Notes to Financial Engineering: Credit Derivatives

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1. The Fundamental Theorem of Asset Pricing

1.1. Introduction

Derivatives require special pricing techniques aside from the traditional discounted cash flow (DCF) approach, as DCF requires an estimate of the appropriate risk-adjusted rate of return. However, the risk of a derivative varies over time, which makes it difficult to estimate the derivative's risk-adjusted return.

As a result, derivatives pricing turns to the no-arbitrage approach (NA), which eliminates the need to build risk into the model.

1.2. Trading Strategy and Derivative Pricing Definitions

A **trading strategy** is a dynamically-rebalanced portfolio.

A trading strategy is **self-financing** if it generates no intermediate cash inflows and requires no intermediate outflows between the time the portfolio is initiated and the time it is liquidated. This implies that

i. All dividends are reinvestd.

ii. Value of the assets sold at a rebalance time must equal the value of the assets bought.

A trading strategy is **strictly positive** if the value of the traded portfolio can never become zero or negative.

Let N be the value of a *strictly positive*, *self-financing* trading strategy. Then N is a **numeraire process** or, simply, a **numeraire**. Here are a few examples:

- Price of Dividend paying asset: NO, as there are intermediate cash outflows, violating self-financing condition.
- The price of a forward contract: NO, as it can go negative, violating the strictly positive condition.
- Price of a Foreign Currency: NO, as it is equivalent to a dividend paying asset because you think of it as an investment in an interest-bearing account.
- Price of a non-defaultable zero-coupon bond: YES.
- Value of a money market account earning the risk free rate, where there are no interim deposits or withdrawals: YES.

1.3. Martingales and Change of Measure

A **martingale** is a stochastic process X with the property

$$E_t[X(T) - X(t)] = 0 \;\Leftrightarrow\; E_t[X(T)] = X(t), \qquad T > t.$$

A **probability measure** is a specification of the probabilities of all the possible states of the words, mapping states to real numbers.

Suppose that ξ is a nonnegative random variable on $(\Omega, \mathcal{F}, P)$ with $E_P[\xi] = 1$. (The subscript P highlights that the last expectation is with respect to measure P.) Then define a new measure

$$Q : \mathcal{F} \to [0,1]$$

$$Q(A) = E\left[1_A \xi\right] = \int_A \xi(\omega) dP(\omega), \qquad A \in \mathcal{F} \tag{1}$$

Clearly, Q is a probability measure on $(\Omega, \mathcal{F})$ and it is absolutely continuous with respect to P—i.e. we have

$$Q(A) > 0 \Rightarrow P(A) > 0.$$

Note that it is common to write the random variable ξ as

$$\xi = \frac{dQ}{dP},$$

and we often refer to ξ as the *Radon-Nikodym derivative* or the *likelihood ratio* of Q with respect to P.

Radon-Nikodym Theorem If P and Q are two probability measures on $(\Omega, \mathcal{F})$, then there *will exist* such a random variable ξ so that Expression 1 holds.

1.4. Fundamental Theorem of Asset Pricing (No Dividends)

1.4.1. Statement of Theorem

Suppose we have n non-dividend-paying assets with price processes $S_1, S_2, \ldots, S_n$. Let N be some numeraire process. Then, barring market imperfection, there are no arbitrage opportunities among these assets if and only if there exists a strictly positive probability measure Q_N (so it's dependent upon the numeraire, N) under which each of the processes S_i/N is a martingale.

- Note that S_i/N is the price of asset i in units of the numeraire N. Therefore, we call S_i/N the **normalized price process**.

- The probability measure Q_N will, in general depend on the numeraire. Therefore, we call Q_N the **martingale measure** or **pricing measure** associated with the numeraire N.

- We can paraphrase FTAP by saying that, if the is no arbitrage or market imperfections, then given *any* numeraire process N, there must exist a corresponding martingale measure Q_N under which the normalized price of any non-dividend paying asset is a martingale:

$$\frac{S_i(t)}{N(t)} = E_t^{Q_N}\left[\frac{S_i(T)}{N(T)}\right], \qquad T > t.$$

  From this, we see that changing N will generally change Q_N as well.

1.4.2. Consequences for Derivative Pricing

Let V denote the price of a derivative with payoff $V(T)$ at time T. Then we can apply FTAP to get

$$V(t) = N(t)E_t^{Q_N}\left[\frac{S_i(T)}{N(T)}\right], \qquad T > t.$$

Note, the price we get for a deriative is *invariant* to the choice of the numeraire.

1.4.3. Special Numeraires and Martingale Measures

T-forward measure Let $P(t,T)$ be the price of a non-defaultable zero-coupon bond with unit face value. Then

$$N(t) = P(t,T)$$

is our numeraire. The associated martingale measure, denoted Q_T, is called the *T-forward martingale measure*. This yields a derivative price of

$$V(t) = E_t^{Q_B}\left[e^{-\int_t^T r(s)ds}V(T)\right]$$

Risk Neutral Measure Let's consider the value of a money market account with unit initial value as our numeraire. Then

$$N(t) = B(t) = e^{\int_0^t r(s)ds}$$

where r is the instantaneous risk-free rate. The associated martingale measure, denoted by Q_B, is called the *risk-neutral martingale measure.* This yields a derivative price of

$$V(t) = P(t,T)\ E_t^{Q_T}\left[V(T)\right] \ = \ e^{-r(t,T)(T-t)} E_t^{Q_T}\left[V(T)\right]$$

$$r(t,T) = -\ln P(t,T)/(T-t)$$

If interest rates are stochastic (and they probably are), then this measure isn't as convenient a tthe T-forward measure.

