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Blatt: 04
Gruppe: 7
Aufgaben: alle

10/10*30

A10)

a.) (i) $\sum_{k=0}^{\infty} k\cdot q^k \cdot \sum_{k=0}^{\infty} q^k \qquad |q|<1$

$$\left(\sum_{k=0}^{\infty} k\cdot q^k\right)\cdot\left(\sum_{k=0}^{\infty} q^k\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} k\cdot q^k \cdot q^{n-k}\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} k\cdot q^n\right) =$$

$$= \sum_{n=0}^{\infty} q^n \cdot \frac{n(n+1)}{2} = \sum_{n=0}^{\infty} q^n\left(\frac{n^2}{2}+\frac{n}{2}\right)$$

(ii) $\sum_{k=0}^{\infty} q^k \cdot \frac{k(k+1)}{2} = \sum_{k=0}^{\infty} k\, q^k \cdot \sum_{k=0}^{\infty} q^k$

$$\Leftrightarrow \frac{1}{2}\sum_{k=0}^{\infty} q^k (k^2+k) = \sum_{k=0}^{\infty} k\cdot q^k \cdot \sum_{k=0}^{\infty} q^k$$

$$\Leftrightarrow \sum_{k=0}^{\infty} q^k k^2 = 2\cdot\left(\sum_{k=0}^{\infty} k\cdot q^k\right)\left(\sum_{k=0}^{\infty} q^k\right) - \sum_{k=0}^{\infty} q^k\cdot k$$

$$\Leftrightarrow \sum_{k=0}^{\infty} q^k k^2 = 2\cdot\frac{q}{(1-q)^2}\cdot\frac{1}{1-q} - \frac{q}{(1-q)^2} = \frac{q}{(1-q)^2}\left(\frac{2}{1-q}-1\right) =$$

$$= \frac{q}{(1-q)^2}\,\frac{1+q}{1-q} = \frac{q\,(1+q)}{(1-q)^3}$$

b.) $\sum_{k=0}^{\infty}\frac{1}{(k+1)(k+2)} = \sum_{k=0}^{\infty}\left(\frac{1}{k+1}-\frac{1}{k+2}\right) = \sum_{k=0}^{\infty}\frac{1}{k+1} - \sum_{k=0}^{\infty}\frac{1}{k+2} =$

$$= \sum_{k=1}^{\infty}\frac{1}{k} - \sum_{k=2}^{\infty}\frac{1}{k} = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \sum_{k=2}^{n}\frac{1}{k}\right) = \lim_{n\to\infty}\left(\frac{1}{1}+\frac{1}{2}+\dots+\frac{1}{n}-\frac{1}{2}-\frac{1}{3}-\dots-\frac{1}{n}\right) = 1$$

A11)

a.) (i) $\lim_{k\to\infty}\sqrt[k]{\left|\frac{5^k}{k}\right|} = \lim_{k\to\infty}\frac{\sqrt[k]{5^k}}{\sqrt[k]{k}} = 5 \Rightarrow R = \frac{1}{5}$

(ii) $\lim_{k\to\infty}\sqrt[k]{\left|(\sqrt{k+1}-\sqrt{k-\sqrt{k}})^{2k}\right|} = \lim_{k\to\infty}(\sqrt{k+1}-\sqrt{k-\sqrt{k}})^2 = \lim_{k\to\infty}\left(\frac{k+1-k+\sqrt{k}}{\sqrt{k+1}+\sqrt{k-\sqrt{k}}}\right)^2 =$

$$= \lim_{k\to\infty}\left(\frac{\sqrt{k}+1}{\sqrt{k+1}+\sqrt{k-\sqrt{k}}}\right)^2 = \lim_{k\to\infty}\left(\frac{1+\frac{1}{\sqrt{k}}}{\sqrt{1+\frac{1}{k}}+\sqrt{1-\frac{1}{\sqrt{k}}}}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4} \Rightarrow R = 4$$

(iii) $\lim_{k\to\infty}\sqrt[k]{|k!+2|} = \lim_{k\to\infty}\sqrt[k]{k!+2} > \underbrace{\lim_{k\to\infty}\sqrt[k]{k!}}_{=+\infty} \Rightarrow R = 0$

(iv) $\sum_{k=0}^{\infty}\frac{2^k}{k^2}x^{4k} = \sum_{k=0}^{\infty}\frac{2^k}{k^2}y^k \quad (y = x^4)$

$\lim_{k\to\infty}\sqrt[k]{\left|\frac{2^k}{k^2}\right|} = \lim_{k\to\infty}\frac{\sqrt[k]{2^k}}{\sqrt[k]{k^2}} = 2 \Rightarrow R_y = \frac{1}{2}$

$\Rightarrow R_x = \sqrt[4]{\frac{1}{2}}$

b.) $y = \frac{1}{x+3}$; $S(y) = \sum_{k=0}^{\infty}\left(\sqrt[k]{3^k} + \frac{4}{\sqrt[k]{k!}} + 1\right)^k y^k$

$\lim_{k\to\infty}\sqrt[k]{\left|\left(\sqrt[k]{3^k}+\frac{4}{\sqrt[k]{k!}}+1\right)\right|^k} = \lim_{k\to\infty}\left(\sqrt[k]{3^k}+\frac{4}{\sqrt[k]{k!}}+1\right) = 1+0+1 = 2 \Rightarrow R(y) = \frac{1}{2}$

$\Rightarrow S(y)$ ist absolut konvergent für alle $y\in\left(-\frac{1}{2};\frac{1}{2}\right)$

Rücksubstitution $y = \frac{1}{x+3}$ $S(x)$ ist absolut konvergent für alle $|x+3| > 2$

$$\Leftrightarrow \left[\begin{array}{l} x+3 > 2 \\ x+3 < -2 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x > -1 \\ x < -5 \end{array}\right.$$

