Asset Pricing

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1. Introduction

If all of Asset Pricing needed to be reduced to one simple, general, yet wholly appropriate epithet, a decent starting point would be "Price equals expected discounted payoff."

To segment the discussion further, John Cochrane suggests the following two classes of pricing approaches in his Asset Pricing book:

1. *Absolute Pricing*: "Pricing assets by exposure to fundamental sources of macroeconomic risk." This approach includes general equilibrium models.
2. *Relative Pricing*: This type of pricing considers only how an asset should be priced relative to other assets that can be bought and sold in the market. Black-Scholes and any arbitrage arguments would fall under this category.

However, many asset pricing approaches are a blend of the two.

To be more specific, we can reduce asset pricing to two equations:

$$p_t = E[m_{t+1}x_{t+1}] \tag{1}$$

$$m_{t+1} = f(\text{data, parameters}) \tag{2}$$

where p_t is the asset price, x_{t+1} is the asset's payoff, and m_{t+1} is the stochastic discount factor. This approach joins together what used to be separate theories so that pricing stocks, bonds, and options now represent special cases of this more general framework summarized by the equations above.

2. Asset Pricing Facts

There are a number of empirical observations that have been made over the years, and this section will detail their conclusions and main insights.

2.1. Equity Premium and Risk

We'll beging by thinking about the *Equity Premium*, which is the returns to stocks above and beyond bonds (so if you borrow in bonds and lend in stocks): $E[R^{\text{stocks}} - R^{\text{bonds}}]$. Here are the facts:

1. The equity premium is *big*, about 7%.
2. There's a lot of risk in stocks: there's a greater standard deviation in returns (although we'll think about risk a bit differently). Stock returns are also correlated with macro variables.
3. Compared with stocks returns, GDP and Consumption growth are much more stable, though they do vary a bit (with the standard deviation being much closer to bond returns' volatility than stock returns).

2.2. Time-Varying Risk Premium

Now we want to consider what explains the Equity Premium. For that, we regress returns on some signal we hope will explain future returns. Here's a few regressions. The main result will be that *excess returns* are forecastable—that is, the time-varying reward for risk can be forecast. We're not forecasting raw rate. Now onto facts:

1. *Old View*: Regression of returns on lagged returns:

$$R_{t+1} = a + bR_t + \varepsilon_{t+1}$$

   This regresion typically yields a value for b around 0, in which case future returns given returns today are more or less *random*. The main result:

$$ER_{t+1} = E[a + 0 \cdot R_t + \varepsilon_{t+1}] = a \tag{3}$$

   in which case *expected returns* are large and constant.

2. *New View*: The old view, encapsulated in Equation 3, drove theory for many years. But now, we have a different picture: there are variables that forecast stock returns. Now, we run the regression

$$R^e_{t\to t+k} = a + b\left(\frac{D_t}{P_t}\right) + \varepsilon_{t+k} \tag{4}$$

   where R^e is the excess return over bonds, D_t is current dividends, and P_t is current price. From this regression, we get a large, positive regression coefficient. So we actually *can* predict returns over the business cycle via this dividend/price ratio (but it's still tough in the short run).

3. *New View Interpretation*: The dividend yield (D_t/P_t), which is the inverse of prices, turns out to vary a lot over time. Moreover, if you buy when the dividend yield ratio is *high* (so that prices are low), you will earn high returns in expectation. Vice versa for a low dividend yield ratio (so high prices).

   Moreover, we can get the standard deviation in the expected returns over time, which is the fitted value $ER^e_{t\to t+k} = a + bE\left(D_t/P_t\right)$ from Equation 4. It turns out that expected returns *also vary*, and they vary *substatially*, at about 6%.

   Recall that it was puzzling that stocks should earn 7% over bonds. Now we're saying that they earn 7% over bonds *and* vary with a standard deviation of about 6% a year. Average *changes* in $ER^e_{t\to t+k}$ are about as much as the average confounding *level*.[1]

4. *Why do prices vary?*: We know that the dividend/price ratio varies a lot, but it's mostly driven by prices. Why? Well, we used to think

$$\text{High Prices} \Rightarrow \text{High Future Growth} \Rightarrow \text{High E[Future Dividends]} \tag{5}$$

[1]Of course, *actual* returns vary even more than expected returns—recall about 17%.

