# Notes to Financial Engineering: Equity Derivatives

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# 1. The Fundamental Theorem of Asset Pricing

## 1.1. Introduction

Derivatives require special pricing techniques aside from the traditional discounted cash flow (DCF) approach, as DCF requires an estimate of the appropriate risk-adjusted rate of return. However, the risk of a derivative varies over time, which makes it difficult to estimate the derivative's risk-adjusted return.

As a result, derivatives pricing turns to the no-arbitrage approach (NA), which eliminates the need to build risk into the model.

## 1.2. Trading Strategy and Derivative Pricing Definitions

A **trading strategy** is a dynamically-rebalanced portfolio.

A trading strategy is **self-financing** if it generates no intermediate cash inflows and requires no intermediate outflows between the time the portfolio is initiated and the time it is liquidated. This implies that

i. All dividends are reinvestd.

ii. Value of the assets sold at a rebalance time must equal the value of the assets bought.

A trading strategy is **strictly positive** if the value of the traded portfolio can never become zero or negative.

Let $N$ be the value of a *strictly positive*, *self-financing* trading strategy. Then $N$ is a **numeraire process** or, simply, a **numeraire**. Here are a few examples:

- Price of Dividend paying asset: NO, as there are intermediate cash outflows, violating self-financing condition.
- The price of a forward contract: NO, as it can go negative, violating the strictly positive condition.
- Price of a Foreign Currency: NO, as it is equivalent to a dividend paying asset because you think of it as an investment in an interest-bearing account.
- Price of a non-defaultable zero-coupon bond: YES.
- Value of a money market account earning the risk free rate, where there are no interim deposits or withdrawals: YES.

## 1.3. Martingales and Change of Measure

A **martingale** is a stochastic process $X$ with the property

$$E_t[X(T) - X(t)] = 0 \;\Leftrightarrow\; E_t[X(T)] = X(t), \qquad T > t.$$

A **probability measure** is a specification of the probabilities of all the possible states of the words, mapping states to real numbers.

Suppose that $\xi$ is a nonnegative random variable on $(\Omega, \mathcal{F}, P)$ with $E_P[\xi] = 1$. (The subscript $P$ highlights that the last expectation is with respect to measure $P$.) Then define a new measure

$$Q : \mathcal{F} \to [0,1]$$

$$Q(A) = E\left[1_A \xi\right] = \int_A \xi(\omega) dP(\omega), \qquad A \in \mathcal{F} \tag{1}$$

Clearly, $Q$ is a probability measure on $(\Omega, \mathcal{F})$ and it is absolutely continuous with respect to $P$—i.e. we have

$$Q(A) > 0 \Rightarrow P(A) > 0.$$

Note that it is common to write the random variable $\xi$ as

$$\xi = \frac{dQ}{dP},$$

and we often refer to $\xi$ as the *Radon-Nikodym derivative* or the *likelihood ratio* of $Q$ with respect to $P$.

**Radon-Nikodym Theorem** If $P$ and $Q$ are two probability measures on $(\Omega, \mathcal{F})$, then there *will exist* such a random variable $\xi$ so that Expression 1 holds.

## 1.4. Fundamental Theorem of Asset Pricing (No Dividends)

### 1.4.1. Statement of Theorem

Suppose we have $n$ non-dividend-paying assets with price processes $S_1, S_2, \ldots, S_n$. Let $N$ be some numeraire process. Then, barring market imperfection, there are no arbitrage opportunities among these assets if and only if there exists a strictly positive probability measure $Q_N$ (so it's dependent upon the numeraire, $N$) under which each of the processes $S_i/N$ is a martingale.

- Note that $S_i/N$ is the price of asset $i$ in units of the numeraire $N$. Therefore, we call $S_i/N$ the **normalized price process**.
- The probability measure $Q_N$ will, in general depend on the numeraire. Therefore, we call $Q_N$ the **martingale measure** or **pricing measure** associated with the numeraire $N$.
- We can paraphrase FTAP by saying that, if the is no arbitrage or market imperfections, then given *any* numeraire process $N$, there must exist a corresponding martingale measure $Q_N$ under which the normalized price of any non-dividend paying asset is a martingale:

  $$\frac{S_i(t)}{N(t)} = E_t^{Q_N}\left[\frac{S_i(T)}{N(T)}\right], \qquad T > t.$$

  From this, we see that changing $N$ will generally change $Q_N$ as well.

### 1.4.2. Consequences for Derivative Pricing

Let $V$ denote the price of a derivative with payoff $V(T)$ at time $T$. Then we can apply FTAP to get

$$V(t) = N(t)E_t^{Q_N}\left[\frac{S_i(T)}{N(T)}\right], \qquad T > t.$$

Note, the price we get for a deriative is *invariant* to the choice of the numeraire.

### 1.4.3. Special Numeraires and Martingale Measures

**T-forward measure** Let $P(t,T)$ be the price of a non-defaultable zero-coupon bond with unit face value. Then

$$N(t) = P(t,T)$$

is our numeraire. The associated martingale measure, denoted $Q_T$, is called the *T-forward martingale measure*. This yields a derivative price of

$$V(t) = E_t^{Q_B}\left[e^{-\int_t^T r(s)ds}V(T)\right]$$

**Risk Neutral Measure** Let's consider the value of a money market account with unit initial value as our numeraire. Then

$$N(t) = B(t) = e^{\int_0^t r(s)ds}$$

where $r$ is the instantaneous risk-free rate. The associated martingale measure, denoted by $Q_B$, is called the *risk-neutral martingale measure*. This yields a derivative price of

$$V(t) = P(t,T)\, E_t^{Q_T}\left[V(T)\right] \;=\; e^{-r(t,T)(T-t)} E_t^{Q_T}\left[V(T)\right]$$

$$r(t,T) = -\ln P(t,T)/(T-t)$$

If interest rates are stochastic (and they probably are), then this measure isn't as convenient a tthe $T$-forward measure.

## 1.5. FTAP for Dividend-Paying Assets

Consider an asset with price process $S$ and let $D(t)$ denote the cumulative dividend paid by the asset from time 0 up to time $t$. We can consider the undiscounted cash flows from holding an asset from $t$ to $T$:

$$S(T) - S(t) + D(T) - D(t) = GP(T) - GP(t)$$

where $GP(t) = S(t) + D(t)$ is the asset's gain process.

Given a numeraire $N$, the asset's *normalized gain process*, denoted NGP, measures the gains from holding the assets in units of $N$:

$$NGP(t) = \frac{S(t)}{N(t)} + \int_0^t \frac{dD(s)}{N(s)}$$

where $dD(s)$ is the dividend paid by the asset at time $s$.

