Chi-Square and Non-Central Chi-Square Distributions

1 Chi-Squared Random Variable

Density Function and Descriptive Statistics Suppose that Y has a χ^2 distribution with ν degrees of freedom. Then the distribution is a special case of the gamma distribution, where $\alpha = \nu/2$ and $\beta = 1/2$ with the pdf and descriptive statistics

$$f_Y(y) = \frac{1}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)} x^{\frac{\nu}{2}-1} e^{-\frac{x}{2}} \qquad \text{Mean} = \nu, \qquad \text{Variance} = 2\nu \tag{1}$$

Derivation of Density Function As stated in the properties section (below), we can generate a $\chi^2(\nu)$ random variable by summing the square of ν independent N(0,1) random variables. This suggests that we can use an induction argument to get the distribution.

Properties The χ^2 distribution has several nice properties that we'll want to discuss:

1. If Y is chi-squared distributed with ν degrees of freedom ($Y \sim \chi^2(\nu)$), then we can generate Y by
$$Y = \sum_{i=1}^{\nu} Z_i^2, \qquad Z_i \sim N(0,1)$$

2. By the way we just defined a χ^2 RV, it's immediately clear that sums of χ^2 RVs are also χ^2 distributed.
$$Y_i \sim \chi^2(\nu_i) \quad \Rightarrow \quad \sum_{i=1}^{n} Y_i \sim \chi^2\left(\sum_{i=1}^{n} \nu_i\right)$$

3. Suppose that $Y_i \sim \text{NID}(\mu, \sigma^2)$. Then $\sum_{i=1}^{n} \frac{(Y_i-\mu)^2}{\sigma^2} \sim \chi^2(n)$.

4. Now let's suppose only the sample mean is available and see what we get
$$\begin{aligned}
\sum_{i=1}^{n} \frac{(Y_i-\mu)^2}{\sigma^2} &= \frac{1}{\sigma^2}\sum_{i=1}^{n}(Y_i - \bar{Y} + \bar{Y} - \mu)^2 \\
&= \frac{1}{\sigma^2}\sum_{i=1}^{n}(Y_i - \bar{Y})^2 + \frac{2}{\sigma^2}\sum_{i=1}^{n}(Y_i - \bar{Y})(\bar{Y} - \mu) + \frac{1}{\sigma^2}\sum_{i=1}^{n}(\bar{Y} - \mu)^2 \\
&= \frac{1}{\sigma^2}\sum_{i=1}^{n}(Y_i - \bar{Y})^2 + \frac{2(\bar{Y} - \mu)}{\sigma^2}\sum_{i=1}^{n}(Y_i - \bar{Y}) + \frac{n(\bar{Y} - \mu)^2}{\sigma^2} \\
&= \sum_{i=1}^{n}\frac{(Y_i - \bar{Y})^2}{\sigma^2} + 0 + \left(\frac{\bar{Y} - \mu}{\sigma/\sqrt{n}}\right)^2
\end{aligned}$$
Now it's clear that the left hand side is a sum of squared normal random variables, so it has a $\chi^2(n)$ distribution. On the right hand side, the third term has a $\chi^2(1)$ distribution, as it is a squared normal random variable. Rearranging, we see
$$\sum_{i=1}^{n}\frac{(Y_i - \bar{Y})^2}{\sigma^2} \sim \chi^2(n-1)$$

2 Non-Central Chi-Squared Random Variable