Bearbeitete Aufgaben: A18, A19, A20

19.5/20*30=29

A18

a)

$$
\begin{aligned}
f(x) &= x^2 + x + \sqrt{x} + 1 + \frac{1}{\sqrt{x}} + \frac{1}{x} + \frac{1}{x^2} \\
f'(x) &= 2x + 1 + \frac{1}{2 \cdot \sqrt{x}} + 0 - \frac{1}{2 \cdot \sqrt{x^3}} - \frac{1}{x^2} - \frac{2}{x^3} \\
&= 2x + 1 + \frac{1}{2 \cdot \sqrt{x}} - \frac{1}{2 \cdot \sqrt{x^3}} - \frac{1}{x^2} - \frac{2}{x^3}
\end{aligned}
$$

b)

$$
\begin{aligned}
f(x) &= (x^2 + \sqrt{2x})^4 \qquad \text{für } x > 0 \\
f'(x) &= (x^2 + \sqrt{2x})^3 \cdot \left(2x + \sqrt{2}\frac{1}{\sqrt{2x}}\right) \cdot 4
\end{aligned}
$$

c)

$$
\begin{aligned}
f(x) &= x\ e^{x^2}\ \ln(2 + 3x) \qquad \text{für } x > -\frac{2}{3} \\
f'(x) &= e^{x^2} \ln(2 + 3x) + 2x^2\ e^{x^2} \ln(2 + 3x) + e^{x^2} \frac{3x}{2 + 3x} \\
&= e^{x^2} \left(\ln(2 + 3x) \left(1 + 2x^2\right) + \frac{3x}{2 + 3x} \right)
\end{aligned}
$$

d)

$$
\begin{aligned}
f(x) &= \arccos(\sqrt{x}) \qquad \text{für } 0 < x < 1 \\
f'(x) &= -\frac{1}{\sqrt{1 - x}} \cdot \frac{1}{2\sqrt{x}} \qquad \text{Nach Aufgabe P24} \\
&= -\frac{1}{2\sqrt{1 - x}\sqrt{x}}
\end{aligned}
$$

e)

$$
\begin{aligned}
f(x) &= \frac{\sin 2x}{\ln(x^2 + 1)} \\
f'(x) &= \frac{2 \ln(x^2 + 1) \cos(2x) - \sin(2x) \frac{2x}{x^2+1}}{\ln(x^2 + 1)}
\end{aligned}
$$

$$= 2\cos(2x) - \frac{2x\sin(2x)}{(x^2+1)\cdot\ln(x^2+1)}$$

f)

$$\begin{aligned} f(x) &= x^\alpha \qquad \text{für } x > 0, \alpha \in \mathbb{R}\setminus\{0\} \\ &= e^{\alpha\cdot\ln x} \\ f'(x) &= e^{\alpha\cdot\ln x}\cdot\frac{\alpha}{x} = \frac{\alpha\cdot x^\alpha}{x} = \alpha\cdot x^{\alpha-1} \end{aligned}$$

g)

$$\begin{aligned} f(x) &= x^{-x^2} \qquad \text{für } x > 0 \\ &= e^{-x^2\cdot\ln x} \\ f'(x) &= e^{-x^2\cdot\ln x}\cdot\left(-2x\ln x - x^2\cdot\frac{1}{x}\right) \\ &= x^{-x^2}\cdot(-2x\ln x - x) \\ &= -x^{-x^2+1}\cdot(2\ln x + 1) \end{aligned}$$

h)

$$\begin{aligned} f(x) &= \ln(x + \ln(2\ln x)) \\ f'(x) &= \frac{1}{x+\ln(2\ln x)}\cdot\left(1 + \frac{1}{2\ln x}\cdot\frac{2}{x}\right) \\ &= \frac{2}{x^2 + x\ln(2\ln x)} + \frac{2}{x^2 + x\ln(2\ln x)}\cdot\frac{1}{2\ln x} \\ &= \frac{2}{x+\ln(2\ln x)} + \frac{1}{x\ln x(x+\ln(2\ln x))} \end{aligned}$$

A19

a)

$$\begin{aligned} \frac{d}{dx}\cos x &= \lim_{h\to0}\frac{\cos(x+h)-\cos(x)}{h} \\ &= \lim_{h\to0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h} \\ &= \lim_{h\to0}\cos x\frac{\cos h}{h} - \sin x\frac{\sin h}{h} - \frac{\cos x}{h} \end{aligned}$$

$$= \lim_{h\to 0} \cos x \frac{\cos h - 1}{h} + \frac{\cos x}{h} - \sin x \frac{\sin h}{h} - \frac{\cos x}{h}$$
$$= \lim_{h\to 0} \underbrace{\cos x}_{\text{beschränkt}} \underbrace{\frac{\cos h - 1}{h}}_{\to 0} - \sin x \underbrace{\frac{\sin h}{h}}_{\to 1} \xrightarrow{(h\to 0)} -\sin x$$

b)

