Continuous Time Finance

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1 Random Walk

We will start with an approximation of Brownian Motion, the **random walk**, denoted by W_n. This random walk will be the driving force behind the random variable X_t, which is approximated by by X_t^n, to be defined below. This second random variable is similar to its driver, the random walk, but it provides a little more flexibility on the time and value dimensions as we shall see.

1.1 The Random Walk, W_n

So to begin, the random walk is a stochastic process, defined and denoted as follows:

$$\{W_n | n \in \mathbb{N}\}$$

$$W_n = \xi_1 + \xi_2 + \cdots + \xi_n$$
$$W_0 = 0$$

where the ξ_i have distribution

$$P(\xi_i = 1) = p$$
$$P(\xi_i = -1) = 1 - p.$$

1.2 Generalized Process, X_t

To handle time, we will pick a $T \in (0, \infty)$, and partition our time interval into n equal segments

$$\Delta t = \frac{T}{n} \quad \Rightarrow t_i = i\Delta t.$$

In addition, the random variable, X_t, will be will approximated by the random variable X_t^n, which depends on how finely we partition our time interval. Eventually, we will let n tend to infinity, so that $X_t^n \to X_t$. But before we get there, a little more about X_t^n.

We will restrict the approximation X_t^n to movements of only Δx, a constant, between time t_{i-1} to time t_i. Eventually, as our time increments become small, Δx will shrink too.

Therefore, at each time $t_i = i\Delta t$, we can define the process as a function of the underlying, driving random walk:

$$X_{t_i}^n = \Delta x W_i = \Delta x(\xi_1 + \cdots + \xi_i), \qquad i \in 0, 1, \ldots, n. \tag{1}$$

At intermediate times, $t \in [t_{i-1}, t_i]$, we will define X_t^n as the linear interpolation between $X_{t_{i-1}}^n$ and $X_{t_i}^n$:

$$X_{t_i}^n = \frac{t_i - t}{\Delta t} X_{t_{i-1}}^n + \frac{t - t_{i-1}}{\Delta t} X_{t_i}^n$$

allowing X_t^n to be defined for all $t \in [0, T]$.

2 The Limiting Case

Ideally, we want to let n tend to infinity, so that $\Delta t \to 0$ and so that $X_t^n \to X_t$. But for this process X^n to converge to something and be useful, we're going to need the first and second moments, and hence EX_T^n and $Var(X_T^n)$ to be bounded.

From Equation 1, we see that

$$EX_T^n = \Delta x \cdot n \cdot E\xi_i = \Delta x \cdot n(2p - 1)$$

$$Var(X_T^n) = (\Delta x)^2 \cdot n \cdot Var(\xi_i) = (\Delta x)^2 \cdot n \cdot 4p(1 - p).$$

And using the fact that $n = \frac{T}{\Delta t}$, we can rewrite the expectation and the variance as

$$EX_T^n = T \cdot \frac{\Delta x}{\Delta t}(2p - 1) \tag{2}$$

$$Var(X_T^n) = T \cdot \frac{(\Delta x)^2}{\Delta t} \cdot 4p(1 - p). \tag{3}$$

Our goal is to *keep these finite* for all fixed t. So let's force the things we have control over in the variance term. Specifically, to prevent the variance from being 0 (and our process deterministic) and to prevent the process from blowing up to infinity, we choose Δx so that

$$\frac{(\Delta x)^2}{\Delta t} = 2D \tag{4}$$

for some positive constant D, which we will call the **diffusion coefficient**. This is the only logical and feasible way to keep it bounded.

This forces our hand with the expectation, because it will blow up to infinity as $\Delta x \to 0$, which we see by plugging 4 into equation 2, which gives

$$EX_T^n = T \cdot \frac{2D}{\Delta x}(2p - 1).$$

Now since we want the expectation bounded, we choose the parameter p so that

$$\frac{2p-1}{\Delta x} = \frac{c}{2D}$$

$$\Rightarrow p = \frac{1}{2} + \frac{c}{2D} \tag{5}$$

for some constant c, which we call the **drift coefficient**.

