Pendulum Lab
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28 October 2022

1. Introduction

The purpose of this study is to analyze collected data on a pendulum and test how well its motion can be modeled using established relationships and best fits. Below are the equations that will be tested:

$$T = 2\pi\sqrt{\frac{L}{g}} \qquad \text{Equation 1 (Mazur 513)}$$

$$T = T_0(1 + B\theta_0 + C\theta_0^2) \qquad \text{Equation 2 (“Pendulum”)}$$

$$x_{max}(t) = \Theta_0 e^{-t/\tau} \qquad \text{Equation 3 (Mazur 514)}$$

$$Q = \frac{\pi\tau}{T} \qquad \text{Equation 4 (Mazur 514)}$$

Data relevant to these equations was gathered with a pendulum made up of light string and a dense metal bob, fixed to a relatively unmoving object. The first experiment modified the angle of release of the pendulum and recorded its period for each angle of release in a series of trials. This showed a symmetrical quadratic relationship (0.9764 correlation) between length of pendulum and period with upward curvature, validating the power series model of Equation 2, and importantly confirming that the period is dependent on the angle of release of the pendulum. Small angles of release (>0.3 radians) would be used in future tests to avoid confounding from this relationship.

In Experiment 2, two methods were used to analyze the behavior of the pendulum's amplitude over time by finding the damping constant Q-factor: one derived from fitting data for amplitude over time with the form of Equation 3 and then solving for Q from Equation 4, the other from counting the periods until the pendulum reached the initial period times $e^{-\pi/4}$ and then multiplying it by 4 (Mazur 511). With resulting Q-factors of 461 ± 5 and 456 ± 4 for Method 1 and 2 respectively, this experiment showed that both methods were precise and accurate compared to each other, but Method 2 would be more viable for future testing because of higher efficiency.

Experiment #3 aimed to validate or disprove the current model for length of period variation with different lengths of pendulum given by Equation 1. Once the data was fit to a power function, the experimental values of a and b were found to be 2.01 ± 0.01 and 0.495 ± 0.006 respectively. This agrees with the expected values of a = 2 and b = 0.5 from Equation 1. The log axes linear best fit gave values of m = 0.51 ± 0.01 and 10^b = 2.04 ± 0.05, which again agree with expected values of m = 0.5 and 10^b = 2. Both of these pieces of evidence validate the current model.

Lastly, Experiment #4 attempted to build a model for the Q-factor of the pendulum used at different lengths and determine if there is a relationship. Using the second method from Experiment #3 at different lengths, a quadratic relationship was found to exist for the domain of lengths tested with a high correlation of 0.9970. Further data would be required in future studies to get a better idea of the behavior of Q-factor with higher lengths and multiple trials for each length would help reduce uncertainties in the data.

2. Pendulum Design

The pendulum used in this lab was designed with a dense metal weight called a plumb bob with a weight of 5 ounces at the end with the goal to both reduce the effects of air resistance and to render the mass of the string experimentally zero. The pendulum string was fit snuggly to a hole in a piece of wood to ensure a single swinging point. This piece of wood was then affixed to a cabinet with a large mass with the pendulum close to the edge to prevent as much reactive motion from the pendulum as possible, and a protractor was attached. All measurements were made parallel to the plane of Figure 1.

Figure 1: Front View

Figure 2: Side View

Figure 3: View of Top

3. Experiment #1: Angle of Release vs. Period

3.1. Methods and Procedures

The length to the center of mass for this experiment was chosen to be long enough to make the period easy to measure accurately. Shown by Equation 1 above, increasing the length to the center of gravity of the bob increases the period. A larger period helps reduce the percent error in measuring the period given fixed errors in measuring time. A manageable value was determined to be 42 cm to the center of mass of the pendulum which would give an approximate period length of 1.3 seconds.

For the experiment, the pendulum was dropped at angles of -90 to 90 degrees with steps of 10 degrees, excluding 0. For each angle of drop, a stopwatch was used to measure one full period of the pendulum, starting from when the pendulum first hit the bottom of its arc. This choice of data measurement was used to avoid as much decay due to the Q-factor, keeping the amplitude practically constant for each trial. This was repeated 3 times for each angle to help account for the higher uncertainty in hand measurement, and the average period was recorded. All angles were converted into radians for the final data.

