1a $y' = 2y, \quad y(0) = 1 = U_0$

$$U_{n+1} = U_n + hf(U_n)$$

$$f(y) = 2y, \quad y(t) = e^{2t}, \quad h = 0.1$$

$$U_1 = U_0 + hf(U_0)$$

$$= U_0(1 + 2h) = 1 \cdot (1 + 0.2) = 1.2$$

$$U_2 = U_1 + hf(U_1)$$

$$= U_1(1 + 2h) = 1.2 \cdot (1 + 0.2) = 1.44$$

$h = 0.05$

$$U_1 = 1 \cdot (1 + 0.1) = 1.1$$

$$U_2 = 1.1(1 + 0.1) = 1.21$$

$h = 0.001$

$$U_1 = 1(1 + 0.002) = 1.002$$

$$U_2 = 1.002(1 + 0.002) \approx 1.004$$

```
%oneStepEuler
%Matt Zeller
%PHYS 428
%12/3/2018

h=0.001;
n=1/h
un=zeros(n,1);
j=1;
for i=0:h:1

    un(j,1)=(1+2*h)^j;
    j=j+1;

end
format long
un(n+1,1)
```

1b $y' = 2y$, $y(0) = y_0 = 1 = u_0$

$= f(y)$, $h = 0.1$

$k_1 = f(u_n)$

$k_2 = f(u_n + hk_1)$

$u_{n+1} = u_n + \frac{h}{2}(k_1 + k_2)$

$$u_1 = u_0 + \frac{h}{2}(2u_0 + 2(u_0 + 2hu_0))$$

$$= u_0 + hu_0 + hu_0 + 2h^2u_0$$

$$= u_0(1 + 2h + 2h^2)$$

$$= 1(1 + 2\cdot 0.1 + 2\cdot(0.1)^2) = 1.22$$

$$u_2 = u_1(1 + 2h + 2h^2)$$

$$= 1.22(1 + 2\cdot 0.1 + 2\cdot(0.1)^2) = 1.4884$$

1b $y' = 2y, \quad y(0) = y_0 = 1 = v_0$

$= f(y), \quad h = 0.05$

$v_1 = v_0(1 + 2h + 2h^2)$

$= (1 + 2\cdot(0.05) + 2\cdot(0.05)^2) = 1.105$

$v_2 = 1.105(1 + 2\cdot(0.05) + 2\cdot(0.05)^2 = 1.221025$

$h = 0.001$

$v_1 = (1 + 2\cdot(0.001) + 2\cdot(0.001)^2 \approx 1.002002$

$v_2 \approx 1.002002^2 \approx 1.004008$

```
%oneStepModEuler
%Matt Zeller
%PHYS 428
%12/3/2018

h=0.01;
n=1/h
un=zeros(n,1);
j=1;
un(1,1)=1;
for i=0:h:1
    un(j+1,1)=un(j,1)*(1+2*h+2*h^2);
    j=j+1;
end
format long
un(n+1,1)
```

1c $y' = 2y$, $y(0) = 1 = U_0$

$$U_{n+1} = U_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

$$U_{n+1} = U_n\left(1 + 2h + 2h^2 + \frac{4}{3}h^3 + \frac{2}{3}h^4\right)$$

$h = 0.1$

$$U_1 = U_0\left(1 + 2\cdot 0.1 + 2\cdot 0.1^2 + \frac{4}{3}\cdot 0.1^3 + \frac{2}{3}\cdot 0.1^4\right)$$

$$= 1.2214$$

$$U_2 = 1.2214^2 \approx 1.49181796$$

$h = 0.05$

$$U_1 = U_0\left(1 + 2\cdot 0.05 + 2\cdot 0.05^2 + \frac{4}{5}\cdot 0.05^3 + \frac{2}{3}\cdot 0.05^4\right)$$

$$\approx 1.105170833$$

$$U_2 = U_1^2 \approx 1.221402571$$

$h = 0.001$

$$U_1 = U_0\left(1 + 2\cdot 0.001 + 2\cdot 0.001^2 + \frac{4}{3}\cdot 0.001^3 + \frac{2}{3}\cdot 0.001^4\right)$$

$$\approx 1.002002001$$

$$U_2 = U_1^2 \approx 1.004008011$$

```
%oneStepRungeOrderFour
%Matt Zeller
%PHYS 428
%12/3/2018

h=0.001;
n=1/h
un=zeros(n,1);
j=1;
un(1,1)=1;
for i=0:h:1
    un(j+1,1)=un(j,1)*(1+2*h+2*h^2+(4/3)*h^3+(2/3)*h^4);
    j=j+1;
end
format long
un(n+1,1)
```

