Analysis

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1 The Riemann-Stieltjes Integral

This definition of the integral was made rigorous in the 1800s by Riemann, Darboux, and Stieltjes. It's an intuitive way to define the area under a curve, and it works well with numerical integration (approximations). *However*, it is incomplete in the sense that there are functions of interest that we cannot integrate in a Riemann sense but can in a Lebesgue sense.

Throughout this section, we'll stick to functions that are univariate from a compact interval to $\mathbb{R}$:

$$f : [a,b] \to \mathbb{R}$$

We'll begin by discussing *partitions* of that interval $[a,b]$ into smaller pieces, from which we'll construct sums that approximate the area under the curve. This will lead us to a definition of the Riemann Integral. Then, we'll generalize and allow the *weight* we place on the sub-intervals (when summing over the entire interval) to vary, which will give us the Riemann-Stieltjes integral. From there, we discuss the relationships between the approximating sums and the integral.

1.1 Partitions

Definition 1.1. A *partition*, P, is an ordered tuple representing a finite sequence on the interval $[a, b]$,

$$a = x_0 < x_1 < \cdots < x_n = b \qquad \text{with} \quad \Delta x_i := x_i - x_{i-1}$$

Definition 1.2. The *norm* of a partition P, sometimes called "mesh P" represents

$$||P|| = \text{norm}(P) := \max_i |x_i - x_{i-1}| = \max_i |\Delta x_i|$$

Definition 1.3. Q is a *refinement* of P if $Q \supset P$ where Q and P are both partitions of $[a, b]$. Q the intervals *finer*.

Definition 1.4. For two partitions, P_1 and P_2, their *common refinement* is $P_1 \cup P_2$.

Definition 1.5. A *tagged partition* is a couplet (P, T), where P is some partition $\{x_0, \ldots, x_n\}$ and T is a set of evaluation points, $\{t_1, \ldots, t_n\}$, for the function f such that

$$x_{i-1} \leq t_i \leq x_i$$

Note. We will now generalize to allow weighting of the sub-intervals within the partition, defined for an *increasing* function $\alpha : [a, b] \to \mathbb{R}$, where

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) > 0$$

This is the main difference between the plain Riemann sum and integral, versus the Riemann-Stieltjes (RS) sum and integral. The latter retains the former as a special case by taking $\alpha(x) = x$. Therefore, the RS version is just a generalization of Riemann, weighting the contribution of the sub-intervals to the total sum/integral by the function α, *not* by the length of the sub-interval.

1.2 Sum Definitions

We now define the various sums approximating the Riemann and RS integrals.

Definition 1.6. We define the upper and lower *Darboux Sums*, respectively, as follows

$$U(f, P) := \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}) \quad \text{where} \quad M_i(f) := \sup_{x \in [x_i, x_{i-1}]} f(x)$$

$$L(f, P) := \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}) \quad \text{where} \quad m_i(f) := \inf_{x \in [x_i, x_{i-1}]} f(x)$$

Definition 1.7. Given f (bounded) and tagged partition (P, T) we define the *Riemann Sum* as

$$S(f, P, T) := \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \tag{1}$$

Definition 1.8. We define the upper and lower *RS-Darboux Sums*, respectively, as follows

$$U_\alpha(f,P) := \sum_{i=1}^{n} M_i(f)\Delta\alpha_i \quad \text{where} \quad M_i(f) := \sup_{x\in[x_i,x_{i-1}]} f(x)$$
$$L_\alpha(f,P) := \sum_{i=1}^{n} m_i(f)\Delta\alpha_i \quad \text{where} \quad m_i(f) := \inf_{x\in[x_i,x_{i-1}]} f(x)$$

Definition 1.9. Given f (bounded) and tagged partition (P,T) we define the *Riemann-Stieltjes Sum* as

$$S_\alpha(f,P,T) := \sum_{i=1}^{n} f(t_i)\Delta\alpha_i \tag{2}$$

1.3 Sum Relations

Remark. Clearly, by Definitions 1.8 and 1.9, for all T associated with P

$$L_\alpha(f,P) \le S_\alpha(f,P,T) \le U_\alpha(f,P)$$

Theorem 1.10. *If $Q \supset P$, i.e. if Q refines P, then*

$$L_\alpha(f,P) \le L_\alpha(f,Q) \le U_\alpha(f,Q) \le U_\alpha(f,P)$$

Proof. The proof proceeds by induction. Assume that $Q = P \cup \{x^*\}$, a single point. Then $x^* \in [x_{i-1}, x_i]$ for some interval, and it's easy show the relation from there. □