3.2. Data Presentation

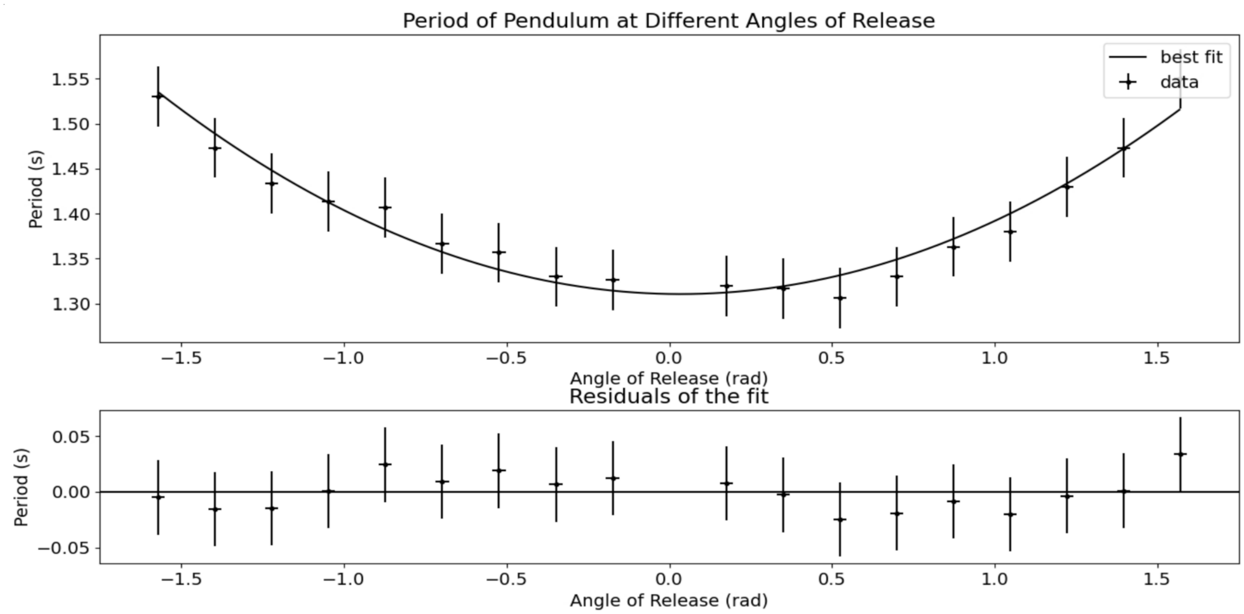

Figure 4: Period of Pendulum at Different Angles of Release

Fit to power series $T = T_0(1 + B\theta_0 + C\theta_0^2)$

Simplified to $T = a\theta_0^2 + b\theta_0 + c$

a	0.871 ± .005
b	-0.006 ± 0.004
c	1.311 ± 0.006

Table 1: Fit Parameters for Figure 4

3.3. Value and Uncertainty Analysis

The largest source of error for angle measurement was due to the dropper being inaccurate with the position of the drop due to visibility. The string used in the experiment was .017 radians wide at the measurement radius of the protractor. The string could have covered the proper angle if the edge was anywhere within 0.017 radians of it. Therefore, the uncertainty for angle would be ±0.02 radians.

The largest factor of period uncertainty was human measurement ability. The tester in this experiment was found to be able to measure 2 seconds on a stopwatch within ±0.1 seconds reliably, so this would be the uncertainty in measurement. This was repeated 3 times and averaged, reducing the uncertainty to ±0.03 seconds.

The values and uncertainties for the power series fit in Figure 4 are given in Table 1. The b-value of the function given in Table 1 is within two uncertainties of zero and therefore it can be considered experimentally zero. This means that the function is symmetric.

3.4. Results and Conclusions

The most significant finding from this experiment is that the period of the pendulum is not independent from the angle. From Figure 4 and Table 1, a second order function is shown to represent the relationship between period and amplitude and fits the data much better than the straight line that would be expected if the period and amplitude were independent, i.e., the correlation was much higher (0.9764 compared to -0.0807). This shows that, as the values of the angle increase in absolute value, the period also increases quadratically.

