Homework 1

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1. (a) We want to show that, for all P,

$$U_\alpha(f,P) - L_\alpha(f,P) \leq w_f\left(||P||\right)\left(\alpha(b)-\alpha(a)\right)$$

By the definition of the Upper and Lower Darboux sums,

$$\begin{aligned} U_\alpha(f,P) - L_\alpha(f,P) &= \sum_{i=1}^{n} M_i(f)\Delta\alpha_i - \sum_{i=1}^{n} m_i(f)\Delta\alpha_i \\ &= \sum_{i=1}^{n} \left[M_i(f) - m_i(f)\right]\Delta\alpha_i \qquad (1) \end{aligned}$$

Now let $\overline{x}_i$ and $\underline{x}_i$ be the points such that

$$\begin{aligned} f(\underline{x}_i) = m_i(f) &= \inf_{x\in[x_i,x_{i-1}]} f(x) \\ f(\overline{x}_i) = M_i(f) &= \sup_{x\in[x_i,x_{i-1}]} f(x) \end{aligned}$$

Now since $\underline{x}_i$ and $\overline{x}_i$ are both in $[x_{i-1},x_i]$, we can combine this with the definition of $||P||$ to get

$$|\underline{x}_i - \overline{x}_i| \leq |x_i - x_{i-1}| \leq ||P||$$

By the definition of the modulus of continuity,

$$\begin{aligned} |\underline{x}_i - \overline{x}_i| \leq ||P|| \quad &\Rightarrow \quad |f(\underline{x}_i) - f(\overline{x}_i)| \leq w_f\left(||P||\right) \\ &\Leftrightarrow \quad |M_i(f) - m_i(f)| \leq w_f\left(||P||\right) \end{aligned}$$

Substituting this fact back into 1, and using the properties of telescoping sums, we get that

$$\begin{aligned} \sum_{i=1}^{n} \left[M_i(f) - m_i(f)\right]\Delta\alpha_i &\leq \sum_{i=1}^{n} w_f\left(||P||\right)\Delta\alpha_i = w_f\left(||P||\right)\sum_{i=1}^{n}\Delta\alpha_i \\ &\leq w_f\left(||P||\right)\left(\alpha(b)-\alpha(a)\right) \end{aligned}$$

(b) We know f is monotone on $[a, b]$. Suppose that f is monotone increasing.[1]Then given any partition, P,

$$f(x_{i-1}) \leq f(c) \leq f(x_i) \quad \forall i, c \in [x_{i-1}, x_i]$$

This implies that

$$M_i = f(x_i) \quad m_i = f(x_{i-1}) \qquad \forall i \tag{2}$$

Writing $U_\alpha - L_\alpha$ as above in Equation 1, we see that the expression reduces to

$$\sum_{i=1}^{n} [M_i(f) - m_i(f)] \, \Delta\alpha_i = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] \, \Delta\alpha_i \tag{3}$$

Now let's consider $w_\alpha(||P||)$. By definition, it is

$$w_\alpha(||P||) := \sup_{|x-y| \leq ||P||} |\alpha(x) - \alpha(y)|$$

It's clear that for each sub-interval, $[x_{i-1}, x_i]$, we have

$$|x_i - x_{i-1}| \leq ||P|| \quad \forall i$$

This implies that

$$|\Delta\alpha_i| = |\alpha(x_i) - \alpha(x_{i-1})| \leq w_\alpha(||P||) \qquad \forall i$$

Substituting this into the right-hand side of equation 3, we can simplify using the properties of telescoping sums to get

$$\begin{aligned}
\sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] \, \Delta\alpha_i &\leq \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] \, w_\alpha(||P||) \\
&\leq w_\alpha(||P||) \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] \\
\Leftrightarrow U_\alpha(f, P) - L_\alpha(f, P) &\leq w_\alpha(||P||) \, [f(b) - f(a)] \qquad (4)
\end{aligned}$$

Finally, since α is assumed continuous on the compact interval $[a, b]$, α is uniformly continuous on $[a, b]$. That means

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \text{s.t.} \quad |x - y| \leq \delta \quad \Rightarrow \quad |\alpha(x) - \alpha(y)| \leq \epsilon$$

So we can make $w_\alpha(||P||) < \epsilon$, by choosing our partition P such that $||P|| \leq \delta$. With Equation 4, we therefore ensure that

$$U_\alpha(f, P) - L_\alpha(f, P) \leq \epsilon[f(b) - f(a)] \tag{5}$$

for any arbitrary ϵ. And so by Riemann's Condition, $f \in \mathscr{R}_\alpha([a, b])$ because for any $\epsilon > 0$, we can ensure the upper and lower sums are within that distance of each other by choosing a sufficiently fine partition.

