Mathe Blatt 3

9/10*30=27

A7 a) i) $a_n = \frac{5 + (-1)^n + \frac{1}{n}\sin n}{n^2}$

$$\frac{5 - 1 + \left(\frac{1}{n} \cdot (-1)\right)}{n^2} \leq \frac{5 + (-1)^n + \frac{1}{n}\sin n}{n^2} \leq \frac{5 + 1 + \frac{1}{n} \cdot 1}{n^2}$$

$$\frac{4 - \left(\frac{1}{n}\right)}{n^2} \leq \frac{5 + (-1)^n + \frac{1}{n}\sin n}{n^2} \leq \frac{6 + \frac{1}{n}}{n^2}$$

$\frac{1}{n} \xrightarrow{n\to\infty} 0$, $n^2 \xrightarrow{n\to\infty} +\infty$, $\to 0$

$$\Rightarrow \lim_{n\to\infty} \frac{5 + (-1)^n + \frac{1}{n}\sin n}{n^2} = \underline{\underline{0}}$$

b) $b_n = \frac{n}{n^2+1} \cdot \frac{5\sin(2n) - 2\sin(3n)}{6 + \cos(4n) - \cos(5n)}$

$$\frac{n}{n^2+1} \cdot \frac{-5 - 2}{6 - 1 - 1} \leq \frac{5\sin(2n) - 2\sin(3n)}{6 + \cos(4n) - \cos(5n)} \cdot \frac{n}{n^2+1} \leq \frac{n}{n^2+1} \cdot \frac{5 + 2}{6 - 1 - 1}$$

$$\frac{n}{n^2+1} \cdot \frac{-7}{4} \leq \frac{n}{n^2+1} \cdot \frac{5\sin(2n) - 2\sin(3n)}{6 + \cos(4n) - \cos(5n)} \leq \frac{n}{n^2+1} \cdot \frac{7}{4}$$

$$\frac{\frac{1}{n}}{n + \frac{1}{n}} \cdot \frac{-7}{4} \leq \frac{n}{n^2+1} \cdot \frac{5\sin(2n) - 2\sin(3n)}{6 + \cos(4n) - \cos(5n)} \leq \frac{\frac{1}{n}}{n + \frac{1}{n}} \cdot \frac{7}{4}$$

$\frac{1}{n} \to 0$, $n + \frac{1}{n} \to \infty$, $\to 0$; $\to 0$

$$\lim_{n\to\infty} \frac{n}{n^2+1} \cdot \frac{\sin(2n) - 2\sin(3n)}{6 + \cos(4n) - \cos(5n)} = \underline{\underline{0}}$$

A7b)

i) Menge d. HP $M = \{0, "+\infty"\}$

$\liminf_{n \to \infty} = 0$

$\limsup_{n \to \infty} = +\infty$

ii) ~~Men~~ $M = \{-1, 1\}$

$\liminf_{n \to \infty} = -1$

$\limsup_{n \to \infty} = 1$

iii) $M = \{"+\infty"\}$

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$\limsup_{n \to \infty} = +\infty$

iv) Fallunterscheidung:

$-1 < q < 1$ $\quad M = \{0\}$

$q = -1$ $\quad M = \{-1, 1\}$

$q = 1$ $\quad M = \{1\}$

$q < -1$ $\quad M = \{"-\infty", "+\infty"\}$

$q > 1$ $\quad M = \{"+\infty"\}$

~~$0 < q \leq 1$~~ ~~Men~~

A8

a) $\sum_{k=0}^{\infty} \frac{k}{2+k}$

Divergenzkrit. $\Rightarrow$ a_n $\left(\frac{k}{2+k}\right)$ konvergiert nicht gegen 0 (sondern 1!)

$\Rightarrow$ Reihe $\sum_{k=0}^{\infty} \frac{k}{2+k}$ ist divergent!

b) $\sum_{k=2}^{\infty} \left(\frac{k-1}{3k^2+2k}\right)^{\frac{k}{2}}$

Wurzelkrit. $\rightarrow$ I: $\sqrt[k]{|(a_k)|} \geq 1 \ \forall k \rightarrow$ Reihe div.

II: $\sqrt[k]{|(a_k)|} \leq q \ \forall k$ und $q < 1$

$\rightarrow$ Reihe konv.

$\sqrt[k]{|(a_k)|} = \sqrt[k]{\left|\left(\frac{k-1}{3k^2+2k}\right)^{\frac{k}{2}}\right|}$

$= \left|\left(\frac{k-1}{3k^2+2k}\right)^{\frac{1}{2}}\right| = \left|\left(\frac{\frac{1}{k}-\frac{1}{k^2}}{k(3+\frac{2}{k})}\right)^{\frac{1}{2}}\right|$

$\limsup_{k\to\infty} \sqrt{\left|\left(\frac{\frac{1}{k}-\frac{1}{k^2}}{k(3+\frac{2}{k})}\right)\right|} = \underline{\underline{0}} < 1 \Rightarrow$ II $\rightarrow$ Reihe konvergiert!

c) $\sum_{k=0}^{\infty} \frac{\sin k}{k^k}$ Wurzelkrit.

$\sqrt[k]{|(a_k)|} = \sqrt[k]{\left|\left(\frac{\sin k}{k^k}\right)\right|} = \frac{\overbrace{\sqrt[k]{|\sin k|}}^{\leq 1}}{k} \leq \frac{1}{k} \leq \frac{1}{2} \ \forall k \geq 2$

$=: q < 1$

$\Rightarrow$ Reihe ist konvergent!

d) $\sum_{k=1}^{\infty} \frac{\sqrt{k+2} - \sqrt{k-1}}{2^k}$

$a_k = \frac{k+2-(k-1)}{2^k(\sqrt{k+2}+\sqrt{k-1})} = \frac{3}{2^k(\sqrt{k+2}+\sqrt{k-1})}$

Quotientenkrit.: $\left|\frac{a_{n+1}}{a_n}\right|$ $< 1 \rightarrow$ Reihe konvergent
$= 1 \rightarrow$?
$> 1 \rightarrow$ Reihe divergent

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{3\cdot 2^k(\sqrt{k+2}+\sqrt{k-1})}{3\cdot 2^{k+1}(\sqrt{k+3}+\sqrt{k})}\right| = \frac{1\left(\sqrt{\frac{1}{k}+\frac{2}{k^2}}+\sqrt{\frac{1}{k}-\frac{1}{k^2}}\right)}{2\left(\sqrt{\frac{1}{k}+\frac{3}{k^2}}+\sqrt{\frac{1}{k}}\right)} \xrightarrow{k\to\infty} < \underline{\underline{1}}$

woher siehst du das?

$\rightarrow$ Reihe ist konvergent!