Homework 3

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PHYS 428

1.a

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

$\|x\|_1 \geq 0$ FOR $x \in \mathbb{R}^n$ & $\|x\|_1 = 0$ IFF $x = 0$

$$\sum_{i=1}^{n} |x_i| = |x_1| + |x_2| + |x_3| + \dots + |x_n| \geq 0 \text{ FOR } x \in \mathbb{R}^n$$

& $\|x\|_1 = 0$ IFF $x = 0$ $\therefore$

1. b

$\|\lambda x\|_1 = |\lambda| \|x\|_1$ FOR $\lambda \in \mathbb{R}$ & $x \in \mathbb{R}^n$

$$\|\lambda x\|_1 = \sum_{i=1}^{n} |\lambda x_i| = |\lambda x_1| + |\lambda x_2| + \dots + |\lambda x_3| = \dots$$

$$\dots = \lambda \left(|x_1| + |x_2| + \dots + |x_3| \right) = |\lambda| \|x\|_1 \therefore$$

1.c $\|x\|_1 = \sum_{i=1}^{n} |x_i|$

$\|x+y\|_1 \leq \|x\|_1 + \|y\|_1$ FOR $x, y \in \mathbb{R}^n$

$\|x+y\|_1 = \sum_{i=1}^{n} |x_i + y_i| = |x_1 + y_1| + |x_2 + y_2| + \cdots$

$\cdots + |x_n + y_n|$

$\|x\|_1 + \|y\|_1 = \left[|x_1| + \cdots + |x_n| \right] + \left[|y_1| + \cdots + |y_n| \right]$

SINCE x OR y COULD BE LESS THAN ZERO, IT FOLLOWS THAT $\|x+y\|_1 \leq \|x\|_1 + \|y\|_1$ FOR $\forall$ $x, y \in \mathbb{R}^n$.

2

a

$$A = LU$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{12} & l_{22} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$$

$l_{11} u_{11} = 0 \to u_{11} = 0$

$l_{12} u_{11} = 1 \to u_{11} \neq 0$

$l_{11} u_{12} = 1$

BY CONTRADICTION, A MUST NOT HAVE AN LU DECOMP. $\therefore$

2.b THE SYSTEM DOES NOT HAVE A UNIQUE SOLUTION BECAUSE A IS SINGULAR, IT HAS A ZERO IN A DIAGONAL ENTRY

2.c IF $A \xrightarrow{R_1 \leftarrow R_2} A$ THEN

$Ax = b$ HAS AN LU DECOMP.

3.a

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -5 & 2 & -1 \\ 1 & 0 & 3 \\ 3 & 1 & 6 \end{bmatrix} = \begin{bmatrix} -5 & 2 & -1 \\ 3 & 1 & 6 \\ 1 & 0 & 3 \end{bmatrix}$$

$$E_1 PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/5 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 & 2 & -1 \\ 3 & 1 & 6 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 2 & -1 \\ 3 & 1 & 6 \\ 0 & 2/5 & 14/5 \end{bmatrix}$$

$$E_2 E_1 PA = \begin{bmatrix} 1 & 0 & 0 \\ 3/5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 & 2 & -1 \\ 3 & 1 & 6 \\ 0 & 2/5 & 14/5 \end{bmatrix} = \begin{bmatrix} -5 & 2 & -1 \\ 0 & 11/5 & 27/5 \\ 0 & 2/5 & 14/5 \end{bmatrix}$$

3.a

$$E_3 E_2 E_1 PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2/11 & 1 \end{bmatrix} \begin{bmatrix} -5 & 2 & -1 \\ 0 & 11/5 & 27/5 \\ 0 & 2/5 & 14/5 \end{bmatrix} = \begin{bmatrix} -5 & 2 & -1 \\ 0 & 11/5 & 27/5 \\ 0 & 0 & 20/11 \end{bmatrix} = U$$

$$U = E_3 P_3 E_2 P_2 E_1 P_1 A, \quad P_3 \,\&\, P_2 = I$$

$$\tilde{E}_3 \tilde{E}_2 \tilde{E}_1 P_3 P_2 P_1 A = U$$

$$P_3 P_2 P_1 A = (\tilde{E}_3 \tilde{E}_2 \tilde{E}_1)^{-1} U$$

$$\begin{bmatrix} -5 & 2 & -1 \\ 3 & 1 & 6 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3/5 & 1 & 0 \\ -1/5 & 2/11 & 1 \end{bmatrix} \begin{bmatrix} -5 & 2 & -1 \\ 0 & 11/5 & 27/5 \\ 0 & 0 & 20/11 \end{bmatrix}$$