[1] Note that I'll lose no generality in assuming that the function f was monotone increasing. If it's monotone decreasing instead, swap the values of M_i and m_i in Equation 2, and proceed in exactly the same way.

2. **Exercise 51.15**: First, let's establish some basic building blocks for the proof. We have α increasing on $[a, b]$ and $f, g \in \mathscr{R}_\alpha([a, b])$. By Riemann's condition, it's clear that for all $\delta > 0$, there exist partitions P_1 and P_2 such that

$$U_\alpha(f, P_1) - L_\alpha(f, P_1) < \delta \tag{6}$$
$$U_\alpha(g, P_2) - L_\alpha(g, P_2) < \delta \tag{7}$$

Now take the common refinement, $P^* = P_1 \cup P_2$. By Lemma 51.5 and Corollary 51.6 (in FoMA), we know that

$$L_\alpha(f, P_1) \leq L_\alpha(f, P^*) \leq U_\alpha(f, P^*) \leq U_\alpha(f, P_1)$$
$$L_\alpha(g, P_2) \leq L_\alpha(g, P^*) \leq U_\alpha(g, P^*) \leq U_\alpha(g, P_2)$$

Combining this with the result from Riemann's condition above (and rewriting as in Question 1), we see that we must also have

$$\sum_{i=1}^{n}[M_i(f) - m_i(f)]\Delta\alpha_i = U_\alpha(f, P^*) - L_\alpha(f, P^*) \leq U_\alpha(f, P_1) - L_\alpha(f, P_1) < \delta$$
$$\sum_{i=1}^{n}[M_i(g) - m_i(g)]\Delta\alpha_i = U_\alpha(g, P^*) - L_\alpha(g, P^*) \leq U_\alpha(g, P_2) - L_\alpha(g, P_2) < \delta$$

Now since α is assumed increasing, we know that $\Delta\alpha_i \geq 0$ for all i. Also, $M_i(\cdot) \geq m_i(\cdot)$ for all i. Therefore, we can conclude

$$0 \leq M_i(f) - m_i(f) < \delta \qquad \forall i \tag{8}$$
$$0 \leq M_i(g) - m_i(g) < \delta \qquad \forall i \tag{9}$$

Now, using all of this as groundwork, let's show the main result.

(a) We'll start by showing $h(x) = \max\{f, g\} \in \mathscr{R}_\alpha([a,b])$ using Riemann's Condition. So for all $\epsilon > 0$, we need to find a partition P such that

$$U_\alpha(h, P) - L_\alpha(h, P) < \epsilon \tag{10}$$

To do so, take $\delta = \epsilon/[\alpha(b) - \alpha(a)]$ and use Riemann's condition to find P_1 and P_2 as above, in Equations 6 and 7. Then take their common refinement to find P^*. **This will be our partition P such that Equation 10 holds**.

To formally show this, consider any arbitrary interval defined by the partition P^*. Over any interval $[x_{i-1}, x_i]$ in P^*,

$$M_i(h) = \sup_{x \in [x_{i-1}, x_i]} \max\{f(x), g(x)\}$$
$$m_i(h) = \inf_{x \in [x_{i-1}, x_i]} \max\{f(x), g(x)\}$$

It's clear that we can narrow the list of candidates for $M_i(h)$ and $m_i(h)$:

$$M_i(h) = \max\{M_i(f), M_i(g)\}$$
$$m_i(h) = \max\{m_i(f), m_i(g)\}$$

Let's consider the cases:

i. Suppose $M_i(h) = M_i(f)$ and $m_i(h) = m_i(f)$. Then by Equation 8 and our choice of δ, we have

$$0 \leq \leq M_i(h) - m_i(h) = M_i(f) - m_i(f) = \leq \frac{\epsilon}{\alpha(b) - \alpha(a)}$$

ii. Similarly, if $M_i(h) = M_i(g)$ and $m_i(h) = m_i(g)$. Then by Equation 9 and our choice of δ, we have

$$0 \leq M_i(h) - m_i(h) = M_i(g) - m_i(g) \leq \frac{\epsilon}{\alpha(b) - \alpha(a)}$$

iii. Next, suppose that $M_i(h) = M_i(f)$ and $m_i(h) = m_i(g)$. In that case,

$$m_i(g) > m_i(f) \quad \Rightarrow \quad M_i(f) - m_i(g) < M_i(f) - m_i(f)$$

We also know that $M_i(f) - m_i(g)$ is bounded below by zero because if not, then $m_i(g) > M_i(f)$, implying that we didn't choose $M_i(h)$ correctly, as $M_i(g)$ would certainly have been larger than $m_i(g)$ and, thus also $M_i(f)$.

