Asset Pricing

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1 Introduction

If all of Asset Pricing needed to be reduced to one simple, general, yet wholly appropriate epithet, a decent starting point would be "Price equals expected discounted payoff."

To segment the discussion further, John Cochrane suggests the following two classes of pricing approaches in his Asset Pricing book:

1. *Absolute Pricing*: "Pricing assets by exposure to fundamental sources of macroeconomic risk." This approach includes general equilibrium models.

2. *Relative Pricing*: This type of pricing considers only how an asset should be priced relative to other assets that can be bought and sold in the market. Black-Scholes and any arbitrage arguments would fall under this category.

However, many asset pricing approaches are a blend of the two.

To be more specific, we can reduce asset pricing to two equations:

$$p_t = E[m_{t+1}x_{t+1}] \tag{1}$$

$$m_{t+1} = f(\text{data, parameters}) \tag{2}$$

where p_t is the asset price, x_{t+1} is the asset's payoff, and m_{t+1} is the stochastic discount factor. This approach joins together what used to be separate theories so that pricing stocks, bonds, and options now represent special cases of this more general framework summarized by the equations above.

2 Discrete Time Review

2.1 Basic Building Block

The basic building block is random iid noise:

$$\varepsilon_t \sim \text{iid} \qquad E_t\varepsilon_{t+1} = 0 \qquad E_t\varepsilon^2_{t+1} = \sigma^2_\varepsilon$$

where $E_t(\cdot)$ represents the expectation *conditional* on the information at time t. In practice, that means any variable with a t subscript inside a expectation, variance, covariance, etc. operator can be taken out because it is *known* at time t.

Often ε_t will be draws from a normal distribution, but they don't have to be.

2.2 Canonical AR(1) Model

More complicated than simple random noise, we have the AR(1) process, which is written in one of either two standard ways, with the latter being more common as we move to continuous time:

$$\begin{aligned} x_{t+1} &= \rho x_t + \varepsilon_{t+1} \qquad E_t\varepsilon_{t+1} = 0 \qquad E_t\varepsilon^2_{t+1} = \sigma^2_\varepsilon \\ x_{t+1} &= \rho x_t + \sigma_\varepsilon\varepsilon_{t+1} \qquad E_t\varepsilon_{t+1} = 0 \qquad E_t\varepsilon^2_{t+1} = 1 \end{aligned}$$

From there, it's easy to show via the linearity property of the expectation operator and by successive substitution that

$$\begin{aligned} E_t x_{t+1} &= E_t[\rho x_t + \varepsilon_{t+1}] \\ &= \rho x_t \\ E_t x_{t+k} &= \rho^k x_t \end{aligned}$$

As for the variance conditional on information at time t, denoted by Var_t, we get

$$\begin{aligned} \text{Var}_t(x_{t+1}) &= \text{Var}_t(\rho x_t + \varepsilon_{t+1}) \\ &= \sigma^2_\varepsilon \\ \text{Var}_t(x_{t+2}) &= \text{Var}_t(\rho x_{t+1} + \varepsilon_{t+2}) \\ &= \text{Var}_t(\rho x_{t+1}) + \text{Var}_t(\varepsilon_{t+2}) + 2\,\text{Cov}_t(\rho x_{t+1},\ \varepsilon_{t+2}) \\ &= \rho^2\,\text{Var}_t(x_{t+1}) + \text{Var}_t(\varepsilon_{t+2}) + 0 = \rho^2\sigma^2_\varepsilon + \sigma^2_\varepsilon \\ &= (1+\rho^2)\sigma^2_\varepsilon \\ \text{Var}_t(x_{t+k}) &= (1 + \rho^2 + \cdots + \rho^{2(k-1)})\sigma^2_\varepsilon \end{aligned}$$

The last line we prove by induction.

Proof. In the case of $k = 3$, which we will take as the base case,

$$\begin{aligned}
\mathrm{Var}_t(x_{t+3}) &= \mathrm{Var}_t(\rho x_{t+2} + \varepsilon_{t+3}) \\
&= \rho^2 \,\mathrm{Var}_t(x_{t+2}) + \mathrm{Var}_t(\varepsilon_{t+3}) + \mathrm{Cov}_t(x_{t+2},\ \varepsilon_{t+3}) \\
\mathrm{Var}(x_{t+2}) \text{ from above} \qquad &= \rho^2(1+\rho^2)\sigma_\varepsilon^2 + \sigma_\varepsilon^2 + 0 \\
&= (1+\rho^2+\rho^4)\sigma_\varepsilon^2
\end{aligned}$$

So the base case holds. Now assume it holds for arbitrary k, and show that the condition also holds for $k+1$:

$$\begin{aligned}
\mathrm{Var}_t(x_{t+k+1}) &= \mathrm{Var}_t(\rho x_{t+k} + \varepsilon_{t+k+1}) \\
&= \rho^2 \,\mathrm{Var}_t(x_{t+k}) + \mathrm{Var}_t(\varepsilon_{t+k+1}) + \mathrm{Cov}_t(x_{t+k},\ \varepsilon_{t+k+1}) \\
&= \rho^2(1+\rho^2+\cdots+\rho^{2(k-1)})\sigma_\varepsilon^2 + \sigma_\varepsilon^2 \\
&= (1+\rho^2+\cdots+\rho^{2([k+1]-1)})\sigma_\varepsilon^2
\end{aligned}$$

so it holds for $k+1$. □

2.3 Canonical MA(1) Model

The basic form of the moving average MA(1) model, with conditional means:

$$x_{t+1} = \varepsilon_{t+1} + \theta\varepsilon_t \qquad E_t\varepsilon_{t+1} = 0, \quad E_t^2\varepsilon_{t+1} = \sigma_\varepsilon^2$$