But if we run the regression implied by Equation 5,

$$\frac{D_{t+k}}{D_t} = a + b\left(\frac{D_t}{P_t}\right) + \varepsilon_{t+k} \tag{6}$$

we find out that the logic of 5 is **wrong**. The regression coefficient in Equation 6 is negligible and there is no explanatory power (as reflected in the R^2). So actaully

$$\text{High Prices} \Rightarrow \text{Low Future E[Returns]} \tag{7}$$

So it's not that dividends will adjust if prices are high and low—dividends are remarkably stable relative to prices! Instead, *prices* will adjust.

In short: high prices are correlated with a good economy, when people are willing to bear more risk. So returns will be low. Vice versa for low prices.

2.3. The Cross Section of Stock Returns

We talked about returns over time in the previous sections. Now let's consider the facts about how returns vary *across assets* at a given time.

1. *Market Cap*: Small stocks earn more on average than large stocks. They are also riskier.

2. *Growth vs. Value*: Value stocks (high book to value) tend to earn more.

3. *Capital Asset Pricing Model*: This model predicts returns based on the correlation between the asset and the market portfolio.

4. *Multifactor Model*: This model was put forward by Fama and French to capture the extra variation in returns that the Capital Asset Pricing Model failed to explain. This is the modern view. Specifically, Fama and French posit

$$ER^{e,i} = \alpha_i + b_i E[R^m - R^f] + h_i E(\text{hml}) + s_i E(\text{smb}) \tag{8}$$

   where $E(\text{hml})$ represents "high minus low" (for value minus growth stocks) and $E(\text{smb})$ for "small minus big" (for market cap).

5. *Stock Market Volatility:* We used to think volatility was pretty constant over time. Now we know that the variance of returns changes over time.

3. Asset Pricing Theory Overview

This section is about understanding the fundamental formula for all of asset pricing, where an *asset* is anything that promises a future stream of payments:

$$P_t = E_t[M_{t+1}X_{t+1}] \tag{9}$$

But first, since we'll start talking about preferences below, we'll eventually need a utility function at some point. So we first consider that.

3.1. Utility Detour

So first, we'll consider a simple utility function for two-period consumption, which covers t and $t+1$—the relevant time periods in Equation 9. Spefically, we'll define

$$U(C_t, C_{t+1}) = U(C_t) + \beta E_t\left[U(C_{t+1})\right] \tag{10}$$

where β is the discount factor with respect to *time*, **not** risk which depends upon the shape of the utility function. Equation 10 above is a very simple form which we can generalize further if we so choose.

For the *internal*, one-period utility function, $U(\cdot)$, we'll want the normal properties of utility, $U' > 0$ and $U'' < 0$, which leads us to define the simple *power function utility*

$$U(C) = \frac{C^{1-\gamma} - 1}{1-\gamma} \quad \Rightarrow \quad U'(C) = C^{-\gamma} \tag{11}$$

This utility function also has the property that as $\gamma \to 1$, $U(C) = \ln C$.

3.2. Understanding the Components

Now let's take each component of Equation 9 in turn before we derive it:

1. X_{t+1}: This is the *payoff* (in gross terms, not net, percentage, etc.) at time $t+1$. Often, X_{t+1} will be random—it's a random variable dependent upon the state of nature.

2. P_t: To get this payoff, X_t, you have to pay some price P_t today. Note that it is possible for P_t to equal 0, like it might in a bet (nothing today, $+1$ tomorrow if I win, -1 if I lose). No money changes hands.

3. M_{t+1}: Recall Equation 10 above. It left us free parameters, β and γ, which allow us to tweak "impatience," or time sensitivity of consumption (β), as well as risk-aversion via curvature (γ). This gives us everything we need to charactertize the discount factor, M_{t+1}.

3.3. Deriving P=E(MX)

We'll first consider the two-period case for simplicity and exposition. Then we'll extend to an infinite-horizon. We'll use each at some point going forward.