**Theorem** Now, let's restate the fundamental theorem of asset pricing allowing for dividend paying assets. So again, consider assets with price processes $S_1, \ldots, S_n$ and cumulative dividend processes $D_1, \ldots, D_n$, letting $N$ be any numeraire process. Then there are no arbitrage opportunities across these assets if and only if there exists a strictly positive probability measure $Q_N$ under which each

$$\frac{S_i(t)}{N(t)} + \int_0^t \frac{dD_i(s)}{N(s)}$$

is a martingale. This implies

$$\frac{S_i(t)}{N(t)} = E_t^{Q_N}\left[\frac{S_i(T)}{N(T)} + \int_t^T \frac{dD_i(s)}{N(s)}\right]$$

And so the normalized price of any asset is equalt to the conditional expectation under the martingale measure of the assets normalized payoffs (including future dividends).

# 2. Continuous Time Stochastic Processes

Here, we develop the necessary machinery in continuous time stochastic processes to model asset price evolution properly and with sufficient richness and generality. In particular, we discuss a heirarchy of model classes that includes Brownian Motion $\subset$ Generalized Brownian Motion $\subset$ Diffusions $\subset$ Ito Processes.

## 2.1. Introduction

A *stochastic process* $X$ is a collection of random variables indexed by time: $X = \{X_t : t \in \mathcal{T}\}$.

- *Discrete Time*: $\mathcal{T}$ countable, and process changes only at discrete time intervals.
- *Continuous Time*: $\mathcal{T}$ uncountable.

**Definition** A process $X$ has stationary increments if $X_T - X_t$ has the same distribution as $X_{T'} - X_{t'}$ provided that $T - t = T' - t'$.

## 2.2. Brownian Motion

**Definition** The most basic continuous-time process is *Brownian Motion* (or the *Wiener Process*). It has three defining properties:

i. $W(0) = O$.

ii. $W(t)$ is continuous, so no jumps.

iii. Given any two times, $T > t$, the increment $W(T) - W(t)$ is independent of all previous history and normally distributed with mean 0 and variance $T - t$.

A few consequences of the definition of $W(t)$:

- Brownian motion has independent stationary increments.
- $W(t)$ is normally distributed with $\mu = 0$, $\sigma^2 = t$.

## 2.3. Generalized Brownian Motion

**Definition** A *generalized Brownian motion* is a continuous-time process $X$ with the following property:

$$X(t) = X(0) + \mu t + \sigma W(t)$$

where $\mu$ is the *drift*, $\sigma$ is the *volatility*, and $W$ is simple Brownian motion. The differential equation equivalent is written:

$$dX(t) = \mu\, dt + \sigma\, dW(t).$$

It follows immediately from the definition that

- $X(t)$ is continuous, so no jumps.
- $X(t)$ is normally distributed with mean $X(0) + \mu t$, variance $\sigma^2 t$.
- Given any two times, $T > t$, the increment $X(T) - X(t)$ is independent of all previous history and normally distributed with mean $\mu(T-t)$ and variance $\sigma^2(T-t)$.
- $X$ is a martingale if and only if $\mu = 0$.

**Theorem** It also happens that Generalized Brownian motions are the only continuous time processes with continuous sample paths and stationary increments.

## 2.4. Ito Processes

Even more general than Brownian Motion (which is retained as a special case), an *Ito Process* is a stochastic process $X$ defined by one of two equivalent formulations:

$$dX(t) = \mu(t)\ dt + \sigma(t)\ dW(t)$$

$$X_t = X_0 + \int_0^t \mu(s)\ ds + \int_0^t \sigma(s)\ dW(s)$$

for any arbitrary stochastic processes $\mu$ (the drift) and $\sigma$ (the volatility) along with some Brownian Motion $W(t)$. Here are some properties

- Has continuous sample paths and is a martingale if and only if $\mu(t) = 0$.
- Increments are not necessarily stationary, as $\mu$ and $\sigma$ can change *randomly* with time.

**Definition** If the drift and volatility of an Ito process depend only upon the current value of the process and time, then $X$ is a *diffusion*. Mathematically, $X$ is a *diffusion* if

$$dX(t) = \mu(X(t), t)\ dt + \sigma(X(t), t)\ dW(t)$$

for some functions $\mu$ and $\sigma$.

## 2.5. Ito's Lemma

Suppose that $X$ is an Ito process defined by

$$dX(t) = \mu(t)\ dt + \sigma(t)\ dW(t) \tag{2}$$

and we define a new process $Y(t) = f(X(t), t)$ where $f$ is some function that's twice differentiable in $X$ and once in $t$. Then we have that

$$dY(t) = f_X(X(t), t)\ dX(t) + f_t(X(t), t)\ dt + \frac{1}{2} f_{XX}(X(t), t)\sigma(t)^2\ dt. \tag{3}$$

where subscripts on $f$ denote the partial derivatives.[1] Subbing Equation 2 into Equation 3, we get that

$$dY(t) = \left( f_X(X(t),t)\mu(t) + f_t(X(t),t) + \frac{1}{2} f_{XX}(X(t),t)\sigma(t)^2 \right)\, dt + f_X(X(t),t)\sigma(t)\, dW(t)$$

Thus, it is clear that $Y$ is also an Ito Process by the statement above with the drift and volatility given by the coefficients on $dt$ and $dW(t)$ as always.

**Using Ito's Lemma** In practice, we use Ito's Lemma whenever we have a (typically complicated) Ito Process that we want to solve. Given the process $X$ and its corresponding Ito Process, we posit a function $f$ that could could help. Then we write a new Ito Process using Ito's lemma with $dY(t) = df(X(t),t)$ on the LHS. From there, hopefully we can integrate $dY(t)$ easily on the left and solve out for $X(t)$.

## 2.6. Multi-dimensional Ito's Lemma

For the sake of completeness, let's generalize Ito's Lemma to consider the case of a finite number of Ito processes, $X_1, X_2, \dots, X_n$,

$$dX_i(t) = \mu_i(t)\, dt + \sigma_i(t)\, dW_i(t).$$

Next, define $Y(t) = f(X_1(t), \dots, X_n(t), t)$ for some differentiable function $f$. Then multi-dimensional Ito's Lemma says

$$\begin{aligned} dY(t) = &\sum_{i=1}^{n} f_{X_i}(X_1(t), \dots, X_n(t), t)\, dX_i(t) \\ &+ f_t(X_1(t), \dots, X_n(t), t)\, dt \\ &+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} f_{X_i,X_j}(X_1(t), \dots, X_n(t), t)\, \rho_{ij}\sigma_i(t)\sigma_j(t)\, dt \end{aligned}$$

where $\rho_{ij}$ is the correlation coefficient between $dW_i$ and $dW_j$. Note that you'll have to plug back in for the $dX_i$ in the first sum.