The first moments for the process, conditional on information at time t, are as follows:

$$\begin{aligned}
\Rightarrow \qquad E_t x_{t+1} &= E_t\varepsilon_{t+1} + E_t[\theta\varepsilon_t] \\
&= \theta\varepsilon_t \\
E_t x_{t+2} &= E_t[\varepsilon_{t+2} + \theta\varepsilon_{t+1}] = E_t[\varepsilon_{t+2}] + E_t[\theta\varepsilon_{t+1}] \\
&= 0 \\
E_t x_{t+k} &= 0
\end{aligned}$$

Now for the conditional variances, dropping the covariance terms which we included above since innovations at different times are always independent:

$$\begin{aligned}
\mathrm{Var}_t(x_{t+1}) &= \mathrm{Var}_t(\varepsilon_{t+1} + \theta\varepsilon_t) \\
&= \mathrm{Var}_t(\varepsilon_{t+1}) + \theta^2\,\mathrm{Var}_t(\varepsilon_t) \\
&= \sigma_\varepsilon^2 \\
\mathrm{Var}_t(x_{t+k}) &= \mathrm{Var}_t(\varepsilon_{t+k} + \theta\varepsilon_{t+k-1}) \\
&= \mathrm{Var}_t(\varepsilon_{t+k}) + \theta^2\mathrm{Var}_t(\varepsilon_{t+k-1}) \\
&= (1+\theta^2)\sigma_\varepsilon^2
\end{aligned}$$

2.4 Unconditioanl Moments

So far, all of the means and variances above were *conditional* on information at time t. So that implies computing $E_t x_{t+1}$ and $\text{Var}_t(x_{t+1})$ via the formulas above requires knowledge of x_t.

Mean: However, suppose you wanted to generate the series from scratch. What is the *unconditional* mean and variance so that you can draw x_t *without* prior values $x_{t-1}, x_{t-2}, \ldots$?

$$\begin{aligned} Ex_t &= E[\rho x_{t-1}] + E\varepsilon_t \\ &= \rho E x_{t-1} + 0 \\ \Rightarrow \quad Ex_t &= 0 \end{aligned}$$

The last line follows because the unconditional average Ex_t must equal the unconditional average Ex_{t-1}, forcing the expecation to be zero.

Variance: Now we do the same thing for the variance. We'll also use the fact that the unconditional variance for x_t must equal that for x_{t-1}, which allows us to solve:

$$\begin{aligned} \text{Var}(x_t) &= \text{Var}(\rho x_{t-1} + \varepsilon_t) \\ &= \rho^2 \text{ Var}(x_{t-1}) + \text{Var}(\varepsilon_t) \\ \text{Using } \text{Var}(x_{t-1}) = \text{Var}(x_t): \quad &= \rho^2 \text{ Var}(x_t) + \sigma_\varepsilon^2 \\ \Rightarrow \quad \text{Var}(x_t) &= \frac{\sigma_\varepsilon^2}{1-\rho^2} \end{aligned}$$

2.5 Translating Between AR and MA Models

AR to MA: First, you can transform an AR(1) model into a MA(∞) model by "solving" the AR model through successive substitution:

$$\begin{aligned} x_t &= \rho x_{t-1} + \varepsilon_t \\ &= \rho(\rho x_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\ &= \rho^2 x_{t-2} + \rho\varepsilon_{t-1}) + \varepsilon_t \\ &= \quad \vdots \quad\quad \vdots \\ &= \sum_{j=1}^{\infty} \rho^j \varepsilon_{t-j} \end{aligned}$$

MA to AR: Similarly, you can transform an MA(1) model to an AR(∞) model as follows:

3 Continuous Time

3.1 Brownian Motion

We define Brownian Motion, the continuous version of the random walk, as follows:

1. Normal Increments for all $\Delta \in \mathbb{R}$:

$$z_{t+\Delta} - z_t \sim \mathrm{N}(0,\ \Delta)$$

2. Independence of non-overlapping increments.

3. Non-overlapping increments of the same length are identically (and normally) distributed.

3.2 Fundamental Building Block, dz_t

Just as random noise, ε_t was the basic building block for discrete time, the analogous building block in continuous time is

$$dz_t = \lim_{\Delta \to 0} z_{t+\Delta} - z_t \tag{3}$$

Note, d is *forward* difference operator.

We also have the result that dz is of size (or order) $\sqrt{dt}$ (which is equivalent to the limit of Δ), since recall

$$\begin{aligned} \mathrm{Var}(z_{t+\Delta} - z_t) &= \Delta \\ \Leftrightarrow \quad \mathrm{Sd}(z_{t+\Delta} - z_t) &= \sqrt{\Delta} \end{aligned}$$

So now we can see why z_t is not differentiable. Namely, if we tried to consider dz_t/dt, we know that dz_t is of order $\sqrt{dt}$; therefore, the ratio dz_t/dt diverges to $\pm\infty$ since dt is close to zero, while the numerator, dz_t of order $\sqrt{dt}$, is larger. So we can talk about *the change*, dz_t, because that change is going to be very small over short time horizons. But we can't talk about the *rate of change*, because compared to "the change" in the process, the change in time is smaller still, such that the ratio diverges.

Moments: Next, we have

$$\begin{aligned} E_t dz_t &= 0 \\ \mathrm{Var}_t(dz_t) &= dt \\ \mathrm{Cov}(dz_t,\ dz_s) &= 0 \qquad s \neq t \end{aligned}$$

From this, we get the very useful result:

$$\mathrm{Var}_t(dz_t) = E_t(dz_t^2) - \tag{4}$$