \frac{2}{7} - \frac{1}{3}\cdot\frac{2}{5} = \frac{16}{105}$$

$$\|P_2\|^2 = \int_{-1}^{1}\left(x^2 - \frac{1}{3}\right)^2 dx = \frac{2}{5} - \frac{4}{9} + \frac{2}{9} = \frac{8}{45}$$

$$\alpha_{42} = -\frac{16}{105}\cdot\frac{45}{8} = -\frac{90}{105} = -\frac{6}{7}$$

<u>6a</u>

$$\langle x^4, P_3 \rangle = \int_{-1}^{1} x^4\left(x^3 - \frac{3}{5}x\right)dx = 0$$

$$\longrightarrow P_4 = x^4 - \frac{6}{7}P_2 - \frac{1}{5}P_0$$

$$= \boxed{x^4 - \frac{6}{7}\left(x^2 - \frac{1}{3}\right) - \frac{1}{5}}$$

<u>6b</u>

$$x^4 = P_4 - \alpha_{43}P_3 - \alpha_{42}P_2 - \alpha_{41}P_1 - \alpha_{40}P_0$$

$$= P_4 - \alpha_{42}P_2 - \alpha_{40}P_0$$

8a $\langle L_0, L_1 \rangle = 0$

$$\int_0^\infty L_0 L_1 \, dx = 0$$

$$\int_0^\infty (1-x)e^{-x} \, dx = 0$$

$$\int_0^\infty e^{-x} \, dx - \int_0^\infty x e^{-x} \, dx = 0$$

$$\lim_{b \to \infty} \left[\int_0^b e^{-x} \, dx - \int_0^b x e^{-x} \, dx \right] = 0$$

$$\lim_{b \to \infty} \left[-e^{-x} \Big|_0^b + e^{-x}(x+1) \Big|_0^b \right] = 0$$

8a

$$\lim_{b\to\infty}\left[-e^{-b}+e^{0}+e^{-b}(b+1)-e^{0}\right]=0$$

$$1-1=0$$

$$\langle L_0, L_2\rangle = 0$$

$$\int_0^{\infty} L_0 L_2 e^{-x}\,dx = 0$$

$$\lim_{b\to\infty}\left[\int_0^{b} x^2e^{-x}-4xe^{-x}+2e^{-x}\,dx\right]=0$$

8b $\langle L_0, L_2 \rangle = \int_0^\infty (x^2 - 4x + 2)e^{-x}\,dx$

$$= \lim_{b\to\infty} \underbrace{\int_0^b x^2 e^{-x}\,dx}_{\alpha} - 4\underbrace{\int_0^b x e^{-x}\,dx}_{\beta} + 2\underbrace{\int_0^b e^{-x}\,dx}_{\Gamma}$$

$$\left\{\begin{matrix} u = x^2 & dv = e^{-x}dx \\ du = 2x\,dx & v = -e^{-x}\end{matrix}\right\} \rightarrow \alpha = -x^2 e^{-x}\Big|_0^b + 2\int_0^b x e^{-x}\,dx = \cdots$$

$$\left\{\begin{matrix} u = x & dv = e^{-x}dx \\ du = dx & v = -e^{-x}\end{matrix}\right\}$$

$$2\left[-xe^{-x}\Big|_0^b + \int_0^b e^{-x}\,dx\right]$$

$$2\left[-be^{-b} - e^{-x}\Big|_0^b\right]$$

$$2\left[-be^{-b} - e^{-b} + 1\right]$$

$$\cdots = -b^2 e^{-b} + 2\left[-be^{-b} - e^{-b} + 1\right]$$

$$\lim_{b\to\infty} \alpha = 0 + 2[0 + 0 + 1] = 2$$

$$\left\{\lim_{b\to\infty} \frac{-b^2}{e^b} \underset{\text{RULE}}{\overset{\text{LH's}}{=}} \lim_{b\to\infty} \frac{-2b}{e^b} \underset{\text{RULE}}{\overset{\text{LH's}}{=}} \lim_{b\to\infty} \frac{-2}{e^b} = 0\right\}$$