1d $y' = 2y, \quad y(0) = 1 = U_0$

$$U_{n+1} = \frac{h}{2}\left(3f(U_n) - f(U_{n-1})\right) + U_n$$

$$= \frac{h}{2}\left(6U_n - 2U_{n-1}\right) + U_n$$

$f(y) = 2y, \quad y = e^{2x}$ EXACT SOLUTION IS e^2

$f(y_{n-1}) = 2y_{n-1} = 2e^2$

$$= \frac{h}{2}\left(6U_n - 2e^2\right) + U_n$$

$h = 0.1$

$$U_1 = \frac{0.1}{2}\left(6 - 2e^2\right) + 1 \approx 0.561094$$

$$U_2 = \frac{0.1}{2}\left(6 \cdot 0.561094 - 2\right) + 0.561094 \approx 0.629422$$

```
%oneStepAdamBashforth
%Matt Zeller
%PHYS 428
%12/3/2018

h=0.001;
n=1/h
un=zeros(n,1);
j=1;
un(1,1)=exp(2);
un(2,1)=1;
for i=0:h:1
    un(j+2,1)=(h/2)*(6*un(j+1,1)-2*un(j,1)) + un(j+1,1);
    j=j+1;
end
format long
un(n+1,1)
```

EULER METHOD

n	h	un	y(1) - un	y(1) - un / h
10	0.1	7.430083707	0.041027608	0.410276079
20	0.05	7.400249944	0.011193845	0.223876907
1000	0.001	7.389061015	0.000004916	0.004916205

MODIFIED EULER METHOD

n	h	un	y(1) - un	y(1) - un / h^2
10	0.1	7.304631415	0.084424684	8.44246835
20	0.05	7.366234842	0.022821257	9.128502802
1000	0.001	7.389046262	0.000009837	9.83730151

RUNGE-KUTTA FOURTH ORDER METHOD

n	h	un	y(1) - un	y(1) - un / h^3
10	0.1	7.388889242	0.000166857	0.016685727
20	0.05	7.389044767	0.000011332	0.004532622
1000	0.001	7.389056099	0.000000000	1.37046E-06

ADAM-BASHFORTH METHOD

n	h	un	y(1) - un	y(1) - un / h^3
10	0.1	2.486007995	4.903048104	490.3048104
20	0.05	4.605845655	2.783210444	1113.284178
1000	0.001	7.327186151	0.061869948	61869.94771

$\underline{2}$ $\quad U_{n+1} = U_n(2h^2 + 2h + 1)$

$U_n(h) = U_{n+1} - U_n 2h - U_n 2h^2$

$U_n(h/2) = U_{n+1} - U_n h - U_n \frac{h^2}{2}$

$$\frac{U_n(h) - 2U_n(h/2)}{-1} = U_{n+1} + U_n h^2 = R_1$$

$U_n(h) = U_n(0.1) \approx 7.304631$

$U_n(h/2) = U_n(0.05) \approx 7.366235$

$$\frac{7.304631 - 2 \cdot 7.366235}{-1} \approx 7.427838$$

$|R_1 - y(1)| \approx 0.038782 < |U_n(0.1) - y(1)| \approx 0.084425$

3a $y' = 2y, \quad y(0) = 1 = U_0$

$$U^0_{n+1} = U_n + hf(U_n)$$

$$U^{k+1}_{n+1} = U_n + \frac{h}{2}\left(f(U_n) + f(U^k_{n+1})\right)$$

$k=0, \ n=0, \ h=0.2$:

$$U^1_1 = U_0 + \frac{h}{2}(2U_0 + 2U^0_1)$$

$$U^0_1 = U_0 + h2U_0 = 1+2h$$

$$U^1_1 = 1 + \frac{h}{2}(2 + 2(1+2h)) = 1.48 = U_1$$

$n=1$:

$$U^1_2 = U_1 + \frac{h}{2}(2U_1 + 2U^0_2)$$

$$U^0_2 = U_1 + h2U_1 = 2.072$$

$$U^1_2 = 2.1904$$

3a

$n = 2$:

$$U_3^1 = U_2 + \frac{h}{2}(2U_2 + 2U_3^0)$$

$$U_3^0 = U_2 + h2U_2 = 3.06656$$

$$U_3^1 \approx 3.24179 = U_3$$

$n = 3$:

