# Homework 1

Matthew Cocci

February 9, 2014

1. (a) We want to show that, for all $P$,

$$U_\alpha(f,P) - L_\alpha(f,P) \leq w_f\left(||P||\right)\left(\alpha(b)-\alpha(a)\right)$$

By the definition of the Upper and Lower Darboux sums,

$$\begin{aligned} U_\alpha(f,P) - L_\alpha(f,P) &= \sum_{i=1}^n M_i(f)\Delta\alpha_i - \sum_{i=1}^n m_i(f)\Delta\alpha_i \\ &= \sum_{i=1}^n \left[M_i(f) - m_i(f)\right]\Delta\alpha_i \qquad (1) \end{aligned}$$

Now let $\overline{x}_i$ and $\underline{x}_i$ be the points such that

$$\begin{aligned} f(\underline{x}_i) &= m_i(f) = \inf_{x\in[x_i,x_{i-1}]} f(x) \\ f(\overline{x}_i) &= M_i(f) = \sup_{x\in[x_i,x_{i-1}]} f(x) \end{aligned}$$

Now since $\underline{x}_i$ and $\overline{x}_i$ are both in $[x_{i-1},x_i]$, we can combine this with the definition of $||P||$ to get

$$|\underline{x}_i - \overline{x}_i| \leq |x_i - x_{i-1}| \leq ||P||$$

By the definition of the modulus of continuity,

$$\begin{aligned} |\underline{x}_i - \overline{x}_i| \leq ||P|| \quad &\Rightarrow \quad |f(\underline{x}_i) - f(\overline{x}_i)| \leq w_f\left(||P||\right) \\ &\Leftrightarrow \quad |M_i(f) - m_i(f)| \leq w_f\left(||P||\right) \end{aligned}$$

Substituting this fact back into 1, and using the properties of telescoping sums, we get that

$$\begin{aligned} \sum_{i=1}^n \left[M_i(f) - m_i(f)\right]\Delta\alpha_i &\leq \sum_{i=1}^n w_f\left(||P||\right)\Delta\alpha_i = w_f\left(||P||\right)\sum_{i=1}^n \Delta\alpha_i \\ &\leq w_f\left(||P||\right)\left(\alpha(b)-\alpha(a)\right) \end{aligned}$$

(b) We know $f$ is monotone on $[a, b]$. Suppose that $f$ is monotone increasing.[1]Then given any partition, $P$,

$$f(x_{i-1}) \leq f(c) \leq f(x_i) \quad \forall i, c \in [x_{i-1}, x_i]$$

This implies that

$$M_i = f(x_i) \quad m_i = f(x_{i-1}) \qquad \forall i \tag{2}$$

Writing $U_\alpha - L_\alpha$ as above in Equation 1, we see that the expression reduces to

$$\sum_{i=1}^{n} [M_i(f) - m_i(f)] \Delta\alpha_i = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] \Delta\alpha_i \tag{3}$$

Now let's consider $w_\alpha(||P||)$. By definition, it is

$$w_\alpha(||P||) := \sup_{|x-y| \leq ||P||} |\alpha(x) - \alpha(y)|$$

It's clear that for each sub-interval, $[x_{i-1}, x_i]$, we have

$$|x_i - x_{i-1}| \leq ||P|| \quad \forall i$$

This implies that

$$|\Delta\alpha_i| = |\alpha(x_i) - \alpha(x_{i-1})| \leq w_\alpha(||P||) \qquad \forall i$$

Subbing this into the right-hand side of equation 3, we can simplify using the properties of telescoping sums to get

$$\begin{aligned}
\sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] \Delta\alpha_i &\leq \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] w_\alpha(||P||) \\
&\leq w_\alpha(||P||) \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] \\
\Leftrightarrow U_\alpha(f, P) - L_\alpha(f, P) &\leq w_\alpha(||P||) [f(b) - f(a)]
\end{aligned} \tag{4}$$

Finally, since $\alpha$ is assumed continuous on the compact interval $[a, b]$, $\alpha$ is uniformly continuous on $[a, b]$. That means

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \text{s.t.} \quad |x - y| \leq \delta \quad \Rightarrow \quad |\alpha(x) - \alpha(y)| \leq \epsilon$$

[1] Note that I'll lose no generality in assuming that the function $f$ was monotone increasing. If it's monotone decreasing instead, swap the values of $M_i$ and $m_i$ in Equation 2, and proceed in exactly the same way.

So if we can make $w_\alpha\left(||P||\right) < \epsilon$, by choosing our partition $P$ such that $||P|| \leq \delta$. With Equation 4, we therefore ensure that

$$U_\alpha(f, P) - L_\alpha(f, P) \leq \epsilon[f(b) - f(a)] \tag{5}$$

for any arbitrary $\epsilon$. And so by Riemann's Condition, $f \in \mathscr{R}_\alpha([a, b])$ because for any $\epsilon > 0$, we can ensure the upper and lower sums are within that distance of each other by choosing a sufficiently fine partition.

2.

3.

4.

5.