3.3.1. Two-Period Case

Let's now figure out the price an investor assigns to a risky payoff. It reduces to a utility maximization problem:

$$\max_{\xi} \quad U(C_t) + \beta E_t\left[U(C_{t+1})\right]$$
$$C_t = a_t - \xi P_t$$
$$C_{t+1} = a_{t+1} + \xi X_{t+1}$$

$$\text{f.o.c.} \qquad 0 = -P_t U'(C_t) + \beta E_t\left[X_{t+1} U'(C_{t+1})\right]$$

This maximization problem, in words, means "Start at some initial consumption, a_t. Give up some fraction ξ of that consumption at price P_t today (ξP_t) in exchange for some fraction of of the uncertain payoff tomorrow (ξX_{t+1}). We can rearrange the first order condition above to get to the fundamental pricing equation:

$$P_t = E_t\left[\beta \frac{U'(C_{t+1})}{U'(C_t)} X_{t+1}\right] = E_t\left[M_{t+1} X_{t+1}\right] \tag{12}$$

where M_{t+1} is the *stochastic discount factor*. Note that the price, P_t, is the *marginal* price—what an investor would willingly pay at the margin for an extra piece of the asset.

3.3.2. Infinite-Period Case

Similarly, we set up an infinite-period, discrete-time optimization problem:

$$\max_{\xi} \; E_t \sum_{s=0}^{\infty} \beta^s U(C_{t+s}) \tag{13}$$
$$C_t = a_t - \xi P_t$$
$$C_{t+s} = a_{t+s} + \xi X_{t+s}$$
$$\text{f.o.c.} \quad 0 = -P_t U'(C_t) + E_t \sum_{s=1}^{\infty} \beta^s U'(C_{t+s}) X_{t+s} \tag{14}$$

This yields us a formula for the price:

$$P_t = E_t \sum_{s=1}^{\infty} \beta^s \frac{U'(C_{t+s})}{U'(C_t)} X_{t+s} \tag{15}$$

Lastly, in continuous time, the objective function has the following form:

$$\int_{s=0}^{\infty} e^{-\delta s} U(C_{t+s}) \, ds \tag{16}$$

4. Classic Issues in Finance

First some definitions and terminology, relating our fundamental pricing formula $P = E(mX)$ to common concepts such as returns, present value, etc.

4.1. Meet the Players

4.1.1. Returns

An *rate of return* is a price 1 security (you put in \$1 today and get \$1 and then some tomorrow):

$$1 = E_t[M_{t+1}R_{t+1}] \tag{17}$$

And of course, prices and returns are inversely proportion. Now the return concept we're using is *gross returns*, including price appreciation and dividends, not expressed in percentages or rates:

$$R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} \tag{18}$$

Now we have enough to derive the *risk-free*, which gives a certain payoff, allowing us to write

$$\begin{aligned} 1 &= E_t[M_{t+1}R^f_{t+1}] = E_t[M_{t+1}]R^f_{t+1} \\ \Rightarrow \quad R^f_{t+1} &= \frac{1}{E_t[M_{t+1}]} \end{aligned} \tag{19}$$

Here, we see that the mean of the discount factor determines the interest rate as well as prices of *zero-coupon, pure discount bonds*, which pay \$1 tomorrow for sure, implying

$$P^{\text{zcb}}_t = E_t[M_{t+1}1] = E_t[M_{t+1}] \tag{20}$$

Next, we consider *excess return*, which is the difference of a return a risk free rate or the difference of any two returns:

$$\begin{aligned} R^e_{t+1} &= R_{t+1} - R^f_{t+1} \\ \text{or} \quad &= R^i_{t+1} - R^j_{t+1} \end{aligned} \tag{21}$$

where R^i and R^j are the returns on any two assets i and j. Note this asset (the excess return asset) is build by *borrowing* in one asset and *lending* in the other. Therefore, it has *price zero*,

$$\text{i.e.} \quad 0 = E_t[M_{t+1}R^e_{t+1}] \tag{22}$$

4.1.2. Present Value

Suppose we let $\{D_t\}$ denote a stream of dividends.[2] What is the price of that stream of dividends? Apply the formula:

$$P_t = E_t \sum_{j=1}^{\infty} \beta^j \frac{U'(C_{t+j})}{U'(C_t)} D_{t+j} = E_t \sum_{j=1}^{\infty} M_{t,t+j} D_{t+j} \tag{23}$$

where $M_{t,t+j}$ is the stochastic discount factor from time t to time $t+j$.

