Math/Engr/Phys 428 and Math 529/Phys 528: Final Exam

Fall 2018

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Due by Thursday, December 13, 2018

NAME: ______________________________

For each problem clearly $\boxed{\textbf{BOX}}$ your final answer. If you need additional space, continue your work on the back of the page or extra sheet at the end of the exam. There is a **crib sheet** attached at the end.

Problem	**Points Possible**	**Points Earned**
1	25	
2	15	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total	100	
Bonus/Math 529/Phys 528	5	

1. [**True/False, 25 pts**]. Indicate whether the following statements are true or false? Justify your answer for full credit, prove/show if true and give a counterexample if false.

   (a) [5 pts] The purpose of pivoting in Gaussian elimination is to reduce the operation count.

   (b) [5 pts] The polynomial $p_6(x)$ of degree ≤ 6 that interpolates the function $f(x) = 4x^3 - 3x^2 + 2.5x - \pi$ at the points $x = -3, -2, -1, 0, 1, 2, 3$ is the function $f(x)$ itself, i.e., $p_6(x) = f(x)$.

(c) [5 pts] Let $f(x)$ be a continuous function on the interval $[0,1]$ and let $p_n(x)$ be the polynomial of degree $\leq n$ that interpolates $f(x)$ at the $n+1$ distinct points $x_i = i/n$, $i = 0, \ldots, n$. Then $p_n(x)$ converges to $f(x)$ as $n \to \infty$ for every $x \in [0,1]$.

(d) [5 pts] Given $n+1$ distinct points $\{x_0, \ldots, x_n\}$ and $n+1$ data values $\{f_0, \ldots, f_n\}$, there is a unique cubic spline $S(x)$ which interpolates the data, i.e. such that $S(x_i) = f_i$, $i = 0, \ldots n$.

(e) [5 pts] Power method can be used to approximate any eigenvalue-eigenvector pair with eigenvectors converging faster than eigenvalues.

2. **[Fixed Point Iterations, 15 pts].** Let $g(x) = -x^2 + 3ax + a - 2a^2$, where a is a parameter.

   (a) Show that a is a fixed point of $g(x)$.

   (b) For what values of a does the iteration scheme $x_{n+1} = g(x_n)$ converge *linearly* to the fixed point a (provided x_0 is chosen sufficiently close to a)?

   (c) Is there a value of a for which convergence is quadratic?

3. [**Iterative methods for linear systems, 10 pts**] Consider the system of linear equations,

$$\begin{array}{rcrcr} 2x_1 & + & x_2 & = & 1 \\ x_1 & + & 2x_2 & = & -1 \end{array}$$

Starting from the initial guess $(x_1, x_2)_0 = (0, 0)$, perform one step of Gauss-Seidel iteration.

4. [**Newton's Method, 10 pts**] Consider the system of two nonlinear equations

$$2x - \cos y = 0 \qquad\qquad 2y - \sin x = 0.$$

(a) Write down the vector-valued function $\mathbf{F}(x, y) = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}$ whose zero we seek.

(b) Calculate the Jacobian of $\mathbf{F}$.

(c) Apply one step of Newton's Method to initial iterate $\mathbf{p}_0 = [0\ 0]^T$ to obtain the next iterate $\mathbf{p}_1$.

5. [**Splines, 10 pts**]. Show that the following function is a natural cubic spline with nodes $\{-1, 0, 1\}$.

$$S(x) = \begin{cases} S_0(x) = (x+1)\big(3 - (x+1)^2\big), & -1 \le x \le 0, \\ S_1(x) = (1-x)\big(3 - (1-x)^2\big), & 0 \le x \le 1 \end{cases}$$

6. [**Numerical Integration, 10 pts**]**.** The following data is known to lie on a cubic curve. Evaluate $\int_{-4}^{4} f(x)dx$ <u>exactly</u> and explain how you are sure that your answer is correct.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-9	0	2	0	-3	-4	0	12	35

7. [**Numerical Methods for Solving ODEs, 10 pts**].

Consider the initial value problem,

$$y' = -y, \quad y(0) = 1.$$

Find an approximation to $y(2)$ with a step size $h = 1$ by using modified Euler's method.

8. **[Richardson Extrapolation, 10 pts].** Let f be a function with four continuous derivatives, and fix a point x. Recall the central difference formula

$$D(h) = \frac{f(x+h) - f(x-h)}{2h},$$

which satisfies an error estimate of the form

$$f'(x) = D(h) + Ch^2 + O(h^4)$$

where C is an (unknown) constant.