We'll mostly consider with the two-dimensional case for the two specific instances below:

i. $Y(t) = X_1(t)X_2(t)$, which gives us

$$dY(t) = X_2(t)\, dX_1(t) + X_1(t)\, dX_2(t) + \rho_{12}\sigma_1(t)\sigma_2(t)\, dt$$

Note that you'll have to plug back in for $dX_1$ and $dX_2$. This is a type of integration-by-parts formula because (after rearranging terms) it relates $X_1\, dX_2$ to $X_2\, dX_1$.

[1] Note that Equation 3 almost looks like the chain rule from traditional calculus, except for that extra term with $f_{XX}$ partial derivative. That arises from the additional variability due to the inclusion of stochastic factors like $W(t)$ in the original Ito Process.

ii. $Y(t) = X_1(t)/X_2(t)$, which gives us

$$dY(t) = \frac{1}{X_2(t)} dX_1(t) - \frac{X_1(t)}{X_2(t)^2} dX_2(t) + \frac{X_1(t)}{X_2(t)^3} \sigma_2(t)^2 \, dt - \frac{1}{X_2(t)^2} \, \rho_{12}\sigma_1(t)\sigma_2(t) \, dt.$$

Note that you'll have to plug back in for $dX_1$ and $dX_2$. This is a type of integration-by-parts formula because (after rearranging terms) it relates $X_1 \, dX_2$ to $X_2 \, dX_1$.

## 2.7. Geometric Brownian Motion

Let's consider the process $X$ governed by

$$dX(t) = X(t)\mu(t) \, dt + X(t)\sigma(t) dW(t).$$

To solve, let us consider the process $Y(t) = \log X(t)$. We compute the partials and apply Ito's Lemma:

$$f_X = \frac{1}{X(t)}, \qquad f_{XX} = -\frac{1}{X(t)^2}, \qquad f_t = 0$$

$$dY(t) = \frac{1}{X(t)} dX(t) - \frac{1}{2}\frac{1}{X(t)^2}(\sigma(t)X(t))^2) \, dt$$

which simplifies (after subbing in for $dX(t)$) into the expression

$$dY(t) = \left(\mu(t) - \frac{1}{2}\sigma(t)^2\right) + \sigma(t) \, dW(t).$$

Next, integrating both sides and substituting back in with $Y(t) = \log X(t)$, we get

$$Y(t) = Y(0) + \int_0^t \left(\mu(s) - \frac{1}{2}\sigma(s)^2\right) ds + \int_0^t \sigma(s) \, dW(s)$$

$$X(t) = X(0) e^{\int_0^t \left(\mu(s) - \frac{1}{2}\sigma(s)^2\right) ds + \int_0^t \sigma(s) \, dW(s)}$$

where it also follows that $X$ is strictly positive.

**Special Case** Suppose that $\mu$ and $\sigma$ are constant, in which case $X$ follows a *geometric Brownian motion*. Then it follows that

$$\log X(t) \sim N\left(\log X(0) + \left(\mu - \frac{1}{2}\sigma^2 t\right), \sigma^2 t\right)$$

so that we say $X(t)$ is *lognormally distributed.*

## 2.8. Girsanov's Theorem

**Theorem** Suppose that $X$ is an Ito process

$$dX(t) = X(t)\mu(t)\ dt + X(t)\sigma(t)\ dW(t),$$

where $\mu$ an $\sigma$ are stochastic processes and $W$ is Brownian Motion under some probability measure $P$—like maybe the real world measure. Then if $Q$ is any other strictly positive probability measure, then $X$ is also an Ito process under $Q$—i.e., there exist processes $\hat{\mu}$, $\hat{\sigma}$, and $\hat{W}$ with the property that

$$dX(t) = X(t)\hat{\mu}(t)\ dt + X(t)\hat{\sigma}(t)\ d\hat{W}(t). \tag{4}$$

Even better, $\hat{\sigma} = \sigma$.

This is particularly useful because, in general, we will have to work with two different probability measures: the true/historical $P$ and the martingale probability measure $Q_N$.

# 3. The Black-Scholes Model

## 3.1. Deriving the Governing Process Under $Q_B$

Let's put to use everything we've developed so far and derive the Black-Scholes Model.

First, we start by specifying the assumptions of the model:

1. The price of the underlying asset, the stock, follows an Ito process with volatility proportional to the price level:
$$dS(t) = S(t)\mu(t)\ dt + S(t)\sigma\ dW(t). \tag{5}$$
This is the movement under the real-world probibility measure $P$.
2. It pays dividends continuously at a constant rate $\delta$, implying that the cumulative dividend at time $t$ is
$$D(t) = \int_0^t S(s)\delta\ ds.$$
3. The instantaneous risk-free rate is *constant* and equal to $r$.
4. There are no arbitrage opportunties or market imperfections.

Now, we can apply FTAP to say that the value of a Path-Independent Option (PID) with payoff $V(T) = \varphi(S(T))$ at time $T$ can be expressed as

$$V(t) = e^{-r(T-t)} E_t^{Q_B} \left[\varphi(S(T))\right].$$

To compute the expression on the RHS, we will need to determine the distribution of $S(T)$ under the risk-neutral measure $Q_B$, or we will need to simulate $S(T)$ under $Q_B$. In that case, we will employ Girsanov's Theorem to our SDE from Equation 5 to rewrite the evolution of the process now under the risk neutral measure

$$dS(t) = S(t)\hat{\mu}(t)\ dt + S(t)\sigma\ d\hat{W}(t) \tag{6}$$

where $\hat{W}$ is Brownian motion under $Q_B$. We know everything written here *except* $\hat{\mu}$, whose computation will now be our main concern.

