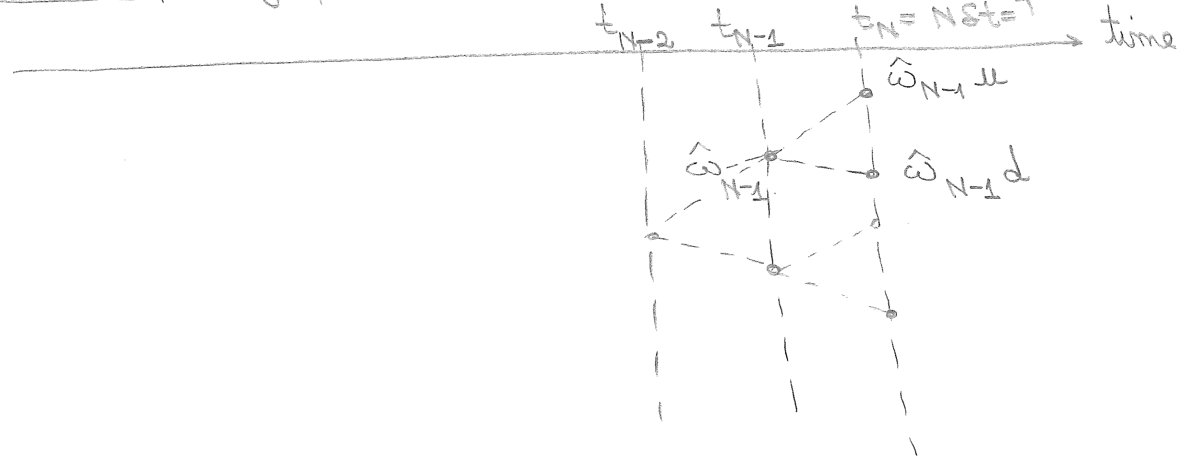
Risk-neutral pricing formula in the binomial model:



In the 1-period model, we defined the risk-neutral probabilities:

$$q = \frac{e^{r\delta t} S_0 - S_1^d}{S_1^u - S_1^d} = \frac{e^{r\delta t} - d}{u-d} = \text{prob. of an "up" move}$$

$$1-q = \frac{S_1^u - e^{r\delta t} S_0}{S_1^u - S_1^d} = \frac{u - e^{r\delta t}}{u-d} = \text{prob. of a "down" move}$$

(see formula (3) on page 20 of the hand-written lecture notes)

In the 1-period model, if V_1 is the payoff of a derivative security at $t_1 = \delta t$, we have shown that the value at t_0 of the derivative security is:

$$V_0 = e^{-r\delta t} \cdot \mathbb{E}_{\mathbb{Q}}[V_1] = e^{-r\delta t}\left(q V_1^u + (1-q) V_1^d\right)$$

(see formula (4) on page 21 of the hand-written lecture notes).

We apply these formulas for the binomial tree:

For the 1-period model corresponding to $[t_{N-1}, t_N]$, we obtain

$$V_{N-1}(\omega) = V_{N-1}(\hat{\omega}_{N-1}) = e^{-r\delta t}\left[q \cdot V_N(\hat{\omega}_{N-1}u) + (1-q) V_N(\hat{\omega}_{N-1}d)\right]. \quad (1)$$

Next, for the 1-period model corresponding to $[t_{N-1}, t_N]$, we obtain

$$(2) \quad V_{N-2}(\omega) = e^{-r\delta t}\left[q V_{N-1}(\hat{\omega}_{N-2}u) + (1-q) V_{N-1}(\hat{\omega}_{N-2}d)\right]$$

$$\underset{\text{using (1)}}{=} e^{-2r\delta t} \cdot \left[q^2 V_N(\hat{\omega}_{N-2}uu) + q(1-q) V_N(\hat{\omega}_{N-2}ud) + (1-q) q V_N(\hat{\omega}_{N-2}du) + (1-q)^2 V_N(\hat{\omega}_{N-2}dd)\right]$$

Iterating this formula we obtain, for all $i = 0, 1, \dots, N-1$,

$$(3) \quad V_i(\omega) = V_i(\hat{\omega}_i) = e^{-(N-i)r \cdot \delta t} \cdot \sum_{\substack{\xi = (\xi_{i+1}, \dots, \xi_N) \\ \xi_{i+1}, \dots, \xi_N \in \{u, d\}}} q^{\mathcal{J}(\xi)} (1-q)^{N-i-\mathcal{J}(\xi)} V_N(\hat{\omega}_i \xi)$$

where $\mathcal{J}(\xi) = \#$ of u in $\xi = (\xi_{i+1}, \dots, \xi_N)$.

