# Linear Algebra Review

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# Contents

# 1 Fundamental Subspaces of a Matrix

Let $A$ be a $mxn$ matrix. The four fundamental subspaces of $A$ include the column space (also known as range), row space, null space, and left null space.

The column space is defined as the span of the columns of $A$, that is

$$Col(A) = \{v : v = Ax, x \in \mathbb{R}^n\}$$

The row space is defined as the span of the rows of $A$, that is

$$Row(A) = Col(A^T) = \{v : v = A^T x, x \in \mathbb{R}^m\}$$

The null space is the vector subspace defined by

$$Null(A) = \{v : Av = 0\}$$

Recall that for a matrix to be invertible, the null space must only contain the trivial solution $v = 0$.

Finally, the left null space is defined as follows

$$Null(A^T) = \{v : A^T v = 0\}$$

The Fundamental Theorem of Linear Algebra also states that $Col(A)$ and $Null(A^T)$ are orthogonal to each other. Similarly, $Col(A^T)$ and $Null(A)$ are orthogonal to each other. To understand this, let $x \in Null(A)$ so that $Ax = 0$.

$$Ax = \begin{pmatrix} -a_1^T- \\ -a_2^T- \\ ... \\ -a_m^T- \end{pmatrix} x = \begin{pmatrix} a_1^T x \\ a_2^T x \\ ... \\ a_m^T x \end{pmatrix} = \vec{0}$$

The inner product of any row of $A$ with $x$ is 0, so it follows that the innder product of $x$ with any linear combination of rows is also 0. In other words, $x$ and any vector in $Row(A)$ are orthogonal. That is, $Null(A)$ and $Row(A)$ are orthogonal spaces.

# 2 Eigenvalues and Eigenvectors

An eigenvector of an nxn square matrix A is a non-zero vector $\vec{e} \in \mathbb{R}^n$ such that

$$Ae = \lambda e$$

for some scalar $\lambda$ called an eigenvalue of $A$. We say that $e$ is an eigenvector corresponding to $\lambda$.
Note that $e = 0$ is not a valid eigenvector (since the above relationship will be trivial) but $\lambda = 0$ is a valid eigenvalue. In this case,

$$Ae = 0e$$
$$Ae = 0$$

which only has a solution when A is not invertible (i.e. A is singular). To understand intuitively why this is the case, consider $Ax$ as a linear transformation of the vector $x$. If $x$ is mapped to the origin (the zero vector), there is no way to recover information from the transformation, and therefore undo it.

To find the eigenvalues of $A$, we can simply write

$$Ae - \lambda e = 0$$
$$(A - \lambda I)e = 0$$

which has a solution when $A - \lambda I$ is singular, which also indicates that the determinant of $A - \lambda I$ is 0. Thus, we want to solve for $\lambda$ such that

$$det(A - \lambda I) = 0$$

This is known as the characteristic equation. To solve for the corresponding eigenvectors, we only need to solve for $e$ after evaluating $(A - \lambda I)e = 0$ for $\lambda$.

Additionally note that the eigenvalues of an diagonal matrix are simply the diagonal elements themselves with eigenvectors corresponding to the standard basis vectors.

# 3 Diagonalization

Suppose the $nxn$ matrix $A$ has $n$ real eigenvalues $\lambda_1, ..., \lambda_n$ and $n$ associated eigenvectors $e_1, .., e_n$. Then note that all eigenvector/eigenvalue equations can be expressed with

$$AQ = Q\Lambda$$

where $Q = [e_1; , ..; e_n] \in \mathbb{R}^{nxn}$ consists of the eigenvectors of $A$ in its columns and $\Lambda = diag(\lambda_1, ..., \lambda_n)$. Then we can write

$$A = AQQ^{-1} = Q\Lambda Q^{-1}$$

We call such $A$ as diagonalizable, and $A$ has the following properties

1. $tr(A) = tr(Q\Lambda Q^{-1}) = tr(QQ^{-1}\Lambda) = \sum_{i=1}^{n} \lambda_i$
2. $det(A) = \prod_{i=1}^{n} \lambda_i$
3. if $A$ is nonsingular ($A^{-1}$ exists), then $1/\lambda_i$ is an eigenvalue of $A^{-1}$ with associated eigenvector $e_i$

Note that here, we assume that the inverse of $Q$ exists. Thus, for the matrix $A$ to be diagonalizable, $A$ must have $n$ linearly independent eigenvectors. That is, the $n$ eigenvectors form a basis in $\mathbb{R}^n$ for a diagonalizable $A$. This is not to be confused with $A$ being nonsingular, when A is singular there are eigenvectors that correspond to $\lambda = 0$ which represent the null space, where $Ae = 0$.

