

Analysis

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1 Riemann-Stieltjes Integral

This definition of the integral was made rigorous in the 1800s by Riemann, Darboux, and Stieltjes. It's an intuitive way to define the area under a curve, and it works well with numerical integration (approximations). However, it is incomplete in the sense that there are functions of interest that we cannot integrate in a Riemann sense but can in a Lebesgue sense.

Throughout this section, we'll stick to functions that are univariate from a compact interval to \mathbb{R} :

$$f : [a, b] \rightarrow \mathbb{R}$$

We'll begin by discussing partitions of that interval $[a, b]$ into smaller pieces, from which we'll construct sums that approximate the area under the curve. This will lead us to a definition of the Riemann Integral. Then, we'll generalize and allow the weight we place on the sub-intervals (when summing over the entire interval) to vary, which will give us the Riemann-Stieltjes integral. From there, we discuss the relationships between the approximating sums and the integral.

1.1 Operational Definitions

Definition 1.1. A partition, P , is an ordered tuple representing a finite sequence on the interval $[a, b]$,

$$a = x_0 < x_1 < \cdots < x_n = b \quad \text{with} \quad \Delta x_i := x_i - x_{i-1}$$

Definition 1.2. The *norm* of a partition P , sometimes called “mesh P ” represents

$$\|P\| = \text{norm}(P) := \max_i |x_i - x_{i-1}| = \max_i |\Delta x_i|$$

Definition 1.3. Q is a refinement of P if $Q \supset P$ where Q and P are both partitions of $[a, b]$. Q the intervals *finer*.

Definition 1.4. For two partitions, P_1 and P_2 , their *common refinement* is $P_1 \cup P_2$.

Definition 1.5. A tagged partition is a couplet (P, T) , where P is some partition $\{x_0, \dots, x_n\}$ and T is a set of evaluation points, $\{t_1, \dots, t_n\}$, for the function f such that

$$x_{i-1} \leq t_i \leq x_i$$

Note. We will now generalize to allow weighting of the sub-intervals within the partition, defined for an *increasing* function $\alpha : [a, b] \rightarrow \mathbb{R}$, where

$$\Delta \alpha_i = \alpha(x_i) - \alpha(x_{i-1}) > 0$$

This is the main difference between the plain Riemann sum and integral, versus the Riemann-Stieltjes (RS) sum and integral. The latter retains the former as a special case by taking $\alpha(x) = x$. Therefore, the RS version is just a generalization of Riemann, weighting the contribution of the sub-intervals to the total sum/integral by the function α , not by the length of the sub-interval.

1.2 Darboux Sum and RS-Darboux Sum Definitions

We now define the various sums approximating the Riemann and RS integrals.

Definition 1.6. We define the upper and lower *Darboux Sums*, respectively, as follows

$$U(f, P) := \sum_{i=1}^n M_i(f)(x_i - x_{i-1}) \quad \text{where} \quad M_i(f) := \sup_{x \in [x_i, x_{i-1}]} f(x)$$

$$L(f, P) := \sum_{i=1}^n m_i(f)(x_i - x_{i-1}) \quad \text{where} \quad m_i(f) := \inf_{x \in [x_i, x_{i-1}]} f(x)$$

Definition 1.7. Given f (bounded) and tagged partition (P, T) we define the Riemann Sum as

$$S(f, P, T) := \sum_{i=1}^n f(t_i)(x_i - x_{i-1}) \tag{1}$$

Definition 1.8. We define the upper and lower *RS-Darboux Sums*, respectively, as follows

$$U_\alpha(f, P) := \sum_{i=1}^n M_i(f) \Delta\alpha_i \quad \text{where} \quad M_i(f) := \sup_{x \in [x_i, x_{i-1}]} f(x)$$

$$L_\alpha(f, P) := \sum_{i=1}^n m_i(f) \Delta\alpha_i \quad \text{where} \quad m_i(f) := \inf_{x \in [x_i, x_{i-1}]} f(x)$$

Definition 1.9. Given f (bounded) and tagged partition (P, T) we define the Riemann-Stieltjes Sum as

$$S_\alpha(f, P, T) := \sum_{i=1}^n f(t_i) \Delta\alpha_i \quad (2)$$

1.3 Sum Relations

Remark. Clearly, by Definitions 1.8 and 1.9, for all T associated with P

$$L_\alpha(f, P) \leq S_\alpha(f, P, T) \leq U_\alpha(f, P)$$

Theorem 1.10. If $Q \supset P$, i.e. if Q refines P , then

$$L_\alpha(f, P) \leq L_\alpha(f, Q) \leq U_\alpha(f, Q) \leq U_\alpha(f, P)$$

Proof. The proof proceeds by induction. Assume that $Q = P \cup \{x^*\}$, a single point. Then $x^* \in [x_{i-1}, x_i]$ for some interval, and it's easy to show the relation from there. \square