So in order to determine $\hat{\mu}$, we will need to compute the dynamics of the normalized stock gain process:

$$NGP(t) = \frac{S(t)}{B(t)} + \int_0^t \frac{dD(s)}{B(s)} = \frac{S(t)}{B(t)} + \int_0^t \frac{S(s)\delta\ ds}{B(s)}. \tag{7}$$

Now it follows from Equation 7 that

$$dNGP(t) = d\left(\frac{S(t)}{B(t)}\right) + \frac{S(t)}{B(t)}\delta\ dt. \tag{8}$$

Next, since we assume $r$ to be constant, by the way we defined $B(t)$, it follows that

$$B(t) = e^{\int_0^t r(s)ds} \quad \Rightarrow \quad dB(t) = B(t)r\ dt$$

Now if we apply Multi-dimensional Ito's Lemma for the ratio of the two processes (a special case we discussed above), we get

$$d\left(\frac{S(t)}{B(t)}\right) = \frac{S(t)}{B(t)}\left(\hat{\mu}(t) - r\right)\ dt + \frac{S(t)}{B(t)}\sigma(t)\ d\hat{W}(t). \tag{9}$$

Substituting Equation 9 into Equation 8, we finally get that

$$dNGP(t) = \frac{S(t)}{B(t)}\left(\hat{\mu}(t) + \delta - r\right)\ dt + \frac{S(t)}{B(t)}\sigma(t)\ d\hat{W}(t). \tag{10}$$

Now since FTAP tells us that the normalized gain process must be a martingale, we know that the the drift term in Equation 10 will need to be 0, forcing

$$\hat{\mu}(t) = r - \delta.$$

So substituting this result back into Equation 6, which we got through Girsanov's theorem, we obtain

$$dS(t) = S(t)(r - \delta)\ dt + S(t)\sigma\ d\hat{W}(t)$$

which, you'll note, forces the stock price to follow a geometric Brownian Motion under $Q_B$. GBM leads to the following expression for $S(T)$:

$$S(T) = S(t)e^{\left(r-\delta-\frac{1}{2}\sigma^2\right)(T-t)+\sigma(\hat{W}(T)-\hat{W}(t))}$$

As a consequence, $\ln S(T)$ is normally distributed with mean

$$\ln S(t) + \left(r - \delta - \frac{1}{2}\sigma^2\right)(T - t)$$

and standard deviation $\sigma\sqrt{T-t}$, *under the measure* $Q_B$. From there, we can easily simulate samples from the distribution of $S(T)$ for Monte Carlo simulation, which will be covered in later sections.

## 3.2. Arriving at a Closed-Form Solution

With a little bit of statistics, we can work out explicit formulas for digital and vanilla options under the Black-Scholes model and assumptions, so let's build up to that.

Consider $X$, where $\ln X$ is normally distributed with mean $\mu_X$ and standard deviation $\sigma_X$. Then if $1_{\{X \geq K\}}$ is the indicator function for $X \geq K$, then we have that

$$E\left[1_{\{X \geq K\}}\right] = P(X \geq K) = \Phi\left(\frac{\mu_X - \ln K}{\sigma_X}\right) \tag{11}$$

$$E\left[X \cdot 1_{\{X \geq K\}}\right] = e^{\mu_X + \frac{1}{2}\sigma_X^2}\ \Phi\left(\frac{\mu_x + \sigma_X^2 - \ln K}{\sigma_X}\right) \tag{12}$$

Coming back to the Black-Scholes model, instead of $X$, we'll take $S(T)$, where we recall that $\ln S(T)$ is normally distributed under $Q_B$ with mean $\ln S(t) + (r - \delta - (1/2)\sigma^2)(T - t)$ and standard deviation $\sigma\sqrt{T-t}$. From there, we apply Result 11 to say that

$$E_t^{Q_B}\left[1_{\{S(T) \geq K\}}\right] = \Phi\left(\frac{\ln S(t) + \left(r - \delta - \frac{1}{2}\sigma^2\right)(T-t) - \ln K}{\sigma\sqrt{T-t}}\right)$$

We can use this result to price a *cash-or-nothing* (CON) call which pays $\varphi(S(T)) = 1_{\{S(T) \geq K\}}$ at $T$:

$$\begin{aligned} c^{\mathrm{CON}}(t) &= e^{-r(T-t)} E_t^{Q_B}\left[1_{\{S(T) \geq K\}}\right] \\ &= e^{-r(T-t)} \Phi\left(\frac{\ln S(t) + \left(r - \delta - \frac{1}{2}\sigma^2\right)(T-t) - \ln K}{\sigma\sqrt{T-t}}\right) \end{aligned} \tag{13}$$

Now let's adapt Result 12 to the Black Scholes model and use it to price another type of derivative—the so-called *all-or-nothing* (AON) call, which has payoff $\varphi(S(T)) = S(T)1_{\{S(T) \geq K\}}$ at time $T$. The price of this option at $T$ is

$$\begin{aligned} c^{\mathrm{AON}}(t) &= e^{-r(T-t)} E_t^{Q_B}\left[1_{\{S(T) \geq K\}}\right] \\ &= e^{-r(T-t)}\ S(t)\ e^{(r-\delta)(T-t)}\ \Phi\left(\frac{\ln S(t) + \left(r - \delta + \frac{1}{2}\sigma^2\right)(T-t) - \ln K}{\sigma\sqrt{T-t}}\right) \\ &= S(t) e^{-\delta(T-t)} \Phi\left(\frac{\ln S(t) + \left(r - \delta + \frac{1}{2}\sigma^2\right)(T-t) - \ln K}{\sigma\sqrt{T-t}}\right) \end{aligned} \tag{14}$$

From there, we price a *vanilla call* by expressing it as a portfolio of cash-or-nothing and all-or-nothing calls:

$$c(t) = c^{\mathrm{AON}}(t) - K c^{\mathrm{CON}}(t)$$

We then get the value by using Formulas 13 and 14.

## 3.3. General Logic

So let's just recap the general steps and intuition that we employed at arriving at the final solution of a PID, being a bit more general than the specific Black-Scholes case:

1. First, we posit some governing process for the stock price under the real-world probability measure, $P$.

2. Next, use Girsanov's Theorem to assert the existence of a governing process under the probability measure $Q$. As per Girsanov's Theorem, this new process will have the same variance as the original process, but *not* the same drift (note that the drift under $P$ is largely irrelevant in how we price derivatives.)

3. Then, figure out how to express the normalized gain process for the underlying. According to FTAP, this must be a Martingale under $Q$, implying a drift of 0.

4. Using the fact that the drift of $dNGP$ must be 0, solve out for $\hat{\mu}$, and plug that value into the governing SDE under $Q$.

5. Finally, solve out the SDE under $Q$, and simulate many sample paths from the solution, computing the value of the derivative at time $T$ under each. Take the average of all of these values to get the approximate expectation under $Q$.

