FOURIER SERIES

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1. Introduction

A Fourier series, not to be confused with a Fourier transform or its many types, is a representation of a periodic function comprising of sine and cosine functions. Fourier transforms are very similar, however use complex functions whereas a Fourier series sticks to harmonics. Although trivial at first sight, Fourier series can be used to describe the periodic variation of gas pressure in an internal combustion engine, are used to calculate heat transfer, and are used extensively in signal analysis. Particularly intriguing is the way Fourier series are used to represent other functions, which allows for the wide application of this concept. Fourier's breakthrough meant that functions could now be displayed as a sum of simple oscillations. More importantly, that discontinuous periodic functions can be analyzed using continuous periodic functions. On a more personal note, what interests me most about Fourier series is the designs and artwork made by summing periodic functions. As a student currently in aerospace engineering, Fourier series are used in various different applications related to my field of study. These stretch from heat transfer, all the way to radar and digital communication.

2. Theory

A Fourier series is defined by

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \ \cos(2\pi nx) \ + \ b_n \ \sin(2\pi nx))$

Where a_0, a_n, b_n and are all constants which can be defined as:

$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \ dx$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \ \cos(2\pi nx) \ dx$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \ \sin(2\pi nx) \ dx$

For this paper, 2π will be used as the period for all the simple harmonic functions, but in reality this can be changed to a different frequency. Notice that n is some whole number multiple of the base frequency. $f(x)$ is called an odd function when there is no cosine terms in the series, and will be called even when there is no sine terms in the series (Bocher). Notice also how it must be a repeating pattern or periodic, therefore,

$f(x) = f(x + 2\pi)$

must be true for the function $f(x)$ for it to be represented by a Fourier series.

As an example, take a square wave, where the y value instantaneously jumps from 1 to -1 and back.

$f(x) = \begin{Bmatrix} 1 \text{ if } x \leq 0.5 \\ -1 \text{ if } x > 0.5 \end{Bmatrix}$

This can also be represented as the following series

$\sum_{n=1}^{\infty} \frac{\frac{2}{2n-1}}{\pi} \sin((2n-1)2\pi x)$

The following is two graphs made in MATLAB using a for loop using the previous series.

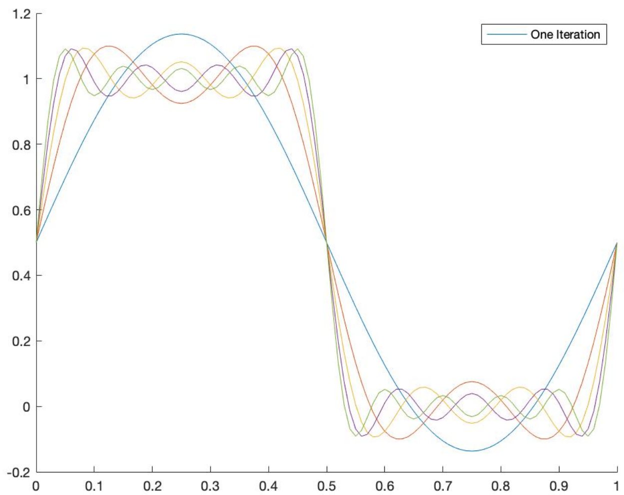

FIGURE 1. 10 Iterations (n=10)

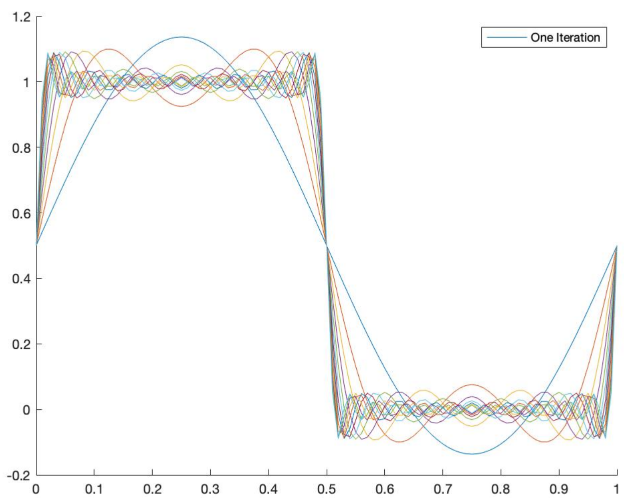

FIGURE 2. 25 Iterations (n=25)

As demonstrated in these graphs, as n increases the sum of functions gets closer and closer to a step function. However it is important to keep in mind that no finite sum of sine waves will never give a discontinuous function. Fourier series are good for approximating, as more iterations are calculated the answer will be closer and closer to the real value. Fourier series can only be applied to periodic functions. Still, the true value in a Fourier series is simplifying a complex function to a combination of simple and easy to use sine or cosine functions (Serov).