1.5. FTAP for Dividend-Paying Assets

Consider an asset with price process S and let $D(t)$ denote the cumulative dividend paid by the asset from time 0 up to time t. We can consider the undiscounted cash flows from holding an asset from t to T:

$$S(T) - S(t) + D(T) - D(t) = GP(T) - GP(t)$$

where $GP(t) = S(t) + D(t)$ is the asset's gain process.

Given a numeraire N, the asset's *normalized gain process*, denoted NGP, measures the gains from holding the assets in units of N:

$$NGP(t) = \frac{S(t)}{N(t)} + \int_0^t \frac{dD(s)}{N(s)}$$

where $dD(s)$ is the dividend paid by the asset at time s.

Theorem Now, let's restate the fundamental theorem of asset pricing allowing for dividend paying assets. So again, consider assets with price processes $S_1, \ldots, S_n$ and cumulative dividend processes $D_1, \ldots, D_n$, letting N be any numeraire process. Then there are no arbitrage opportunities across these assets if and only if there exists a strictly positive probability measure Q_N under which each

$$\frac{S_i(t)}{N(t)} + \int_0^t \frac{dD_i(s)}{N(s)}$$

is a martingale. This implies

$$\frac{S_i(t)}{N(t)} = E_t^{Q_N}\left[\frac{S_i(T)}{N(T)} + \int_t^T \frac{dD_i(s)}{N(s)}\right]$$

And so the normalized price of any asset is equalt to the conditional expectation under the martingale measure of the assets normalized payoffs (including future dividends).

2. Continuous Time Stochastic Processes

Here, we develop the necessary machinery in continuous time stochastic processes to model asset price evolution properly and with sufficient richness and generality. In particular, we discuss a heirarchy of model classes that includes Brownian Motion $\subset$ Generalized Brownian Motion $\subset$ Diffusions $\subset$ Ito Processes.

2.1. Introduction

A *stochastic process* X is a collection of random variables indexed by time: $X = \{X_t : t \in \mathcal{T}\}$.

- *Discrete Time*: $\mathcal{T}$ countable, and process changes only at discrete time intervals.
- *Continuous Time*: $\mathcal{T}$ uncountable.

Definition A process X has stationary increments if $X_T - X_t$ has the same distribution as $X_{T'} - X_{t'}$ provided that $T - t = T' - t'$.

2.2. Brownian Motion

Definition The most basic continuous-time process is *Brownian Motion* (or the *Wiener Process*). It has three defining properties:

i. $W(0) = O$.

ii. $W(t)$ is continuous, so no jumps.

iii. Given any two times, $T > t$, the increment $W(T) - W(t)$ is independent of all previous history and normally distributed with mean 0 and variance $T - t$.

A few consequences of the definition of $W(t)$:

- Brownian motion has independent stationary increments.
- $W(t)$ is normally distributed with $\mu = 0$, $\sigma^2 = t$.

2.3. Generalized Brownian Motion

Definition A *generalized Brownian motion* is a continuous-time process X with the following property:

$$X(t) = X(0) + \mu t + \sigma W(t)$$

where μ is the *drift*, σ is the *volatility*, and W is simple Brownian motion. The differential equation equivalent is written:

$$dX(t) = \mu\ dt + \sigma\ dW(t).$$

It follows immediately from the definition that

- $X(t)$ is continuous, so no jumps.
- $X(t)$ is normally distributed with mean $X(0) + \mu t$, variance $\sigma^2 t$.
- Given any two times, $T > t$, the increment $X(T) - X(t)$ is independent of all previous history and normally distributed with mean $\mu(T-t)$ and variance $\sigma^2(T-t)$.
- X is a martingale if and only if $\mu = 0$.

Theorem It also happens that Generalized Brownian motions are the only continuous time processes with continuous sample paths and stationary increments.

2.4. Ito Processes

Even more general than Brownian Motion (which is retained as a special case), an *Ito Process* is a stochastic process X defined by one of two equivalent formulations:

$$dX(t) = \mu(t)\, dt + \sigma(t)\, dW(t)$$

$$X_t = X_0 + \int_0^t \mu(s)\, ds + \int_0^t \sigma(s)\, dW(s)$$

for any arbitrary stochastic processes μ (the drift) and σ (the volatility) along with some Brownian Motion $W(t)$. Here are some properties

- Has continuous sample paths and is a martingale if and only if $\mu(t) = 0$.
- Increments are not necessarily stationary, as μ and σ can change *randomly* with time.

Definition If the drift and volatility of an Ito process depend only upon the current value of the process and time, then X is a *diffusion.* Mathematically, X is a *diffusion* if

$$dX(t) = \mu(X(t), t)\, dt + \sigma(X(t), t)\, dW(t)$$

for some functions μ and σ.

2.5. Ito's Lemma

Suppose that X is an Ito process defined by

$$dX(t) = \mu(t)\, dt + \sigma(t)\, dW(t) \tag{2}$$

and we define a new process $Y(t) = f(X(t), t)$ where f is some function that's twice differentiable in X and once in t. Then we have that

$$dY(t) = f_X(X(t), t)\, dX(t) + f_t(X(t), t)\, dt + \frac{1}{2} f_{XX}(X(t), t)\sigma(t)^2\, dt. \tag{3}$$

where subscripts on f denote the partial derivatives.[1] Subbing Equation 2 into Equation 3, we get that

$$dY(t) = \left(f_X(X(t),t)\mu(t) + f_t(X(t),t) + \frac{1}{2} f_{XX}(X(t),t)\sigma(t)^2 \right) \, dt + f_X(X(t),t)\sigma(t) \, dW(t)$$

Thus, it is clear that Y is also an Ito Process by the statement above with the drift and volatility given by the coefficients on dt and $dW(t)$ as always.

