Bearbeitete Aufgaben: A10, A11, A12

10/10*30=30

A10

a)

(i) für $|q| < 1$

$$\begin{aligned}\left(\sum_{k=0}^{\infty} k\ q^k\right) \cdot \left(\sum_{k=0}^{\infty} q^k\right) &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} k \cdot q^k \cdot q^{n-k}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} k \cdot q^n\right) = \sum_{n=0}^{\infty} q^n \cdot \left(\sum_{k=0}^{n} k\right) \\ &\overset{*}{=} \sum_{n=0}^{\infty} q^n \cdot \frac{n \cdot (n+1)}{2} = \sum_{n=0}^{\infty} q^n \cdot \frac{n^2+n}{2} \\ &= \frac{q}{(1-q)^2} \cdot \frac{1}{1-q} = \frac{q}{(1-q)^3}\end{aligned}$$

$*$: Gaußsche Summenformel aus 1. Semester

(ii) für $|q| < 1$

$$\begin{aligned}\sum_{k=0}^{\infty} k^2\ q^k &= \sum_{k=0}^{\infty} q^k \cdot k^2 \\ &= \sum_{k=0}^{\infty} q^k \cdot (k^2 + k - k) \\ &= \sum_{k=0}^{\infty} q^k \cdot (k^2 + k) - q^k \cdot k \\ &= \sum_{k=0}^{\infty} \left(q^k \cdot (k^2+k)\right) - \sum_{k=0}^{\infty} \left(q^k \cdot k\right) \\ &= \sum_{k=0}^{\infty} \left(q^k \cdot \frac{k^2+k}{2} \cdot 2\right) - \sum_{k=0}^{\infty} \left(q^k \cdot k\right) \\ &= 2 \cdot \sum_{k=0}^{\infty} \left(q^k \cdot \frac{k^2+k}{2}\right) - \sum_{k=0}^{\infty} \left(q^k \cdot k\right) \\ &= \frac{2q}{(1-q)^3} - \frac{q}{(1-q)^2} = \frac{q+q^2}{(1-q)^3}\end{aligned}$$

b) $\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)}$
n-te Partialsumme p_n:

$$
\begin{aligned}
p_n &= \sum_{k=0}^{n} \frac{1}{(k+1)(k+2)} = \sum_{k=0}^{n} \frac{1}{k+1} - \frac{1}{k+2} = \sum_{k=0}^{n} \frac{1}{k+1} - \frac{1}{k+2} \\
&= \sum_{k=0}^{n} \frac{1}{k+1} - \sum_{k=0}^{n} \frac{1}{k+2} \overset{*}{=} \sum_{k=0}^{n} \frac{1}{k+1} - \sum_{k=1}^{n+1} \frac{1}{k+1} \\
&= \frac{1}{1} - \frac{1}{n+2} \xrightarrow{(n\to\infty)} 1
\end{aligned}
$$

$*$: Indextransformation in der 2. Summe: $k_{neu} = k_{alt} + 1$
Der gesuchte Grenzwert der Reihe ist der Grenzwert der Folge der Partialsummen, also 1.

A11

a)

(i) $\sum_{k=0}^{\infty} \frac{5^k}{k} x^k$

$$\limsup_k \sqrt[k]{\frac{5^k}{k}} = \limsup_k \frac{5}{\sqrt[k]{k}} = \lim_k \frac{5}{\sqrt[k]{k}} = 5 \implies R = \frac{1}{5}$$

(ii) $\sum_{k=0}^{\infty} \left(\sqrt{k+1} - \sqrt{k - \sqrt{k}}\right)^{2k} x^k$

$$
\begin{aligned}
\limsup_k \sqrt[k]{\left(\sqrt{k+1} - \sqrt{k-\sqrt{k}}\right)^{2k}} &= \limsup_k \left(\sqrt{k+1} - \sqrt{k-\sqrt{k}}\right)^2 \\
&= \left(\limsup_k \sqrt{k+1} - \sqrt{k-\sqrt{k}}\right)^2 \\
&= \left(\limsup_k \frac{\left(\sqrt{k+1} - \sqrt{k-\sqrt{k}}\right)\left(\sqrt{k+1} + \sqrt{k-\sqrt{k}}\right)}{\sqrt{k+1} + \sqrt{k-\sqrt{k}}}\right)^2 \\
&= \left(\limsup_k \frac{k+1-k+\sqrt{k}}{\sqrt{k+1} + \sqrt{k-\sqrt{k}}}\right)^2
\end{aligned}
$$

$$= \left(\limsup_k \frac{\sqrt{k}\left(\frac{1}{\sqrt{k}}+1\right)}{\sqrt{k}\left(\sqrt{1+\frac{1}{k}}+\sqrt{1-\frac{1}{\sqrt{k}}}\right)} \right)^2$$
$$= \left(\limsup_k \frac{\frac{1}{\sqrt{k}}+1}{\sqrt{1+\frac{1}{k}}+\sqrt{1-\frac{1}{\sqrt{k}}}} \right)^2$$
$$= \left(\lim_k \frac{\frac{1}{\sqrt{k}}+1}{\sqrt{1+\frac{1}{k}}+\sqrt{1-\frac{1}{\sqrt{k}}}} \right)^2$$
$$= \left(\frac{1}{2}\right)^2 = \frac{1}{4} \implies R = 4$$

(iii) $\displaystyle\sum_{k=0}^{\infty}(k!+2)x^k$

$$\limsup_k \sqrt[k]{(k!+2)} = +\infty \implies R = 0$$

Da $\sqrt[k]{k!}$ wie aus der Vorlesung bekannt gegen $+\infty$ divergiert, divergiert auch der größere Term $\sqrt[k]{(k!+2)}$ gegen $+\infty$ (Minorantenkriterium).