## 3.4. Relaxing Some Assumptions

Recall that the Black-Scholes model assumes that $r$, $\delta$, and $\sigma$ are all *constants*. However, we can easily adapt in the case that each one is a *non-random* function of time.[2]. Specifically, we replace $\theta(T-t)$[3] with $\bar{\theta}(T-t)$:

$$\bar{r} = \frac{1}{T-t}\int_t^T r(s)ds, \qquad \bar{\delta} = \frac{1}{T-t}\int_t^T \delta(s)ds, \qquad \bar{\sigma^2} = \frac{1}{T-t}\int_t^T \sigma(s)^2 ds$$

[2]So no stochastic factors can influence any of those parameters

[3]$\theta$ is any one of the three parameters.

# 4. Monte Carlo Pricing of PIDs

Let's consider a path-independent derivative (PID) that provides the payoff $\varphi(S(T))$ at expiration date $T$. FTAP tells us that

$$V(0) = e^{-rT} E^{Q_B} \left[ \varphi(S(T)) \right], \tag{15}$$

where $E^{Q_B}$ is the expectation under the risk-neutral probability.

Next, if we believe the assumptions of the Black-Scholes model, then

$$S(T) = S(0)\, e^{\left(r-\delta-\frac{1}{2}\sigma^2\right)T+\sigma \hat{W}(T)} \tag{16}$$

If we wish to compute the price $V(0)$ numerically (as opposed to analytically), then we use Monte Carlo simulation:

1. Draw a random value from a $N(0, (\sqrt{T})^2)$ distribution.
2. Plug that value into Expression 16 to get a random draw from the distribution of $S(T)$ under $Q_B$.
3. Compute the payoff, $\varphi(S(T))$.
4. Repeat those steps to get many draws of $\varphi(S(T))$ under $Q_B$, then average them to get an unbiased estimate of $E^{Q_B} [\varphi(S(T))]$.
5. Multiply that average by the discount factor $e^{-rT}$ to get an estimate of the PID price.

Now that we've figured out a way to compute a numeric approximation, it's naturual to ask how accurate this is. To do so, we use the Central Limit Theorem to assert

$$EX - \bar{X} \sim N\left(0,\ \frac{\sigma_X}{\sqrt{N}}\right)$$

as the number of draws gets very large, and where $EX$ is the true mean, and $\bar{X}$ is the sample mean from the Monte Carlo simulation. Getting even more specific,

$$\hat{V}_0 \sim N\left(V_0,\ e^{-rT}\frac{\sigma_\varphi}{\sqrt{N}}\right)$$

where $\sigma_\varphi$ is the standard deviation of the payoff $\varphi(S(T))$, $V_0$ is the true price, and $\hat{V}_0$ is the Monte Carlo estimate.

# 5. Monte Carlo Pricing of PDDs

Next, consider a generic *discretely-sampled* European path-dependent derivative with the payoff dependent upon the price of the underlying at a set of dates $\{t_1, \ldots, t_n\}$ where $t_i \in [t, T]$, i.e.

$$V(T) = \varphi(S(t_1), \ldots, S(t_n)).$$

Recalling FTAP and the path of stock prices under the Black-Scholes assumption,

$$V(t) = e^{-r(T-t)} E_t^{Q_B} \left[ \varphi(S(t_1), \ldots, S(t_n)) \right]$$

$$S(t_i) = S(t_{i-1}) e^{\left(r-\delta-\frac{1}{2}\sigma^2\right)(t_i - t_{i-1}) + \sigma\left(\hat{W}(t_i) - \hat{W}(t_{i-1})\right)}$$

It's easy to see that reapplying the $S(t_i)$ formula again and again allows us to simulate sample paths under $Q_B$.

**Note** While path independent derivatives likely allow analytical solutions, computing solutions to PDDs analytically would be computationally slow because of the dependence upon the value of the underlying at several maturity dates. Therefore, Monte Carlo simulation is a much better alternative.

# 6. Forwards and Futures Prices

Since forward and futures prices are *not* assets prices, we cannot immediately apply FTAP and its resulting properties to them. Instead, let's examine more closely how they are priced in a sufficiently general setitng with few restrictions.

## 6.1. Forward Prices

We start by examining *forward prices*, which are effectively fixed delivery prices agreed upon at initiation of a forward contract. They differ from the *value* of a forward contract, which is the resale price or value of an already-initiated contract. While the forward price is somewhere in the ballpark of the current spot price, $S(t)$, the value of a forward contract will begin at 0 and fluctuate with changes in the value of $S$ over time.

### 6.1.1. No Dividend Case

We will start with forward prices, denoed by $G(t, T)$. We start with the fact that the payoff at the delivery date $T$ of a forward contract with delivery price $K$ will be

$$V(T) = S(T) - K. \tag{17}$$

Next, it follows from FTAP that the *value* of the forward contract—note, I'm not yet talking about the forward *price* or *rate*—at any time $t < T$ will be, under the $T$-forward measure,

$$V(t) = P(t, T)\left(E_t^{Q_T}[S(T)] - K\right) = P(t, T)E_t^{Q_T}[V(T)] \tag{18}$$

This follows because the payoff $V(T)$ in Equation 17 is a payoff at time $T$ that we can price exactly like a PID.

Next, since we want $V(t) = 0$, when we initiate a forward contract, this forces

$$G(t, T) = E_t^{Q_T}[S(T)] = K$$

where $G(t, T)$ is the delivery price set at time $t$ such for delivery at time $T$. But, by the fact that $G(T, T) = S(T)$ for forward contracts we have that

$$\begin{aligned} G(t, T) &= E_t^{Q_T}[S(T)] = E_t^{Q_T}[G(T, T)] \\ \Rightarrow \quad & E_t^{Q_T}[G(T, T) - G(t, T)] = 0 \end{aligned}$$

In other words, the forward price for delivery at time $T$ is a *martingale* under the $T$-forward measure $Q_T$. We can use this result along with a few others,[4] to get the current forward price

$$G(t, T) = E_t^{Q_T}[G(T, T)] = E_t^{Q_T}[S(T)] = E_t^{Q_T}\left[\frac{S(T)}{P(T, T)}\right] = \frac{S(t)}{P(t, T)}$$

[4]Namely, that the discounted process for $S(t)$ must be a martingale and that $S(T) = G(T, T)$.

which agrees with the simple formula for forward pricing.

From Equation 18 we also get that the value of a *already-initiated* forward contract at time $t$ with delivery price $K$ equals

$$P(t,T)(G(t,T)-K).$$

### 6.1.2. Incorporating Dividends

Now let's consider an asset paying dividends *continuously* and at *non-random* rate, $\delta$.