(i)

$$(\tan)'(x) = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

(ii)

$$(\tan)'(x) = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = 1 + \tan^2 x$$

c) $*$: Verweis auf b) (ii)

(i)

aus Vorlesung bekannt: $(f^{-1})'(y) = \dfrac{1}{f'(f^{-1}(y))}$

$$(\tan^{-1})'(y) = (\arctan)'(y) = \frac{1}{\tan'(\tan^{-1}(y))} \overset{*}{=} \frac{1}{1 + \tan^2(\tan^{-1}(y))} = \frac{1}{1+y^2}$$

(ii)

$$\tan''(x) = (\tan')' \overset{*}{=} (1 + \tan^2 x)' \overset{*}{=} 0 + 2\tan(x)\cdot(1+\tan^2 x) = \underline{2\tan(x) + 2\tan^3(x)}$$
$$\tan'''(x) = (\tan'')' = \big(2\tan(x) + 2\tan^3(x)\big)' = 2 + 2\tan^2(x) + 6\tan^2(x)(1+\tan^2 x)$$
$$= \underline{2 + 8\tan^2(x) + 6\tan^4(x)}$$

A20

a) für $x > 0$ (unter Zuhilfenahme von A18 f):

$$f'(x) = \left(x^\alpha \sin\frac{1}{x^2}\right)' = \alpha \cdot x^{\alpha-1} \cdot \sin\left(\frac{1}{x^2}\right) + x^\alpha \cdot \cos\left(\frac{1}{x^2}\right) \cdot \frac{-2}{x^3}$$

$$= \alpha \cdot x^{\alpha-1} \cdot \sin\left(\frac{1}{x^2}\right) - 2 \cdot x^{\alpha-1} \cdot \cos\left(\frac{1}{x^2}\right)\frac{1}{x^2}$$
$$= x^{\alpha-1}\left(\alpha \cdot \sin\left(\frac{1}{x^2}\right) - \frac{2}{x^2} \cdot \cos\left(\frac{1}{x^2}\right)\right)$$

b) $\alpha \in (0, \infty)$

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{h^\alpha \sin\left(\frac{1}{h^2}\right) - 0}{h} \overset{*}{=} \lim_{h\searrow 0} \frac{h^\alpha \sin\left(\frac{1}{h^2}\right)}{h}$$
$$= \lim_{h\searrow 0} h^{\alpha-1} \sin\left(\frac{1}{h^2}\right)$$

$*$: da $x \geq 0 \rightsquigarrow h > 0$

Falls $\alpha > 1$:

$$f'(0) = \lim_{h\searrow 0} \overbrace{h^{\alpha-1}}^{\to 0} \overbrace{\sin\left(\frac{1}{h^2}\right)}^{\text{beschränkt}} = 0 \qquad \rightsquigarrow f'(0) \text{ existiert für } \alpha > 1$$

Falls $\alpha = 1$:

$$f'(0) = \lim_{h\searrow 0} \overbrace{h^{\alpha-1}}^{\to 1} \overbrace{\sin\left(\frac{1}{h^2}\right)}^{\text{unbestimmt div.}} \qquad \rightsquigarrow f'(0) \text{ existiert nicht für } \alpha = 1$$

Falls $\alpha < 1$:

$$f'(0) = \lim_{h\searrow 0} \overbrace{h^{\alpha-1}}^{\to \infty} \overbrace{\sin\left(\frac{1}{h^2}\right)}^{\text{unbestimmt div.}} \qquad \rightsquigarrow f'(0) \text{ existiert nicht für } \alpha < 1$$

c) $f'(0) = 0$ für $\alpha > 1$

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} \alpha \cdot x^{\alpha-1} \cdot \sin\left(\frac{1}{x^2}\right) - 2 \cdot x^{\alpha-1} \cdot \cos\left(\frac{1}{x^2}\right)\frac{1}{x^2}$$
$$= \lim_{x\searrow 0} x^{\alpha-1} \cdot \alpha \cdot \sin\left(\frac{1}{x^2}\right) - x^{\alpha-3} \cdot 2\cos\left(\frac{1}{x^2}\right)$$

