Mathe Blatt 2

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A4)

a) $a_1 := 1 \qquad a_{n+1} := \frac{1}{2}a_n + \sqrt{a_n}$

I.A.: $n = 1$

$a_1 = 1$ ✓

~~$a_{1+1} = \frac{1}{2}\cdot 1 + \sqrt{1} = 1\frac{1}{2} \leq 4$~~ ✓

I.V.: $a_n \in (0, 4)$ z.z. für alle $n \in \mathbb{N}$

$n \to n+1$

$a_{n+1} = \frac{1}{2} \cdot "(0,4)" + \sqrt{"(0,4)"} = "(0,2)" + "(0,2)"$

$\Rightarrow \quad 0 < "(0,2)" + "(0,2)" < 4 \quad \square$

b) z.z.: $a_{n+1} - a_n \geq 0$

$a_{n+1} - a_n = \frac{1}{2}a_n + \sqrt{a_n} - a_n = -\frac{1}{2}a_n + \sqrt{a_n} \qquad a_n \in (0,4)$

Für Werte zw. 0 und 4 liegt der Graph der Wurzelfunktion $f(x) = \sqrt{x}$ oberhalb des Funktionsgraphen der Funktion $g(x) = \frac{1}{2}x$.
Bei $x = 4$ schneiden sich die Graphen und g liegt ab 4 oberhalb von f. Somit ist $a_{n+1} - a_n$ immer größer 0!

c) Es gilt eine obere Schranke + monoton wachsend

$\Rightarrow$ Folge konvergent

Grenzwert a $\widehat{=}$ Supremum

$\sup(a_n) = a$

$a = \frac{1}{2}a + \sqrt{a}$

$\frac{1}{2}a = \sqrt{a} \qquad \Rightarrow \qquad \underline{\underline{a = 4}}$

A5)

a) $a_0 = 0, \quad a_1 = 1, \qquad a_{n+1} = \alpha a_n + \beta a_{n-1} \quad \forall n \in \mathbb{N}$

$x^2 = \alpha x + \beta \qquad a_n = \frac{x_1^n - x_2^n}{x_1 - x_2}$

Was ist I.V?

$$a_{n+1} = \alpha a_n + \beta a_{n-1} \overset{IV}{=} \alpha \frac{x_1^n - x_2^n}{x_1 - x_2} + \beta \frac{x_1^{n-1} - x_2^{n-1}}{x_1 - x_2}$$

$$= \frac{\alpha x_1^n + \beta x_1^{n-1} - \alpha x_2^n - \beta x_2^{n-1}}{x_1 - x_2} = \frac{x_1^{n-1}(\alpha x_1 + \beta) - x_2^{n-1}(\alpha x_2 + \beta)}{x_1 - x_2}$$

$$= \frac{x_1^{n-1} \cdot x_1^2 - x_2^{n-1} \cdot x_2^2}{x_1 - x_2} = \frac{x_1^{n+1} - x_2^{n+1}}{x_1 - x_2} \qquad \square$$

b) $0 = x^2 - \alpha x - \beta$

$$x_{1/2} = \frac{+\alpha \pm \sqrt{\alpha^2 + 4\beta}}{2}$$

i) $\alpha^2 + 4\beta < 0 \Rightarrow$ neg. Diskriminante, evtl. komplexer Zahlenbereich!

ii) $\alpha^2 + 4\beta = 0 \Rightarrow x_1 = x_2$ ↯ durch 0 teilen!

$\Rightarrow$ Aussage gilt nicht!

A5)

c) i) $\alpha = 1, \quad \beta = 1$

$$0 = x^2 - x - 1$$

$$x_{1/2} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \boxed{\sqrt{5}}}{\boxed{2}}$$

$$a_n = \frac{\left(\frac{1}{2} + \sqrt{5}\right)^n - \left(\frac{1}{2} - \boxed{\sqrt{5}}\right)^n}{\left(\frac{1}{2} + \sqrt{5}\right) - \left(\frac{1}{2} - \sqrt{5}\right)} = \frac{\left(\frac{1}{2} + \sqrt{5}\right)^n - \left(\frac{1}{2} - \sqrt{5}\right)^n}{2\sqrt{5}}$$

ii) $\alpha = 4, \quad \beta = 7$

$$0 = x^2 - 4x - 7$$

$$x_{1/2} = \frac{4 \pm \sqrt{16+28}}{2} = \frac{4 \pm \sqrt{44}}{2} = 2 \pm \sqrt{11}$$

$$a_n = \frac{(2+\sqrt{11})^n - (2-\sqrt{11})^n}{(2+\sqrt{11}) - (2-\sqrt{11})} = \frac{(2+\sqrt{11})^n - (2-\sqrt{11})^n}{2\sqrt{11}}$$

iii) $\cancel{0 = x^2 + 1}$ $\quad x^2 = -1$

$x_{1/2} = \pm i \quad \Rightarrow \quad a_n = \dfrac{i^n - (-i)^n}{2i}$

Fall 1 : $n = 0 \bmod 4 \Rightarrow a_n = \dfrac{1-1}{2i} = 0$

Fall 2 : $n = 1 \bmod 4 \Rightarrow a_n = \dfrac{i+i}{2i} = 1$

Fall 3 : $n = 2 \bmod 4 \Rightarrow a_n = \dfrac{-1-(-1)}{2i} = 0$

Fall 4 : $n = 3 \bmod 4 \Rightarrow a_n = \dfrac{-i-i}{2i} = -1$

$\Rightarrow$ keine Folgenglieder reell!