The interval on which the C-value of the curve in Figure 4 can be ignored is the area of the graph where the function is roughly linear, i.e., close to the y-axis. Roughly, amplitudes up to 0.3 radians in magnitude could be considered part of this linear section, although the trend will generally be more linear as the amplitude gets closer to zero.

4. Experiment #2: Amplitude Over Time For Decay and Q-factor

4.1. Methods and Procedures

The same length of pendulum from Experiment #1 was used for this experiment. Shown in the previous section, it is necessary to work with small angles (>0.3 radians) to have a near constant period, so the pendulum will be dropped from 15 degrees (0.26 radians).

Two methods were used to measure the Q-factor. Method 1: the pendulum was dropped at 15 degrees (0.26 radians) once and recorded with a cell phone camera. The maximum amplitude for each period and the time that it occurred was measured, starting after the first swing of the pendulum. This data was fit to an exponential and Equations 3 and 4 were used to calculate the Q-factor. Method 2: the pendulum was dropped at 15 degrees (0.26 radians), and the number of oscillations the pendulum took to get below 15 degrees * $e^{-\pi/4}$ = 6.8 degrees (0.12 radians) from an initial amplitude of 15 degrees (0.26 radians) was recorded. This was then multiplied by 4 to find the Q-factor (Mazur 511).