Using Ito's Lemma In practice, we use Ito's Lemma whenever we have a (typically complicated) Ito Process that we want to solve. Given the process X and its corresponding Ito Process, we posit a function f that could could help. Then we write a new Ito Process using Ito's lemma with $dY(t) = df(X(t),t)$ on the LHS. From there, hopefully we can integrate $dY(t)$ easily on the left and solve out for $X(t)$.

2.6. Multi-dimensional Ito's Lemma

For the sake of completeness, let's generalize Ito's Lemma to consider the case of a finite number of Ito processes, $X_1, X_2, \dots, X_n$,

$$dX_i(t) = \mu_i(t) \, dt + \sigma_i(t) \, dW_i(t).$$

Next, define $Y(t) = f(X_1(t), \dots, X_n(t), t)$ for some differentiable function f. Then multi-dimensional Ito's Lemma says

$$\begin{aligned} dY(t) = &\sum_{i=1}^{n} f_{X_i}(X_1(t), \dots, X_n(t), t) \, dX_i(t) \\ &+ f_t(X_1(t), \dots, X_n(t), t) \, dt \\ &+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} f_{X_i, X_j}(X_1(t), \dots, X_n(t), t) \, \rho_{ij}\sigma_i(t)\sigma_j(t) \, dt \end{aligned}$$

where ρ_{ij} is the correlation coefficient between dW_i and dW_j. Note that you'll have to plug back in for the dX_i in the first sum.

We'll mostly consider with the two-dimensional case for the two specific instances below:

i. $Y(t) = X_1(t)X_2(t)$, which gives us

$$dY(t) = X_2(t) \, dX_1(t) + X_1(t) \, dX_2(t) + \rho_{12}\sigma_1(t)\sigma_2(t) \, dt$$

Note that you'll have to plug back in for dX_1 and dX_2. This is a type of integration-by-parts formula because (after rearranging terms) it relates $X_1 \, dX_2$ to $X_2 \, dX_1$.

[1] Note that Equation 3 almost looks like the chain rule from traditional calculus, except for that extra term with f_{XX} partial derivative. That arises from the additional variability due to the inclusion of stochastic factors like $W(t)$ in the original Ito Process.

ii. $Y(t) = X_1(t)/X_2(t)$, which gives us

$$dY(t) = \frac{1}{X_2(t)}dX_1(t) - \frac{X_1(t)}{X_2(t)^2}dX_2(t) + \frac{X_1(t)}{X_2(t)^3}\sigma_2(t)^2\ dt - \frac{1}{X_2(t)^2}\ \rho_{12}\sigma_1(t)\sigma_2(t)\ dt.$$

Note that you'll have to plug back in for dX_1 and dX_2. This is a type of integration-by-parts formula because (after rearranging terms) it relates $X_1\ dX_2$ to $X_2\ dX_1$.

2.7. Geometric Brownian Motion

Let's consider the process X governed by

$$dX(t) = X(t)\mu(t)\ dt + X(t)\sigma(t)dW(t).$$

To solve, let us consider the process $Y(t) = \log X(t)$. We compute the partials and apply Ito's Lemma:

$$f_X = \frac{1}{X(t)}, \qquad f_{XX} = -\frac{1}{X(t)^2}, \qquad f_t = 0$$

$$dY(t) = \frac{1}{X(t)}dX(t) - \frac{1}{2}\frac{1}{X(t)^2}(\sigma(t)X(t))^2)\ dt$$

which simplifies (after subbing in for $dX(t)$) into the expression

$$dY(t) = \left(\mu(t) - \frac{1}{2}\sigma(t)^2\right)\ + \sigma(t)\ dW(t).$$

Next, integrating both sides and substituting back in with $Y(t) = \log X(t)$, we get

$$Y(t) = Y(0) + \int_0^t \left(\mu(s) - \frac{1}{2}\sigma(s)^2\right) ds + \int_0^t \sigma(s)\ dW(s)$$

$$X(t) = X(0)e^{\int_0^t \left(\mu(s) - \frac{1}{2}\sigma(s)^2\right)ds + \int_0^t \sigma(s)\ dW(s)}$$

where it also follows that X is strictly positive.

Special Case Suppose that μ and σ are constant, in which case X follows a *geometric Brownian motion.* Then it follows that

$$\log X(t) \sim N\left(\log X(0) + \left(\mu - \frac{1}{2}\sigma^2 t\right), \sigma^2 t\right)$$

so that we say $X(t)$ is *lognormally distributed.*

2.8. Girsanov's Theorem

Theorem Suppose that X is an Ito process

$$dX(t) = X(t)\mu(t)\ dt + X(t)\sigma(t)\ dW(t),$$

where μ an σ are stochastic processes and W is Brownian Motion under some probability measure P—like maybe the real world measure. Then if Q is any other strictly positive probability measure, then X is also an Ito process under Q—i.e., there exist processes $\hat{\mu}$, $\hat{\sigma}$, and $\hat{W}$ with the property that

$$dX(t) = X(t)\hat{\mu}(t)\ dt + X(t)\hat{\sigma}(t)\ d\hat{W}(t). \tag{4}$$

Even better, $\hat{\sigma} = \sigma$.