Another way to better comprehend Fourier series is as an addition of vectors. If each individual sine or cosine function is viewed as a vector, graphing the y value of the tip of the sum of vectors gives the same result as before. Take the Fourier series from earlier which converges to a square wave, as each iteration is calculated, the frequency and length of the vector changes. With each new sin function the frequency is increasing as the length decreases. As each vector rotates a specific frequency times a constant based on the iteration number, the smaller vectors complete their full rotations when at the top of the first vector's trajectory. This results in a motion similar to that of a whip when the sum of vectors is moving from high y values to low y values. The rapid motion is then seen on the graph as a sharp change in y value. This motion can be seen in a .gif file found in the appendix.

This concept of vectors moving in circular motion with their centre moving on the circumference of a separate circle is called an epicycle. Epicycles were used as far back as the ancient Greeks, with Ptolemy using them to map orbits of celestial objects. Although not perfect as Kepler later discovered most orbits are in fact elliptical, in an ideal world where orbital planes are equivalent and orbits are circular, epicycles do a good job depicting orbits.

The definition of a Fourier series can also be simplified to a sum of a single function using Euler's formula (Serov).

$\sum_{n=0}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} i\ b_n\ \sin(nx) = \sum_{n=-\infty}^{\infty} c_n\ e^{inx}$

This notation will converge to the integral for the inverse Fourier transform is commonly seen in engineering. Fourier transforms which build upon the idea of a Fourier series, and in short are Fourier series who's period extends up to infinity. Moreover, this notation shows the applicability of the Fourier series. How does this work? Start by considering $c \cdot e^{i\ n\ t}$ and the graph it will produce on the complex plane, with n and c being constants. As the value of t increases, it traces the motion of the unit circle at a frequency of n with a vector length of c (Niu). The complex plane contains one axis which is the real number line with the other axis being the complex number line. The complex plane helps visualize complex numbers by representing them geometrically on what is essentially a modified cartesian plane.

3. **Applications**

Since a Fourier series can be used to represent an approximation of a discontinuous function as a continuous function, Fourier series are very useful in analyzing discontinuous functions. This in itself was a breakthrough for the math world. Joseph Fourier first considered this a possibility while working on his theory for heat flow (Daileda). For example, it is possible to calculate how long it will take for two metal bars of different temperatures to reach equilibrium. By viewing the metal bars as one on a graph with temperature on the y axis and position on the bar on the x axis, the moment when the two bars are first connected can be seen as a step function. Slowly, the heat will dissipate to the colder parts of the new bar as time progresses. This can then be expressed by simple harmonics with frequencies that are whole number multiple of a base frequency. Importantly, if you were to take one fact away from the previous example of the two bars, it should be that continuous function can be represented as a sum of cosine or sine functions if it is periodic. Moreover, that these sine and cosine functions are much simpler to use or analyze than the original function. This concept is known as Fourier analysis.

Taking a closer look at the heat equation, the following solution will show how the Fourier series is related to the heat equation.

$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$

With initial conditions of $u(0,t) = u(l,t) = 0$

$u(x,t) = X(x)T(t)$ is assumed using separation of variables

Giving $\frac{\partial u}{\partial t} = X(x)\ T'(t)$ and $\frac{\partial^2 u}{\partial x^2} = X''(x)\ T(t)$

$X(x)\ T'(t) = \alpha^2\ X''(x)\ T(t)\ \frac{X''(x)}{X(x)} = \frac{T'(x)}{\alpha^2 T(x)} = -\lambda$

Which gives two ordinary differential equations instead of a partial one.

$X''(x) + \lambda X(x) = 0, \quad X(0) = 0, \quad X(l) = 0, \quad T'(t) + \alpha^2 \lambda T(t)$

The solution to the first is $X_n(x) = \sin(\sqrt{\lambda}_n x) = \sin(\frac{n\pi}{l} x)$

And the solution to the second is $T_n(t) = e^{-\frac{\alpha^2 n^2 \pi^2}{l^2} t} = e^{-\alpha^2 \lambda^2 t}$

Therefore the solution to the original equation is

$u_n(x,t) = X_n(x) T_n(t) = \sin \frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2}{l^2} t}$

Plugging in the original value of t=0 gives $f(x) = u_n(x,0) = \sin \frac{n\pi x}{l}$

$f(x)$ is then equal to $f(x) = \sum c_n \sin \frac{n\pi x}{l}$

$u(x,t) = \sum c_n \sin \frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2}{l^2} t}$ solves the heat equation (Daileda).

So taking a bar of length with a piecewise temperature function as seen in the earlier example can be represented as the following Fourier series.

