Homework 2

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1. **Exercise 51.17, FoMA**: We consider f, a positive continuous function where $f \in \mathscr{R}([a,b])$. Letting
$$M = \max_{x\in[a,b]} f(x)$$
we want to prove
$$M = \lim_{n\to\infty} \left[\int_a^b [f(x)]^n \; dx\right]^{1/n} \tag{1}$$
Also, let x^* be the x value such that $f(x^*) = M$. By the intermediate value theorem, we know such an x^* exists.

   Now because f is continuous on a compact interval, f is bounded, implying that M does exist. And since f is positive, $M > 0$ as well. Given that, we can show an equivalent statement to Equation 1 by dividing through by M:
$$\begin{aligned} 1 &= \lim_{n\to\infty} \left[\frac{1}{M^n}\int_a^b [f(x)]^n \; dx\right]^{1/n} \\ 1 &= \lim_{n\to\infty} \left[\int_a^b \left[\frac{f(x)}{M}\right]^n \; dx\right]^{1/n} \end{aligned} \tag{2}$$
This will be our equivalent statement to prove, rather than Equation 1.

   So first, we know that that the integral in Equation 2 over the function h exists (i.e. $h \in \mathscr{R}([a,b])$—where
$$h(x) = \left[\frac{f(x)}{M}\right]^n \quad \text{with} \quad \begin{cases} h(x) = 1 & x = x^* \\ h(x) < 1 & x \neq x^* \end{cases}$$
This integral exists because both $f(x)$ is continuous and $g(x) = (x/M)^n$ is continuous. And because $h = g \circ f$, a composition of continuous functions, $h \in \mathscr{R}_\alpha([a,b])$ by Theorem 51.12 in FoMA.

It's also clear that we must have

$$\lim_{n\to\infty} h(x) = \lim_{n\to\infty} \left[\frac{f(x)}{M}\right]^n = \begin{cases} 1 & x = x^* \\ 0 & x \neq x^* \end{cases} \tag{3}$$

Finally, since h is a continuous function in $\mathscr{R}_\alpha([a,b])$, we can use Theorem 52.5 in FoMA to assert that

$$\int_a^b h\ dx = \lim_{||P||\to 0} S(h, P, T)$$

for any T. Choosing T so that x^* is one of the evaluation points in T, we can then

2. **Exercise 52.1, FoMA**: We want to prove, for $f \in \mathscr{R}([0,1])$ that

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(k/n)\ \frac{1}{n} = \int_0^1 f(x)\ dx \tag{4}$$

Note that we're taking $\alpha(x) = x$, a continuous function.

We construct the proof by treating the lefthand side as a Riemman Sum. Namely, we construct a partition

$$P = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\right\} \qquad \Rightarrow \qquad \Delta x_i = \frac{1}{n}$$

We also consider the evaluation points within each interval

$$T = \left\{\frac{1}{n}, \frac{2}{n}, \ldots, 1\right\} \tag{5}$$

Combining our partition and evaluation points into one sum, we have a Riemann Sum

$$S(f, P, T) = \sum_{k=1}^{n} f(t_i)\Delta x_i = \sum_{k=1}^{n} f(k/n)\ \frac{1}{n}$$

which is exactly the lefthand side of Equation 4. Now, we can invoke Theorem 52.5 in FoMA, as we satisfy the conditions that

- $f \in \mathscr{R}([0,1])$, which is assumed.
- α continuous, as $\alpha(x) = x$.

Thus, we can assert, since $||P|| = 1/n$, which goes to zero as n grows, that

$$\begin{aligned} \int_0^1 f dx = \lim_{||P||\to 0} S(f, P, T) &= \lim_{||P||\to 0} \sum_{k=1}^{n} f(t_i)\Delta x_i \\ &= \sum_{k=1}^{n} f(k/n)\ \frac{1}{n} \end{aligned}$$

which is exactly what we wanted to show.

3. Recall the Lipshitz Condition: A function f is Lipshitz at x if for some $C, \delta > 0$

$$|x - y| \leq \delta \quad \Rightarrow \quad |f(x) - f(y)| \leq C|x - y| \tag{6}$$

We want to show that for $f : [0, 1] \to \mathbb{R}$

$$\left| \int_0^1 f \; dx - \frac{1}{n} \sum_{k=1}^{n} f(k/n) \right| \leq \frac{C}{n} \qquad \forall n \tag{7}$$