Let $V(t) = S(t)e^{\int_0^t \delta(s)\,ds}$ be the value of a portfolio that begins with one share of the underlying and reinvests all dividends. Also, let $G_V(t,T)$ denote the forward price at time $t$ for the delivery of this just-defined portfolio $V$ at time $T$. Since $V$ itself pays no dividends (they are all reinvested), we set the forward price on this portfolio using the results from the last section:

$$G_V(t,T) = \frac{V(t)}{P(t,T)} = \frac{S(t)e^{\int_0^t \delta(s)\,ds}}{P(t,T)}$$

Since $V(T) = S(T)e^{\int_0^T \delta(s)\,ds}$, it's clear that the forward price for delivery of $S$ only at time $T$ must equal

$$G(t,T) = \frac{G_V(t,T)}{e^{\int_0^T \delta(s)\,ds}} = \frac{S(t)e^{\int_0^t \delta(s)\,ds}}{P(t,T)e^{\int_0^T \delta(s)\,ds}} = \frac{S(t)e^{-\int_t^T \delta(s)\,ds}}{P(t,T)} \tag{19}$$

And under the Black-Scholes assumptions of constant interest rates and dividends, we get that

$$G(t,T) = S(t)e^{(r-\delta)(T-t)}$$

which we know as the cost *cost-of-carry* formula. Note that we call Equation 19 the *generalized cost-of-carry* formula.

# 7. Black Model

We cause the martingale property of forward prices to obtain an option-pricing model that is far more general in its assumptions than the Black-Scholes model. We call this new model the *Black Model*. It links forward prices, $G$, to spot prices, $S$, by using the fact that $G(t, T)$ will converge to $S(T)$ as $t \to T$.

## 7.1. Assumptions

First, we assume that the *forward price* of the underlying asset follows an Ito process

$$dG(t, T) = G(t, T)\mu(t)\ dt + G(t, T)\sigma\ dW(t). \tag{20}$$

We also assume that there are no arbitrage opportunities or market imperfections.

**Note** We make no restrictive assumptions about dividends or interest rates. That gives us much more general results. We do, however, retain the constant proportional volatility specification. BUT, the volatility term, $\sigma$, is the volatility of the *forward price*, not the spot price volatility.

## 7.2. Pricing PID's Under the Black Model

Let's price the arbitrary PID's under the Black Model using Monte Carlo simulation, where the PID's have payoff $V(T) = \varphi(S(T))$ at time $T$:[5]

$$V(t) = P(t, T)E_t^{Q_T}[\varphi(S(T))] = P(t, T)E_t^{Q_T}[\varphi(G(T, T))] \tag{21}$$

In order to compute this expectation, we'll need to identify the stochastic process followed by the forward price under $Q_T$.

So apply Girsanov's Theorem to Equation 20, we get a new SDE under $Q_T$:

$$dG(t, T) = G(t, T)\hat{\mu}(t)\ dt + G(t, T)\sigma\ d\hat{W}(t). \tag{22}$$

And since we showed in the section about forwards that $G$ must be a martingale under $Q_T$, it follows that in Equation 22 that $\hat{\mu}$ must equal 0. As a result, $G$ follows Geometric Brownian Motion without drift. More generally, we get

$$S(T) = G(t, T)e^{-\frac{1}{2}\sigma^2(T-t)+\sigma(\hat{W}(T)-\hat{W}(t))} \tag{23}$$

which we can easily simulate and plug into Equation 21.

[5]We use the $T$-forward measure, $Q_T$, as it is much easier to compute given stochastic interest rates. This is in keeping with the more general framework of the Black Model.

## 7.3. Results and Computational Notes

Given the form of $S(T)$ described in Equation 23, we have a few immediate results:

1. Conditional on the information at time $t$,

$$\ln S(T) \sim N\left(\ln G(t,T) - \frac{1}{2}\sigma^2(T-t),\ \ (\sigma\sqrt{T-t})^2\right)$$

   under the measure $Q_T$.

2. We can compute the price of a generic PID from Equation 21 after we simulate values of $S(T)$ using Equation 23.

3. We can price options analytically under the Black Model by using the closed-form solution Black-Scholes model and making the following substitutions:

$$G(t,T) \to S(t), \qquad r(t,T) \to r, \qquad r(t,T) \to \delta, \qquad \sigma_{spot} \to \sigma_{fwd}$$

So we can price options with the following explicit formulas, where

$$y = \frac{\ln\left(\frac{G(t,T)}{K}\right) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

and where $\sigma$ is the volatility of the forward price:

- *Cash-or-Nothing*:

$$c^{\text{CON}} = P(t,T)\Phi(y)$$

- *All-or-Nothing*:

$$c^{\text{AON}} = P(t,T)G(t,T)\ \Phi(y + \sigma\sqrt{T-t})$$

- *Vanilla Call*:

$$c(t) = P(t,T)\left[G(t,T)\ \Phi(y + \sigma\sqrt{T-t}) - K\Phi(y)\right]$$

- *Vanilla Put*:

$$c(t) = P(t,T)\left[K\Phi(-y) - G(t,T)\ \Phi(-y - \sigma\sqrt{T-t})\right]$$

The above formulas are called the *Black Formulas.* They retain the Black-Scholes Formulas as a special case under the additional assumptions that the instantaneous interest rate is constant and that the stock pays dividends continuously at a constant rate $\delta$. But the Black Formulas are far more general and allow for stochastic interest rates and stochastic or discrete dividends.[6]

[6] We can extend this framework to price PDD's as well, which is covered in Notes 3.

## 7.4. Pricing PIDs in the Black Model

Pricing path-independent options is a bit less straightforward than pricing PIDs in the Black model. This arises because the Black model simulates *forward rates*, $G(t,T)$. Now, we know that we can simulate forward rates easily by using

$$G(t_i, T) = G(t_{i-1}, T)e^{-\frac{1}{2}\sigma^2(t_i - t_{i-1}) + \sigma(\hat{w}(t_j) - \hat{w}(t_{j-1}))}$$

and iterating over the sampling times, $\{t = t_0, t_1, \ldots, t_n = T\}$.

The complication arises, however, if the derivative's payoff is a function of the *stock price*, not the forward rate. Then, you have two options.

**Option 1** Simulate multiple forward prices with delivery dates corresponding to each sampling date, which requires lots of calibrations for volatilities and correlation coefficients, but requires no assumptions about interest rates and dividends.