(a) Show how to use the two estimates $D(h)$ and $D(h/3)$ to obtain a new estimate $R_1(h)$ for $f'(x)$ that has error $O(h^4)$.

(b) Use the central difference approximation $D(h)$ to estimate $f'(0)$ for $f(x) = \ln(1+x)$ with $h = 0.3$ and $h = 0.1$.

(c) Perform one step of Richardson extrapolation on the values obtained in part (b) to get a new estimate R_1.

Bonus Problem /Additional Problem for Math 529/Phys 528 students: Stability. [**5 pts**]

Suppose that Euler's method is used to solve the initial value problem $y' = Ay$, $y(0) = y_0$ where

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

It is known that $y(t) \to 0$ as $t \to \infty$ for any initial vector y_0. What restriction on the step-size h is needed to ensure that the numerical solution u_n remains bounded for all $n \geq 0$ and all initial vectors?

Number Systems

Base β

$$x = \pm(.a_1 a_2 a_3 \ldots a_i \ldots)_\beta \beta^e, \;\; 1 \le a_i < \beta$$

Chopping

$$\widetilde{x} = \pm(.a_1 a_2 a_3 \ldots a_i)_\beta \beta^e$$

Rounding

$$\widetilde{x} = \begin{cases} \pm(.a_1 \ldots a_i)_\beta \beta^e, & a_{i+1} < \frac{\beta}{2} \\ \pm[(.a_1 \ldots a_i)_\beta \beta^e + (.0 \ldots 1)_\beta \beta^e], & a_{i+1} \ge \frac{\beta}{2} \end{cases}$$

Error

Error: $e(\tilde{x}) = |x - \tilde{x}|$

Relative Error: $re(\tilde{x}) = \left|\frac{x-\tilde{x}}{x}\right|$

Linear Systems, $Ax = b$

THM: Given a matrix A, the following are equivalent

1. The Equation $Ax = b$ has a unique solution
2. A is invertible.
3. $\det(A) \neq 0$
4. $Ax = 0$ has a unique solution, $x = 0$
5. The columns of A are linearly independent
6. The eignevalues, λ, of A are non-zero.

Gaussian Elimination: $A = LU$
Gaussian Elimination with pivoting: $PA = LU$

Norms

Properties of Vector Norms:

$\|x\| \ge 0, \; \|x\| = 0 \Rightarrow x = 0$

$\|\lambda x\| = |\lambda| \; \|x\|, \quad \lambda$ scalar

$\|x + y\| \le \|x\| \; + \; \|y\|$

Vector Norms:

l_∞ : $\|x\|_\infty = \max_{1 \le i \le n} |x_i|$

l_1 : $\|x\|_1 = \sum_{i=1}^n |x_i|$

l_2 : $\|x\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$

Matrix Norm:

$$\|A\| = \max_{\|u\| \neq 0} \{\|Au\|/\|u\| : u \in \mathbf{R^n}\}$$

Properties of Matrix Norms:

$\|A\| \ge 0, \; \|A\| = 0 \Leftrightarrow A = 0$

$\|\lambda A\| = |\lambda| \; \|A\|$

$\|A + B\| \le \|A\| + \|B\|$

$\|Ax\| \le \|A\| \;\; \|x\|$

$\|AB\| \le \|A\| \;\; \|B\|$

Examples of Matrix Norms:

l_∞ Matrix Norm: $\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^n |a_{i,j}|$

l_1 Matrix Norm: $\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^n |a_{i,j}|$

l_2 Matrix Norm: $\|A\|_2 = \sqrt{\rho(A^* A)}$

Stability

Condition Number: $\kappa(A) = \|A^{-1}\| \; \|A\|$
Residual: $r = b - A\widetilde{x}$
THM:

$$\frac{1}{\kappa(A)} \frac{\|r\|}{\|b\|} \le \frac{\|e\|}{\|x\|} \le \kappa(A) \frac{\|r\|}{\|b\|}$$

Iterative Methods

$A = (L + D + U)$ where D is a diagonal matrix,L is lower triangular and U is upper triangular.

Jacobi: $Dx^{n+1} = -(L + U)x^n + b$

Gauss-Seidel: $Dx^{n+1} = -(Lx^{n+1} + Ux^n) + b$

SOR: $(D + \omega L)x^{n+1} = ((1 - \omega)D - \omega U)x^n + \omega b$

Root Finding Methods

Newton's Methods: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Secant Methods: $x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$

Error Bound for Bisection Method:

$$|\alpha - x_n| \le \left(\frac{1}{2}\right)^n |b_0 - a_0|$$

Polynomial Interpolation

Let f be defined on $[a, b]$; $x_0, x_1, \ldots, x_n$: $n+1$ distinct points in $[a, b]$. Let p_n be the interpolating polynomial of degree $\leq n$. Then

$$f(x) = p_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)\ldots(x - x_n)$$

for some $\xi \in [a, b]$.