# 4 Diagonalization of Symmetric Matrices

Now suppose that the matrix $A$ is symmetric (that is $A^T = A$). It can be shown that any two eigenvectors (corresponding to different eigenvalues/ different eigenspaces) are orthogonal to each other for a symmetric matrix. The proof of this is simple. Let $v_1$ and $v_2$ be eigenvectors that correspond to distinct eigenvalues $\lambda_1$ and $\lambda_2$. Then

$$\lambda_1 v_1 \cdot v_2 = (\lambda_1 v_1)^T v_2$$

since $v_1$ is an eigenvector, we can write the above as

$$(Av_1)^T v_2 = v_1^T A^T v_2 = v_1^T (Av_2)$$

since $v_2$ is an eigenvector, we can write the above as

$$v_1^T(\lambda_2 v_2)$$

This implies that

$$\begin{aligned} \lambda_1 v_1 \cdot v_2 &= \lambda_2 v_1 \cdot v_2 \\ (\lambda_1 - \lambda_2) v_1 \cdot v_2 &= 0 \end{aligned}$$

Since we have that $\lambda_1 \neq \lambda_2$, we have shown that $v_1 \cdot v_2 = 0$ so $v_1$ and $v_2$ are orthogonal. Thus, distinct eigenspaces are mutually orthogonal.

A symmetric nxn matrix $A$ is special because we can show that

1. $A$ has $n$ real eigenvalues (counting multiplicities)
2. The geometric multiplicity of $\lambda$, $dim(Null(A-\lambda I))$ , that is, the dimension or number of basis vectors in the eigenspace of $\lambda$, is equal to the algebraic multiplicity of $\lambda$ (the order corresponding to $\lambda$ in the characteristic equation)
3. The eigenspaces are mutually orthogonal (as we have shown above)

Note that it is possible that an eigenvalue may have larger multiplicity (put plainly, the case where multiple eigenvectors corresponding to the same eigenvalue). Utilizing fact 2 above, we can simply choose for a specific eigenvalue $\lambda$ an orthonormal basis for the eigenspace $\{v : Av = \lambda v\}$, for example, using the Gram-Schmidt process. We do this so that all chosen eigenvectors $e_1, ..., e_n$ are mutually orthogonal, whether they correspond to the same eigenspace or not. Such a diagonalization of a symmetric $A$ is called an *orthogonal* diagonalization. In fact, $A$ is orthogonally diagonalizable *if and only if* it is symmetric.

Specifically, we can now take an orthonormal matrix $Q$ (we have chosen eigenvectors such that every column of $Q$ has unit length and are mutually orthogonal). Since we now have $QQ^T = Q^TQ = I$, we see that $Q^T = Q^{-1}$. Then we can write the diagonalization of $A$ as

$$
\begin{aligned}
A &= Q\Lambda Q^{-1} = Q\Lambda Q^T \\
A &= \lambda_1 e_1 e_1^T + ... + \lambda_n e_n e_n^T
\end{aligned}
$$

This is known as the spectral decomposition of the symmetric matrix $A$.

# 5 Quadratic Forms

A quadratic form on $\mathbb{R}^n$ is a function written as

$$f(x) = x^T Ax$$

where A is a nxn *symmetric* matrix. For example, in $\mathbb{R}^2$

$$
\begin{aligned}
x^T Ax &= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -2 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= 3x_1^2 - 2x_1x_2 - 2x_1x_2 + 7x_2^2
\end{aligned}
$$

Note that the diagonals of $A$ represent the coefficients for the quadratic terms, and the off diagonals represent the cross product terms.