4.2. Data Presentation

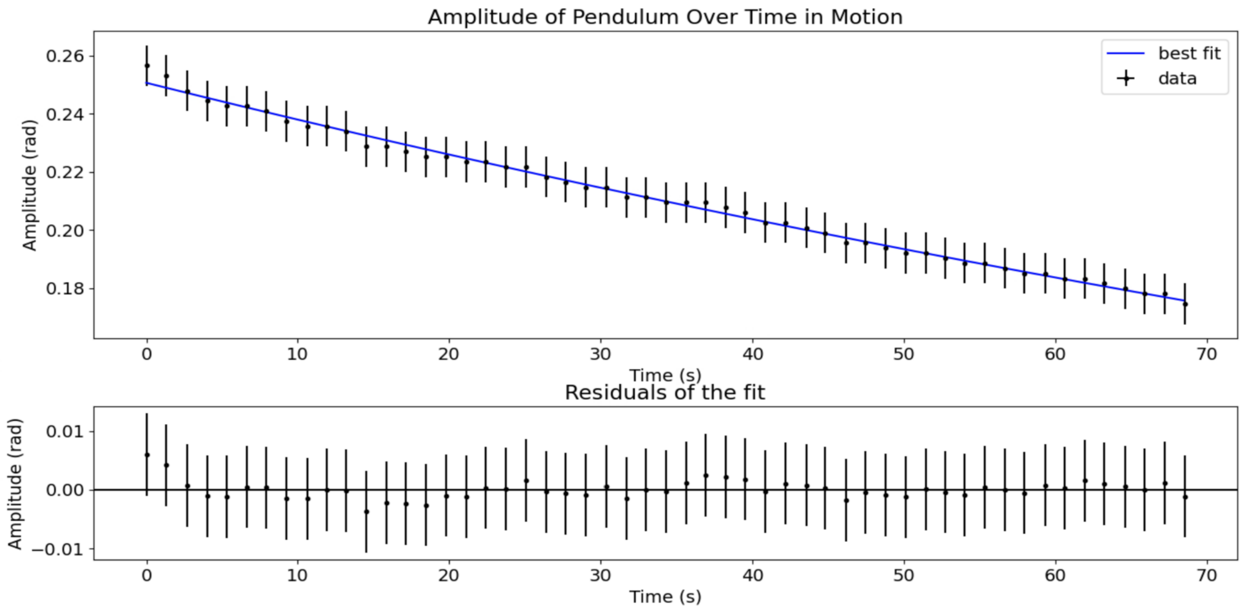

Figure 5: Exponential Decay of Amplitude of Pendulum Over Time in Motion

Data Fit to Exponential to $x_{max}(t) = \Theta_0 e^{-t/\tau}$

Θ_0	0.2506 ± 0.0005
τ	193 ± 2

Table 2: Fit Parameter for Figure 5

4.3. Value and Uncertainty Analysis

The three variables measured in this experiment were time, maximum amplitude, and number of periods of the pendulum. The camera used refreshed frames every 0.04 seconds, so the time uncertainty is ±0.04 seconds. The uncertainty for maximum amplitude is ±.004 radians because the string was approximately .017 radians wide at the measuring radius and it was possible to get within the center one fifth of the string in measuring the angle using video playback. The number of periods measured to get below an amplitude 6.8 degrees has an uncertainty of ±1 due to the fact that the amplitude was only measured every full oscillation, so the amplitude could have reached 6.8 degrees any time in the oscillation. In the future, uncertainties could be reduced by using a more accurate protractor or a thinner string.

Uncertainties are given for the exponential fit in Table 2. Using Equation 4, the value of Q can be calculated by multiplying τ by π and dividing by the average period which is the total trial time (68.40 ± 0.04 seconds) divided by the number of periods (52). This yields an average period of 1.3154 ± 0.0008 seconds. From this calculation, the value of the Q-factor is 461 ± 5. The uncertainty is derived from the percent uncertainty of τ. Using the other method, the number of oscillations recorded (114 ± 1) is multiplied by 4, yielding a Q-factor of 456 ± 4. Uncertainty for this value could be reduced in the future by instead counting oscillations all the way until the pendulum reaches $e^{-\pi}$ times the initial amplitude which would not have to be multiplied and would give a Q-factor with an uncertainty of only ±1.

4.4. Results and Conclusions

The main goals of this experiment were to both develop a model of the amplitude over time and determine the Q-factor using two methods. Method 1 used the curve fit from the experimental data of Figure 5 and yielded a Q-factor value of 461 ± 5. Method 2 yielded an incredibly similar result of 456 ± 4 from counting the oscillations until the amplitude of the pendulum reached 6.8 degrees (0.12 radians) from 15 degrees (0.26 radians). The fact that both of these are so similar, with each value within one uncertainty of the other, gives strong evidence to the values discovered and shows that both methods are equally precise for determining Q-factor.

5. Experiment #3: Length vs. Period

5.1. Methods and Procedures

With the same logic from Experiment #2, the angle of release for all drops was 15 degrees (0.26 radians). The pendulum was dropped three times for each length to the center of mass of the pendulum from 10 cm to 60 cm in steps of 5 cm. Starting from the bottom of its arc, two periods were measured with a stopwatch for each drop to decrease measurement error due to very fast periods for some trials. This was then divided by two and all three trials were averaged to produce an average period for each length.