**Option 2** Things simplify a bit, however, if you assume continuous dividends along with deterministic evolution of the dividend and instantaneous interest rate. In that case, we can invert the cost of carry formula to recover the stock price:

$$G(t,T) = S(t)e^{\int_t^T (r(s) - \delta(s))\, ds} \qquad \Leftrightarrow \qquad S(t) = G(t,T)e^{\int_t^T (\delta(s) - r(s))\, ds}$$

Note that assuming non-random interest rates means assuming future interest rates equal current forward rates.

# 8. Put-Call Parity

Now let's consider *Put-Call Parity* in it's most general case, namely where we don't restrict ourselves to constant interest or dividend rates.

To start, we know that long a call and short a put—with identical strike, $K$, and maturity, $T$—will put you in a situation where you own the asset and owe the strike at time $T$. Mathematically, at expiry time $T$, we will have a payoff of

$$c(T) - p(T) = S(T) - K \tag{24}$$

But, the RHS of Equation 24 is simply the payoff of a forward contract, $G$. So if we consider that same portfolio today at time $t$, we know that the portfolio must be worth the same as the the corresponding forward on the underlying. So we get

$$c(t) - p(t) = P(t,T)\left(G(t,T) - K\right). \tag{25}$$

This is the most general form of Put-Call Parity, and it only requires the assumption of perfect markets.

# 9. Volatility Surfaces

In the Black formulas, everything you need to value a simple or vanilla option is likely to be observable—save the estimated volatility parameter, $\sigma$. The current forward price $(G)$, the discount factor $(P)$, the strike $(K)$, and the time to maturity $(T-t)$ are all there for you to use.

As a result, there is a one-to-one mapping between the only disputable parameter, $\sigma$, and the option price. Consequently, by simply inferring the volatility from the market prices of options, we can get at an options *implied volatility* (IV), which we'll now discuss.

If we let $c^M$ and $p^M$ denote the market value of the call and put, and if we let $c^B(\sigma)$ and $p^B(\sigma)$ denote the call and put prices obtained from the Black formula, then the *implied volatility* of the call price and the put price, respectively, is that $\sigma$ that solves

$$c^B(\sigma) = c^M, \qquad p^B(\sigma) = p^M$$

First, we show that the IV for the put equals the IV for the call:

$$c^B(\sigma) - p^B(\sigma) = P(t,T)(G(t,T) - K)$$

$$c^M(\sigma) - p^M(\sigma) = P(t,T)(G(t,T) - K)$$

where the first line follows from put-call parity, and the second line follows if we assume no arbitrage. So equating and rearranging, we get that

$$c^B(\sigma) - c^M = p^B(\sigma) - p^M,$$

which proves that IV of the call must equal that of the put to preclude arbitrage.

Because of the one-to-one correspondence between IV and option prices, it's common for OTC dealers to quote the IV rather than the price. The IV can then be plugged into the Black Formula to get the prices.[7]

**Results** We see that while the market quotes Black-implied volatilities, it does not *price* according to the Black model. This leads to non-constant volatilities across different strikes and over time, with the most notable features being

- *Volatility Smile*: Short term options give rise to a convex pattern across strikes.
- *Volatility Skew*: The smile gradually flattens out for longer term options, although low strike IVs are higher in general.

[7]Note: That doesn't mean dealers are using the Black Model to price the options. In fact, they're probably using somethining more sophisticated. But this is a simple way to quote options, and the convention is used nonetheless.

# 10. Local Volatility Models

Local volatility models are one class of models which attempts to generate prices in line with the market, which is equivalent to a model whose prices have Black-implied volatilities that match the observed volatility smile and skew. To do so, this model relaxes that assumption that the volatility of the spot (or forward) price is a constant that's proportional to the price level of the asset.

Ideally, we'd also like to incorporate some stylistic facts into our model about regarding the behavior of volatility. For example, we know that there is a strong negative correlation between changes in the VIX and changes in the S&P. In practice, this means generalizing the BS and Black models to allow the volatility of the underlying asset to vary and be negatively correlated with price.

# A. Antithetic Variates: A Variance Reduction Technique

The technique of *antithetic variates* reduces the sample size needed to achieve a given precision (in the form of standard error) in Monte Carlo simulations. Let's go through the steps:

1. Suppose that we want to estimate $EY$ using MC simulation. Instead of simulating a single sample of size $N$, we simulate $N/2$ pairs, $(Y_1, Y_2)_i$, where each pair is i.i.d. for all pairs $i = 1, \ldots, N/2$.

2. We can get an unbiased estimator of $EY$ by first computing the average value $\bar{Y}_i = \frac{1}{2}(Y_{i1} + Y_{i2})$ then averaging across the $\bar{Y}_i$.

3. If we simulate the $N/2$ pairs independently, then

$$\begin{aligned} Var(\bar{Y}_i) &= Var\left(\frac{1}{2}(Y_{i1} + Y_{i2})\right) = \frac{1}{4}\left[Var(Y_{i1}) + Var(Y_{i2}) + 2Cov(Y_{i1}, Y_{i2})\right] \\ &= \frac{(1+\rho)}{2} Var(Y), \quad \rho \text{ is the correlation between } Y_{i1},\, Y_{i2} \end{aligned}$$

   So then if we want the total variance of $Y$ over our entire simulation, then we get

$$Var(\bar{Y}) = \frac{(1+\rho)Var(Y)}{2(N/2)} = \frac{(1+\rho)Var(Y)}{N}$$

4. Clearly, by making the correlation negative between the terms $Y_1$ and $Y_2$ in each given pair, we can lower the variance of our estimator for $EY$ as a whole. (Note that the pairs themselves are still independent from pair to pair.)

Note that some of the negative correlation will wash out and we won't get perfect -1 correlation when we jump from random draws to prices, as typically the prices are some convoluted *function* of the random draws.

Also, the gain—while significant—for path *dependent* derivatives is not quite as large due to the dependence of the payoffs upon the entire path.

# B. Extras from the Textbook

This entire section aims to be a little more rigorous and specific about the approaches to asset pricing. Especially in the second section, I hope to be a little more specific about what's going in the background of the Fundamental Theorem of Asset Pricing.

## B.1. Replication Approach

One way to price a derivative involves a replicating portfolios. Suppose you know that one must exist (because, say, the market is complete), and you know the value of the derivative, $V(S(t), t)$, is a relatively nicely function from the standpoint of Ito's Lemma.