We say that a matrix $A$ is *positive definite* if for all $x \in \mathbb{R}^n$

$$x^T Ax > 0$$

and *positive semdefinite* if for all $x \in \mathbb{R}^n$

$$x^T Ax \geq 0$$

To check if a symmetric matrix is positive semidefinite, it is sufficient to check if all the eigenvalues of $A$ are non-negative, that is

$$\lambda_1, .., \lambda_n \geq 0$$

To understand this, we can substitute the spectral decomposition of $A$ into the definition of positive semidefinite

$$
\begin{aligned}
x^T Ax &= x^T(\lambda_1 e_1 e_1^T + ... + \lambda_n e_n e_n^T)x \\
&= \lambda_1 x^T e_1 e_1^T x + ... + \lambda_n x^T e_n e_n^T x \\
&= \lambda_1 (x^T e_1)^2 + ... + \lambda_n (x^T e_n)^2
\end{aligned}
$$

which is non-negative for all $x$ if and only if all the eigenvalues $\lambda_1, .., \lambda_n \geq 0$. Additionally, using a similar argument, we can show that if a matrix $A$ is positive definite, we have that its eigenvalues $\lambda_1, .., \lambda_n > 0$.

We write

$$A \succeq 0$$

denoting $A$ as positive semidefinite. If $A$ is positive definite we write

$$A \succ 0$$

If we write

$$A \succeq B$$

it denotes that $A - B$ is positive semidefinite.

# 6 Projection Matrix

A projection matrix $P$ onto a vector subspace defines a mapping from an arbitrary vector onto the closest point in that vector subspace. Below, we show that a matrix of the form $P = A(A^TA)^{-1}A^T \in \mathbb{R}^{n,n}$ defines an orthogonal projection onto $Col(A)$ where $A \in \mathbb{R}^{n,m}$.

Let $y$ be an arbitrary vector and $Ax$ be its respective projection onto $Col(A)$. By construction of an orthogonal projection, we have that $b - Ax$ is orthogonal to $Col(A)$. Since a vector orthogonal to the column space is in the left null space, we can write

$$\begin{aligned} A^T(b - Ax) &= 0 \\ A^Tb - A^TAx &= 0 \\ A^Tb &= A^TAx \\ (A^TA)^{-1}A^Tb &= x \\ A(A^TA)^{-1}A^Tb &= Ax \\ Pb &= Ax \end{aligned}$$

so $P$ is a projection onto $Col(A)$.

**Example**. Show that $P$ is symmetric and idempotent.

$$P^T = \left(A(A^TA)^{-1}A^T\right)^T = A(A^TA)^{-1}A^T = P$$

so $P$ is symmetric.

$$P^2 = PP = A(A^TA)^{-1}A^TA(A^TA)^{-1}A^T = A(A^TA)^{-1}A^T = P$$

so $P$ is idempotent.

**Example**. Show that $Rank(P) = Tr(P) = m$.
We first need to show that a projection matrix only has 0 or 1 eigenvalues. Let $v$ be a eigenvector of $P$ corresponding to eigenvalue $\lambda$. Since $P$ is idempotent,

$$\lambda v = Av = AAv = \lambda Av = \lambda^2 v$$

Noting that $v \neq 0$, $\lambda^2 = \lambda$ so $\lambda$ can only be 0 or 1.

Let $P = Q\Lambda Q^T$ be the spectral decomposition of $P$. Then we have

$$Tr(P) = Tr(Q\Lambda Q^T) = Tr(\Lambda Q^TQ) = Tr(\Lambda) = \sum_{i=1}^{n} \lambda_i$$

By the Rank-Nullity theorem, we know that $Rank(P) + Nullity(P) = n$, the number of columns in $P$. The nullity of $P$ is simply the dimension of the null space $P$, which is equal to the number of zero eigenvalues (recall that the algebraic and geometric multiplicity of $\lambda = 0$ are the same since $P$ is square and symmetric). Conversely, we have that $Rank(P) = n - Nullity(P)$, which is the number of non-zero eigenvalues since $P$ has $n$ real eigenvalues. Since the number of nonzero eigenvalues of $P$ is precisely $Tr(\Lambda)$ because all the eigenvalues are either 0 or 1, we have that

$$Rank(P) = Tr(\Lambda) = Tr(P)$$

Finally, we have that

$$Rank(P) = Tr(P) = Tr(A(A^TA)^{-1}A^T) = Tr(I_m) = m$$