[Example : $i = N-2$, then $\xi = (\xi_{N-1}, \xi_N) \in \{u, d\}$, and the preceding formula (3) coincides with (2).]

In particular, when $i = 0$, we obtain the price at t_0 of the derivative security

$$(4) \quad V_0 = e^{-Nr\delta t} \sum_{\xi = (\xi_1, \dots, \xi_N) \in \Omega} q^{\mathcal{J}(\xi)} (1-q)^{N-\mathcal{J}(\xi)} V_N(\xi).$$

By defining the risk-neutral probabilities in the binomial model:

$$(5) \quad \mathbb{Q}(\xi) = q^{\mathcal{J}(\xi)} (1-q)^{N-\mathcal{J}(\xi)}, \text{ for all } \xi = (\xi_1, \dots \xi_N) \in \Omega,$$

where $\mathcal{J}(\xi) = \#$ of u in $\xi = (\xi_1 \dots \xi_N)$,

we can write (4) in the following form

$$(6) \quad \boxed{V_0 = e^{-Nr\delta t} \cdot \mathbb{E}_{\mathbb{Q}} [V_N]},$$

and we can write

$$V_i = e^{-(N-i)r\delta t} \cdot \mathbb{E}_{\mathbb{Q}} [V_N | \mathcal{F}_i],$$

or by multiplying by $e^{-ir\delta t}$, we have

$$(7) \quad \boxed{e^{-ir\delta t} V_i = \mathbb{E}_{\mathbb{Q}} [e^{-Nr\delta t} V_N | \mathcal{F}_i].}$$

Next, we introduce the concept of <u>martingale</u>, and we will see that the processes $\{e^{-ir\delta t} V_i, e^{-ir\delta t} S_i : i = 0, 1, \dots, N\}$ are martingales with respect to the filtration $\{\mathcal{F}_i : i=0,1,\dots,N\}$.

Basic Martingale Theory

def: Let $\{M_n : n\in\mathbb{N}\}$ be a stochastic process adapted to the filtration $\{\mathcal{F}_n : n\in\mathbb{N}\}$. We say that M_n is a martingale with respect to the filtration $\mathcal{F}_n$ if:

(a) $\mathbb{E}[|M_n|] < \infty$, for all $n\in\mathbb{N}$;

(b) $\mathbb{E}[M_n | \mathcal{F}_m] = M_m$, for all $m \leq n$.

def (Conditional expectation with respect to a σ-algebra) Let $(\Omega, \mathbb{P})$ a prob. space. Let $X : \Omega \to \mathbb{R}$ be a random variable and $\mathcal{F}$ a σ-algebra on $(\Omega, \mathbb{P})$. Then $\mathbb{E}[X|\mathcal{F}]$ is the unique random variable Z, adapted to $\mathcal{F}$, such that, for all sets $A\in\mathcal{F}$,

$$\int_A X(\omega)\, d\mathbb{P}(\omega) = \int_A Z(\omega)\, d\mathbb{P}(\omega).$$

def: (σ-algebra generated by random variables)
Let $X_0, X_1, \dots, X_n$ be random variables on a probability space $(\Omega, \mathbb{P})$. Then, the σ-algebra generated by $X_0, \dots, X_n$ is denoted by

$$\sigma(X_0, X_1, \dots, X_n)$$

and is generated by the sets of the form

$$\{ X_i^{-1}([a,b]) \ , i=0,\dots,N \},$$

where $[a,b]$ is any interval in $\mathbb{R}$.

