Linear and Discrete Optimization

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1 Introduction to Linear Optimization

1.1 Definition of a Linear Program

Linear optimization, also known as *linear programming*, consists of a linear *objective function* and *linear inequalities*:

$$\begin{aligned} \max \quad & c_1x_1 + \cdots + c_nx_n \\ \text{s.t.} \quad & a_{11}x_1 + \cdots + a_{1n}x_n \leq b_n \\ & \quad\quad \vdots \\ & a_{m1}x_1 + \cdots + a_{mn}x_n \leq b_m \end{aligned}$$

So we want to find the values of the *variables* $x_1, \ldots, x_n \in \mathbb{R}$ such that the objective function is maximized while still satisfying the linear inequalities. Using more convenient matrix notation, our linear program above can be rewritten more compactly as

$$\max\{c^T x \;:\; x \in \mathbb{R}^n,\; Ax \leq b\} \tag{1}$$

If we want to find the minimum instead of the max, we can easily use the fact that the minimum of some set S ($\min S$) is equivalent to $-\max S$.

1.2 Feasability, Optimality, Boundedness

We call a solution $x \in \mathbb{R}^n$ *feasible* if x satisfies all the linear inequalities. We call a linear program feasible if it has a feasible solution.

A feasible solution $s \in \mathbb{R}^n$ is an *optimal* solution of a linear program if $c^T x \geq c^T y$ for all $y \in \mathbb{R}^n$.

A linear program is *bounded* if there exists a constant $M \in \mathbb{R}$ such that $c^T x \leq \mathbb{R}$ for all feasible $x \in \mathbb{R}^n$.

1.3 Linear Algebra Defintions

Recall that for $A \in \mathbb{R}^{m\times n}$, the *kernel* of A, the *image* of A, and the *rowspace* of A are defined

$$\begin{aligned}\ker(A) &= \{x \in \mathbb{R}^n \mid Ax = 0\} \subset \mathbb{R}^n \\ \mathrm{im}(A) &= \{Ax \mid x \in \mathbb{R}^n\} \subset \mathbb{R}^m \\ \mathrm{rowspace}(A) &= \{y \in \mathbb{R}^n \mid \exists \lambda \in \mathbb{R}^m \text{ s.t. } A^T\lambda = y\}\end{aligned}$$

Note, the rowspace is just the set of all linear combinations of the rows of A.

Result: With these definitions in hand, we can show that a solution to the linear program in Equation 1 is feasible and unbounded if $b \in \mathrm{im}(A)$ and if $c \in \ker A \setminus \{0\}$.

Proof. First, $b \in \mathrm{im}(A)$ proves feasibility. It also implies that $\exists\ x^* \in \mathbb{R}^n$ such that $Ax^* = b$. This and the fact that $c \in \ker(A) \setminus \{0\}$ allows us to write

$$\begin{aligned}A(x^* + \lambda c) &= Ax^* + \lambda Ac \\ &= b + 0 = b\end{aligned}$$

which proves $x^* + \lambda c$ is a feasible solution, satisfying all of the linear inequalities.

So to finish up the unboundedness proof, we suppose the opposite: that the linear program is bounded:

$$c^T y \leq M \qquad \forall y \in \mathbb{R}^n$$

Since $x^* + \lambda c$ is feasible, then it should also be bounded:

$$\begin{aligned}\Rightarrow \quad c^T(x^* + \lambda c) &\leq M \\ c^T x^* + \lambda c^T c &\leq M\end{aligned}$$

Now since $c \neq 0$, we know that $c^T c$ will be greater than 0. Now let's rearrange and choose

$$\lambda \geq \frac{M - c^T x^*}{c^T c}$$

in which case $c^T(x^* + \lambda c) > M$. $\square$

2 The Geometry of Linear Programming

2.1 General Definitions

Polyhedron: A set P of vectors in $\mathbb{R}^n$ is a polyhedron if $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ for some matrix A and some vector b. By our definition of a linear program in Equation 1, we see that the set of feasible solutions is a polyhedron.[1]

Half space: This is defined as $\{x \in \mathbb{R}^n \mid a^T x \leq \beta\}$, where $a \in \mathbb{R}^n \setminus \mathbf{0}$.

Hyperplane: This is defined as $\{x \in \mathbb{R}^n \mid a^T x = \beta\}$, where $a \in \mathbb{R}^n \setminus \mathbf{0}$.

Valid: An inequality $a^T x \leq \beta$ is valid for a polyhedron, P, if each $x^* \in P$ satisfies $a^T x^* \leq \beta$.

Active: An inequality $a^T x \leq \beta$ is active at $x^* \in \mathbb{R}^n$ if $a^T x^* = \beta$.

2.2 Vertices

We'll offer three equivalent definitions, each of which offers some different intuition or an additional operational advantage:

1. A point $x^* \in P$ is a *vertex* of P if there exists an inequality $a^T x \leq \beta$ such that
   a) $a^T x \leq \beta$ is valid for P.
   b) $a^T x \leq \beta$ is active at x^* and not active at any other point in P.

2. There's also an equivalent definition which says $x^* \in P$ is a vertex if and only if there exists a vector $c \in \mathbb{R}^n$ such that x^* is the unique optimal solution of the linear program $\max\{c^T x \mid x \in P\}$.

3. Suppose we have $x^* \in P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ as a vertex of P. Then this vertex satisfies a smaller number of active contraints (relative to all of the constraints that define the linear program and P). Therefore, we express those active constraints that we want to enforce with $\bar{A}x \leq \bar{b}$.

   Now since x^* is a vertex satisfying those certain active constraints, that means x^* is the unique solution of $\bar{A}x = \bar{b}$. That's equivalent to the statement that $\operatorname{rank}(\bar{A}) = n$, which is equivalent to the columns of $\bar{A}$ being linearly independent.

   So if we suspect that x^* is a vertex of our linear program, then we examine the subsystem, $\bar{A}x \leq \bar{b}$, of our LP that x^* satisfies with equality, and we look at the rank of $\bar{A}$. If x^* really is a vertex of P, the the rank of $\bar{A}$ equals n and the columns will be linearly independent.

[1] Just one quick note for generality: The empty set, $\emptyset$, is also a polyhedron as $\emptyset = \{x \in \mathbb{R}^n \mid \mathbf{0}^T x \leq -1\}$, satisfying the definition of a polyhedron.