5.2. Data Presentation

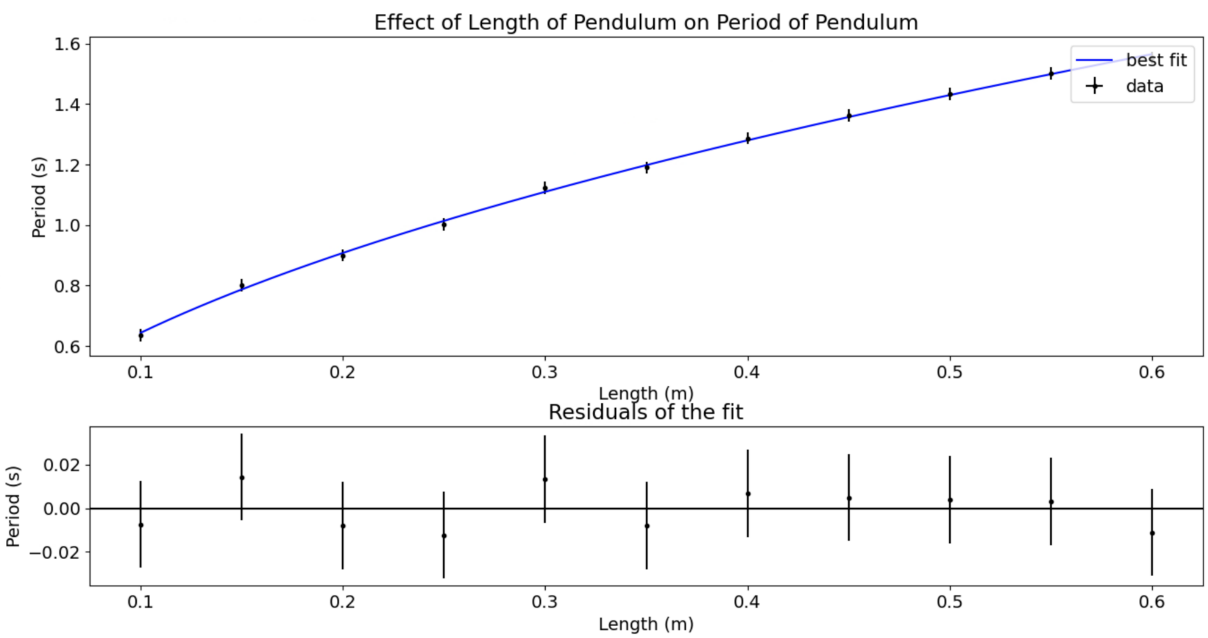

Figure 7: Effect of Length of Pendulum on Period of Pendulum

Data Fit to $T = 2\pi\sqrt{\frac{L}{g}} = aL^b$

a	2.01 ± 0.01
b	0.495 ± 0.006

Table 3: Fit Parameters for Figure 7

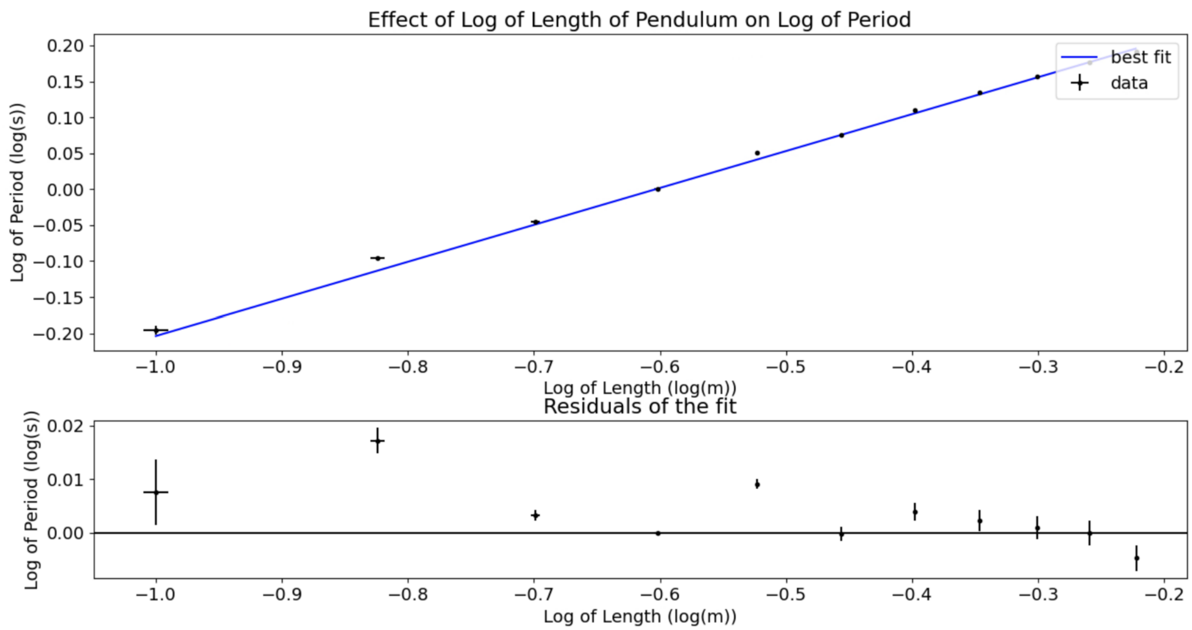

Figure 8: Effect of Log of Length of Pendulum on Log of Period

Data Fit to Linear Function $y = mx + b$

m	0.51 ± 0.01
b	0.310 ± 0.007

Table 4: Fit Parameters for Figure 8

5.3. Value and Uncertainty Analysis

The length to the center of mass was measured with a measuring tape that had increments of millimeters so the error in length would be ±0.001 m.

