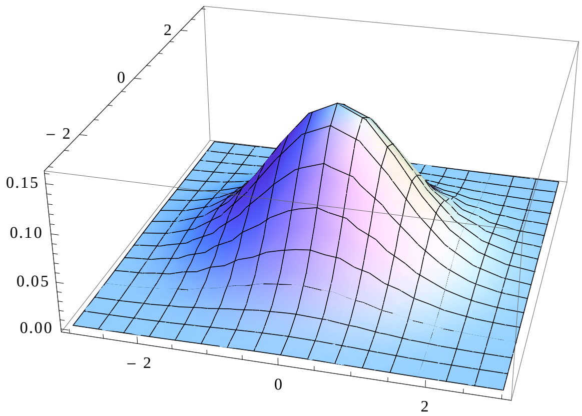
# The Multivariate Normal Distribution

The univariate normal distribution is an extremely familiar concept where some random variable $X$ can take values along the real with probabilities that match the famouse bell-curve. Recall the probability density function of

$$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2\sigma^2}(y-\mu)^2}$$

However, that's limited to only one dimension, and we would like to generalize to higher dimensions. In the next-simplest 2-dimensional case, we'd like a distribution that actually looks like a bell—where potentional values can range over the real plane, $\mathbb{R}$, where the density is clustered around some mean before tapering off in all directions, as seen below.




This figure has mean zero for both $X_1$ and and $X_2$, and which are independent, implying $\sigma = I_2$, the identity matrix. It's easy to see that any vertical cuts parallel to $xz$ or $yz$ planes will yield a traditional normal random variable. This of course generalizes to higher dimensions, although we can't display it so nicely.

## 1 Notation

In this note, the multivariate distribution will apply to a $n$-dimensional random vector

$$\mathbf{X} = \begin{pmatrix} X_1 & X_2 & \dots & X_n \end{pmatrix}, \qquad \mathbf{X} \sim N_n(\mu, \Sigma)$$

where $\mu$ is the $n$-dimensional *mean vector*,

$$\mu = \begin{pmatrix} EX_1 & EX_2 & \dots & EX_n \end{pmatrix},$$

and where $\Sigma$ is the $n \times n$ *covariance matrix*, which is defined and has in its $i, j$ entry

$$\sigma = E\left[(\mathbf{X}-\mu)(\mathbf{X}-\mu)'\right] \in \mathbb{R}^{N\times N}$$

$$\Sigma_{ij} = Cov(X_i, X_j), \qquad i, j = 1, \dots, n$$

## 2 Definition

A random vector $\mathbf{X}$ has a *multivariate normal* distribution if every linear combination of its components,

$$Y = a_1 X_1 + \ldots + a_n X_n$$
$$\Leftrightarrow Y = \mathbf{a}'\mathbf{X}, \qquad \mathbf{a} \in \mathbb{R}^n$$

is *normally distributed*, with a corresponding mean and variance. This gives a joint density function of

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \, e^{-\frac{1}{2}(\mathbf{x}-\mu)' \, \Sigma^{-1}(\mathbf{x}-\mu)}, \qquad |\Sigma| = \det \Sigma$$

## 3 Joint Normality

Suppose that $X$ and $Y$ are normally distributed and indepdent. This then implies that they are *jointly normally distrubed.* In other words, $\begin{pmatrix} X & Y \end{pmatrix}$ must have a multivariate normal distribution. Note, however, that $X$ and $Y$ must be independent for this to hold, not just uncorrelated.[1]

[1]Note that uncorrelated does not, in general, imply independence. Moreover, a lot of confusions exist on this concept, so be very careful when considering it.