Homework 2

Matt Zeller

June 25, 2018

Phys 428

1a

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

$$f(x) = \sin \frac{\pi}{2} x, \qquad a = 0$$

$$f(a) = \sin 0 = 0$$

$$f'(a) = \frac{\pi}{2} \cos 0 = \frac{\pi}{2}$$

$$f''(a) = -\frac{\pi^2}{4} \sin 0 = 0$$

$$f'''(a) = -\frac{\pi^3}{16} \cos 0 = -\frac{\pi^3}{16}$$

$$f^{(4)}(a) = 0$$

$$f^{(5)}(a) = \frac{\pi^5}{64} \cos 0 = \frac{\pi^5}{64}$$

$$\vdots$$

$$\vdots$$

$$f^{(6)}(a) = 0$$

$$f^{(7)}(a) = \frac{\pi^7}{256} \cos 0 = \frac{\pi^7}{256}$$

1.a $$\boxed{\sin\frac{\pi}{2}x = \frac{\pi}{2}x - \frac{\pi^3}{8}\frac{x^3}{3!} + \frac{\pi^5}{32}\frac{x^5}{5!} - \frac{\pi^7}{256}\frac{x^7}{7!} + \cdots}$$

1.b $$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

$$R_7(x) = \frac{f^{(8)}(\xi)}{8!}(x-a)^8 \quad , \quad f^{(8)}(x) = -\frac{\pi^8}{512}\sin\left(\frac{\pi}{2}x\right)$$

$$\boxed{R_7(x) = -\frac{\pi^8}{512}\frac{\sin\left(\frac{\pi}{2}\xi\right)}{8!}x^8}$$

1.c WE WANT $R_n(x) \leq 10^{-5}$

SAY $x = 0.1$, $\xi = 0.001$

THEN $R_7(x) = -\frac{\pi^8}{512}\frac{\sin\left(\frac{\pi}{2}0.001\right)}{8!}(0.1)^8 \approx 7.220 \times 10^{-15} < 10^{-5}$

SO $n = 7$ WOULD BE ACCURATE ENOUGH

2.a $x_{n+1} = -16 + 6x_n + \frac{12}{x_n}$, $\alpha = 2$

$g(x) = -16 + 6x + \frac{12}{x_n}$

I g IS CONTINUOUS ON $[1, 3]$

II $g : [1, 3] \longrightarrow [1, 3]$

III $|g'(x)| = \left|6 - \frac{12}{x^2}\right|$

FOR CONVERGENCE TO α w/ $P_0 \in [1, 3]$

$|g'(1)| < 1$

BUT

$|g'(1)| = |6 - 12| = 6 > 1$

SO $g(x)$ DOES NOT CONVERGE FOR $\forall x \in [1, 3]$

2.b $x_{n+1} = \frac{2}{3}x_n + \frac{1}{x_n^2}$, $\alpha = 3^{1/3}$

$$g'(x) = \frac{2}{3} - \frac{2}{x^3}$$

$$|g'(\alpha)| = \frac{2}{3} - \frac{2}{3} = 0 \longrightarrow \text{CONVERGENCE}$$

$$|g''(\alpha)| = \left|\frac{6}{3^{4/3}}\right| \approx 1.387 > 0 \longrightarrow \text{QUADRATIC CONVERGENCE}$$

2.c $x_{n+1} = \dfrac{12}{1 + x_n}$, $\alpha = 3$

$$g'(x) = \frac{d\,g(x)}{dx} = -\frac{12}{(1+x)^2}$$

$$|g'(\alpha)| = \frac{12}{16} \left\{ \begin{matrix} < 1 \\ \neq 0 \end{matrix} \right\} \text{LINEAR CONVERGENCE}$$

6

$$\begin{bmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftarrow \frac{1}{a}R_1} \begin{bmatrix} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftarrow R_2 - cR_1} \begin{bmatrix} 1 & b/a & 1/a & 0 \\ 0 & d-\frac{cb}{a} & -\frac{c}{a} & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftarrow \frac{1}{d-\frac{cb}{a}}R_2} \begin{bmatrix} 1 & b/a & 1/a & 0 \\ 0 & 1 & -\frac{cb}{a}\frac{1}{d-\frac{cb}{a}} & \frac{1}{d-\frac{cb}{a}} \end{bmatrix}$$

$$\left\{ -\frac{cb}{a}\frac{1}{d-\frac{cb}{a}} = -\frac{cb}{ad-\frac{abc}{a}} = \frac{cb}{ad-bc} \right\}$$

$$\begin{bmatrix} 1 & b/a & 1/a & 0 \\ 0 & 1 & \frac{cb}{ad-bc} & \frac{1}{d-\frac{cb}{a}} \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - \frac{b}{a}R_2} \begin{bmatrix} 1 & 0 & \frac{1}{a}-\frac{b}{a}\frac{cb}{ad-bc} & -\frac{b}{a}\left(d-\frac{cb}{a}\right)^{-1} \\ 0 & 1 & \frac{cb}{ad-bc} & \left(d-\frac{cb}{a}\right)^{-1} \end{bmatrix}$$

6.2

FOR $Ax_1 = e_1$ & $Ax_2 = e_2$ ...

$$\ldots \boxed{x_1 = \begin{bmatrix} \frac{1}{a}\left(1 - \frac{1}{ad-bc}\right) \\ \frac{cb}{ad-bc} \end{bmatrix}, \quad x_2 = \begin{bmatrix} -\frac{b}{ad-bc} \\ \frac{1}{d - \frac{cb}{a}} \end{bmatrix}}$$