(iv) $\displaystyle\sum_{k=0}^{\infty}\frac{2^k}{k^2}x^{4k} \rightsquigarrow y := x^4 \rightsquigarrow \sum_{k=0}^{\infty}\frac{2^k}{k^2}y^k$

$$\limsup_k \sqrt[k]{\frac{2^k}{k^2}} = \limsup_k \frac{2}{\sqrt[k]{k^2}} = \limsup_k \frac{2}{\sqrt[k]{k}\cdot\sqrt[k]{k}} = \lim_k \frac{2}{\underbrace{\sqrt[k]{k}}_{\xrightarrow{(k\to\infty)}1} \cdot \underbrace{\sqrt[k]{k}}_{\xrightarrow{(k\to\infty)}1}} = 2 \implies R_y = \frac{1}{2}$$
$$\implies R_x = \sqrt[4]{R_y} = \frac{1}{\sqrt[4]{2}}$$

b)

$$S(x) := \sum_{k=0}^{\infty}\left(\sqrt[k]{3k}+\frac{4}{\sqrt[k]{k!}}+1\right)^k\left(\frac{1}{x+3}\right)^k \quad \rightsquigarrow y := \frac{1}{x+3} \quad \rightsquigarrow \sum_{k=0}^{\infty}\left(\sqrt[k]{3k}+\frac{4}{\sqrt[k]{k!}}+1\right)^k y^k$$
$$\limsup_k \sqrt[k]{\left(\sqrt[k]{3k}+\frac{4}{\sqrt[k]{k!}}+1\right)^k} = \limsup_k \sqrt[k]{3k}+\frac{4}{\sqrt[k]{k!}}+1 = \lim_k \underbrace{\sqrt[k]{3k}}_{*}+\frac{4}{\sqrt[k]{k!}}+1 = 2 \implies R_y = \frac{1}{2}$$

$*$: $\underbrace{\sqrt[k]{k}}_{\xrightarrow{(k\to\infty)}1} < \sqrt[k]{3k} \leq \underbrace{\sqrt[k]{k^2}}_{\xrightarrow{(k\to\infty)}1} = \underbrace{\sqrt[k]{k}}_{\xrightarrow{(k\to\infty)}1} \cdot \underbrace{\sqrt[k]{k}}_{\xrightarrow{(k\to\infty)}1} \qquad \forall k \geq 3 \qquad \implies \lim_k \sqrt[k]{3k} = 1$

Die gegebene Reihe ist konvergent, für $x \in \mathbb{R} : \left|\frac{1}{x+3}\right| < R_y = \frac{1}{2} \implies 2 < |x+3| \wedge x \neq -3$

$\rightsquigarrow 2 < x+3 \wedge 0 < x+3 \implies -1 < x$

$\rightsquigarrow 2 < (-1)\cdot(x+3) \wedge 0 \geq x+3 \implies x < -5$

Also muss $x \in (-\infty, -5) \cup (-1, +\infty)$ gelten, damit die Reihe konvergent ist.

A12

a)

(i)

$$\begin{aligned}
&\exp(3ix) = \exp(ix)^3 \\
\overset{*}{\iff}\ &\cos(3x) + i\sin(3x) = (\cos(x) + i\sin(x))^3 \\
\iff\ &\cos(3x) + i\sin(3x) = \cos(x)^3 - 3\cos(x)\sin(x)^2 + 3i\cos(x)^2\sin(x) - i\sin(x)^3 \\
&\rightsquigarrow \cos(3x) = \cos(x)^3 - 3\cos(x)\sin(x)^2 \\
&\rightsquigarrow \sin(3x) = 3\cos(x)^2\sin(x) - \sin(x)^3
\end{aligned}$$

$*$: Eulersche Formel

(ii)

$$\begin{aligned}
\cos(3x) &= \cos(x)\cos(2x) - \sin(x)\sin(2x) \\
&\overset{*}{=} \cos(x)\big(\cos(x)\cos(x) - \sin(x)\sin(x)\big) - \sin(x)\big(\sin(x)\cos(x) + \cos(x)\sin(x)\big) \\
&= \cos(x)^3 - \cos(x)\sin(x)^2 - 2\cos(x)\sin(x)^2 \\
&= \cos(x)^3 - 3\cos(x)\sin(x)^2 \\
\sin(3x) &= \sin(x)\cos(2x) + \cos(x)\sin(2x) \\
&= \sin(x)\big(\cos(x)\cos(x) - \sin(x)\sin(x)\big) + \cos(x)\big(\sin(x)\cos(x) + \cos(x)\sin(x)\big) \\
&= \cos(x)^2\sin(x) - \sin(x)^3 + \cos(x)^2\sin(x) \\
&= 3\cos(x)^2\sin(x) - \sin(x)^3
\end{aligned}$$