Now let's tackle continuous time, which turns the sum into an integral:

$$P_t = E_t \int_{s=0}^{\infty} e^{-\delta s} \frac{U'(C_{t+s})}{U'(C_t)} D_{t+s}\, ds \equiv E_t \int_{s=0}^{\infty} \frac{\Lambda_{t+s}}{\Lambda_t} D_{t+s}\, ds \tag{24}$$

where we define the Λ terms which define the *levels* of marginal utilities.

4.1.3. Relating Returns to Present Value

Returns are, in one sense, a "first difference" of the price. To derive the returns in discrete time, group the terms in Equation 23 a bit differently and rewrite

$$\begin{aligned} P_t &= E_t \sum_{j=1}^{\infty} M_{t,t+j} D_{t+j} = E_t \left[M_{t,t+1} D_{t+1} + M_{t,t+1} \sum_{j=2}^{\infty} \frac{M_{t,t+j}}{M_{t,t+1}} D_{t+j} \right] \\ &= E_t \left[M_{t,t+1} D_{t+1} + M_{t,t+1} P_{t+1} \right] \\ \Rightarrow \quad 1 &= E_t \left[M_{t+1} \, \frac{(D_{t+1} + P_{t+1})}{P_t} \right] \\ 1 &= E_t \left[M_{t+1} R_{t+1} \right] \end{aligned}$$

Now let's do the same thing, but for continuous time. In this case, our starting point is Equation 24

[2] Don't take the term "dividends" *too* literally here. We really just mean future cashflow streams to shareholders, in whatever form that may be (stock repurchases, liquidation of the firm, and so on).

A. Price-Dividend and Return Linearizations

We'll derive an approximation to an identity that says the following: there are three potential reasons the price-dividend ratio might be high:

1. Investors expect dividends to rise.
2. Investors expect low future returns, so future cashflows are discounted at a lower than usual rate. This leads to higher prices.
3. Investors expect prices to rise forever, giving an adequate return even if there are no dividends.

Now, we asserted above that Option 2 is the correct observation. But how can we test that? We'll, let's derive the identity that lays out the theory behind Options 1-3.

A.1. The Identity

Start with the identity, and do some rearranging, letting R equal gross returns (price increases plus dividends), D be dividends, and P be price.

$$1 = R_{t+1}^{-1} R_{t+1} = R_{t+1}^{-1} \frac{P_{t+1} + D_{t+1}}{P_t}$$

$$\Leftrightarrow \quad \frac{P_t}{D_t} = R_{t+1}^{-1} \left(1 + \frac{P_{t+1}}{D_{t+1}}\right) \frac{D_{t+1}}{D_t} \tag{25}$$

$$\Leftrightarrow \quad P_t = R_{t+1}^{-1} \left(D_{t+1} + P_{t+1}\right) \tag{26}$$

Now, we can iterate forward Equation 26 and take a conditional expectation to get

$$P_t = E_t \sum_{j=1}^{\infty} \left(\prod_{k=1}^{j} R_{t+k}^{-1} \right) D_{t+j} \tag{27}$$

Now look at the previous line, and do the reverse of what we did in going from Equation 25 to Equation 26 (defining $\Delta D_t := D_t / D_{t-1}$:

$$\Rightarrow \quad \frac{P_t}{D_t} = E_t \sum_{j=1}^{\infty} \left(\prod_{k=1}^{j} R_{t+k}^{-1} \right) \frac{D_{t+j}}{D_t}$$

$$\Leftrightarrow \quad \frac{P_t}{D_t} = E_t \sum_{j=1}^{\infty} \left(\prod_{k=1}^{j} R_{t+k}^{-1} \right) \left(\prod_{k=1}^{j} \frac{D_{t+k}}{D_{t+k-1}} \right) \quad \text{Telescoping product trick!}$$

$$\Leftrightarrow \quad \frac{P_t}{D_t} = E_t \sum_{j=1}^{\infty} \left(\prod_{k=1}^{j} R_{t+k}^{-1} \Delta D_{t+k} \right) \tag{28}$$