IF $\det(A) = 0$ THEN $ad - bc = 0$, AND A^{-1} CONTAINS TERMS DIVIDED BY $ad-bc$, SO A^{-1} CANNOT EXIST IF $\det(A) = 0$, ... FOR A^{-1} TO EXIST

$\det(A) \neq 0$

7

$$EA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/4 & 1 & 0 & 0 \\ 0 & -4/15 & 1 & 0 \\ 0 & 0 & -15/56 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 15/4 & 1 & 0 \\ 0 & 0 & 56/15 & 1 \\ 0 & 0 & 0 & 209/56 \end{bmatrix} = U$$

$$A = E^{-1}U = LU$$

$$Ly = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 0 & 4/15 & 1 & 0 \\ 0 & 0 & 15/56 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 3 \\ 2 \end{bmatrix} = b \quad \left\{ \begin{array}{l} y_1 = 2 \\ \frac{1}{4}y_1 + y_2 = -3 \\ \frac{4}{15}y_2 + y_3 = 3 \\ \frac{15}{56}y_3 + y_4 = -2 \end{array} \right.$$

$$y_2 = -3 - \frac{1}{2} = -3.5$$

$$y_3 = 3 - \frac{4}{15}(-3.5) \cong 3.933$$

$$y_4 \cong -2 - \frac{15}{56}(3.933) \cong -3.053$$

7.2

$$Ux = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 15/4 & 1 & 0 \\ 0 & 0 & 56/15 & 1 \\ 0 & 0 & 0 & 209/56 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cong \begin{bmatrix} 2 \\ -3.5 \\ 3.933 \\ -3.053 \end{bmatrix}$$

$$4x_1 + x_2 = 2$$

$$\frac{15}{4}x_2 + x_3 = -3.5$$

$$\frac{56}{15}x_3 + x_4 \cong 3.933$$

$$\frac{209}{56}x_4 \cong -3.053$$

$$x_4 \cong 6.785$$

$$x_3 \cong \frac{3.933 - 6.785}{56/15} \cong -0.7639$$

$$x_2 \cong \frac{-3.5 + 0.7639}{15/4} \cong -0.7296$$

$$x_1 \cong \frac{2 + 0.7296}{4} \cong 0.6824$$

$$\boxed{x \cong \begin{bmatrix} 0.6824 \\ -0.7296 \\ -0.7639 \\ 6.785 \end{bmatrix}}$$

9.1 $\quad Ly = f$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_2 & 1 & 0 \\ 0 & l_3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

$$y_1 = f_1$$

$$l_2 y_1 + y_2 = f_2 \qquad y_2 = f_2 - l_2 y_1$$

$$l_3 y_2 + y_3 = f_3 \qquad y_3 = f_3 - l_3 y_2$$

n = 3,
2 sub's
2 mult's

9.1

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ l_2 & 1 & 0 & 0 \\ 0 & l_3 & 1 & 0 \\ 0 & 0 & l_4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$

$y_1 = f_1$

$l_2 y_1 + y_2 = f_2$ $\qquad y_2 = f_2 - l_2 y_1$

$l_3 y_2 + y_3 = f_3$ $\qquad \vdots$

$l_4 y_3 + y_4 = f_4$

$n = 4$,
3 SUB.'S,
3 MULT.'S,

$\longrightarrow$ $n-1$ SUB.'S, $n-1$ MULT.'S } FOR FORWARD ELIM., THE OPERATION COUNT IS $O(n)$

9.2

$$Uu = y$$

$$\begin{bmatrix} v_1 & c_1 & 0 \\ 0 & v_2 & c_2 \\ 0 & 0 & v_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$v_3 u_3 = y_3$ $\qquad$ $u_3 = \dfrac{y_3}{v_3}$

$v_2 u_2 + c_2 u_3 = y_2$ $\qquad$ $u_2 = \dfrac{1}{v_2}\left(y_2 - c_2 u_3\right)$

$v_1 u_1 + c_1 u_2 = y_3$ $\qquad$ $u_1 = \dfrac{1}{v_1}\left(y_3 - c_1 u_2\right)$

$\}$ $n = 3$,
3 DIVIDES,
2 MULT.'S,
2 SUB.'S

MOST DIFFICULT COURSE

TAKE THE TIME...

9.2 $Uv = y$

$$\begin{bmatrix} v_1 & c_1 & 0 & 0 \\ 0 & v_2 & c_2 & 0 \\ 0 & 0 & v_3 & c_3 \\ 0 & 0 & 0 & v_4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$v_4 u_4 = y_4 \qquad u_4 = \frac{y_4}{v_4}$$

$$v_3 u_3 + c_3 u_4 = y_3 \qquad u_3 = \frac{1}{v_3}(y_3 - c_3 u_4)$$

$$v_2 u_2 + c_2 u_3 = y_2 \qquad u_2 = \frac{1}{v_2}(y_2 - c_2 u_3)$$

$$v_1 u_1 + c_1 u_2 = y_1 \qquad u_1 = \frac{1}{v_1}(y_1 - c_1 u_2)$$

$n = 4$,
4 DIVIDES,
3 MULT'S,
3 SUB'S

$\longrightarrow$ n DIVIDES,
$n-1$ MULTIPLICATIONS,
$n-1$ SUBTRACTIONS

FOR BACK. SUB., THE OPERATION COUNT IS $O(n)$