So if we take equations 2 and 5, we find

$$EX_T^n = cT$$

while taking equations 3, 4, and 5 gives us

$$Var(X_T^n) = 2DT\left(1 - \frac{c^2}{D^2}(\Delta x)^2\right).$$

Final Result So now let's take the limit of the mean and variance expressions, noting that Δx goes to 0 as $n \to \infty$:

$$\lim_{n\to\infty} EX_T^n = cT$$

$$\lim_{n\to\infty} Var(X_T^n) = 2DT$$

both of which are bounded!

Finally, we can apply the Central Limit Theorem to conclude that,

$$\lim_{n\to\infty} \frac{X_T^n - EX_T^n}{\sqrt{Var(X_T^n)}} \to Z$$

where Z is the standard normal random variable.

3 Brownian Motion

Now we can take the limiting case to define our result. So for any fixed $T \in (0,\infty)$, we can define the limit of X_T^n as X_T, which is a normally distributed random variable defined by parameters

$$\mu_T = cT$$

$$\sigma_T = \sqrt{2DT}.$$

If we do this for each time $T \in (0,\infty)$, we obtain a continuous process

$$\{X_T \mid T \geq 0\}$$

$$p(x,T) = \frac{1}{\sqrt{4\pi DT}}\, e^{\frac{-(x-cT)^2}{4DT}}, \quad x \in \mathbb{R}.$$

Definition If we set the parameters at $c = 0$ and $D = 1/2$, we get as the corresponding process **Brownian Motion**, also called the **Wiener Process**, denoted by $\{W_t \mid t \geq 0\}$:

4 Properties of Brownian Motion

Brownian Motion, as just defined, satisfies certain properties:

i. $W_0 = 0$ with probability 1.

ii. W_t is a continuous function of t with probability 1.

iii. For all $u < s < t$, the increments $W_t - W_s$ and $W_s - W_u$ are independent.

iv. For all $s < t$, $W_t - W_s$ is normally distributed with mean 0 and variance $t - s$. The pdf is

$$p(x, t-s) = \frac{1}{\sqrt{2\pi(t-s)}}\, e^{-\frac{x^2}{2(t-s)}}.$$

4.1 Brownian Motion Starting at y

Let W_t^y denote Brownian motion starting at y, rather than 0. Much remains the same, but let's run through just to be clear.

i. $W_0^y = y$, since it has a non-zero starting point.

ii. Doesn't change.

iii. Non-overlapping increments remain independent, and an increment like $W_t - W_s$ will have expectation 0, and variance $t - s$.

iv. W_t^y is normally distributed with density function

$$f(x, t|y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}.$$

4.2 Additional Properties

Before we prove the following results, let's just state them quickly:

1. The process W_t is **nowhere differentiable**.

2. W_t is not of **bounded first order variation**.

3. The function W_t has **bounded quadratic variation**.

4. $(dW_t)^2 = dt$.

Now we can prove these statements. Many of the properties of the Wiener Process, the Itō Integral, and Stochastic Differential Equations will follow from these results. So let's get to the proofs.

4.2.1 W_t is Nowhere Differentiable

Proof. We'll want to show that

$$\lim_{h\to 0} P\left(\left|\frac{W_{t+h}-W_t}{h}\right| \geq n\right) = 1$$

which would imply W_t cannot be differentiated.

So let's rewrite, using the properties that result from the definition of the Wiener Process:

$$\begin{aligned} P\left(\left|\frac{W_{t+h}-W_t}{h}\right| \geq n\right) &= P(|W_h| \geq n|h|) \\ &= 1 - P(|W_h| < n|h|) \end{aligned} \tag{6}$$

As h goes to zero, the second term in Equation 6 also goes zero, so the probability as a whole tends to unity. To see this, assume that $h > 0$ (without loss of generality) and carry out the rest of the proof:

$$1 - P(|W_h| < nh) = 1 - \frac{1}{\sqrt{2\pi h}} \int_{-nh}^{nh} e^{-\frac{x^2}{2t}}\,dx$$

