Stochastic Processes: Discrete Space, Continuous Time

Matthew Cocci

December 10, 2012

1 Basic Infrastructure

Let $X = \{X_t\}$ be a process that can assume only discrete values. We will, however, let t vary continuously. The process is therefore defined by

$$T = (\tau_1, \tau_2, \dots)$$

$$J = (j_1, j_2, j_3, \dots)$$

where J is a sequence of discrete values (either finite or countably infinite) that the process can assume, and T is a sequence of jump times–times when the process transitions from one particular to $j_i \in J$ to another $j_k \in J$.

For each $j \in J$, there is an accompanying variable F_j that describes how long the process, starting at j will jump. So we define, for the first jump

$$F_j(t) = P(\tau_1 \leq t | X_0 = j)$$

Second, We will also need to describe the **transition probabilities**. Specifically, we need the probability that, when the process jumps from j, it moves to k. And given that it's pretty stupid to define a probability that the position will jump from itself *to* itself, we're not even going to consider $Q_{j,j}$. (We'll see why this helps in a minute.) So here are a few useful facts:

$$\sum_{k \in J} Q_{jk} = 1$$

$$Q_{jj} = 0.$$

Because of how we defined F_j and $Q_{j,k}$ (and since we let $Q_{j,j} = 0$), we can assume independence between *when* the process jumps and *where* to define

$$P(\tau_1 < t, X_{\tau_1} = k \mid X_0 = j) = F_j(t) Q_{jk}.$$

This is the probability that process, which started at j, had its first jump to value k at by time t.

Employing the full force of the Markov assumption and stationarity, we also recognize that

$$P(\tau_1 < s; X_{\tau_1} = k; \tau_2 - \tau_1 < t; X_{\tau_2} = l \mid X_0 = j) = F_j(s)Q_{jk}F_k(t)Q_{kl}.$$

In fact, what we really want to know is a similar, but somewhat different, value: the probability that the process, X, is at a specific value, k, at some time t, given that it started at j. Mathematically, we express this is as

$$P_{jk}(t) = P(X_t = k|X_0 = j).$$

Now we defined this value, given that the process started at j *at time 0*. But if we employ the Markov and stationarity assumptions, then we can shift this probability through time. This means,

$$P(X_t = k|X_s = j) = P_{jk}(t - s).$$

1.1 Definitions

Just to make things a little more precise, I define the following terms for the stochastic process X_t.

Definition The process X_t has the **Markov Property** if, for all $s_i \leq t$, we have that

$$P(X_t = k \mid X_{s_1} = j_1, \ldots, X_{s_n} = j_n, X_s = j) = P(X_t = k \mid X_s = j)$$

In words, the process is Markov if only the last observed value matters.

Definition The process X_t is stationary if, for all $s \leq t$, we have that

$$P(X_t = k \mid X_s = j) = P(X_{t-s} = k \mid X_0 = j)$$

This allows us to shift the process through time.

2 The Exponential Distribution

Given our stationarity assumption, we've put a pretty tight constraint on the jump-time distribution. Specifically, stationarity tells us that we need

$$P_j(\tau_1 > t + s \mid \tau_1 > s) = P_j(\tau_1 > t)$$

where $P_j = P(\ldots|X_0 = j)$. Then, after applying Bayes' Rule on to get the left hand side, our requirement becomes

$$\frac{1 - F_j(t + s)}{1 - F_j(s)} = 1 - F_j(t).$$

A standard distribution which obeys this constraint is the exponential distribution. So we'll assume that

$$f_j(t) = q_j e^{-q_j t}, \ \ t \geq 0$$

$$F_j(t) = 1 - e^{-q_j t}$$

where q_j is a paramter specific to the starting point $X_0 = j$. In practice, it makes sense that the time until a jump could differ for different starting points of the process.

3 Chapman-Kolmogorov Equations

Second, as a result of our Markov and stationarity assumptions, we get the Chapman-Kolmogorov Equatios

$$P_{jk}(t+s) = \sum_{l \in J} P_{jl}(t) P_{lk}(s).$$

So if take some intermediate time $t \in (0, s)$, to find the probability of going from j to k in time $t + s$, we need only sum over all the possible ways of getting from j to k via any position l at the intermediate time t.