First, you use the condition that the portfolio must be self-financing:

$$\theta(t)^T S(t) - \theta(0)^T S(0) = \int_0^t \theta(u)^T \, dS(u), \tag{26}$$

which essentially says that a portfolio process, $\theta \in \mathbb{R}^d$, should be structured such that the total incremental gains from trading over the time $(0, T)$ (the right-hand side) should be equal to the total gains (the left-hand side). Note that $S(t)$ without the subscript is also a vector for the value of the $d$ assets.

Next, we suppose that the value of the derivative over time, $V(S(t), t)$, is relatively well-behaved in the sense that we can apply Ito's Lemma to it. So let's do that to get

$$\begin{aligned} V(S(t), t) &= V(S(0), 0) + \sum_{i=1}^{d} \int_0^t \frac{\partial V(S(u), u)}{\partial S_i} dS_i(u) \\ &+ \int_0^t \left[ \frac{\partial V(S(u), u)}{\partial u} + \frac{1}{2} \sum_{i,j=1}^{d} S_i(u) S_j(u) \Sigma_{ij}(S(u), u) \frac{\partial^2 V(S(u), u)}{\partial S_i \partial S_j} \right] du \end{aligned} \tag{27}$$

where $d$ is the number of assets and $\sigma_{ij}$ is the covariance between instantaneous returns on assets $i$ and $j$.

So now let's adapt our self-financing consraint to the derivative. We'll assume that the market is complete so a replicating a portfolio does exist, allowing us to swap out some elements in Equation 26, rearrange, and get

$$V(S(t), t) = V(S(0), 0) + \sum_{i=1}^{d} \int_0^t \theta_i(u) \, dS_i(u) \tag{28}$$

So now that we have two equations in terms of $V(S(t), t)$, we'll eventually want to equate them. But let's just take note now that in order for them to be equal, we'll need everything on the second line of Equation 27 to equal 0:

$$\Rightarrow \frac{\partial V(S(u), u)}{\partial u} + \frac{1}{2} \sum_{i,j=1}^{d} S_i(u) S_j(u) \Sigma_{ij}(S(u), u) \frac{\partial^2 V(S(u), u)}{\partial S_i \partial S_j} = 0 \tag{29}$$

Next, we'll also need

$$\theta_i(u) = \frac{\partial V(S(u), u)}{\partial S_i}, \qquad i = 1, \ldots, d \tag{30}$$

But given the fact that $\theta(t)^T S(t)$ is the value of the replicating portfolio at time $t$ (and, as a consequence, the derivative as well), we can write

$$V(S,t) = \sum_{i=1}^{d} \frac{\partial V(S,t)}{\partial S_i} S_i \tag{31}$$

Finally, note that we also have the boundary condition

$$V(S(T),T) = f(S(T)) \tag{32}$$

where $f$ is the payoff function for the derivative.

**Note**, that the drifts, $\mu_i(t)$ of the assets, $S_i(t)$, do not appear anywhere in our foundational Equations 31 and 32. They are already implicit in the stock prices, $S_i(t)$, but they don't show up explicitly because our argument had *nothing* to say about risk preferences. We merely described a self-financing portfolio that replicates the derivative, and $V(S(0),0)$ is simply the cost to initiate that strategy—regardless of the investor's risk preferences.

**Solving the PDE** Let's just clarify how everything relates when trying to implement this approach:

1. Equation 29 is the PDE that we have to solve out, subject to the boundary condition.

2. Equation 32 is that boundary condition.

3. Equation 31 actually gives us the value of the derivative security.

## B.2. Risk-Neutral and Expectation-Type Pricing

It may be that the price dynamics are very complex or that the derivative is path-dependent. In such cases, a PDE may not be feasible or even apply. In such cases, Monte Carlo methods provide a valuable and manageable alternative.

Suppose we have a risk free asset governed by

$$\frac{dN(t)}{N(t)} = r\,dt \qquad \Rightarrow \qquad N(t) = N(0)e^{rt}$$

Next, suppose that the market is arbitrage free, so it admits a stochastic discount factor, $Z(t)$. Then since $N(t)$ is attainable, the process $N(t)/Z(t)$ will be a martingale. Note that $N(t)$ clearly corresponds to the numeraire, $N$, that we defined earlier in FTAP. Moreover, the stochastic discount factor $Z(t)$ was sort of swept under the table in the above definition of FTAP, where I said "there will be a measure." Here $Z(t)$ is integral to the converting between measures and translating expectations under $P$ to expectations under $Q$ or $Q_N$.

Now, if we fix $N(0)$ at 1, and if we normalize the stochastic discount factor so that $Z(0) = 1$, then we see that $N(0)/Z(0) = 1$. Since we already said this is a Martingale, it has expectation

$$E_P[N(0)/Z(0)] = 1, \qquad \text{where } P \text{ is the real-world probability measure.}$$

Then this fact allows us to perform a change of measure to, say, $Q$. Specifically, this gives us

$$\left(\frac{dQ}{dP}\right)_t = \frac{N(t)}{Z(t)}$$
$$\Rightarrow \quad Q(A) = E_P\left[1_A \cdot \left(\frac{dQ}{dP}\right)_t\right] = E_0\left[1_A \frac{N(t)}{Z(t)}\right]$$

So now if we want to take an expectation under the new measure, as opposed to the real world one, we compute

$$E_Q[Y] = E_P\left[Y\frac{N(t)}{Z(t)}\right]$$

where $Q$ is called the *risk-neutral measure*—a particular choice of *equivalent martingale measure*. So now let's go back to the fact that with a arbitrage-free marget, there's a stochastic discount factor—i.e. a positive process $Z(t)$ such that every attainable price process is a martingale, which states mathematically and implies

$$\frac{V(t)}{Z(t)} = E_P\left[\frac{V(T)}{Z(T)}\right] \Rightarrow \quad V(t) = E_P\left[V(T)\frac{Z(t)}{Z(T)}\right]$$
$$\Rightarrow \quad V(0) = E_P\left[\frac{V(T)}{Z(T)}\right].$$

If we want to rewrite in terms of the new probability measure, $Q$, we can say that

$$V(t) = \beta(t)E_Q\left[\frac{V(T)}{N(T)}\right]$$

and this should look precisely analogous to the statement of FTAP above, just allow $N$ to be a little more general then I treated it in this section. So now it's clear that we're discounting the terminal value under under the *risk-free rate*, rather than under that funky stochastic discount factor (which is likely to be a bitch to work with and think about).

Next, since we know that the normalized asset price processes, $S_i(t)/N(t)$ will be martingales under $Q$, we will have to specify the dynamics that make it so. This will mean that we will need to find governing processes such that the drift term is 0.