With an eye towards using the Squeeze Theorem, we can set up the inequality

$$\begin{aligned} 1 - \frac{1}{\sqrt{2\pi h}} \int_{-nh}^{nh} e^{-\frac{x^2}{2t}}\,dx &\geq 1 - \frac{1}{\sqrt{2\pi h}} \int_{-nh}^{nh} 1dx \\ \Rightarrow 1 - \frac{1}{\sqrt{2\pi h}} \int_{-nh}^{nh} e^{-\frac{x^2}{2t}}\,dx &\geq 1 - \frac{2hn}{\sqrt{2\pi h}} = 1 - \sqrt{\frac{2hn}{\pi}} \end{aligned}$$

and taking the limit of both sides of the inequality,

$$\lim_{h\to 0}\left[1 - P(|W_h| \leq nh)\right] \geq \lim_{h\to 0}\left[1 - \sqrt{\frac{2hn}{\pi}}\right] = 1$$

$$\Rightarrow \lim_{h\to 0} 1 - P(|W_h| \leq nh) \geq 1.$$

□

4.2.2 W_t is not of Bounded First Order Variation

Definition A function $f : [0,T] \to \mathbb{R}$ has **bounded variation** if

$$BV(f) = \lim_{||\Delta||\to 0} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|$$

exists and is finite, where

$$\Delta = (t_0, t_1, \ldots, t_n = T)$$

is any partition of the interval $[0,T]$ and

$$||\Delta|| = \max_{i=1,\ldots,n} |t_i - t_{i-1}|.$$

Proof. To prove that the function $t \mapsto W_t$ does not have bounded variation, we can compute something related instead, given that $\Delta = (t_0, \dots, t_n)$ is a partition of $[0, T]$:

$$E\left[\sum_{i=1}^{n} |W_{t_i} - W_{t_{i-1}}|\right] = \sum_{i=1}^{n} E(|W_{t_i} - W_{t_{i-1}}|).$$

For if this is not bounded, then $BV(W_t)$ is surely not bounded. So let's compute the expectation for a single time interval

$$E(|W_{t_i} - W_{t_{i-1}}|) = \int_{-\infty}^{+\infty} |x| \frac{1}{\sqrt{2\pi\Delta t}} e^{-\frac{x^2}{2\Delta t}} dx$$

where $\Delta t = t_i - t_{i-1}$. Using the symmetry of the normal distribution, and then changing variables by letting $u = \frac{x^2}{2\Delta t}$ and $du = \frac{x}{\Delta t} dx$, we get

$$E(|W_{t_i} - W_{t_{i-1}}|) = 2\int_{0}^{+\infty} x \frac{1}{\sqrt{2\pi\Delta t}} e^{-\frac{x^2}{2\Delta t}} dx$$

$$= \int_{0}^{+\infty} \sqrt{\frac{2\Delta t}{\pi}} e^{-u} du = \left[-\sqrt{\frac{2\Delta t}{\pi}} e^{-u}\right]_{0}^{+\infty}$$

$$= \sqrt{\frac{2}{\pi}(t_i - t_{i-1})} = c\sqrt{t_i - t_{i-1}}$$

So for the expectation, we get

$$E\left[\sum_{i=1}^{n} |W_{t_i} - W_{t_{i-1}}|\right] = c\sum_{i=1}^{n} \sqrt{t_i - t_{i-1}}.$$

Recalling that $t_i = i \cdot (\frac{T}{n})$, we can simplify

$$E\left[\sum_{i=1}^{n} |W_{t_i} - W_{t_{i-1}}|\right] = c\sum_{i=1}^{n} \sqrt{\frac{T}{n}} = cn\sqrt{\frac{T}{n}} = c\sqrt{nT}.$$

Therefore,

$$\lim_{||\Delta|| \to 0} E\left[\sum_{i=1}^{n} |W_{t_i} - W_{t_{i-1}}|\right] = \lim_{n \to \infty} c\sqrt{nT} = \infty$$