$*$: Additionstheorem

b)

(i) $x := \frac{\pi}{3}$

$$\begin{aligned}
\sin\left(3\cdot\frac{\pi}{3}\right) &= 3\cos\left(\frac{\pi}{3}\right)^2\sin\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right)^3 \\
\iff \sin(\pi) &= \sin\left(\frac{\pi}{3}\right)\left(3\cos\left(\frac{\pi}{3}\right)^2 - \sin\left(\frac{\pi}{3}\right)^2\right) \\
\overset{*}{\iff} 0 &= \sin\left(\frac{\pi}{3}\right)\left(3 - 3\sin\left(\frac{\pi}{3}\right)^2 - \sin\left(\frac{\pi}{3}\right)^2\right) \\
\iff 0 &= \underbrace{\sin\left(\frac{\pi}{3}\right)}_{\neq 0}\left(3 - 4\sin\left(\frac{\pi}{3}\right)^2\right) \\
\iff 0 &= 3 - 4\sin\left(\frac{\pi}{3}\right)^2 \\
\iff 3 &= 4\sin\left(\frac{\pi}{3}\right)^2 \\
\iff \frac{3}{4} &= \sin\left(\frac{\pi}{3}\right)^2 \\
\overset{**}{\implies} \frac{\sqrt{3}}{2} &= \sin\left(\frac{\pi}{3}\right)
\end{aligned}$$

$*$: $\cos(x)^2 = 1 - \sin(x)^2$
$**$: laut Angabe ist $\sin\frac{\pi}{3} \geq 0$

$$\begin{aligned}
\cos(x)^2 &= 1 - \sin(x)^2 \\
\implies \cos\left(\frac{\pi}{3}\right)^2 &= 1 - \sin\left(\frac{\pi}{3}\right)^2 \\
\iff \cos\left(\frac{\pi}{3}\right)^2 &= 1 - \left(\frac{\sqrt{3}}{2}\right)^2 \\
\iff \cos\left(\frac{\pi}{3}\right)^2 &= 1 - \frac{3}{4} \\
\iff \cos\left(\frac{\pi}{3}\right)^2 &= \frac{1}{4} \\
\overset{***}{\implies} \cos\left(\frac{\pi}{3}\right) &= \frac{1}{2}
\end{aligned}$$

$***$: laut Angabe ist $\cos\frac{\pi}{3} \geq 0$

(ii)

$$\cos(2x) = \cos(x)^2 - \sin(x)^2 = \cos(x)^2 - 1 + \cos(x)^2 = 2\cos(x)^2 - 1$$

$$\iff \cos(2x) + 1 = 2\cos(x)^2 \iff \frac{1}{2}\cos(2x) + \frac{1}{2} = \cos(x)^2$$
$$\implies \cos\left(\frac{\pi}{6}\right)^2 = \frac{1}{2}\cos\left(\frac{\pi}{3}\right) + \frac{1}{2}$$
$$\iff \cos\left(\frac{\pi}{6}\right)^2 = \frac{3}{4} \overset{*}{\implies} \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$*$: laut Angabe ist $\cos\frac{\pi}{6} \geq 0$

$$1 = \cos(x)^2 + \sin(x)^2 \iff \sin(x)^2 = 1 - \cos(x)^2$$
$$\implies \sin\left(\frac{\pi}{6}\right)^2 = 1 - \cos\left(\frac{\pi}{6}\right)^2 = 1 - \left(\frac{\sqrt{3}}{2}\right)^2 = 1 - \frac{3}{4} = \frac{1}{4}$$
$$\overset{**}{\implies} \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$**$: laut Angabe ist $\sin\frac{\pi}{6} \geq 0$

(iii)

$$\cos(x)^2 = \frac{1}{2}\cos(2x) + \frac{1}{2}$$
$$\implies \cos\left(\frac{\pi}{12}\right)^2 = \frac{1}{2}\cos\left(\frac{\pi}{6}\right) + \frac{1}{2} = \frac{2+\sqrt{3}}{4}$$
$$\overset{*}{\implies} \cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{2+\sqrt{3}}}{2}$$

$*$: laut Angabe ist $\cos\frac{\pi}{12} \geq 0$

$$\sin(x)^2 = 1 - \cos(x)^2$$
$$\implies \sin\left(\frac{\pi}{12}\right)^2 = 1 - \cos\left(\frac{\pi}{12}\right)^2 = 1 - \left(\frac{\sqrt{2+\sqrt{3}}}{2}\right)^2 = 1 - \frac{2+\sqrt{3}}{4} = \frac{2-\sqrt{3}}{4}$$
$$\overset{**}{\implies} \sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{2-\sqrt{3}}}{2}$$

$**$: laut Angabe ist $\sin\frac{\pi}{12} \geq 0$