$$PA = LU$$

3.b

$$L\,y = P\,b$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{5} & 1 & 0 \\ -\frac{1}{5} & \frac{2}{11} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$y_1 = 2$$

$$-\frac{3}{5} y_1 + y_2 = 1 \qquad y_2 = \frac{5}{5} + \frac{6}{5} = \frac{11}{5}$$

$$-\frac{1}{5} y_1 + \frac{2}{11} y_2 + y_3 = -2 \qquad y_3 = -2 + \frac{1}{5} y_1 - \frac{2}{11} y_2$$

$$= -\frac{10}{5} + \frac{2}{5} - \frac{2}{5} = -2$$

3.b $Ux = y$

$$\begin{bmatrix} -5 & 2 & -1 \\ 0 & 11/5 & 27/5 \\ 0 & 0 & 20/11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 11/5 \\ -2 \end{bmatrix}$$

$$x_3 = \frac{11}{20} \cdot -2 = -\frac{22}{20}$$

$$\frac{11}{5}x_2 + \frac{27}{5}x_3 = \frac{11}{5}, \qquad x_2 = \frac{5}{11}\left(\frac{11}{5} + \frac{27}{5} \cdot \frac{22}{20}\right) = 3.7$$

$$-5x_1 + 2x_2 - x_3 = 2, \qquad x_1 = -\frac{1}{5}\left(2 - 2 \cdot 3.7 - \frac{22}{20}\right)$$

$$= 1.3$$

$$\boxed{x = \begin{bmatrix} 1.3 \\ 3.7 \\ -1.1 \end{bmatrix}}$$

3.c

$$Ly = Pb$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -3/5 & 1 & 0 \\ -1/5 & 2/11 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}$$

$$y_1 = 0$$

$$-\frac{3}{5}y_1 + y_2 = 5 \qquad y_2 = 5$$

$$-\frac{1}{5}y_1 + \frac{2}{11}y_2 + y_3 = 1 \qquad y_3 = 1 - \frac{2}{11} \cdot 5 = \frac{11}{11} - \frac{10}{11} = \frac{1}{11}$$

3.c

$$Ux = y$$

$$\begin{bmatrix} -5 & 2 & -1 \\ 0 & 11/5 & \frac{27}{5} \\ 0 & 0 & 20/11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 1/11 \end{bmatrix}$$

$$\frac{20}{11} x_3 = 1/11 \quad , \quad \boxed{x_3 = \frac{1}{20}}$$

$$\frac{11}{5} x_2 + \frac{27}{5} x_3 = 5 \qquad x_2 = \frac{5}{11}\left(5 - \frac{27}{5} \cdot \frac{1}{20}\right) = \boxed{2.15}$$

$$-5x_1 + 2x_2 - x_3 = 0 \qquad x_1 = - (-2*2.15 + 1/20) / 5$$

$$= \boxed{0.85}$$

5.a

$$A = \begin{bmatrix} 2 & -3 & 1 \\ -4 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}$$

$$\|A\|_\infty = |-4| + |1| + |2| = \boxed{7}$$

$$= \frac{\|Ax\|_\infty}{\|x\|_\infty} \longrightarrow \|Ax\|_\infty = 7, \quad \|x\|_\infty = 1$$

$$Ax = \begin{bmatrix} 2x_1 - 3x_2 + x_3 \\ -4x_1 + x_2 + 2x_3 \\ 5x_1 + x_3 \end{bmatrix}$$

$$\boxed{x = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}$$

$$Ax = \begin{bmatrix} -2 - 3 + 1 \\ 4 + 1 + 2 \\ -5 + 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 7 \\ -4 \end{bmatrix}$$

$$\frac{\|Ax\|_\infty}{\|x\|_\infty} = \boxed{7}$$

5.b

$$A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}, \quad \|A\|_\infty = 1$$

SAY $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ THEN $\det(A - \lambda I) = 0$

$$\det \begin{bmatrix} 1-\lambda & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$$0 = 0$$