$$U_4^1 = U_3 + \frac{h}{2}(2U_3 + 2U_4^0)$$

$$U_4^0 = U_3 + h2U_3 \approx 4.53851$$

$$U_4^1 \approx 4.79785 = U_4$$

$n = 4$:

$$U_5^1 = U_4 + \frac{h}{2}(2U_4 + 2U_5^0)$$

$$U_5^0 = U_4 + h2U_4 \approx 6.71699$$

$$U_5^1 \approx 7.10092$$

3b

$u_1^1 = 1.48$ $\quad n=0, k=0$

$u_1^2 = u_0 + \frac{h}{2}(2u_0 + 2u_1^1)$ $\quad n=0, k=1$

$= 1.496 = u_1'$

$u_2^{1'} = u_1' + \frac{h}{2}(2u_1' + 2u_2^{0'})$ $\quad n=1, k=0$

$u_2^{0'} = u_1' + 2hu_1' = 2.0944$

$u_2^{1'} = 2.21408$

$u_2^2 = u_1' + \frac{h}{2}(2u_1' + 2u_2^{1'})$ $\quad n=1, k=1$

$= 2.23802 = u_2'$

$u_3^{1'} = u_2' + \frac{h}{2}(2u_2' + 2u_3^{0'})$ $\quad n=2, k=0$

$u_3^{0'} = u_2' + 2hu_2' \approx 3.13323$

$u_3^{1'} \approx 3.31227$

$u_3^2 = u_2' + \frac{h}{2}(2u_2' + 2u_3^{1'})$ $\quad n=2, k=1$

≈ 3.34808 = u3'

3b

$U_4^{1\prime} = U_3' + \frac{h}{2}(2U_3' + 2U_4^{0\prime})$ $\quad n = 3,\ k = 0$

$U_4^{0\prime} = U_3' + 2hU_3' \approx 4.68731$

$U_4^{1\prime} \approx 4.95516$

$U_4^{2\prime} = U_3' + \frac{h}{2}(2U_3' + 2U_4^{1\prime})$ $\quad n = 3,\ k = 1$

$\approx 5.00873 = U_4'$

$U_5^{1\prime} = U_4' + \frac{h}{2}(2U_4' + 2U_5^{0\prime})$ $\quad n = 4,\ k = 0$

$U_5^{0\prime} = U_4' + 2hU_4' \approx 7.01222$

$U_5^{1\prime} \approx 7.41292$

4

$$y''(t) + \sin(y(t)) = 0, \quad y(0)=1, \quad y'(0)=0$$

$$y_1 = y, \quad y_2 = y' \longrightarrow y_2' = y'', \quad y_1' = y_2$$

$$y_2'(t) = -\sin(y(t))$$

$$y_1' = y_2$$

$$y_2'(t) = -\sin(y(t)) = f(y)$$

$$k_1 = f(u_0) = f(y(0)) = -\sin(u_0)$$

$$k_2 = f(u_0 + \tfrac{h}{2}k_1) = -\sin(u_0 - \tfrac{h}{2}\sin(u_0))$$

$$k_3 = f(u_0 + \tfrac{h}{2}k_2) = -\sin(u_0 - \tfrac{h}{2}\sin(u_0 - \tfrac{h}{2}\sin(u_0)))$$

$$k_4 = f(u_0 + hk_3) = -\sin(u_0 - h\sin(u_0 - \tfrac{h}{2}\sin(u_0 - \tfrac{h}{2}\sin(u_0))))$$

4

$$v_{n+1} = v_n + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$v_1 = v_0 + \frac{h}{6}\Big(-\sin(v_0) - 2\sin(v_0 - \tfrac{h}{2}\sin(v_0)) - \cdots$$

$$\cdots - 2\sin(v_0 - \tfrac{h}{2}(\sin(v_0 - \tfrac{h}{2}(\sin v_0)))) - \cdots$$

$$\cdots - 2\sin(v_0 - \tfrac{h}{2}(\sin(v_0 - \tfrac{h}{2}\sin(v_0 - \tfrac{h}{2}\sin v_0)))) - \cdots$$

$$\cdots - \sin(v_0 - \tfrac{h}{2}\sin(v_0 - \tfrac{h}{2}\sin(v_0 - \tfrac{h}{2}\sin(v_0 - \tfrac{h}{2}\sin v_0)))))$$

$$= 0$$

$$y'(1) \approx v_{n+1}(1) \approx 0.94956$$

4

$y_1' = y_2$

$k_1 = U_n$

$k_2 = \left(1 + \frac{h}{2}\right) U_n$

$k_3 = \left(1 + \frac{h}{2} + \frac{h^2}{4}\right) U_n$

$k_4 = \left(1 + h + \frac{h^2}{2} + \frac{h^3}{4}\right) U_n$

$U_{n+1} = \left(1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24}\right) U_n$

