1. **[True/False, 25 pts].** Indicate whether the following statements are true or false? Justify your answer for full credit, prove/show if true and give a counterexample if false.

   (a) [5 pts] The purpose of pivoting in Gaussian elimination is to reduce the operation count.

   FALSE, PIVOTING IS EMPLOYED WHEN THE LEADING DIAGONAL ENTRY OF A MATRIX IS ZERO. GAUSSIAN ELIMINATION FAILS IN THIS CASE, PIVOTING REMEDIES THE SITUATION

   (b) [5 pts] The polynomial $p_6(x)$ of degree ≤ 6 that interpolates the function $f(x) = 4x^3 - 3x^2 + 2.5x - \pi$ at the points $x = -3, -2, -1, 0, 1, 2, 3$ is the function $f(x)$ itself, i.e., $p_6(x) = f(x)$.

(c) [5 pts] Let $f(x)$ be a continuous function on the interval $[0,1]$ and let $p_n(x)$ be the polynomial of degree $\leq n$ that interpolates $f(x)$ at the $n+1$ distinct points $x_i = i/n$, $i = 0,\ldots,n$. Then $p_n(x)$ converges to $f(x)$ as $n \to \infty$ for every $x \in [0,1]$.

TRUE, $\lim_{n\to\infty} P_n(x) = \lim_{n\to\infty}\left[f(x) + \frac{f^{(n+1)}(\xi_n)}{(n+1)!}(x-x_0)\cdots(x-x_n)\right]$

$= f(x) + \lim_{n\to\infty}\left[\frac{f^{(n+1)}(\xi_n)}{(n+1)!}(x-x_0)\cdots(x-x_n)\right]$

$\approx f(x) + \frac{f^{(\infty)}(\xi_n)\,g(x)}{\infty}$

$= f(x) + 0$

$\boxed{\lim_{n\to\infty} P_n(x) = f(x)}$

(d) [5 pts] Given $n+1$ distinct points $\{x_0,\ldots,x_n\}$ and $n+1$ data values $\{f_0,\ldots,f_n\}$, there is a unique cubic spline $S(x)$ which interpolates the data, i.e. such that $S(x_i) = f_i$, $i = 0,\ldots n$.

(e) [5 pts] Power method can be used to approximate any eigenvalue-eigenvector pair with eigenvectors converging faster than eigenvalues.

FALSE, POWER METHOD WILL FAIL IF THE EIGENVECTOR IS COMPLEX & THE MATRIX & STARTING VECTOR ARE REAL.

ADDITIONALLY, EIGENVALUES WILL CONVERGE FASTER THAN EIGENVECTORS.

2. **[Fixed Point Iterations, 15 pts].** Let $g(x) = -x^2 + 3ax + a - 2a^2$, where a is a parameter.

   (a) Show that a is a fixed point of $g(x)$.

   (b) For what values of a does the iteration scheme $x_{n+1} = g(x_n)$ converge *linearly* to the fixed point a (provided x_0 is chosen sufficiently close to a)?

   (c) Is there a value of a for which convergence is quadratic?

a) $g(a) = -a^2 + 3a^2 + a - 2a^2$

$= a$

b) $g'(x) = -2x + 3a$

$g'(a) = a$

FOR $a < 1$, THE SCHEME CONVERGES LINEARLY

c) FOR $a = 0$, THE SCHEME CONVERGES QUADRATICALLY

3. **[Iterative methods for linear systems, 10 pts]** Consider the system of linear equations,

$$\begin{array}{rcrcr} 2x_1 & + & x_2 & = & 1 \\ x_1 & + & 2x_2 & = & -1 \end{array} \Bigg\} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Starting from the initial guess $(x_1, x_2)_0 = (0, 0)$, perform one step of Gauss-Seidel iteration.

$$x^{(1)} = (D+L)^{-1}(b - Ux^{(0)})$$

$$= \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}^{-1} \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right)$$

$$= \frac{1}{4} \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix} \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1/2 & 0 \\ -1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -3/4 \end{bmatrix}$$