So by this fact, Equation 8, and our choice of δ:

$$0 \leq M_i(h) - m_i(h) = M_i(f) - m_i(g) \leq M_i(f) - m_i(f) \leq \frac{\epsilon}{\alpha(b) - \alpha(a)}$$

iv. Finally, suppose that $M_i(h) = M_i(g)$ and $m_i(h) = m_i(f)$. In that case,

$$m_i(f) > m_i(g) \quad \Rightarrow \quad M_i(g) - m_i(f) < M_i(g) - m_i(g)$$

We also know that $M_i(g) - m_i(f)$ is bounded below by zero because if not, then $m_i(f) > M_i(g)$, implying that we didn't choose $M_i(h)$ correctly, as $M_i(f)$ would certainly have been larger than $m_i(f)$ and, thus also $M_i(g)$.

So by this fact, Equation 9, and our choice of δ:

$$0 \leq M_i(h) - m_i(h) = M_i(g) - m_i(f) \leq M_i(g) - m_i(g) \leq \frac{\epsilon}{\alpha(b) - \alpha(a)}$$

Putting it all together, we managed to bound

$$M_i(h) - m_i(h) \leq \frac{\epsilon}{\alpha(b) - \alpha(a)} \qquad \forall i$$

This implies that

$$\begin{aligned} U_\alpha(h, P^*) - L_\alpha(h, P^*) &= \sum_{i=1}^{n} [M_i(h) - m_i(h)] \Delta\alpha_i \leq \sum_{i=1}^{n} \frac{\epsilon}{\alpha(b) - \alpha(a)} \Delta\alpha_i \\ &\leq \frac{\epsilon}{\alpha(b) - \alpha(a)} \sum_{i=1}^{n} \Delta\alpha_i \\ &\leq \frac{\epsilon}{\alpha(b) - \alpha(a)} \cdot (\alpha(b) - \alpha(a)) = \epsilon \end{aligned}$$

So we have a process to get the partition to satisfy Riemann's Condition for integrability for any ϵ implying $h \in \mathscr{R}_\alpha([a, b])$.

(b) Next, we show that $h(x) = \min\{f, g\} \in \mathscr{R}_\alpha([a, b])$ using Riemann's Condition. So for all $\epsilon > 0$, we need to again find a sufficient partition P.

Also again, take $\delta = \epsilon/[\alpha(b) - \alpha(a)]$ and use Riemann's condition to find P_1 and P_2 as above, in Equations 6 and 7. Then take their common refinement to find P^*. Again, **this will be our partition P such that Equation 10 holds**.

To formally show this, consider any arbitrary interval defined by the partition P^*. Over any interval $[x_{i-1}, x_i]$ in P^*,

$$
\begin{aligned}
M_i(h) &= \sup_{x \in [x_{i-1}, x_i]} \min\{f(x), g(x)\} \\
m_i(h) &= \inf_{x \in [x_{i-1}, x_i]} \min\{f(x), g(x)\}
\end{aligned}
$$

Narrowing down the list of candidates for $M_i(h)$ and $m_i(h)$:

$$
\begin{aligned}
M_i(h) &= \min\{M_i(f), M_i(g)\} \\
m_i(h) &= \min\{m_i(f), m_i(g)\}
\end{aligned}
$$

The cases are then *exactly analogous* to what we saw for the case of the max. The result is that we again managed to bound

$$
M_i(h) - m_i(h) \leq \frac{\epsilon}{\alpha(b) - \alpha(a)} \qquad \forall i
$$

This implies that

$$
\begin{aligned}
U_\alpha(h, P^*) - L_\alpha(h, P^*) = \sum_{i=1}^{n} [M_i(h) - m_i(h)] \Delta\alpha_i &\leq \sum_{i=1}^{n} \frac{\epsilon}{\alpha(b) - \alpha(a)} \Delta\alpha_i \\
&\leq \frac{\epsilon}{\alpha(b) - \alpha(a)} \sum_{i=1}^{n} \Delta\alpha_i \\
&\leq \frac{\epsilon}{\alpha(b) - \alpha(a)} \cdot (\alpha(b) - \alpha(a)) = \epsilon
\end{aligned}
$$

So we have a process to get the partition to satisfy Riemann's Condition for integrability for any ϵ implying $h \in \mathscr{R}_\alpha([a, b])$.