Remark: Usually, we will consider martingales with respect to a filtration $\mathcal{F}_n = \sigma(X_0, X_1, \dots, X_n)$ generated by $X_0, \dots, X_n$. In this case

$$\mathbb{E}[X | \mathcal{F}_n] = \mathbb{E}[X | X_0, X_1, \dots, X_n],$$

that is the definition of conditional expectation with respect to a σ-algebra, $\mathcal{F}_n$, and the cond. exp. with respect to a sequence of r.v. coincide. (32)

Example 1: (about filtration)

We consider the binomial model with N periods. Let $\{\mathcal{F}_i : i=0,1,\dots,N\}$ be the filtration introduced on page 23 of the hand-written lecture notes, and let $\{S_i : i=0,1,\dots,N\}$ be the binomial model for stock prices introduced on page 22 of the h-w. lecture notes. We want to show that

$$\mathcal{F}_i = \sigma(S_0, S_1, \dots, S_i), \text{ for all } i=0,1,\dots,N.$$

$i=0$: $\mathcal{F}_0 = \{\phi, \Omega\}$, $S_0 =$ constant. Recall from the example on page 16 of the hand-written lect. notes that $\sigma(S_0) = \{\phi, \Omega\}$, and so $\mathcal{F}_0 = \sigma(S_0)$

For arbitrary i, $\mathcal{F}_i$ is generated by the sets $A_{\hat{\omega}_i} = \{(\hat{\omega}_i\, \xi_{i+1} \cdots \xi_N) : \xi_{i+1}, \dots, \xi_N \in \{u,d\}\}$. Notice that any set $A_{\hat{\omega}_i}$ can be obtained from $S_0, S_1, \dots, S_i$ in the following way:

$$\{\omega \in \Omega : S_1 = \hat{\omega}_1 S_0,\ S_2 = \hat{\omega}_1\hat{\omega}_2 S_0, \dots, S_i = \hat{\omega}_1\hat{\omega}_2\cdots\hat{\omega}_i S_0\}$$

$$= \{S_1^{-1}(\hat{\omega}_1 S_0),\ S_2^{-1}(\hat{\omega}_1\hat{\omega}_2 S_0), \dots, S_i^{-1}(\hat{\omega}_1\hat{\omega}_2\cdots\hat{\omega}_i S_0)\},$$

where $\hat{\omega}_i = (\hat{\omega}_1 \dots \hat{\omega}_i)$. This shows that $\boxed{\mathcal{F}_i \subseteq \sigma(S_0, S_1, \dots, S_i)}$.

The reverse inclusion, $\sigma(S_0, S_1, \dots, S_i) \subseteq \mathcal{F}_i$, follows from the following

Lemma 2: If $X_0, X_1, \dots, X_m : \Omega \to \mathbb{R}$ are random variables adapted to the σ-algebra $\mathcal{F}$, then

$$\sigma(X_0, X_1, \dots, X_m) \subseteq \mathcal{F}.$$

In the example on the bottom of page 24 we have shown that S_1 is adapted to $\mathcal{F}_1$, but any S_i is adapted to $\mathcal{F}_i$. Therefore, any $S_0, S_1, \dots, S_i$ is adapted to $\mathcal{F}_i$, and the preceding lemma shows that $\boxed{\sigma(S_0, S_1, \dots, S_i) \subseteq \mathcal{F}}$ ∎

Lemma 3: Let $\{M_n : n \in \mathbb{N}\}$ be a stochastic process adapted to the filtration $\{\mathcal{F}_n : n \in \mathbb{N}\}$. Then M_n is a martingale if and only if

(a) $\mathbb{E}[|M_n|] < \infty$, for all n.

(b') $\mathbb{E}[M_{n+1} | \mathcal{F}_n] = M_n$, for all n.

Example 4: $\{e^{-ir\delta t} S_i : i = 0, \ldots, N\}$ is a martingale w.r.t. $\{\mathcal{F}_i : i = 0, 1, \ldots, N\}$, where S_i and $\mathcal{F}_i$ are as in the binomial model. We check (b') with $n = i$: Let

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-(i+1)r\delta t} S_{i+1} \,|\, \mathcal{F}_i\right] = Z$$

Recall $S_{i+1}(\omega) = S_{i+1}(\hat{\omega}_{i+1}) = \begin{cases} u\, S_i(\hat{\omega}_i), & \text{if } \omega_{i+1} = u, \\ d\, S_i(\hat{\omega}_i), & \text{if } \omega_{i+1} = d. \end{cases}$

Then,
$$Z(\hat{\omega}_i) = \left(u \cdot S_i(\hat{\omega}_i) \cdot \mathbb{Q}(u) + d \cdot S_i(\hat{\omega}_i) \cdot \mathbb{Q}(d)\right) \cdot e^{-(i+1)r\delta t}$$
$$= S_i(\hat{\omega}_i)\,(u \cdot q + d(1-q)) \cdot e^{-r\delta t} \cdot e^{-ir\delta t}.$$