29TH ANNE, 1:00PM

WATCH 4 CLAIMS (M

8b

$$\langle L_0, L_2 \rangle = \int_0^{\infty} (x^2 - 4x + 2) e^{-x}\, dx$$

$$= \lim_{b \to \infty} \overbrace{\int_0^b x^2 e^{-x}\, dx}^{\alpha} \overbrace{- 4\int_0^b x e^{-x}\, dx}^{\beta} + \overbrace{2\int_0^b e^{-x}\, dx}^{\Gamma}$$

$$\left\{ \begin{array}{ll} u = x & dv = e^{-x} dx \\ du = dx & v = -e^{-x} \end{array} \right\}$$

$$\beta = -4\left[-x e^{-x} \Big|_0^b + \int_0^b e^{-x} dx \right] = -4\left[-be^{-b} - e^{-x}\Big|_0^b \right] = -4\left[-be^{-b} - e^{-b} + 1 \right]$$

$$\lim_{b \to \infty} \beta = -4$$

$$\Gamma = -2 e^{-x} \Big|_0^b = -2e^{-b} + 2$$

$$\lim_{b \to \infty} \Gamma = 2$$

$$\langle L_0, L_2 \rangle = \lim_{b \to \infty} (\alpha - \beta + \Gamma) = 2 - 4 + 2 = 0$$

8a $\langle L_1, L_2 \rangle = \int_0^\infty (1-x)(x^2-4x+2)e^{-x}\,dx = 0$

$$= \int_0^\infty (x^2-4x+2 \quad -x^3+4x^2-2x)e^{-x}\,dx$$

$$= \int_0^\infty -x^3e^{-x} + 5x^2e^{-x} - 6xe^{-x} + 2e^{-x}\,dx$$

$$= \lim_{b\to\infty} \int_0^b \overbrace{-x^3e^{-x}}^{\alpha} + \overbrace{5x^2e^{-x}}^{\beta} - \overbrace{6xe^{-x}}^{\gamma} + \overbrace{2e^{-x}}^{\delta}\,dx$$

$$\left\{\begin{matrix} u=-x^3 & dv=-e^{-x}dx \\ du=-3x^2dx & v=e^{-x} \end{matrix}\right\}$$

$$\alpha = -x^3e^{-x}\Big|_0^b + 3\int_0^b x^2e^{-x}\,dx$$

$$\left\{\begin{matrix} u=x^2 & dv=e^{-x}dx \\ du=2x\,dx & v=-e^{-x} \end{matrix}\right\}$$

$$3\left[-x^2e^{-x}\Big|_0^b + 2\int_0^b xe^{-x}\,dx\right]$$

$$\left\{\begin{matrix} u=x & dv=e^{-x}dx \\ du=dx & v=-e^{-x} \end{matrix}\right\}$$

$$2\left[-xe^{-x}\Big|_0^b + \int_0^b e^{-x}\,dx\right]$$

<u>8a</u> $\langle L_1, L_2 \rangle = \int_0^\infty (1-x)(x^2-4x+2)e^{-x}\,dx = 0$

$$= \int_0^\infty (x^2 - 4x + 2 - x^3 + 4x^2 - 2x)e^{-x}\,dx$$

$$= \int_0^\infty -x^3e^{-x} + 5x^2e^{-x} - 6xe^{-x} + 2e^{-x}\,dx$$

$$= \lim_{b\to\infty} \int_0^b -x^3e^{-x} + 5x^2e^{-x} - 6xe^{-x} + 2e^{-x}\,dx$$

$$= 6 + 10 - 6 + 2 \neq 0 \ \ldots ?$$

8b $\quad c_1 = \dfrac{2-\sqrt{2}}{4} \qquad c_2 = \dfrac{2+\sqrt{2}}{4}$