Theorem 1.11. For all partitions P_1, P_2 ,

$$L_\alpha(f, P_1) \leq U_\alpha(f, P_2)$$

Proof. Let $Q = P_1 \cup P_2$. Then by Theorem 1.10,

$$L_\alpha(f, P_1) \leq L_\alpha(f, Q) \leq U_\alpha(f, Q) \leq U_\alpha(f, P_2)$$

\square

1.4 Riemann-Stieltjes Integral

1.4.1 Definition and Characterization

Definition 1.12. We define the upper and lower Riemann-Stieltjes integrals, respectively, in terms of the RS-Darboux Sums

$$\overline{\int_a^b} f d\alpha := \inf_P U_\alpha(f, P)$$

$$\underline{\int_a^b} f d\alpha := \sup_P L_\alpha(f, P)$$

From Theorem 1.11, it's clear that $\underline{\int} f d\alpha \leq \overline{\int} f d\alpha$.

Definition 1.13. We say f is Riemann-Stieltjes integrable on $[a, b]$ —i.e. $f \in \mathcal{R}_\alpha([a, b])$ —if

$$\overline{\int_a^b f d\alpha} = \int_a^b f d\alpha := \int_a^b f d\alpha$$

Example 1.14. A case where $f \notin \mathcal{R}_\alpha([a, b])$ is where

$$f(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$$

for $x \in [0, 1]$. In this case, the upper integral is always 1, while the lower integral is always zero.

Theorem 1.15. (Riemann's Condition) $f \in \mathcal{R}_\alpha([a, b])$ if and only if there exists a partition P such that the upper and lower RS-Darboux sums can be made arbitrarily close given that P , i.e.

$$U_\alpha(f, P) - L_\alpha(f, P) \leq \varepsilon$$

Proof. First, the \Leftarrow direction. Use Theorems 1.10 and 1.11. It's obvious. Next, for the \Rightarrow direction. By the definition of the RS integral and the RS-Darboux sums,

$$U_\alpha(f, P_1) < \int_a^b f d\alpha + \varepsilon/2 \quad L_\alpha(f, P_2) < \int_a^b f d\alpha + \varepsilon/2 \quad (3)$$

Taking the common refinement, and using Theorem 1.10, we get that

$$\begin{aligned} U_\alpha(f, P_1 \cup P_2) - L_\alpha(f, P_1 \cup P_2) &\leq U_\alpha(f, P_1) - L_\alpha(f, P_2) \\ &= \left(U_\alpha(f, P_1) - \int_a^b f d\alpha \right) - \left(L_\alpha(f, P_2) - \int_a^b f d\alpha \right) \\ \text{By Expression 3} \quad &\leq \varepsilon/2 + \varepsilon/2 \end{aligned}$$

□

Theorem 1.16. The set of all continuous functions on $[a, b]$, denoted $C([a, b])$, is a subset of $\mathcal{R}([a, b])$.

Proof. By Theorem 1.15 we want to show that, for all $\epsilon > 0$, there exists a partition P such that

$$\begin{aligned} U_\alpha(f, P) - L_\alpha(f, P) &< \epsilon \\ \Leftrightarrow \sum_{i=1}^n (M_i(f) - m_i(f)) \Delta\alpha_i &< \epsilon \end{aligned}$$

Now since f is continuous on a compact interval, $[a, b]$, f is uniformly continuous on $[a, b]$. That means, given our ϵ from above,

$$\exists \delta > 0 \quad \text{s.t.} \quad |x_i - x_{i-1}| < \delta \quad \Rightarrow \quad |f(x_i) - f(x_{i-1})| < \frac{\epsilon}{\alpha(b) - \alpha(a)}$$

So we can choose P such that that $\|P\| < \delta$. This means that

$$\begin{aligned} \sum_{i=1}^n [M_i(f) - m_i(f)] \Delta\alpha_i &\leq \sum_{i=1}^n \frac{\epsilon}{\alpha(b) - \alpha(a)} \Delta\alpha_i = \frac{\epsilon}{\alpha(b) - \alpha(a)} \sum_{i=1}^n \Delta\alpha_i \\ &= \frac{\epsilon}{\alpha(b) - \alpha(a)} \cdot [\alpha(b) - \alpha(a)] = \epsilon \end{aligned}$$

□

1.4.2 Properties of $\mathcal{R}_\alpha([a, b])$

Now for some useful properties of the set of Riemann-Stieltjes integrable functions. Consider $f, g \in \mathcal{R}_\alpha([a, b])$ and $c \in \mathbb{R}$.

- **Linearity:** $f + g \in \mathcal{R}_\alpha([a, b])$ and $cf \in \mathcal{R}_\alpha([a, b])$, with

$$\int_a^b c f d\alpha = c \int_a^b f d\alpha \quad \text{and} \quad \int_a^b f + g d\alpha = \int_a^b f d\alpha + \int_a^b g d\alpha$$

- **Subsets:** If $[c, d] \subset [a, b]$, then $f \in \mathcal{R}_\alpha([c, d])$.
- **Splitting the Interval:** If $c \in [a, b]$, then

$$\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$$