$y(1) \approx U_{n+1}(1) \approx 1.64844$

```
%y2
%Matt Zeller
%PHYS 428
%12/5/2018

%u2 is solution to y'(t), v2 is solution to y(t)
function y2(u1,v1)
format long
u2=u1+(0.5/6)*((  -sin(u1)  )  -  2*(  sin(u1 - 2*(  sin(u1)  )))  -  2*(  sin(u1 - 2*(  ( ↙
sin(u1 - 2*sin(u1))  )))  )  -  (  (sin(u1 - 2*(  sin(u1 - 2*(sin(u1 - 2*sin(u1)))))))  ))
v2=v1*((0.5^4)/24+(0.5^3)/6+(0.5^2)/2+0.5+1)
end
```

```
>> y2(1,1)

u2 =

   0.949559376650841


v2 =

   1.648437500000000
```

5a

FROM MATLAB PROGRAM 'pmeth':

$\boxed{24}$ ITERATIONS NEEDED FOR CONVERGENCE OF EIGENVECTOR

$n_L = 22$ IS NUMBER OF ITERATIONS FOR EIGENVALUE CONVERGENCE

$n_c > n_L$ WHERE n_c IS NUMBER OF ITERATIONS FOR EIGENVECTOR

CONVERGENCE TOLERANCE VALUES ARE

$T_c \approx 3.1690 \times 10^{-5}$ $T_L \approx 1.6368 \times 10^{-5}$ WHERE T_c, T_L ARE FROM THE EIGENVECTOR CASE AND

$'T_c \approx 8.8066 \times 10^{-5}$ $'T_L \approx 4.5486 \times 10^{-5}$ $'T_c, 'T_L$ ARE FROM THE EIGENVALUE CASE

```matlab
%pmeth
%Matt Zeller
%12/3/2018
%PHYS 428
%This program finds eigenvector and dominant eigenvalue of matrix A
%v2 is the leading eigenvector, v1 follows, and so on
%tolVec is convergence tolerance for the vector, tolVal--the value
%c2 is leading 'convergence' as defined on page 280 of A Friendly Introduction to
%Numerical Analysis--it is  "an estimate for the asymptotic rate of linear convergence" of the sequence towards the dominant eigenvalue
%c1 follows c2
A = [1 4 5; 4 -3 0; 5 0 7];
v00 = ones(3,1);
v0 = ones(3,1);
v1 = ones(3,1);
v2 = ones(3,1);
v1 = (1/sqrt(3))*v1;
tolVec = 1;
tolVal = 1;
n = 0;
format long
%change tolVec to tolVal to evaluate convergence of the eigenvalue
while tolVec> 5*10^-5
   n=n+1;
   v2 = A*v1;
   c2 = (v2(3,1)-v1(3,1))/(v1(3,1)-v0(3,1));
   c1 = (v1(3,1)-v0(3,1))/(v0(3,1)-v00(3,1));
   tolVec = abs(c2-c1);
   tolVal = abs((v2(3,1)/v1(3,1))-(v0(3,1)/v00(3,1)));
   v00 = v0;
   v0 = v1;
   v1 = v2;
end
domEig = v2(3,1)/v1(3,1)
n
```

```
tolVec
tolVal
v2
```

```
domEig =

     1


n =

    24


tolVec =

     3.169036606109899e-05


tolVal =

     1.636784771186228e-05


v2 =

   1.0e+24 *

   0.662237996490559
   0.201428976555814
   1.050874143638659
```

*results for the case when eigenvalue convergence is used for convergence tolerance are listed in HW problem 5a. Alternatively, results can be reproduced by pmeth.m.

```
%inversepmethod
%Matt Zeller
%12/3/2018
%PHYS 428

%This program finds dominant eigenvalue of matrix A using
%inverse power method
A = [1 4 5; 4 -3 0; 5 0 7];
Ainv = inv(A);
v1 = (1/sqrt(3))*ones(3,1);
v2 = ones(3,1);
format long
disp(['n','    ','Estimate at n','      ','Reciprocal'])
disp(' ')
for n=1:10
   v2 = Ainv*v1;
   en = norm(v2,inf);
   disp([num2str(n),'    ',num2str(en),'                   ',num2str(1/en)])
   v2 = v2 / norm(v2,inf);
   v1 = v2;
end
disp(' ')
disp(' ')
disp(['The approximate eigenvalue of A nearest to q=1 is ',num2str(1/en),])
```