The uncertainty for time measurement is ±0.1 seconds using the same logic in Experiment #1. However, this quantity is then divided by two to find the period and averaged over three trials, leading to a final uncertainty of ±0.02 seconds.

5.4. Results and Conclusions

This experiment aimed to validate or disprove Equation 1. From this equation, the expected values of the constants for the data fit given in Table 3 were $a = 2\pi\sqrt{1/g} \approx 2$ and b = 0.5 given by the root in the equation. The actual values measured in the experiment were a = 2.01 ± 0.01 and b = 0.495 ± 0.006. The fit was very accurate, with very small residuals and a correlation of 0.995. The expected values for both are within the uncertainties of the actual values measured, and therefore this data heavily supports the current models.

This data was also graphed with log axes shown in Figure 8. Theoretically, the slope m of this fit should also equal 0.5 and raising 10 to the power of the y-intercept b should equal 2. The experimental values of this experiment shown in Table 4 were m = 0.51 ± 0.01 and b = 0.310 ± 0.007. The value of 10 to the power of b = 2.04 ± 0.05. Both of the theoretical values are within the uncertainties of the values of the experimentally gathered values. This once again strongly supports the current model given by Equation 1.

6. Experiment #4: Length vs. Q-Factor

6.1. Methods and Procedures

From the results of Experiment #2, Method 1 and Method 2 are both viable for precise and accurate results. Using Method 2 is much more efficient, so Method 2 will be used again. Using the logic from Experiment #2, the angle of release for all drops was 15 degrees (0.26 radians). For each length to the center of mass from 10 cm to 60 cm with steps of 5 cm, the pendulum was dropped and Q-factor was calculated with Method 2.