Chebyshev Points

$$x_k = \cos(2k+1)\pi 2n,\ k = 0, 1, \ldots, n-1$$

or

$$x_k = -\cos \pi k n,\ k = 0, 1, \ldots, n$$

Hermite Interpolation

Given f, $x_0, x_1, \ldots, x_n$: $n+1$ distinct points, the Hermite interpolating polynomial $p(x)$ ($\deg p \leq 2n+1$) is

$$p(x) = \sum_{i=0}^{n} \left(f(x_i)h_i(x) + f'(x_i)\tilde{h}_i(x) \right)$$

where

$$h_i(x) = (1 - 2(x - x_i)l_i')\, l_i^2(x), \quad \tilde{h}_i(x) = (x - x_i)l_i^2(x)$$

If $f \in C^{(2n+2)}[a, b]$, $p(x)$ is the Hermite interpolating polynomial, then

$$f(x) = p(x) + \frac{f^{(2n+2)}(\xi)}{(2n+2)!}(x - x_0)^2 \ldots (x - x_n)^2$$

Splines

Let $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$. A spline of degree m is a function $S(x)$ which satisfies the following conditions:

1) For $x \in [x_i, x_{i+1}]$, $S(x) = S_i(x)$: polynomial of degree $\leq m$

2) $S^{(m-1)}(x)$ exists and is continuous at the interior points $x_1, \ldots, x_{n-1}$, i.e. $\lim_{x \to x_i^-} S_{i-1}^{(m-1)}(x) = \lim_{x \to x_i^+} S_i^{(m-1)}(x)$

Let f be defined on $[a, b]$, $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ and let S be the natural cubic spline interpolant of f. Then

$$1)\ |f(x) - S(x)| \leq \frac{5}{384} \max_{a \leq x \leq b} |f^{(4)}(x)| h^4$$

where $h = \max_i |x_{i+1} - x_i|$

$$\int_a^b (S''(x))^2\, dx \leq \int_a^b (f''(x))^2\, dx$$

Numerical Integration

$$\int_a^b f(x)dx \sim \sum_{i=0}^{n} c_i f(x_i)$$

Trapezoid Rule:

$$T(h) = h\left(\frac{1}{2}f(x_0) + f(x_1) + \cdots + f(x_{n-1}) + \frac{1}{2}f(x_n) \right)$$

Local Error Estimate:

$$\int_a^{a+h} f(x)dx = h\frac{f(a) + f(a+h)}{2} - \frac{h^3}{12}f''(\xi)$$

Global Error Estimate:

$$\int_a^b f(x)dx = T(h) - \frac{f''(\xi)}{12}h^2(b - a)$$

Simpson's Rule:

$$S(h) = h\left(\frac{1}{3}f(x_0) + \frac{4}{3}f(x_1) + \frac{2}{3}f(x_2) + \cdots \right.$$

$$\left. \cdots + \frac{2}{3}f(x_{n-2}) + \frac{4}{3}f(x_{n-1}) + \frac{1}{3}f(x_n) \right)$$

Error:

$$\int_a^b f(x)dx = S(h) - \frac{f^{(4)}(\xi)}{180}h^4(b - a)$$

Orthogonal Polynomials:

The *inner product* of two functions f and g on $[a, b]$ with the weighting function $w(x)$ is

$$\langle f, g \rangle = \int_a^b f(x)g(x)w(x)dx$$

Properties:

$$1)\ \langle f, f \rangle \geq 0, \langle f, f \rangle = ||f||^2 = 0 \Leftrightarrow f = 0$$

$$2)\ \langle f, \alpha g + h \rangle = \alpha \langle f, g \rangle + \langle f, h \rangle$$

Gaussian Quadrature:

$$\int_{-1}^{1} f(x)dx \sim \sum_{i=1}^{n} c_i f(x_i)$$

where x_i, $i = 1, \ldots, n$ are roots of Legendre polynomial $P_n(x)$.