5b

n	Estimate at n	Reciprocal
1	0.33845	2.9547
2	0.88235	1.1333
3	1.0414	0.96026
4	1.0684	0.93595
5	1.0652	0.9388
6	1.0658	0.93828
7	1.0657	0.93835
8	1.0657	0.93834
9	1.0657	0.93834
10	1.0657	0.93834

The approximate eigenvalue of A nearest to q=1 is 0.93834

5c

RALEIGH QUOTIENT ITERATION METHOD CONVERGES AROUND $n = 10$, WHEREAS THE OTHER TWO METHODS CONVERGE AROUND $n = 20$

```
%inversepmethod
%Matt Zeller
%12/3/2018
%PHYS 428

%This program finds dominant eigenvalue of matrix A using
%Rayleigh Quotient
A = [1 4 5; 4 -3 0; 5 0 7];
v1 = (1/sqrt(3))*ones(3,1);
v2 = ones(3,1);
S = v1'*A/v1'*v1;
As = A-eye(3)*S;

disp(['n','    ','Estimate at n','                              ','Reciprocal'])
disp(' ')
format long
for n=1:20
   v2 = As'*v1;
   en = norm(v2,inf);
   disp([num2str(n),'    ',num2str(en,'%1.10e'),'                   ',num2str(1/en,'%↙
1.10e')])
   v2 = v2 / norm(v2,inf);
   v1 = v2;
end
disp(' ')
disp(' ')
disp(['The approximate dominant eigenvalue of A is ',num2str(1/en),])
disp(['The associated eigenvector is '])
disp(' ')
disp(v2)
```

```
>> raleigh

S =

   4.426352063787133
   4.426352063787133
   4.426352063787133

n    Estimate at n                 Reciprocal

1    7.0893163975e+00              1.4105732400e-01
2    8.0012810548e+00              1.2497998672e-01
3    9.7102721492e+00              1.0298372534e-01
4    8.8451560967e+00              1.1305622977e-01
5    9.2406199623e+00              1.0821784730e-01
6    9.0503974112e+00              1.1049238553e-01
7    9.1397218001e+00              1.0941252063e-01
8    9.0972759654e+00              1.0992301474e-01
9    9.1173272199e+00              1.0968126688e-01
10    9.1078262667e+00              1.0979568239e-01
11    9.1123207990e+00              1.0974152711e-01
12    9.1101926319e+00              1.0976716305e-01
13    9.1111997506e+00              1.0975502978e-01
14    9.1107229720e+00              1.0976077344e-01
15    9.1109486247e+00              1.0975805497e-01
16    9.1108418062e+00              1.0975934181e-01
17    9.1108923642e+00              1.0975873274e-01
18    9.1108684321e+00              1.0975902105e-01
19    9.1108797596e+00              1.0975888458e-01
20    9.1108743977e+00              1.0975894918e-01


The approximate dominant eigenvalue of A is 0.10976
The associated eigenvector is
```