Falls $\alpha > 3$

$$\lim_{x\to 0} f'(x) = \lim_{x\searrow 0} \overbrace{x^{\alpha-1}}^{\to 0} \cdot \overbrace{\alpha \cdot \sin\left(\frac{1}{x^2}\right)}^{\text{beschränkt}} - \overbrace{x^{\alpha-3}}^{\to 0} \cdot \overbrace{2\cos\left(\frac{1}{x^2}\right)}^{\text{beschränkt}}$$
$$= 0 = f'(0) \rightsquigarrow f'(x) \text{ ist für } \alpha > 3 \text{ stetig an der Stelle } x = 0$$

Falls $\alpha = 3$

$$\lim_{x\to 0} f'(x) = \lim_{x\searrow 0} \overbrace{x^{\alpha-1}}^{\to 0} \cdot \overbrace{\alpha \cdot \sin\left(\frac{1}{x^2}\right)}^{\text{beschränkt}} - \overbrace{x^{\alpha-3}}^{\to 1} \cdot \overbrace{2\cos\left(\frac{1}{x^2}\right)}^{\text{unbestimt div.}}$$
$$\rightsquigarrow \text{Grenzwert existiert nicht } \rightsquigarrow f'(x) \text{ ist für } \alpha = 3 \text{ nicht stetig an der Stelle } x = 0$$

Falls $\alpha < 3$

$$\lim_{x\to 0} f'(x) = \lim_{x\searrow 0} \overbrace{x^{\alpha-1}}^{\to 0} \cdot \overbrace{\alpha \cdot \sin\left(\frac{1}{x^2}\right)}^{\text{beschränkt}} - \overbrace{x^{\alpha-3}}^{\to \infty} \cdot \overbrace{2\cos\left(\frac{1}{x^2}\right)}^{\text{unbestimt div.}}$$
$$\rightsquigarrow \text{Grenzwert existiert nicht } \rightsquigarrow f'(x) \text{ ist für } \alpha < 3 \text{ nicht stetig an der Stelle } x = 0$$

d)

$$
\begin{aligned}
f'' =& (\alpha-1)\cdot x^{\alpha-2}\cdot\left(\alpha\cdot\sin\left(\frac{1}{x^2}\right)-\frac{2}{x^2}\cdot\cos\left(\frac{1}{x^2}\right)\right)\\
&+x^{\alpha-1}\cdot\left(-\frac{2}{x^3}\alpha\cos\left(\frac{1}{x^2}\right)+\frac{4}{x^3}\cos\left(\frac{1}{x^2}\right)-\frac{2}{x^2}\sin\left(\frac{1}{x^2}\right)\cdot\frac{2}{x^3}\right)\\
=& (\alpha-1)\cdot x^{\alpha-2}\cdot\alpha\cdot\sin\left(\frac{1}{x^2}\right)-2(\alpha-1)\cdot x^{\alpha-4}\cdot\cos\left(\frac{1}{x^2}\right)\\
&-2\alpha\cdot x^{\alpha-4}\cdot\cos\left(\frac{1}{x^2}\right)+x^{\alpha-1}\cdot\left(\frac{4}{x^3}\cos\left(\frac{1}{x^2}\right)-\frac{4}{x^5}\sin\left(\frac{1}{x^2}\right)\right)\\
=& (\alpha-1)\cdot x^{\alpha-2}\cdot\alpha\cdot\sin\left(\frac{1}{x^2}\right)-2\cdot x^{\alpha-4}\cdot\cos\left(\frac{1}{x^2}\right)(\alpha-1+\alpha)\\
&+x^{\alpha}\cdot\left(\frac{4}{x^4}\cos\left(\frac{1}{x^2}\right)-\frac{4}{x^6}\sin\left(\frac{1}{x^2}\right)\right)\\
=& (\alpha-1)\cdot x^{\alpha-2}\cdot\alpha\cdot\sin\left(\frac{1}{x^2}\right)-4\alpha\cdot x^{\alpha-4}\cdot\cos\left(\frac{1}{x^2}\right)\\
&-2\cdot x^{\alpha-4}\cdot\cos\left(\frac{1}{x^2}\right)+x^{\alpha}\cdot\left(\frac{4}{x^4}\cos\left(\frac{1}{x^2}\right)-\frac{4}{x^6}\sin\left(\frac{1}{x^2}\right)\right)\\
=& (\alpha-1)\cdot x^{\alpha-2}\cdot\alpha\cdot\sin\left(\frac{1}{x^2}\right)-4\alpha\cdot x^{\alpha-4}\cdot\cos\left(\frac{1}{x^2}\right)\\
&+x^{\alpha}\cdot\left(\frac{6}{x^4}\cos\left(\frac{1}{x^2}\right)-\frac{4}{x^6}\sin\left(\frac{1}{x^2}\right)\right)
\end{aligned}
$$