This is particularly useful because, in general, we will have to work with two different probability measures: the true/historical P and the martingale probability measure Q_N.

3. The Market for Credit Derivatives

Credit Derivatives are financial contracts between a buyer and a seller with payoffs contingent upon credit events affecting a third party, the *reference credit.*

The most popular form if credit derivatives is the *Credit Default Swap* (CDS), which includes as special cases single-name CDS's, basket CSD's, CDS indices, and CDS index tranches. While CDS's were introduced in the early 1990s, CDS indices were introduced in 2003, and growth exploded between 2001 and the financial crisis in 2008.[2]

3.1. Single-Name Credit Default Swaps

In such arrangements, the *protection seller* agrees to make a payment to the *protection buyer* if a qualifying credit event affects the third party, *reference entity* before the maturity of the contract.[3]

Payment Structure: Payments are as follows.

- Should a qualifying credit event occur, the payment from protection seller to buyer is the difference between the par and post-default makret value of *deliverable obligations* issued by the reference entity having total face value equal to the notional of the CDS.

- In order to get this protection, the protection buyer agrees to make a series of quarterly payments until the credit event or maturity, whatever comes first.[4]

- Prior to April 2009, convention dictated that there was no upfront payment from CDS buyer to seller. Instead, the the buyer just made periodic payments at the quoted annualized rate, the *CDS Par Spread.*[5]

- Since April 2009, the standard North-American CDS requires an up-front payment in addition to standardized quarterly coupon payments at a fixed annual rate of 100 basis points for investment grade credits, and 500 basis points for high-yield credits.[6] Depending on the risk of the reference credit, this means that the upfront payment can either be positive or negative.

[2]However, some of the substantial growth in notional amounts has been overstated as offsetting trades couldn't properly cancel each other out prior to subsequent standardization

[3]The "qualifying credit events" usually include bankruptcy, default on outstanding bonds, and debt restructuring.

[4]Should the credit event occur in between payment dates, the buyer still must pay the fraction of the next payment that has accrued since the last payment date.

[5]Note that since buying a defaultable bond and a single-name CDS results in a position roughly equivalent to buying a risk-free bond, the CDS spread should be approximately equal to the credit spread on the defaultable bond.

[6]Payments due on the 20th of March, June, September, and December.

Market Quotes: There are two conventions, depending on the credit quality of the reference entity.

- Quotes for high-yield credits are in terms of the up-front fee.
- Quotes for investment-grade credits are in terms of a *conventional spread*, in which case the up-front fee is the present value of the difference between the quoted spread and the coupon rate.

Settlement: There are two ways to settle the payment from protection seller to buyer should a qualifying credit event occur:

1. Physical Delivery: The protection buyer delivers obligations with face value equal to the notional amount and receives the notional amount.
2. Cash Settlement: The protection seller pays an amount equal to the notional times *loss given default* (LGD), which is defined as 1 minus the *recovery rate*, which is the percentage market value of the deliverable obligation. This recovery rate is determined through a *credit auction*, which is determined by polling a number of bond dealers.

3.2. Basket Credit Default Swaps

Such swaps are equivalent to a portfolio of single-name CDS's. They are strctured so that

1. A notional is specified for each reference credit in the basket.
2. If one of the reference credits experiences a credit event priot to maturity, then the protection seller pays the notional for that particular credit times the LGD.
3. Following a credit event, the affected reference entity is removed from the basket and the CDS continues to its original maturity, coverin the remaining reference entities in the basket, with periodic payments from protection buyer to seller computed on the residual notional amount.

Nth to Default (N2D) Basket CDS: These are a variation of the straight Basket CDS. Here, the event that triggers the protection payment is the Nth credit event affecting a basket of reference entities after which the CDS is terminated.

- Cleary, a CDS on a basket of M credits is equivalent to a portfolio with a 1st to Defaults, 2nd to Default, . . . , and an Mth to default CDS on the basket.
- The primary advantage of an Nth to default CDS is that they allow a protection buyer to buy only partial protection on a given basket of reference entities.
- Unlike pricing of straight basket CDS's, the pricing of an N2D CDS depedends crucially on the correlation between the default times of the different credits in the basket. In general, the spread on a 1st to Default CDS decreases as correlation increases. The opposite is true for the Last to Default.

3.3. CDS Indices

CDS Indicees allow investors to buy or sell protection on *standardized* baskets of credits. The main indices cover different geographical regions, credit quality, and industry type, and they incude the credit entities with the most actively traded single-name CDS's in a given segment. Overall, an index typically includes 20-125 equally weighted credits. Here's how they work

- Once formed the composition of an index reamins static over its lifetime, except in the case of credit events where you drop the offending entity from the index.[7]
- Each index series specifies a quartly coupon to be paid by the buyer to the protection seller.
- Indices are quoted on either a spread or price basis:
  - Price of an index: $100 \times [1 - (\text{up-front fee})]$.
  - Up-front Fee: Present value of the difference between the quoted spread and coupon.

  Note that the par spread on an index is roughly equal to a weighted average of the par spreads on the individual credits in the index.
- If one of the reference entities in the index defaults, the protection seller must pay the protection seller a value equal to

$$\text{Payment} = (\text{Notional}) \times (\text{Trade Weight of Credit in Index}) \times LGD$$

  After the default, notional is reduced appropriately, and the protection buyer pays a quarterly coupon based on this lower notional.

CDS Index Tranches: TO FINISH

[7]Every six months a new index series is launched with updated components. Although the older index still trades, focus concentrates upon the on-the-run series.