And since

$$E\left[\sum_{i=1}^{n} |W_{t_i} - W_{t_{i-1}}|\right] \leq \sum_{i=1}^{n} |W_{t_{i-1}} - W_{t_i}|$$

we see that $BV(W_t)$ diverges to infinity with probability 1. $\square$

4.2.3 W_t has Bounded Quadratic Variation

I'll omit the proof because it's rather straightforward to show that the following converges to 0:

$$\lim_{||\Delta|| \to 0} E\left[\left(\sum_{i=1}^{n} |W_{t_i} - W_{t_{i-1}}|^2 - T\right)^2\right] = 0.$$

Definition A function $f : [0, t] \to \mathbb{R}$ has **bounded quadratic variation** if

$$BV^2(f) = \lim_{||\Delta|| \to 0} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^2 < \infty.$$

In fact, for Brownian Motion, with probability 1:

$$BV^2(W_t) = T.$$

4.2.4 Slogan of Itō Calculus

Finally, that last property says that

$$(dW_t)^2 = dt$$

and we can show that $(dW_t)^p = 0$ for all $p > 2$.

5 Application to Stock Process

Assume that W_t is the value of a stock at time t, and that ϕ_t is the number of shares held at time t. So the change in the value of our portfolio between time t_{i-1} and t_i is

$$\phi_{t_i}\left(W_{t_i} - W_{t_{i-1}}\right).$$

So the change in the value of the portfolio over the time interval $[0, T]$ is the **Martingale Transform**:

$$\sum_{i=1}^{n} \phi_{t_i}\left(W_{t_i} - W_{t_{i-1}}\right).$$

If we take the time intervals to be infinitesimal, then we get

$$\int_0^T \phi_t \, dW_t$$

which follows the form of the **Riemann-Stieltjes Integral**, which we can make sense of, provided that W_t has bounded second order variation (and we just showed it does).

6 Riemann-Stieltjes Integral

We'll want to review a few properties of the Riemann and Riemann-Stieltjes integrals before we define a similar animal, the Itō Integral.

Theorem (Riemann's Theorem) Let f be a function such that the probability of choosing a point of discontinuity is 0. Let P be a partition of the interval $[a, b]$:

$$P = \{t_1 = a, t_2, \ldots, t_{n-1}, t_n = b\}.$$

And define $||\Delta||$ as follows

$$||\Delta|| = max\{|t_i - t_{i-1}| \, |i = 1, \ldots, n\}.$$

Define the Riemann sum

$$RS(f, P) = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}), \;\; \xi_i \in (t_{i-1}, t_i).$$

Then the following expression exists and is independent of the choice of ξ_i within the respective intervals.

$$\lim_{||\Delta|| \to 0} RS(f, P) = \int_a^b f(x)dx$$

Definition We can further generalize the Riemann Integral into the **Riemann-Stieltjes Integral**, which is again a limit of sums. The sums are defined

$$RSS(f, P, \phi) = \sum_{i=1}^{n} f(\xi_i)(\phi(t_i) - \phi(t_{i-1})), \;\; \xi_i \in (t_{i-1}, t_i).$$

where ϕ is a *non-decreasing* function, and where P and ξ_i have their usual meaning. This leads to

$$\lim_{||\Delta|| \to 0} RSS(f, P, \phi) = \int_a^b f(t)\; d\phi(t).$$

Theorem If ϕ is differentiable, then

$$\int_a^b f(t)\; d\phi(t) = \int_a^b f(t)\phi'(t)\; dt.$$

7 Stochastic Integrals

Why did we go through these steps? Well, unsurprisingly, a **stochastic integral** is very much like a Riemann-Stieltjes integral. However, when X_t and ϕ_t are stochastic processes, the resulting integral

$$\int_a^b \phi_t\; dX_t$$

is a random variable.

We will like working with integrals because they allow us to solve stochastic differential equations. Moreover, integrals are more well-behaved than derivaves, so they will tend to smooth out bumps. For that reason, we'll often try to convert from differential equations to integral equations below.

8 The Itō Integral

The ultimate representation that we want for the **Itō Integral**,

$$\int_0^T \phi_t \, dW_t,$$

is the result of the limit of specific sums, which must first be defined.