6.2. Data Presentation

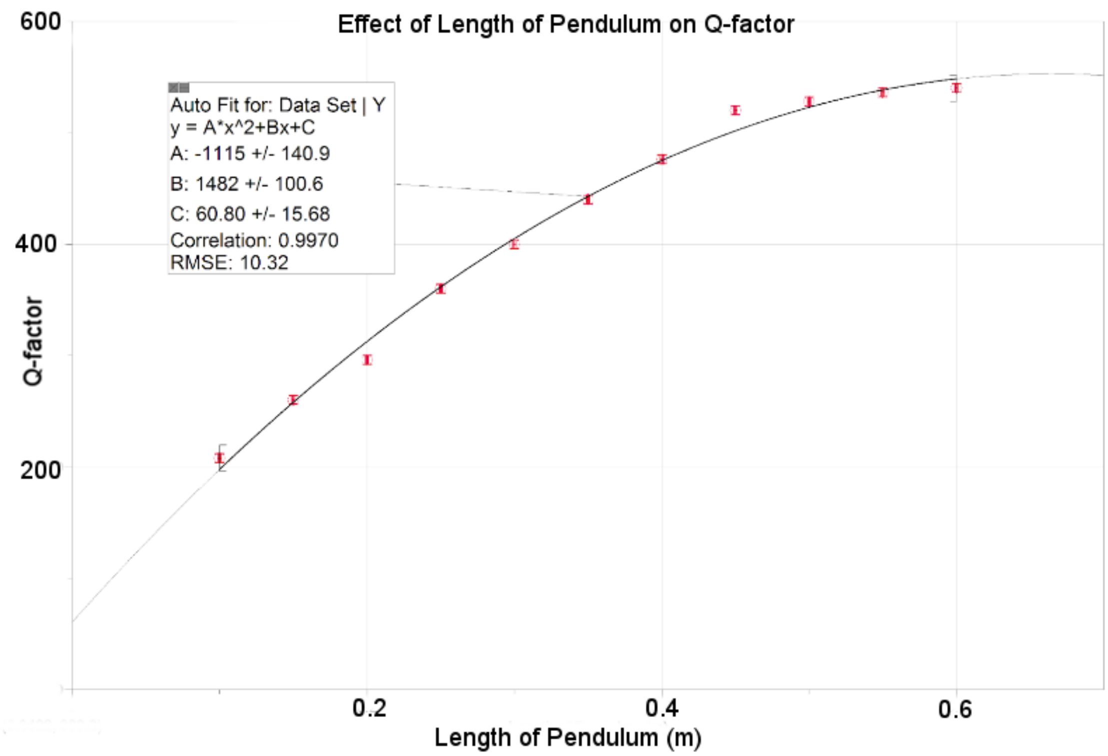

Figure 9: Effect of Length of Pendulum on Q-factor
Fit to Quadratic $Q = Ax^2 + Bx + C$

A	-1100 ±100
B	1500 ± 100
C	60 ± 20

Table 5: Fit Parameters for Figure 9

6.3. Value and Uncertainty Analysis

From the same logic as Experiment #3, the error in length to the center of mass would be ±0.001 m. From Experiment #2, the uncertainty of the Q-factor is ±4. Uncertainties for root fit given in Figure 9.

6.4 Results and Conclusions

This experiment aimed to develop a model for the Q-factor based on the length of the pendulum. The model that best fit the data was found to be a quadratic polynomial, given in Figure 9. This model had the highest correlation of the functions tested (0.9970). This also means that there is definitely a strong positive relationship initially between length of pendulum and Q-factor. This makes sense that the trend would go up initially, because the pendulum's momentum increases as the string gets longer, and therefore is harder to stop.

My theory for the Q-factor's negative concavity is that as the pendulum gets longer, the air resistance from the string and mass of the string starts to have a bearing on the system. The center of mass starts to move up from where it was initially and the stopping force from air resistance increases with the higher surface area. It would be a bad extrapolation to say that this fit is accurate for all values of length because the Q-factor cannot be negative as described by the quadratic fit, so the data most likely approaches zero once the pendulum is long enough. Ideally in the future, extending this experiment to higher lengths of the pendulum would help to get a better idea of the long-term behavior of Q-factor when compared to length. Taking more trials and increasing measurement precision would also help to establish more certainty in the data and results.

7. Conclusion

The first significant finding of this study is that the period of the pendulum is not independent from the angle. A symmetric quadratic relationship was found to have a higher correlation than a straight line (0.9764 compared to -0.0807), validating Equation 2. Because of this, future testing necessitated working in the roughly linear space where this relationship is insignificant, i.e., less than 0.3 radians in amplitude.

Secondly, the two methods of measuring Q-factor used in this study were deemed to be equally precise and were accurate when compared to each other. Method 1 yielded a Q-factor of 461 ± 5 and Method 2 yielded a Q-factor of 456 ± 4. The fact that these values' uncertainty intervals overlap shows that both methods are viable and accurate. Method 2 was deemed to be more useful for further tests due to higher efficiency and equal results.

Thirdly, using both a power fit and logarithmic best fit, Equation 1 was validated. The expected values given by Equation 1 for the power function in Figure 7 were a = 2 and b = 0.5. The actual values measured in the experiment were a = 2.01 ± 0.01 and b = 0.495 ± 0.006, which both are within the uncertainties of the actual values measured, heavily supporting the current model. With log axes, the theoretical slope m the linear fit should equal 0.5 and raising 10 to the power of the y-intercept b should theoretically equal 2. The values from the experiment were m = 0.51 ± 0.01 and 10^b = 2.04 ± 0.05. Both of the theoretical values are within the uncertainties of the values of the experimentally gathered values, once again strongly supporting the current model.

Finally, the Q-factor was determined to be dependent on length. A quadratic polynomial was found to have the highest correlation of the functions tested (0.9970), with the values tested approaching a theoretical maximum. Unfortunately, this model can not be extrapolated to higher values of length, so testing higher lengths in the future would be valuable in understanding the full relationship.

Works Cited

Mazur, Eric. *Principles & Practice of Physics, 2e*. Pearson, 2006, pp. 511, 513-514.

"Pendulum." Wikipedia. Wikimedia Foundation, 15 August, 2022, https://en.wikipedia.org/wiki/Pendulum.