0.036956950753800
1.000000000000000
0.377007561885549

6

$$P_n(x) = \sum_{i=0}^{n} L_{ni}(x) f_i(x)$$

$$P_2(t) = \sum_{i=0}^{2} L_{2i}(t) f_i(t)$$

$$= L_{20}(t) f_0(t) + L_{21}(t) f_1(t) + L_{22}(t) f_2(t)$$

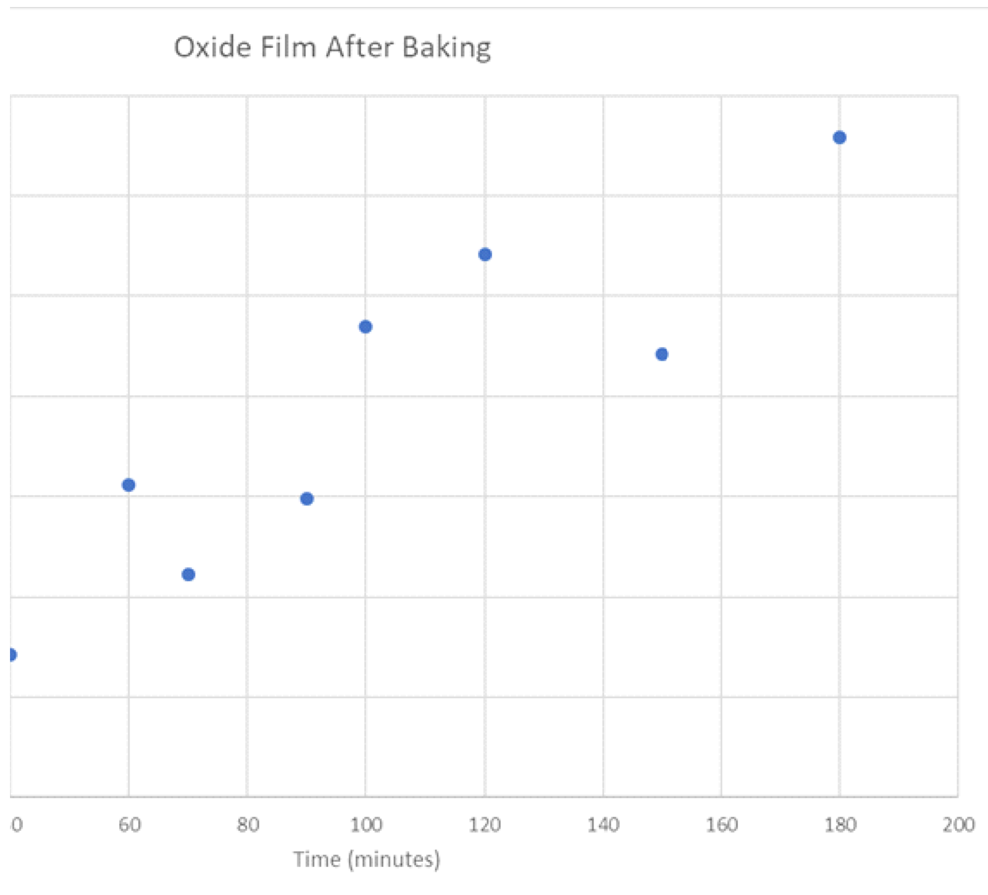
$$= L_{20}(t) f(t_0, y_0) + L_{21}(t) f(t_1, y_1) + L_{22}(t) f(t_2, y_2)$$

$$\rightarrow \int_{t_1}^{t_2} P_2(t)\,dt = f(t_0, y_0) \int_{t_1}^{t_2} L_{20}(t)\,dt + f(t_1, y_1) \int_{t_1}^{t_2} L_{21}(t)\,dt + \cdots$$

$$\cdots + f(t_2, y_2) \int_{t_1}^{t_2} L_{22}(t)\,dt$$

HERE I AM STUCK,
INTEGRALS RESULT
IN BIZZARE
TERMS

8a



Oxide Film After Baking
0
60
80
100
120
140
160
180
200
Time (minutes)

8b

$$A = \begin{bmatrix} 1 & 20 \\ 1 & 30 \\ 1 & 40 \\ 1 & 60 \\ 1 & 70 \\ 1 & 90 \\ 1 & 100 \\ 1 & 120 \\ 1 & 150 \\ 1 & 180 \end{bmatrix}, \vec{b} = \begin{bmatrix} 3.5 \\ 7.4 \\ 7.1 \\ 15.6 \\ 11.1 \\ 14.9 \\ 23.5 \\ 27.1 \\ 22.1 \\ 32.9 \end{bmatrix}, \vec{z} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

$$y = a_0 + a_1 x$$

$$A\vec{z} = \vec{b} \longrightarrow \cdots$$

$$A^T A \vec{z} = A^T \vec{b}$$

$$A^T A = \begin{bmatrix} 10 & 860 \\ 860 & 98800 \end{bmatrix} \qquad A^T \vec{b} = \begin{bmatrix} 165.20 \\ 1946.9 \end{bmatrix}$$

$$\cdots \longrightarrow \vec{z} = (A^T A)^{-1} A^T \vec{b} = \begin{bmatrix} 1.7650 \\ 0.17157 \end{bmatrix}$$

$$\boxed{y = 1.7650 + 0.17157x}$$

EVERYTHING COMPUTED IN MATLAB

8b

a_0 IS A PREDICTION OF OXIDE THICKNESS BEFORE ANY BAKING,
a_1 IS AN ESTIMATE OF THE RATE OF OXIDE GROWTH AND HAS THE DIMENSION OF DISTANCE PER TIME

8c

$$1.7650 + (45)\,0.1757 \approx 9.6\,\text{Å}$$