So, what do we have? The identitiy in Equation 27 tells us that prices will increase if the discount rate *falls* or if Expected future dividends rise. *However*, prices aren't stationary, which is why we went to the trouble of deriving the identity in Equation 28, which *is* a stationary variable that captures the same logic as 27, and can be estimated properly via traditional time series approaches.

A.2. The Linearization

To make the identities derived above easier to handle, we take logs to linearize! Letting lowercase letters represent the logs of uppercase letters, we get from Equation 25

$$
\begin{aligned}
\ln \frac{P_t}{D_t} &= \ln \left\{ R_{t+1}^{-1} \left(1 + \frac{P_{t+1}}{D_{t+1}} \right) \frac{D_{t+1}}{D_t} \right\} \\
p_t - d_t &= -r_{t+1} + \ln(1 + e^{p_{t+1} - d_{t+1}}) + \Delta d_{t+1} \qquad (29)
\end{aligned}
$$

Now do Taylor Expansion of the last term about a point $P/D = e^{p-d}$ (we'll use both the left- and righthand side of that point to simplify, so watch out for that). Also implicit in the deriviation, I'll be evaluating the derivative in the second term at $P/D = e^{d-d}$, so watch out for that too:

$$
\begin{aligned}
f(p_t - d_t) &= \ln(1 + e^{p_{t+1} - d_{t+1}}) \\
&\approx \ln(1 + e^{p-d}) + \frac{d\left[\ln(1 + e^{p_{t+1} - d_{t+1}})\right]}{d(p_t - d_t)} \cdot \{(p_{t+1} - d_{t+1}) - (p - d)\} \\
&= \ln(1 + e^{p-d}) + \frac{e^{p_{t+1} - d_{t+1}}}{1 + e^{p_{t+1} - d_{t+1}}} \cdot \{(p_{t+1} - d_{t+1}) - (p - d)\} \\
&= \ln(1 + P/D) + \frac{P/D}{1 + P/D} \cdot \{(p_{t+1} - d_{t+1}) - (p - d)\} \\
&= \ln(1 + P/D) + \rho \{(p_{t+1} - d_{t+1}) - (p - d)\}
\end{aligned}
$$

So now let's just plug this log-linear approximation into Equation 29:

$$
p_t - d_t \approx -r_{t+1} + \Delta d_{t+1} + k + \rho \{(p_{t+1} - d_{t+1}) - (p - d)\} \qquad (30)
$$

where k equals $\ln(1 + P/D)$, which is a constant.[3] Now iterating forward is even more straightforward:

$$
p_t - d_t = C + \sum_{j=1}^{\infty} \rho^{j-1} (\Delta d_{t+j} - r_{t+j}) \qquad (31)
$$

From this, it's clear that an ex-post high dividend-price ratio will result only in the presence of high dividend growth or low subsequent returns. To turn Equation 31 into an ex ante price-dividend ratio, take expectations of everything on the right.

[3] Note, ρ is just a constant of approximation that can be simplified using the fact that dividend yields roughly 4% on average, so the Price/Dividend ratio is about 25:

$$
\rho = \frac{P/D}{1 + P/D} = \frac{1}{1 + D/P} \approx 1 - D/P = 0.96
$$

A.3. Decomposing the Variance

To answer the question of what drives a high dividend-price ratio, we'll decompose the variance in Equation 31 as follows:

$$\text{Var}(p_t - d_t) = E\left[(p_t - d_t) - E(p_t - d_t)\right]^2 p_t - d_t = C + \sum_{j=1}^{\infty} \rho^{j-1}(\Delta d_{t+j} - r_{t+j})$$