8.1 Itō Sums

So let us define **Itō's Sums**, for stochastic processes ϕ_t and W_t as

$$IS(\phi, \Delta) = \sum_{i=1}^{n} \phi_{t_{i-1}}(W_{t_i} - W_{t_{i-1}}).$$

where $\Delta = (t_0, t_1, \ldots, t_n)$ is some partition of the time inteval, $[0, T]$. Note, the Itō Integral is a bit more general in that we could substitute any stochastic process, X_t, in for W_t. But the Wiener Process has some nice properties, so we'll stick to it.

Note Very important is the subscript on ϕ. Notice that we evaluate at the *left* endpoint of our chopped-up time intervals: t_{i-1}. This turns out to make a difference, because it means that the process, ϕ, doesn't overlap the difference in the Brownian Motion terms, allowing us to assert independence.

Moreover, when we do this, if W_t (or X_t more generally) is a Martingale, then the resulting Itō Sum will be as well. There's even an added benefit in that the choice of the left endpoint imposes what looks like a "previsibility condition"—which will be very useful for stock prices and portfolio determination.

8.2 Itō's Integral

We define the **Itō Integral** as the limit of of Itō Sums:

$$\lim_{||\Delta|| \to 0} IS(\phi, \Delta) = \lim_{||\Delta|| \to 0} \sum_{i=1}^{n} \phi_{t_{i-1}}(W_{t_i} - W_{t_{i-1}}) = \int_0^T \phi_t \, dW_t$$

Theorem 4.3.7 If ϕ_s is adapted to W_s (definition below), then

$$\int_a^t \phi_s \, (dW_s)^2 = \int_a^t \phi_s \, ds.$$

In addition, for $n > 2$,

$$\int_a^t \phi_s \, (dW_s)^2 = 0.$$

8.3 Properties of the Itō Integral

Given processes ϕ_s and ψ_s which are adapted (either to X_s or W_s), we obtain the following results:

1. $\int_a^t dX_s = X_t - X_a$.
2. $\int_a^t (\phi_s + \psi_s) \, dX_s = \int_a^t \phi_s \, dX_s + \int_a^t \psi_s \, dX_s$.
3. $E\left[\int_a^t \phi_s \, dW_s\right] = 0$.
4. $E\left[\left(\int_a^t \phi_s \, dW_s\right)^2\right] = E\left[\int_a^t \phi_s^2 \, dW_s\right]$.

8.4 Existence Conditions

For the limit to exist—and, therefore, the Itō Integral—we must satisfy certain conditions.

(i) ϕ_t is *adapted* to W_t.

(ii) ϕ_t is *square-integrable*.

Definition We say that a process, ϕ_t is **adapted** to W_t if, for any $s > t$, ϕ_t is independent of $W_s - W_t$.

Definition We say that a random variable process, ϕ_t is **square-integrable** if, for any $T > 0$,

$$E\left[\int_0^T |\phi_t|^2 \, dt\right] < \infty.$$

This is equivalent to checking that the first and second moments are finite. And as a side note, you can take the expectation operater inside the integral because $|\phi_t|^2$ is a positive random variable.

8.5 Mode of Convergence

If ϕ_t satisfies the two conditions above, then the sequence of sums $IS(\phi, \Delta)$ will converge as $||\Delta|| \to 0$ to a random variable, denoted by the Itō integral of ϕ_t against W_t. What's more, it so happens that the Itō Sums converge to the Itō Integral in mean square.

Definition We say that a sequence of random variables, $\{X_n | n \in \mathbb{N}\}$ converges to a random variable X in **mean-square** if

$$\lim_{n\to\infty} E\left[|X_n - X|^2\right] = 0,$$

which, as can be seen upon expanding, checks that $EX_n \to EX$ and $Var(X_n) \to Var(X)$—i.e. that the the two random variables are the same. In the case of the Itō Integral, this means that

$$\lim_{||\Delta||\to 0} E\left[\left|IS(\phi, \Delta) - \int_0^T \phi_t dW_t\right|^2\right] = 0.$$

8.6 Useful Examples

1. In the derivation of the following result, one uses the property of convergence in mean square to obtain:
$$\int_0^T W_t\, dW_t = \frac{1}{2}W_T^2 - \frac{1}{2}T$$