$\Rightarrow x \in (-\infty;-5) \vee (-1;+\infty)$

A12)

a.) (i) $\exp(3ix) = \exp(ix)^3$

$\exp(3ix) = \cos(3x) + i\sin(3x)$

$\exp(ix)^3 = (\cos(x) + i\sin(x))^3$

(ii) $\sin(3x) = \sin(2x+x) = \sin 2x \cos x + \sin x \cos 2x = 2\sin x \cdot \cos x \cdot \cos x + \sin x\,(\cos^2 x - \sin^2 x) =$
$= 2\sin x \cos^2 x + \sin x \cos^2 x - \sin^3 x = 3\sin x \cos^2 x - \sin^3 x = 3\sin x - 4\sin^3 x$

$\cos(3x) = \cos(2x+x) = \cos 2x \cdot \cos x - \sin 2x \sin x = (\cos^2 x - \sin^2 x)\cdot \cos x - 2\sin^2 x \cos x =$
$= \cos^3 x - 3\sin^2 x \cos x = 4\cos^3 x - 3x$

(i) $\exp(3ix) = \cos(3x) + i\sin(3x)$ ①

$\exp(ix)^3 = (\cos(x) + i\sin(x))^3 = (\cos x + i\sin x)(\cos x + i\sin x)(\cos x + i\sin x) =$
$= (\cos^2 x - \sin^2 x + i\,2\sin x \cos x)(\cos x + i\sin x) = \cos^3 x - \sin^2 x \cdot \cos x - 2\sin^2 x \cos x$
$+ i\,2\sin x \cos^2 x + i\sin x \cos^2 x - i\sin^3 x = (\cos^3 x - 3\sin^2 x \cos x) + i\,(3\sin x \cos^2 x - \sin^3 x)$ ②

aus ① und ②

$$\Rightarrow \begin{cases} \cos 3x = \cos^3 x - 3\sin^2 x \cos x = 4\cos^3 x - 3\cos x \\ \sin 3x = 3\sin x \cos^2 x - \sin^3 x = 3\sin x - 4\sin^3 x \end{cases}$$

b.) (i) $\sin 3x = 3\sin x - 4\sin^3 x$; setze $x = \frac{\pi}{3}$ $\Rightarrow \sin\pi = 3\sin\frac{\pi}{3} - 4\sin^3\frac{\pi}{3} = 0$

$\Leftrightarrow \sin\frac{\pi}{3}\left(3 - 4\sin^2\frac{\pi}{3}\right) = 0$

$\Leftrightarrow \left[\begin{array}{l} \sin\frac{\pi}{3} = 0 \\ \sin^2\frac{\pi}{3} = \frac{3}{4} \end{array}\right.$ geht nicht laut Aufgabenstellung

$\Leftrightarrow \left[\begin{array}{l} \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \\ \sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2} \end{array}\right.$ geht nicht laut Aufgabenstellung

$\sin^2\frac{\pi}{3} + \cos^2\frac{\pi}{3} = 1$

$\Leftrightarrow \frac{3}{4} + \cos^2\frac{\pi}{3} = 1 \Leftrightarrow \cos^2\frac{\pi}{3} = \frac{1}{4}$

$\Leftrightarrow \cos\frac{\pi}{3} = \frac{1}{2} \quad (\cos\frac{\pi}{3} > 0)$

(ii) $\cos(\frac{\pi}{3}) = \cos(2\cdot\frac{\pi}{6}) \Leftrightarrow \frac{1}{2} = 1 - 2\sin^2\frac{\pi}{6} \Leftrightarrow \sin^2\frac{\pi}{6} = \frac{1}{4}$

$\Leftrightarrow \sin\frac{\pi}{6} = \frac{1}{2} \quad (\sin\frac{\pi}{6} > 0) \Leftrightarrow \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \quad (1 - \sin^2\frac{\pi}{6} \wedge \cos\frac{\pi}{6} > 0)$

$\cos^2\frac{\pi}{6} + \sin^2\frac{\pi}{6} = 1 \Leftrightarrow \cos^2\frac{\pi}{6} + \frac{1}{4} = 1 \Leftrightarrow \cos^2\frac{\pi}{6} = \frac{3}{4} \Leftrightarrow \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

(iii) $\cos(\frac{\pi}{6}) = 1 - 2\sin^2\frac{\pi}{12} \Leftrightarrow \frac{\sqrt{3}}{2} = 1 - 2\sin^2\frac{\pi}{12} \Leftrightarrow \sin^2\frac{\pi}{12} = \frac{2-\sqrt{3}}{4}$

$\Leftrightarrow \sin\frac{\pi}{12} = \sqrt{\frac{2-\sqrt{3}}{4}} \quad (\sin\frac{\pi}{12} > 0) \Leftrightarrow \sin\frac{\pi}{12} = \sqrt{\frac{8-4\sqrt{3}}{16}} = \sqrt{\frac{(\sqrt{6}-\sqrt{2})^2}{16}} = \frac{\sqrt{6}-\sqrt{2}}{4}$

$\cos\frac{\pi}{12} = \sqrt{1 - \sin^2\frac{\pi}{12}} = \sqrt{1 - \frac{2-\sqrt{3}}{4}}$

$= \sqrt{\frac{2+\sqrt{3}}{4}} = \sqrt{\frac{8+4\sqrt{3}}{16}} = \sqrt{\frac{6 + 2\sqrt{6}\sqrt{2} + 2}{16}} = \frac{\sqrt{(\sqrt{6}+\sqrt{2})^2}}{4} =$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$

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