$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-(\frac{\alpha n \pi}{l})^2 t}$

When the rate at which something changes is proportional to the value itself, e should come to mind. Let's take a temperature function of a cosine wave and say we wanted to find when the temperature of a certain part would reach zero (Serov). The solution to this would be that it diminishes by a factor of $e^{-at}\cos(x)$. In other words as time progresses each point on the wave is approaching zero at a rate of e^{-at}. Returning to the idea of Fourier series, if a wave temperature function were made up of four sine functions, then the sum of the solutions to the heat equation of those sine functions would the overall solution to the original wave temperature function. The more one thinks about this problem and its solution, the crazier this concept seems. The wildest part of all of this is that Joseph Fourier published his work on heat flow in 1822.

As Fourier analysis is a good way of analyzing different functions, it is highly used in signal processing. Although signal processing is more commonly associated with Fourier transforms, it is still possible to use Fourier series to analyze signals. However, as Fourier transform are an extension of Fourier series, this essay will take a small look into signal processing. Signal processing involves the examination of different signals describing the change of some physical object or quantity (Niu). This is used in electronics, global positioning systems, cell communication, medical imaging, and many more. For example, Fourier analysis is used to correct data by eliminating noise to view the overall trend of the signal. Another example could be measuring radar reflection on aircraft or other objects at long distances. Most students in STEM majors probably have some experience with electric square waves and its relation to direct current.

One of the most important algorithms of all time is the fast Fourier transform. Heavily used in computing and coding, the fast Fourier transform aims to reduce the complexity of a discrete Fourier transform to create more efficient computing. A discrete Fourier transform is a Fourier transform with summations instead of integrals. Without going too in depth on fast Fourier transforms, it allows computations to be done using roughly half the ram. This is done by calculating $n\ log(n)$, rather than calculating n^2 (Niu). The larger the order of magnitude, the more efficient the fast Fourier transform becomes, and yet both the discrete Fourier transform and fast Fourier transform give the exact same answer. Most digital communication and compression of data relies on this concept. What is even more interesting is that although it was first used in 1965, it turns out Gauss had concluded a similar algorithm back in 1805.

As a second year aerospace engineering student, electronics and electrical engineering play a big role in my field of study. The aforementioned heat equation is central to many of the fundamentals of flight and thermodynamics. It is mind boggling that the reason photos and other large files can be sent in what seems instantaneous today is all thanks to a French man from the early 1800s.

4. **Extra**

In the introduction to this paper, the idea that images made using Fourier series was proposed. Although true, it was very difficult to try and create complex images with my current knowledge of coding. Therefore I simply made code that animates the tip of two vectors added together, mimicking a moon's orbit around a central body. See Figure (3).

While doing research on Fourier series being implemented into MATLAB, I stumbled across someone who did manage to replicate an image using MATLAB and complex Fourier transforms. This is more to show the potential of Fourier analysis. The reason this can be done is because the image is periodic. Therefore the beginning of the "drawing" is connected to the end of the function, making it periodic. Just as before, the image becomes sharper and sharper as the number of iterations is increased. All of which is impressive when considering that this is simply vectors of different constant lengths and frequencies added together. See Figure (4) (Ikegami).