A6)

a) $a_n = \frac{2n^3 - n}{n(3n^2+2)} = \frac{2n^2-1}{3n^2+2} = \frac{2-\frac{1}{n^2}}{3+\frac{2}{n^2}}$

$$\lim_{n\to\infty} \frac{2-\frac{1}{n^2}}{3+\frac{2}{n^2}} = \frac{\lim_{n\to\infty} 2 - \frac{1}{n^2} \to 0}{\lim_{n\to\infty} 3 + \frac{2}{n^2} \to 0} = \frac{2-0}{3+0} = \frac{2}{3}$$

b) $b_n = \left(\frac{5+2n}{1+n}\right)^3 = \left(\frac{\frac{5}{n}+2}{\frac{1}{n}+1}\right)^3$

$$\left(\frac{\lim_{n\to\infty} \frac{5}{n}+2}{\lim_{n\to\infty} \frac{1}{n}+1}\right)^3 = 8$$

c) $c_n = \sqrt{2n^2+n+1} - \sqrt{2n^2+9n}$

$$= \frac{\left(\sqrt{2n^2+n+1} - \sqrt{2n^2+9n}\right)\left(\sqrt{2n^2+n+1} + \sqrt{2n^2+9n}\right)}{\left(\sqrt{2n^2+n+1} + \sqrt{2n^2+9n}\right)}$$

$$= \frac{2n^2+n+1-2n^2-9n}{\sqrt{2n^2+n+1} - \sqrt{2n^2+9n}} = \frac{-8n+1}{\sqrt{2n^2+n+1} + \sqrt{2n^2+9n}}$$

$$= \frac{-8n+1}{n\sqrt{2+\frac{1}{n}+\frac{1}{n^2}} + n\sqrt{2+\frac{9}{n}}} = \frac{-8n+1}{n\left(\sqrt{2+\frac{1}{n}+\frac{1}{n^2}} + \sqrt{2+\frac{9}{n}}\right)}$$

$$= \frac{-8+\frac{1}{n}}{\sqrt{2+\frac{1}{n}+\frac{1}{n^2}} + \sqrt{2+\frac{9}{n}}}$$

$$\lim_{n\to\infty} \frac{-8+\frac{1}{n}}{\sqrt{2+\frac{1}{n}+\frac{1}{n^2}} + \sqrt{2+\frac{9}{n}}} = \frac{-8}{2\sqrt{2}} = -2\sqrt{2}$$

A6)

d) $$d_n = n^3 - \sqrt{n^6+n^2+1} = \frac{\left(n^3-\sqrt{n^6+n^2+1}\right)\left(n^3+\sqrt{n^6+n^2+1}\right)}{n^3+\sqrt{n^6+n^2+1}}$$

$$= \frac{n^6-\left(n^6+n^2+1\right)}{n^3+\sqrt{n^6+n^2+1}} = \frac{n^2+1}{n^3+n^3\sqrt{1+\frac{1}{n^4}+\frac{1}{n^6}}}$$

$$= \frac{n^3\left(\frac{1}{n}+\frac{1}{n^3}\right)}{n^3\left(1+\sqrt{1+\frac{1}{n^4}+\frac{1}{n^6}}\right)}$$

$$\lim_{n\to\infty} \frac{\frac{1}{n}+\frac{1}{n^3}}{1+\sqrt{1+\frac{1}{n^4}+\frac{1}{n^6}}} = \underline{\underline{0}}$$

e) $$e_n = \sqrt[4]{n^4+n^3} - \sqrt[4]{n^4-n^3}$$

$$= \frac{\sqrt[2]{n^4+n^3}-\sqrt[2]{n^4-n^3}}{\sqrt[4]{n^4+n^3}+\sqrt[4]{n^4-n^3}} = \frac{(n^4+n^3)-(n^4-n^3)}{\left(n\sqrt[4]{1+\frac{1}{n}}+n\sqrt[4]{1-\frac{1}{n}}\right)\left(n^2\sqrt{1+\frac{1}{n}}+n^2\sqrt{1-\frac{1}{n}}\right)}$$

$$\lim_{n\to\infty}\left(\frac{2\cancel{n^3}}{\underbrace{\left(\cancel{n^3}\sqrt{1+\frac{1}{n}}\sqrt{1+\frac{1}{n}}\right)}_{\text{für } n\to\infty \to 1\cdot 1=1}+2\left(\cancel{n^3}\sqrt{1+\frac{1}{n}}\sqrt{1-\frac{1}{n}}\right)+\left(\cancel{n^3}\sqrt{1-\frac{1}{n}}\sqrt{1-\frac{1}{n}}\right)}\right) =$$

$$= \frac{2}{4} = \underline{\underline{\frac{1}{2}}}$$

f) $$f_n = \frac{n^2}{n+2} - \frac{n^2}{n+1} = \frac{n^2(n+1)-n^2(n+2)}{(n+2)(n+1)}$$

$$= \frac{n^3+n^2-n^3-2n^2}{n^2+n+2n+2} = \frac{-n^2}{n^2+3n+2}$$

$$\lim_{n\to\infty} \frac{-1}{\left(1+\frac{3}{n}+\frac{2}{n^2}\right)} = \underline{\underline{-1}}$$