4 Backward and Forward Equations

So why define all this? Well, they allow us to compute two very useful systems of differential equations for asset pricing: the backward and the forward equations. First, the **backward equation**.

$$P'_{jk}(t) = \sum_{\ell \in J} q_{j\ell} P_{\ell k}(t) \tag{1}$$

and the **forward equation**

$$P'_{jk}(t) = \sum_{\ell \in J} q_{\ell k} P_{j\ell}(t) \tag{2}$$

where the define that q_{mn} coefficient as

$$q_{mn} = P'_{mn}(0)$$

$$\Rightarrow q_{mn} = \begin{cases} -q_m & m = n \\ q_m Q_{mn} & m \neq n \end{cases}$$

recalling that q_m is the coefficient on the exponential distribution and density functions.

4.1 Proof of the Backward Equation

To prove the backward equation, we set up the following formula which we will then explain

$$P_{jk}(t) = \delta_{jk} e^{-q_j t} + \int_0^t q_j e^{-q_j s} \left(\sum_{\ell \neq j} Q_{j\ell} P_{\ell k}(t-s) \right) ds.$$

Wow, what a bear of an equation. Let's break it down component by component to figure out what the hell it means.

- δ_{jk}, **Kronecker Delta Symbol**: The variable is defined as follows:

$$\delta_{jk} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

  So once we put this together into $\delta_{jk}e^{-q_j t}$, we see that in the case where $j = k$, we get

$$\delta_{jj}e^{-q_j t} = e^{-q_j t} = 1 - F_j(t)$$

  which represents the probability that the process hasn't jumped yet at time t, i.e. that $\tau_1 > t$. Otherwise, it's zero.

- Next, we consider the integral. Let's consider the first term in the integral, along with the differenital

$$q_j e^{-q_j s} ds = f_j(s) ds$$

  which is roughly the probability that the first jump occurs between time s and $s + ds$.

  With that in mind, we can consider the probability that the process jumps from j to some ℓ between time s and $s + ds$:

$$q_j e^{-q_j s} Q_{j\ell} ds.$$

  But recall, we want the probability that we get to k, not to ℓ. And given that we jumped at s, an intervening time between 0 and t, we have $t - s$ units of time to get there. So including the probability of getting to k, we get

$$q_j e^{-q_j s} Q_{jl} P_{lk}(t - s) ds$$

  which, in total, is the approximate probability for

  1. Jumping at time s: $q_j e^{-q_j s}$.
  2. Given a jump, jumping from j to l at time s: $Q_{j\ell}$.
  3. Then somehow getting from ℓ to k in the time remaining in the interval of interest, $[0, t]$, in any number of ways. (This is the term $P_{\ell k}(t - s)$, which will be the right-hand side version of the variable on the left-hand side that we will use to solve a differential equation.)

  But, and here's the key, that was specific to jumping *at time* s and *to* ℓ in the interim when going from j to k. But s and ℓ are not unique. There's any number of times that we could jump from j to ℓ. So we integrate over the different possible times, s.

  Second, there's any number of intermediate steps steps that we could take between j and k. So we should use the Chapman-Kolmogorov Equation to sum over *all possible* intermediate steps, ℓ, hence the sum inside the integral.

Awesome. Now that we have a good representation for the process, we can solve it to get a differential equation, the backward equation, which we are after.

Proof. First step, let's replace $t-s$ by s in the integral, changing the variables—note that we can do this because t is a constant, not a variable here. It's harmless enough and we can factor out a term to get:

$$P_{jk}(t) = \delta_{jk}e^{-q_j t} + \int_{t-0}^{t-t} q_j e^{-q_j(t-s)} \left(\sum_{\ell \neq j} Q_{j\ell} P_{\ell k}(s) \right) (-1) ds.$$

$$\Rightarrow P_{jk}(t) = e^{-q_j t} \left[\delta_{jk} + q_j \int_0^t e^{q_j s} \left(\sum_{\ell \neq j} Q_{j\ell} P_{\ell k}(s) \right) ds. \right]$$

And by more technical arguments involving the boundedness of the integral, it just so happens that $P_{jk}(t)$ is a continuous function. So we can differentiate with respect to t, using the product rule, then use the fundamental theorem of calculus on the integral to finally get

$$P'_{jk}(t) = -q_j P_{jk}(t) + q_j \sum_{\ell \neq j} Q_{j\ell} P_{\ell k}(t)$$

Finally, to get the equation in it's final form, we note that

$$X_0 = j \Rightarrow P_{jk}(0) = \delta_{jk}$$

allowing us to write

$$P'_{jk}(0) = -q_j \delta_{jk} + q_j \sum_{\ell \neq j} Q_{j\ell} \delta \ell k$$

$$P'_{jk}(0) = -q_j \delta_{jk} + q_j Q_{jk}.$$

Therefore we define $q_{jj} = -q_j$ and $q_{jk} = q_j Q_{jk}$ whenever $j \neq k$ to write the final form of the backward equation:

$$P'_{jk}(t) = \sum_{\ell \in S} q_{j\ell} P_{\ell k}(t).$$

□

Phew. Now just another 10 pages to prove Forward Equation.