Using the identity (check this!)

$$(uq + d(1-q))\, e^{-r\delta t} = 1,$$

we obtain that

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-(i+1)r\delta t} S_{i+1} \,|\, \mathcal{F}_i\right] = Z = e^{-ir\delta t} S_i$$

and so, $\{e^{-ir\delta t} S_i : i = 0, \ldots, N\}$ is a martingale w.r.t. $\{\mathcal{F}_i : i = 0, \ldots, N\}$.

Theorem 5: Let M_n be a martingale with respect to $\mathcal{F}_n$ and let $\{\phi_n : n \in \mathbb{N}\}$ be a predictable process w.r.t. $\mathcal{F}_n$, then

$$G_n = G_0 + \sum_{k=1}^{n} \phi_k (M_k - M_{k-1})$$

is a martingale w.r.t. $\mathcal{F}_n$.

Example 6: We use Theorem 5 to show that $\{e^{-ir\delta t} V_i : i=0,1,\dots,N\}$ (the discounted value of the price of a deriv. security at time t_i) is a martingale w.r.t. $\{\mathcal{F}_i : i=0,\dots,N\}$ in the binomial model.

We know that $e^{-ir\delta t} S_i$ and $e^{-ir\delta t} B_i \equiv 1$ (because $B_i = e^{ir\delta t}$) are martingales.

Let ϕ_i^1 = # of share of S_i, and ϕ_i^1 = # of units of B_i, be the predictable process introduced on page 25 of the hand-written notes. Then, we have shown that

$$V_i(\omega) = \phi_i^1(\omega)\, S_i(\omega) + \phi_i^2(\omega)\, B_i(\omega) \quad \Big|\, e^{-ir\delta t} \Rightarrow$$

$$e^{-ir\delta t} V_i = \phi_i^1 \; e^{-ir\delta t} S_i + \phi_i^2 \cdot e^{-ir\delta t} B_i$$

By construction, the trading strategy is self-financing, and so

$$V_{i-1} = \phi_i^1 \; S_{i-1} + \phi_i^2 \; B_{i-1} \quad \Big|\cdot e^{-(i-1)r\delta t} \Rightarrow$$

$$e^{-(i-1)r\delta t} V_{i-1} = \phi_i^1 \; e^{-(i-1)r\delta t} + \phi_i^2 \; e^{-(i-1)r\delta t} B_{i-1}$$

Therefore, by denoting $G_i = e^{-ir\delta t} V_i$, $M_i = e^{-ir\delta t} S_i$, and $N_i = e^{-ir\delta t} B_i$, we have

$$G_i = G_{i-1} + \phi_i^1 (M_i - M_{i-1}) + \phi_i^2 (N_i - N_{i-1}), \text{ which implies}$$

$$G_i = G_0 + \sum_{k=1}^{i} \phi_k^1 (M_k - M_{k-1}) + \sum_{k=1}^{i} \phi_k^2 (N_k - N_{k-1})$$

Since $\{\phi_i^{1,2}\}$ is predictable and $\{M_i\}, \{N_i\}$ are martingales, it follows by Theorem 5 that $G_i = e^{-ir\delta t} V_i$ is a martingale.

Dynamic portfolio selection

We consider a market consisting of a stock, S, and a bond, B, with trading times t_i, $i=0,1,\dots,N$, $t_i = i\,\delta t$, and time horizon $T = N\,\delta t$.

We have an initial endowment, W_0, and we want to invest it in the stock and bond, so that we maximize the expected value of such a portfolio under the real-world measure.

Let $\mathbb{P}$ be the real-world probability and $\mathbb{Q}$ be the risk-neutral probability.

Let Π_i be the value of our portfolio at time t_i, such that $\Pi_0 = W_0$.

We want to maximize not quite $\mathbb{E}_{\mathbb{P}}[\Pi_N]$ (the expected value of the portfolio at expiry $t_N = T$, under the real-world measure), but we want to maximize $\mathbb{E}_{\mathbb{P}}[U(\Pi_N)]$, where U is an utility function which accounts for the risk-exposure when we invest in the market.

