Homework 2

Matthew Cocci

February 19, 2014

1. **Exercise 51.17, FoMA:** We consider f, a positive continuous function where $f \in \mathscr{R}([a,b])$. Letting

$$M = \max_{x \in [a,b]} f(x)$$

we want to prove

$$M = \lim_{n \to \infty} \left[\int_a^b [f(x)]^n \ dx \right]^{1/n} \tag{1}$$

Also, let x^* be the x value such that $f(x^*) = M$. By the intermediate value theorem, we know such an x^* exists.

Now because f is continuous on a compact interval, f is bounded, implying that M does exist. And since f is positive, $M > 0$ as well. Given that, we can show an equivalent statement to Equation **??** by dividing through by M:

$$1 = \lim_{n \to \infty} \left[\frac{1}{M^n} \int_a^b [f(x)]^n \ dx \right]^{1/n}$$

$$1 = \lim_{n \to \infty} \left[\int_a^b \left[\frac{f(x)}{M} \right]^n \ dx \right]^{1/n} \tag{2}$$

This will be our equivalent statement to prove, rather than Equation **??**.

So first, we know that that the integral in Equation **??** over the function h exists (i.e. $h \in \mathscr{R}([a,b])$—where

$$h(x) = \left[\frac{f(x)}{M} \right]^n \quad \text{with} \quad \begin{cases} h(x) = 1 & x = x^* \\ h(x) < 1 & x \neq x^* \end{cases}$$

This integral exists because both $f(x)$ is continuous and $g(x) = (x/M)^n$ is continuous. And because $h = g \circ f$, a composition of continuous functions, $h \in \mathscr{R}_\alpha([a,b])$ by Theorem 51.12 in FoMA.

It's also clear that we must have

$$\lim_{n\to\infty} h(x) = \lim_{n\to\infty} \left[\frac{f(x)}{M}\right]^n = \begin{cases} 1 & x = x^* \\ 0 & x \neq x^* \end{cases} \tag{3}$$

Finally, since h is a continuous function in $\mathscr{R}_\alpha([a,b])$, we can use Theorem 52.5 in FoMA to assert that

$$\int_a^b h\,dx = \lim_{||P||\to 0} S(h,P,T)$$

for any T. Choosing T so that x^* is one of the evaluation points in T, we can then

2. **Exercise 52.1, FoMA**: We want to prove, for $f \in \mathscr{R}([0,1])$ that

$$\lim_{n\to\infty} \sum_{k=1}^n f(k/n)\,\frac{1}{n} = \int_0^1 f(x)\,dx \tag{4}$$

Note that we're taking $\alpha(x) = x$, a continuous function.

We construct the proof by treating the lefthand side as a Riemman Sum. Namely, we construct a partition

$$P = \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, 1\right\} \qquad \Rightarrow \qquad \Delta x_i = \frac{1}{n}$$

We also consider the evaluation points within each interval

$$T = \left\{\frac{1}{n}, \frac{2}{n}, \dots, 1\right\} \tag{5}$$

Combining our partition and evaluation points into one sum, we have a Riemann Sum

$$S(f,P,T) = \sum_{k=1}^n f(t_i)\Delta x_i = \sum_{k=1}^n f(k/n)\,\frac{1}{n}$$

which is exactly the lefthand side of Equation ?. Now, we can invoke Theorem 52.5 in FoMA, as we satisfy the conditions that

- $f \in \mathscr{R}([0,1])$, which is assumed.
- α continuous, as $\alpha(x) = x$.

Thus, we can assert, since $||P|| = 1/n$, which goes to zero as n grows, that

$$\int_0^1 f\,dx = \lim_{||P||\to 0} S(f,P,T) = \lim_{||P||\to 0} \sum_{k=1}^n f(t_i)\Delta x_i$$

$$= \sum_{k=1}^n f(k/n)\,\frac{1}{n}$$

which is exactly what we wanted to show.

3. Recall the Lipshitz Condition: A function f is Lipshitz at x if for some $C, \delta > 0$

$$|x - y| \leq \delta \quad \Rightarrow \quad |f(x) - f(y)| \leq C|x - y| \tag{6}$$

We want to show that for $f : [0, 1] \to \mathbb{R}$

$$\left| \int_0^1 f \, dx - \frac{1}{n} \sum_{k=1}^{n} f(k/n) \right| \leq \frac{C}{n} \qquad \forall n \tag{7}$$

Now let's consider the partition,

$$P = \left\{ 0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n} \right\} \quad \Rightarrow \quad \Delta x_i = \frac{1}{n} \tag{8}$$

With this partition in mind, we can rewrite the integral

$$\int_0^1 f \, dx = \underline{\int_0^1} f \, dx$$
$$= L(f, P) + \epsilon$$

where $\epsilon \geq 0$, since

$$L(f, P) \leq \sup_Q L(f, Q)$$

And so we can rewrite the lefthand side of Inequality **??** as

$$\left| \int_0^1 f \, dx - \frac{1}{n} \sum_{k=1}^{n} f(k/n) \right| = \left| [L(f, P) + \epsilon] - \frac{1}{n} \sum_{k=1}^{n} f(k/n) \right|$$
$$\leq \left| L(f, P) - \frac{1}{n} \sum_{k=1}^{n} f(k/n) \right|$$
$$\leq \left| \sum_{i=1}^{n} m_i(f) \Delta x_i - \frac{1}{n} \sum_{k=1}^{n} f(k/n) \right|$$
$$\leq \left| \frac{1}{n} \sum_{i=1}^{n} [m_i(f) - f(k/n)] \right|$$

The integral is the lower sum at the same intervals as the sum on the right, plus some epsilon.

Remove the epsilon to get the inequality.

Use the lipshitz to bound.