4.2 Proof of the Forward Equation

Kidding! This is substantially easier.

Proof. Recall the Chapman-Kolmogorov equation:

$$P_{jk}(t+s) = \sum_{\ell \in J} P_{j\ell}(t) P_{\ell k}(s)$$

Well, let's differentiate both sides with respect to s to get

$$P'_{jk}(t+s) = \sum_{\ell \in J} P_{j\ell}(t) P'_{\ell k}(s)$$

and set $s = 0$, using our definition of q_{jk} just defined to get the forward equation:

$$P'_{jk}(t) = \sum_{\ell \in J} q_{\ell k} P_{j\ell}(t).$$

$\square$

5 Poisson Process

This is a very special case of the birth-and-death process with the restriction that the process can only jump one unit at a time, and in one direction. Specifically, we define the transition probabilities

$$Q_{jk} = \begin{cases} 1 & k = j+1 \\ 0 & o.w. \end{cases}$$

In addition, the inter-jump times—τ_1, $\tau_2 - \tau_1$, etc.— are exponentially distributed with parameter $\lambda > 0$.

$$F_j(t) = P(\tau_1 \le t \mid X_0 = j) = 1 - e^{-\lambda t}.$$

Namely, we want to show—using the Forward or Backward Equation—that

$$P_{jk}(t) = \begin{cases} \frac{(\lambda t)^{k-j}}{(k-j)!} e^{-\lambda t} & k \ge j, \ t > 0 \\ 0 & k < j \end{cases}$$

Proof. We're going to use the Forward Equation, which states

$$P'_{jk}(t) = \sum_{\ell \in J} q_{\ell k} P_{j\ell}(t).$$

But first, the probability $P_{jj}(t)$ is really easy, as it represents the probability that the process didn't jump yet. So

$$P_{jj}(t) = P(\tau_1 > t) = 1 - F_j(t) = e^{-\lambda t}$$

Next, because of the way we structured the Poisson Process, things are going to simplify *considerably.* Specifically, if we look at that coefficient q_k, we really only have two cases:

1. $k = j$ implies, by our definition of the coefficient

$$q_{jj} = -q_j = -\lambda.$$

2. $k = j+1$, which implies

$$q_{j,j+1} = q_j Q_{jk} = \lambda \cdot 1 = \lambda.$$

So we can rewrite the entire forward equation as

$$P'_{jk}(t) = \sum_{\ell \in J} q_{\ell k} P_{j\ell}(t).$$

$$= q_{kk} P_{jk}(t) + q_{k-1,k} P_{j,k-1}(t)$$
$$= \lambda P_{j,k-1}(t) - \lambda P_{jk}(t).$$

This we can solve like a traditional ODE:

$$P'_{jk}(t) = \lambda P_{j,k-1}(t) - \lambda P_{jk}(t)$$
$$e^{\lambda t} \left(P'_{jk}(t) + \lambda P_{jk}(t)\right) = \lambda P_{j,k-1}(t)$$
$$\Big(e^{\lambda t} P_{jk}(t)\Big)' = \lambda e^{\lambda t} P_{j,k-1}(t)$$
$$e^{\lambda t} P_{jk}(t) - P_{jk}(0) = \int_0^t \lambda e^{\lambda s} P_{j,k-1}(s) ds$$

and recognizing that $P_{jk}(0)$ is just δ_{jk}, we can rewrite this as

$$P_{jk}(t) = e^{-\lambda t} \delta_{jk} + \int_0^t \lambda e^{\lambda(s-t)} P_{j,k-1}(s) ds$$

Then, we can use the fact that $P_{jj}(t) = e^{-\lambda t}$ to plug into the equation just stated, and we can start solving inductively to get the general expression we wanted:

$$P_{jk}(t) = \begin{cases} \frac{(\lambda t)^{k-j}}{(k-j)!} \, e^{-\lambda t} & k \geq j, \; t > 0 \\ 0 & k < j \end{cases}$$

$\square$