Properties of utility function:

(1) non-decreasing: $U' \geq 0$ (the larger Π_N, the better)

(2) concavity: $U'' \leq 0$.

Examples:
$U(x) = \ln x$;
$U(x) = \frac{x^p}{p}$,
for $0 \neq p < 1$

Problem:

maximize $\mathbb{E}_{\mathbb{P}}[U(\Pi_N)]$ over all self-financing portfolios $\{\Pi_i : i=0,1,\dots,N\}$ subject to the constraint $\Pi_0 = W_0$ (fixed $\to$ initial endowment)

Solution: Assume $U(x) = \ln x$.

Because $\{\Pi_i : i=0,1,\dots,N\}$ is a self-financing strategy, we may view Π_N the payoff of a deriv. sec. subject to the constraint $\Pi_0 = W_0$.

Then, from the risk-neutral pricing formula we know

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-rT}\Pi_N\right] = \Pi_0 = W_0.$$

Now, denote Π_N by h. So we need to solve:

$$\text{maximize } \mathbb{E}_{\mathbb{P}}[U(h)] \text{ where } \mathbb{E}_{\mathbb{Q}}\left[e^{-rT}h\right] = W_0.$$

We want to replace $\mathbb{E}_{\mathbb{Q}}$ by $\mathbb{E}_{\mathbb{P}}$, and for that we use the following

Theorem 1: Assume $\mathbb{P}$ and $\mathbb{Q}$ are probability measures on the same sample space, and for any event E on the sample space we have:

$$\text{if } \mathbb{P}(E)=0, \text{ then } \mathbb{Q}(E)=0.$$

Then there is a random variable ξ ($\xi := \frac{d\mathbb{Q}}{d\mathbb{P}}$ the Radon Nikodym deriv.) such that for any r.v. X we have

$$\mathbb{E}_{\mathbb{Q}}[X] = \mathbb{E}_{\mathbb{P}}[\xi X].$$

Therefore, we may write $\mathbb{E}_{\mathbb{Q}}[e^{-rT} h] = \mathbb{E}_{\mathbb{P}}[e^{-rT} \xi h] = W_0$.

To solve the maximization problem we apply the method of Lagrange multipliers. Consider the Lagrange function:

$$\varphi(h,\lambda) = \mathbb{E}_{\mathbb{P}}[\ln h] + \lambda \cdot \left(\mathbb{E}_{\mathbb{P}}[e^{-rT}\xi \cdot h] - W_0\right).$$

where λ = Lagrange multiplier. The solution to the maximization problem will be a stationary point, (h_0,λ_0), of φ, that is

$$\begin{cases} \frac{\partial}{\partial h}\varphi(h_0,\lambda_0) = 0 \\ \frac{\partial}{\partial \lambda}\varphi(h_0,\lambda_0) = 0 \end{cases}$$

To find "$\frac{\partial}{\partial h}\varphi(h,\lambda)$", take any r.v. v and $\varepsilon > 0$ and compute:

$$\lim_{\varepsilon\to 0} \frac{\varphi(h+\varepsilon v,\lambda) - \varphi(h,\lambda)}{\varepsilon} \overset{\text{use } U = \ln}{=} \lim_{\varepsilon \to 0} \mathbb{E}_{\mathbb{P}}\left[\frac{\ln(h+\varepsilon v) - \ln h}{\varepsilon}\right] +$$

$$+ \lambda\, \mathbb{E}_{\mathbb{P}}\left[e^{-rT}\xi\, \frac{(h+\varepsilon v - h)}{\varepsilon}\right] = \mathbb{E}_{\mathbb{P}}\left[\frac{1}{h}\cdot v\right] + \lambda\, \mathbb{E}_{\mathbb{P}}\left[e^{-rT}\xi \cdot v\right] =$$

$$= \mathbb{E}_{\mathbb{P}}\left[\left(\frac{1}{h} + \lambda e^{-rT}\xi\right)\cdot v\right] = \bar{0}$$

Hence, $h_0 = -\dfrac{e^{rT}}{\lambda_0 \xi}$.

Using the constraint $\mathbb{E}_{\mathbb{P}}[e^{-rT}\xi h_0] = W_0$, we obtain $\lambda_0 = -\frac{1}{W_0}$, and so

$$\boxed{h_0 = W_0 e^{rT}/\xi.}$$