SO, $\rho(A) = \{ \max |\lambda| \} = 0$

6.a

$$2 \cdot 1.2969 - 2 \cdot 0.8648 = 0.8642$$

$$2 \cdot 0.2161 - 2 \cdot 0.1441 = 0.1440$$

6.b

$$e_1 = x - x_1$$

$$= \begin{bmatrix} 2 - 0 \\ 2 - 1 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

$$r_1 = b - Ax_1 = \begin{bmatrix} 0.8642 \\ 0.1440 \end{bmatrix} - \begin{bmatrix} 1.2969 & 0.8648 \\ 0.2161 & 0.1441 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \ldots$$

$$\ldots = \begin{bmatrix} 0.8642 - 0.8648 \\ 0.1440 - 0.1441 \end{bmatrix} = \boxed{\begin{bmatrix} -6 \times 10^{-4} \\ -1 \times 10^{-4} \end{bmatrix}}$$

6.b $e_2 = x - x_2$

$$= \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 0.9911 \\ -0.4870 \end{bmatrix} = \boxed{\begin{bmatrix} 1.009 \\ 2.487 \end{bmatrix}}$$

$$r_2 = b - Ax_2 = \begin{bmatrix} 0.8642 \\ 0.1440 \end{bmatrix} - \begin{bmatrix} 1.2969 & 0.8648 \\ 0.2161 & 0.1441 \end{bmatrix} \begin{bmatrix} 0.9911 \\ -0.4870 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8642 \\ 0.1440 \end{bmatrix} - \begin{bmatrix} 0.8642 \\ 0.1440 \end{bmatrix} \overset{*}{=} \boxed{\begin{bmatrix} \text{1e-8} \\ \text{1e-8} \end{bmatrix}}$$

*with greater precision

6.d $\dfrac{\|e_1\|_\infty}{\|x\|_\infty} \le K_\infty(A) \dfrac{\|r_1\|_\infty}{\|b\|_\infty}$

$$\frac{2}{2} \le 3.2707 \times 10^8 \cdot \frac{6 \times 10^{-4}}{8.642 \times 10^{-1}}$$

$1 < 2$ [redacted] $\times 10^5$ SO THE THEOREM HOLDS

$$\frac{\|e_2\|}{\|x\|_\infty} \le K(A) \frac{\|r_2\|_\infty}{\|b\|_\infty}$$

$$\frac{2.487}{2} \le 3.2707 \times 10^8 \cdot \frac{1 \times 10^{-8}}{8.642 \times 10^{-1}}$$

$1.244 < 3.$ THE RESULT FITS W/ THE THEOREM

8. a THE POSSIBLE MAX ROW SUMS ARE...

$$S_1 = \frac{1}{a_{11}}\left(|a_{12}| + |a_{13}|\right)$$

$$S_2 = \frac{1}{a_{22}}\left(|a_{21}| + |a_{23}|\right)$$

$$S_3 = \frac{1}{a_{33}}\left(|a_{31}| + |a_{32}|\right)$$

$$\sum_{j=1\ j\neq i}^{3} |a_{ij}| = |a_{12}| + |a_{13}| + |a_{21}| + |a_{23}| + \cdots$$

$$\cdots + |a_{31}| + |a_{32}| < |a_{ii}|$$

8. a

$|a_{12}| + |a_{13}| < a_{11}$

$|a_{21}| + |a_{23}| < a_{22}$

$|a_{31}| + |a_{32}| < a_{33}$

SO $S_1, S_2, S_3 < 1$

IT FOLLOWS THAT THIS IS TRUE FOR ANY n

8.b

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad {}^{1}\|B\|_\infty = \left|\frac{a_{12}}{a_{21}}\right|, \quad {}^{2}\|B\|_\infty = \left|\frac{a_{21}\,a_{12}}{a_{11}\,a_{22}}\right|$$

$$\sum_{j=1,\, j\neq i}^{2} |a_{ij}| = |a_{12}| + |a_{21}| < a_{ii}$$

$$\left.\begin{array}{l} |a_{21}| < |a_{11}|, \\ |a_{12}| < |a_{22}| \end{array}\right\} \quad |a_{21}|\,|a_{12}| < |a_{11}|\,|a_{22}| \qquad \left|\frac{a_{21}\,a_{12}}{a_{11}\,a_{22}}\right| < 1$$

$${}^{2}\|B\|_\infty < 1$$

... ONE OF TWO POSSIBLE MAX ROW SUMS IS

< 1

9

$34x + 55y - 21 = 0 \longrightarrow f(x, y) = 0$

$55x + 89y - 34 = 0 \longrightarrow g(x, y) = 0$

DoGood174's ANSWER IS CLOSER TO ZERO THAN MightDo11's, SO THE FORMER IS BETTER