Theorem 1.11. *For all partitions P_1, P_2,*

$$L_\alpha(f,P_1) \le U_\alpha(f,P_2)$$

Proof. Let $Q = P_1 \cup P_2$. Then by Theorem 1.10,

$$L_\alpha(f,P_1) \le L_\alpha(f,Q) \le U_\alpha(f,Q) \le U_\alpha(f,P_2)$$

□

1.4 Integrability and $\mathscr{R}_\alpha([a,b])$

Definition 1.12. We define *the upper and lower Riemann-Stieltjes integrals*, respectively, in terms of the RS-Darboux Sums

$$\overline{\int_a^b} f\,d\alpha := \inf_P U_\alpha(f,P)$$
$$\underline{\int_a^b} f\,d\alpha := \sup_P L_\alpha(f,P)$$

From Theorem 1.11, it's clear that $\underline{\int} f\,d\alpha \le \overline{\int} f\,d\alpha$.

Definition 1.13. We say f is *Riemann-Stieltjes integrable* on $[a,b]$—i.e. $f \in \mathscr{R}_\alpha([a,b])$—if

$$\overline{\int_a^b} f d\alpha = \underline{\int_a^b} f d\alpha := \int_a^b f d\alpha$$

Example 1.14. A case where $f \notin \mathscr{R}_\alpha([a,b])$ is where

$$f(x) = \begin{cases} 1 & \text{x rational} \\ 0 & \text{x irrational} \end{cases}$$

for $x \in [0,1]$. In this case, the upper integral is always 1, while the lower integral is always zero.

Theorem 1.15. (Riemann's Condition) $f \in \mathscr{R}_\alpha([a,b])$ *if and only if there exists a partition* P *such that the upper and lower RS-Darboux sums can be made arbitrarily close given that* P*, i.e.*

$$U_\alpha(f,P) - L_\alpha(f,P) \leq \varepsilon$$

Proof. First, the $\Leftarrow$ direction. Use Theorems 1.10 and 1.11. It's obvious. Next, for the $\Rightarrow$ direction. By the definition of the RS integral and the RS-Darboux sums,

$$U_\alpha(f,P_1) < \int_a^b f d\alpha + \varepsilon/2 \qquad L_\alpha(f,P_2) < \int_a^b f d\alpha + \varepsilon/2 \tag{3}$$

Taking the common refinement, and using Theorem 1.10, we get that

$$\begin{aligned} U_\alpha(f, P_1 \cup P_2) - L_\alpha(f, P_1 \cup P_2) &\leq U_\alpha(f,P_1) - L_\alpha(f,P_2) \\ &= \left(U_\alpha(f,P_1) - \int_a^b f d\alpha\right) - \left(L_\alpha(f,P_2) - \int_a^b f d\alpha\right) \\ \text{By Expression 3} \qquad &\leq \varepsilon/2 + \varepsilon/2 \end{aligned}$$

□

Theorem 1.16. *The set of all continuous functions on* $[a,b]$*, denoted* $C([a,b])$*, is a subset of* $\mathscr{R}([a,b])$.

Proof. By Theorem 1.15, we want to show that, for all $\epsilon > 0$, there exists a partition P such that

$$U_\alpha(f,P) - L_\alpha(f,P) < \epsilon$$
$$\Leftrightarrow \sum_{i=1}^{n} (M_i(f) - m_i(f))\Delta\alpha_i < \epsilon$$

Now since f is continuous on a compact interval, $[a, b]$, f is *uniformly continuous* on $[a, b]$. That means, given our ϵ from above,

$$\exists\, \delta > 0 \quad \text{s.t.} \quad |x_i - x_{i-1}| < \delta \quad \Rightarrow \quad |f(x_i) - f(x_{i-1})| < \frac{\epsilon}{\alpha(b) - \alpha(a)}$$