4. **[Newton's Method, 10 pts]** Consider the system of two nonlinear equations

$$2x - \cos y = 0 \qquad 2y - \sin x = 0.$$

(a) Write down the vector-valued function $\mathbf{F}(x, y) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$ whose zero we seek.

$$F(x, y) = \begin{bmatrix} 2x - \cos y \\ 2y - \sin x \end{bmatrix}$$

(b) Calculate the Jacobian of $\mathbf{F}$.

$$J_F(x, y) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix}$$

$$= \begin{bmatrix} 2 & \sin y \\ -\cos x & 2 \end{bmatrix}$$

(c) Apply one step of Newton's Method to initial iterate $\mathbf{p}_0 = [0\ 0]^T$ to obtain the next iterate $\mathbf{p}_1$.

$$F(P_0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad J_F(P_0) = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$$

$$S_0 = -J_F^{-1}(P_0) F(P_0) = -\begin{bmatrix} 1/2 & 0 \\ 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/4 \end{bmatrix}$$

$$P_1 = P_0 + S_0 = \begin{bmatrix} 1/2 \\ 1/4 \end{bmatrix}$$

6. **[Numerical Integration, 10 pts]**. The following data is known to lie on a cubic curve. Evaluate $\int_{-4}^{4} f(x)dx$ exactly and explain how you are sure that your answer is correct.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-9	0	2	0	-3	-4	0	12	35

SIMPSON'S RULE IS EXACT FOR POLYNOMIALS OF DEGREE ≤ 3, SO,

$$\int_{-4}^{4} f(x)dx = 1\cdot\left(\frac{1}{3}f(-4) + \frac{4}{3}f(-3) + \frac{2}{3}f(-2) + \cdots + \frac{4}{3}f(3) + \frac{1}{3}f(4)\right)$$

$$= \frac{1}{3}(-9) + \frac{4}{3}(0) + \frac{2}{3}(2) + \frac{4}{3}(0) + \frac{2}{3}(-3) + \dots$$

$$\dots + \frac{4}{3}(-4) + \frac{2}{3}(0) + \frac{4}{3}(12) + \frac{1}{3}(35)$$

$$= 56/3$$

7. **[Numerical Methods for Solving ODEs, 10 pts].**

Consider the initial value problem,

$$y' = -y, \quad y(0) = 1.$$

Find an approximation to $y(2)$ with a step size $h = 1$ by using modified Euler's method.

$n = 2/h = 2$

$\cancel{U_{n+1} =}$

$U_1 = U_0 + \frac{h}{2}\left(-U_0 + (-U_0 + k_1)\right)$

$= U_0 + \frac{h}{2}\left(-U_0 - U_0 - U_0\right)$

$= \left(1 - \frac{3h}{2}\right)U_0 = -\frac{1}{2}$

$U_2 = \left(1 - \frac{3h}{2}\right)U_1 = -\frac{1}{4} \approx y(2)$

8. **[Richardson Extrapolation, 10 pts].** Let f be a function with four continuous derivatives, and fix a point x. Recall the central difference formula

$$D(h) = \frac{f(x+h) - f(x-h)}{2h},$$

which satisfies an error estimate of the form

$$f'(x) = D(h) + Ch^2 + O(h^4)$$

where C is an (unknown) constant.

(a) Show how to use the two estimates $D(h)$ and $D(h/3)$ to obtain a new estimate $R_1(h)$ for $f'(x)$ that has error $O(h^4)$.

(b) Use the central difference approximation $D(h)$ to estimate $f'(0)$ for $f(x) = \ln(1+x)$ with $h = 0.3$ and $h = 0.1$.

(c) Perform one step of Richardson extrapolation on the values obtained in part (b) to get a new estimate R_1.

a) $D(h) = f'(x) - Ch^2 - O(h^4)$

$D(h/3) = f'(x) - C\frac{h^2}{9} - O(h^4)$

$-\dfrac{D(h) - 9D(h/3)}{8} = f'(x) - O(h^4) = R_1(h)$