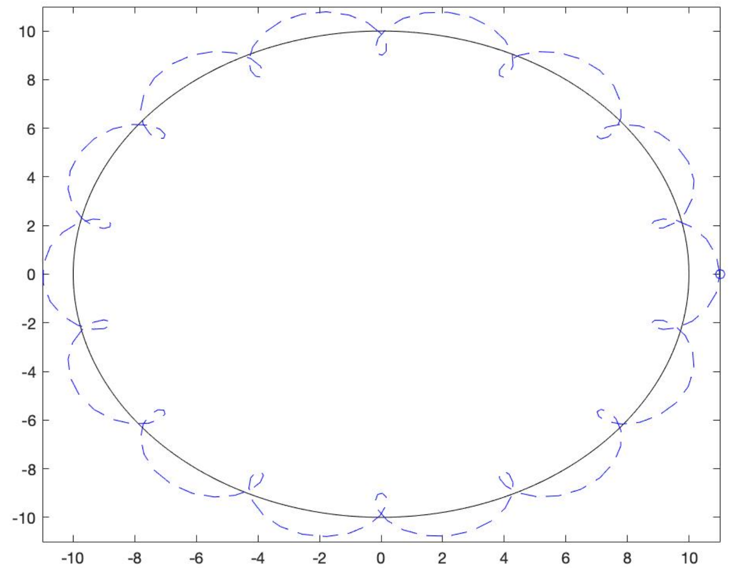

FIGURE 3. Personal attempt at making an image

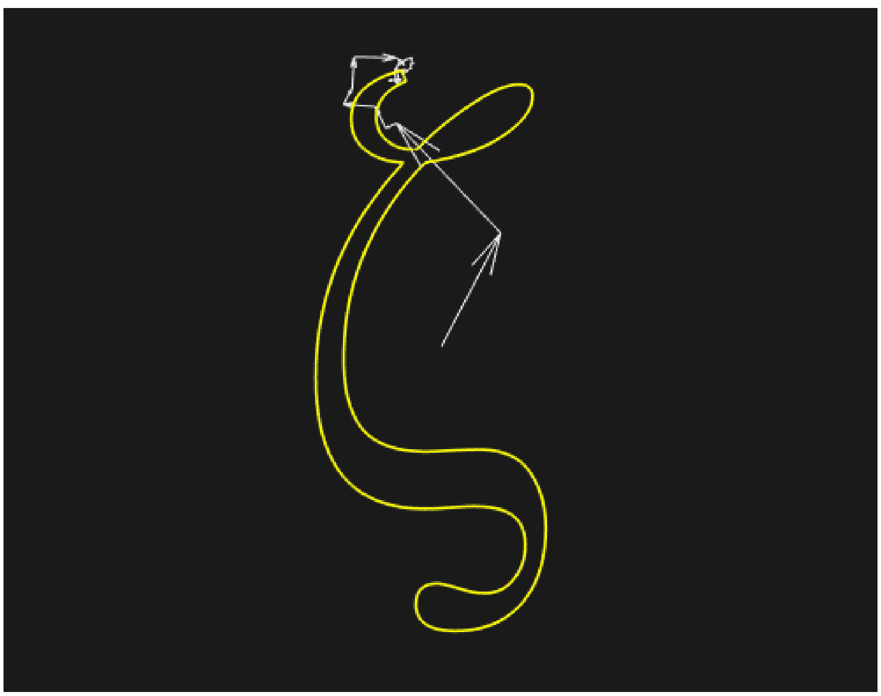

FIGURE 4. Greek letter *zeta* created using Fourier transforms by Tomihiro Ikegami

5. **Conclusion**

In Conclusion a Fourier series is a sum of simple harmonic functions representing a greater generic periodic function. Using Euler's formula, this can also be used to describe complex equations. This can then be used to approximate discontinuous functions by analyzing continuous ones. Fourier transforms, an extension of Fourier series, are used to today in digital communication and signal analysis. The fast Fourier transform, a more efficient manner of computing a Fourier transform, is one of the most important algorithms in computing today. Some go as far as to say that it is one of the most important advances of the 20th century. A solution to the heat equation was calculated showing how a partial derivative was simplified to a series involving sin functions. This then shows heat transfer and its relation to Euler's number. MATLAB was used to illustrate graphs and other visualizations to better understand the idea of Fourier series and what it does. Personally, I enjoyed learning about the applications of Fourier series. Yes, the images and graphs showing the summation of sine functions are cool, however it does not feel useful to know as a skill. I have always wondered how large data files such as photos can be sent rather quickly, and after writing this paper, I better understand how that is done. Throughout the entire paper, my mind could not get over the fact that this was all theorized in the early 19th century. I find it difficult at times to understand or visualize certain aspects, and Joseph Fourier did this all with just pen and paper. Still, this project is only a small view into the world of Fourier analysis. There is still much for me to learn out there.

6. **Bibliography**

References

[1] Bocher, Maxime. Annals of Mathematics. Vol. 7, Mathematics Department, Princeton University, 1906.

[2] Daileda, R.C. "The One-Dimensional Heat Equation." *Math Department Trinity University*, Trinity University, ramanujan.math.trinity.edu/rdaileda/teach/s17/m3357/lectures/lecture9.pdf.

[3] Ikegami, Tomihiro. "2D Line Art Fourier Transform Animation." Mathworks, 20 July 2019.

[4] Rui, Niu. Fourier Series and Their Applications. Massachusetts Institute of Technology, 2006.

[5] Serov, Valery. Fourier Series, *Fourier Transform and Their Applications to Mathematical Physics*. Springer International, 2018.

7. Appendix

Link showing whip like motion: https://imgur.com/gallery/GUvxR6y

```
N=25; %Number of iterations
x=linspace(0,1,101); %Want to plot 100 points
y=zeros(1,101)*1/2; %Creating array
for i=1:2:N
    bn=2/pi/i;
    y=y+bn*sin(2*pi*i*x);
    figure(1)
    hold on
    plot(x,y)
    legend('One Iteration')
end
hold off
```

Figure 5. MATLAB code for step function represented by a sum of sine functions

```
for i=1:N
    plot(xMainOrbit,yMainOrbit,'k');
    hold on
    plot(x(i),y(i),'ob');
    plot(x(1:i),y(1:i),'b--');
    axis([-a a -a a]);
    hold off
    pause(0.1);
    drawnow;
end
```

Figure 6. MATLAB code for orbit of an orbiting object (my attempt at making some image using epicycles)

Lastly, a fun interactive site by Jez Swanson showing the makeup of various Fourier transforms and allows the user to create some of their own: http://www.jezzamon.com/fourier/index.html