Gamma Distribution and Special Cases

1 General Gamma Function

The gamma function has many special cases, but let's consider it in its most general form, along with the mean and variance:

$$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \qquad x > 0 \tag{1}$$

$$\text{Mean} = \frac{\alpha}{\beta} \qquad \text{Variance} = \frac{\alpha}{\beta^2}$$

Gamma random variables also have the nice property that

$$X_i \sim \text{Gamma}(\alpha_i,\ \beta) \quad \Rightarrow \quad \sum_{i=1}^{n} X_i \sim \text{Gamma}\left(\sum_{i=1}^{n} \alpha_i,\ \beta\right)$$

2 Special Case: Exponential Random Variable

This family has $\alpha = 1$ with β unrestricted. Changing the way we denote the variables (set $\beta = \theta$), this simplifies to

$$f_X(x) = \theta\, e^{-\theta}$$

This is particularly useful for waiting times, as it is a memoryless distribution. In addition, as it is technically a Gamma random variable, sums of exponential variables will have a Gamma distribution:

$$X_i \sim \text{Exp}(\theta) \quad \Rightarrow \quad \sum_{i=1}^{n} X_i \sim \text{Gamma}(n, \theta)$$

Another attraction is the intimate connection this distribution has with the Poisson process. Namely, suppose $N(t)$ is a Poisson process with parameter θ, in which case $N(t)$ is Poisson distributed with parameter θt, then the interarrival times are exponentially distributed with parameter θ.

3 Special Case: Chi-Squared Random Variable

3.1 Density Function and Descriptive Statistics

Suppose that Y has a χ^2 distribution with ν degrees of freedom. Then the distribution is a special case of the gamma distribution, where $\alpha = \nu/2$ and $\beta = 1/2$ with the pdf and descriptive statistics

$$f_Y(y) = \frac{1}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)} x^{\frac{\nu}{2}-1} e^{-\frac{x}{2}} \tag{2}$$

$$\text{Mean} = \nu, \qquad \text{Variance} = 2\nu$$

3.2 Properties

The χ^2 distribution has several nice properties that we'll want to discuss:

- If Y is chi-squared distributed with ν degrees of freedom ($Y \sim \chi^2(\nu)$), then we can generate Y by

$$Y = \sum_{i=1}^{\nu} Z_i^2$$

  where Z_i is a standard, normal random variable.

- By the way we just defined a χ^2 RV, it's immediately clear that

$$Y_i \sim \chi^2(\nu_i) \quad \Rightarrow \quad \sum_{i=1}^{n} Y_i \sim \chi^2\left(\sum_{i=1}^{n} \nu_i\right)$$

  So sums of χ^2 RVs are also χ^2 distributed.

- Suppose that $Y_i \sim \text{NID}(\mu, \sigma^2)$. Then

$$\sum_{i=1}^{n} \frac{(Y_i - \mu)^2}{\sigma^2} \sim \chi^2(n)$$

- Now let's consider a similar situation, but where only the sample mean is avalable.

$$\begin{aligned}
\sum_{i=1}^{n} \frac{(Y_i - \mu)^2}{\sigma^2} &= \frac{1}{\sigma^2} \sum_{i=1}^{n} (Y_i - \bar{Y} + \bar{Y} - \mu)^2 \\
&= \frac{1}{\sigma^2} \sum_{i=1}^{n} (Y_i - \bar{Y})^2 + \frac{2}{\sigma^2} \sum_{i=1}^{n} (Y_i - \bar{Y})(\bar{Y} - \mu) + \frac{1}{\sigma^2} \sum_{i=1}^{n} (\bar{Y} - \mu)^2 \\
&= \frac{1}{\sigma^2} \sum_{i=1}^{n} (Y_i - \bar{Y})^2 + \frac{2(\bar{Y} - \mu)}{\sigma^2} \sum_{i=1}^{n} (Y_i - \bar{Y}) + \frac{n(\bar{Y} - \mu)^2}{\sigma^2} \\
&= \sum_{i=1}^{n} \frac{(Y_i - \bar{Y})^2}{\sigma^2} + \left(\frac{\bar{Y} - \mu}{\sigma/\sqrt{n}}\right)^2
\end{aligned}$$

  Now it's clear that the left hand side is a sum of squared normal random variables, so it has a $\chi^2(n)$ distribution. On the right hand side, the second term has a $\chi^2(1)$ distribution, as it is a squared normal random variable. We can rearrange to assert that

$$\sum_{i=1}^{n} \frac{(Y_i - \bar{Y})^2}{\sigma^2} \sim \chi^2(n-1)$$