Interest Rate Models

1 Vasicek Model

The classic short rate model is the Vasicek Model, adapted from the Ornstein-Uhlenbeck process:

$$dr(t) = \lambda(\mu - r(t))\ dt + \sigma\ dW(t)$$

There also exists an even simpler model, the Ho-Lee Model, written

$$dr(t) = g(t)\ dt + \sigma\ dW(t)$$

Both are part of an even more general class of models called the *general Gaussian Markov Process* specified by

$$dr(t) = [g(t) + h(t)r(t)]\ \ dt + \sigma(t)\ dW(t) \tag{1}$$

where g, h, and σ are all deterministic functions of time. It has solution

$$r(t) = e^{H(t)}r(0) + \int_0^t e^{H(t)-H(s)}g(s)\ ds + \int_0^t e^{H(t)-H(s)}\sigma(s)\ dW(s)$$

$$H(t) = \int_0^t h(s)\ ds$$

Vasicek Model with Time-Varying μ Again, this is a special case of Equation 1, and it's solution simplifies to

$$r(t) = e^{-\lambda t}r(0) + \lambda\int_0^t e^{-\lambda(t-s)}\mu(s)\ ds + \sigma\int_0^t e^{-\lambda(t-s)}dW(s).$$

Supposing that we want to simulate, we set

$$r(t_{i+1}) = e^{-\lambda(t_{i+1}-t_i)}r(t_i) + \lambda\int_{t_i}^{t_{i+1}} e^{-\lambda(t_{i+1}-s)}\mu(s)\ ds + \frac{\sigma^2}{2\lambda}\left(1 - e^{-2\lambda(t_{i+1}-t_i)}\right)Z_{i+1}$$

Ho-Lee with Time-Varying Non-Constant $g(t)$ This is also a special case of Equation 1, and—at any time t—the short rate is given by

$$r(t) = r(0) + \int_0^t g(s)\ ds + \sigma W(t).$$

It's integral is given by

$$\int_0^T r(u)\ du = r(0)T + \int_0^T\int_0^u g(s)\ ds\ du + \sigma\int_0^T W(u)\ du$$

This integral has mean and variance

$$m = r(0)T + \int_0^T\int_0^u g(s)\ ds\ du, \qquad s^2 = \frac{1}{3}\sigma^2T^3$$

2 Forward Rate Models: Continuous Rates