Convergence

1 Definitions

Suppose we are working on a probability space $(\Omega, \mathcal{F}, P)$ and we consider random variables $\{X_n\}$, where $n = 1, 2, \ldots$ each with distributions functions F_n:

Almost Surely Used in the *Strong Law of Large Numbers.* Defined

$$P\left(\lim_{n\to\infty} X_n = X\right) = 1.$$

In Probability Note, this is the type of convergence established by the *Weak Law of Large Numbers.*

$$lim_{n\to\infty} P\left(|X_n - X| > \epsilon\right) = 0, \qquad \forall\epsilon$$

In p-norm All X_n and X have finite pth moment and

$$\lim_{n\to\infty} E\left[|X_n - X|^p\right] = 0, \qquad 0 < p < \infty$$

In Distribution We say X_n with distributions F_n converge *In Distribution* to X with distribution F if

$$\lim_{n\to\infty} F_n(x) = F(x)$$

for all $x \in \mathbb{R}$ at which F is continuous. Also called **Weak Convergence**.

2 Relationships

The concepts just defined are related in the following way:

- Almost surely $\Rightarrow$ In Probability.
- In Probability $\Rightarrow$ there's a deterministic subsequence that converges Almost Surely.
- In p-norm $\Rightarrow$ In Probability.
- Almost Surely and In p-norm, undecidable.
- Almost Surely, In Probability, and In p-norm each $\Rightarrow$ In Distribution.

3 Related Concepts

Consistency A sequence of estimators $\{\hat{\theta}_n\}$ where $n = 1, 2, \ldots$ is *consistent* for parameter θ if $\hat{\theta}_n$ converges *In Probability* to θ.

Strongly Consistent If convergence of $\hat{\theta}_n$ to θ holds with probability 1.