4. Pricing Credit Derivatives

4.1. Survival Probabilities and Hazard Rates

The price of single-name cDS reflects the probability distribution of the time of default under the martingale measure. So, letting τ be the random time of default of the reference entity, we have

$$\text{Cumulative Default Prob.}: \quad G(t,s) = Q_B(\tau \leq s | I_t) \tag{5}$$

$$\text{Cumulative Survival Prob.}: \quad 1 - G(t,s) = Q_B(\tau > s | I_t) \tag{6}$$

where Equation 5 is the risk-neutral CDF for τ and where Equation 6 is also under the risk-neutral measure. We also note that $G(t,t) = 0$.

Hazard Rates: Often, it is convenient to express the probability distriution of default time time inters of *hazard rates* (or the *forward default rate*) defined

$$h(t,s) = \frac{g(t,s)}{1 - G(t,s)}, \qquad g(t,s) = \frac{\partial}{\partial s} G(t,s) = \text{Density of } G(t,T) \tag{7}$$

In other words, $h(t,s)$ is the risk-neutral conditional probability that, at time t, the reference entity will default between times s and $s + ds$ given that it has survived until time s. We can also express survival probabilities in terms of hazard rates, shown by

$$\frac{\partial}{\partial s} \ln(1 - G(t,s)) = -\frac{g(t,s)}{1 - G(t,s)} = -h(t,s)$$

$$\text{Integrate} \quad \ln(1 - G(t,s)) = -\int_t^s h(t,u)\, du$$

$$\Rightarrow 1 - G(t,s) = e^{-\int_t^s h(t,u)\, du}$$

$$\text{By Definition:} \quad Q_B(\tau > s | I_t) = 1 - G(t,s) = e^{-\int_t^s h(t,u)\, du}$$

4.2. Pricing Defaultable ZCB's

Now that we defined hazard rates, we can express the price of defaultable ZCB's and CDS's in terms of hazard rates. To do so, first price two more basic credit derivatves.

First Security, $\mathcal{P}_0(t,T)$: This asset pays \$1 at time T if $\tau > T$ and nothing otherwise. So this is a zero-recovery defaultable ZCB. Now if we assume the time of default is independent of the level of interest rates, then

$$\begin{aligned}
\mathcal{P}_0(t,T) &= B(t)E_t^{Q_B}\left[\frac{1_{\{\tau>T\}}}{B(T)}\right] \\
\text{By indepedence} \qquad &= B(t)E_t^{Q_B}\left[\frac{1}{B(T)}\right]E_t^{Q_B}\left[1_{\{\tau>T\}}\right] \\
&= P(t,T)(1-G(t,T)) \\
&= P(t,T)e^{-\int_t^T h(t,u)\,du}
\end{aligned}$$

where $P(t,T)$ is the value of a riskless ZCB. So if we assume independence of time of default and the level of rates, the value of a zero-recovery defaultable ZCB is the value of a riskless ZCB times the probability of the issuer not defaulting before maturity. We also have the nice result, if we expand out $P(t,T)$, that

$$\begin{aligned}
\mathcal{P}_0(t,T) &= P(t,T)e^{-\int_t^T h(t,u)\,du} = E_t^{Q_B}\left[e^{-\int_t^T r(u)\,du}\right]e^{-\int_t^T h(t,u)\,du} \\
&= E_t^{Q_B}\left[e^{-\int_t^T (r(u)+h(t,u))\,du}\right]
\end{aligned}$$

So that the price of zero-recovery ZCB is computing by discounting the promised payment at the risk free rate *plus* the hazard rate.

Second Security, $\mathcal{D}(t,T)$: This asset pays \$1 at time τ if $\tau \le T$, nothing otherwise. Then the value of this security, again assuming that the level of interest rates and the time of default are independent, is

$$\begin{aligned}
\mathcal{D}(t,T) &= B(t)E_t^{Q_B}\left[\frac{1_{\{t\le\tau\le T\}}}{B(\tau)}\right] = B(t)E_t^{Q_B}\left[\int_t^T \frac{1_{\{\tau\in ds\}}}{B(s)}\,ds\right] \\
&= \int_t^T B(t)E_t^{Q_B}\left[\frac{1}{B(s)}\right]E_t^{Q_B}\left[1_{\{\tau\in ds\}}\right] \\
&= \int_t^T P(t,s)g(t,s)\,ds \\
&= \int_t^T P(t,s)h(t,s)e^{-\int_t^s h(t,u)\,du}\,ds
\end{aligned}$$

From here, we have have nough to price a defaultable ZCB with random recovery rate.

Pricing Defaultable ZCB, Random Recovery: Assuming that interest rates, time of default, and recovery value are mutually independent, we can express the value at time t of a defaultable ZCB with random recory, R, as

$$\begin{aligned}\mathcal{P}(t,T) &= B(t)E_t^{Q_B}\left[\frac{1_{\{\tau>T\}}}{B(T)} + R\cdot\frac{1_{\{t<\tau\leq T\}}}{B(\tau)}\right]\\ &= \mathcal{P}_0(t,T) + E_t^{Q_B}[R]\mathcal{D}(t,T)\end{aligned}$$

Thus, it's clear that it's possible to recover hazard rates from the prices of defaultable bonds.