So we can choose P such that that $||P|| < \delta$. This means that

$$\begin{aligned}\sum_{i=1}^{n}[M_i(f) - m_i(f)]\Delta\alpha_i &\leq \sum_{i=1}^{n} \frac{\epsilon}{\alpha(b) - \alpha(a)}\Delta\alpha_i = \frac{\epsilon}{\alpha(b) - \alpha(a)} \sum_{i=1}^{n} \Delta\alpha_i \\ &\leq \frac{\epsilon}{\alpha(b) - \alpha(a)} \cdot [\alpha(b) - \alpha(a)] = \epsilon\end{aligned}$$

□

1.5 Properties of $\mathscr{R}_\alpha([a, b])$

Now for some useful properties of the set of Riemann-Stieltjes integrable functions. Consider $f, g \in \mathscr{R}_\alpha([a, b])$ and $c \in \mathbb{R}$.

- **Linearity**: $f + g \in \mathscr{R}_\alpha([a, b])$ and $cf \in \mathscr{R}_\alpha([a, b])$, with

$$\int_a^b cf d\alpha = c\int_a^b f d\alpha \qquad \text{and} \qquad \int_a^b f + g d\alpha = \int_a^b f d\alpha + \int_a^b g d\alpha$$

- **Subsets**: If $[c, d] \subset [a, b]$, then $f \in \mathscr{R}_\alpha([c, d])$.

- **Splitting the Interval**: If $c \in [a, b]$, then

$$\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$$

- **Monotonicity**: If $f, g, h \in \mathscr{R}_\alpha([a, b])$, $f \geq 0$, and $g \leq h$ on $[a, b]$, then

$$\int_a^b f d\alpha \geq 0 \qquad \int_a^b g\, d\alpha \leq \int_a^b h\, d\alpha$$

- **Absolute Value Relations**: If we have $f \in \mathscr{R}_\alpha([a, b])$, then both

$$|f| \in \mathscr{R}_\alpha([a, b]) \qquad \left|\int_a^b f d\alpha\right| \leq \int_a^b |f|\, d\alpha$$

- **Compositions**: Suppose that $f \in \mathscr{R}_\alpha([a, b])$ and $g : \mathbb{R} \to \mathbb{R}$ is continuous. Then $g \circ f \in \mathscr{R}_\alpha([a, b])$

Proof. Let $m := \inf_{[a,b]} f$ and $M := \sup_{[a,b]} f$. Then since g is continuous on the on the compact interval $[m, M]$, g is uniformly continuous. As a result, for all $\epsilon > 0$ there then exists $\delta > 0$ such that

$$|u - v| \leq \delta \quad \Rightarrow \quad |g(u) - g(v)| \leq \epsilon \tag{4}$$

We also know that because $f \in \mathscr{R}_\alpha([a, b])$, wich implies that there exists a partion P such that

$$U_\alpha(f, P) - L_\alpha(f, P) \leq \epsilon \cdot \delta \tag{5}$$

As a result, for any i,

$$M_i(f) - m_i(f) < \delta \quad \Rightarrow \quad M_i(g \circ f) - m_i(g \circ f) \leq \epsilon \tag{6}$$

Now we're interested in showing Riemann integrability for $g \circ f$, which requires us to show that the upper and lower RS-Darboux sums are arbitrarily close. We will do so by breaking the sums into two parts based on the following characteristics of the original function f:

$$\begin{aligned} A &= \{i \mid M_i(f) - m_i(f) \leq \delta\} \\ B &= \{i \mid M_i(f) - m_i(f) > \delta\} \end{aligned}$$

Now let's work out the sums for $g \circ f$:

$$\begin{aligned} U_\alpha(g \circ f, P) - L_\alpha(g \circ f, P) &= \sum_{i=1}^{n} \left[M_i(g \circ f) - m_i(g \circ f)\right] \Delta\alpha_i \\ &= \sum_{i \in A} + \sum_{i \in B} \\ &\leq \delta \sum_{i : M_i(f) - m_i(f) \leq \delta} + \sum_{i : M_i(f) - m_i(f) > \delta} \end{aligned}$$

$\square$

A Additional Definitions

Modulus of Continuity