Homework 2

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1. **Exercise 51.17, FoMA**: We consider f, a positive continuous function where $f \in \mathscr{R}([a,b])$. Letting
$$M = \max_{x \in [a,b]} f(x)$$
we want to prove
$$M = \lim_{n\to\infty} \left[\int_a^b [f(x)]^n \; dx \right]^{1/n} \tag{1}$$
Also, let x^* be the x value such that $f(x^*) = M$. By the intermediate value theorem, we know such an x^* exists.

   Now because f is continuous on a compact interval, f is bounded, implying that M does exist. And since f is positive, $M > 0$ as well. Given that, we can show an equivalent statement to Equation 1 by dividing through by M:
$$1 = \lim_{n\to\infty} \left[\frac{1}{M^n} \int_a^b [f(x)]^n \; dx \right]^{1/n}$$
$$1 = \lim_{n\to\infty} \left[\int_a^b \left[\frac{f(x)}{M} \right]^n \; dx \right]^{1/n} \tag{2}$$
This will be our equivalent statement to prove, rather than Equation 1.

   So first, we know that that the integral in Equation 2 over the function h exists (i.e. $h \in \mathscr{R}([a,b])$—where
$$h(x) = \left[\frac{f(x)}{M} \right]^n \quad \text{with} \quad \begin{cases} h(x) = 1 & x = x^* \\ 0 < h(x) < 1 & x \neq x^* \end{cases}$$
This integral exists because both $f(x)$ is continuous and $g(x) = (x/M)^n$ is continuous. And because $h = g \circ f$, a composition of continuous functions, $h \in \mathscr{R}_\alpha([a,b])$ by Theorem 51.12 in FoMA.

Now because $h \in \mathscr{R}_\alpha([a,b])$, we can write

$$\begin{aligned}\int_a^b \left[\frac{f(x)}{M}\right]^n \, dx = \overline{\int_a^b} \left[\frac{f(x)}{M}\right]^n \, dx &= \inf_P U(h,P) \\ &= \sum_{i=1}^m M_i(h)\Delta x_i\end{aligned}$$

But over all intervals, $M_i(h)$ will be greater than 0 (since $h(x)$ is positive), but less than or equal to one, as mentioned above. Thus, we can put bounds on the integral by putting bounds on the sum:

$$\begin{aligned}&\sum_{i=1}^m 0\Delta x_i < \sum_{i=1}^m M_i(h)\Delta x_i \leq \sum_{i=1}^m 1\Delta x_i \\ \Leftrightarrow \qquad &\epsilon \leq \sum_{i=1}^m M_i(h)\Delta x_i \leq b-a \\ \Rightarrow \qquad &\epsilon \leq \int_a^b h(x)\, dx \leq b-a\end{aligned}$$

for some small, non-zero value ϵ. The value isn't strictly crucial—only that it's non-zero.

Now using these inequalities, we can squeeze the limit in Equation 2:

$$\begin{aligned}\epsilon^{1/n} &\leq \left[\int_a^b h(x)\, dx\right]^{1/n} \leq (b-a)^{1/n} \\ \lim_{n\to\infty} \epsilon^{1/n} &\leq \lim_{n\to\infty} \left[\int_a^b h(x)\, dx\right]^{1/n} \leq \lim_{n\to\infty} (b-a)^{1/n} \\ 1 &\leq \lim_{n\to\infty} \left[\int_a^b h(x)\, dx\right]^{1/n} \leq 1 \\ \Rightarrow \qquad 1 &= \lim_{n\to\infty} \left[\int_a^b h(x)\, dx\right]^{1/n}\end{aligned}$$

which is what we wanted to prove.

2. **Exercise 52.1, FoMA**: We want to prove, for $f \in \mathscr{R}([0,1])$ that

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(k/n)\, \frac{1}{n} = \int_0^1 f(x)\, dx \tag{3}$$

Note that we're taking $\alpha(x) = x$, a continuous function.

We construct the proof by treating the lefthand side as a Riemman Sum. Namely, we construct a partition

$$P = \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, 1\right\} \qquad \Rightarrow \quad \Delta x_i = \frac{1}{n}$$

We also consider the evaluation points within each interval

$$T = \left\{\frac{1}{n}, \frac{2}{n}, \dots, 1\right\} \tag{4}$$

Combining our partition and evaluation points into one sum, we have a Riemann Sum

$$S(f, P, T) = \sum_{k=1}^{n} f(t_i)\Delta x_i = \sum_{k=1}^{n} f(k/n)\, \frac{1}{n}$$

which is exactly the lefthand side of Equation 3. Now, we can invoke Theorem 52.5 in FoMA, as we satisfy the conditions that

- $f \in \mathscr{R}([0,1])$, which is assumed.
- α continuous, as $\alpha(x) = x$.

Thus, we can assert, since $||P|| = 1/n$, which goes to zero as n grows, that

$$\begin{aligned}\int_0^1 f dx &= \lim_{||P||\to 0} S(f, P, T) = \lim_{||P||\to 0} \sum_{k=1}^{n} f(t_i)\Delta x_i \\ &= \sum_{k=1}^{n} f(k/n)\, \frac{1}{n}\end{aligned}$$

which is exactly what we wanted to show.

3. Recall the Lipshitz Condition: A function f is Lipshitz at x if for some $C, \delta > 0$

$$|x-y| \leq \delta \quad \Rightarrow \quad |f(x)-f(y)| \leq C|x-y| \tag{5}$$

We want to show that for $f : [0,1] \to \mathbb{R}$

$$\left| \int_0^1 f \; dx - \frac{1}{n} \sum_{k=1}^{n} f(k/n) \right| \leq \frac{C}{n} \qquad \forall n \tag{6}$$

Now let's define our partition, which we'll use throughout the problem:

$$P = \left\{ 0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n} \right\} \qquad \Rightarrow \qquad \Delta x_i = \frac{1}{n} \tag{7}$$

We next use the modulus of continuity result from HW1 that

$$U(f,P) - L(f,P) \leq w_f\left(||P||\right)(b-a) \tag{8}$$

Given that f is Lipshitz, we see

$$w_f\left(||P||\right) = \sup_{|x-y| \leq ||P||} |f(x)-f(y)| \leq C \cdot \frac{1}{n}$$

Substituting into Equation 8 for this particular function, f:

$$U(f,P) - L(f,P) \leq \frac{C}{n}$$

Now clearly, we must have

$$L(f,P) \leq \int_0^1 f \; dx \leq U(f,P) \tag{9}$$

since the integral must lie between the upper and lower sums always.

Next, the evaluation points k/n for each interval must fall in between the sup and the inf on the interval, which implies (by our choice of partition)

$$\begin{aligned} \sum_{k=1}^{n} m_k(f)\frac{1}{n} \leq \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\frac{1}{n} \leq \sum_{k=1}^{n} M_k(f)\frac{1}{n} \\ L(f,P) \leq \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \leq U(f,P) \end{aligned} \tag{10}$$

Now combining Inequalties 9 and 10, it's clear that we must have

$$\left| \int_0^1 f \; dx - \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \right| \leq U(f,P) - L(f,P) \leq \frac{C}{n}$$

since both terms on the left lie in between the upper and lower sums.