4.3. Pricing Single-Name CDS's

Suppose we have a CDS with unit notional, tenor T_n, coupon dates $\{T_1, \ldots, T_n\}$, up-front fee K_u, coupon rate K_c, and par spread K_s. Assuming the recovery rate, R, is independent of the time of default and the level of interes rates, we can price

- *Protection Leg:* This is the time 0 value of a payment equal to the loss given default (LGD) at the time of default:

$$\begin{aligned} V_{ps}(0) &= B(0)E^{Q_B}\left[\frac{(1-R)1_{\{\tau \le T_n\}}}{B(\tau)}\right] \\ &= (1 - E^{Q_B}[R])B(0)E^{Q_B}\left[\frac{1_{\{\tau \le T_n\}}}{B(\tau)}\right] \\ &= (1 - E_t^{Q_B}[R])\mathcal{D}(0, T_n) \end{aligned} \tag{8}$$

- *Premium Leg:* This is the value at time 0 of the up-front payment plus the value of a series of fixed coupon payments, $K_c(T_i - T_{i-1})$ at each coupon date T_i (provided that default has not occured at time T_i), plus the fractional coupon payment of $K_c(\tau - T_{i-1})$ if default occurs between T_{i-1} and T_i:

$$V_{pb}(0) = K_u + K_c\mathcal{A}(0, T_1, T_n) \tag{9}$$

where $$\mathcal{A}(0, T_1, T_n) = \sum_{i=1}^{n}(T_i - T_{i-1})\mathcal{P}_0(0, T_i) + \sum_{i=1}^{n}\int_{T_{i-1}}^{T_i}(s - T_{i-1})P(0, s)g(0, s)\, ds$$

  If we re-evaluate the premium leg in terms of the spread, rather than the up-front fee and coupon, then we get

$$V_{pb}(0) = K_s\mathcal{A}(0, T_1, T_n) \tag{10}$$

- Since the initial value of a CDS must be 0, we know that Equation 8, 9, and 10 must equal each other. This allows us to deduce

$$K_u = (K_s - K_c)\mathcal{A}(0, T_1, T_n) \tag{11}$$

$$K_s = \frac{(1 - E_t^{Q_B}[R])\mathcal{D}(0, T_n)}{\mathcal{A}(0, T_1, T_n)} \tag{12}$$

  Equation 11 means that the upfront fee is the PV of the difference between the spread and the coupon, while Equation 12 shows that the spread is equal to the value of the protection leg divided by an annuity factor.

ISDA CDS Standard Model: Note that market quotes are typically the spread, which we must convert to an up-front fee. By using Equations 8, 10, and 11, we can compute the upfront for CDS's that are quoted in terms of spread using the conventions specified by the ISDA Model:

1. We assume that $E_t^{Q_B}[R]$ is equal to the *conventional recovery rate*, R_c.[8] We also also assume that the hazard rate curve is flat, so that $h(0,T) = h(0,0)$ for all T.

2. Then, we solve for the value of $h(0,0$ that equalizes Equations 8 and 10. That means solving

$$(1 - R_c) \int_0^{T_n} P(0,s) h(0,0) e^{-h(0,0)s}\, ds = K_s \sum_{i=1}^{n} (T_i - T_{i=1}) P(0,T_i) e^{-h(0,0)T_i}$$
$$+ K_s \sum_{i=1}^{n} \int_{T_{i-1}}^{T_i} (s - T_{i-1}) P(0,s) h(0) e^{-h(0,0)s}\, ds$$

   Then, we substitute $h(0,s) = h(0,0)$ in Equation 11 to compute the up-front.

Alternative Approach: Alternatively, we can take a set of CDS market quotes and an estimated recovery rate, use Equations 8, 9, and 10 to bootstrap and infer an implied (piecewise constant) hazard rate curve, and hence an implied default time distribution.[9]

[8]For single-name CDS's, $R_c = 40\%$ for senior unsecured obligations, $R_c = 20\%$ for subordinated obligations, and $R_c = 25\%$ for emerging markets (both senior and subordinated).

[9]This is very clearly a departure from the ISDA CDS Standard Model which assumes a flat hazard rate.

4.4. Pricing Multi-Name CDS's

4.4.1. Background and Motivation

Straight basket CDS's or CDS indicex spreads can be derived directly from the spreads for individual single-name CDS's. However, the determination of the spreads for N2D CDS's or CDS index tranches is a bit more complication, for the following reasons:

1. The structure of N2D CDS's and CDS index tranches require us to specify the joint probability distribution of the default times of the reference entities in the basket or index under the martingale measure.

2. In order for multi-name CDS's to be priced consistently with single-name CDS's, the assumed joint default distribution must have *marginals* that are consistent with the default time distributions used for pricing CDS's on the individual reference entities.

Therefore, we will be starting from arbitrary default distributions for the individual credits, and we'll have to construct a joint default distribution consistent with those individual distributions.

The most flexible way to do so uses **copulas**, whose characteristics are detailed in the appendix. But note that pricing is tremendously sensitive to the choice of copula. It's no trivial matter and should be taken very seriously.

4.4.2. Implementation

Once the individual default probability distributions, $F_i(T) = G_i(0, T)$, and a copula, C, have been calibrated, the default times τ_i on a basket of credits can be simulated as follows:

1. Draw random variables $(U_1, \ldots, U_m)$ having joint distribution, C.

2. Set $\tau_i = F_i^{-1}(U_i)$, which sets τ_i so that

$$U_i = F(\tau_i) = G_i(0, \tau_i) = 1 - e^{-\int_0^{\tau_i} h_i(0,s)\, ds}$$

   where h_i is the hazard rate for credit i.

In the special case that selected copula is Gaussian with base correlatoin matrix ρ, the random variables $(U_1, \ldots, U_m)$ for step 1 above can be simulated by

1. Drawing $(X_1, \ldots, X_m)$ from a multivariate standard normal distribution with correlation matrix ρ.

2. Setting $U_i = N(X_i)$.

Once we have simulated default times, we can price multi-name CDS's via Monte Carlo simulation.