3. **Exercise 51.18**: We want an example of an increasing α on $[a, b]$ and a bounded function f such that $|f| \in \mathscr{R}_\alpha([a, b])$ but $f \notin \mathscr{R}_\alpha([a, b])$.

   Consider the interval $[0, 1]$ and the functions

$$f(x) = \begin{cases} -1 & x \in \mathbb{Q} \subset \mathbb{R} \\ 1 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \qquad \alpha(x) = x \tag{11}$$

   First, for $|f| \in \mathscr{R}_\alpha([0, 1])$, we a situation similar to Example 51.9 in the book, as

$$|f(x)| = 1$$

   a constant. Then for any partition P, we have

$$M_i(|f|) = m_i(|f|) = 1$$

   which implies (after telescoping the sum) that

$$\begin{aligned} U_\alpha(|f|, P) &= \sum_{i=1}^{n} M_i(|f|)\Delta\alpha(i) = \sum_{i=1}^{n} 1 \cdot \Delta\alpha(i) \\ &= \alpha(b) - \alpha(a) = 1 - 0 = 1 \\ L_\alpha(|f|, P) &= \sum_{i=1}^{n} m_i(|f|)\Delta\alpha(i) = \sum_{i=1}^{n} 1 \cdot \Delta\alpha(i) \\ &= \alpha(b) - \alpha(a) = 1 - 0 = 1 \end{aligned}$$

   And so we satisfy Riemann's Condition as

$$U_\alpha(|f|, P) - L_\alpha(|f|, P) = 0 \leq \epsilon$$

   for all $\epsilon > 0$, which implies $|f| \in \mathscr{R}_\alpha([a, b])$

   Show $f \notin \mathscr{R}_\alpha([0, 1])$: For this, we want to show that there does not exist a partition such that Riemann's condition holds.

   So start with the fact that for any partition P, all intervals $[x_{i-1}, x_i]$ will contain a rational and an irrational number. Thus by our definition of f,

$$M_i(f) = 1, \quad m_i(f) = -1 \qquad \forall i$$

   Therefore, for all P

$$\begin{aligned} U_\alpha(f, P) - L_\alpha(f, P) &= \sum_{i=1}^{n} [M_i(f) - m_i(f)]\Delta\alpha_i = \sum_{i=1}^{n} [1 - (-1)]\Delta\alpha_i \\ &= 2\sum_{i=1}^{n} \Delta\alpha_i = 2(\alpha(1) - \alpha(0)) = 2(1 - 0) \\ &= 2 \not\leq \epsilon \qquad \forall \epsilon > 0 \end{aligned}$$

   Thus, by Riemann's condition $f \notin \mathscr{R}_\alpha([0, 1])$

4. We want to show

$$U_\alpha(|f|,P) - L_\alpha(|f|,P) \leq U_\alpha(f,P) - L_\alpha(f,P) \tag{12}$$

and then give an alternate proof that

$$f \in \mathscr{R}_\alpha([a,b]) \quad \Rightarrow \quad |f| \in \mathscr{R}_\alpha([a,b]) \tag{13}$$

Now it's clear that if we can show Relation 12, then Relation 13 follows immediately. This is because if $f \in \mathscr{R}_\alpha([a,b])$, then (by Riemann's condition) there exists a partition P^* such that

$$U_\alpha(f,P^*) - L_\alpha(f,P^*) \leq \epsilon$$

And if this is true, by Relation 12

$$U_\alpha(|f|,P^*) - L_\alpha(|f|,P^*) \leq \epsilon$$

and so by Riemann's condition in reverse, we have that $|f| \in \mathscr{R}_\alpha([a,b])$.

So now, we just need to prove Relation 12. Rewriting as we did in the problems above, we need to show that

$$\sum_{i=1}^{n}[M_i(|f|) - m(|f|)]\Delta\alpha_i \leq \sum_{i=1}^{n}[M_i(f) - m(f)]\Delta\alpha_i$$

$$\text{By } \alpha \text{ increasing } \Leftrightarrow \quad M_i(|f|) - m(|f|) \leq M_i(f) - m(f) \qquad \forall i \tag{14}$$

Now first, by our definition of $M_i(f)$ and $m_i(f)$, we have that

$$f(x) - f(y) \leq M_i(f) - m_i(f) \qquad \forall x,y \in [x_{i-1},x_i] \tag{15}$$

This is true because $[m_i(f), M_i(f)]$ bound the range of f on $[x_{i-1},x_i]$ such that $f(x) \in [m_i(f), M_i(f)]$ for all x.[2] Since $M_i(f) - m_i(f)$ denotes the maximum distance between points in the image of f on the interval $[x_{i-1},x_i]$, Relation 15 must hold.

