1 The accuracy of learning decision boundaries

This problem exercises your basic probability skills in the context of understanding why lots of training data helps to improve the accuracy of learning outcomes.

For each $\theta \in (1/3, 2/3)$, define $f_\theta : [0,1] \to \{0,1\}$, such that

$$f_\theta(x) = \begin{cases} 1 \text{ if } x > \theta \\ 0 \text{ otherwise.} \end{cases}$$

We draw samples $X_1, X_2, \ldots, X_n$ uniformly at random and i.i.d. from the interval $[0,1]$. Our goal is to learn an estimate for θ from n random samples $(X_1, f_\theta(X_1)), (X_2, f_\theta(X_2)), \ldots, (X_n, f_\theta(X_n))$.

Let $T_{min} = \max(\{\frac{1}{3}\} \cup \{X_i | f_\theta(X_i) = 0\})$. We know that the true θ must be larger than T_{min}.

Let $T_{max} = \min(\{\frac{2}{3}\} \cup \{X_i | f_\theta(X_i) = 1\})$. We know that the true θ must be smaller than T_{max}.

The gap between T_{min} and T_{max} represents the uncertainty we will have about the true θ given the training data that we have received.

(a) **What is the probability that $T_{max} - \theta > \epsilon$ as a function of ϵ? And what is the probability that $\theta - T_{min} > \epsilon$ as a function of ϵ?**

(b) Suppose that you would like to have an estimate for θ that is ϵ-close (defined as $|\hat{\theta} - \theta_0| < \epsilon$, where $\hat{\theta}$ is the estimation and θ_0 is the true value) with probability at least $1 - \delta$. Both ϵ and δ are some small positive numbers. **Please bound or estimate how big of an n do you need?**

(c) Let us say that instead of getting random samples $(X_i, f(X_i))$, we were allowed to choose where to sample the function, but you had to choose all the places you were going to sample in advance. **Propose a method to estimate θ. How many samples suffice to achieve an estimate that is ϵ-close as above?** (**Hint:** You need not use a randomized strategy.)

(d) Suppose that you could pick where to sample the function adaptively — choosing where to sample the function in response to what the answers were previously. **Propose a method to estimate θ. How many samples suffice to achieve an estimate that is ϵ-close as above?**

(e) In the three sampling approaches above: random, deterministic, and adaptive, **compare the scaling of n with ϵ (and δ as well for the random case).**

(f) **Why do you think we asked this series of questions? What are the implications of those results in a machine learning application?**