Monte Carlo Pricing of Multi-Name N2D CDS: Suppose that the Nth to default CDS has tenor T_n and m credits in the basket. We simulate the normalized cash flows on the protection leg and on the annuity, $\mathcal{A}$, as follows:

1. Simulate the default times $(\tau_1, \ldots, \tau_m)$ using the copula.
2. Determine the time τ of the Nth default.
3. The normalized cash flow on the protection leg is
$$\frac{1-R}{B(\tau)} \cdot 1_{\{\tau \leq T_n\}} \tag{13}$$
The value R used here can either be constant or drawn from some calibrated distribution.
4. The normalized cash flow on the annuity is
$$\sum_{i=1}^{n} \frac{T_i - T_{i-1}}{B(T_i)} \cdot 1_{\{\tau \geq T_i\}} + \sum_{i=1}^{n} \frac{\tau - T_{i-1}}{B(\tau)} \cdot 1_{\{T_{i-1} < \tau < T_i\}} \tag{14}$$
5. The mean of the simulated values of the normalized cash flows on the premium leg and the annuity are then the Monte Carlo estimates of $V_{ps(0)}$ and $\mathcal{A}(0, T_1, T_n)$, respectively.
6. Given V_{ps} and $\mathcal{A}(0, T_1, T_n)$, the value of the premium leg is
$$V_{ps} = K_s \mathcal{A}(0, T_1, T_n)$$
while the value of the CDS par spread is
$$K_s = \frac{V_{ps}(0)}{\mathcal{A}(0, T_1, T_n)}$$

4.5. More General Credit Derivatives

We saw that the pricing of a CDS depends on the default time distribution under Q_B as perceived at the valuation date, t. Moreover, this distribution can be expressed easily in terms of hazard rates. But suppose instead that we wanted to price a credit swaption with expiration T. Then the value of this swaption will depend upon the value of the underlying CDS at T, wihch depends upon the hazard rates (and thus the distribution of default times) at time T under Q_B.

So in order to price general credit derivatives, we need a model of how the probability distribution of the time of default (or, equivalently, hazard rates) evolves over time. Therefore, we need a **Dynamic Credit Risk Model**.

5. Dynamic Credit Risk Models

Dynamic Credit Risk Models can come in two flavors:

1. *Structural* (or *Firm Value*) Models: A relatively older form of model, these model default-times as the time when the value of firm assets, V, falls below some some *default trigger level*. They make explicit assumptions about the stochastic process, V, and the conditional default probabilities are then determined by the distance of $V(t)$ from the default trigger level.

2. *Intensity* (or *Reduced Form*) Models: This relatively newer class of models avoids modeling the mechanism that triggers default. Instead, it directly postulates a stochastic process for the default intensity, which is the conditional probability of default over the next instant. This is calibrated to market data.

5.1. Structural Models

5.1.1. Merton Model

This model assumes that the value of a firms assets follows a geometric Brownian motion, as $r(t)$, $\delta(t)$, and $\sigma(t)$ are constant:

$$dV(t) = (r(t) - \delta(t))V(t)\ dt + V(t)\sigma(t)\ d\hat{w}(t) \tag{15}$$

The model also assumes that the firm is financed by equity and a single issue of ZCBs maturing at T with face value K so that default occurs if $V(T) < K$.

Pricing Debt: Because we assume that default can only (!) happen at time T,[10]the value of the firm's debt at time T is

$$D(T) = \min\{K,\ V(T)\} = K - \max\{0,\ K - V(T)\}$$

Therefore, the value of the debt at any time $t < T$ equals the value of a riskless ZCB less the value of a put option on the firms value, which (because of the assumed model in Equation 15) can be priced by the Black-Scholes Equation.

Credit Spreads: If we plot the curves for the credit spread as a function of time to maturity, we get a hump shaped pattern because the value of the firm cannot jump (so low probability of default in the short term), a slightly higher probability of default exists in the medium term, but in the longer term, the drift is positive, so spreads decay to zero.

[10]This is a pretty big assumption. Specifically, we're assuming that there is no uncertainty regarding *when* default will occur, just *if*.

Default Probabilities: If we want to to get the cumulative probability of default, we have to measure the number of times that the value of the assets of the firm end up below the default trigger level at time T. In other words, we want

$$\begin{aligned} G(t,T) &= Q_B(V(T) < K|V(t)) = N(-y(t)) \\ \text{where} \quad y(t) &= \frac{\ln\left(\frac{V(t)e^{-\delta(T-t)}}{Ke^{-r(T-t)}}\right)}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t} \end{aligned}$$

Note, this allows us to get the distribution $G(t,T)$ and, thus, the hazard rates.[11] From there we, can price CDS's. Also, just a quick side note that $y(t)$ is often called the *distance to default*, as the probability of default is inversely related to $y(t)$.

Modeling Defaults for Multiple Firms: We can assume that the value, V_i, of each firm follows its own process

$$dV_i(t) = V_i(t)(r-\delta_i)\;dt + V_i(t)\sigma_i d\hat{w}_i(t)$$

where the $\hat{w}_i(t)$ are Brownian motions under Q_B with some correlation ρ. Therefore, we have the cumulative probability of default is

$$\begin{aligned} &Q_B(\tau_1 \leq T_1, \tau_2 \leq T_2|I_t) = C_\rho(Q_B(\tau_1 \leq T_1|I_t),\; Q_B(\tau_2 \leq T_2|I_t)) \\ \text{where} \quad &C_\rho(x,y) = N_\rho(N^{-1}(x), N^{-1}(y)) \end{aligned}$$

and N^{-1} denotes the inverse standard normal distribution function. Thus, the merton model assumes the joint default probabilities are related to the individual default probabilities by the Gaussian copula.