2. Follows because it's a telescoping sum:
$$\int_a^b dX_t = X_b - X_a$$
$$\int_0^T dW_t = W_T - W_0 = W_T$$

3. This next result follows from the fact that $M_t = \int_0^T \phi_s dW_s$ is a martingale:
$$E\left[\int_0^T \phi_t\, dW_t\right] = 0$$

4. The second order variation comes into play here:
$$E\left[\left(\int_0^T \phi_t\, dW_t\right)^2\right] = E\left[\int_0^T \phi_t^2\, dW_t\right]$$

9 Stochastic Differential Equations

Our models for stock prices will be solutions to **stochastic differential equations**, which are expressions of the form

$$dX_t = a(X_t, t)\, dt + b(X_t, t)\, dW_t. \tag{7}$$

A stochastic process X_t that solves this SDE will be of the form

$$X_t - X_a = \int_a^t dX_s = \int_a^t a(X_s, s)\, ds + b(X_s, s)\, W_s$$

Now as a word of caution, such equations don't always have solutions, and if they do, there's no guarantee that the solution is unique, unless we put some assumptions on the coefficients. Respectively, the coefficients $a(x, s)$ and $b(x, s)$ are functions for the *drift* coefficient and the *diffusion* coefficient.

9.1 Itō's Lemma

Lemma Let $f : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be a function that is twice differentiable in x, and once differentiable in t. Also, let X_t be a soltuion to the equation

$$dX_t = a(X_t, t)\, dt + b(X_t, t)\, dW_t.$$

Then $f(X_t, t)$ is a stochastic precess that satisfies the SDE

$$df(X_t, t) = \left[f_t(X_t, t) + a(X_t, t) \cdot f_x(X_t, t) + \frac{1}{2}\, b^2(X_t, t) \cdot f_{xx}(X_t, t) \right] dt$$
$$+ b(X_t, t) \cdot f_x(X_t, t)\, dW_t.$$

9.2 Solving SDE Problems

Stochastic diffeernital equations offer two possibilities:

1. Solving stochastic differential equations that are given by finding a function for the variable X_t of interest.
2. Solving stochastic integrals by transforming the problem into an SDE and solving an easier expression

We'll consider both of those now—albeit in the reverse order from how they were stated.

9.2.1 Solving Stochastic Integrals with SDE's

Suppose that you're given a stochastic integral, $\int \phi_t dW_t$, to compute. How can you simplify?

1. The ultimate goal is to find a function $f(x)$ such that
$$\phi_t dW_t = d(f(\phi_t)).$$
If we have that function f, we can use the properties of the Itō Integral to compute easily
$$\int_a^b \phi_t \, dW_t = \int_a^b d(f(\phi_t)) = f(\phi_b) - f(\phi_a).$$

2. But first, we have to find the SDE that this corresponds to. Since we're applying Itō's Lemma by finding $d(f(\phi_t))$, we should really find the parameters such that
$$d\phi_t = a(\phi_t, t) \, dt + b(\phi_t, t) \, dW_t. \tag{8}$$
Since we know ϕ and we dt and dW_t are already in there, this amounts to finding the values $a(\phi_t, t)$ and $b(\phi_t, t)$ that make Equation 8 hold.

3. Once we have $a(\phi_t, t)$, $b(\phi_t, t)$, and a suitable guess for $f(x)$, we can apply Itō's Lemma. Hopefully, the original integral, $\int \phi_t dW_t$, is in the derived equation, and everything else is simple enough for us to solve.

Example 1 We hope to solve $\int_a^t W_s dW_s$. I will solve this problem in steps mirroring the numbered steps listed above.

1. Since our integral looks somewhat like $\int f(x)dx$, an educated guess would be the equation $f(x) = \frac{1}{2}x^2$, which is the integral of the traditional calculus problem.

2. Next, we realize that $\phi_t = W_t$ in our problem, and so set up the corresponding, implicit SDE:
$$dW_t = a(W_t, t)dt + b(W_t, t)dW_t.$$
The only values for a and b that make this hold are $a = 0$ and $b = 1$.