IVP for ODEs:

$$y' = f(t, y),\ y(a) = \alpha,\ a \leq t \leq b$$

Euler's Method:

$$u_{n+1} = u_n + hf(t_n, u_n),\ u_0 = \alpha$$

Local Truncation Error: $\tau_n = \frac{h^2}{2}y''(\tilde{t}_n)$

Global Error:

$$|y_n - u_n| \leq \frac{hM}{2L}\left(e^{L(t_n - a)} - 1 \right)$$

where L is a Lipschitz constant, $M = \max|y''(t)|$.

Modified Euler's Method:

$$k_1 = f(t_n, u_n),\ k_2 = f(t_n + h, u_n + hk_1)$$

$$u_{n+1} = u_n + \frac{h}{2}(k_1 + k_2)$$

4th Order Runge-Kutta Method:

$$k_1 = f(t_n, u_n),\ k_2 = f(t_n + \frac{h}{2}, u_n + \frac{h}{2}k_1)$$

$$k_3 = f(t_n + \frac{h}{2}, u_n + \frac{h}{2}k_2),\ k_4 = f(t_n + h, u_n + hk_3)$$

$$u_{n+1} = u_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Backward Euler's Method:

$$u_{n+1} = u_n + hf(u_{n+1})$$

System of ODEs:

$$y' = Ay,\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Exact Solution:

$$y(t) = \alpha_1(0)\, \mathrm{e}^{\lambda_1 t}\, p_1 + \alpha_2(0)\, \mathrm{e}^{\lambda_2 t}\, p_2$$

Forward Euler:

$$u_n = \alpha_1(0)\,(1 + h\lambda_1)^n\, p_1 + \alpha_2(0)\,(1 + h\lambda_2)^n\, p_2$$

Backward Euler:

$$u_n = \alpha_1(0)\left(\frac{1}{1 - h\lambda_1}\right)^n p_1 + \alpha_2(0)\left(\frac{1}{1 - h\lambda_2}\right)^n p_2$$

Multistep Methods:

General 2-Step Method:

$$\alpha_0 u_{n+1} + \alpha_1 u_n + \alpha_2 u_{n-1} = h\left[\beta_0 f(u_{n+1}) + \beta_1 f(u_n) + \beta_2 f(u_{n-1})\right]$$

Adams-Bashforth:

$$u_{n+1} = u_n + \frac{h}{2}\left[3f(u_n) - f(u_{n-1})\right]$$

Adams-Moulton

$$u_{n+1} = u_n + \frac{h}{12}\left[5f(u_{n+1}) + 8f(u_n) - f(u_{n-1})\right]$$

Leap-Frog

$$u_{n+1} = u_{n-1} + 2hf(u_n)$$

BDF: Backward Differentiation Formula — Gear's Method

$$\frac{3}{2}u_{n+1} - 2u_n + \frac{1}{2}u_{n-1} = hf(u_n) \qquad 15$$

Computing eigenvalues and eigenvectors

$$Ax = \lambda x, \quad x \neq 0$$

$\lambda :$ eigenvalue

$x :$ associated eigenvector

Power method:

Idea: v, Av, A^2v, $\ldots$

1. $v^{(0)}$: given, $||v^{(0)}||_2 = 1$
2. for $k = 1, 2, \ldots$
3. $\quad w = Av^{(k-1)}$
4. $\quad v^{(k)} = w/||w||_2$
5. $\quad \lambda^{(k)} = \left(v^{(k)}\right)^T Av^{(k)}$

Inverse iteration:

Idea: apply power method to A^{-1}, $(A-\mu I)^{-1}$, μ: shift

1. $v^{(0)}$: given, $||v^{(0)}||_2 = 1$
2. for $k = 1, 2, \ldots$
3. $\quad$ solve $(A - \mu I)w = v^{(k-1)}$
4. $\quad v^{(k)} = w/||w||_2$
5. $\quad \lambda^{(k)} = \left(v^{(k)}\right)^T Av^{(k)}$

Rayleigh quotient iteration:

Idea: update μ

1. $v^{(0)}$: given, $||v^{(0)}||_2 = 1$, $\lambda^{(0)} = \left(v^{(0)}\right)^T Av^{(0)}$
2. for $k = 1, 2, \ldots$
3. $\quad$ solve $(A - \lambda^{(k-1)} I)w = v^{(k-1)}$
4. $\quad v^{(k)} = w/||w||_2$
5. $\quad \lambda^{(k)} = \left(v^{(k)}\right)^T Av^{(k)}$

Least Squares:

$A\vec{z} = \vec{b}$: $m \times n$ system with $m \geq n$

$A^T A\vec{z} = A^T\vec{b}$: normal equations