Limitations However, the Merton Model has a number of shortcomings. Among them,

1. There are often safety covenants that trigger investors the right to reorganize a firm if the value falls below a given level prior to maturity of the bonds.
2. The firm must typically make coupon payments; the debt is not usually financed by ZCBs.
3. Finally, the firm typically has a mix of short- and long-term debt. Moreover, the value of the firm must remain above the face value of short-term debt to refinance.
4. Credit spreads convert to zero for very short maturities.

[11] It just so happens that default can only occur at time T as stated above. As a result, the distribution is discountinous at T, the density function ($g(t,s)$) and hazard rates ($h(t,s)$) equal zero for all $s < T$ and both are not defined for $s = T$.

5.1.2. Black-Cox Model

First-Passage Models generalize Merton's model by assume that default can happen if

1. The value of the assets of the firm drops below a default trigger level $H(t)$ at any time t, in which case the bondholders receive $V(t) = H(t)$.[12]

2. The value of the assets of the firm at the long-term debt's maturity T is below the debt's face value K, in which case bondholders receive $V(T)$.

Specifying $H(t)$: We start by noting that if $H(t) = Ke^{-r(T-t)}$, the debt becomes riskless, as the bondholders will always receive the PV of the promised payment in the case of default. Therefore, we assume that $H(t)$ will be less than that. In particular, Black and Cox assumed that the default boundary has form

$$H(t) = \alpha K e^{-\eta(T-t)} \tag{16}$$

[12]It's, therefore, clear that the Merton Model is a special case of the Black-Cox Model where $H(t)$ is set to 0 for all t.

5.2. Intensity Models

Recall that, unlike structural models, *intensity models* don't try to model the mechanism that triggers default. Rather, they directly postulated a probability model for the arrival of default. Specifically, they assume that in each infinitesimal interval, dt, there is some risk-neutral probability $\lambda(t)$ of default occuring, where λ is a given stochastic process.

So we start by postulating that default occurs at the first time of a Poisson process, N, with stochastic intensity λ. In this case, it follows that the time τ of the first jump has the expontential distribution

$$\begin{aligned} Q_B(\tau > t) &= Q_B(N(t) = 0) = e^{-\lambda t} \\ \Rightarrow 1 - G(t,T) &= Q_B(\tau > T|I_t) = Q_B(\tau > T|\tau > t) \\ &= Q_B(N(T-t) = 0) = e^{-\lambda(T-t)} \end{aligned}$$

where we recall that N is the value of the Poisson process. Note that if the Poisson process has stochastic intensity, the above expression becomes

$$1 - G(t,T) = E_t^{Q_B}\left[e^{-\int_t^T \lambda(s)\, ds}\right] \tag{17}$$

Incorporating Hazard Rates: We recall that we can rewrite the survival probability as a function of hazard rates, which allows us to redefine Expression 17 as

$$e^{-\int_t^T h(t,s)\, ds} = E_t^{Q_B}\left[e^{-\int_t^T \lambda(s)\, ds}\right] \tag{18}$$

So once a process for the default intensity has been specified, we can easily recover the hazard rates at any time t. Moreover, if we differential both sides of Equation 18 with respect to T and then send $T \to t$, it follows that

$$\lambda(t) = h(t,t), \qquad \forall t \text{ including } t = 0$$

Specifiying a Process for λ: We obviously want λ to be non-negative and we want a closed form solution for survival probabilities in Equation 17, which looks very must like the price of a ZCB in a short-rate model, but with λ instead of r. So we'll use the CIR process, which has the non-negativity property and a closed-form solution:

$$d\lambda(t) = \kappa_\lambda(\theta_\lambda - \lambda(t))\, dt + \sigma_\lambda\sqrt{\lambda(t)}\, d\hat{w}(t) \tag{19}$$

We can refer back to the Fixed Income notes for the solution.

A. Copulas

First, we need a simple result from statistics. If X_i is a continuous RV with cumulative distribution function F_i, then the RV $U_i = F_i(X_i)$, called the *probability integral transform* of X_i, has a Unif(0,1) distribution.

Because the value of each X_i is uniquely determined by the value of it's probability integral transform, U_i, then in order to specify the jdf of $(X_1, \ldots, X_m)$, it's enough to specify the jdf of $(U_1, \ldots, U_m)$.[13] Specifically, if C is the jdf of $X_1, \ldots, X_m$, then

$$\begin{aligned} F(x_1, \ldots, x_m) &= P(X_1 \leq x_1, \ldots, X_m \leq x_m) \\ &= P(U_1 \leq F_1(x_1), \ldots, U_m \leq F_m(x_m)) \\ &= C(F_1(x_1), \ldots, F_m(x_m)) \end{aligned}$$

Therefore the jdf, C, of $(U_1, \ldots, U_m)$ must be a multivariate distribution on $[0,1]^m$ with uniform marginals. This is called an **m-dimensional copula**.

Copulas are unique and awesome because a copula, together with the marginal distributions, *completely* describes the statistical dependence between random variables, while the correlation matrix and marginal distributions only describe *linear* dependence.

[13] This result is called Sklar's Theorem—namely, that a unique m dimensional copula exists provided that $X_1, \ldots, X_m$ are continuous RVs.