3. Next, we apply Itō's Lemma, which simplifies to
$$d(f(W_t)) = \frac{1}{2} \, dt + W_t \, dW_t.$$
From there, we can shift around to get our original problem
$$W_t \, dW_t = d(f(W_t)) - \frac{1}{2} \, dt$$
and integrate to solve the SDE:
$$\int_a^t W_t \, dW_t = \int_a^t d(f(W_t)) - \frac{1}{2} \, dt$$
$$\int_a^t W_t \, dW_t = \frac{1}{2} \, W_t^2 - \frac{1}{2} \, W_a^2 - \frac{1}{2} \, (t - a).$$

Example 2 We hope to solve $\int_a^t W_s^2 \, dW_s$. I will solve this problem in steps mirroring the numbered steps listed above.

1. Since our integral looks somewhat like $\int f^2 dx$, an educated guess would be the equation $f(x) = \frac{1}{3}x^3$, which is the integral of the traditional calculus problem.

2. Need help with this.

9.2.2 Solving SDE's with Integrals

The general method involves finding a suitable function to use in applying Ito's Lemma. Then solve this easier problem.

9.3 Geometric Brownian Motion

The basic stochastic differential equation representation is

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t$$

We start by considering a function we can use in order to apply Itō's Lemma:

$$f(x) = \ln(x)$$

$$f_t = 0, \quad f_x = \frac{1}{x}, \quad f_{xx} = -\frac{1}{x^2}$$

Using the Lemma, we ccan write

$$d(\ln(X_t)) = \left[0 + \mu X_t \cdot \frac{1}{X_t} - \frac{1}{2} \, \sigma^2 X_t^2 \cdot \frac{1}{X_t^2} \right] \, dt + \sigma \frac{1}{X_t} X_t \, dW_t$$

$$d(\ln(X_t)) = \left(\mu - \frac{1}{2}\sigma^2 \right) dt + \sigma dW_t$$

We can now solve this easily for X_t since all terms on the righthand side are known, and independentof X_t:

$$\int_0^t d(\ln(X_s)) \, ds = \int_0^t \left(\mu - \frac{1}{2}\sigma^2 \right) ds + \sigma dW_s$$

$$\ln(X_t) - ln(X_0) = \left(\mu - \frac{1}{2}\sigma^2 \right) t - 0 + \sigma W_t - \sigma W_0$$

$$\ln(X_t) = ln(X_0) + \left(\mu - \frac{1}{2}\sigma^2 \right) t + \sigma W_t$$

$$X_t = X_0 \, e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}$$

9.4 Ornstein-Uhlenbeck Process

This process is defined by the SDE

$$dX_t = \mu X_t \; dt + \sigma \; dW_t$$

This case differs from geometric Brownian motion in that there is no X_t in front of dW_t.

For this case, choose $f(x,t) = xe^{-\mu t}$. Applying Itō's Lemma gives us

$$d(X_t e^{-\mu t}) = \left(-X_t \; \mu e^{-\mu t} + X_t \mu e^{-\mu t}\right) + \sigma e^{-\mu t} dW_t$$

$$d(X_t e^{-\mu t}) = \sigma e^{-\mu t} dW_t$$

Integrating this expression yields

$$\int_0^t d(X_s e^{-\mu s}) = \int_0^t \sigma e^{-\mu t} dW_t$$

$$X_t e^{-\mu t} - X_0 e^{-\mu 0} = \int_0^t \sigma e^{-\mu t} dW_t$$

$$X_t = X_0 e^{\mu t} + \int_0^t \sigma e^{-\mu(t-s)} dW_t$$

9.5 Cox-Ingersoll-Ross Model

This model, which exhhibits mean reversion and volatility that grows with r_t is defined

$$dr_t = a(b - r_t) \; dt + \sigma \sqrt{r_t} \; dW_t.$$

9.6 Vasicek Model

This interest rate model has mean reversion, where b is the mean. It's defined by the following SDE:

$$dr_t = a(b - r_t) \; dt + \sigma \; dW_t.$$