Applied Linear Algebra

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1 Properties of Matrices

Here are some words we use to describe matrices.

1. *Symmetric*: $A = A^T$
2. *Sparse*: A matrix consisting of mostly zeros.
3. *Tridiagonal*: A matrix with three diagonals, something like

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

4. *Invertible*: Matrix A is invertible if there exists a matrix A^{-1} such that $AA^{-1} = I$.
5. *Rank*: For an arbitrary $m \times n$ matrix A, the rank is the number of linearly independent columns or, alternatively, rows of A.[1] The rank r is such that $r \leq n$ and $r \leq m$.

Now, one of the most important concepts for describing an $n \times n$ matrix A. The singular vs. nonsingular distinction entails a whole host of consequences, detailed below:

Nonsingular	**Singular**
A invertible	A not invertible
Columns independent	Columns dependent
Rows independent	Rows dependent
$\det(A) \neq 0$	$\det(A) = 0$
$Ax = 0$ has one solution, $x = 0$	$Ax = 0$ has infinitely many solutions
$Ax = b$ has one solution, $x = A^{-1}b$	$Ax = b$ has no solution or infinitely many
A has n (nonzero) pivots	A has $r < n$ pivots
A has full rank	A has rank $r < n$
Reduced row echelon form is $R = I$	R has at least one zero row
Column space is all of $\mathbb{R}^n$	Column space has dimension $r < n$
Row space is all of $\mathbb{R}^n$	Row space has dimension $r < n$
All eigenvalues are non-zero	Zero is an eigenvalue of A
A^TA is symmetric positive definite	A^TA is only semidefinite
A has n (positive) singular values	A has $r < n$ singular values

[1]Whether you check rows or columns doesn't matter, those numbers will be equal.

2 Matrix Multiplication and the Nullspace

Matrix multiplication Ax is the combination of the columns of A. Each column scaled by the corresponding element of x. This is, hands down, one of the simplest, most important, and most practical ways of thinking about matrix multiplication.

Being more explicit, and letting $A_{\cdot i}$ denote the ith column of A (an $m \times n$ matrix), and letting x_j denote the jth element of $n \times 1$ vector x, we can express

$$Ax = x_1 A_{\cdot 1} + \cdots + x_n A_{\cdot n}$$

Since the result is a combination of the columns of A—each an $m \times 1$ vector—we can easily see that the result will also be an $m \times 1$ column vector.

This intuition also provides a convenient way to "pick out" the ith column of matrix A via matrix multiplication: just right-multiply A by an $n \times 1$ vector whose elements are all zero, except for the ith element, which equals 1. Example,

$$\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 8 \\ 3 \\ 4 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix} + 0 \begin{bmatrix} 6 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix}$$

Definition 2.1. The *nullspace* of a matrix A $(m \times n)$, which is denoted $N(A)$, is the set of all $x \in \mathbb{R}^n$ such that $Ax = 0$. Intuitively, the nullspace consists of all vectors that can linearly combine the columns of A to get the zero vector.

It is *always* the case that the zero vector is in $N(A)$, so the set can never be empty.