Next, we'll use a simple property of norms:

$$|\,|p| - |q|\,| \leq |p - q| \tag{16}$$

Putting Relations 15 and 16 together, we see that

$$|\,|f(x)| - |f(y)|\,| \leq |f(x) - f(y)| \leq |M_i(f) - m_i(f)| \qquad x,y \in [x_{i-1},x_i]$$

This proves Relation 14 because

$$M_i(|f|) = |f(x)| \qquad \text{for some } x \in [x_{i-1},x_i]$$
$$m_i(|f|) = |f(y)| \qquad \text{for some } y \in [x_{i-1},x_i]$$

[2] If that weren't the case, then either $f(x) > M_i(f)$ or $f(x) < m_i(f)$, in which case we clearly chose the wrong $m_i(f)$ or $M_i(f)$.

5. Without loss of generality, suppose the set Ω has one accumulation point. Let $a \in [0,1]$ denote this set accumulation point. Our goal is to find a P such that

$$U_\alpha(\chi_\Omega, P) - L_\alpha(\chi_\Omega, P) \leq \epsilon \quad \Rightarrow \quad \chi_\Omega \in \mathscr{R}_\alpha([a,b])$$

Do do so, we'll make the intervals $[x_{i-1}, x_i]$ around the points in Ω (the jumps where $M_i(\chi_\Omega) - m_i(\chi_\Omega) = 1$ all very small. And recall that this problem assumes $\alpha(x) = x$

So consider the accumulation point a and take $\delta = \epsilon/4$ (where ϵ is as above). By definition, there exists a sequence in $\{a_m\}$ in Ω such that

$$\lim_{m \to \infty} a_m = a$$

Then, by the definition of limit points, there is a number M such that

$$m > M \Rightarrow |a_m - a| \leq \delta$$

Thus there are only finitely many points outside of the interval

$$[a - \delta, a + \delta] \tag{17}$$

—M points outside, to be exact. Now let

$$B = \{b_1, \ldots, b_M\}$$

be that set of points.

So given this setup, let's choose the intervals for our partition, P. Around the point a, let,

$$[y_0, y_1] = [a - \delta, a + \delta] \qquad \text{from 17} \tag{18}$$

Next, for each $b_k \in B$, set

$$[y_{2 \cdot k}, y_{2 \cdot k+1}] = [b_k - \delta/M, b_k + \delta/M] \tag{19}$$

Now we have a set of points, $\{y_0, \ldots, y_{2 \cdot M+1}\}$ on the interval $[0,1]$ which can be sorted, relabeled as x's, and re-indexed to form our partition $\{x_0, \ldots, x_N\}$.

Given this partition, P, let i^* denote the index for the interval formed in Equation 18. Let I denote the indices for the intervals from Equation 19. Let J denote the other interval indices. Note that the intervals with index i^* or $i \in I$ will have the characteristic-function jumps where $M_i - m_i = 1$. Other intervals will be flat at 0, by design.

Now, finally, let's rewrite our difference of Darboux sums, noting that the problem assumes $\alpha(x) = x$:

$$\begin{aligned}
U_\alpha(\chi_\Omega, P) - L_\alpha(\chi_\Omega, P) &= \sum_{i=1}^{n} [M_i(\chi_\Omega) - m_i(\chi_\Omega)]\Delta x_i \\
&= [M_{i^*}(\chi_\Omega) - m_{i^*}(\chi_\Omega)]\Delta x_{i^*} \\
&\quad + \sum_{i \in I} [M_i(\chi_\Omega) - m_i(\chi_\Omega)]\Delta x_i \\
&\quad + \sum_{j \in J} [M_j(\chi_\Omega) - m_j(\chi_\Omega)]\Delta x_j \\
&\leq [1 - 0]\,(2 \cdot \delta) + \sum_{i \in I} [1 - 0]\,(2 \cdot \delta / M) \\
&\quad + \sum_{j \in J} 0(x_j - x_{j-1}) \\
&\leq 2\delta + M(2\delta/M) = 4\delta \\
\Rightarrow U_\alpha(\chi_\Omega, P) - L_\alpha(\chi_\Omega, P) &\leq 4\delta = 4(\epsilon/4) = \epsilon
\end{aligned}$$

So by Riemann's condition $\chi_\Omega \in \mathscr{R}_\alpha([a, b])$.

Finally, I didn't lose any generality by assuming that there was only one accumulation point. If there were N_a accumulation points, divide δ above by N_a and proceed in exactly the same way for each accumulation point.